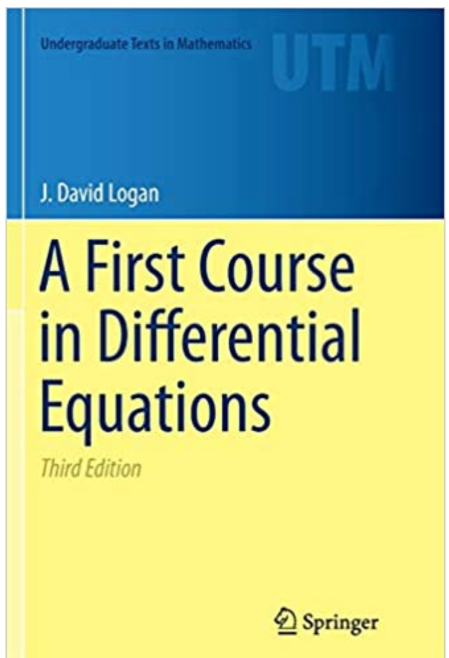
A Solution Manual For

A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.



Nasser M. Abbasi

May 15, 2024

Contents

1 Chapter 1, First order differential equations. Section 1.1 First order equations. Exercises page 10 3

2 Chapter 1, First order differential equations. Section 1.1.3 Geometric. Exercises page 15 93

3 Chapter 1, First order differential equations. Section 1.2 Antiderivatives. Exercises page 19 101

4 Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26 171

5 Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41 501

6 Chapter 2, Second order linear equations. Section 2.2.2 Real eigenvalues. Exercises page 90 929

7 Chapter 2, Second order linear equations. Section 2.2.3 Complex eigenvalues. Exercises page 94 1032

8 Chapter 2, Second order linear equations. Section 2.2.4. Applications. Exercises page 99 1101

9 Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110 1124

10 Chapter 2, Second order linear equations. Section 2.3.2 Resonance Exercises page 114 1424

11 Chapter 2, Second order linear equations. Section 2.4.1 Cauchy-Euler equations. Exercises page 120 1467

12 Chapter 2, Second order linear equations. Section 2.4.2 Variation of parameters. Exercises page 124 1619

13 Chapter 2, Second order linear equations. Section 2.4.3 Reduction of order. Exercises page 125 1765

14 Chapter 2, Second order linear equations. Section 2.5 Higher order equations. Exercises page 130 **1787**

15 Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156 **1835**

16 Chapter 3, Laplace transform. Section 3.3 The convolution property. Exercises page 162 **1910**

17 Chapter 3, Laplace transform. Section 3.4 Impulsive sources. Exercises page 173 **1923**

18 Chapter 4, Linear Systems. Exercises page 190 **1966**

19 Chapter 4, Linear Systems. Exercises page 202 **2037**

20 Chapter 4, Linear Systems. Exercises page 218 **2118**

21 Chapter 4, Linear Systems. Exercises page 225 **2171**

22 Chapter 4, Linear Systems. Exercises page 237 **2189**

23 Chapter 4, Linear Systems. Exercises page 244 **2277**

1 Chapter 1, First order differential equations. Section 1.1 First order equations. Exercises page 10

1.1 problem 1(a) 4
1.2 problem 1(b) 19
1.3 problem 3 34
1.4 problem 4 37
1.5 problem 5 45
1.6 problem 6 48
1.7 problem 7 61
1.8 problem 8 76
1.9 problem 9 85

1.1 problem 1(a)

1.1.1 Solving as separable ode 4
1.1.2 Solving as linear ode 6
1.1.3 Solving as homogeneousTypeD2 ode 7
1.1.4 Solving as first order ode lie symmetry lookup ode 9
1.1.5 Solving as exact ode 13
1.1.6 Maple step by step solution 17

Internal problem ID [11349]
Internal file name [`OUTPUT/10331_Wednesday_May_17_2023_07_49_17_PM_75656588/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.1 First order equations. Exercises page 10
Problem number: 1(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' - \frac{2x}{t} = 0}$$

1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= f(t)g(x) \\ &= \frac{2x}{t} \end{aligned}$$

Where $f(t) = \frac{2}{t}$ and $g(x) = x$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{x}\,dx &= \frac{2}{t}\,dt \\
\int \frac{1}{x}\,dx &= \int \frac{2}{t}\,dt \\
\ln(x) &= 2\ln(t) + c_1 \\
x &= \mathrm{e}^{2\ln(t)+c_1} \\
&= c_1 t^2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1 t^2 \tag{1}
$$

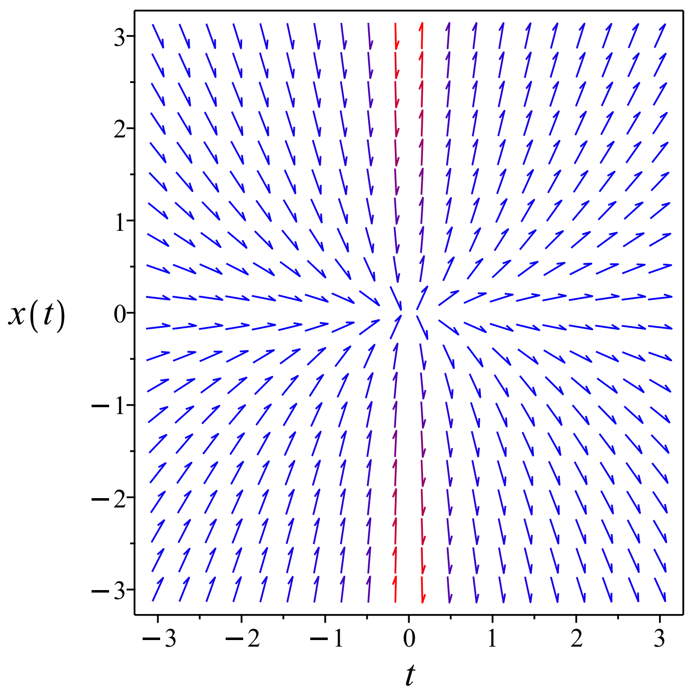


Figure 1: Slope field plot

Verification of solutions

$$
x = c_1 t^2
$$

Verified OK.

1.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= -\frac{2}{t} \\ q(t) &= 0 \end{aligned}$$

Hence the ode is

$$x' - \frac{2x}{t} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{2}{t}dt} \\ &= \frac{1}{t^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mu x &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{t^2}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{x}{t^2} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$x = c_1 t^2$$

Summary
The solution(s) found are the following

$$x = c_1 t^2 \tag{1}$$

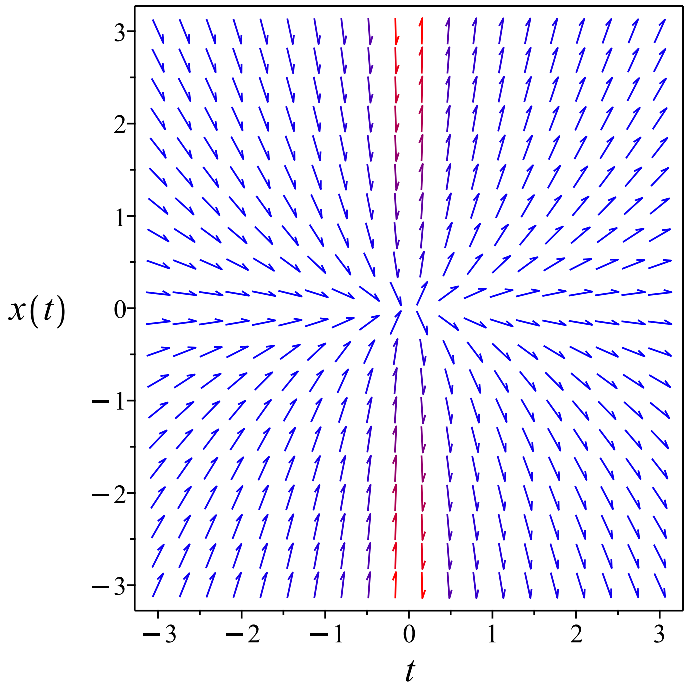


Figure 2: Slope field plot

Verification of solutions

$$x = c_1 t^2$$

Verified OK.

1.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $x = u(t)\, t$ on the above ode results in new ode in $u(t)$

$$u'(t)\, t - u(t) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t) g(u) \\ &= \frac{u}{t} \end{aligned}$$

Where $f(t) = \frac{1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{1}{t}\,dt \\
\int \frac{1}{u}\,du &= \int \frac{1}{t}\,dt \\
\ln(u) &= \ln(t) + c_2 \\
u &= \mathrm{e}^{\ln(t)+c_2} \\
&= c_2 t
\end{aligned}$$

Therefore the solution x is

$$\begin{aligned}
x &= tu \\
&= t^2 c_2
\end{aligned}$$

Summary

The solution(s) found are the following

$$x = t^2 c_2 \tag{1}$$

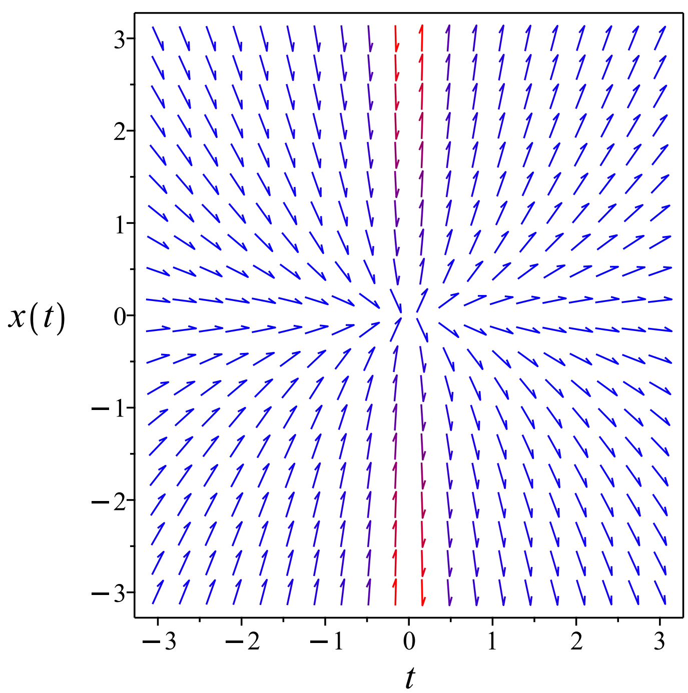


Figure 3: Slope field plot

Verification of solutions

$$x = t^2 c_2$$

Verified OK.

1.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = \frac{2x}{t}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= t^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{t^2} dy \end{aligned}$$

Which results in

$$S = \frac{x}{t^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = \frac{2x}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= -\frac{2x}{t^3} \\ S_x &= \frac{1}{t^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\frac{x}{t^2} = c_1$$

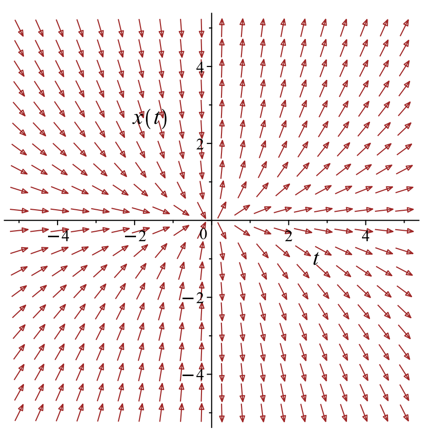
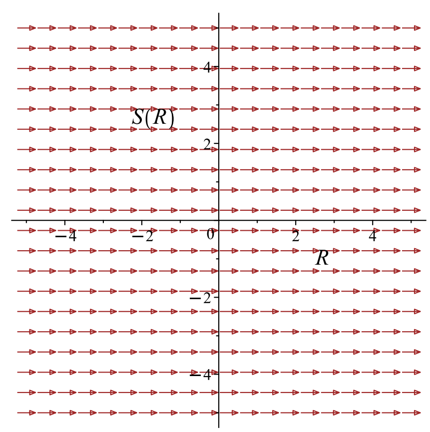
Which simplifies to

$$\frac{x}{t^2} = c_1$$

Which gives

$$x = c_1 t^2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \frac{2x}{t}$ 	$R = t$ $S = \frac{x}{t^2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$x = c_1 t^2 \tag{1}$$

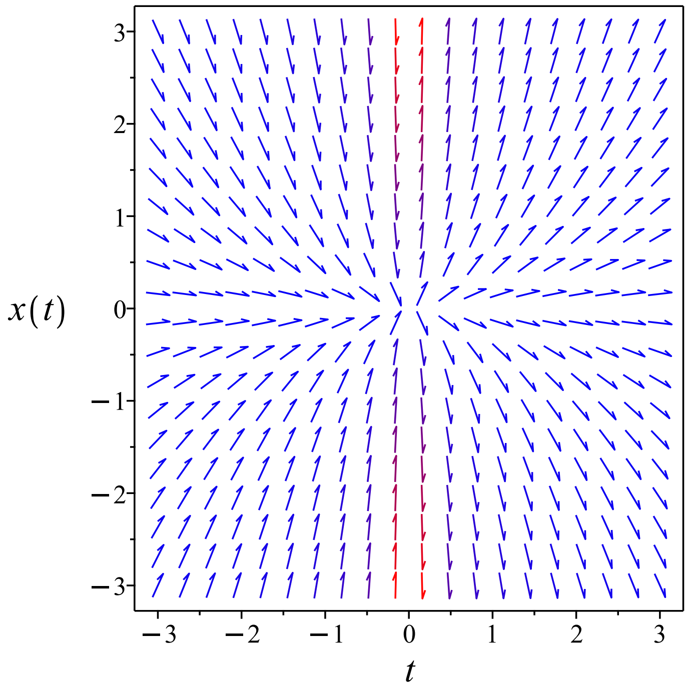


Figure 4: Slope field plot

Verification of solutions

$$x = c_1 t^2$$

Verified OK.

1.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{2x}\right)\mathrm{d}x &= \left(\frac{1}{t}\right)\mathrm{d}t\\ \left(-\frac{1}{t}\right)\mathrm{d}t + \left(\frac{1}{2x}\right)\mathrm{d}x &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, x) &= -\frac{1}{t}\\ N(t, x) &= \frac{1}{2x}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{t}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{2x}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{1}{t}\,\mathrm{d}t\\ \phi &= -\ln(t) + f(x) \qquad (3)\end{aligned}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{2x}$. Therefore equation (4) becomes

$$\frac{1}{2x} = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = \frac{1}{2x}$$

Integrating the above w.r.t x gives

$$\begin{aligned}\int f'(x)\,\mathrm{d}x &= \int \left(\frac{1}{2x}\right)\mathrm{d}x\\ f(x) &= \frac{\ln(x)}{2} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\ln(t) + \frac{\ln(x)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(t) + \frac{\ln(x)}{2}$$

The solution becomes

$$x = e^{2c_1} t^2$$

<u>Summary</u>
The solution(s) found are the following

$$x = e^{2c_1} t^2 \tag{1}$$

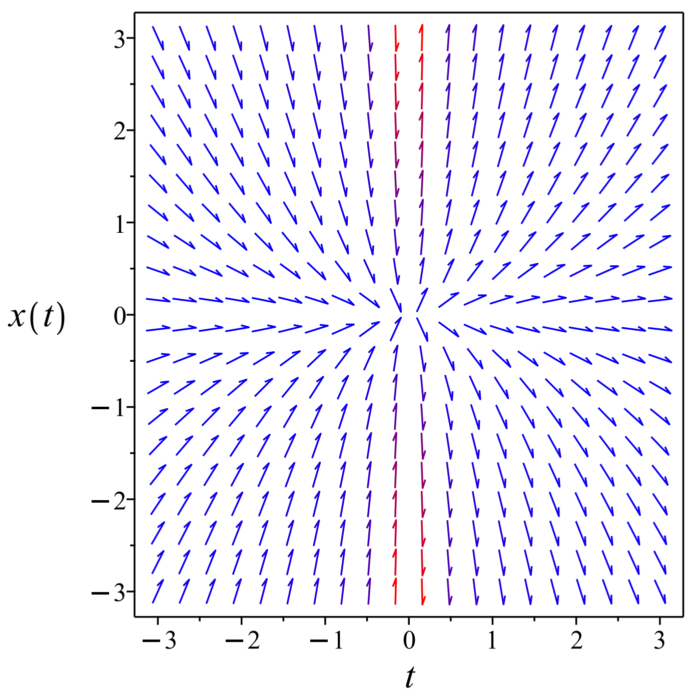


Figure 5: Slope field plot

<u>Verification of solutions</u>

$$x = e^{2c_1} t^2$$

Verified OK.

1.1.6 Maple step by step solution

Let's solve

$x' - \frac{2x}{t} = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x} = \frac{2}{t}$

- Integrate both sides with respect to t

  $\int \frac{x'}{x} dt = \int \frac{2}{t} dt + c_1$

- Evaluate integral

  $\ln(x) = 2\ln(t) + c_1$

- Solve for x

  $x = e^{c_1} t^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(x(t),t)=2*x(t)/t,x(t), singsol=all)
```

$$x(t) = c_1 t^2$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 16

```
DSolve[x'[t]==2*x[t]/t,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 t^2$$
$$x(t) \to 0$$

1.2 problem 1(b)

1.2.1 Solving as separable ode 19
1.2.2 Solving as homogeneousTypeD2 ode 21
1.2.3 Solving as differentialType ode 23
1.2.4 Solving as first order ode lie symmetry lookup ode 24
1.2.5 Solving as exact ode 28
1.2.6 Maple step by step solution 32

Internal problem ID [11350]
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Section: Chapter 1, First order differential equations. Section 1.1 First order equations. Exercises page 10
Problem number: 1(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{x' + \frac{t}{x} = 0}$$

1.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= f(t)g(x) \\ &= -\frac{t}{x} \end{aligned}$$

Where $f(t) = -t$ and $g(x) = \frac{1}{x}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{x}}\,dx &= -t\,dt \\
\int \frac{1}{\frac{1}{x}}\,dx &= \int -t\,dt \\
\frac{x^2}{2} &= -\frac{t^2}{2} + c_1
\end{aligned}
$$

Which results in

$$
\begin{aligned}
x &= \sqrt{-t^2 + 2c_1} \\
x &= -\sqrt{-t^2 + 2c_1}
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$x = \sqrt{-t^2 + 2c_1} \tag{1}$$
$$x = -\sqrt{-t^2 + 2c_1} \tag{2}$$

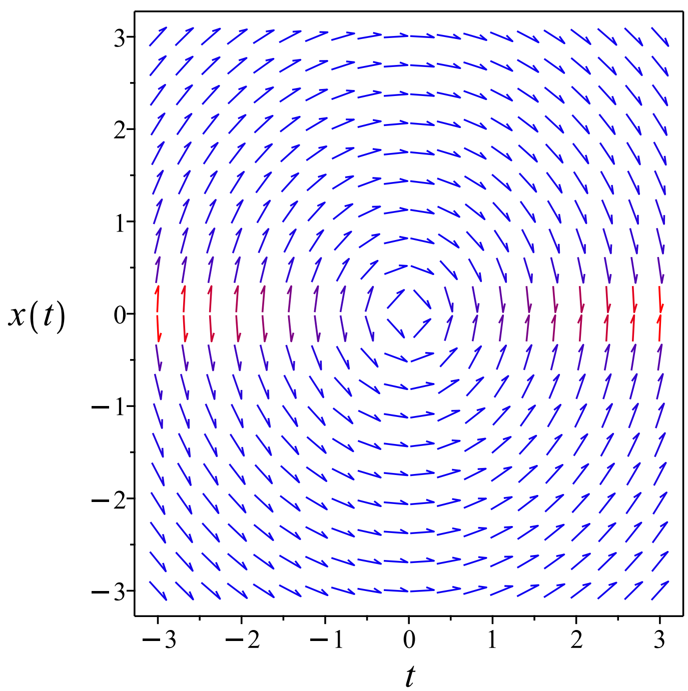


Figure 6: Slope field plot

Verification of solutions

$$x = \sqrt{-t^2 + 2c_1}$$

Verified OK.

$$x = -\sqrt{-t^2 + 2c_1}$$

Verified OK.

1.2.2 Solving as homogeneousTypeD2 ode

Using the change of variables $x = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u'(t)\,t + u(t) + \frac{1}{u\,(t)} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{u^2 + 1}{tu} \end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(u) = \frac{u^2+1}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2+1}{u}}\,du &= -\frac{1}{t}\,dt \\ \int \frac{1}{\frac{u^2+1}{u}}\,du &= \int -\frac{1}{t}\,dt \\ \frac{\ln\left(u^2+1\right)}{2} &= -\ln\left(t\right) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\sqrt{u^2 + 1} = \mathrm{e}^{-\ln(t) + c_2}$$

Which simplifies to

$$\sqrt{u^2 + 1} = \frac{c_3}{t}$$

Which simplifies to

$$\sqrt{u\,(t)^2 + 1} = \frac{c_3 \mathrm{e}^{c_2}}{t}$$

The solution is

$$\sqrt{u(t)^2+1} = \frac{c_3 e^{c_2}}{t}$$

Replacing $u(t)$ in the above solution by $\frac{x}{t}$ results in the solution for x in implicit form

$$\sqrt{\frac{x^2}{t^2}+1} = \frac{c_3 e^{c_2}}{t}$$
$$\sqrt{\frac{x^2+t^2}{t^2}} = \frac{c_3 e^{c_2}}{t}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{x^2+t^2}{t^2}} = \frac{c_3 e^{c_2}}{t} \tag{1}$$

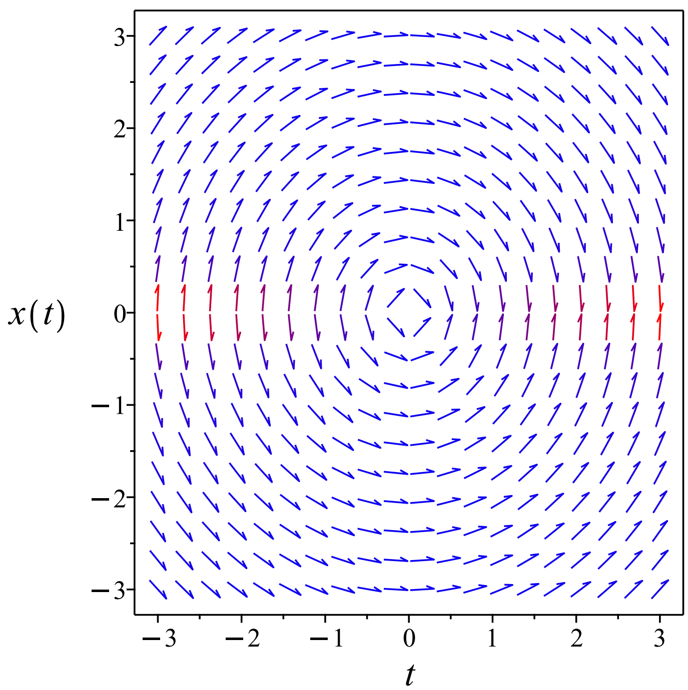


Figure 7: Slope field plot

Verification of solutions

$$\sqrt{\frac{x^2+t^2}{t^2}} = \frac{c_3 e^{c_2}}{t}$$

Verified OK.

1.2.3 Solving as differentialType ode

Writing the ode as

$$x' = -\frac{t}{x} \tag{1}$$

Which becomes

$$(x)\,dx = (-t)\,dt \tag{2}$$

But the RHS is complete differential because

$$(-t)\,dt = d\left(-\frac{t^2}{2}\right)$$

Hence (2) becomes

$$(x)\,dx = d\left(-\frac{t^2}{2}\right)$$

Integrating both sides gives gives these solutions

$$x = \sqrt{-t^2+2c_1}+c_1$$
$$x = -\sqrt{-t^2+2c_1}+c_1$$

<u>Summary</u>
The solution(s) found are the following

$$x = \sqrt{-t^2+2c_1}+c_1 \tag{1}$$
$$x = -\sqrt{-t^2+2c_1}+c_1 \tag{2}$$

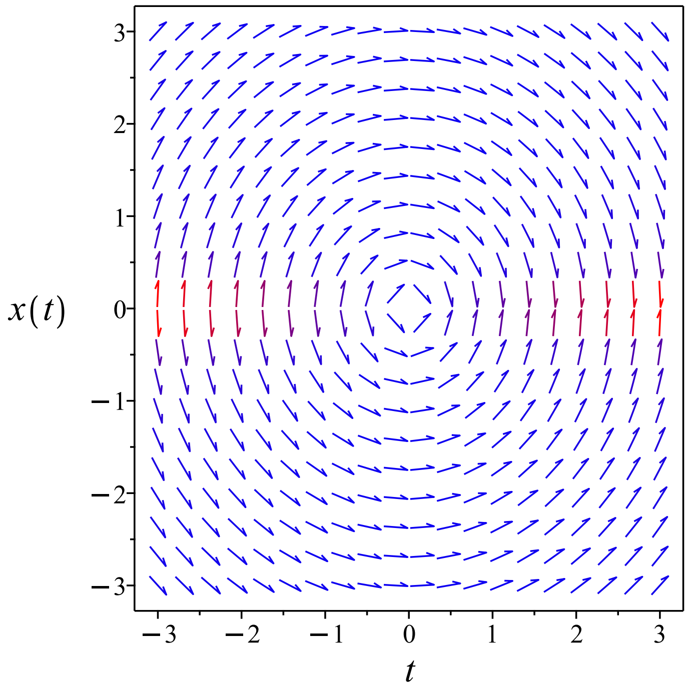


Figure 8: Slope field plot

<u>Verification of solutions</u>

$$
x = \sqrt{-t^2 + 2c_1} + c_1
$$

Verified OK.

$$
x = -\sqrt{-t^2 + 2c_1} + c_1
$$

Verified OK.

1.2.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= -\frac{t}{x} \\
x' &= \omega(t, x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 4: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= -\frac{1}{t} \\
\eta(t, x) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{-\frac{1}{t}} dt \end{aligned}$$

Which results in

$$S = -\frac{t^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -\frac{t}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_x &= 1 \\ S_t &= -t \\ S_x &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

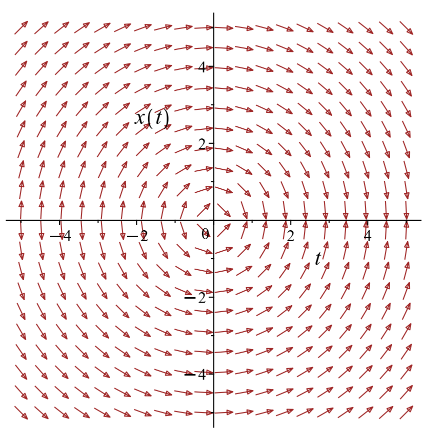
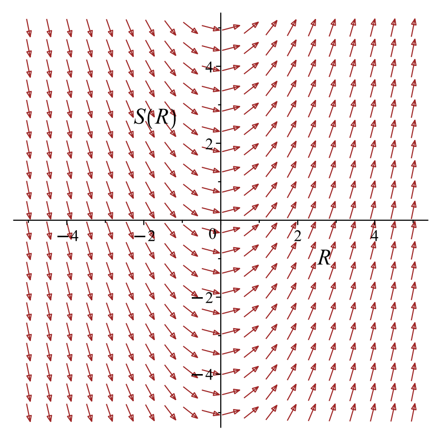
To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$-\frac{t^2}{2} = \frac{x^2}{2} + c_1$$

Which simplifies to

$$-\frac{t^2}{2} = \frac{x^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\frac{t}{x}$ 	$R = x$ $S = -\frac{t^2}{2}$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$-\frac{t^2}{2} = \frac{x^2}{2} + c_1 \tag{1}$$

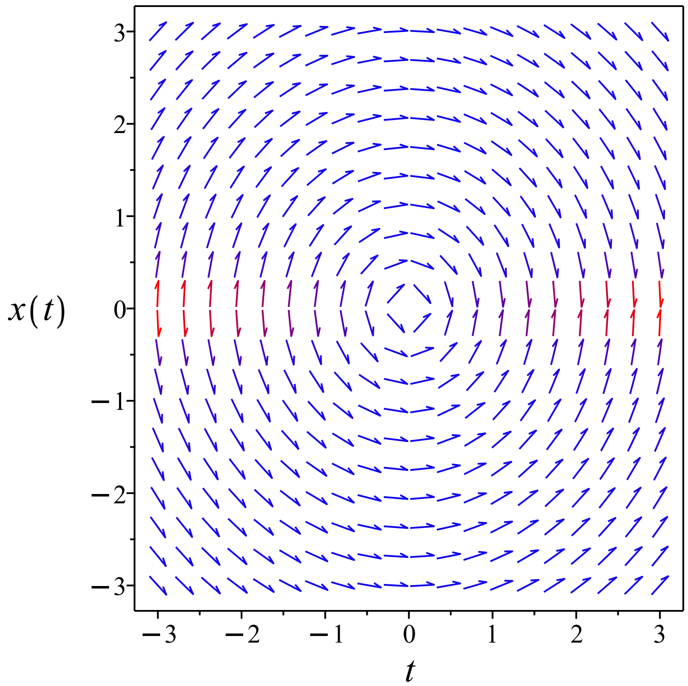


Figure 9: Slope field plot

<u>Verification of solutions</u>

$$-\frac{t^2}{2} = \frac{x^2}{2} + c_1$$

Verified OK.

1.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, x)\, \mathrm{d}t + N(t, x)\, \mathrm{d}x = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(-x)\, \mathrm{d}x &= (t)\, \mathrm{d}t \\
(-t)\, \mathrm{d}t + (-x)\, \mathrm{d}x &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, x) &= -t \\
N(t, x) &= -x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-t) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(-x) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int M\, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int -t\, \mathrm{d}t$$

$$\phi = -\frac{t^2}{2} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = -x$. Therefore equation (4) becomes

$$-x = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -x$$

Integrating the above w.r.t x gives

$$\int f'(x)\, \mathrm{d}x = \int (-x)\, \mathrm{d}x$$

$$f(x) = -\frac{x^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} - \frac{x^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} - \frac{x^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{t^2}{2} - \frac{x^2}{2} = c_1 \tag{1}$$

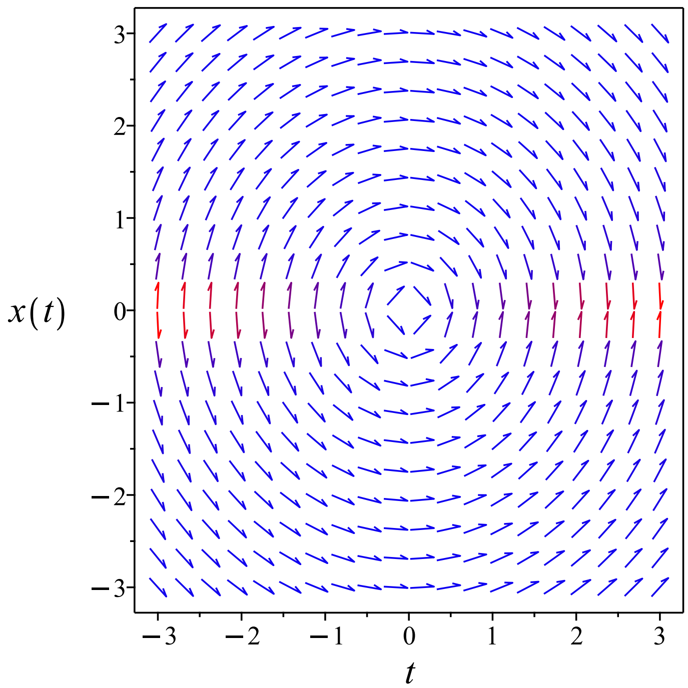


Figure 10: Slope field plot

Verification of solutions

$$-\frac{t^2}{2} - \frac{x^2}{2} = c_1$$

Verified OK.

1.2.6 Maple step by step solution

Let's solve

$x' + \frac{t}{x} = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $x'x = -t$

- Integrate both sides with respect to t

  $\int x'x dt = \int -t dt + c_1$

- Evaluate integral

  $\frac{x^2}{2} = -\frac{t^2}{2} + c_1$

- Solve for x

  $\left\{x = \sqrt{-t^2 + 2c_1}, x = -\sqrt{-t^2 + 2c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(x(t),t)=-t/x(t),x(t), singsol=all)
```

$$x(t) = \sqrt{-t^2 + c_1}$$
$$x(t) = -\sqrt{-t^2 + c_1}$$

✓ Solution by Mathematica

Time used: 0.139 (sec). Leaf size: 39

```
DSolve[x'[t]==-t/x[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\sqrt{-t^2 + 2c_1}$$
$$x(t) \to \sqrt{-t^2 + 2c_1}$$

1.3 problem 3

1.3.1 Solving as quadrature ode 34
1.3.2 Maple step by step solution 35

Internal problem ID [11351]
Internal file name [`OUTPUT/10333_Wednesday_May_17_2023_07_49_20_PM_54653337/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.1 First order equations. Exercises page 10
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' + x^2 = 0}$$

1.3.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{x^2} dx = t + c_1$$
$$\frac{1}{x} = t + c_1$$

Solving for x gives these solutions

$$x_1 = \frac{1}{t + c_1}$$

<u>Summary</u>
The solution(s) found are the following

$$x = \frac{1}{t + c_1} \tag{1}$$

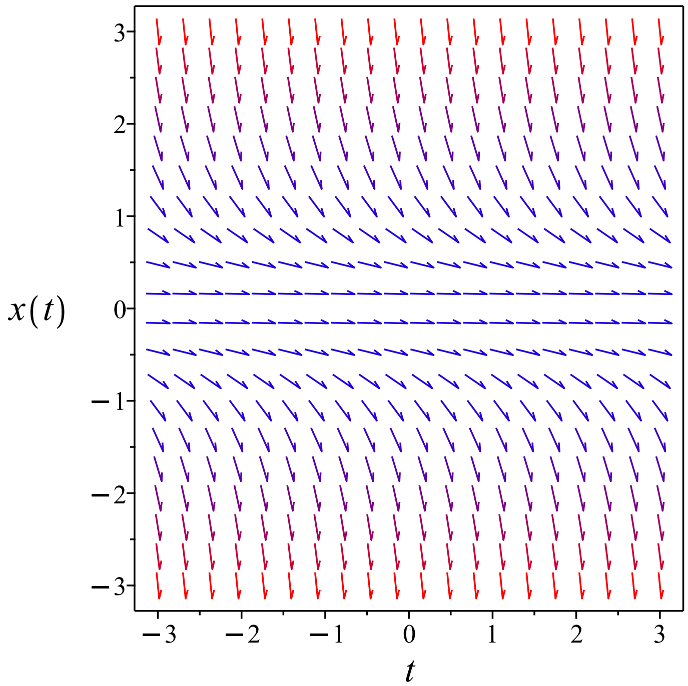


Figure 11: Slope field plot

<u>Verification of solutions</u>

$$x = \frac{1}{t + c_1}$$

Verified OK.

1.3.2 Maple step by step solution

Let's solve

$x' + x^2 = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x^2} = -1$

- Integrate both sides with respect to t

  $\int \frac{x'}{x^2} dt = \int (-1)\, dt + c_1$

- Evaluate integral

$-\frac{1}{x} = -t + c_1$

- Solve for x

$x = -\frac{1}{-t+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(x(t),t)=-x(t)^2,x(t), singsol=all)
```

$$x(t) = \frac{1}{t + c_1}$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 39

```
DSolve[x'[t]==-t/x[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\sqrt{-t^2 + 2c_1}$$
$$x(t) \to \sqrt{-t^2 + 2c_1}$$

1.4 problem 4

1.4.1 Solving as second order linear constant coeff ode 37
1.4.2 Solving using Kovacic algorithm 39
1.4.3 Maple step by step solution 43

Internal problem ID [11352]
Internal file name [`OUTPUT/10334_Wednesday_May_17_2023_07_49_20_PM_27323432/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.1 First order equations. Exercises page 10
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' + 2x' + 2x = 0}$$

1.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 2, C = 2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda\, e^{\lambda t} + 2\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(2)} \\ &= -1 \pm i\end{aligned}$$

Hence

$$\lambda_1 = -1 + i$$
$$\lambda_2 = -1 - i$$

Which simplifies to

$$\lambda_1 = -1 + i$$
$$\lambda_2 = -1 - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-t}(c_1 \cos(t) + c_2 \sin(t))$$

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{-t}(c_1 \cos(t) + c_2 \sin(t)) \tag{1}$$

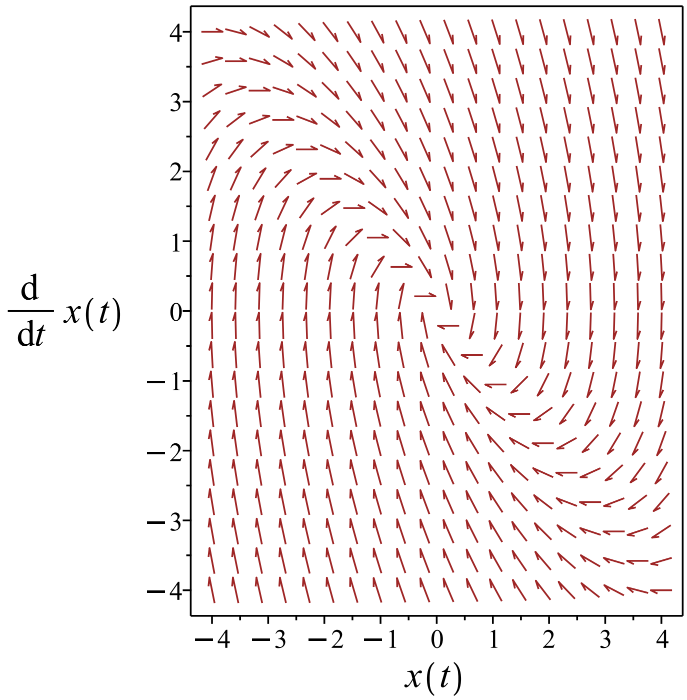


Figure 12: Slope field plot

Verification of solutions

$$x = e^{-t}(c_1 \cos(t) + c_2 \sin(t))$$

Verified OK.

1.4.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 2x' + 2x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 8: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt} \\ &= z_1 e^{-t} \\ &= z_1\left(\mathrm{e}^{-t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t}\cos(t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{-2t}}{(x_1)^2}\,dt \\ &= x_1(\tan(t)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-t} \cos(t)\right) + c_2 \left(\mathrm{e}^{-t} \cos(t) (\tan(t))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1 \mathrm{e}^{-t} \cos(t) + c_2 \mathrm{e}^{-t} \sin(t) \tag{1}
$$

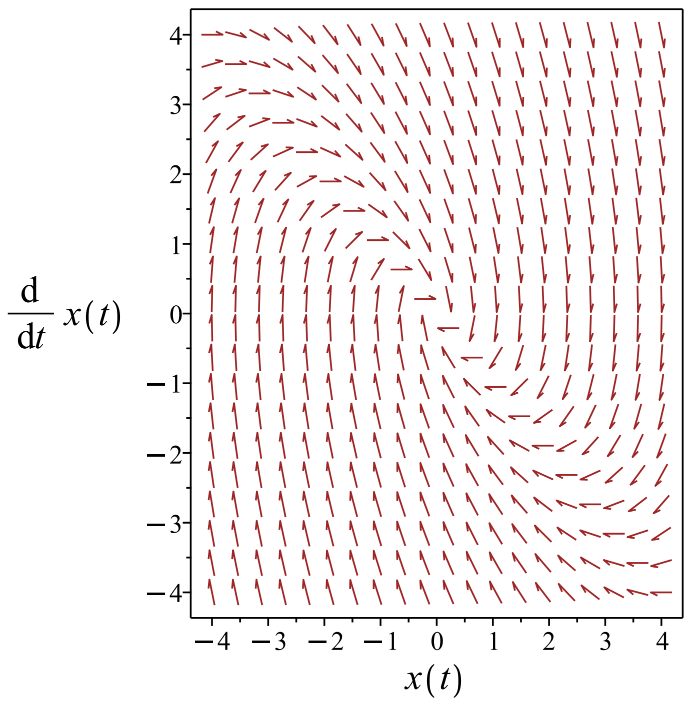


Figure 13: Slope field plot

Verification of solutions

$$
x = c_1 \mathrm{e}^{-t} \cos(t) + c_2 \mathrm{e}^{-t} \sin(t)
$$

Verified OK.

1.4.3 Maple step by step solution

Let's solve

$x'' + 2x' + 2x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - \mathrm{I}, -1 + \mathrm{I})$

- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^{-t} \cos(t)$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{-t} \sin(t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 \mathrm{e}^{-t} \cos(t) + c_2 \mathrm{e}^{-t} \sin(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(x(t),t$2)+2*diff(x(t),t)+2*x(t)=0,x(t), singsol=all)
```

$$x(t) = e^{-t}(c_1 \sin(t) + c_2 \cos(t))$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 22

```
DSolve[x''[t]+2*x'[t]+2*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-t}(c_2 \cos(t) + c_1 \sin(t))$$

1.5 problem 5

1.5.1 Solving as quadrature ode 45
1.5.2 Maple step by step solution 46

Internal problem ID [11353]
Internal file name [`OUTPUT/10335_Wednesday_May_17_2023_07_49_21_PM_79597876/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.1 First order equations. Exercises page 10
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' - \mathrm{e}^{-x} = 0}$$

1.5.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int \mathrm{e}^{x} dx &= t + c_1 \\ \mathrm{e}^{x} &= t + c_1\end{aligned}$$

Solving for x gives these solutions

$$x_1 = \ln(t + c_1)$$

Summary
The solution(s) found are the following

$$x = \ln(t + c_1) \tag{1}$$

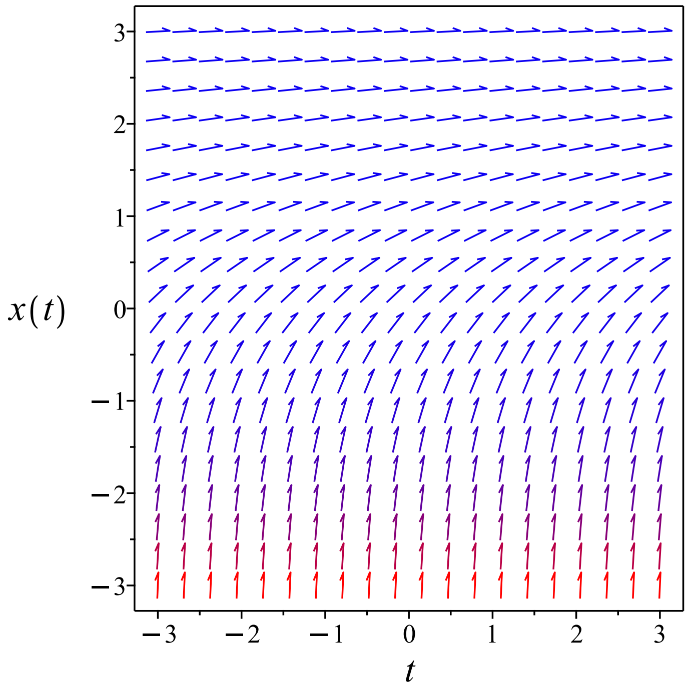


Figure 14: Slope field plot

Verification of solutions

$$x = \ln(t + c_1)$$

Verified OK.

1.5.2 Maple step by step solution

Let's solve

$x' - \mathrm{e}^{-x} = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{\mathrm{e}^{-x}} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{\mathrm{e}^{-x}} dt = \int 1 dt + c_1$

- Evaluate integral

$$\frac{1}{e^{-x}} = t + c_1$$

- Solve for x

$$x = \ln(t + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(x(t),t)=exp(-x(t)),x(t), singsol=all)
```

$$x(t) = \ln(t + c_1)$$

✓ Solution by Mathematica

Time used: 0.394 (sec). Leaf size: 10

```
DSolve[x'[t]==Exp[-x[t]],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \log(t + c_1)$$

1.6 problem 6

1.6.1 Solving as linear ode 48
1.6.2 Solving as first order ode lie symmetry lookup ode 50
1.6.3 Solving as exact ode 54
1.6.4 Maple step by step solution 58

Internal problem ID [11354]
Internal file name [`OUTPUT/10336_Wednesday_May_17_2023_07_49_22_PM_5441983/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.1 First order equations. Exercises page 10
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{x' + 2x = t^2 + 4t + 7}$$

1.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = 2$$
$$q(t) = t^2 + 4t + 7$$

Hence the ode is

$$x' + 2x = t^2 + 4t + 7$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 2dt} \\
&= e^{2t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu x) &= (\mu)\left(t^2+4t+7\right) \\
\frac{d}{dt}\left(e^{2t}x\right) &= \left(e^{2t}\right)\left(t^2+4t+7\right) \\
d\left(e^{2t}x\right) &= \left(\left(t^2+4t+7\right)e^{2t}\right)\, dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{2t}x &= \int \left(t^2+4t+7\right)e^{2t}\, dt \\
e^{2t}x &= \frac{(2t^2+6t+11)\, e^{2t}}{4} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{2t}$ results in

$$
x = \frac{e^{-2t}(2t^2+6t+11)\, e^{2t}}{4} + c_1 e^{-2t}
$$

which simplifies to

$$
x = \frac{t^2}{2} + \frac{3t}{2} + \frac{11}{4} + c_1 e^{-2t}
$$

Summary

The solution(s) found are the following

$$
x = \frac{t^2}{2} + \frac{3t}{2} + \frac{11}{4} + c_1 e^{-2t} \tag{1}
$$

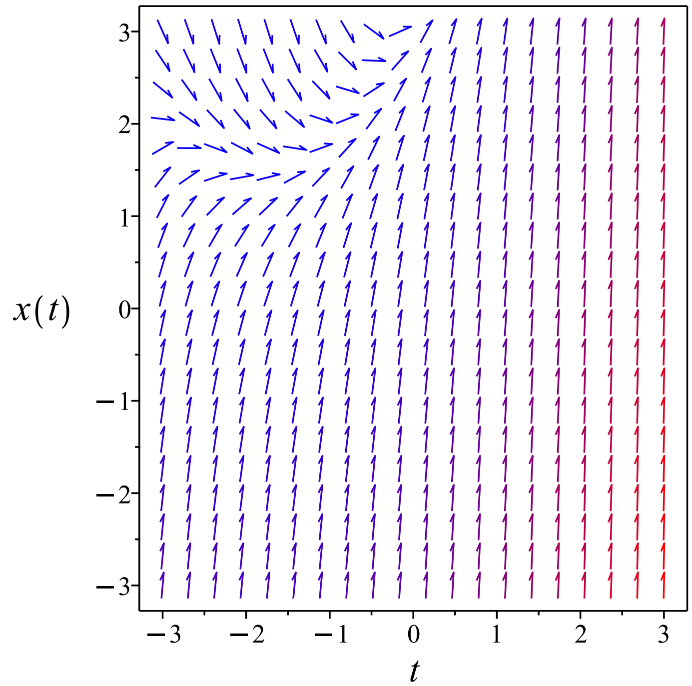


Figure 15: Slope field plot

Verification of solutions

$$x = \frac{t^2}{2} + \frac{3t}{2} + \frac{11}{4} + c_1 e^{-2t}$$

Verified OK.

1.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = t^2 + 4t - 2x + 7$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 11: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \mathrm{e}^{-2t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-2t}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{2t} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = t^2 + 4t - 2x + 7$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= 2\,\mathrm{e}^{2t} x \\ S_x &= \mathrm{e}^{2t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \left(t^2 + 4t + 7\right) \mathrm{e}^{2t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \left(R^2 + 4R + 7\right) \mathrm{e}^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{(2R^2 + 6R + 11)\, e^{2R}}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$e^{2t} x = \frac{(2t^2 + 6t + 11)\, e^{2t}}{4} + c_1$$

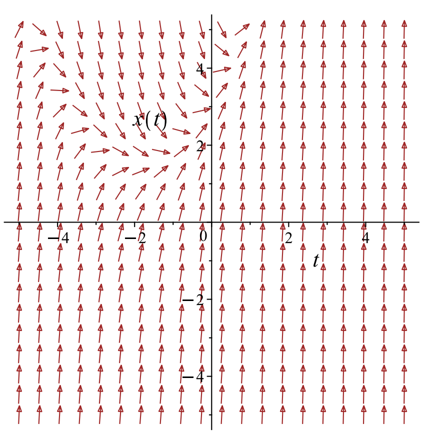
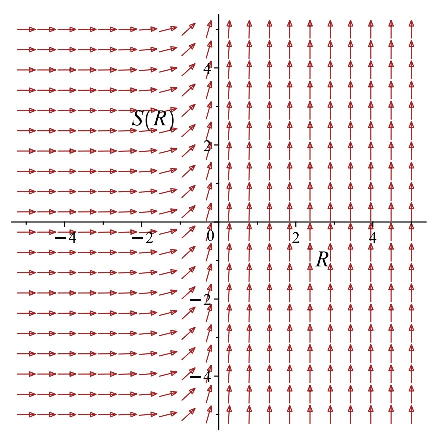
Which simplifies to

$$e^{2t} x = \frac{(2t^2 + 6t + 11)\, e^{2t}}{4} + c_1$$

Which gives

$$x = \frac{(2\, e^{2t} t^2 + 6\, e^{2t} t + 11\, e^{2t} + 4c_1)\, e^{-2t}}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = t^2 + 4t - 2x + 7$ 	$R = t$ $S = e^{2t} x$	$\frac{dS}{dR} = (R^2 + 4R + 7)\, e^{2R}$ 

Summary

The solution(s) found are the following

$$x = \frac{(2\, e^{2t} t^2 + 6\, e^{2t} t + 11\, e^{2t} + 4c_1)\, e^{-2t}}{4} \tag{1}$$

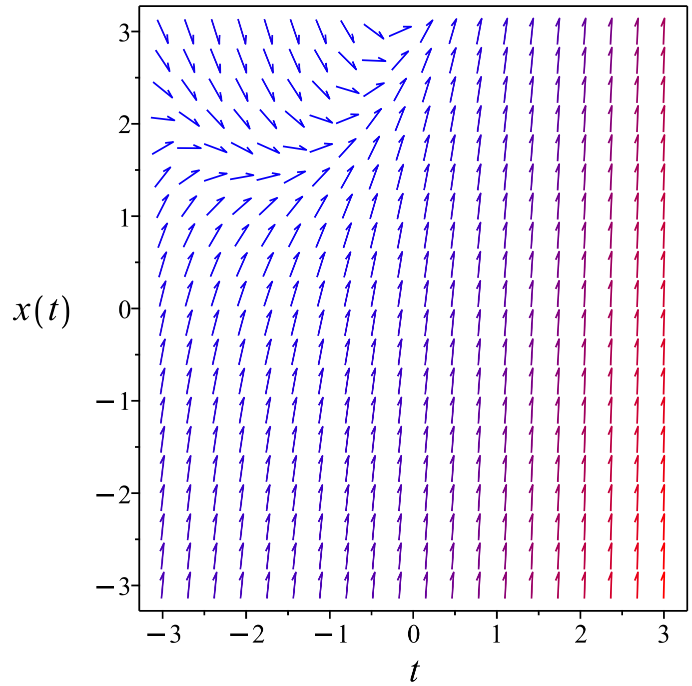


Figure 16: Slope field plot

Verification of solutions

$$x = \frac{(2\,\mathrm{e}^{2t}t^2 + 6\,\mathrm{e}^{2t}t + 11\,\mathrm{e}^{2t} + 4c_1)\,\mathrm{e}^{-2t}}{4}$$

Verified OK.

1.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}x &= \left(t^2 + 4t - 2x + 7\right)\mathrm{d}t \\
\left(-t^2 - 4t + 2x - 7\right)\mathrm{d}t + \mathrm{d}x &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t,x) &= -t^2 - 4t + 2x - 7 \\
N(t,x) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-t^2 - 4t + 2x - 7\right) \\
&= 2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\ &= 1((2) - (0)) \\ &= 2 \end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \,\mathrm{d}t} \\ &= e^{\int 2 \,\mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{2t} \\ &= \mathrm{e}^{2t} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{2t}\left(-t^2 - 4t + 2x - 7\right) \\ &= -\mathrm{e}^{2t}\left(t^2 + 4t - 2x + 7\right) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{2t}(1) \\ &= \mathrm{e}^{2t} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ \left(-\mathrm{e}^{2t}\left(t^2 + 4t - 2x + 7\right)\right) + \left(\mathrm{e}^{2t}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\mathrm{e}^{2t}\left(t^2+4t-2x+7\right)\mathrm{d}t \\
\phi &= -\frac{\mathrm{e}^{2t}\left(t^2+3t-2x+\frac{11}{2}\right)}{2}+f(x) \qquad (3)
\end{aligned}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \mathrm{e}^{2t} + f'(x) \qquad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \mathrm{e}^{2t}$. Therefore equation (4) becomes

$$\mathrm{e}^{2t} = \mathrm{e}^{2t} + f'(x) \qquad (5)$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{\mathrm{e}^{2t}\left(t^2+3t-2x+\frac{11}{2}\right)}{2}+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\mathrm{e}^{2t}\left(t^2+3t-2x+\frac{11}{2}\right)}{2}$$

The solution becomes

$$x = \frac{\left(2\,\mathrm{e}^{2t}t^2+6\,\mathrm{e}^{2t}t+11\,\mathrm{e}^{2t}+4c_1\right)\mathrm{e}^{-2t}}{4}$$

Summary

The solution(s) found are the following

$$x = \frac{(2\,\mathrm{e}^{2t}t^2 + 6\,\mathrm{e}^{2t}t + 11\,\mathrm{e}^{2t} + 4c_1)\,\mathrm{e}^{-2t}}{4} \tag{1}$$

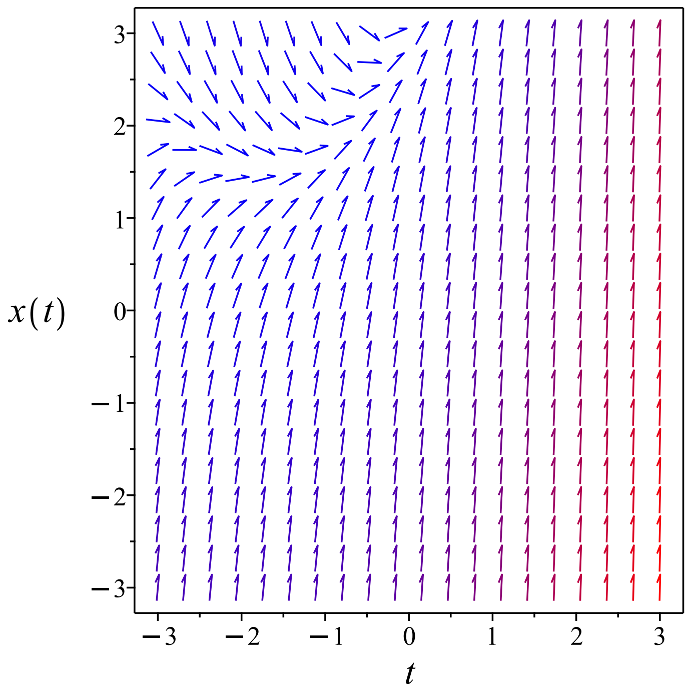


Figure 17: Slope field plot

Verification of solutions

$$x = \frac{(2\,\mathrm{e}^{2t}t^2 + 6\,\mathrm{e}^{2t}t + 11\,\mathrm{e}^{2t} + 4c_1)\,\mathrm{e}^{-2t}}{4}$$

Verified OK.

1.6.4 Maple step by step solution

Let's solve

$x' + 2x = t^2 + 4t + 7$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

$x' = -2x + t^2 + 4t + 7$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + 2x = t^2 + 4t + 7$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(x' + 2x\right) = \mu(t)\left(t^2 + 4t + 7\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}\left(\mu(t)\, x\right)$

  $\mu(t)\left(x' + 2x\right) = \mu'(t)\, x + \mu(t)\, x'$

- Isolate $\mu'(t)$

  $\mu'(t) = 2\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{2t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}\left(\mu(t)\, x\right)\right) dt = \int \mu(t)\left(t^2 + 4t + 7\right) dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, x = \int \mu(t)\left(t^2 + 4t + 7\right) dt + c_1$

- Solve for x

  $x = \frac{\int \mu(t)\left(t^2+4t+7\right) dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{2t}$

  $x = \frac{\int \left(t^2+4t+7\right)\mathrm{e}^{2t} dt + c_1}{\mathrm{e}^{2t}}$

- Evaluate the integrals on the rhs

  $x = \frac{\frac{\left(2t^2+6t+11\right)\mathrm{e}^{2t}}{4} + c_1}{\mathrm{e}^{2t}}$

- Simplify

  $x = \frac{t^2}{2} + \frac{3t}{2} + \frac{11}{4} + c_1 \mathrm{e}^{-2t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(x(t),t)+2*x(t)=t^2+4*t+7,x(t), singsol=all)
```

$$x(t) = \frac{t^2}{2} + \frac{3t}{2} + \frac{11}{4} + e^{-2t}c_1$$

✓ Solution by Mathematica

Time used: 0.119 (sec). Leaf size: 28

```
DSolve[x'[t]+2*x[t]==t^2+4*t+7,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{4}(2t^2 + 6t + 11) + c_1 e^{-2t}$$

1.7 problem 7

1.7.1 Solving as separable ode 61
1.7.2 Solving as linear ode 63
1.7.3 Solving as homogeneousTypeD2 ode 64
1.7.4 Solving as first order ode lie symmetry lookup ode 66
1.7.5 Solving as exact ode 70
1.7.6 Maple step by step solution 74

Internal problem ID [11355]
Internal file name [`OUTPUT/10337_Wednesday_May_17_2023_07_49_23_PM_37563748/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.1 First order equations. Exercises page 10
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{2tx' - x = 0}$$

1.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= f(t)g(x) \\ &= \frac{x}{2t} \end{aligned}$$

Where $f(t) = \frac{1}{2t}$ and $g(x) = x$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{x}\,dx &= \frac{1}{2t}\,dt \\
\int \frac{1}{x}\,dx &= \int \frac{1}{2t}\,dt \\
\ln(x) &= \frac{\ln(t)}{2} + c_1 \\
x &= \mathrm{e}^{\frac{\ln(t)}{2}+c_1} \\
&= c_1\sqrt{t}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
x = c_1\sqrt{t} \tag{1}
$$

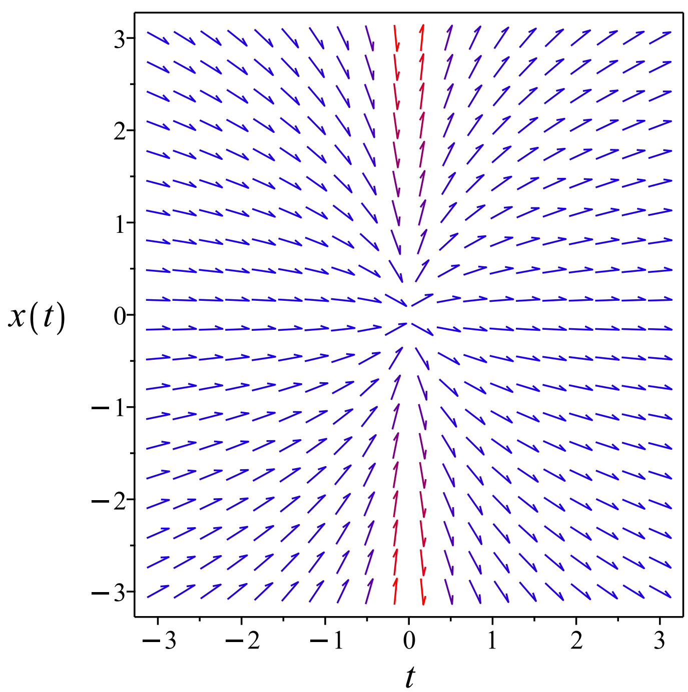


Figure 18: Slope field plot

Verification of solutions

$$
x = c_1\sqrt{t}
$$

Verified OK.

1.7.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned}p(t) &= -\frac{1}{2t}\\ q(t) &= 0\end{aligned}$$

Hence the ode is

$$x' - \frac{x}{2t} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{2t}dt}\\ &= \frac{1}{\sqrt{t}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu x &= 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{\sqrt{t}}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{x}{\sqrt{t}} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{t}}$ results in

$$x = c_1\sqrt{t}$$

Summary

The solution(s) found are the following

$$x = c_1\sqrt{t} \tag{1}$$

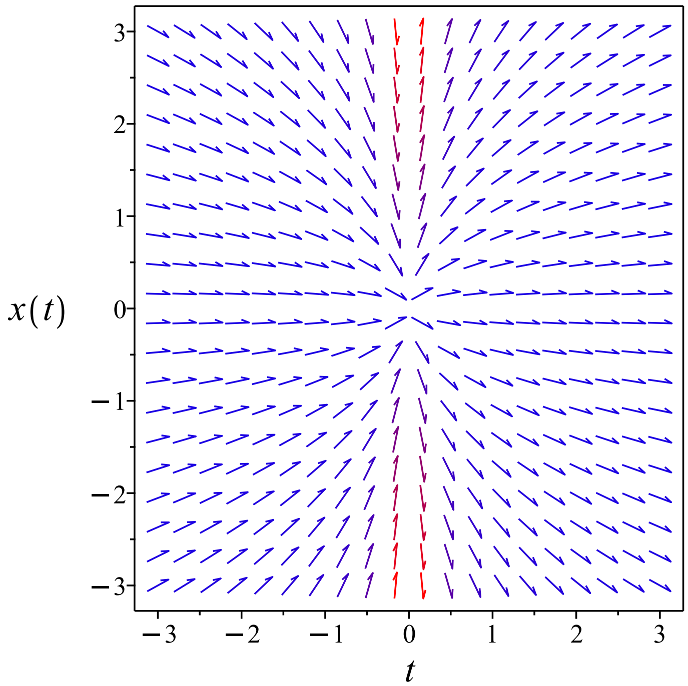


Figure 19: Slope field plot

Verification of solutions

$$x = c_1\sqrt{t}$$

Verified OK.

1.7.3 Solving as homogeneousTypeD2 ode

Using the change of variables $x = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$2t(u'(t)\,t + u(t)) - u(t)\,t = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= -\frac{u}{2t} \end{aligned}$$

Where $f(t) = -\frac{1}{2t}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{1}{2t}\, dt \\
\int \frac{1}{u}\, du &= \int -\frac{1}{2t}\, dt \\
\ln(u) &= -\frac{\ln(t)}{2} + c_2 \\
u &= \mathrm{e}^{-\frac{\ln(t)}{2}+c_2} \\
&= \frac{c_2}{\sqrt{t}}
\end{aligned}
$$

Therefore the solution x is

$$
\begin{aligned}
x &= ut \\
&= \sqrt{t}\, c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = \sqrt{t}\, c_2 \tag{1}
$$

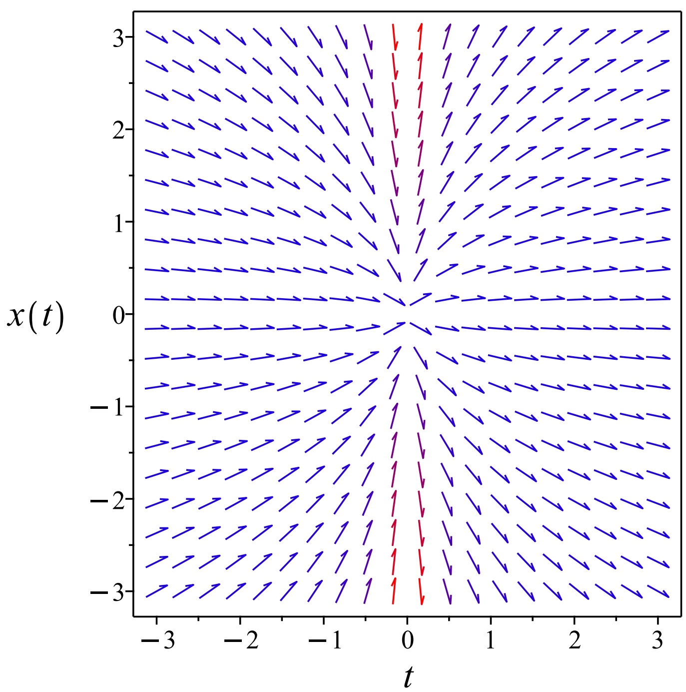


Figure 20: Slope field plot

Verification of solutions

$$x = \sqrt{t}\, c_2$$

Verified OK.

1.7.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = \frac{x}{2t}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 14: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \sqrt{t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{t}} dy \end{aligned}$$

Which results in

$$S = \frac{x}{\sqrt{t}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = \frac{x}{2t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= -\frac{x}{2t^{\frac{3}{2}}} \\ S_x &= \frac{1}{\sqrt{t}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\frac{x}{\sqrt{t}} = c_1$$

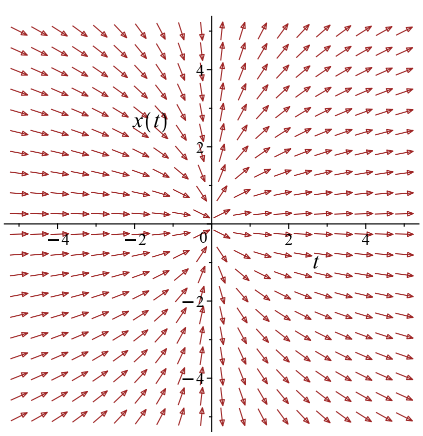
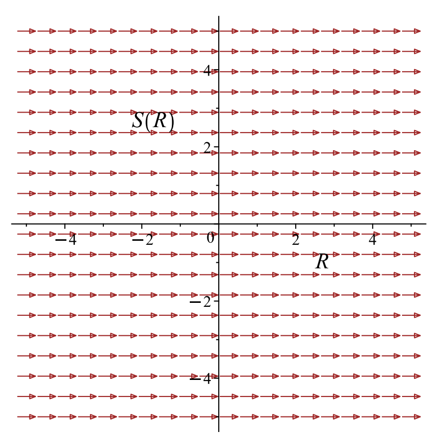
Which simplifies to

$$\frac{x}{\sqrt{t}} = c_1$$

Which gives

$$x = c_1 \sqrt{t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \frac{x}{2t}$ 	$R = t$ $S = \frac{x}{\sqrt{t}}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$x = c_1 \sqrt{t} \tag{1}$$

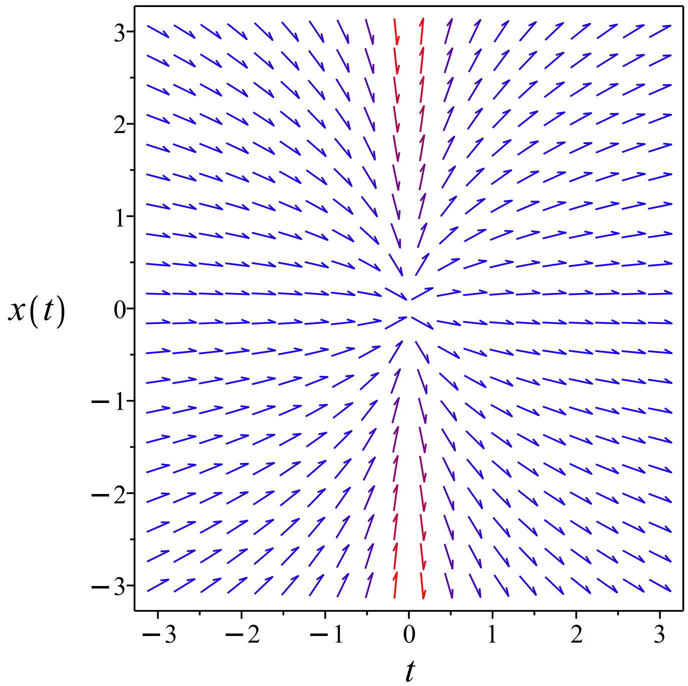


Figure 21: Slope field plot

<u>Verification of solutions</u>

$$x = c_1\sqrt{t}$$

Verified OK.

1.7.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{2}{x}\right) \mathrm{d}x &= \left(\frac{1}{t}\right) \mathrm{d}t\\ \left(-\frac{1}{t}\right) \mathrm{d}t + \left(\frac{2}{x}\right) \mathrm{d}x &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, x) &= -\frac{1}{t}\\ N(t, x) &= \frac{2}{x}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{t}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{2}{x}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{1}{t}\,\mathrm{d}t\end{aligned}$$

$$\phi = -\ln(t) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{2}{x}$. Therefore equation (4) becomes

$$\frac{2}{x} = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = \frac{2}{x}$$

Integrating the above w.r.t x gives

$$\begin{aligned}\int f'(x)\,\mathrm{d}x &= \int \left(\frac{2}{x}\right)\mathrm{d}x\\ f(x) &= 2\ln(x) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\ln(t) + 2\ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(t) + 2\ln(x)$$

The solution becomes

$$x = \mathrm{e}^{\frac{\ln(t)}{2}+\frac{c_1}{2}}$$

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{\frac{\ln(t)}{2}+\frac{c_1}{2}} \tag{1}$$

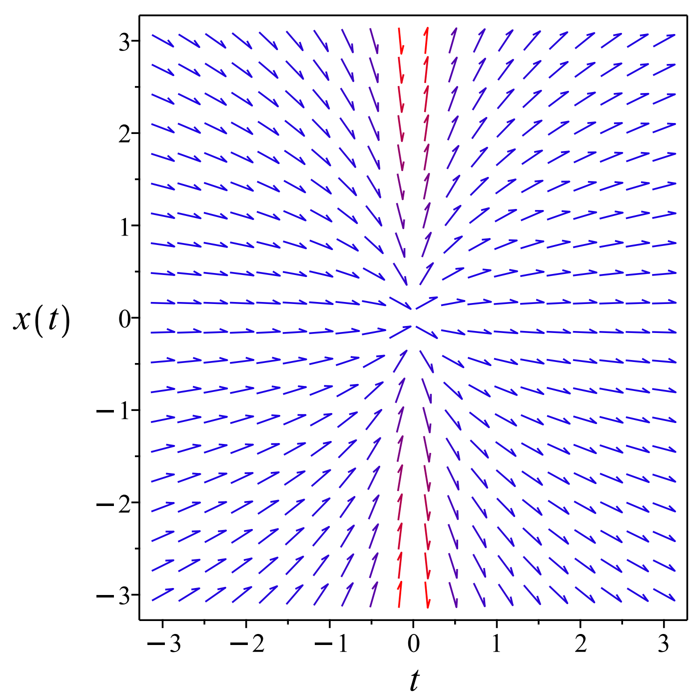


Figure 22: Slope field plot

Verification of solutions

$$x = \mathrm{e}^{\frac{\ln(t)}{2}+\frac{c_1}{2}}$$

Verified OK.

1.7.6 Maple step by step solution

Let's solve

$2tx' - x = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x} = \frac{1}{2t}$

- Integrate both sides with respect to t

  $\int \frac{x'}{x} dt = \int \frac{1}{2t} dt + c_1$

- Evaluate integral

  $\ln(x) = \frac{\ln(t)}{2} + c_1$

- Solve for x

  $\left\{x = \frac{\sqrt{\mathrm{e}^{-2c_1} t}}{\mathrm{e}^{-2c_1}}, x = -\frac{\sqrt{\mathrm{e}^{-2c_1} t}}{\mathrm{e}^{-2c_1}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(2*t*diff(x(t),t)=x(t),x(t), singsol=all)
```

$$x(t) = c_1\sqrt{t}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 18

```
DSolve[2*t*x'[t]==x[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 \sqrt{t}$$
$$x(t) \to 0$$

1.8 problem 8

1.8.1 Solving as second order euler ode ode 76
1.8.2 Solving using Kovacic algorithm 77
1.8.3 Maple step by step solution 82

Internal problem ID [11356]
Internal file name [`OUTPUT/10338_Wednesday_May_17_2023_07_49_24_PM_21883334/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.1 First order equations. Exercises page 10
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{t^2x'' - 6x = 0}$$

1.8.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 0rt^{r-1} - 6t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 0\,t^r - 6t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 0 - 6 = 0$$

Or

$$r^2 - r - 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= -2 \\
r_2 &= 3
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$
x = c_1 x_1 + c_2 x_2
$$

Where $x_1 = t^{r_1}$ and $x_2 = t^{r_2}$. Hence

$$
x = \frac{c_1}{t^2} + c_2 t^3
$$

Summary

The solution(s) found are the following

$$
x = \frac{c_1}{t^2} + c_2 t^3 \tag{1}
$$

Verification of solutions

$$
x = \frac{c_1}{t^2} + c_2 t^3
$$

Verified OK.

1.8.2 Solving using Kovacic algorithm

Writing the ode as

$$
t^2 x'' - 6x = 0 \tag{1}
$$

$$
Ax'' + Bx' + Cx = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= t^2 \\
B &= 0 \\
C &= -6
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = x e^{\int \frac{B}{2A} \, dt}
$$

Then (2) becomes

$$
z''(t) = r z(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{6}{t^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 6 \\
t &= t^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{6}{t^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 17: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6}{t^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6}{t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -2$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -2 - (-2) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{t} + (-)(0) \\ &= -\frac{2}{t} \\ &= -\frac{2}{t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{t}\right)(0) + \left(\left(\frac{2}{t^2}\right) + \left(-\frac{2}{t}\right)^2 - \left(\frac{6}{t^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int -\frac{2}{t} dt} \\
&= \frac{1}{t^2}
\end{aligned}$$

The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
x_1 &= z_1 \\
&= \frac{1}{t^2}
\end{aligned}$$

Which simplifies to

$$x_1 = \frac{1}{t^2}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{x_1^2} \, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2} \, dt \\
&= \frac{1}{t^2} \int \frac{1}{\frac{1}{t^4}} \, dt \\
&= \frac{1}{t^2} \left(\frac{t^5}{5}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\frac{1}{t^2}\right) + c_2\left(\frac{1}{t^2}\left(\frac{t^5}{5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = \frac{c_1}{t^2} + \frac{c_2 t^3}{5} \tag{1}
$$

Verification of solutions

$$
x = \frac{c_1}{t^2} + \frac{c_2 t^3}{5}
$$

Verified OK.

1.8.3 Maple step by step solution

Let's solve

$t^2 x'' - 6x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = \frac{6x}{t^2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' - \frac{6x}{t^2} = 0$

- Multiply by denominators of the ODE

  $t^2 x'' - 6x = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of x with respect to t , using the chain rule

    $x' = \left(\frac{d}{ds}x(s)\right) s'(t)$

- Compute derivative

  $x' = \frac{\frac{d}{ds}x(s)}{t}$

- Calculate the 2nd derivative of x with respect to t , using the chain rule

  $x'' = \left(\frac{d^2}{ds^2}x(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}x(s)\right)$

- Compute derivative

  $x'' = \frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $t^2 \left(\frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}\right) - 6x(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}x(s) - \frac{d}{ds}x(s) - 6x(s) = 0$

- Characteristic polynomial of ODE

  $r^2 - r - 6 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 3)$

- 1st solution of the ODE

  $x_1(s) = e^{-2s}$

- 2nd solution of the ODE

  $x_2(s) = e^{3s}$

- General solution of the ODE

  $x(s) = c_1x_1(s) + c_2x_2(s)$

- Substitute in solutions

  $x(s) = c_1e^{-2s} + c_2e^{3s}$

- Change variables back using $s = \ln(t)$

  $x = \frac{c_1}{t^2} + c_2t^3$

- Simplify

  $x = \frac{c_1}{t^2} + c_2t^3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(t^2*diff(x(t),t$2)-6*x(t)=0,x(t), singsol=all)
```

$$x(t) = \frac{c_1 t^5 + c_2}{t^2}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 18

```
DSolve[t^2*x''[t]-6*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{c_2 t^5 + c_1}{t^2}$$

1.9 problem 9

1.9.1 Solving as second order linear constant coeff ode 85
1.9.2 Solving using Kovacic algorithm 87
1.9.3 Maple step by step solution 91

Internal problem ID [11357]
Internal file name [`OUTPUT/10339_Wednesday_May_17_2023_07_49_25_PM_35823396/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.1 First order equations. Exercises page 10
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2x'' - 5x' - 3x = 0}$$

1.9.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 2, B = -5, C = -3$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$2\lambda^2 \mathrm{e}^{\lambda t} - 5\lambda\, \mathrm{e}^{\lambda t} - 3\, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$2\lambda^2 - 5\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = -5, C = -3$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{5}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{-5^2 - (4)\,(2)\,(-3)} \\
&= \frac{5}{4} \pm \frac{7}{4}
\end{aligned}$$

Hence

$$\lambda_1 = \frac{5}{4} + \frac{7}{4}$$

$$\lambda_2 = \frac{5}{4} - \frac{7}{4}$$

Which simplifies to

$$\lambda_1 = 3$$

$$\lambda_2 = -\frac{1}{2}$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$x = c_1 e^{(3)t} + c_2 e^{\left(-\frac{1}{2}\right)t}$$

Or

$$x = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-\frac{t}{2}}$$

Summary

The solution(s) found are the following

$$x = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-\frac{t}{2}} \tag{1}$$

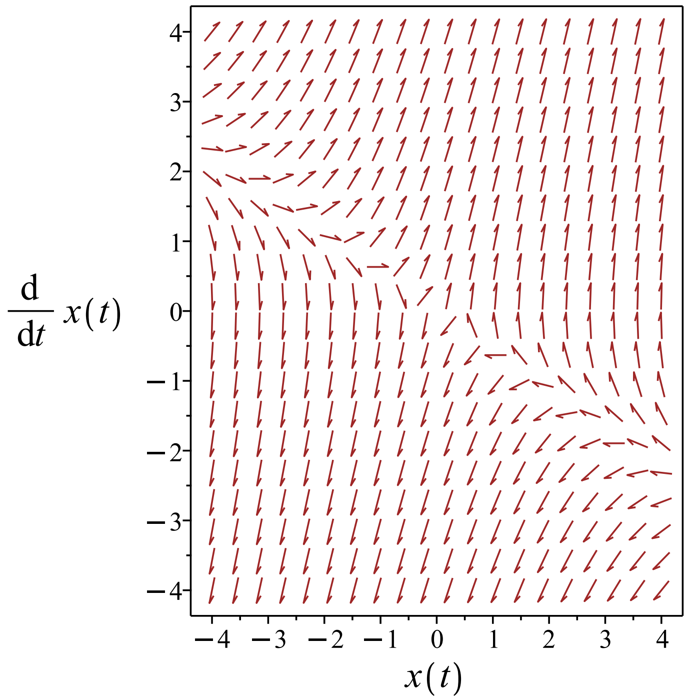


Figure 23: Slope field plot

Verification of solutions

$$x = c_1 e^{3t} + c_2 e^{-\frac{t}{2}}$$

Verified OK.

1.9.2 Solving using Kovacic algorithm

Writing the ode as

$$2x'' - 5x' - 3x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= -5 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{49}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 49 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{49z(t)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 19: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{49}{16}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{7t}{4}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5}{2}\,dt} \\
&= z_1 e^{\frac{5t}{4}} \\
&= z_1\left(\mathrm{e}^{\frac{5t}{4}}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-\frac{t}{2}}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-5}{2}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{\frac{5t}{2}}}{(x_1)^2}\,dt \\
&= x_1\left(\frac{2\,\mathrm{e}^{\frac{7t}{2}}}{7}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{t}{2}} \left(\frac{2\,\mathrm{e}^{\frac{7t}{2}}}{7}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1 \mathrm{e}^{-\frac{t}{2}} + \frac{2c_2 \mathrm{e}^{3t}}{7} \tag{1}
$$

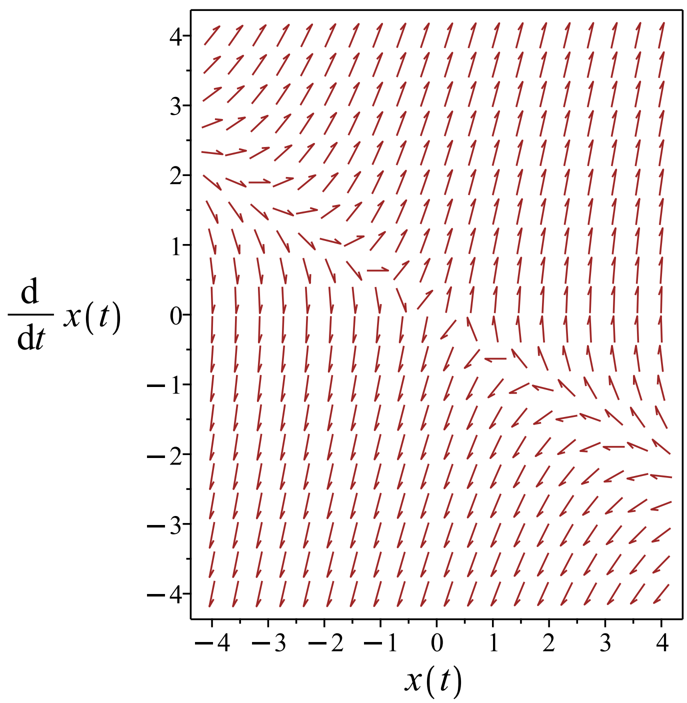


Figure 24: Slope field plot

Verification of solutions

$$
x = c_1 \mathrm{e}^{-\frac{t}{2}} + \frac{2c_2 \mathrm{e}^{3t}}{7}
$$

Verified OK.

1.9.3 Maple step by step solution

Let's solve

$2x'' - 5x' - 3x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = \frac{5x'}{2} + \frac{3x}{2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' - \frac{5x'}{2} - \frac{3x}{2} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{5}{2}r - \frac{3}{2} = 0$

- Factor the characteristic polynomial

  $\frac{(2r+1)(r-3)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(3, -\frac{1}{2}\right)$

- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^{3t}$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{-\frac{t}{2}}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-\frac{t}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(2*diff(x(t),t$2)-5*diff(x(t),t)-3*x(t)=0,x(t), singsol=all)
```

$$x(t) = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-\frac{t}{2}}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 24

```
DSolve[2*x''[t]-5*x'[t]-3*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^{-t/2} + c_2 e^{3t}$$

2 Chapter 1, First order differential equations. Section 1.1.3 Geometric. Exercises page 15

2.1 problem 1 94
2.2 problem 2 97

2.1 problem 1

2.1.1 Solving as quadrature ode 94
2.1.2 Maple step by step solution 95

Internal problem ID [11358]
Internal file name [`OUTPUT/10340_Wednesday_May_17_2023_07_49_26_PM_49721640/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.1.3 Geometric. Exercises page 15
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' - x\left(1 - \frac{x}{4}\right) = 0}$$

2.1.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{4}{x\,(-4+x)}dx = \int dt$$
$$-\ln\left(-4+x\right) + \ln\left(x\right) = t + c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{-\ln(-4+x)+\ln(x)} = \mathrm{e}^{t+c_1}$$

Which simplifies to

$$\frac{x}{-4+x} = c_2\mathrm{e}^{t}$$

Summary

The solution(s) found are the following

$$x = \frac{4c_2 e^t}{-1 + c_2 e^t} \tag{1}$$

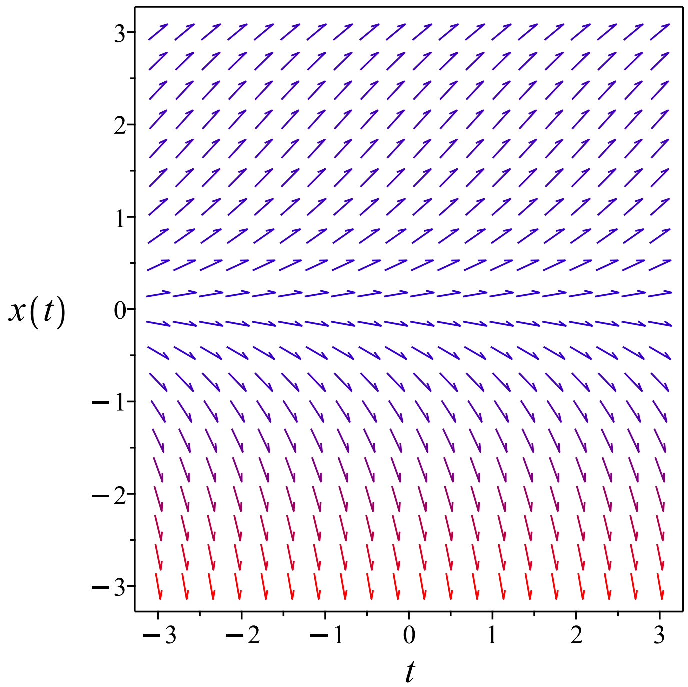


Figure 25: Slope field plot

Verification of solutions

$$x = \frac{4c_2 e^t}{-1 + c_2 e^t}$$

Verified OK.

2.1.2 Maple step by step solution

Let's solve

$x' - x\left(1 - \frac{x}{4}\right) = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

$$\frac{x'}{x\left(1-\frac{x}{4}\right)} = 1$$

- Integrate both sides with respect to t

$$\int \frac{x'}{x\left(1-\frac{x}{4}\right)}dt = \int 1dt + c_1$$

- Evaluate integral

$$-\ln(-4+x) + \ln(x) = t + c_1$$

- Solve for x

$$x = \frac{4\,e^{t+c_1}}{-1+e^{t+c_1}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(diff(x(t),t)=x(t)*(1-x(t)/4),x(t), singsol=all)
```

$$x(t) = \frac{4}{1+4\,e^{-t}c_1}$$

✓ Solution by Mathematica

Time used: 0.439 (sec). Leaf size: 32

```
DSolve[x'[t]==x[t]*(1-x[t]/4),x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{4e^t}{e^t + e^{4c_1}}$$
$$x(t) \to 0$$
$$x(t) \to 4$$

2.2 problem 2

2.2.1 Solving as riccati ode 97

Internal problem ID [11359]
Internal file name [`OUTPUT/10341_Wednesday_May_17_2023_07_49_28_PM_50385180/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.1.3 Geometric. Exercises page 15
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[[_Riccati, _special]]`

$$\boxed{x' - x^2 = t^2}$$

2.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= t^2 + x^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$x' = t^2 + x^2$$

With Riccati ODE standard form

$$x' = f_0(t) + f_1(t)x + f_2(t)x^2$$

Shows that $f_0(t) = t^2$, $f_1(t) = 0$ and $f_2(t) = 1$. Let

$$\begin{aligned} x &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= t^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(t) + t^2 u(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = \left(\mathrm{BesselJ}\left(\frac{1}{4}, \frac{t^2}{2}\right) c_1 + \mathrm{BesselY}\left(\frac{1}{4}, \frac{t^2}{2}\right) c_2\right)\sqrt{t}$$

The above shows that

$$u'(t) = t^{\frac{3}{2}}\left(\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{t^2}{2}\right) c_1 + \mathrm{BesselY}\left(-\frac{3}{4}, \frac{t^2}{2}\right) c_2\right)$$

Using the above in (1) gives the solution

$$x = -\frac{t\left(\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{t^2}{2}\right) c_1 + \mathrm{BesselY}\left(-\frac{3}{4}, \frac{t^2}{2}\right) c_2\right)}{\mathrm{BesselJ}\left(\frac{1}{4}, \frac{t^2}{2}\right) c_1 + \mathrm{BesselY}\left(\frac{1}{4}, \frac{t^2}{2}\right) c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$x = -\frac{t\left(\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{t^2}{2}\right) c_3 + \mathrm{BesselY}\left(-\frac{3}{4}, \frac{t^2}{2}\right)\right)}{\mathrm{BesselJ}\left(\frac{1}{4}, \frac{t^2}{2}\right) c_3 + \mathrm{BesselY}\left(\frac{1}{4}, \frac{t^2}{2}\right)}$$

Summary

The solution(s) found are the following

$$x = -\frac{t\left(\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{t^2}{2}\right) c_3 + \mathrm{BesselY}\left(-\frac{3}{4}, \frac{t^2}{2}\right)\right)}{\mathrm{BesselJ}\left(\frac{1}{4}, \frac{t^2}{2}\right) c_3 + \mathrm{BesselY}\left(\frac{1}{4}, \frac{t^2}{2}\right)} \tag{1}$$

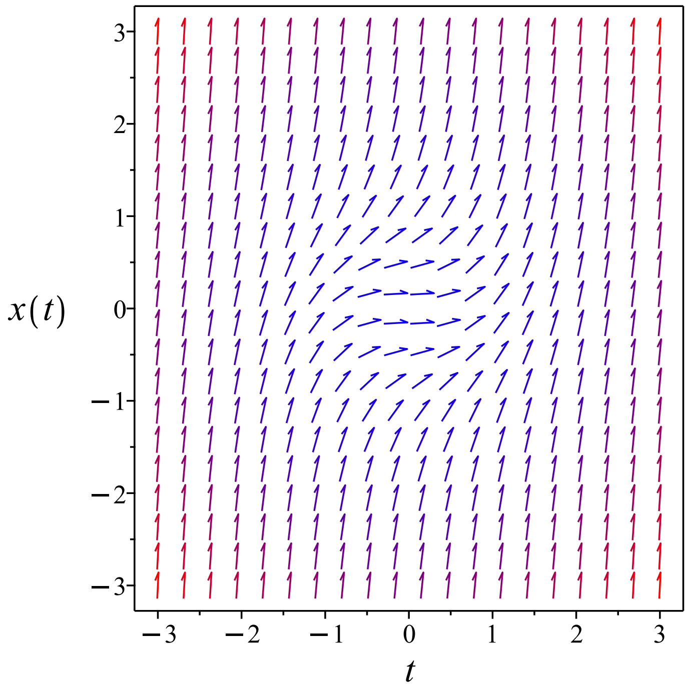


Figure 26: Slope field plot

Verification of solutions

$$x = -\frac{t\left(\text{BesselJ}\left(-\frac{3}{4}, \frac{t^2}{2}\right) c_3 + \text{BesselY}\left(-\frac{3}{4}, \frac{t^2}{2}\right)\right)}{\text{BesselJ}\left(\frac{1}{4}, \frac{t^2}{2}\right) c_3 + \text{BesselY}\left(\frac{1}{4}, \frac{t^2}{2}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 43

```
dsolve(diff(x(t),t)=x(t)^2+t^2,x(t), singsol=all)
```

$$x(t) = -\frac{t\left(\operatorname{BesselJ}\left(-\frac{3}{4}, \frac{t^2}{2}\right) c_1 + \operatorname{BesselY}\left(-\frac{3}{4}, \frac{t^2}{2}\right)\right)}{c_1 \operatorname{BesselJ}\left(\frac{1}{4}, \frac{t^2}{2}\right) + \operatorname{BesselY}\left(\frac{1}{4}, \frac{t^2}{2}\right)}$$

✓ Solution by Mathematica

Time used: 0.203 (sec). Leaf size: 169

```
DSolve[x'[t]==x[t]^2+t^2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{t^2\left(-2\operatorname{BesselJ}\left(-\frac{3}{4}, \frac{t^2}{2}\right) + c_1\left(\operatorname{BesselJ}\left(\frac{3}{4}, \frac{t^2}{2}\right) - \operatorname{BesselJ}\left(-\frac{5}{4}, \frac{t^2}{2}\right)\right)\right) - c_1 \operatorname{BesselJ}\left(-\frac{1}{4}, \frac{t^2}{2}\right)}{2t\left(\operatorname{BesselJ}\left(\frac{1}{4}, \frac{t^2}{2}\right) + c_1 \operatorname{BesselJ}\left(-\frac{1}{4}, \frac{t^2}{2}\right)\right)}$$

$$x(t) \to -\frac{t^2 \operatorname{BesselJ}\left(-\frac{5}{4}, \frac{t^2}{2}\right) - t^2 \operatorname{BesselJ}\left(\frac{3}{4}, \frac{t^2}{2}\right) + \operatorname{BesselJ}\left(-\frac{1}{4}, \frac{t^2}{2}\right)}{2t \operatorname{BesselJ}\left(-\frac{1}{4}, \frac{t^2}{2}\right)}$$

3 Chapter 1, First order differential equations. Section 1.2 Antiderivatives. Exercises page 19

3.1 problem 1 102
3.2 problem 2 106
3.3 problem 3 110
3.4 problem 4(a) 133
3.5 problem 4(b) 136
3.6 problem 4(c) 139
3.7 problem 6 142
3.8 problem 7 146

3.1 problem 1

3.1.1 Existence and uniqueness analysis 102
3.1.2 Solving as quadrature ode 103
3.1.3 Maple step by step solution 104

Internal problem ID [11360]
Internal file name [`OUTPUT/10342_Wednesday_May_17_2023_07_49_30_PM_98842774/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.2 Antiderivatives. Exercises page 19
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' = t\cos\left(t^2\right)}$$

With initial conditions

$$[x(0) = 1]$$

3.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = 0$$
$$q(t) = t\cos\left(t^2\right)$$

Hence the ode is

$$x' = t\cos\left(t^2\right)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = t\cos(t^2)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

3.1.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} x &= \int t\cos(t^2)\, \mathrm{d}t \\ &= \frac{\sin(t^2)}{2} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

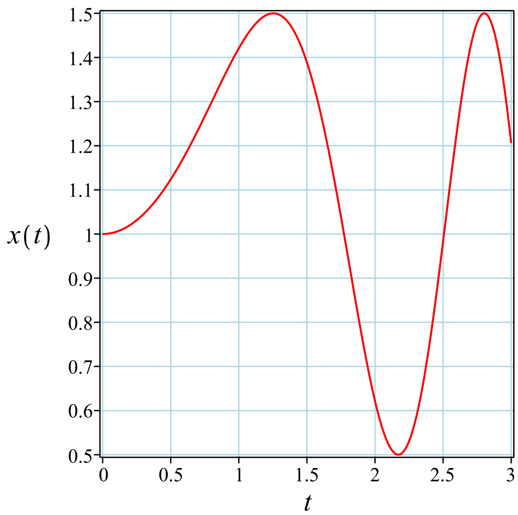
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

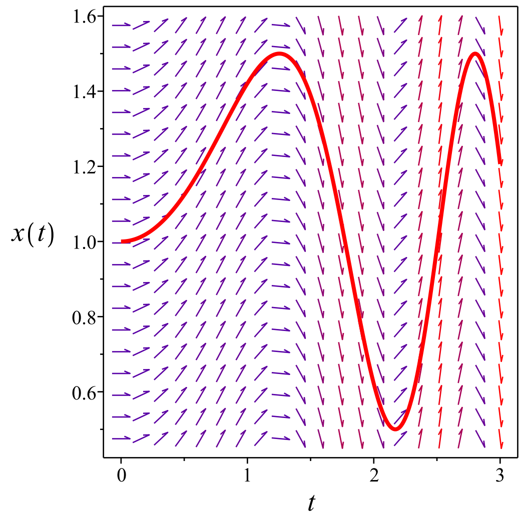
$$x = \frac{\sin(t^2)}{2} + 1$$

Summary
The solution(s) found are the following

$$x = \frac{\sin(t^2)}{2} + 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{\sin(t^2)}{2} + 1$$

Verified OK.

3.1.3 Maple step by step solution

Let's solve

$[x' = t\cos(t^2), x(0) = 1]$

- Highest derivative means the order of the ODE is 1

  x'

- Integrate both sides with respect to t

  $\int x' dt = \int t\cos(t^2)\, dt + c_1$

- Evaluate integral

  $x = \frac{\sin(t^2)}{2} + c_1$

- Solve for x

  $x = \frac{\sin(t^2)}{2} + c_1$

- Use initial condition $x(0) = 1$

  $1 = c_1$

- Solve for c_1

  $c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

  $x = \frac{\sin(t^2)}{2} + 1$

- Solution to the IVP

  $x = \frac{\sin(t^2)}{2} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve([diff(x(t),t)=t*cos(t^2),x(0) = 1],x(t), singsol=all)
```

$$x(t) = \frac{\sin(t^2)}{2} + 1$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 15

```
DSolve[{x'[t]==t*Cos[t^2],{x[0]==1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \rightarrow \frac{1}{2}(\sin(t^2) + 2)$$

3.2 problem 2

3.2.1 Existence and uniqueness analysis 106
3.2.2 Solving as quadrature ode 107
3.2.3 Maple step by step solution 108

Internal problem ID [11361]
Internal file name [`OUTPUT/10343_Wednesday_May_17_2023_07_49_32_PM_21072012/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.2 Antiderivatives. Exercises page 19
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' = \frac{1+t}{\sqrt{t}}}$$

With initial conditions

$$[x(1) = 4]$$

3.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = 0$$
$$q(t) = \frac{1+t}{\sqrt{t}}$$

Hence the ode is

$$x' = \frac{1+t}{\sqrt{t}}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = \frac{1+t}{\sqrt{t}}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

3.2.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} x &= \int \frac{1+t}{\sqrt{t}} \, \mathrm{d}t \\ &= \frac{2\sqrt{t}\,(3+t)}{3} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = \frac{8}{3} + c_1$$

$$c_1 = \frac{4}{3}$$

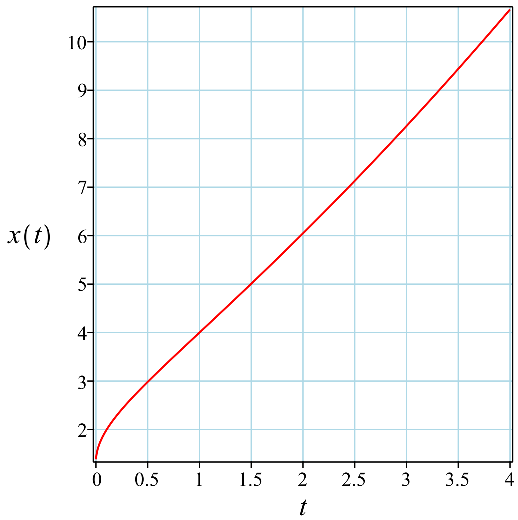
Substituting c_1 found above in the general solution gives

$$x = 2\sqrt{t} + \frac{2t^{\frac{3}{2}}}{3} + \frac{4}{3}$$

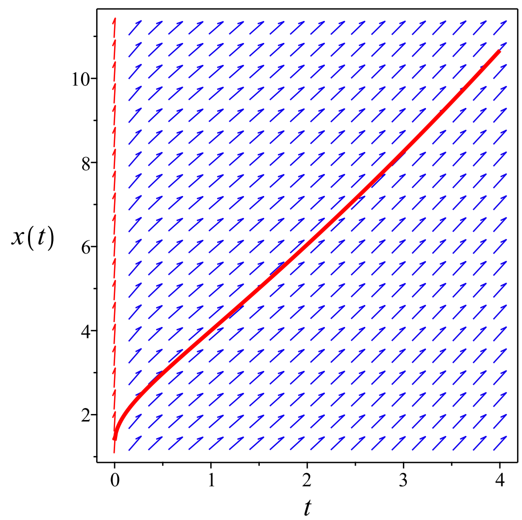
Summary

The solution(s) found are the following

$$x = 2\sqrt{t} + \frac{2t^{\frac{3}{2}}}{3} + \frac{4}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 2\sqrt{t} + \frac{2t^{\frac{3}{2}}}{3} + \frac{4}{3}$$

Verified OK.

3.2.3 Maple step by step solution

Let's solve

$\left[x' = \frac{1+t}{\sqrt{t}}, x(1) = 4\right]$

- Highest derivative means the order of the ODE is 1

  x'

- Integrate both sides with respect to t

  $\int x' dt = \int \frac{1+t}{\sqrt{t}} dt + c_1$

- Evaluate integral

  $x = \frac{2\sqrt{t}\,(3+t)}{3} + c_1$

- Solve for x

  $x = \frac{2t^{\frac{3}{2}}}{3} + 2\sqrt{t} + c_1$

- Use initial condition $x(1) = 4$

$4 = \frac{8}{3} + c_1$

- Solve for c_1

  $c_1 = \frac{4}{3}$

- Substitute $c_1 = \frac{4}{3}$ into general solution and simplify

  $x = 2\sqrt{t} + \frac{2t^{\frac{3}{2}}}{3} + \frac{4}{3}$

- Solution to the IVP

  $x = 2\sqrt{t} + \frac{2t^{\frac{3}{2}}}{3} + \frac{4}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(x(t),t)=(1+t)/sqrt(t),x(1) = 4],x(t), singsol=all)
```

$$x(t) = \frac{2t^{\frac{3}{2}}}{3} + 2\sqrt{t} + \frac{4}{3}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 23

```
DSolve[{x'[t]==(1+t)/Sqrt[t],{x[1]==4}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{2}{3}\left(t^{3/2} + 3\sqrt{t} + 2\right)$$

3.3 problem 3

3.3.1 Existence and uniqueness analysis 111
3.3.2 Solving as second order ode quadrature ode 111
3.3.3 Solving as second order linear constant coeff ode 113
3.3.4 Solving as second order integrable as is ode 117
3.3.5 Solving as second order ode missing y ode 119
3.3.6 Solving using Kovacic algorithm 121
3.3.7 Solving as exact linear second order ode ode 127
3.3.8 Maple step by step solution 130

Internal problem ID [11362]
Internal file name [`OUTPUT/10344_Wednesday_May_17_2023_07_49_34_PM_52810511/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.2 Antiderivatives. Exercises page 19
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{x'' = -3\sqrt{t}}$$

With initial conditions

$$[x(1) = 4, x'(1) = 2]$$

3.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 0 \\ F &= -3\sqrt{t} \end{aligned}$$

Hence the ode is

$$x'' = -3\sqrt{t}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $F = -3\sqrt{t}$ is

$$\{0 \leq t\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

3.3.2 Solving as second order ode quadrature ode

Integrating once gives

$$x' = -2t^{\frac{3}{2}} + c_1$$

Integrating again gives

$$x = -\frac{4t^{\frac{5}{2}}}{5} + c_1 x + c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = -\frac{4t^{\frac{5}{2}}}{5} + c_1 t + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 4$ and $t = 1$ in the above gives

$$4 = -\frac{4}{5} + c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -2t^{\frac{3}{2}} + c_1$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = -2 + c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = \frac{4}{5}$$

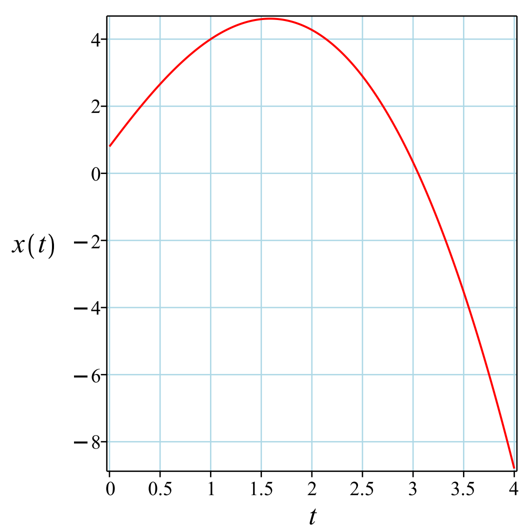
Substituting these values back in above solution results in

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5}$$

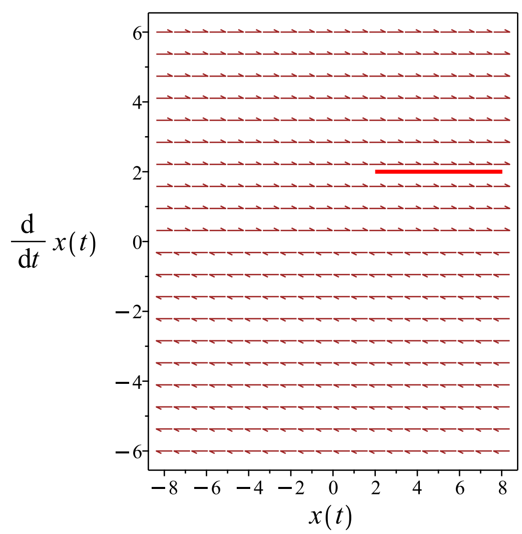
Summary

The solution(s) found are the following

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5}$$

Verified OK.

3.3.3 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = 0, f(t) = -3\sqrt{t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$x = c_1 1 + c_2 t \tag{1}$$

Therefore the homogeneous solution x_h is

$$x_h = c_2 t + c_1$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} x_1 &= 1 \\ x_2 &= t \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & t \\ \frac{d}{dt}(1) & \frac{d}{dt}(t) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix}$$

Therefore

$$W = (1)(1) - (t)(0)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-3t^{\frac{3}{2}}}{1} \, dt$$

Which simplifies to

$$u_1 = -\int -3t^{\frac{3}{2}} dt$$

Hence

$$u_1 = \frac{6t^{\frac{5}{2}}}{5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-3\sqrt{t}}{1} \, dt$$

Which simplifies to

$$u_2 = \int -3\sqrt{t} dt$$

Hence

$$u_2 = -2t^{\frac{3}{2}}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = -\frac{4t^{\frac{5}{2}}}{5}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_2 t + c_1) + \left(-\frac{4t^{\frac{5}{2}}}{5}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_2 t + c_1 - \frac{4t^{\frac{5}{2}}}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 4$ and $t = 1$ in the above gives

$$4 = -\frac{4}{5} + c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_2 - 2t^{\frac{3}{2}}$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = c_2 - 2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{4}{5}$$
$$c_2 = 4$$

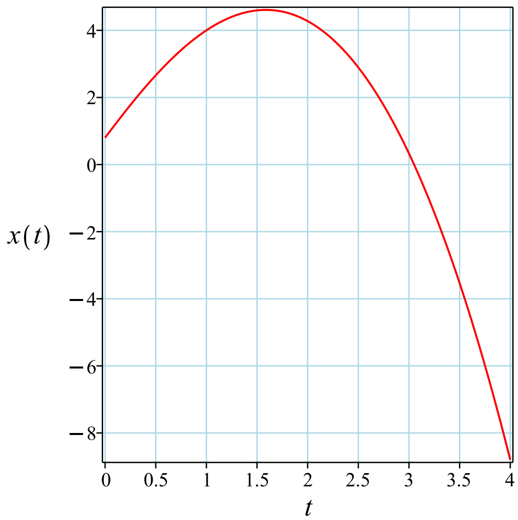
Substituting these values back in above solution results in

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5}$$

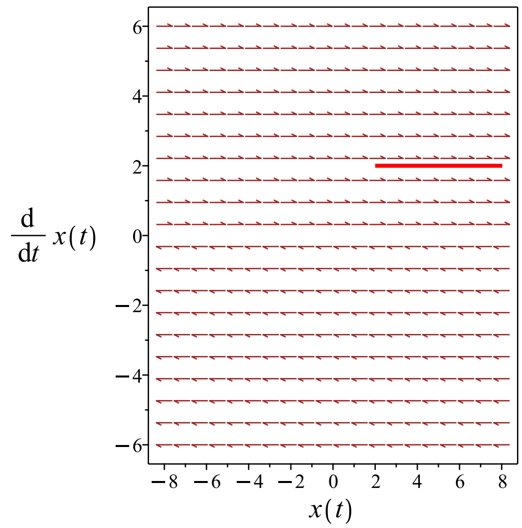
Summary

The solution(s) found are the following

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5}$$

Verified OK.

3.3.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int x'' dt = \int -3\sqrt{t} dt$$
$$x' = -2t^{\frac{3}{2}} + c_1$$

Which is now solved for x. Integrating both sides gives

$$\begin{aligned} x &= \int -2t^{\frac{3}{2}} + c_1 \, \mathrm{d}t \\ &= -\frac{4t^{\frac{5}{2}}}{5} + c_1 t + c_2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = -\frac{4t^{\frac{5}{2}}}{5} + c_1 t + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 4$ and $t = 1$ in the above gives

$$4 = -\frac{4}{5} + c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -2t^{\frac{3}{2}} + c_1$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = -2 + c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = \frac{4}{5}$$

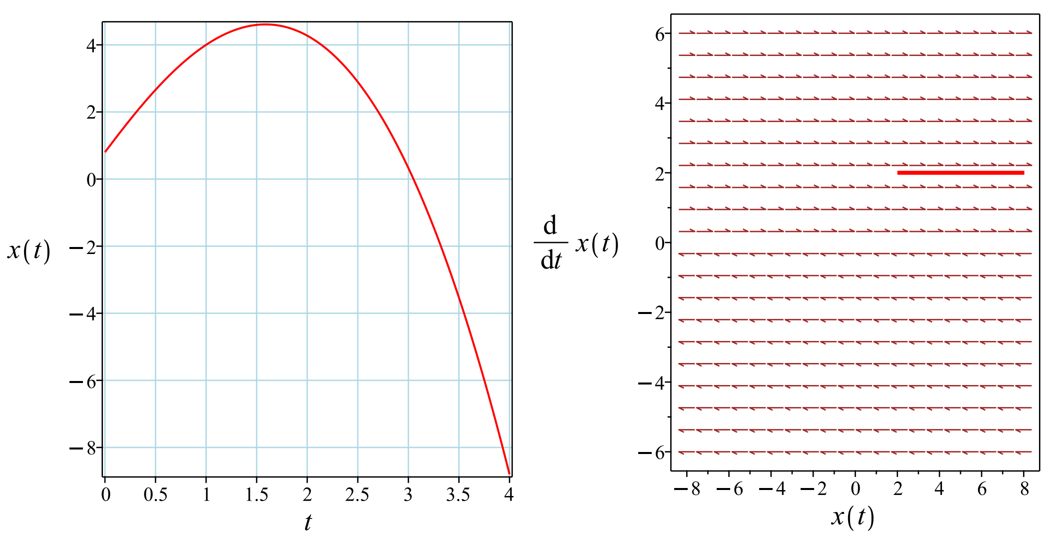
Substituting these values back in above solution results in

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5}$$

Summary

The solution(s) found are the following

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5}$$

Verified OK.

3.3.5 Solving as second order ode missing y ode

This is second order ode with missing dependent variable x. Let

$$p(t) = x'$$

Then

$$p'(t) = x''$$

Hence the ode becomes

$$p'(t) + 3\sqrt{t} = 0$$

Which is now solve for $p(t)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(t) &= \int -3\sqrt{t}\, \mathrm{d}t \\ &= -2t^{\frac{3}{2}} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $p = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -2 + c_1$$

$$c_1 = 4$$

Substituting c_1 found above in the general solution gives

$$p(t) = -2t^{\frac{3}{2}} + 4$$

Since $p = x'$ then the new first order ode to solve is

$$x' = -2t^{\frac{3}{2}} + 4$$

Integrating both sides gives

$$\begin{aligned} x &= \int -2t^{\frac{3}{2}} + 4\, \mathrm{d}t \\ &= -\frac{4t^{\frac{5}{2}}}{5} + 4t + c_2 \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $t = 1$ and $x = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = \frac{16}{5} + c_2$$

$$c_2 = \frac{4}{5}$$

Substituting c_2 found above in the general solution gives

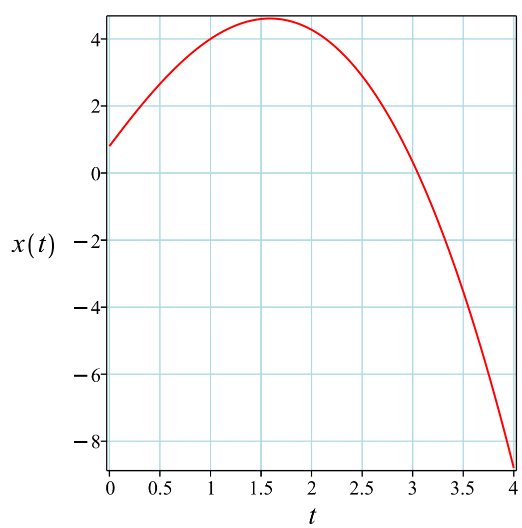
$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5}$$

Initial conditions are used to solve for the constants of integration.

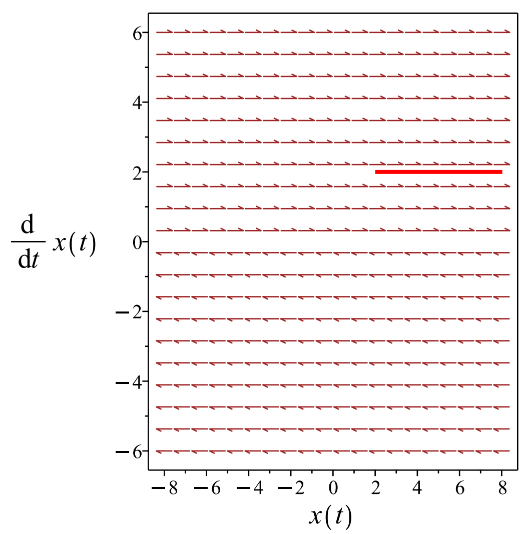
Summary

The solution(s) found are the following

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5}$$

Verified OK.

3.3.6 Solving using Kovacic algorithm

Writing the ode as

$$x'' = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\,e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 24: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
x_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$x_1 = 1$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\ &= 1 \int \frac{1}{1}\, dt \\ &= 1(t) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1(1) + c_2(1(t)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_2 t + c_1$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} x_1 &= 1 \\ x_2 &= t \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & t \\ \frac{d}{dt}(1) & \frac{d}{dt}(t) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix}$$

Therefore

$$W = (1)(1) - (t)(0)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-3t^{\frac{3}{2}}}{1}\, dt$$

Which simplifies to

$$u_1 = -\int -3t^{\frac{3}{2}} dt$$

Hence

$$u_1 = \frac{6t^{\frac{5}{2}}}{5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-3\sqrt{t}}{1}\, dt$$

Which simplifies to

$$u_2 = \int -3\sqrt{t} dt$$

Hence

$$u_2 = -2t^{\frac{3}{2}}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = -\frac{4t^{\frac{5}{2}}}{5}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_2 t + c_1) + \left(-\frac{4t^{\frac{5}{2}}}{5}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_2 t + c_1 - \frac{4t^{\frac{5}{2}}}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 4$ and $t = 1$ in the above gives

$$4 = -\frac{4}{5} + c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_2 - 2t^{\frac{3}{2}}$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = c_2 - 2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

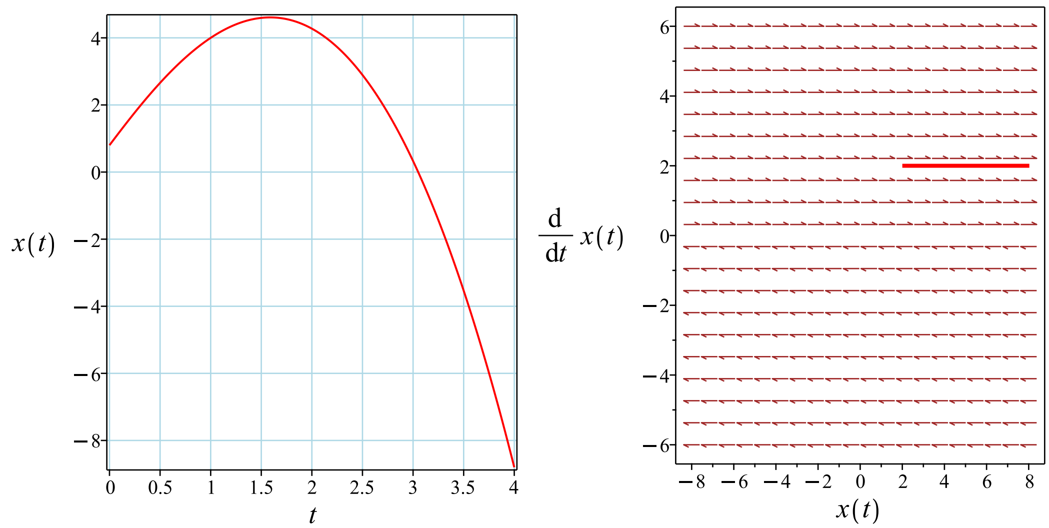
$$\begin{aligned} c_1 &= \frac{4}{5} \\ c_2 &= 4 \end{aligned}$$

Substituting these values back in above solution results in

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5}$$

<u>Summary</u>
The solution(s) found are the following

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5}$$

Verified OK.

3.3.7 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\,x'' + q(t)\,x' + r(t)\,x = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= -3\sqrt{t} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,x'+(q(t)-p'(t))\,x)'=s(x)$$

Integrating gives

$$p(t)\,x'+(q(t)-p'(t))\,x=\int s(t)\;dt$$

Substituting the above values for p,q,r,s gives

$$x'=\int -3\sqrt{t}\,dt$$

We now have a first order ode to solve which is

$$x'=-2t^{\frac{3}{2}}+c_1$$

Integrating both sides gives

$$\begin{aligned}x&=\int -2t^{\frac{3}{2}}+c_1\;\mathrm{d}t\\&=-\frac{4t^{\frac{5}{2}}}{5}+c_1t+c_2\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x=-\frac{4t^{\frac{5}{2}}}{5}+c_1t+c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x=4$ and $t=1$ in the above gives

$$4=-\frac{4}{5}+c_1+c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x'=-2t^{\frac{3}{2}}+c_1$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = -2 + c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = \frac{4}{5}$$

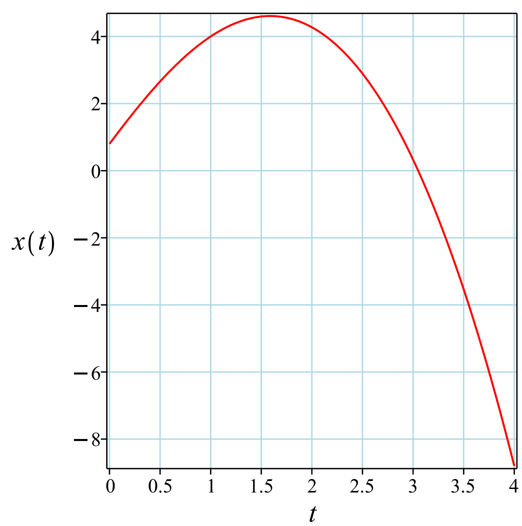
Substituting these values back in above solution results in

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5}$$

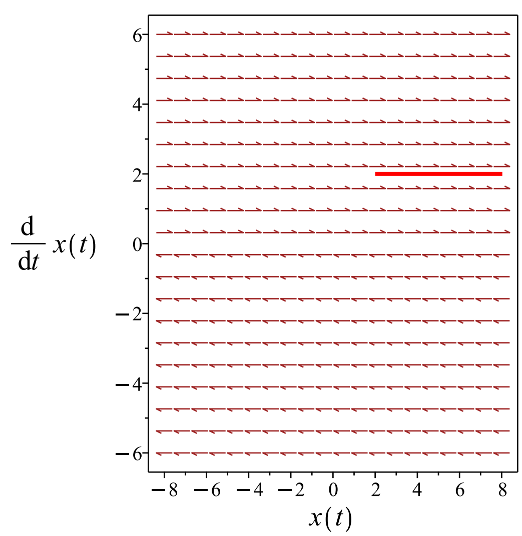
Summary

The solution(s) found are the following

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5}$$

Verified OK.

3.3.8 Maple step by step solution

Let's solve

$\left[x'' = -3\sqrt{t}, x(1) = 4, x'\Big|_{\{t=1\}} = 2\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

  $x_1(t) = 1$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t) = t$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 + c_2 t + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = -3\sqrt{t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 1$

  - Substitute functions into equation for $x_p(t)$

$$x_p(t) = 3\left(\int t^{\frac{3}{2}}dt\right) - 3t\left(\int \sqrt{t}dt\right)$$

- ○ Compute integrals

$$x_p(t) = -\frac{4t^{\frac{5}{2}}}{5}$$

- • Substitute particular solution into general solution to ODE

$$x = c_2 t + c_1 - \frac{4t^{\frac{5}{2}}}{5}$$

- □ Check validity of solution $x = c_2 t + c_1 - \frac{4t^{\frac{5}{2}}}{5}$
  - ○ Use initial condition $x(1) = 4$

$$4 = -\frac{4}{5} + c_1 + c_2$$

  - ○ Compute derivative of the solution

$$x' = c_2 - 2t^{\frac{3}{2}}$$

  - ○ Use the initial condition $x'\Big|_{\{t=1\}} = 2$

$$2 = c_2 - 2$$

  - ○ Solve for c_1 and c_2

$$\left\{c_1 = \frac{4}{5}, c_2 = 4\right\}$$

  - ○ Substitute constant values into general solution and simplify

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5}$$

- • Solution to the IVP

$$x = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([diff(x(t),t$2)=-3*sqrt(t),x(1) = 4, D(x)(1) = 2],x(t), singsol=all)
```

$$x(t) = -\frac{4t^{\frac{5}{2}}}{5} + 4t + \frac{4}{5}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 19

```
DSolve[{x''[t]==-3*Sqrt[t],{x[1]==4,x'[1]==2}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{4}{5}\left(t^{5/2} - 5t - 1\right)$$

3.4 problem 4(a)

3.4.1 Solving as quadrature ode 133
3.4.2 Maple step by step solution 134

Internal problem ID [11363]
Internal file name [`OUTPUT/10345_Wednesday_May_17_2023_07_49_35_PM_51119002/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.2 Antiderivatives. Exercises page 19
Problem number: 4(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' = t\,\mathrm{e}^{-2t}}$$

3.4.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} x &= \int t\,\mathrm{e}^{-2t}\,\mathrm{d}t \\ &= -\frac{(2t+1)\,\mathrm{e}^{-2t}}{4} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$x = -\frac{(2t+1)\,\mathrm{e}^{-2t}}{4} + c_1 \tag{1}$$

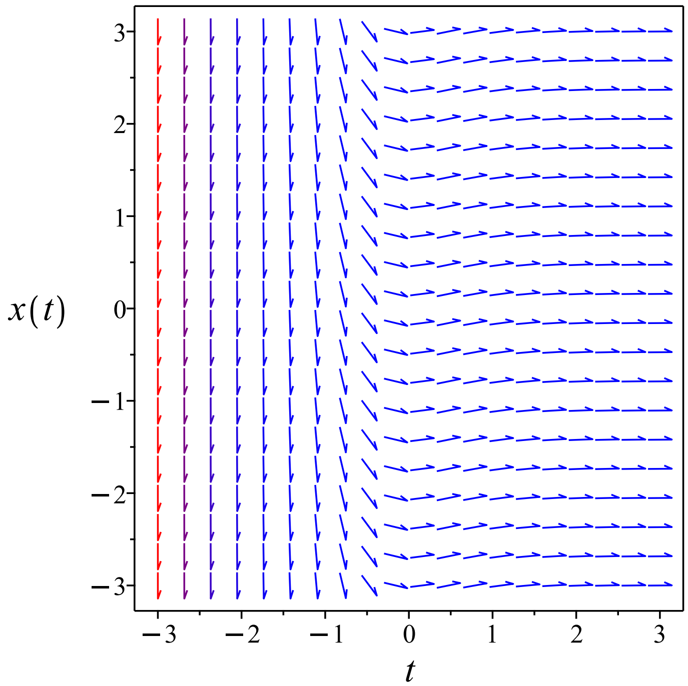


Figure 35: Slope field plot

Verification of solutions

$$x = -\frac{(2t+1)\,\mathrm{e}^{-2t}}{4} + c_1$$

Verified OK.

3.4.2 Maple step by step solution

Let's solve

$x' = t\,\mathrm{e}^{-2t}$

- Highest derivative means the order of the ODE is 1

  x'

- Integrate both sides with respect to t

  $\int x' dt = \int t\,\mathrm{e}^{-2t} dt + c_1$

- Evaluate integral

  $x = -\frac{(2t+1)\mathrm{e}^{-2t}}{4} + c_1$

- Solve for x

$$x = -\frac{t\,\mathrm{e}^{-2t}}{2} - \frac{\mathrm{e}^{-2t}}{4} + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(x(t),t)=t*exp(-2*t),x(t), singsol=all)
```

$$x(t) = \frac{(-2t-1)\,\mathrm{e}^{-2t}}{4} + c_1$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 22

```
DSolve[x'[t]==t*Exp[-2*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{1}{4}e^{-2t}(2t+1) + c_1$$

3.5 problem 4(b)

3.5.1 Solving as quadrature ode 136
3.5.2 Maple step by step solution 137

Internal problem ID [11364]
Internal file name [`OUTPUT/10346_Wednesday_May_17_2023_07_49_37_PM_13773109/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.2 Antiderivatives. Exercises page 19
Problem number: 4(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' = \frac{1}{t \ln(t)}}$$

3.5.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} x &= \int \frac{1}{t \ln(t)}\, \mathrm{d}t \\ &= \ln(\ln(t)) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$x = \ln(\ln(t)) + c_1 \tag{1}$$

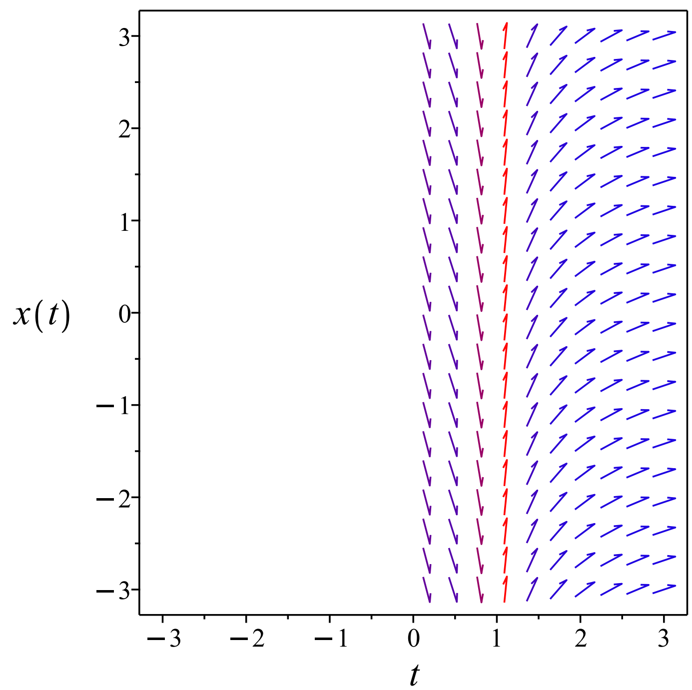


Figure 36: Slope field plot

Verification of solutions

$$x = \ln(\ln(t)) + c_1$$

Verified OK.

3.5.2 Maple step by step solution

Let's solve

$x' = \frac{1}{t\ln(t)}$

- Highest derivative means the order of the ODE is 1

  x'

- Integrate both sides with respect to t

  $\int x' dt = \int \frac{1}{t\ln(t)} dt + c_1$

- Evaluate integral

  $x = \ln(\ln(t)) + c_1$

- Solve for x

$$x = \ln(\ln(t)) + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(x(t),t)=1/(t*ln(t)),x(t), singsol=all)
```

$$x(t) = \ln(\ln(t)) + c_1$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 11

```
DSolve[x'[t]==1/(t*Log[t]),x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \log(\log(t)) + c_1$$

3.6 problem 4(c)

3.6.1 Solving as quadrature ode 139
3.6.2 Maple step by step solution 140

Internal problem ID [11365]
Internal file name [`OUTPUT/10347_Wednesday_May_17_2023_07_49_39_PM_55711235/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.2 Antiderivatives. Exercises page 19
Problem number: 4(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{x'\sqrt{t} = \cos\left(\sqrt{t}\right)}$$

3.6.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} x &= \int \frac{\cos\left(\sqrt{t}\right)}{\sqrt{t}}\,\mathrm{d}t \\ &= 2\sin\left(\sqrt{t}\right) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$x = 2\sin\left(\sqrt{t}\right) + c_1 \tag{1}$$

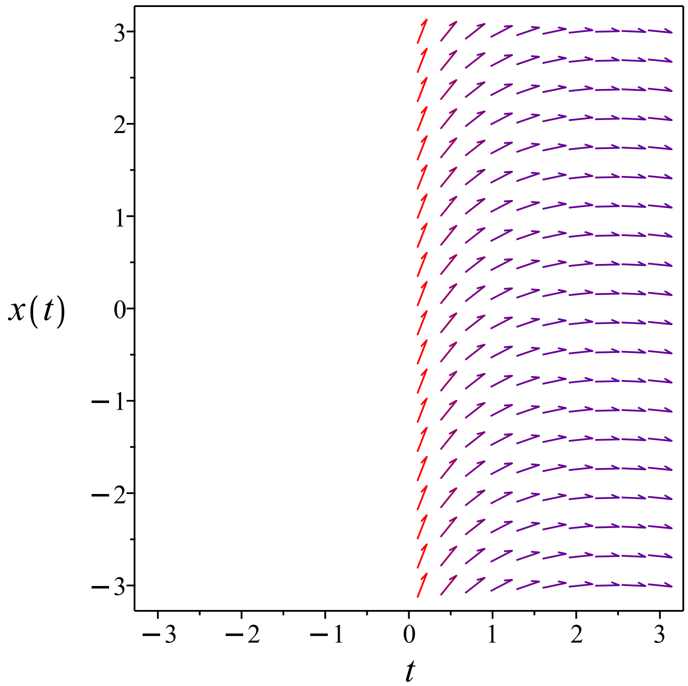


Figure 37: Slope field plot

Verification of solutions

$$x = 2\sin\left(\sqrt{t}\right) + c_1$$

Verified OK.

3.6.2 Maple step by step solution

Let's solve

$x'\sqrt{t} = \cos\left(\sqrt{t}\right)$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $x' = \frac{\cos(\sqrt{t})}{\sqrt{t}}$

- Integrate both sides with respect to t

  $\int x'dt = \int \frac{\cos(\sqrt{t})}{\sqrt{t}}dt + c_1$

- Evaluate integral

$$x = 2\sin\left(\sqrt{t}\right) + c_1$$

- Solve for x

$$x = 2\sin\left(\sqrt{t}\right) + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(sqrt(t)*diff(x(t),t)=cos(sqrt(t)),x(t), singsol=all)
```

$$x(t) = 2\sin\left(\sqrt{t}\right) + c_1$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 16

```
DSolve[Sqrt[t]*x'[t]==Cos[Sqrt[t]],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 2\sin\left(\sqrt{t}\right) + c_1$$

3.7 problem 6

3.7.1 Existence and uniqueness analysis 142
3.7.2 Solving as quadrature ode 143

Internal problem ID [11366]
Internal file name [`OUTPUT/10348_Wednesday_May_17_2023_07_49_41_PM_7363715/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.2 Antiderivatives. Exercises page 19
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{x' = \frac{e^{-t}}{\sqrt{t}}}$$

With initial conditions

$$[x(1) = 0]$$

3.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = 0$$
$$q(t) = \frac{e^{-t}}{\sqrt{t}}$$

Hence the ode is

$$x' = \frac{e^{-t}}{\sqrt{t}}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = \frac{e^{-t}}{\sqrt{t}}$ is

$$\{0 < t\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

3.7.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} x &= \int \frac{e^{-t}}{\sqrt{t}}\, dt \\ &= \sqrt{\pi}\, \text{erf}\left(\sqrt{t}\right) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \text{erf}\,(1)\sqrt{\pi} + c_1$$

$$c_1 = -\,\text{erf}\,(1)\sqrt{\pi}$$

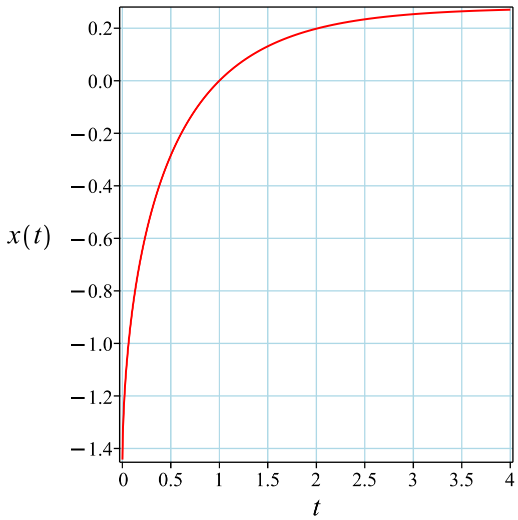
Substituting c_1 found above in the general solution gives

$$x = \sqrt{\pi}\, \text{erf}\left(\sqrt{t}\right) - \text{erf}\,(1)\sqrt{\pi}$$

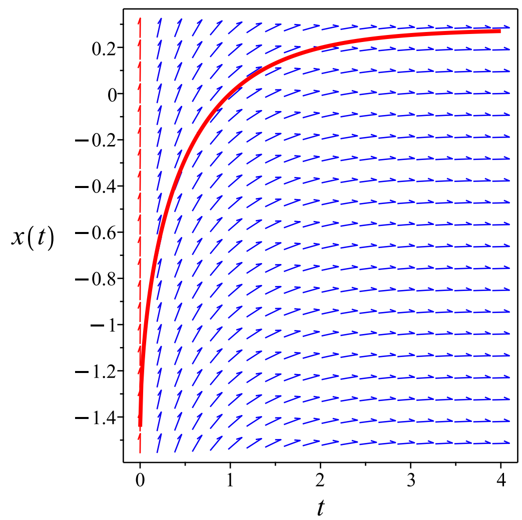
Summary

The solution(s) found are the following

$$x = \sqrt{\pi}\, \text{erf}\left(\sqrt{t}\right) - \text{erf}\,(1)\sqrt{\pi} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \sqrt{\pi}\,\mathrm{erf}\left(\sqrt{t}\right) - \mathrm{erf}(1)\sqrt{\pi}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 17

```
dsolve([diff(x(t),t)=exp(-t)/sqrt(t),x(1) = 0],x(t), singsol=all)
```

$$x(t) = \left(-\operatorname{erf}(1) + \operatorname{erf}\left(\sqrt{t}\right)\right)\sqrt{\pi}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 22

```
DSolve[{x'[t]==Exp[-t]/Sqrt[t],{x[1]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \sqrt{\pi}\left(\operatorname{erf}\left(\sqrt{t}\right) - \operatorname{erf}(1)\right)$$

3.8 problem 7

3.8.1 Existence and uniqueness analysis 147
3.8.2 Solving as second order integrable as is ode 147
3.8.3 Solving as second order ode missing y ode 149
3.8.4 Solving as second order ode non constant coeff transformation on B ode 151
3.8.5 Solving as type second_order_integrable_as_is (not using ABC version) 156
3.8.6 Solving using Kovacic algorithm 158
3.8.7 Solving as exact linear second order ode ode 166
3.8.8 Maple step by step solution 169

Internal problem ID [11367]
Internal file name [`OUTPUT/10349_Wednesday_May_17_2023_07_49_42_PM_84307621/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.2 Antiderivatives. Exercises page 19
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x' + tx'' = 1}$$

With initial conditions

$$[x(1) = 0, x'(1) = 2]$$

3.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{t} \\ q(t) &= 0 \\ F &= \frac{1}{t} \end{aligned}$$

Hence the ode is

$$x'' + \frac{x'}{t} = \frac{1}{t}$$

The domain of $p(t) = \frac{1}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $F = \frac{1}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

3.8.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\begin{aligned} \int (x' + tx'')\, dt &= \int 1 dt \\ tx' &= t + c_1 \end{aligned}$$

Which is now solved for x. Integrating both sides gives

$$\begin{aligned} x &= \int \frac{t + c_1}{t}\, \mathrm{d}t \\ &= t + c_1 \ln(t) + c_2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = t + c_1 \ln(t) + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$0 = 1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = 1 + \frac{c_1}{t}$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = 1 + c_1 \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$x = t + \ln(t) - 1$$

Summary
The solution(s) found are the following

$$x = t + \ln(t) - 1 \tag{1}$$

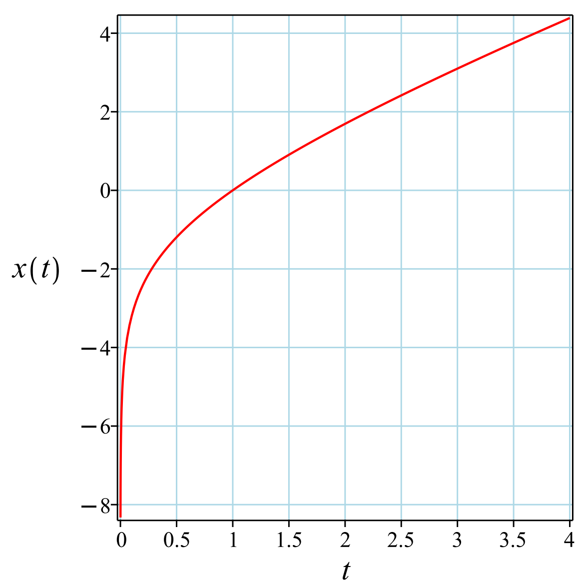


Figure 39: Solution plot

Verification of solutions

$$x = t + \ln(t) - 1$$

Verified OK.

3.8.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable x. Let

$$p(t) = x'$$

Then

$$p'(t) = x''$$

Hence the ode becomes

$$p(t) + tp'(t) - 1 = 0$$

Which is now solve for $p(t)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(t, p) \\ &= f(t)g(p) \\ &= \frac{-p+1}{t} \end{aligned}$$

Where $f(t) = \frac{1}{t}$ and $g(p) = -p + 1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-p+1}\,dp &= \frac{1}{t}\,dt \\ \int \frac{1}{-p+1}\,dp &= \int \frac{1}{t}\,dt \\ -\ln(p-1) &= \ln(t) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{p-1} = e^{\ln(t)+c_1}$$

Which simplifies to

$$\frac{1}{p-1} = c_2 t$$

Which can be simplified to become

$$p(t) = \frac{(c_2 e^{c_1} t + 1) e^{-c_1}}{c_2 t}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $p = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{e^{-c_1} e^{c_1} c_2 + e^{-c_1}}{c_2}$$

$$c_1 = -\ln(c_2)$$

Substituting c_1 found above in the general solution gives

$$p(t) = \frac{1+t}{t}$$

Since $p = x'$ then the new first order ode to solve is

$$x' = \frac{1+t}{t}$$

Integrating both sides gives

$$\begin{aligned} x &= \int \frac{1+t}{t} \, dt \\ &= t + \ln(t) + c_3 \end{aligned}$$

Initial conditions are used to solve for c_3. Substituting $t = 1$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 1 + c_3$$

$$c_3 = -1$$

Substituting c_3 found above in the general solution gives

$$x = t + \ln(t) - 1$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$x = t + \ln(t) - 1 \tag{1}$$

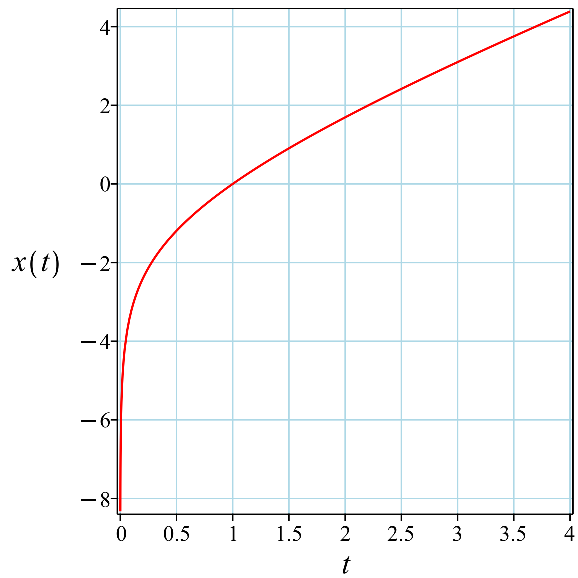


Figure 40: Solution plot

Verification of solutions

$$x = t + \ln(t) - 1$$

Verified OK.

3.8.4 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ax'' + Bx' + Cx = F(t)$$

This method reduces the order ode the ODE by one by applying the transformation

$$x = Bv$$

This results in

$$\begin{aligned}
x' &= B'v + v'B \\
x'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $x = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= t \\ B &= 1 \\ C &= 0 \\ F &= 1 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (t)\,(0) + (1)\,(0) + (0)\,(1) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$tv'' + (1)\,v' = 0$$

Now by applying $v' = u$ the above becomes

$$tu'(t) + u(t) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{u}{t} \end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{1}{t}\,dt \\ \int \frac{1}{u}\,du &= \int -\frac{1}{t}\,dt \\ \ln(u) &= -\ln(t) + c_1 \\ u &= e^{-\ln(t) + c_1} \\ &= \frac{c_1}{t} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{t} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(t) &= \int \frac{c_1}{t}\, \mathrm{d}t \\ &= c_1 \ln(t) + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} x_h(t) &= Bv \\ &= (1)\left(c_1 \ln(t) + c_2\right) \\ &= c_1 \ln(t) + c_2 \end{aligned}$$

And now the particular solution $x_p(t)$ will be found. The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} x_1 &= 1 \\ x_2 &= \ln(t) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(t) \\ \frac{d}{dt}(1) & \frac{d}{dt}(\ln(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(t) \\ 0 & \frac{1}{t} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{t}\right) - (\ln(t))(0)$$

Which simplifies to

$$W = \frac{1}{t}$$

Which simplifies to

$$W = \frac{1}{t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(t)}{1}\,dt$$

Which simplifies to

$$u_1 = -\int \ln(t)\,dt$$

Hence

$$u_1 = -t\ln(t) + t$$

And Eq. (3) becomes

$$u_2 = \int \frac{1}{1}\,dt$$

Which simplifies to

$$u_2 = \int 1dt$$

Hence

$$u_2 = t$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = t$$

Hence the complete solution is

$$\begin{aligned} x(t) &= x_h + x_p \\ &= (c_1 \ln (t) + c_2) + (t) \\ &= t + c_1 \ln (t) + c_2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = t + c_1 \ln (t) + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$0 = 1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = 1 + \frac{c_1}{t}$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = 1 + c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= -1 \end{aligned}$$

Substituting these values back in above solution results in

$$x = t + \ln (t) - 1$$

Summary

The solution(s) found are the following

$$x = t + \ln (t) - 1 \tag{1}$$

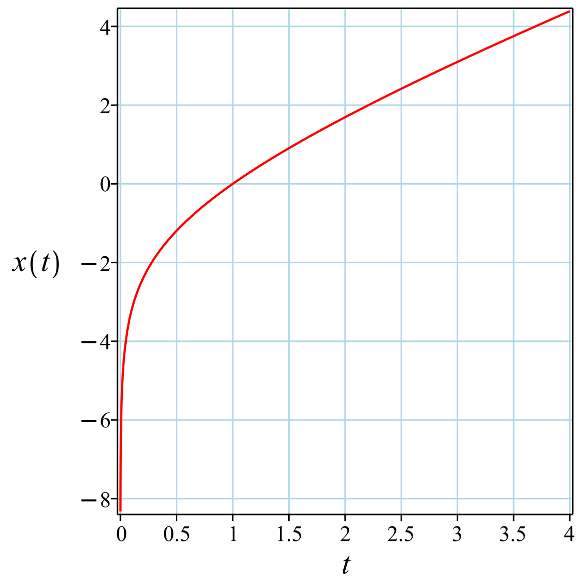


Figure 41: Solution plot

Verification of solutions

$$x = t + \ln(t) - 1$$

Verified OK.

3.8.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x' + tx'' = 1$$

Integrating both sides of the ODE w.r.t t gives

$$\int (x' + tx'')\, dt = \int 1 dt$$
$$tx' = t + c_1$$

Which is now solved for x. Integrating both sides gives

$$x = \int \frac{t + c_1}{t}\, \mathrm{d}t$$
$$= t + c_1 \ln(t) + c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = t + c_1 \ln(t) + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$0 = 1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = 1 + \frac{c_1}{t}$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = 1 + c_1 \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$x = t + \ln(t) - 1$$

Summary
The solution(s) found are the following

$$x = t + \ln(t) - 1 \tag{1}$$

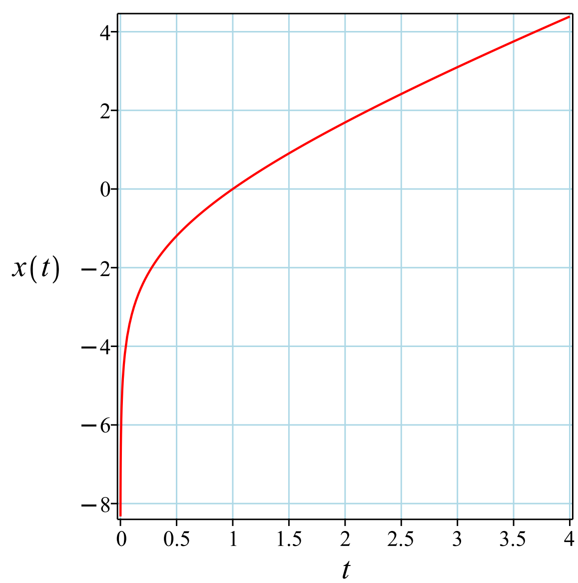


Figure 42: Solution plot

Verification of solutions

$$x = t + \ln(t) - 1$$

Verified OK.

3.8.6 Solving using Kovacic algorithm

Writing the ode as

$$x' + tx'' = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 29: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4t^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2t} + (-)(0) \\
&= \frac{1}{2t} \\
&= \frac{1}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2t}\right)^2 - \left(-\frac{1}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= pe^{\int \omega\, dt} \\
&= e^{\int \frac{1}{2t} dt} \\
&= \sqrt{t}
\end{aligned}
$$

The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{t}\,dt} \\
&= z_1 e^{-\frac{\ln(t)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = 1
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{t}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-\ln(t)}}{(x_1)^2}\,dt \\
&= x_1(\ln(t))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(1) + c_2(1(\ln(t)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x' + tx'' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 + c_2 \ln(t)$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} x_1 &= 1 \\ x_2 &= \ln(t) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(t) \\ \frac{d}{dt}(1) & \frac{d}{dt}(\ln(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(t) \\ 0 & \frac{1}{t} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{t}\right) - (\ln(t))(0)$$

Which simplifies to

$$W = \frac{1}{t}$$

Which simplifies to

$$W = \frac{1}{t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(t)}{1} \, dt$$

Which simplifies to

$$u_1 = -\int \ln(t) \, dt$$

Hence

$$u_1 = -t\ln(t) + t$$

And Eq. (3) becomes

$$u_2 = \int \frac{1}{1} \, dt$$

Which simplifies to

$$u_2 = \int 1 dt$$

Hence

$$u_2 = t$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = t$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1 + c_2 \ln(t)) + (t) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 + c_2 \ln(t) + t \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$0 = 1 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{c_2}{t} + 1$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = 1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -1 \\ c_2 &= 1 \end{aligned}$$

Substituting these values back in above solution results in

$$x = t + \ln(t) - 1$$

Summary

The solution(s) found are the following

$$x = t + \ln(t) - 1 \tag{1}$$

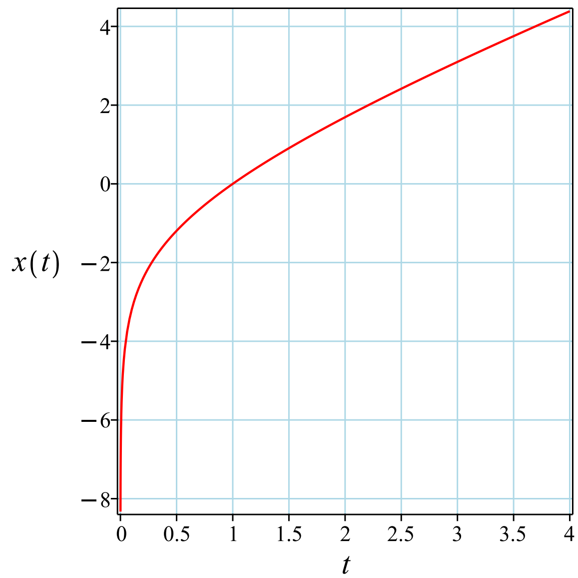


Figure 43: Solution plot

Verification of solutions

$$x = t + \ln(t) - 1$$

Verified OK.

3.8.7 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\, x'' + q(t)\, x' + r(t)\, x = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= t \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= 1 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(t)\,x' + \left(q(t) - p'(t)\right)x\right)' = s(x)$$

Integrating gives

$$p(t)\,x' + \left(q(t) - p'(t)\right)x = \int s(t)\; dt$$

Substituting the above values for p, q, r, s gives

$$tx' = \int 1\, dt$$

We now have a first order ode to solve which is

$$tx' = t + c_1$$

Integrating both sides gives

$$\begin{aligned} x &= \int \frac{t+c_1}{t}\, \mathrm{d}t \\ &= t + c_1 \ln(t) + c_2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = t + c_1 \ln(t) + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$0 = 1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = 1 + \frac{c_1}{t}$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = 1 + c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= -1
\end{aligned}
$$

Substituting these values back in above solution results in

$$x = t + \ln(t) - 1$$

Summary

The solution(s) found are the following

$$x = t + \ln(t) - 1 \tag{1}$$

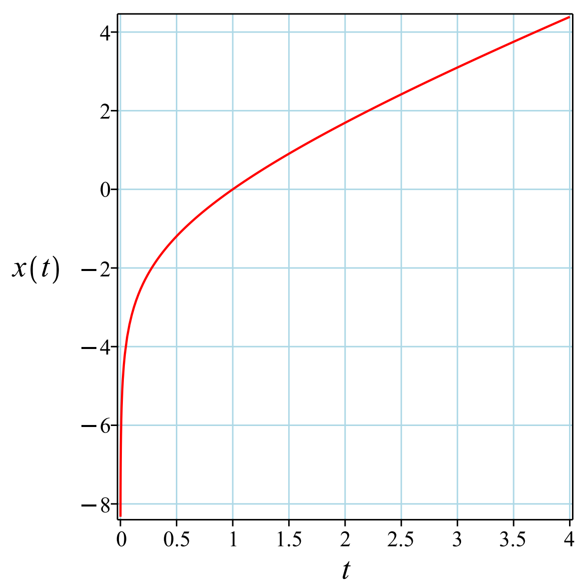


Figure 44: Solution plot

Verification of solutions

$$x = t + \ln(t) - 1$$

Verified OK.

3.8.8 Maple step by step solution

Let's solve

$$\left[x' + tx'' = 1, x(1) = 0, x'\Big|_{\{t=1\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Make substitution $u = x'$ to reduce order of ODE

  $u(t) + tu'(t) = 1$

- Integrate both sides with respect to t

  $\int (u(t) + tu'(t))\, dt = \int 1dt + c_1$

- Evaluate integral

  $tu(t) = t + c_1$

- Solve for $u(t)$

  $u(t) = \frac{t+c_1}{t}$

- Solve 1st ODE for $u(t)$

  $u(t) = \frac{t+c_1}{t}$

- Make substitution $u = x'$

  $x' = \frac{t+c_1}{t}$

- Integrate both sides to solve for x

  $\int x'dt = \int \frac{t+c_1}{t}dt + c_2$

- Compute integrals

  $x = t + c_1 \ln(t) + c_2$

☐ Check validity of solution $x = t + c_1 \ln(t) + c_2$

  - Use initial condition $x(1) = 0$

    $0 = 1 + c_2$

  - Compute derivative of the solution

    $x' = 1 + \frac{c_1}{t}$

  - Use the initial condition $x'\Big|_{\{t=1\}} = 2$

    $2 = 1 + c_1$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = -1\}$

- Substitute constant values into general solution and simplify

  $x = t + \ln(t) - 1$

- Solution to the IVP

  $x = t + \ln(t) - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(_b(_a)-1)/_a, _b(_a)`   *** Sublevel
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 9

```
dsolve([diff(t*diff(x(t),t),t)=1,x(1) = 0, D(x)(1) = 2],x(t), singsol=all)
```

$$x(t) = \ln(t) + t - 1$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 10

```
DSolve[{D[t*x'[t],t]==1,{x[1]==0,x'[1]==2}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to t + \log(t) - 1$$

4 Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26

4.1 problem 1(a) 172
4.2 problem 1(b) 176
4.3 problem 1(c) 181
4.4 problem 1(d) 184
4.5 problem 1(e) 188
4.6 problem 1(f) 191
4.7 problem 1(g) 194
4.8 problem 1(h) 197
4.9 problem 4(a) 200
4.10 problem 4(b) 218
4.11 problem 4(c) 230
4.12 problem 4(d) 247
4.13 problem 4(e) 261
4.14 problem 4(f) 264
4.15 problem 5 278
4.16 problem 6 283
4.17 problem 7 295
4.18 problem 8 311
4.19 problem 9 325
4.20 problem 10(a) 330
4.21 problem 10(b) 344
4.22 problem 10(c) 358
4.23 problem 11 369
4.24 problem 12 383
4.25 problem 13 397
4.26 problem 15 412
4.27 problem 21 424
4.28 problem 23 439
4.29 problem 24 452
4.30 problem 26 468
4.31 problem 28 486

4.1 problem 1(a)

4.1.1 Existence and uniqueness analysis 172
4.1.2 Solving as quadrature ode 173
4.1.3 Maple step by step solution 174

Internal problem ID [11368]
Internal file name [`OUTPUT/10350_Wednesday_May_17_2023_07_49_44_PM_57195971/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 1(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' - \sqrt{x} = 0}$$

With initial conditions

$$[x(0) = 1]$$

4.1.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t, x) \\ &= \sqrt{x} \end{aligned}$$

The x domain of $f(t, x)$ when $t = 0$ is

$$\{0 \leq x\}$$

And the point $x_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(\sqrt{x}) \\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial x}$ when $t = 0$ is

$$\{0 < x\}$$

And the point $x_0 = 1$ is inside this domain. Therefore solution exists and is unique.

4.1.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{\sqrt{x}} dx &= \int dt \\ 2\sqrt{x} &= t + c_1\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

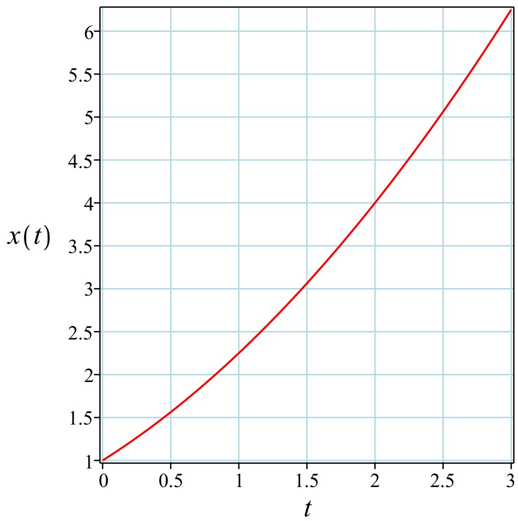
$$2\sqrt{x} = t + 2$$

Solving for x from the above gives

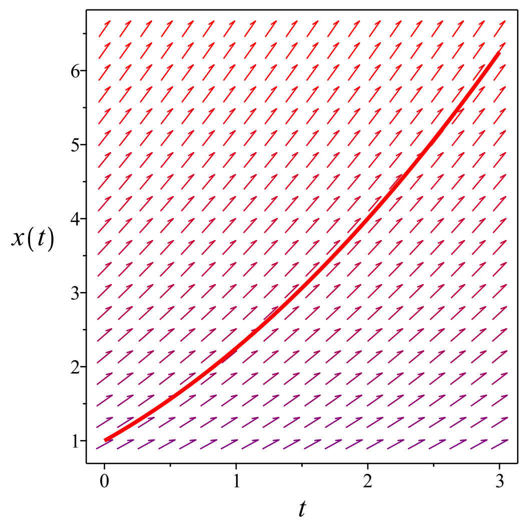
$$x = \frac{(t + 2)^2}{4}$$

Summary
The solution(s) found are the following

$$x = \frac{(t + 2)^2}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$x = \frac{(t+2)^2}{4}$$

Verified OK.

4.1.3 Maple step by step solution

Let's solve

$\left[x' - \sqrt{x} = 0, x(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{\sqrt{x}} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{\sqrt{x}} dt = \int 1 dt + c_1$

- Evaluate integral

  $2\sqrt{x} = t + c_1$

- Solve for x

$x = \frac{1}{4}t^2 + \frac{1}{2}c_1 t + \frac{1}{4}c_1^2$

- Use initial condition $x(0) = 1$

  $1 = \frac{c_1^2}{4}$

- Solve for c_1

  $c_1 = (-2, 2)$

- Substitute $c_1 = (-2, 2)$ into general solution and simplify

  $x = \frac{(t-2)^2}{4}$

- Solution to the IVP

  $x = \frac{(t-2)^2}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 11

```
dsolve([diff(x(t),t)=sqrt(x(t)),x(0) = 1],x(t), singsol=all)
```

$$x(t) = \frac{(t+2)^2}{4}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 16

```
DSolve[{x'[t]==Sqrt[t],{x[0]==1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{2t^{3/2}}{3} + 1$$

4.2 problem 1(b)

4.2.1 Existence and uniqueness analysis 176
4.2.2 Solving as quadrature ode 177
4.2.3 Maple step by step solution 178

Internal problem ID [11369]
Internal file name [`OUTPUT/10351_Wednesday_May_17_2023_07_49_46_PM_51759581/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 1(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' - \mathrm{e}^{-2x} = 0}$$

With initial conditions

$$[x(0) = 1]$$

4.2.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t, x) \\ &= \mathrm{e}^{-2x} \end{aligned}$$

The x domain of $f(t, x)$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(\mathrm{e}^{-2x}\right)\\ &= -2\,\mathrm{e}^{-2x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial x}$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. Therefore solution exists and is unique.

4.2.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int \mathrm{e}^{2x} dx &= t + c_1\\ \frac{\mathrm{e}^{2x}}{2} &= t + c_1\end{aligned}$$

Solving for x gives these solutions

$$x_1 = -\frac{\ln\left(\frac{1}{2c_1+2t}\right)}{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\ln(2)}{2} - \frac{\ln\left(\frac{1}{c_1}\right)}{2}$$

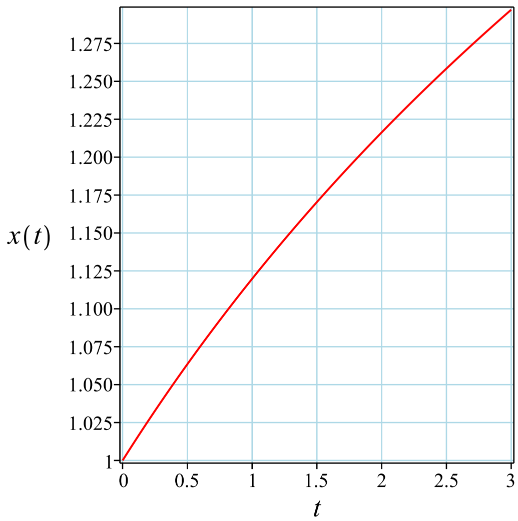
$$c_1 = \frac{\mathrm{e}^2}{2}$$

Substituting c_1 found above in the general solution gives

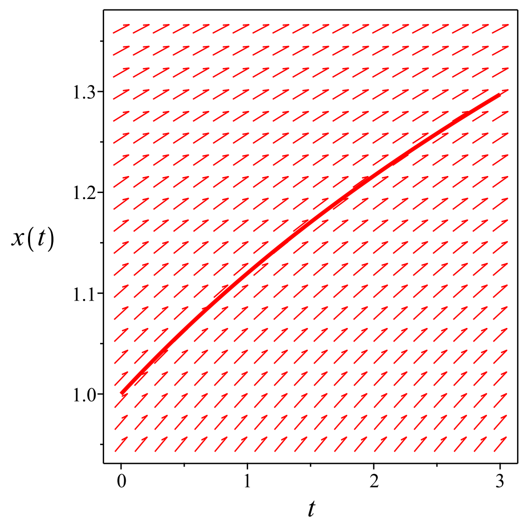
$$x = -\frac{\ln\left(\frac{1}{2t+\mathrm{e}^2}\right)}{2}$$

Summary
The solution(s) found are the following

$$x = -\frac{\ln\left(\frac{1}{2t+\mathrm{e}^2}\right)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$x = -\frac{\ln\left(\frac{1}{2t+e^2}\right)}{2}$$

Verified OK.

4.2.3 Maple step by step solution

Let's solve

$[x' - e^{-2x} = 0, x(0) = 1]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{e^{-2x}} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{e^{-2x}} dt = \int 1 dt + c_1$

- Evaluate integral

  $\frac{1}{2\,e^{-2x}} = t + c_1$

- Solve for x

$$x = -\frac{\ln\left(\frac{1}{2(t+c_1)}\right)}{2}$$

- Use initial condition $x(0) = 1$

$$1 = -\frac{\ln\left(\frac{1}{2c_1}\right)}{2}$$

- Solve for c_1

$$c_1 = \frac{e^2}{2}$$

- Substitute $c_1 = \frac{e^2}{2}$ into general solution and simplify

$$x = -\frac{\ln\left(\frac{1}{2t+e^2}\right)}{2}$$

- Solution to the IVP

$$x = -\frac{\ln\left(\frac{1}{2t+e^2}\right)}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 13

```
dsolve([diff(x(t),t)=exp(-2*x(t)),x(0) = 1],x(t), singsol=all)
```

$$x(t) = \frac{\ln \left(2t + \mathrm{e}^2\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 17

```
DSolve[{x'[t]==Exp[-2*x[t]],{x[0]==1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2} \log \left(2t + e^2\right)$$

4.3 problem 1(c)

4.3.1 Solving as quadrature ode 181
4.3.2 Maple step by step solution 182

Internal problem ID [11370]
Internal file name [`OUTPUT/10352_Wednesday_May_17_2023_07_49_47_PM_12695813/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 1(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 = 1}$$

4.3.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2+1} dy = t + c_1$$
$$\arctan(y) = t + c_1$$

Solving for y gives these solutions

$$y_1 = \tan(t + c_1)$$

Summary
The solution(s) found are the following

$$y = \tan(t + c_1) \tag{1}$$

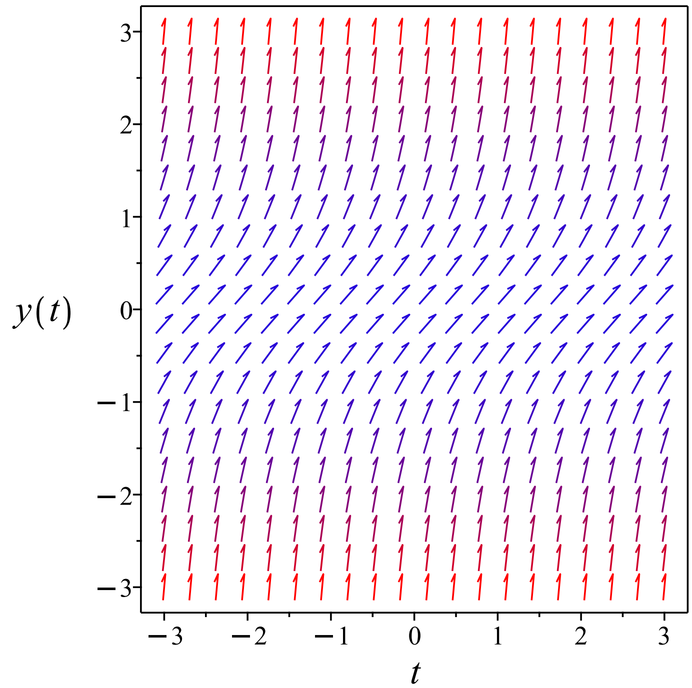


Figure 47: Slope field plot

Verification of solutions

$$y = \tan(t + c_1)$$

Verified OK.

4.3.2 Maple step by step solution

Let's solve

$y' - y^2 = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{1+y^2} dt = \int 1 dt + c_1$

- Evaluate integral

$$\arctan(y) = t + c_1$$

- Solve for y

$$y = \tan(t + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(y(t),t)=1+y(t)^2,y(t), singsol=all)
```

$$y(t) = \tan(t + c_1)$$

✓ Solution by Mathematica

Time used: 0.216 (sec). Leaf size: 24

```
DSolve[y'[t]==1+y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \tan(t + c_1)$$
$$y(t) \to -i$$
$$y(t) \to i$$

4.4 problem 1(d)

4.4.1 Solving as quadrature ode 184
4.4.2 Maple step by step solution 186

Internal problem ID [11371]
Internal file name [`OUTPUT/10353_Wednesday_May_17_2023_07_49_48_PM_88378866/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 1(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{u' - \frac{1}{5-2u} = 0}$$

4.4.1 Solving as quadrature ode

Integrating both sides gives

$$\int (5-2u)\, du = t + c_1$$
$$-u^2 + 5u = t + c_1$$

Solving for u gives these solutions

$$u_1 = \frac{5}{2} - \frac{\sqrt{25-4t-4c_1}}{2}$$
$$u_2 = \frac{5}{2} + \frac{\sqrt{25-4t-4c_1}}{2}$$

Summary

The solution(s) found are the following

$$u = \frac{5}{2} - \frac{\sqrt{25 - 4t - 4c_1}}{2} \tag{1}$$

$$u = \frac{5}{2} + \frac{\sqrt{25 - 4t - 4c_1}}{2} \tag{2}$$

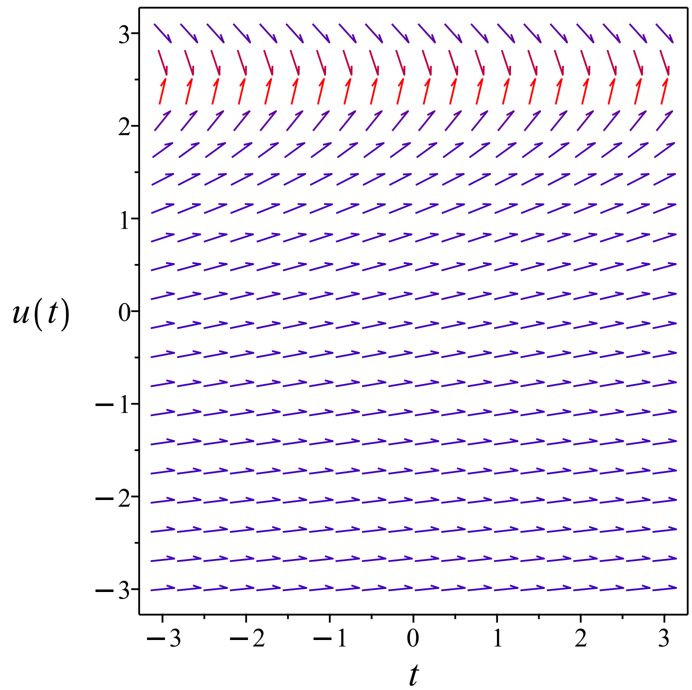


Figure 48: Slope field plot

Verification of solutions

$$u = \frac{5}{2} - \frac{\sqrt{25 - 4t - 4c_1}}{2}$$

Verified OK.

$$u = \frac{5}{2} + \frac{\sqrt{25 - 4t - 4c_1}}{2}$$

Verified OK.

4.4.2 Maple step by step solution

Let's solve

$u' - \frac{1}{5-2u} = 0$

- Highest derivative means the order of the ODE is 1

  u'

- Separate variables

  $(5-2u)\, u' = 1$

- Integrate both sides with respect to t

  $\int (5-2u)\, u' dt = \int 1 dt + c_1$

- Evaluate integral

  $-u^2 + 5u = t + c_1$

- Solve for u

  $\left\{u = \frac{5}{2} - \frac{\sqrt{25-4t-4c_1}}{2}, u = \frac{5}{2} + \frac{\sqrt{25-4t-4c_1}}{2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(u(t),t)=1/(5-2*u(t)),u(t), singsol=all)
```

$$u(t) = \frac{5}{2} - \frac{\sqrt{25 - 4t - 4c_1}}{2}$$
$$u(t) = \frac{5}{2} + \frac{\sqrt{25 - 4t - 4c_1}}{2}$$

✓ Solution by Mathematica

Time used: 0.142 (sec). Leaf size: 49

```
DSolve[u'[t]==1/(5-2*u[t]),u[t],t,IncludeSingularSolutions -> True]
```

$$u(t) \rightarrow \frac{1}{2}\left(5 - \sqrt{-4t + 25 + 4c_1}\right)$$
$$u(t) \rightarrow \frac{1}{2}\left(5 + \sqrt{-4t + 25 + 4c_1}\right)$$

4.5 problem 1(e)

4.5.1 Solving as quadrature ode 188
4.5.2 Maple step by step solution 189

Internal problem ID [11372]
Internal file name [`OUTPUT/10354_Wednesday_May_17_2023_07_49_49_PM_11615872/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 1(e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' - ax = b}$$

4.5.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{ax+b}dx = \int dt$$
$$\frac{\ln(ax+b)}{a} = t + c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{\frac{\ln(ax+b)}{a}} = \mathrm{e}^{t+c_1}$$

Which simplifies to

$$(ax+b)^{\frac{1}{a}} = c_2\mathrm{e}^t$$

Summary
The solution(s) found are the following

$$x = \frac{(c_2\mathrm{e}^t)^a - b}{a} \tag{1}$$

Verification of solutions

$$x=\frac{(c_2\mathrm{e}^t)^a-b}{a}$$

Verified OK.

4.5.2 Maple step by step solution

Let's solve

$x'-ax=b$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{ax+b}=1$

- Integrate both sides with respect to t

  $\int \frac{x'}{ax+b}dt=\int 1dt+c_1$

- Evaluate integral

  $\frac{\ln(ax+b)}{a}=t+c_1$

- Solve for x

  $x=\frac{\mathrm{e}^{c_1a+ta}-b}{a}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(x(t),t)=a*x(t)+b,x(t), singsol=all)
```

$$x(t) = \frac{e^{at} c_1 a - b}{a}$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 30

```
DSolve[x'[t]==a*x[t]+b,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{b}{a} + c_1 e^{at}$$
$$x(t) \to -\frac{b}{a}$$

4.6 problem 1(f)

4.6.1 Solving as quadrature ode 191
4.6.2 Maple step by step solution 192

Internal problem ID [11373]
Internal file name [`OUTPUT/10355_Wednesday_May_17_2023_07_49_50_PM_20057251/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 1(f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{Q' - \frac{Q}{4 + Q^2} = 0}$$

4.6.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{Q^2 + 4}{Q} dQ = t + c_1$$
$$\frac{Q^2}{2} + 4\ln(Q) = t + c_1$$

Solving for Q gives these solutions

Summary
The solution(s) found are the following

$$Q = \mathrm{e}^{-\frac{\operatorname{LambertW}\left(\frac{\mathrm{e}^{\frac{t}{2}+\frac{c_1}{2}}}{4}\right)}{2}+\frac{t}{4}+\frac{c_1}{4}} \tag{1}$$

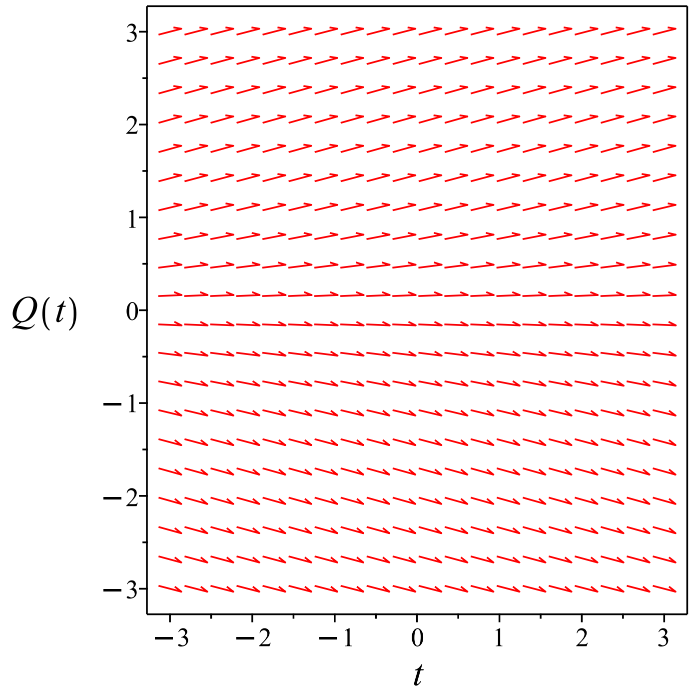


Figure 49: Slope field plot

Verification of solutions

$$Q = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(\frac{\mathrm{e}^{\frac{t}{2}+\frac{c_1}{2}}}{4}\right)}{2}+\frac{t}{4}+\frac{c_1}{4}}$$

Verified OK.

4.6.2 Maple step by step solution

Let's solve

$Q' - \frac{Q}{4+Q^2} = 0$

- Highest derivative means the order of the ODE is 1

  Q'

- Separate variables

  $\frac{Q'(4+Q^2)}{Q} = 1$

- Integrate both sides with respect to t

  $\int \frac{Q'(4+Q^2)}{Q} dt = \int 1 dt + c_1$

- Evaluate integral

  $\frac{Q^2}{2} + 4\ln(Q) = t + c_1$

- Solve for Q

  $Q = e^{-\frac{\mathit{LambertW}\left(\frac{e^{\frac{t}{2}+\frac{c_1}{2}}}{4}\right)}{2}+\frac{t}{4}+\frac{c_1}{4}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(Q(t),t)=Q(t)/(4+Q(t)^2),Q(t), singsol=all)
```

$$Q(t) = \frac{2\,e^{\frac{t}{4}+\frac{c_1}{4}}}{\sqrt{\frac{e^{\frac{t}{2}+\frac{c_1}{2}}}{\mathrm{LambertW}\left(\frac{e^{\frac{t}{2}+\frac{c_1}{2}}}{4}\right)}}}$$

✓ Solution by Mathematica

Time used: 0.092 (sec). Leaf size: 42

```
DSolve[Q'[t]==Q[t]/(4*Q[t]^2),Q[t],t,IncludeSingularSolutions -> True]
```

$$Q(t) \to -\frac{\sqrt{t+4c_1}}{\sqrt{2}}$$
$$Q(t) \to \frac{\sqrt{t+4c_1}}{\sqrt{2}}$$

4.7 problem 1(g)

4.7.1 Solving as quadrature ode 194
4.7.2 Maple step by step solution 195

Internal problem ID [11374]
Internal file name [`OUTPUT/10356_Wednesday_May_17_2023_07_49_52_PM_772028/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 1(g).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{x' - e^{x^2} = 0}$$

4.7.1 Solving as quadrature ode

Integrating both sides gives

$$\int e^{-x^2} dx = \int dt$$
$$\int^x e^{-\_a^2} d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^x e^{-\_a^2} d\_a = t + c_1 \tag{1}$$

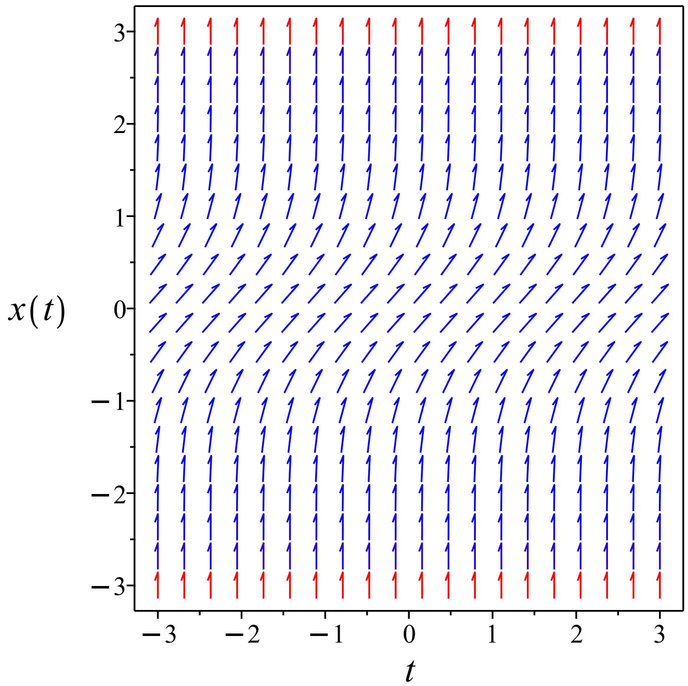


Figure 50: Slope field plot

Verification of solutions

$$\int^x e^{-\_a^2} d\_a = t + c_1$$

Verified OK.

4.7.2 Maple step by step solution

Let's solve

$x' - e^{x^2} = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{e^{x^2}} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{e^{x^2}} dt = \int 1 dt + c_1$

- Evaluate integral

$$\frac{\sqrt{\pi}\,\mathrm{erf}(x)}{2} = t + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(diff(x(t),t)=exp(x(t)^2),x(t), singsol=all)
```

$$t - \frac{\sqrt{\pi}\,\mathrm{erf}\left(x(t)\right)}{2} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.594 (sec). Leaf size: 17

```
DSolve[x'[t]==Exp[x[t]^2],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \mathrm{erf}^{-1}\left(\frac{2(t+c_1)}{\sqrt{\pi}}\right)$$

4.8 problem 1(h)

4.8.1 Solving as quadrature ode 197
4.8.2 Maple step by step solution 198

Internal problem ID [11375]
Internal file name [`OUTPUT/10357_Wednesday_May_17_2023_07_49_53_PM_61469210/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 1(h).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - r(a - y) = 0}$$

4.8.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{r\left(-a+y\right)}dy = \int dt$$
$$-\frac{\ln\left(-a+y\right)}{r} = t + c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{-\frac{\ln(-a+y)}{r}} = \mathrm{e}^{t+c_1}$$

Which simplifies to

$$\left(-a+y\right)^{-\frac{1}{r}} = c_2\mathrm{e}^{t}$$

Summary
The solution(s) found are the following

$$y = \left(c_2\mathrm{e}^{t}\right)^{-r} + a \tag{1}$$

Verification of solutions

$$y = (c_2 e^t)^{-r} + a$$

Verified OK.

4.8.2 Maple step by step solution

Let's solve

$y' - r(a - y) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{a-y} = r$

- Integrate both sides with respect to t

  $\int \frac{y'}{a-y} dt = \int r dt + c_1$

- Evaluate integral

  $-\ln(a - y) = tr + c_1$

- Solve for y

  $y = -e^{-tr-c_1} + a$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(t),t)=r*(a-y(t)),y(t), singsol=all)
```

$$y(t) = a + \mathrm{e}^{-rt} c_1$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 21

```
DSolve[y'[t]==r*(a-y[t]),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to a + c_1 e^{-rt}$$
$$y(t) \to a$$

4.9 problem 4(a)

4.9.1 Solving as separable ode 200
4.9.2 Solving as linear ode 202
4.9.3 Solving as homogeneousTypeD2 ode 203
4.9.4 Solving as homogeneousTypeMapleC ode 205
4.9.5 Solving as first order ode lie symmetry lookup ode 208
4.9.6 Solving as exact ode 212
4.9.7 Maple step by step solution 216

Internal problem ID [11376]
Internal file name [`OUTPUT/10358_Wednesday_May_17_2023_07_49_54_PM_36544411/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 4(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' - \frac{2x}{1+t} = 0}$$

4.9.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= f(t)g(x) \\ &= \frac{2x}{1+t} \end{aligned}$$

Where $f(t) = \frac{2}{1+t}$ and $g(x) = x$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{x}\, dx &= \frac{2}{1+t}\, dt \\
\int \frac{1}{x}\, dx &= \int \frac{2}{1+t}\, dt \\
\ln(x) &= 2\ln(1+t) + c_1 \\
x &= e^{2\ln(1+t)+c_1} \\
&= c_1(1+t)^2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1(1+t)^2 \tag{1}
$$

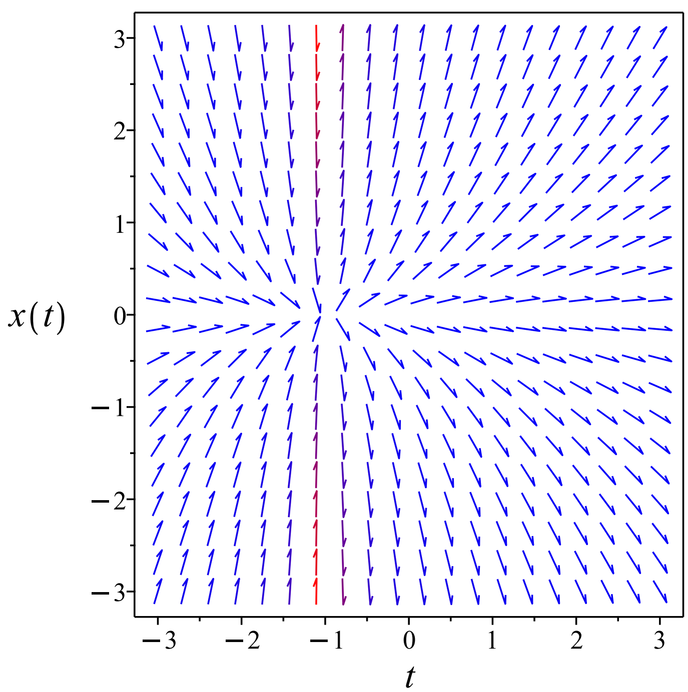


Figure 51: Slope field plot

Verification of solutions

$$
x = c_1(1+t)^2
$$

Verified OK.

4.9.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned}p(t) &= -\frac{2}{1+t}\\ q(t) &= 0\end{aligned}$$

Hence the ode is

$$x' - \frac{2x}{1+t} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{2}{1+t}dt}\\ &= \frac{1}{(1+t)^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu x &= 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{(1+t)^2}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{x}{(1+t)^2} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(1+t)^2}$ results in

$$x = c_1(1+t)^2$$

Summary
The solution(s) found are the following

$$x = c_1(1+t)^2 \tag{1}$$

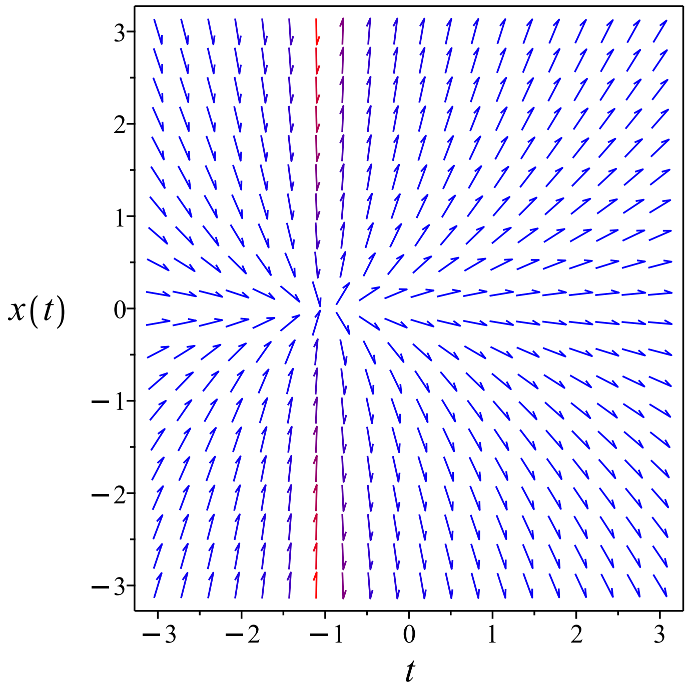


Figure 52: Slope field plot

Verification of solutions

$$x = c_1(1+t)^2$$

Verified OK.

4.9.3 Solving as homogeneousTypeD2 ode

Using the change of variables $x = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u'(t)\,t + u(t) - \frac{2u(t)\,t}{1+t} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= \frac{u(t-1)}{t\,(1+t)} \end{aligned}$$

Where $f(t) = \frac{t-1}{t(1+t)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{t-1}{t\,(1+t)}\,dt \\
\int \frac{1}{u}\,du &= \int \frac{t-1}{t\,(1+t)}\,dt \\
\ln\left(u\right) &= 2\ln\left(1+t\right) - \ln\left(t\right) + c_2 \\
u &= \mathrm{e}^{2\ln(1+t)-\ln(t)+c_2} \\
&= c_2 \mathrm{e}^{2\ln(1+t)-\ln(t)}
\end{aligned}
$$

Which simplifies to

$$
u(t) = c_2\left(t + 2 + \frac{1}{t}\right)
$$

Therefore the solution x is

$$
\begin{aligned}
x &= ut \\
&= tc_2\left(t + 2 + \frac{1}{t}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = tc_2\left(t + 2 + \frac{1}{t}\right) \tag{1}
$$

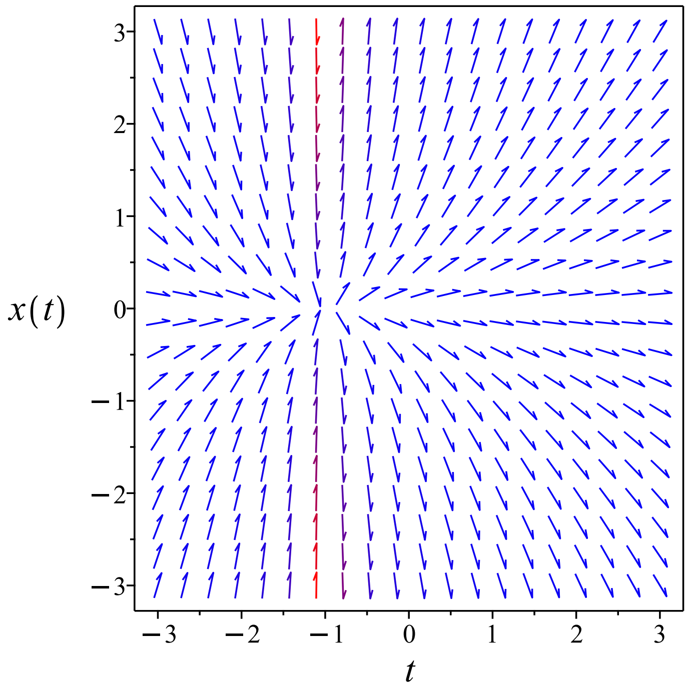


Figure 53: Slope field plot

Verification of solutions

$$x = tc_2\left(t + 2 + \frac{1}{t}\right)$$

Verified OK.

4.9.4 Solving as homogeneousTypeMapleC ode

Let $Y = x + y_0$ and $X = t + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{2Y(X) + 2y_0}{1 + X + x_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -1$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{2Y(X)}{X}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{2Y}{X}
\end{aligned}
\tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = 2Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= 2u \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) - \frac{u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX} u(X)\right) X - u(X) = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= \frac{u}{X}
\end{aligned}
$$

Where $f(X) = \frac{1}{X}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= \frac{1}{X}\, dX \\
\int \frac{1}{u}\, du &= \int \frac{1}{X}\, dX \\
\ln(u) &= \ln(X) + c_2 \\
u &= \mathrm{e}^{\ln(X)+c_2} \\
&= c_2 X
\end{aligned}
$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$
Y(X) = X^2 c_2
$$

Using the solution for $Y(X)$

$$
Y(X) = X^2 c_2
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= x + y_0 \\
X &= t + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= x \\
X &= t - 1
\end{aligned}
$$

Then the solution in x becomes

$$
x = c_2 (1+t)^2
$$

Summary

The solution(s) found are the following

$$
x = c_2 (1+t)^2 \tag{1}
$$

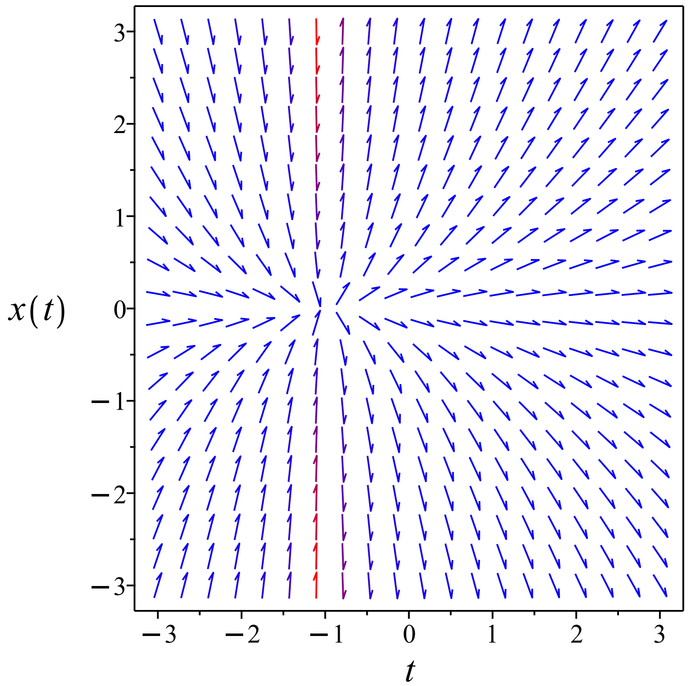


Figure 54: Slope field plot

Verification of solutions

$$x = c_2(1+t)^2$$

Verified OK.

4.9.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = \frac{2x}{1+t}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 39: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= 0 \\
\eta(t, x) &= (1 + t)^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{(1+t)^2} dy \end{aligned}$$

Which results in

$$S = \frac{x}{(1+t)^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = \frac{2x}{1+t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= -\frac{2x}{(1+t)^3} \\ S_x &= \frac{1}{(1+t)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\frac{x}{(1+t)^2} = c_1$$

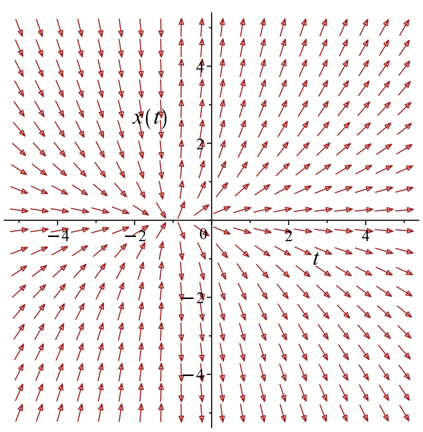
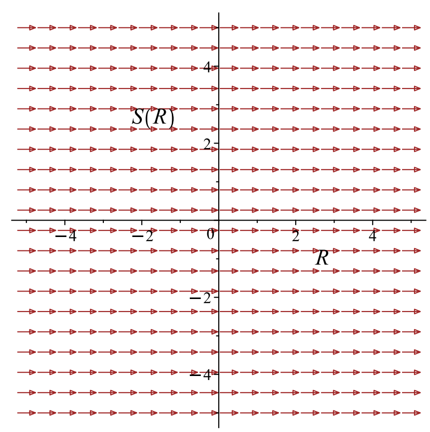
Which simplifies to

$$\frac{x}{(1+t)^2} = c_1$$

Which gives

$$x = c_1(1+t)^2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \frac{2x}{1+t}$ 	$R = t$ $S = \frac{x}{(1+t)^2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$x = c_1(1+t)^2 \tag{1}$$

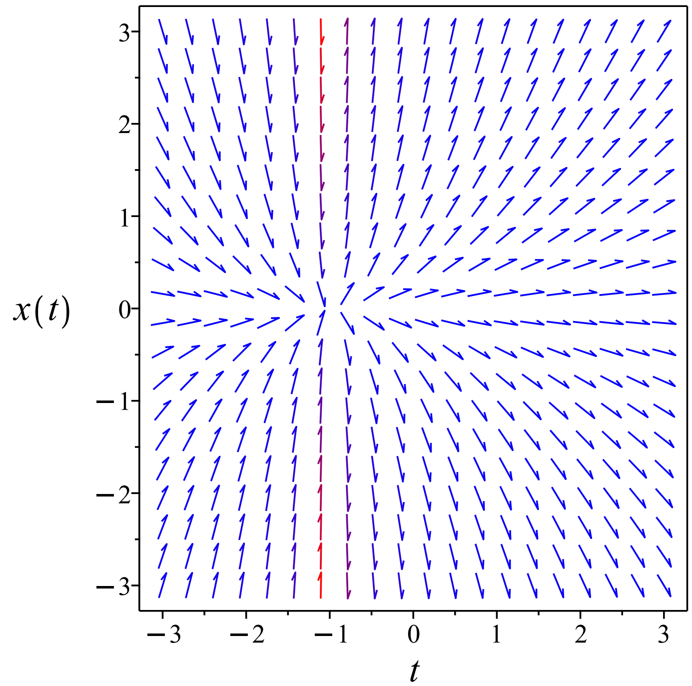


Figure 55: Slope field plot

Verification of solutions

$$x = c_1(1+t)^2$$

Verified OK.

4.9.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, x) \, \mathrm{d}t + N(t, x) \, \mathrm{d}x = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{2x}\right) \mathrm{d}x &= \left(\frac{1}{1+t}\right) \mathrm{d}t \\
\left(-\frac{1}{1+t}\right) \mathrm{d}t + \left(\frac{1}{2x}\right) \mathrm{d}x &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, x) &= -\frac{1}{1+t} \\
N(t, x) &= \frac{1}{2x}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{1+t}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{2x}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x}=\frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,x)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} &= M \qquad (1) \\
\frac{\partial \phi}{\partial x} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t &= \int M\, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t &= \int -\frac{1}{1+t}\, \mathrm{d}t \\
\phi &= -\ln(1+t) + f(x) \qquad (3)
\end{aligned}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x}=\frac{1}{2x}$. Therefore equation (4) becomes

$$
\frac{1}{2x} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = \frac{1}{2x}
$$

Integrating the above w.r.t x gives

$$
\begin{aligned}
\int f'(x)\, \mathrm{d}x &= \int \left(\frac{1}{2x}\right) \mathrm{d}x \\
f(x) &= \frac{\ln(x)}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\ln(1+t) + \frac{\ln(x)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(1+t) + \frac{\ln(x)}{2}$$

The solution becomes

$$x = e^{2c_1}(1+t)^2$$

Summary

The solution(s) found are the following

$$x = e^{2c_1}(1+t)^2 \tag{1}$$

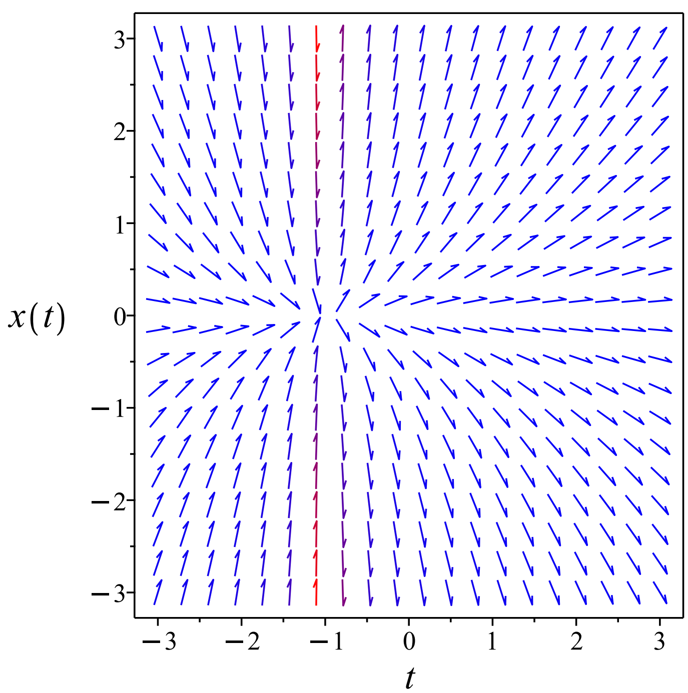


Figure 56: Slope field plot

Verification of solutions

$$x = e^{2c_1}(1+t)^2$$

Verified OK.

4.9.7 Maple step by step solution

Let's solve

$x' - \frac{2x}{1+t} = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x} = \frac{2}{1+t}$

- Integrate both sides with respect to t

  $\int \frac{x'}{x} dt = \int \frac{2}{1+t} dt + c_1$

- Evaluate integral

  $\ln(x) = 2\ln(1+t) + c_1$

- Solve for x

  $x = e^{c_1}(1+t)^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 11

```
dsolve(diff(x(t),t)=2*x(t)/(t+1),x(t), singsol=all)
```

$$x(t) = c_1(t+1)^2$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 18

```
DSolve[x'[t]==2*x[t]/(t+1),x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1(t+1)^2$$
$$x(t) \to 0$$

4.10 problem 4(b)

4.10.1 Solving as separable ode 218
4.10.2 Solving as first order ode lie symmetry lookup ode 220
4.10.3 Solving as exact ode 224
4.10.4 Maple step by step solution 228

Internal problem ID [11377]
Internal file name [`OUTPUT/10359_Wednesday_May_17_2023_07_49_55_PM_79842807/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 4(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\theta' - t\sqrt{t^2+1}\,\sec(\theta) = 0}$$

4.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}\theta' &= F(t,\theta)\\ &= f(t)g(\theta)\\ &= t\sqrt{t^2+1}\,\sec(\theta)\end{aligned}$$

Where $f(t) = t\sqrt{t^2+1}$ and $g(\theta) = \sec(\theta)$. Integrating both sides gives

$$\frac{1}{\sec(\theta)}\,d\theta = t\sqrt{t^2+1}\,dt$$

$$\int \frac{1}{\sec(\theta)}\,d\theta = \int t\sqrt{t^2+1}\,dt$$

$$\sin(\theta) = \frac{(t^2+1)^{\frac{3}{2}}}{3} + c_1$$

Which results in

$$\theta = \arcsin\left(\frac{(t^2+1)^{\frac{3}{2}}}{3} + c_1\right)$$

Summary

The solution(s) found are the following

$$\theta = \arcsin\left(\frac{(t^2+1)^{\frac{3}{2}}}{3} + c_1\right) \tag{1}$$

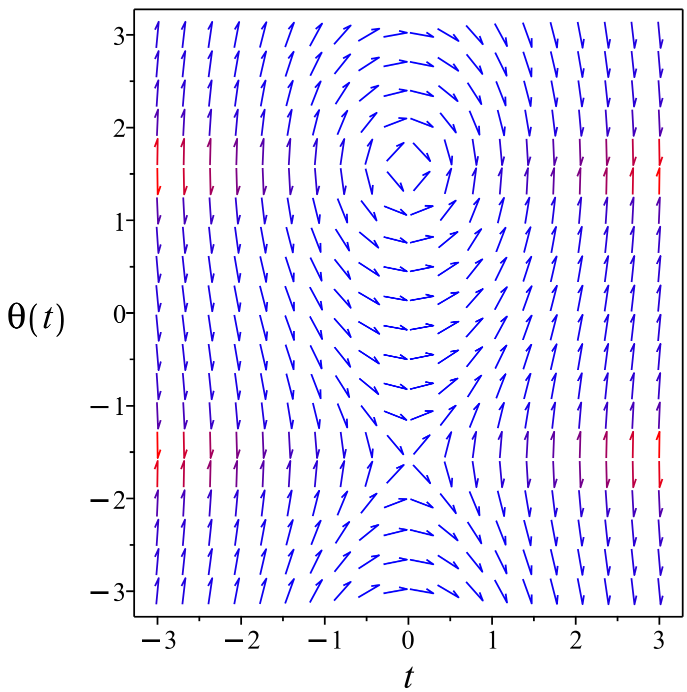


Figure 57: Slope field plot

Verification of solutions

$$\theta = \arcsin\left(\frac{(t^2+1)^{\frac{3}{2}}}{3} + c_1\right)$$

Verified OK.

4.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
\theta' &= t\sqrt{t^2+1}\,\sec(\theta) \\
\theta' &= \omega(t,\theta)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_\theta - \xi_t) - \omega^2 \xi_\theta - \omega_t \xi - \omega_\theta \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 42: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,\theta) &= \frac{1}{t\sqrt{t^2+1}} \\
\eta(t,\theta) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(t,\theta) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{d\theta}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial \theta}\right) S(t,\theta) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = \theta
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dt \\
&= \int \frac{1}{\frac{1}{t\sqrt{t^2+1}}} dt
\end{aligned}
$$

Which results in

$$
S = \frac{(t^2+1)^{\frac{3}{2}}}{3}
$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,\theta)S_\theta}{R_t + \omega(t,\theta)R_\theta} \tag{2}
$$

Where in the above $R_t, R_\theta, S_t, S_\theta$ are all partial derivatives and $\omega(t,\theta)$ is the right hand side of the original ode given by

$$
\omega(t,\theta) = t\sqrt{t^2+1}\, \sec(\theta)
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_\theta &= 1 \\
S_t &= t\sqrt{t^2+1} \\
S_\theta &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \cos(\theta) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for t, θ in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \cos(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \sin(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to t, θ coordinates. This results in

$$
\frac{(t^2+1)^{\frac{3}{2}}}{3} = \sin(\theta) + c_1
$$

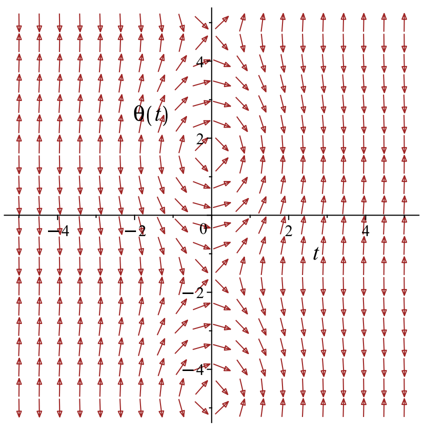
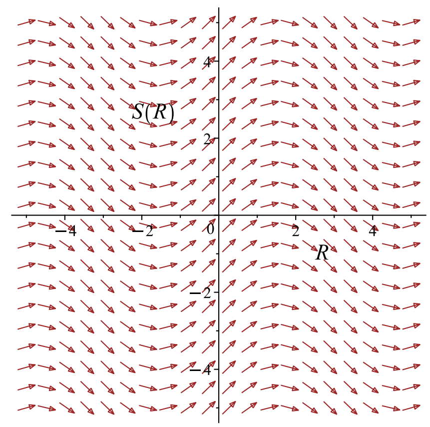
Which simplifies to

$$
\frac{(t^2+1)^{\frac{3}{2}}}{3} = \sin(\theta) + c_1
$$

Which gives

$$
\theta = -\arcsin\left(-\frac{(t^2+1)^{\frac{3}{2}}}{3} + c_1\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, θ coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{d\theta}{dt} = t\sqrt{t^2+1}\,\sec(\theta)$

</td><td>$R = \theta$
$S = \frac{(t^2+1)^{\frac{3}{2}}}{3}$</td><td>$\frac{dS}{dR} = \cos(R)$

</td></tr>
</table>

<u>Summary</u>

The solution(s) found are the following

$$\theta = -\arcsin\left(-\frac{(t^2+1)^{\frac{3}{2}}}{3} + c_1\right) \tag{1}$$

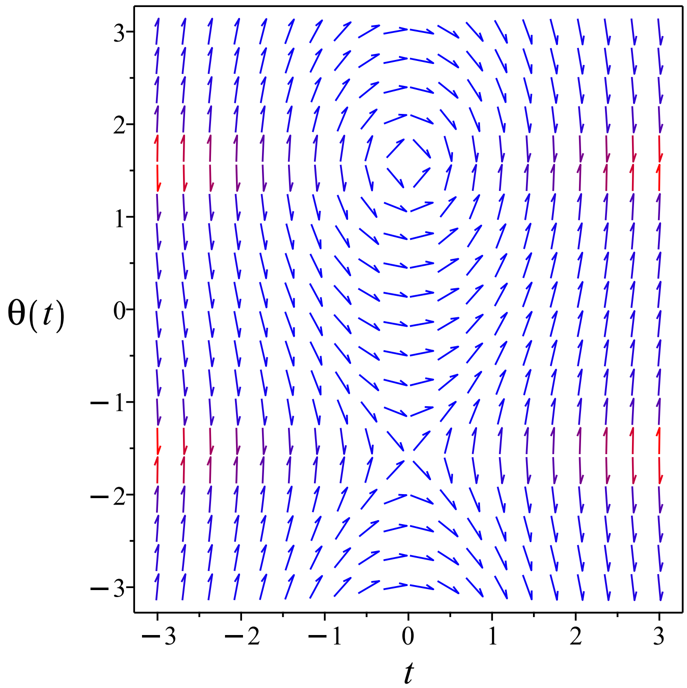


Figure 58: Slope field plot

Verification of solutions

$$\theta = -\arcsin\left(-\frac{(t^2+1)^{\frac{3}{2}}}{3} + c_1\right)$$

Verified OK.

4.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, \theta) \, \mathrm{d}t + N(t, \theta) \, \mathrm{d}\theta = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{\sec(\theta)}\right) \mathrm{d}\theta &= \left(t\sqrt{t^2+1}\right) \mathrm{d}t \\
\left(-t\sqrt{t^2+1}\right) \mathrm{d}t + \left(\frac{1}{\sec(\theta)}\right) \mathrm{d}\theta &= 0 \qquad\qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, \theta) &= -t\sqrt{t^2+1} \\
N(t, \theta) &= \frac{1}{\sec(\theta)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial \theta} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(-t\sqrt{t^2+1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{\sec(\theta)}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial \theta} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,\theta)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial \theta} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int M\,\mathrm{d}t
$$

$$
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int -t\sqrt{t^2+1}\,\mathrm{d}t
$$

$$
\phi = -\frac{(t^2+1)^{\frac{3}{2}}}{3} + f(\theta) \tag{3}
$$

Where $f(\theta)$ is used for the constant of integration since ϕ is a function of both t and θ. Taking derivative of equation (3) w.r.t θ gives

$$
\frac{\partial \phi}{\partial \theta} = 0 + f'(\theta) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial \theta} = \frac{1}{\sec(\theta)}$. Therefore equation (4) becomes

$$
\frac{1}{\sec(\theta)} = 0 + f'(\theta) \tag{5}
$$

Solving equation (5) for $f'(\theta)$ gives

$$
\begin{aligned}
f'(\theta) &= \frac{1}{\sec(\theta)}\\
&= \cos(\theta)
\end{aligned}
$$

Integrating the above w.r.t θ results in

$$\int f'(\theta)\, d\theta = \int (\cos(\theta))\, d\theta$$
$$f(\theta) = \sin(\theta) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(\theta)$ into equation (3) gives ϕ

$$\phi = -\frac{(t^2+1)^{\frac{3}{2}}}{3} + \sin(\theta) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(t^2+1)^{\frac{3}{2}}}{3} + \sin(\theta)$$

Summary
The solution(s) found are the following

$$-\frac{(t^2+1)^{\frac{3}{2}}}{3} + \sin(\theta) = c_1 \tag{1}$$

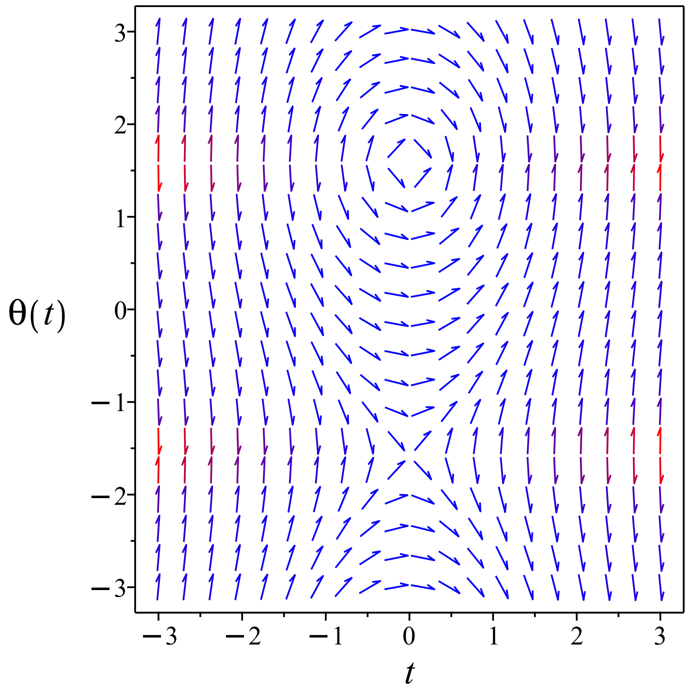


Figure 59: Slope field plot

Verification of solutions

$$-\frac{(t^2+1)^{\frac{3}{2}}}{3}+\sin(\theta)=c_1$$

Verified OK.

4.10.4 Maple step by step solution

Let's solve

$\theta'-t\sqrt{t^2+1}\,\sec(\theta)=0$

- Highest derivative means the order of the ODE is 1

  θ'

- Separate variables

  $\frac{\theta'}{\sec(\theta)}=t\sqrt{t^2+1}$

- Integrate both sides with respect to t

  $\int \frac{\theta'}{\sec(\theta)}dt=\int t\sqrt{t^2+1}dt+c_1$

- Evaluate integral

$$\sin(\theta) = \frac{(t^2+1)^{\frac{3}{2}}}{3} + c_1$$

- Solve for θ

$$\theta = \arcsin\left(\frac{(t^2+1)^{\frac{3}{2}}}{3} + c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(theta(t),t)=t*sqrt(1+t^2)*sec(theta(t)),theta(t), singsol=all)
```

$$\theta(t) = \arcsin\left(\frac{t^2\sqrt{t^2+1}}{3} + \frac{\sqrt{t^2+1}}{3} + c_1\right)$$

✓ Solution by Mathematica

Time used: 5.052 (sec). Leaf size: 91

```
DSolve[theta'[t]==t*Sqrt[1+t^2]*Sec[theta[t]],theta[t],t,IncludeSingularSolutions -> True]
```

$$\theta(t) \to \arcsin\left(\frac{1}{3}\left(\sqrt{t^2+1}t^2 + \sqrt{t^2+1} + 3c_1\right)\right)$$
$$\theta(t) \to \arcsin\left(\frac{1}{3}\left(\sqrt{t^2+1}t^2 + \sqrt{t^2+1} + 3c_1\right)\right)$$
$$\theta(t) \to \arcsin\left(\frac{1}{3}(t^2+1)^{3/2}\right)$$

4.11 problem 4(c)

4.11.1 Solving as separable ode 230
4.11.2 Solving as differentialType ode 232
4.11.3 Solving as homogeneousTypeMapleC ode 234
4.11.4 Solving as first order ode lie symmetry lookup ode 237
4.11.5 Solving as exact ode 241
4.11.6 Maple step by step solution 245

Internal problem ID [11378]
Internal file name [`OUTPUT/10360_Wednesday_May_17_2023_07_49_57_PM_48557998/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 4(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(2u+1)\,u' = 1+t}$$

4.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= \frac{1+t}{2u+1} \end{aligned}$$

Where $f(t) = 1 + t$ and $g(u) = \frac{1}{2u+1}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{2u+1}}\,du &= 1 + t\,dt \\
\int \frac{1}{\frac{1}{2u+1}}\,du &= \int 1 + t\,dt \\
u^2 + u &= \frac{1}{2}t^2 + t + c_1
\end{aligned}
$$

Which results in

$$
\begin{aligned}
u &= -\frac{1}{2} + \frac{\sqrt{2t^2 + 4c_1 + 4t + 1}}{2} \\
u &= -\frac{1}{2} - \frac{\sqrt{2t^2 + 4c_1 + 4t + 1}}{2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
u = -\frac{1}{2} + \frac{\sqrt{2t^2 + 4c_1 + 4t + 1}}{2} \tag{1}
$$

$$
u = -\frac{1}{2} - \frac{\sqrt{2t^2 + 4c_1 + 4t + 1}}{2} \tag{2}
$$

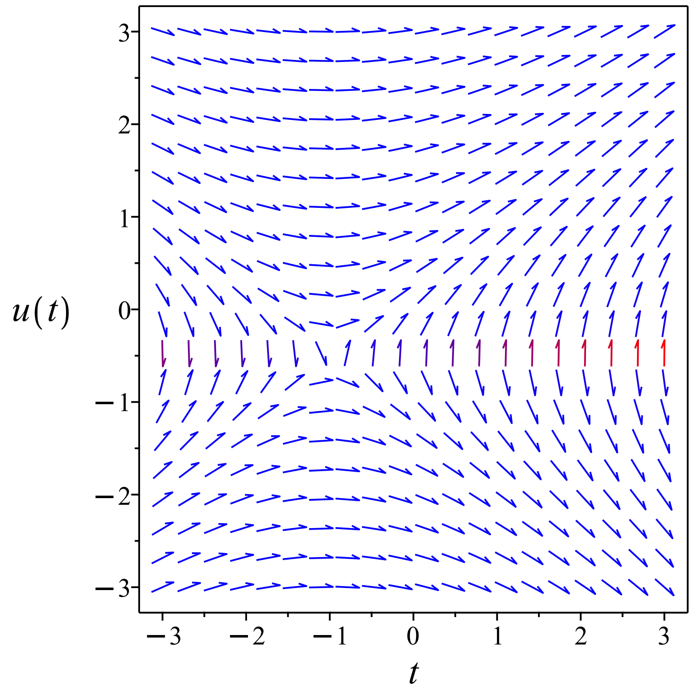


Figure 60: Slope field plot

Verification of solutions

$$u = -\frac{1}{2} + \frac{\sqrt{2t^2 + 4c_1 + 4t + 1}}{2}$$

Verified OK.

$$u = -\frac{1}{2} - \frac{\sqrt{2t^2 + 4c_1 + 4t + 1}}{2}$$

Verified OK.

4.11.2 Solving as differentialType ode

Writing the ode as

$$u' = \frac{1+t}{2u+1} \tag{1}$$

Which becomes

$$(2u+1)\,du = (1+t)\,dt \tag{2}$$

But the RHS is complete differential because

$$(1+t)\,dt = d\left(\frac{1}{2}t^2 + t\right)$$

Hence (2) becomes

$$(2u + 1)\, du = d\left(\frac{1}{2}t^2 + t\right)$$

Integrating both sides gives gives these solutions

$$u = -\frac{1}{2} + \frac{\sqrt{2t^2 + 4c_1 + 4t + 1}}{2} + c_1$$
$$u = -\frac{1}{2} - \frac{\sqrt{2t^2 + 4c_1 + 4t + 1}}{2} + c_1$$

Summary

The solution(s) found are the following

$$u = -\frac{1}{2} + \frac{\sqrt{2t^2 + 4c_1 + 4t + 1}}{2} + c_1 \tag{1}$$
$$u = -\frac{1}{2} - \frac{\sqrt{2t^2 + 4c_1 + 4t + 1}}{2} + c_1 \tag{2}$$

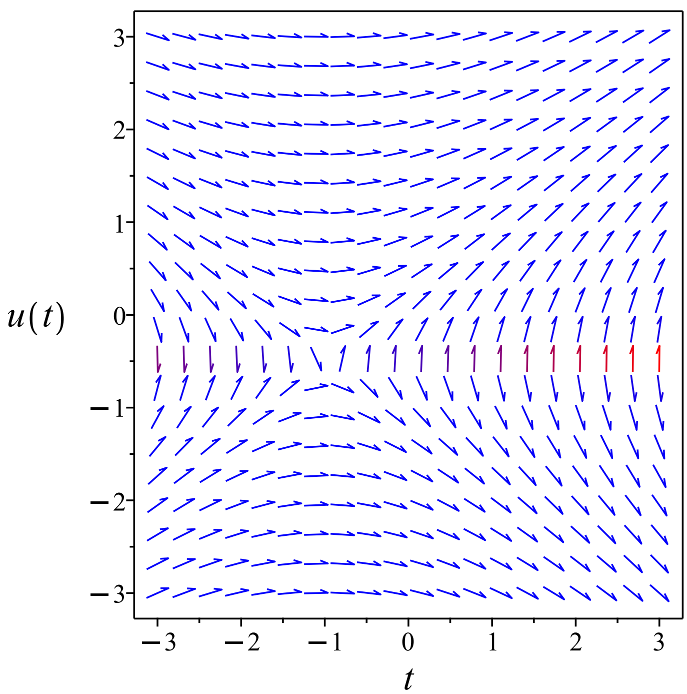


Figure 61: Slope field plot

Verification of solutions

$$
u = -\frac{1}{2} + \frac{\sqrt{2t^2 + 4c_1 + 4t + 1}}{2} + c_1
$$

Verified OK.

$$
u = -\frac{1}{2} - \frac{\sqrt{2t^2 + 4c_1 + 4t + 1}}{2} + c_1
$$

Verified OK.

4.11.3 Solving as homogeneousTypeMapleC ode

Let $Y = u + y_0$ and $X = t + x_0$ then the above is transformed to new ode in $Y(X)$

$$
\frac{d}{dX}Y(X) = \frac{1 + X + x_0}{2Y(X) + 2y_0 + 1}
$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$
\begin{aligned}
x_0 &= -1 \\
y_0 &= -\frac{1}{2}
\end{aligned}
$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$
\frac{d}{dX}Y(X) = \frac{X}{2Y(X)}
$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{X}{2Y} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = X$ and $N = 2Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{1}{2u} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{1}{2u(X)} - u(X)}{X}
\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{1}{2u(X)} - u(X)}{X} = 0$$

Or

$$2\left(\frac{d}{dX}u(X)\right)u(X)\,X + 2u(X)^2 - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(X, u) \\
&= f(X)g(u) \\
&= -\frac{2u^2 - 1}{2uX}
\end{aligned}$$

Where $f(X) = -\frac{1}{2X}$ and $g(u) = \frac{2u^2-1}{u}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{2u^2-1}{u}}\,du &= -\frac{1}{2X}\,dX \\
\int \frac{1}{\frac{2u^2-1}{u}}\,du &= \int -\frac{1}{2X}\,dX \\
\frac{\ln(2u^2-1)}{4} &= -\frac{\ln(X)}{2} + c_2
\end{aligned}$$

Raising both side to exponential gives

$$\left(2u^2 - 1\right)^{\frac{1}{4}} = \mathrm{e}^{-\frac{\ln(X)}{2} + c_2}$$

Which simplifies to

$$\left(2u^2 - 1\right)^{\frac{1}{4}} = \frac{c_3}{\sqrt{X}}$$

Which simplifies to

$$\left(2u(X)^2-1\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{X}}$$

The solution is

$$\left(2u(X)^2-1\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{X}}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\left(\frac{2Y(X)^2}{X^2}-1\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{X}}$$

Which simplifies to

$$\left(-\frac{-2Y(X)^2+X^2}{X^2}\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{X}}$$

Using the solution for $Y(X)$

$$\left(-\frac{-2Y(X)^2+X^2}{X^2}\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{X}}$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= u + y_0 \\ X &= t + x_0 \end{aligned}$$

Or

$$\begin{aligned} Y &= u - \frac{1}{2} \\ X &= t - 1 \end{aligned}$$

Then the solution in u becomes

$$\left(-\frac{-2\left(u+\frac{1}{2}\right)^2+(1+t)^2}{(1+t)^2}\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{1+t}}$$

Summary
The solution(s) found are the following

$$\left(-\frac{-2\left(u+\frac{1}{2}\right)^2+(1+t)^2}{(1+t)^2}\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{1+t}} \tag{1}$$

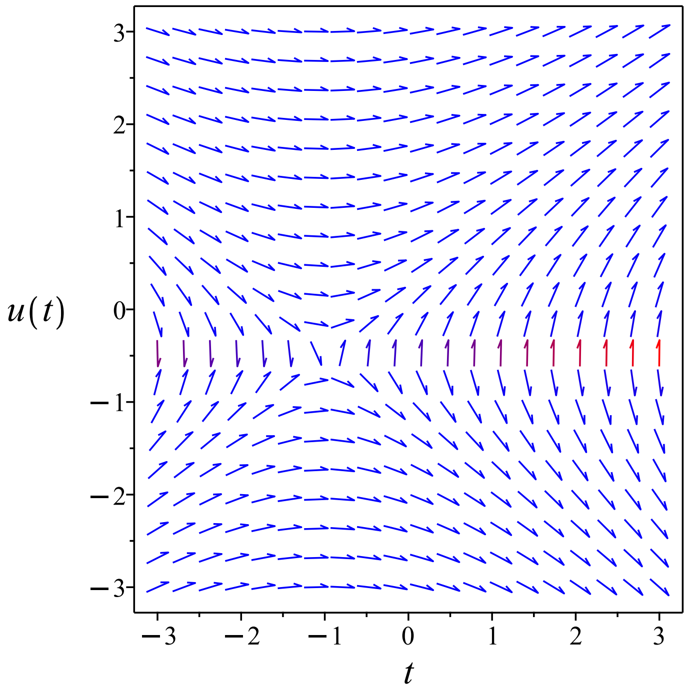


Figure 62: Slope field plot

Verification of solutions

$$\left(-\frac{-2\left(u+\frac{1}{2}\right)^2+(1+t)^2}{(1+t)^2}\right)^{\frac{1}{4}}=\frac{c_3\mathrm{e}^{c_2}}{\sqrt{1+t}}$$

Verified OK.

4.11.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$u'=\frac{1+t}{2u+1}$$
$$u'=\omega(t,u)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t+\omega(\eta_u-\xi_t)-\omega^2\xi_u-\omega_t\xi-\omega_u\eta=0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 45: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t,u) &= \frac{1}{1+t} \\ \eta(t,u) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,u) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{du}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}\right) S(t,u) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = u$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{1+t}} dt \end{aligned}$$

Which results in

$$S = \frac{1}{2}t^2 + t$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,u)S_u}{R_t + \omega(t,u)R_u} \tag{2}$$

Where in the above R_t, R_u, S_t, S_u are all partial derivatives and $\omega(t,u)$ is the right hand side of the original ode given by

$$\omega(t,u) = \frac{1+t}{2u+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_u &= 1 \\ S_t &= 1+t \\ S_u &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2u + 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, u in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R + 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + R + c_1 \tag{4}$$

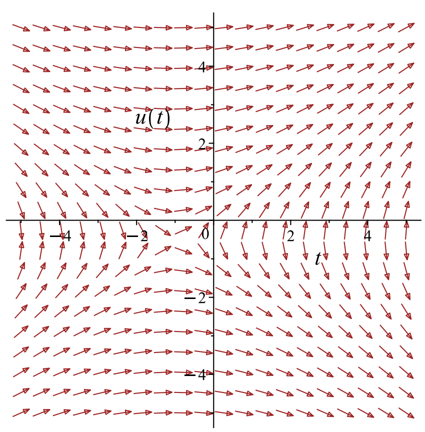
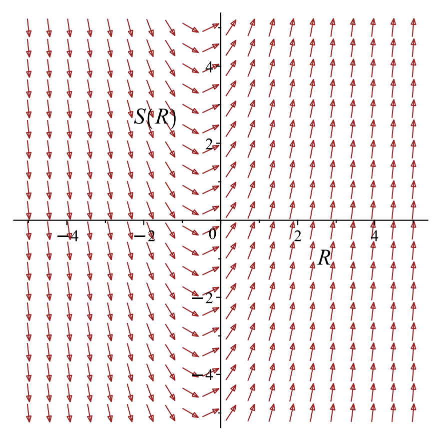
To complete the solution, we just need to transform (4) back to t, u coordinates. This results in

$$\frac{1}{2}t^2 + t = u^2 + c_1 + u$$

Which simplifies to

$$\frac{1}{2}t^2 + t = u^2 + c_1 + u$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, u coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{du}{dt} = \frac{1+t}{2u+1}$ 	$R = u$ $S = \frac{1}{2}t^2 + t$	$\frac{dS}{dR} = 2R + 1$ 

Summary

The solution(s) found are the following

$$\frac{1}{2}t^2 + t = u^2 + c_1 + u \tag{1}$$

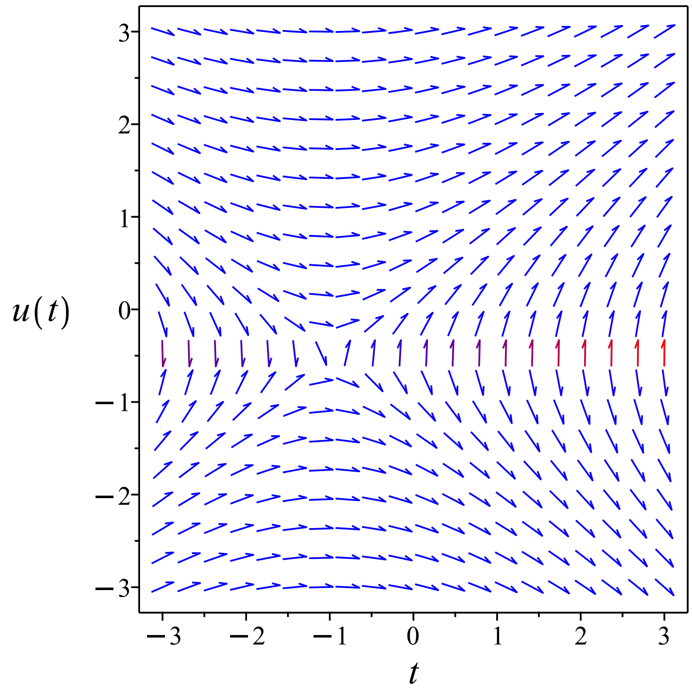


Figure 63: Slope field plot

<u>Verification of solutions</u>

$$\frac{1}{2}t^2 + t = u^2 + c_1 + u$$

Verified OK.

4.11.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,u)\,\mathrm{d}t + N(t,u)\,\mathrm{d}u = 0 \tag{1A}$$

Therefore

$$(2u+1)\,\mathrm{d}u = (1+t)\,\mathrm{d}t$$
$$(-1-t)\,\mathrm{d}t + (2u+1)\,\mathrm{d}u = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t,u) = -1-t$$
$$N(t,u) = 2u+1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial u} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial u} &= \frac{\partial}{\partial u}(-1-t)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(2u+1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial u} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,u)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial u} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int M\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int -1 - t\,\mathrm{d}t$$
$$\phi = -t - \frac{1}{2}t^2 + f(u) \tag{3}$$

Where $f(u)$ is used for the constant of integration since ϕ is a function of both t and u. Taking derivative of equation (3) w.r.t u gives

$$\frac{\partial \phi}{\partial u} = 0 + f'(u) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial u} = 2u + 1$. Therefore equation (4) becomes

$$2u + 1 = 0 + f'(u) \tag{5}$$

Solving equation (5) for $f'(u)$ gives

$$f'(u) = 2u + 1$$

Integrating the above w.r.t u gives

$$\int f'(u)\,\mathrm{d}u = \int (2u + 1)\,\mathrm{d}u$$
$$f(u) = u^2 + u + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(u)$ into equation (3) gives ϕ

$$\phi = -t - \frac{1}{2}t^2 + u^2 + u + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -t - \frac{1}{2}t^2 + u^2 + u$$

Summary

The solution(s) found are the following

$$u^2 - \frac{t^2}{2} + u - t = c_1 \tag{1}$$

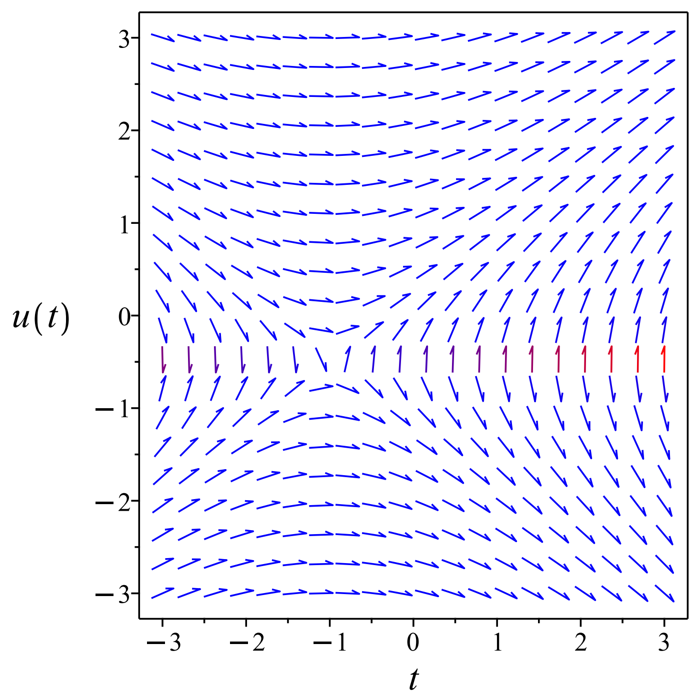


Figure 64: Slope field plot

Verification of solutions

$$u^2 - \frac{t^2}{2} + u - t = c_1$$

Verified OK.

4.11.6 Maple step by step solution

Let's solve

$(2u+1)\,u' = 1+t$

- Highest derivative means the order of the ODE is 1

  u'

- Integrate both sides with respect to t

  $\int (2u+1)\,u' dt = \int (1+t)\,dt + c_1$

- Evaluate integral

  $u^2 + u = \frac{1}{2}t^2 + t + c_1$

- Solve for u

  $\left\{u = -\frac{1}{2} - \frac{\sqrt{2t^2+4c_1+4t+1}}{2}, u = -\frac{1}{2} + \frac{\sqrt{2t^2+4c_1+4t+1}}{2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve((2*u(t)+1)*diff(u(t),t)-(1+t)=0,u(t), singsol=all)
```

$$u(t) = -\frac{1}{2} - \frac{\sqrt{2t^2+4c_1+4t+1}}{2}$$
$$u(t) = -\frac{1}{2} + \frac{\sqrt{2t^2+4c_1+4t+1}}{2}$$

✓ Solution by Mathematica

Time used: 0.171 (sec). Leaf size: 59

```
DSolve[(2*u[t]+1)*u'[t]-(1+t)==0,u[t],t,IncludeSingularSolutions -> True]
```

$$u(t) \to \frac{1}{2}\left(-1 - \sqrt{2t^2 + 4t + 1 + 4c_1}\right)$$
$$u(t) \to \frac{1}{2}\left(-1 + \sqrt{2t^2 + 4t + 1 + 4c_1}\right)$$

4.12 problem 4(d)

4.12.1 Solving as separable ode 247
4.12.2 Solving as first order ode lie symmetry lookup ode 249
4.12.3 Solving as exact ode 253
4.12.4 Solving as riccati ode 257
4.12.5 Maple step by step solution 259

Internal problem ID [11379]
Internal file name [`OUTPUT/10361_Wednesday_May_17_2023_07_49_58_PM_81950458/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 4(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{R' - (1+t)\left(1+R^2\right) = 0}$$

4.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} R' &= F(t,R) \\ &= f(t)g(R) \\ &= (1+t)\left(R^2+1\right) \end{aligned}$$

Where $f(t) = 1+t$ and $g(R) = R^2+1$. Integrating both sides gives

$$\frac{1}{R^2+1}\,dR = 1+t\,dt$$

$$\int \frac{1}{R^2+1}\, dR = \int 1+t\, dt$$
$$\arctan(R) = \frac{1}{2}t^2 + t + c_1$$

Which results in

$$R = \tan\left(\frac{1}{2}t^2 + t + c_1\right)$$

Summary

The solution(s) found are the following

$$R = \tan\left(\frac{1}{2}t^2 + t + c_1\right) \tag{1}$$

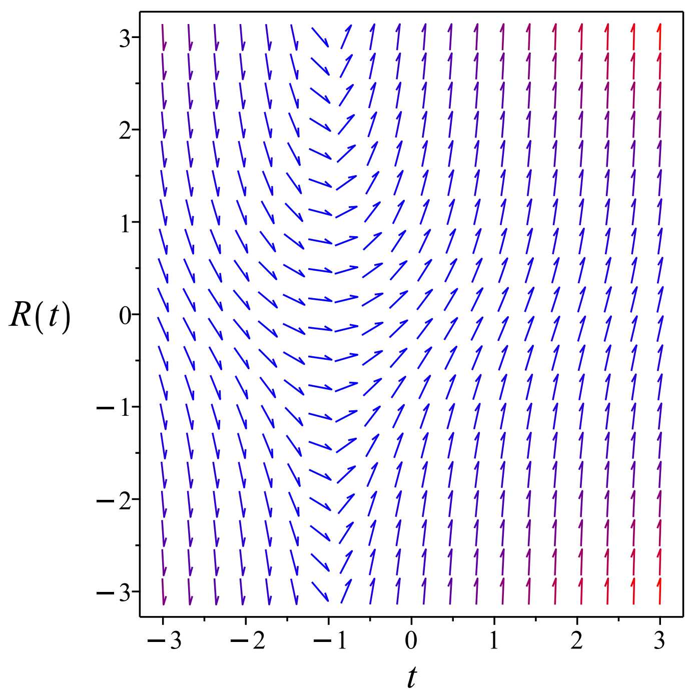


Figure 65: Slope field plot

Verification of solutions

$$R = \tan\left(\frac{1}{2}t^2 + t + c_1\right)$$

Verified OK.

4.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
R' &= (1+t)\left(R^2+1\right) \\
R' &= \omega(t, R)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_R - \xi_t) - \omega^2 \xi_R - \omega_t \xi - \omega_R \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 48: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, R) &= \frac{1}{1+t} \\
\eta(t, R) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, R) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dR}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial R}\right) S(t, R) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = R
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dt \\
&= \int \frac{1}{\frac{1}{1+t}} dt
\end{aligned}
$$

Which results in

$$
S = \frac{1}{2} t^2 + t
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t, R) S_R}{R_t + \omega(t, R) R_R} \tag{2}
$$

Where in the above R_t, R_R, S_t, S_R are all partial derivatives and $\omega(t, R)$ is the right hand side of the original ode given by

$$
\omega(t, R) = (1+t)\left(R^2 + 1\right)
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_R &= 1 \\
S_t &= 1 + t \\
S_R &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{R^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, R in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, R coordinates. This results in

$$\frac{1}{2}t^2 + t = \arctan(R) + c_1$$

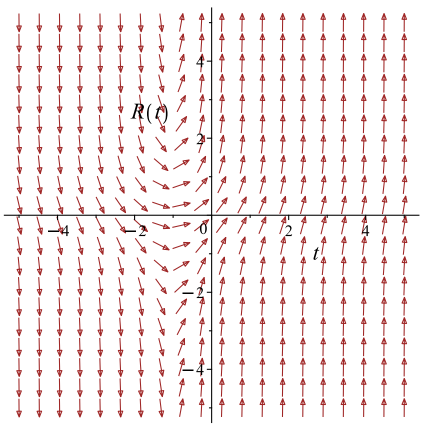
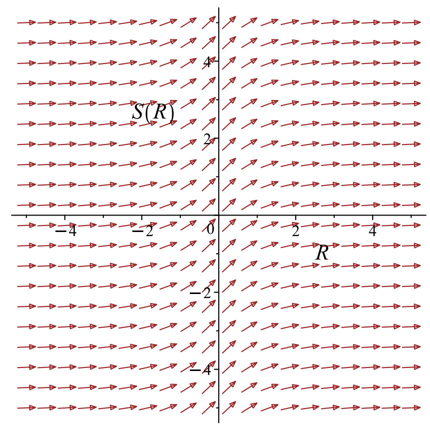
Which simplifies to

$$\frac{1}{2}t^2 + t = \arctan(R) + c_1$$

Which gives

$$R = -\tan\left(-\frac{1}{2}t^2 + c_1 - t\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, R coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dR}{dt} = (1+t)(R^2+1)$ 	$R = R$ $S = \frac{1}{2}t^2 + t$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

Summary

The solution(s) found are the following

$$R = -\tan\left(-\frac{1}{2}t^2 + c_1 - t\right) \tag{1}$$

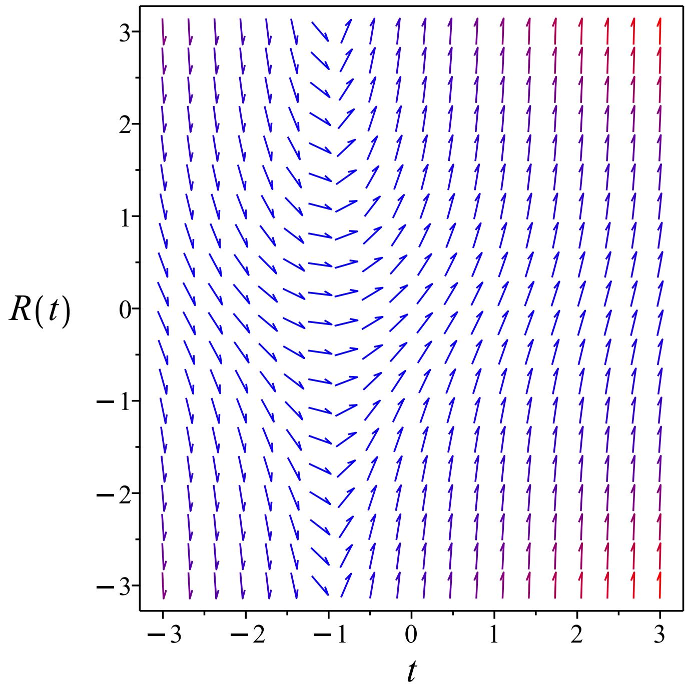


Figure 66: Slope field plot

Verification of solutions

$$R = -\tan\left(-\frac{1}{2}t^2 + c_1 - t\right)$$

Verified OK.

4.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, R)\,\mathrm{d}t + N(t, R)\,\mathrm{d}R = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{R^2+1}\right) \mathrm{d}R = (1+t)\,\mathrm{d}t$$
$$(-1-t)\,\mathrm{d}t + \left(\frac{1}{R^2+1}\right) \mathrm{d}R = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, R) = -1 - t$$
$$N(t, R) = \frac{1}{R^2+1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial R} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial R} = \frac{\partial}{\partial R}(-1-t)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{R^2+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial R} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, R)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial R} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -1 - t \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = -t - \frac{1}{2}t^2 + f(R) \tag{3}
$$

Where $f(R)$ is used for the constant of integration since ϕ is a function of both t and R. Taking derivative of equation (3) w.r.t R gives

$$
\frac{\partial \phi}{\partial R} = 0 + f'(R) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial R} = \frac{1}{R^2+1}$. Therefore equation (4) becomes

$$
\frac{1}{R^2+1} = 0 + f'(R) \tag{5}
$$

Solving equation (5) for $f'(R)$ gives

$$
f'(R) = \frac{1}{R^2+1}
$$

Integrating the above w.r.t R gives

$$
\begin{aligned}
\int f'(R) \, \mathrm{d}R &= \int \left(\frac{1}{R^2+1}\right) \mathrm{d}R \\
f(R) &= \arctan(R) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(R)$ into equation (3) gives ϕ

$$\phi = -t - \frac{t^2}{2} + \arctan(R) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -t - \frac{t^2}{2} + \arctan(R)$$

The solution becomes

$$R = \tan\left(\frac{1}{2}t^2 + t + c_1\right)$$

Summary

The solution(s) found are the following

$$R = \tan\left(\frac{1}{2}t^2 + t + c_1\right) \tag{1}$$

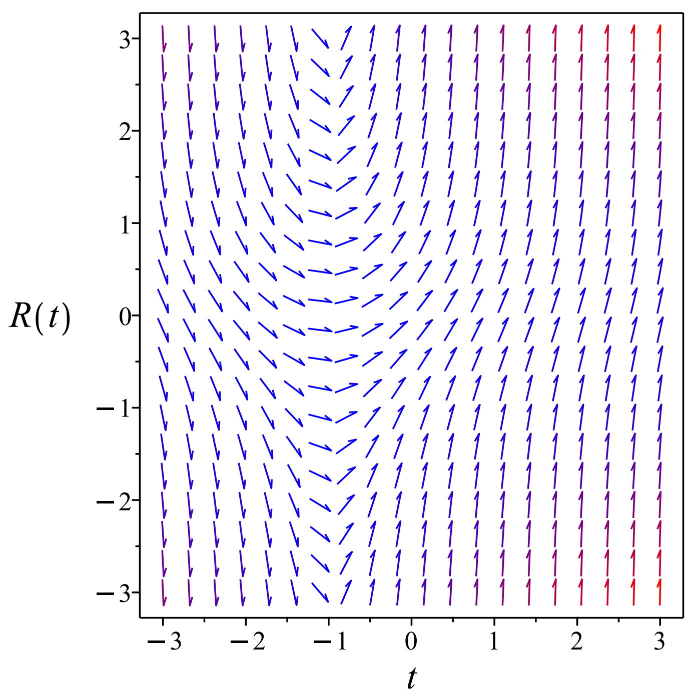


Figure 67: Slope field plot

Verification of solutions

$$R = \tan\left(\frac{1}{2}t^2 + t + c_1\right)$$

Verified OK.

4.12.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}R' &= F(t, R)\\ &= (1+t)\left(R^2+1\right)\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$R' = R^2 t + R^2 + t + 1$$

With Riccati ODE standard form

$$R' = f_0(t) + f_1(t)R + f_2(t)R^2$$

Shows that $f_0(t) = 1+t$, $f_1(t) = 0$ and $f_2(t) = 1+t$. Let

$$\begin{aligned}R &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{(1+t)\,u}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2)\,u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned}f_2' &= 1\\ f_1 f_2 &= 0\\ f_2^2 f_0 &= (1+t)^3\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(1+t)\,u''(t) - u'(t) + (1+t)^3\,u(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 \sin\left(\frac{1}{2}t^2 + t\right) + c_2 \cos\left(\frac{1}{2}t^2 + t\right)$$

The above shows that

$$u'(t) = (1+t)\left(c_1 \cos\left(\frac{1}{2}t^2 + t\right) - c_2 \sin\left(\frac{1}{2}t^2 + t\right)\right)$$

Using the above in (1) gives the solution

$$R = -\frac{c_1 \cos\left(\frac{1}{2}t^2 + t\right) - c_2 \sin\left(\frac{1}{2}t^2 + t\right)}{c_1 \sin\left(\frac{1}{2}t^2 + t\right) + c_2 \cos\left(\frac{1}{2}t^2 + t\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$R = \frac{-c_3 \cos\left(\frac{1}{2}t^2 + t\right) + \sin\left(\frac{1}{2}t^2 + t\right)}{c_3 \sin\left(\frac{1}{2}t^2 + t\right) + \cos\left(\frac{1}{2}t^2 + t\right)}$$

<u>Summary</u>

The solution(s) found are the following

$$R = \frac{-c_3 \cos\left(\frac{1}{2}t^2 + t\right) + \sin\left(\frac{1}{2}t^2 + t\right)}{c_3 \sin\left(\frac{1}{2}t^2 + t\right) + \cos\left(\frac{1}{2}t^2 + t\right)} \tag{1}$$

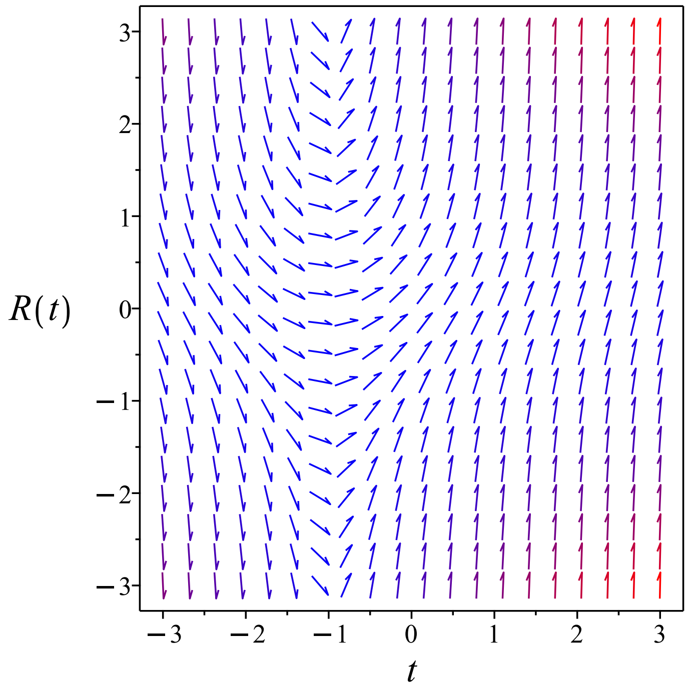


Figure 68: Slope field plot

Verification of solutions

$$R = \frac{-c_3 \cos\left(\frac{1}{2}t^2 + t\right) + \sin\left(\frac{1}{2}t^2 + t\right)}{c_3 \sin\left(\frac{1}{2}t^2 + t\right) + \cos\left(\frac{1}{2}t^2 + t\right)}$$

Verified OK.

4.12.5 Maple step by step solution

Let's solve

$R' - (1 + t)(1 + R^2) = 0$

- Highest derivative means the order of the ODE is 1

  R'

- Separate variables

  $\frac{R'}{1+R^2} = 1 + t$

- Integrate both sides with respect to t

  $\int \frac{R'}{1+R^2} dt = \int (1 + t)\, dt + c_1$

- Evaluate integral

$$\arctan(R) = \frac{1}{2}t^2 + t + c_1$$

- Solve for R

$$R = \tan\left(\frac{1}{2}t^2 + t + c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve(diff(R(t),t)=(t+1)*(1+R(t)^2),R(t), singsol=all)
```

$$R(t) = \tan\left(\frac{1}{2}t^2 + t + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.315 (sec). Leaf size: 31

```
DSolve[R'[t]==(t+1)*(1+R[t]^2),R[t],t,IncludeSingularSolutions -> True]
```

$$R(t) \to \tan\left(\frac{t^2}{2} + t + c_1\right)$$
$$R(t) \to -i$$
$$R(t) \to i$$

4.13 problem 4(e)

4.13.1 Solving as quadrature ode 261
4.13.2 Maple step by step solution 262

Internal problem ID [11380]
Internal file name [`OUTPUT/10362_Wednesday_May_17_2023_07_49_59_PM_52105283/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 4(e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y + \frac{1}{y} = 0}$$

4.13.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{y}{y^2+1} dy = \int dt$$
$$-\frac{\ln(y^2+1)}{2} = t + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{y^2+1}} = e^{t+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{y^2+1}} = c_2 e^t$$

Summary

The solution(s) found are the following

$$y = \operatorname{RootOf}\left(\_Z^2 c_2^2 e^{2t} + c_2^2 e^{2t} - 1\right) \tag{1}$$

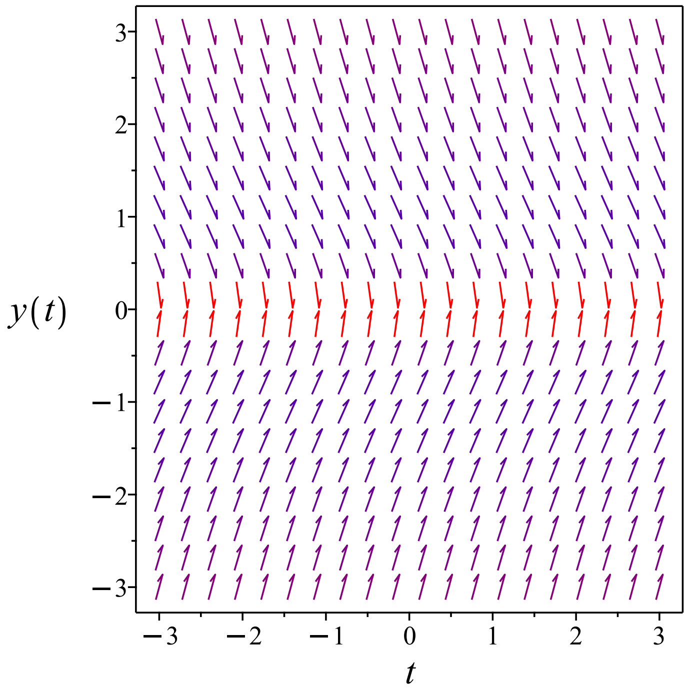


Figure 69: Slope field plot

Verification of solutions

$$y = \operatorname{RootOf}\left(\_Z^2 c_2^2 e^{2t} + c_2^2 e^{2t} - 1\right)$$

Verified OK.

4.13.2 Maple step by step solution

Let's solve

$y' + y + \frac{1}{y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y-\frac{1}{y}} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{-y-\frac{1}{y}}dt = \int 1dt + c_1$

- Evaluate integral

  $-\frac{\ln(1+y^2)}{2} = t + c_1$

- Solve for y

  $\left\{y = \sqrt{-1+\mathrm{e}^{-2t-2c_1}}, y = -\sqrt{-1+\mathrm{e}^{-2t-2c_1}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve(diff(y(t),t)+y(t)+1/y(t)=0,y(t), singsol=all)
```

$$y(t) = \sqrt{\mathrm{e}^{-2t}c_1 - 1}$$
$$y(t) = -\sqrt{\mathrm{e}^{-2t}c_1 - 1}$$

✓ Solution by Mathematica

Time used: 4.571 (sec). Leaf size: 57

```
DSolve[y'[t]+y[t]+1/y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\sqrt{-1+e^{-2t+2c_1}}$$
$$y(t) \to \sqrt{-1+e^{-2t+2c_1}}$$
$$y(t) \to -i$$
$$y(t) \to i$$

4.14 problem 4(f)

4.14.1 Solving as separable ode 264
4.14.2 Solving as first order ode lie symmetry lookup ode 266
4.14.3 Solving as exact ode 270
4.14.4 Solving as riccati ode 274
4.14.5 Maple step by step solution 276

Internal problem ID [11381]
Internal file name [`OUTPUT/10363_Wednesday_May_17_2023_07_50_00_PM_89926492/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 4(f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(1+t)\,x'+x^2=0}$$

4.14.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}x' &= F(t,x)\\ &= f(t)g(x)\\ &= -\frac{x^2}{1+t}\end{aligned}$$

Where $f(t)=-\frac{1}{1+t}$ and $g(x)=x^2$. Integrating both sides gives

$$\frac{1}{x^2}\,dx=-\frac{1}{1+t}\,dt$$

$$\int \frac{1}{x^2}\,dx = \int -\frac{1}{1+t}\,dt$$

$$-\frac{1}{x} = -\ln(1+t) + c_1$$

Which results in

$$x = \frac{1}{\ln(1+t) - c_1}$$

Summary

The solution(s) found are the following

$$x = \frac{1}{\ln(1+t) - c_1} \tag{1}$$

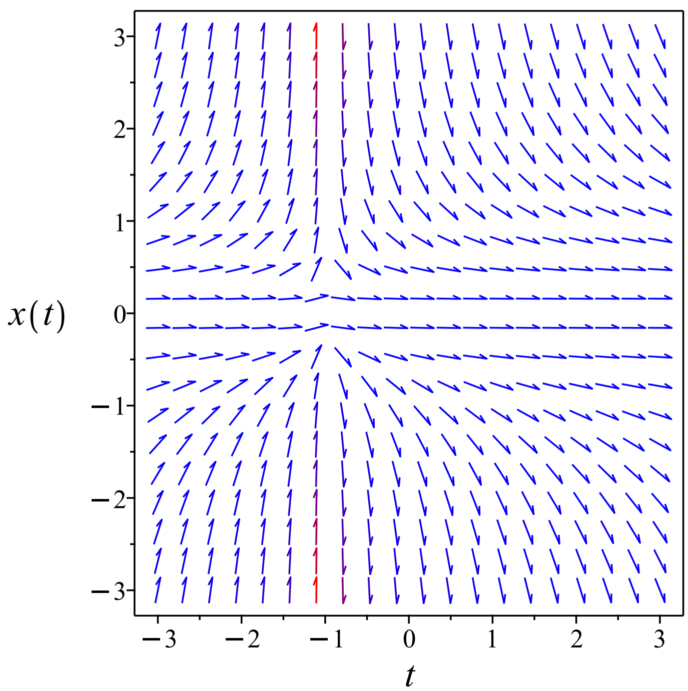


Figure 70: Slope field plot

Verification of solutions

$$x = \frac{1}{\ln(1+t) - c_1}$$

Verified OK.

4.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= -\frac{x^2}{1+t} \\
x' &= \omega(t, x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 52: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t,x) &= -1-t\\ \eta(t,x) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dt\\ &= \int \frac{1}{-1-t} dt\end{aligned}$$

Which results in

$$S = -\ln(-1-t)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -\frac{x^2}{1+t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_t &= 0\\ R_x &= 1\\ S_t &= \frac{1}{-1-t}\\ S_x &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$-\ln(-1-t) = -\frac{1}{x} + c_1$$

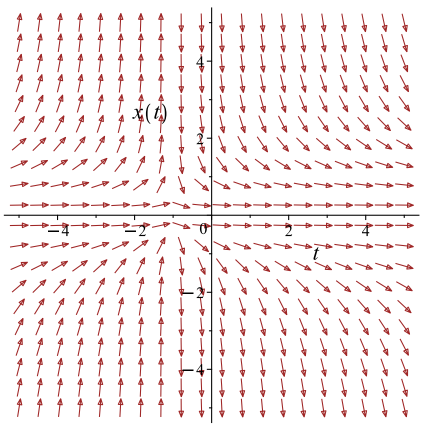
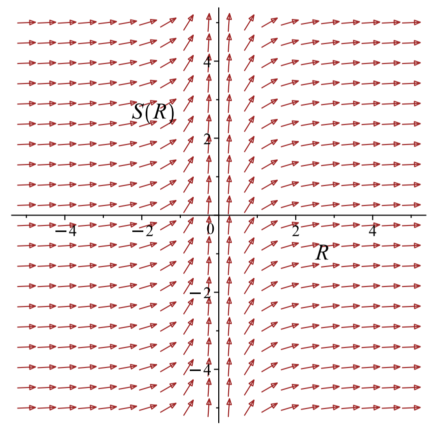
Which simplifies to

$$-\ln(-1-t) = -\frac{1}{x} + c_1$$

Which gives

$$x = \frac{1}{\ln(-1-t) + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\frac{x^2}{1+t}$ 	$R = x$ $S = -\ln(-1-t)$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$x = \frac{1}{\ln(-1-t) + c_1} \tag{1}$$

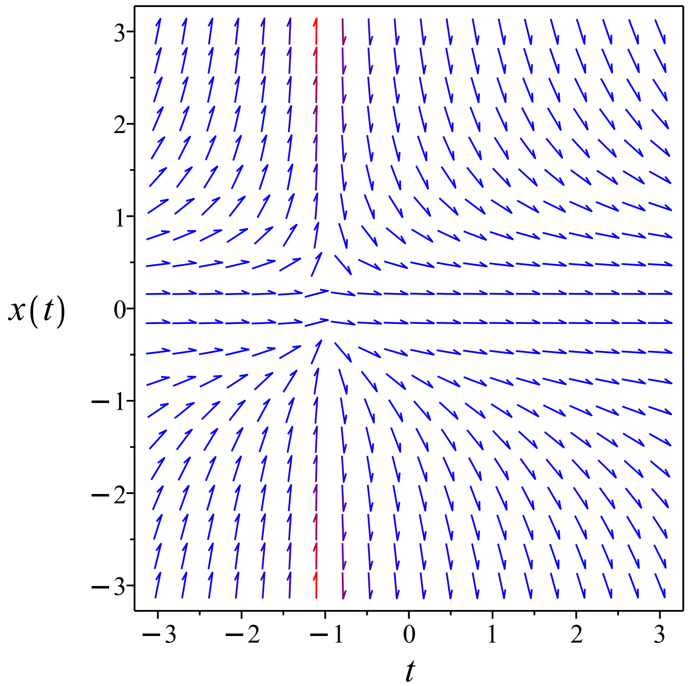


Figure 71: Slope field plot

Verification of solutions

$$x = \frac{1}{\ln(-1-t)+c_1}$$

Verified OK.

4.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{x^2}\right)\mathrm{d}x &= \left(\frac{1}{1+t}\right)\mathrm{d}t \\ \left(-\frac{1}{1+t}\right)\mathrm{d}t + \left(-\frac{1}{x^2}\right)\mathrm{d}x &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x) &= -\frac{1}{1+t} \\ N(t,x) &= -\frac{1}{x^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{1+t}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{1}{x^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{1}{1+t}\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = -\ln(1+t) + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = -\frac{1}{x^2}$. Therefore equation (4) becomes

$$
-\frac{1}{x^2} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = -\frac{1}{x^2}
$$

Integrating the above w.r.t x gives

$$
\begin{aligned}
\int f'(x)\,\mathrm{d}x &= \int \left(-\frac{1}{x^2}\right)\mathrm{d}x \\
f(x) &= \frac{1}{x} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\ln(1+t) + \frac{1}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(1+t) + \frac{1}{x}$$

The solution becomes

$$x = \frac{1}{\ln(1+t) + c_1}$$

Summary

The solution(s) found are the following

$$x = \frac{1}{\ln(1+t) + c_1} \tag{1}$$

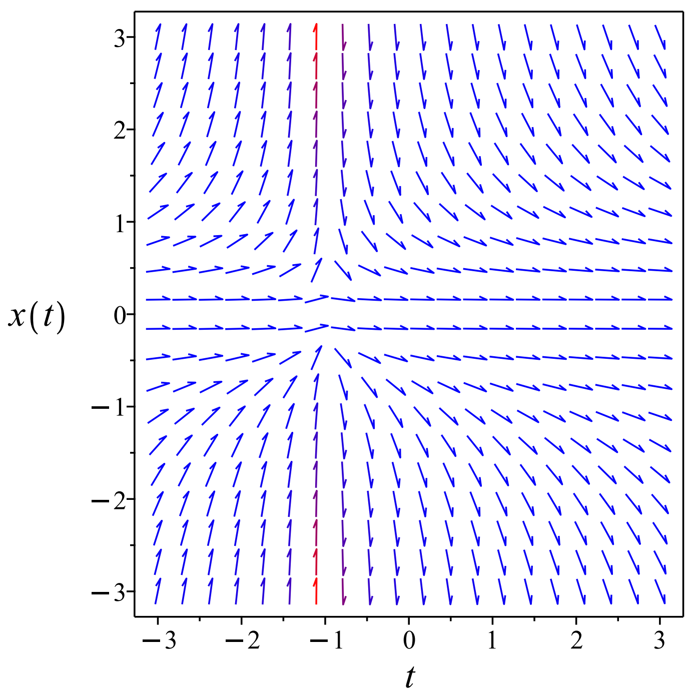


Figure 72: Slope field plot

Verification of solutions

$$x = \frac{1}{\ln(1+t) + c_1}$$

Verified OK.

4.14.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= -\frac{x^2}{1+t} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$x' = -\frac{x^2}{1+t}$$

With Riccati ODE standard form

$$x' = f_0(t) + f_1(t)x + f_2(t)x^2$$

Shows that $f_0(t) = 0$, $f_1(t) = 0$ and $f_2(t) = -\frac{1}{1+t}$. Let

$$\begin{aligned} x &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{1+t}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2) u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{1}{(1+t)^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(t)}{1+t} - \frac{u'(t)}{(1+t)^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_2 \ln(1+t) + c_1$$

The above shows that

$$u'(t) = \frac{c_2}{1+t}$$

Using the above in (1) gives the solution

$$x = \frac{c_2}{c_2 \ln(1+t) + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$x = \frac{1}{\ln(1+t) + c_3}$$

Summary
The solution(s) found are the following

$$x = \frac{1}{\ln(1+t) + c_3} \tag{1}$$

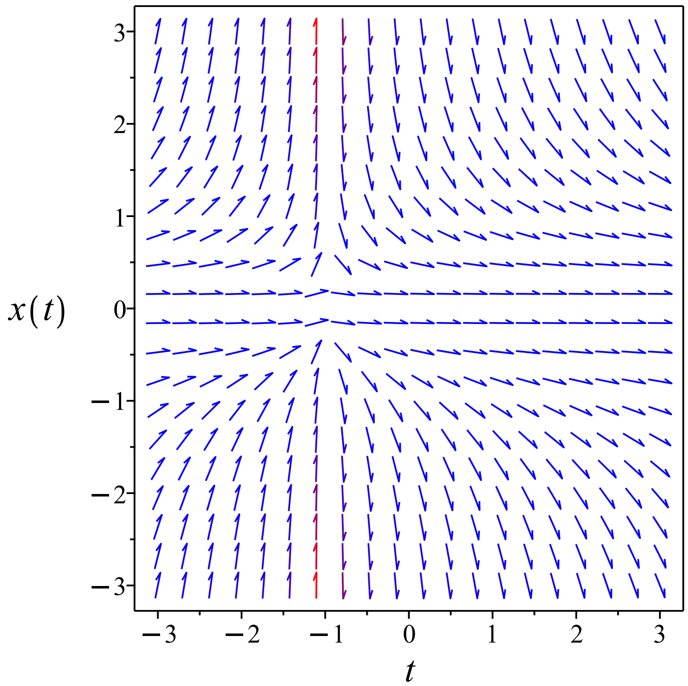


Figure 73: Slope field plot

Verification of solutions

$$x = \frac{1}{\ln(1+t)+c_3}$$

Verified OK.

4.14.5 Maple step by step solution

Let's solve

$(1+t)\,x' + x^2 = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x^2} = -\frac{1}{1+t}$

- Integrate both sides with respect to t

  $\int \frac{x'}{x^2}dt = \int -\frac{1}{1+t}dt + c_1$

- Evaluate integral

$$-\frac{1}{x} = -\ln(1+t) + c_1$$

- Solve for x

$$x = \frac{1}{\ln(1+t) - c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((1+t)*diff(x(t),t)+x(t)^2=0,x(t), singsol=all)
```

$$x(t) = \frac{1}{\ln(t+1) + c_1}$$

✓ Solution by Mathematica

Time used: 0.161 (sec). Leaf size: 21

```
DSolve[(1+t)*x'[t]+x[t]^2==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{\log(t+1) - c_1}$$
$$x(t) \to 0$$

4.15 problem 5

4.15.1 Existence and uniqueness analysis 278
4.15.2 Solving as quadrature ode 279
4.15.3 Maple step by step solution 280

Internal problem ID [11382]
Internal file name [`OUTPUT/10364_Wednesday_May_17_2023_07_50_02_PM_6291675/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \frac{1}{2y+1} = 0}$$

With initial conditions

$$[y(0) = 1]$$

4.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= \frac{1}{2y+1} \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\left\{y < -\frac{1}{2} \vee -\frac{1}{2} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{1}{2y+1}\right)\\ &= -\frac{2}{(2y+1)^2}\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\left\{y < -\frac{1}{2} \vee -\frac{1}{2} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

4.15.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int (2y+1)\,dy &= t + c_1\\ y^2 + y &= t + c_1\end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned}y_1 &= -\frac{1}{2} - \frac{\sqrt{1+4c_1+4t}}{2}\\ y_2 &= -\frac{1}{2} + \frac{\sqrt{1+4c_1+4t}}{2}\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{2} + \frac{\sqrt{1+4c_1}}{2}$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{2} + \frac{\sqrt{9+4t}}{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

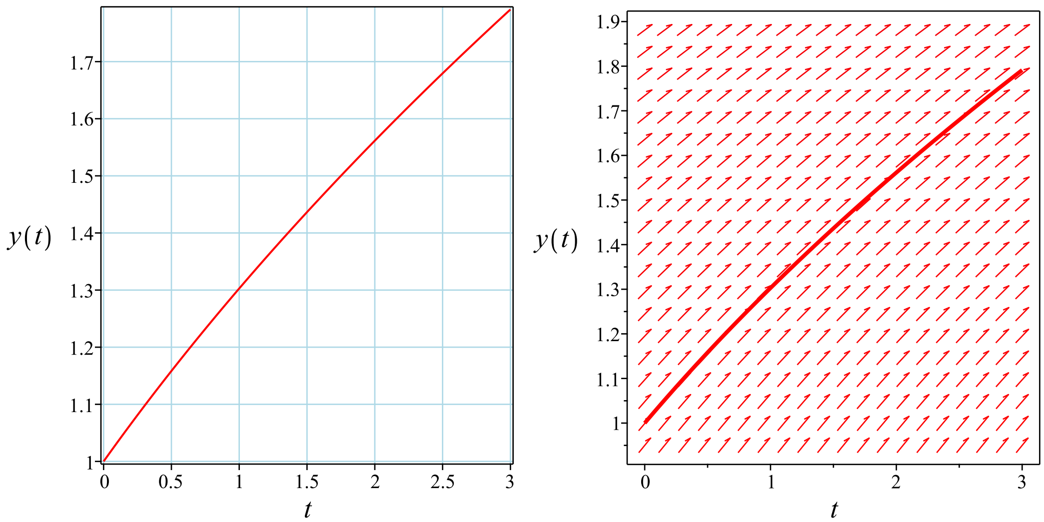
$$1 = -\frac{1}{2} - \frac{\sqrt{1+4c_1}}{2}$$

Warning: Unable to solve for constant of integration.

Summary

The solution(s) found are the following

$$y = -\frac{1}{2} + \frac{\sqrt{9+4t}}{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{2} + \frac{\sqrt{9+4t}}{2}$$

Verified OK.

4.15.3 Maple step by step solution

Let's solve

$\left[y' - \frac{1}{2y+1} = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(2y+1) = 1$

- Integrate both sides with respect to t

$\int y'(2y+1)\,dt = \int 1dt + c_1$

- Evaluate integral

  $y^2 + y = t + c_1$

- Solve for y

  $\left\{y = -\frac{1}{2} - \frac{\sqrt{1+4c_1+4t}}{2}, y = -\frac{1}{2} + \frac{\sqrt{1+4c_1+4t}}{2}\right\}$

- Use initial condition $y(0) = 1$

  $1 = -\frac{1}{2} - \frac{\sqrt{1+4c_1}}{2}$

- Solution does not satisfy initial condition
- Use initial condition $y(0) = 1$

  $1 = -\frac{1}{2} + \frac{\sqrt{1+4c_1}}{2}$

- Solve for c_1

  $c_1 = 2$

- Substitute $c_1 = 2$ into general solution and simplify

  $y = -\frac{1}{2} + \frac{\sqrt{9+4t}}{2}$

- Solution to the IVP

  $y = -\frac{1}{2} + \frac{\sqrt{9+4t}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve([diff(y(t),t)=1/(2*y(t)+1),y(0) = 1],y(t), singsol=all)
```

$$y(t) = -\frac{1}{2} + \frac{\sqrt{9+4t}}{2}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 20

```
DSolve[{y'[t]==1/(2*y[t]+1),{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}\left(\sqrt{4t+9}-1\right)$$

4.16 problem 6

4.16.1 Existence and uniqueness analysis 283
4.16.2 Solving as homogeneousTypeC ode 284
4.16.3 Solving as first order ode lie symmetry lookup ode 286
4.16.4 Solving as riccati ode 291

Internal problem ID [11383]
Internal file name [`OUTPUT/10365_Wednesday_May_17_2023_07_50_03_PM_43331283/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _Riccati]
```

$$\boxed{x' - (4t - x)^2 = 0}$$

With initial conditions

$$[x(0) = 1]$$

4.16.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t,x) \\ &= (-4t + x)^2 \end{aligned}$$

The t domain of $f(t,x)$ when $x = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The x domain of $f(t, x)$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left((-4t + x)^2\right) \\
&= -8t + 2x
\end{aligned}$$

The t domain of $\frac{\partial f}{\partial x}$ when $x = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The x domain of $\frac{\partial f}{\partial x}$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. Therefore solution exists and is unique.

4.16.2 Solving as homogeneousTypeC ode

Let

$$z = 4t - x \tag{1}$$

Then

$$z'(t) = 4 - x'$$

Therefore

$$x' = -z'(t) + 4$$

Hence the given ode can now be written as

$$-z'(t) + 4 = z^2$$

This is separable first order ode. Integrating

$$\begin{aligned}
\int dt &= \int \frac{1}{-z^2 + 4} dz \\
t + c_1 &= -\frac{\ln (z - 2)}{4} + \frac{\ln (z + 2)}{4}
\end{aligned}$$

Replacing z back by its value from (1) then the above gives the solution as

$$x = \frac{(4t-2)\,e^{4t+4c_1} - 4t - 2}{e^{4t+4c_1} - 1}$$

$$x = \frac{(4t-2)\,e^{4t+4c_1} - 4t - 2}{e^{4t+4c_1} - 1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-2\,e^{4c_1} - 2}{e^{4c_1} - 1}$$

$$c_1 = -\frac{\ln(3)}{4} + \frac{i\pi}{4}$$

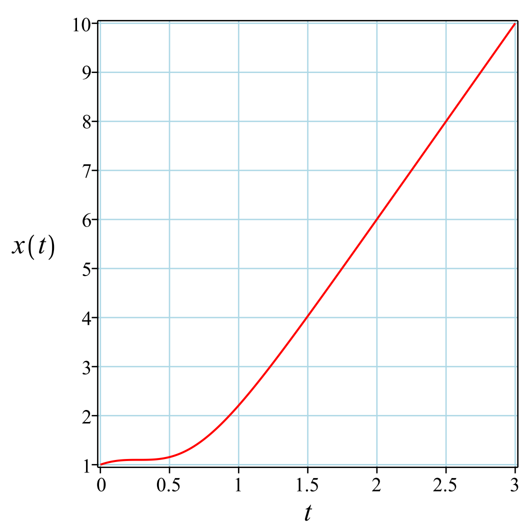
Substituting c_1 found above in the general solution gives

$$x = \frac{4\,e^{4t}t - 2\,e^{4t} + 12t + 6}{e^{4t} + 3}$$

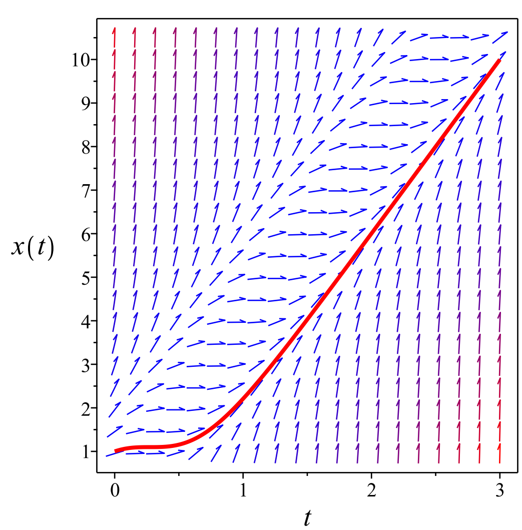
Summary

The solution(s) found are the following

$$x = \frac{4\,e^{4t}t - 2\,e^{4t} + 12t + 6}{e^{4t} + 3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{4\,\mathrm{e}^{4t}t - 2\,\mathrm{e}^{4t} + 12t + 6}{\mathrm{e}^{4t} + 3}$$

Verified OK.

4.16.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} x' &= (-4t + x)^2 \\ x' &= \omega(t, x) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 56: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= 1 \\
\eta(t, x) &= 4
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{\eta}{\xi} \\
&= \frac{4}{1} \\
&= 4
\end{aligned}
$$

This is easily solved to give

$$
x = 4t + c_1
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = -4t + x
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dt}{\xi} \\
&= \frac{dt}{1}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dt}{T} \\
&= t
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}
$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$
\omega(t,x) = (-4t + x)^2
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= -4 \\
R_x &= 1 \\
S_t &= 1 \\
S_x &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{-4 + (-4t + x)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 - 4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R-2)}{4} - \frac{\ln(R+2)}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$t = \frac{\ln(-4t + x - 2)}{4} - \frac{\ln(-4t + x + 2)}{4} + c_1$$

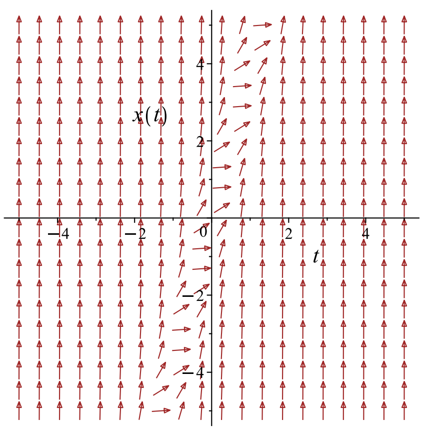
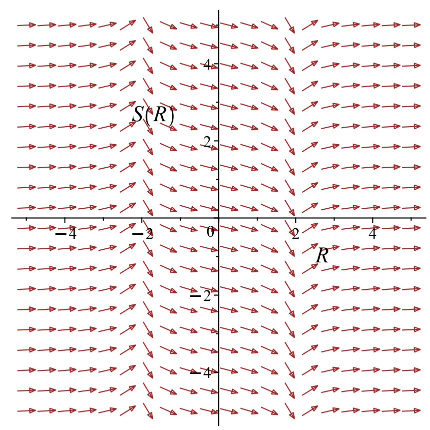
Which simplifies to

$$t = \frac{\ln(-4t + x - 2)}{4} - \frac{\ln(-4t + x + 2)}{4} + c_1$$

Which gives

$$x = \frac{4t\,\mathrm{e}^{-4t+4c_1} + 2\,\mathrm{e}^{-4t+4c_1} - 4t + 2}{\mathrm{e}^{-4t+4c_1} - 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = (-4t + x)^2$ 	$R = -4t + x$ $S = t$	$\frac{dS}{dR} = \frac{1}{R^2-4}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{2\,e^{4c_1} + 2}{e^{4c_1} - 1}$$

$$c_1 = \frac{\ln(3)}{4} + \frac{i\pi}{4}$$

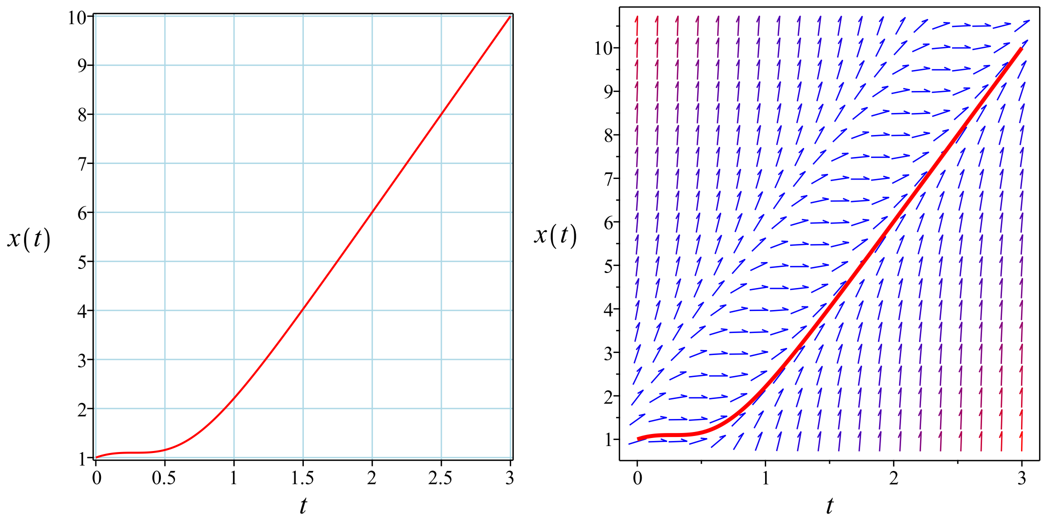
Substituting c_1 found above in the general solution gives

$$x = \frac{12t\,e^{-4t} + 6\,e^{-4t} + 4t - 2}{3\,e^{-4t} + 1}$$

<u>Summary</u>

The solution(s) found are the following

$$x = \frac{12t\,e^{-4t} + 6\,e^{-4t} + 4t - 2}{3\,e^{-4t} + 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$x = \frac{12t\,\mathrm{e}^{-4t} + 6\,\mathrm{e}^{-4t} + 4t - 2}{3\,\mathrm{e}^{-4t} + 1}$$

Verified OK.

4.16.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= (-4t + x)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$x' = 16t^2 - 8tx + x^2$$

With Riccati ODE standard form

$$x' = f_0(t) + f_1(t)x + f_2(t)x^2$$

Shows that $f_0(t) = 16t^2$, $f_1(t) = -8t$ and $f_2(t) = 1$. Let

$$\begin{aligned} x &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(t) - (f_2' + f_1f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= -8t \\ f_2^2 f_0 &= 16t^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(t) + 8tu'(t) + 16t^2u(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1\mathrm{e}^{-2t(t-1)} + c_2\mathrm{e}^{-2(1+t)t}$$

The above shows that

$$u'(t) = c_1(-4t+2)\,\mathrm{e}^{-2t(t-1)} - 4\left(t+\frac{1}{2}\right) c_2\mathrm{e}^{-2(1+t)t}$$

Using the above in (1) gives the solution

$$x = -\frac{c_1(-4t+2)\,\mathrm{e}^{-2t(t-1)} - 4\left(t+\frac{1}{2}\right) c_2\mathrm{e}^{-2(1+t)t}}{c_1\mathrm{e}^{-2t(t-1)} + c_2\mathrm{e}^{-2(1+t)t}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$x = \frac{(4t-2)\, c_3\mathrm{e}^{-2t(t-1)} + (4t+2)\,\mathrm{e}^{-2(1+t)t}}{c_3\mathrm{e}^{-2t(t-1)} + \mathrm{e}^{-2(1+t)t}}$$

Initial conditions are used to solve for c_3. Substituting $t = 0$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-2c_3 + 2}{1 + c_3}$$

$$c_3 = \frac{1}{3}$$

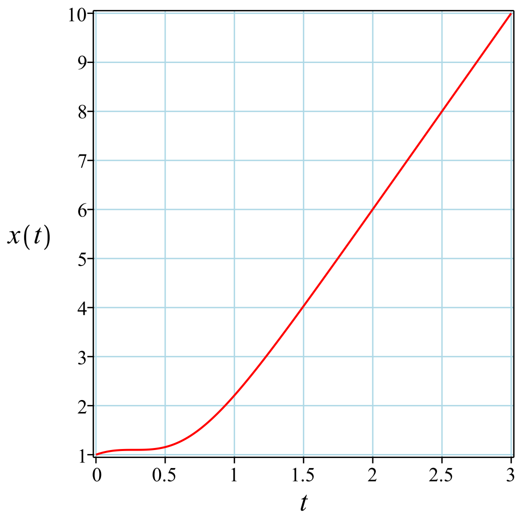
Substituting c_3 found above in the general solution gives

$$x = \frac{4\,\mathrm{e}^{-2t(t-1)}t + 12\,\mathrm{e}^{-2(1+t)t}t - 2\,\mathrm{e}^{-2t(t-1)} + 6\,\mathrm{e}^{-2(1+t)t}}{\mathrm{e}^{-2t(t-1)} + 3\,\mathrm{e}^{-2(1+t)t}}$$

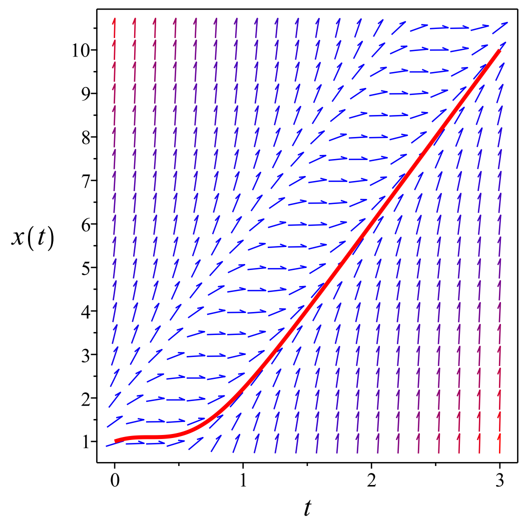
Summary

The solution(s) found are the following

$$x = \frac{4\,\mathrm{e}^{-2t(t-1)}t + 12\,\mathrm{e}^{-2(1+t)t}t - 2\,\mathrm{e}^{-2t(t-1)} + 6\,\mathrm{e}^{-2(1+t)t}}{\mathrm{e}^{-2t(t-1)} + 3\,\mathrm{e}^{-2(1+t)t}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{4\,\mathrm{e}^{-2t(t-1)}t + 12\,\mathrm{e}^{-2(1+t)t}t - 2\,\mathrm{e}^{-2t(t-1)} + 6\,\mathrm{e}^{-2(1+t)t}}{\mathrm{e}^{-2t(t-1)} + 3\,\mathrm{e}^{-2(1+t)t}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = 4, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 28

```
dsolve([diff(x(t),t)=(4*t-x(t))^2,x(0) = 1],x(t), singsol=all)
```

$$x(t) = \frac{(4t-2)\,e^{4t}+12t+6}{3+e^{4t}}$$

✓ Solution by Mathematica

Time used: 0.271 (sec). Leaf size: 31

```
DSolve[{x'[t]==(4*t-x[t])^2,{x[0]==1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{12t+e^{4t}(4t-2)+6}{e^{4t}+3}$$

4.17 problem 7

4.17.1 Existence and uniqueness analysis 295
4.17.2 Solving as separable ode 296
4.17.3 Solving as first order ode lie symmetry lookup ode 298
4.17.4 Solving as exact ode 302
4.17.5 Solving as riccati ode 306
4.17.6 Maple step by step solution 308

Internal problem ID [11384]
Internal file name [`OUTPUT/10366_Wednesday_May_17_2023_07_50_04_PM_32131591/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' - 2tx^2 = 0}$$

With initial conditions

$$[x(0) = 1]$$

4.17.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t,x) \\ &= 2t\,x^2 \end{aligned}$$

The t domain of $f(t,x)$ when $x = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The x domain of $f(t, x)$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(2t\,x^2)\\ &= 4tx\end{aligned}$$

The t domain of $\frac{\partial f}{\partial x}$ when $x = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The x domain of $\frac{\partial f}{\partial x}$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. Therefore solution exists and is unique.

4.17.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}x' &= F(t, x)\\ &= f(t)g(x)\\ &= 2t\,x^2\end{aligned}$$

Where $f(t) = 2t$ and $g(x) = x^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{x^2}\,dx &= 2t\,dt\\ \int \frac{1}{x^2}\,dx &= \int 2t\,dt\\ -\frac{1}{x} &= t^2 + c_1\end{aligned}$$

Which results in

$$x = -\frac{1}{t^2 + c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{c_1}$$

$$c_1 = -1$$

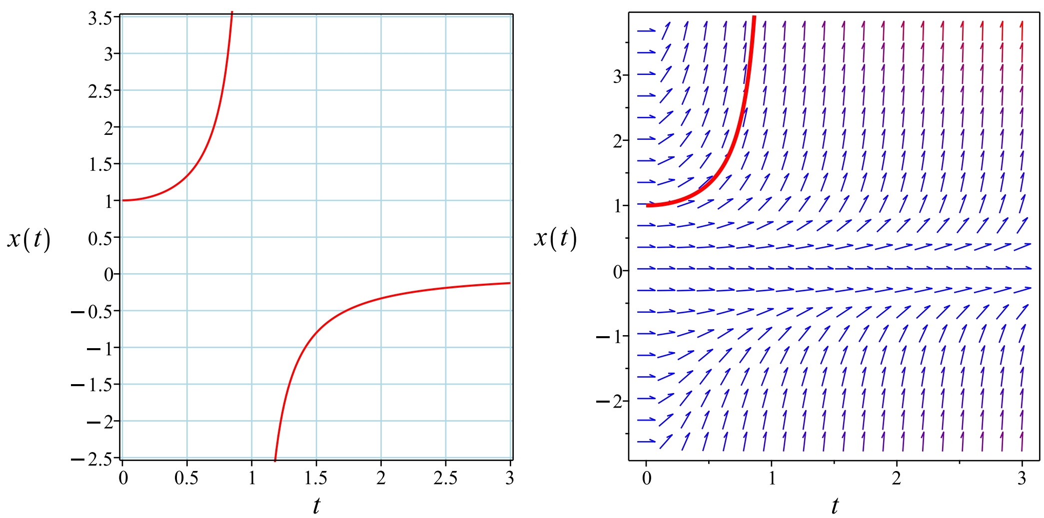
Substituting c_1 found above in the general solution gives

$$x = -\frac{1}{t^2 - 1}$$

Summary

The solution(s) found are the following

$$x = -\frac{1}{t^2 - 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = -\frac{1}{t^2 - 1}$$

Verified OK.

4.17.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= 2t\,x^2 \\
x' &= \omega(t,x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 58: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= \frac{1}{2t} \\
\eta(t,x) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(t,x)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dt \\
&= \int \frac{1}{\frac{1}{2t}} dt
\end{aligned}
$$

Which results in

$$
S = t^2
$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}
$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$
\omega(t,x) = 2t\,x^2
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_x &= 1 \\
S_t &= 2t \\
S_x &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$t^2 = -\frac{1}{x} + c_1$$

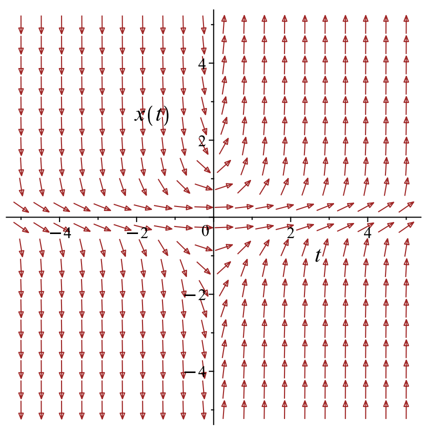
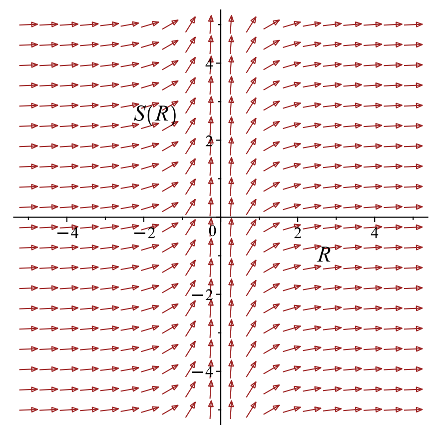
Which simplifies to

$$t^2 = -\frac{1}{x} + c_1$$

Which gives

$$x = \frac{1}{-t^2 + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = 2t\,x^2$ 	$R = x$ $S = t^2$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{c_1}$$

$$c_1 = 1$$

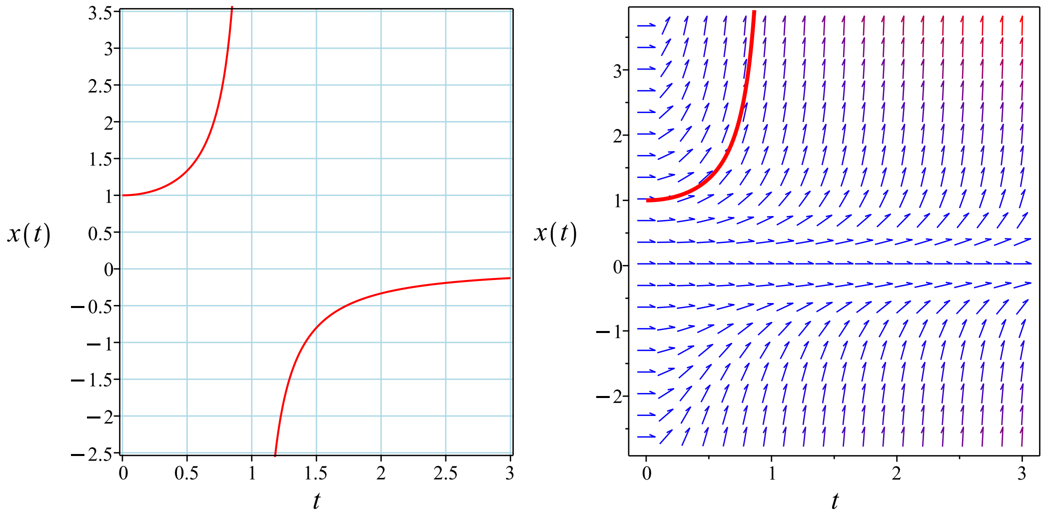
Substituting c_1 found above in the general solution gives

$$x = -\frac{1}{t^2 - 1}$$

Summary

The solution(s) found are the following

$$x = -\frac{1}{t^2 - 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = -\frac{1}{t^2 - 1}$$

Verified OK.

4.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{2x^2}\right)\mathrm{d}x &= (t)\,\mathrm{d}t\\ (-t)\,\mathrm{d}t + \left(\frac{1}{2x^2}\right)\mathrm{d}x &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, x) &= -t\\ N(t, x) &= \frac{1}{2x^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-t)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{2x^2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, x)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} &= M & (1)\\
\frac{\partial \phi}{\partial x} &= N & (2)
\end{aligned}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -t \, \mathrm{d}t \\
\phi &= -\frac{t^2}{2} + f(x) & (3)
\end{aligned}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{2x^2}$. Therefore equation (4) becomes

$$
\frac{1}{2x^2} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = \frac{1}{2x^2}
$$

Integrating the above w.r.t x gives

$$
\begin{aligned}
\int f'(x) \, \mathrm{d}x &= \int \left(\frac{1}{2x^2}\right) \mathrm{d}x \\
f(x) &= -\frac{1}{2x} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} - \frac{1}{2x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} - \frac{1}{2x}$$

The solution becomes

$$x = -\frac{1}{t^2 + 2c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{2c_1}$$

$$c_1 = -\frac{1}{2}$$

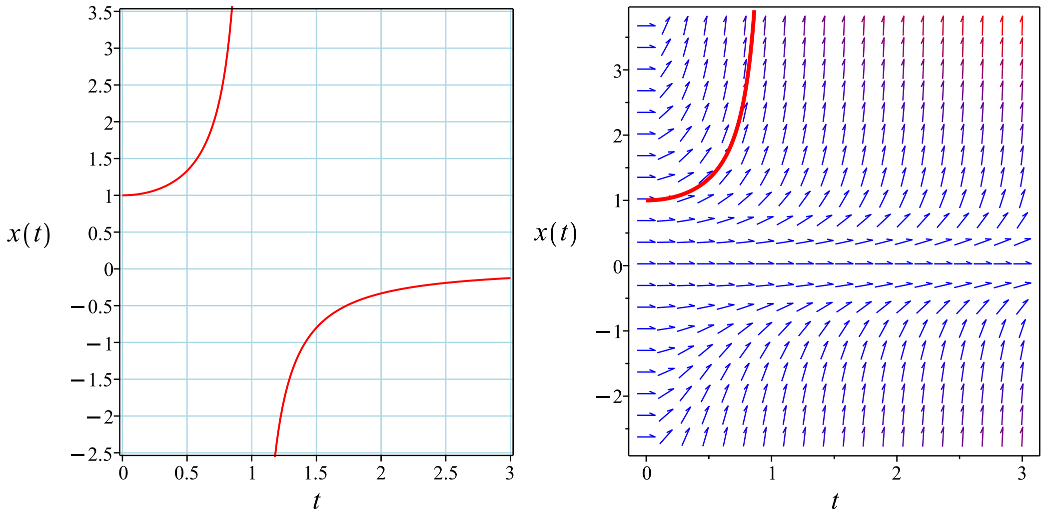
Substituting c_1 found above in the general solution gives

$$x = -\frac{1}{t^2 - 1}$$

Summary

The solution(s) found are the following

$$x = -\frac{1}{t^2 - 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = -\frac{1}{t^2 - 1}$$

Verified OK.

4.17.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= 2t\,x^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$x' = 2t\,x^2$$

With Riccati ODE standard form

$$x' = f_0(t) + f_1(t)x + f_2(t)x^2$$

Shows that $f_0(t) = 0$, $f_1(t) = 0$ and $f_2(t) = 2t$. Let

$$\begin{aligned} x &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{2tu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - \left(f_2' + f_1 f_2\right) u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$2tu''(t) - 2u'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = t^2 c_2 + c_1$$

The above shows that

$$u'(t) = 2c_2 t$$

Using the above in (1) gives the solution

$$x = -\frac{c_2}{t^2 c_2 + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$x = -\frac{1}{t^2 + c_3}$$

Initial conditions are used to solve for c_3. Substituting $t = 0$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{c_3}$$

$$c_3 = -1$$

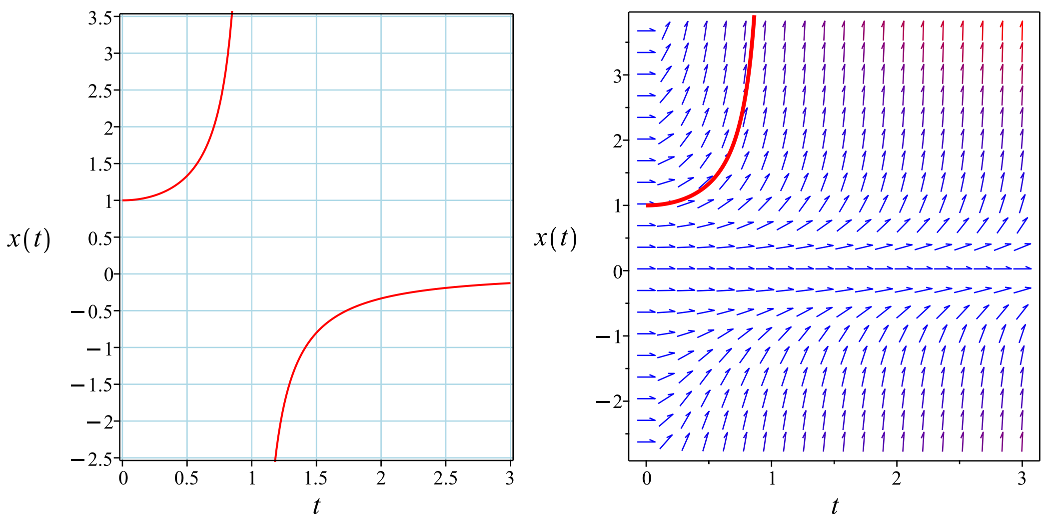
Substituting c_3 found above in the general solution gives

$$x = -\frac{1}{t^2 - 1}$$

Summary

The solution(s) found are the following

$$x = -\frac{1}{t^2 - 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = -\frac{1}{t^2 - 1}$$

Verified OK.

4.17.6 Maple step by step solution

Let's solve

$[x' - 2tx^2 = 0, x(0) = 1]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

$\frac{x'}{x^2} = 2t$

- Integrate both sides with respect to t

$\int \frac{x'}{x^2} dt = \int 2t dt + c_1$

- Evaluate integral

$-\frac{1}{x} = t^2 + c_1$

- Solve for x

$x = -\frac{1}{t^2 + c_1}$

- Use initial condition $x(0) = 1$

$1 = -\frac{1}{c_1}$

- Solve for c_1

$c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

$x = -\frac{1}{t^2 - 1}$

- Solution to the IVP

$x = -\frac{1}{t^2 - 1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 13

```
dsolve([diff(x(t),t)=2*t*x(t)^2,x(0) = 1],x(t), singsol=all)
```

$$x(t) = -\frac{1}{t^2 - 1}$$

✓ Solution by Mathematica

Time used: 0.183 (sec). Leaf size: 14

```
DSolve[{x'[t]==2*t*x[t]^2,{x[0]==1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{1-t^2}$$

4.18 problem 8

4.18.1 Existence and uniqueness analysis 311
4.18.2 Solving as separable ode 312
4.18.3 Solving as first order special form ID 1 ode 314
4.18.4 Solving as first order ode lie symmetry lookup ode 315
4.18.5 Solving as exact ode 319
4.18.6 Maple step by step solution 323

Internal problem ID [11385]
Internal file name [`OUTPUT/10367_Wednesday_May_17_2023_07_50_06_PM_28518811/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' - t^2 \mathrm{e}^{-x} = 0}$$

With initial conditions

$$[x(0) = \ln(2)]$$

4.18.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t, x) \\ &= t^2 \mathrm{e}^{-x} \end{aligned}$$

The t domain of $f(t, x)$ when $x = \ln(2)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The x domain of $f(t, x)$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \ln(2)$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(t^2 \mathrm{e}^{-x}\right) \\ &= -t^2 \mathrm{e}^{-x}\end{aligned}$$

The t domain of $\frac{\partial f}{\partial x}$ when $x = \ln(2)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The x domain of $\frac{\partial f}{\partial x}$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \ln(2)$ is inside this domain. Therefore solution exists and is unique.

4.18.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}x' &= F(t, x) \\ &= f(t)g(x) \\ &= t^2 \mathrm{e}^{-x}\end{aligned}$$

Where $f(t) = t^2$ and $g(x) = \mathrm{e}^{-x}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\mathrm{e}^{-x}}\, dx &= t^2\, dt \\ \int \frac{1}{\mathrm{e}^{-x}}\, dx &= \int t^2\, dt \\ \mathrm{e}^{x} &= \frac{t^3}{3} + c_1\end{aligned}$$

Which results in

$$x = -\ln\left(\frac{3}{t^3 + 3c_1}\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = \ln(2)$ in the above solution gives an equation to solve for the constant of integration.

$$\ln(2) = -\ln\left(\frac{1}{c_1}\right)$$

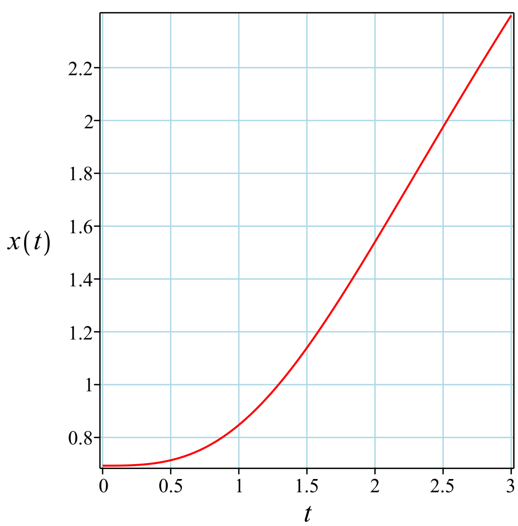
$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

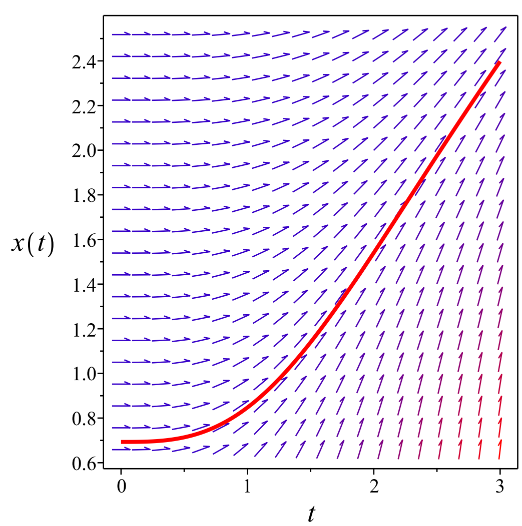
$$x = -\ln(3) - \ln\left(\frac{1}{t^3 + 6}\right)$$

Summary
The solution(s) found are the following

$$x = -\ln(3) - \ln\left(\frac{1}{t^3 + 6}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -\ln(3) - \ln\left(\frac{1}{t^3 + 6}\right)$$

Verified OK.

4.18.3 Solving as first order special form ID 1 ode

Writing the ode as

$$x' = t^2 \mathrm{e}^{-x} \tag{1}$$

And using the substitution $u = \mathrm{e}^{x}$ then

$$u' = x' \mathrm{e}^{x}$$

The above shows that

$$\begin{aligned}x' &= u'(t)\,\mathrm{e}^{-x}\\ &= \frac{u'(t)}{u}\end{aligned}$$

Substituting this in (1) gives

$$\frac{u'(t)}{u} = \frac{t^2}{u}$$

The above simplifies to

$$u'(t) = t^2 \tag{2}$$

Now ode (2) is solved for $u(t)$ Integrating both sides gives

$$\begin{aligned}u(t) &= \int t^2\, \mathrm{d}t\\ &= \frac{t^3}{3} + c_1\end{aligned}$$

Substituting the solution found for $u(t)$ in $u = \mathrm{e}^{x}$ gives

$$\begin{aligned}x &= \ln (u(t))\\ &= \ln \left(\frac{t^3}{3} + c_1\right)\\ &= -\ln (3) + \ln \left(t^3 + 3c_1\right)\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = \ln (2)$ in the above solution gives an equation to solve for the constant of integration.

$$\ln (2) = \ln (c_1)$$

$$c_1 = 2$$

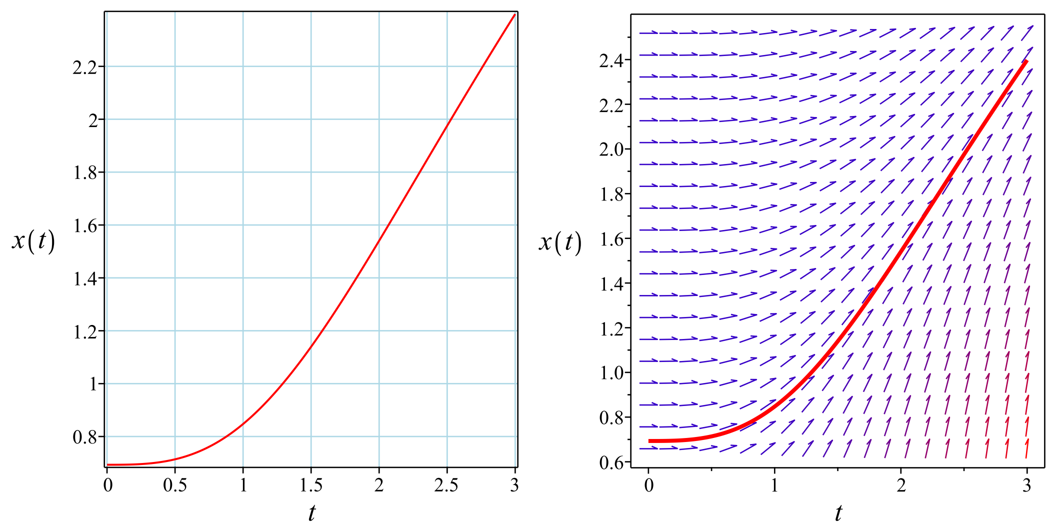
Substituting c_1 found above in the general solution gives

$$x = -\ln(3) + \ln(t^3 + 6)$$

Summary

The solution(s) found are the following

$$x = -\ln(3) + \ln(t^3 + 6) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = -\ln(3) + \ln(t^3 + 6)$$

Verified OK.

4.18.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = t^2 e^{-x}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 61: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= \frac{1}{t^2} \\
\eta(t, x) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{t^2}} dt \end{aligned}$$

Which results in

$$S = \frac{t^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = t^2 \mathrm{e}^{-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_x &= 1 \\ S_t &= t^2 \\ S_x &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \mathrm{e}^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\frac{t^3}{3} = \mathrm{e}^{x} + c_1$$

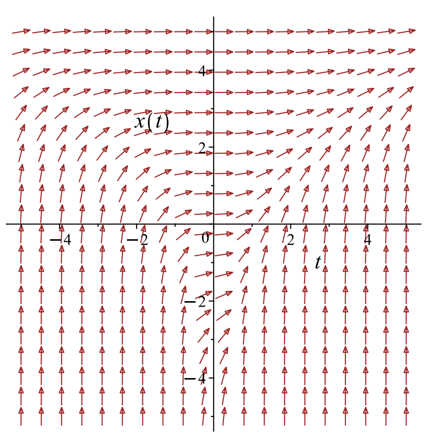
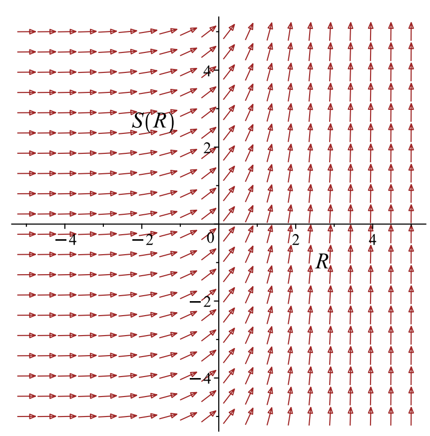
Which simplifies to

$$\frac{t^3}{3} = \mathrm{e}^{x} + c_1$$

Which gives

$$x = \ln\left(\frac{t^3}{3} - c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = t^2 \mathrm{e}^{-x}$ 	$R = x$ $S = \frac{t^3}{3}$	$\frac{dS}{dR} = \mathrm{e}^{R}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = \ln(2)$ in the above solution gives an equation to solve for the constant of integration.

$$\ln(2) = \ln(-c_1)$$

$$c_1 = -2$$

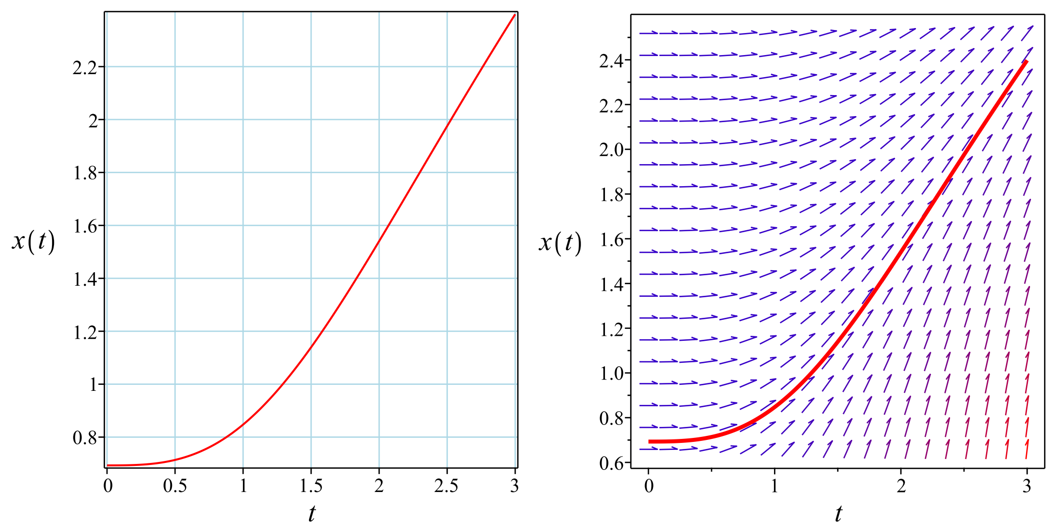
Substituting c_1 found above in the general solution gives

$$x = -\ln(3) + \ln(t^3 + 6)$$

Summary

The solution(s) found are the following

$$x = -\ln(3) + \ln(t^3 + 6) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = -\ln(3) + \ln(t^3 + 6)$$

Verified OK.

4.18.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x) \, \mathrm{d}t + N(t, x) \, \mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^x) \, \mathrm{d}x &= \left(t^2\right) \mathrm{d}t \\ \left(-t^2\right) \mathrm{d}t + (\mathrm{e}^x) \, \mathrm{d}x &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, x) &= -t^2 \\ N(t, x) &= \mathrm{e}^x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-t^2\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(\mathrm{e}^{x}) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,x)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int M \,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int -t^{2} \,\mathrm{d}t \\
\phi &= -\frac{t^{3}}{3} + f(x) \qquad (3)
\end{aligned}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$
\mathrm{e}^{x} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
\begin{aligned}
f'(x) &= \mathrm{e}^{x} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

Integrating the above w.r.t x results in

$$
\begin{aligned}
\int f'(x) \,\mathrm{d}x &= \int (\mathrm{e}^{x}) \,\mathrm{d}x \\
f(x) &= \mathrm{e}^{x} + c_{1}
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{t^3}{3} + e^x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^3}{3} + e^x$$

The solution becomes

$$x = \ln\left(\frac{t^3}{3} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = \ln(2)$ in the above solution gives an equation to solve for the constant of integration.

$$\ln(2) = \ln(c_1)$$

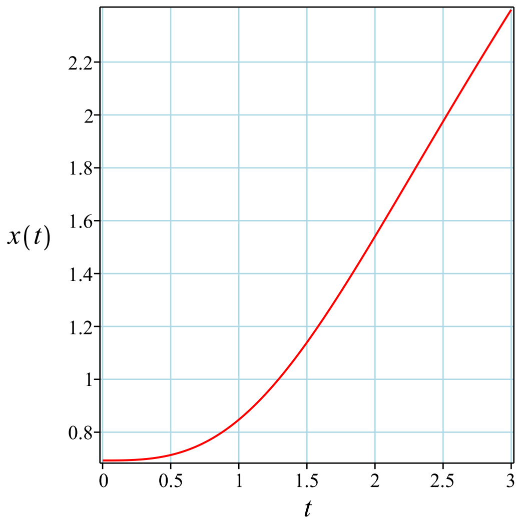
$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

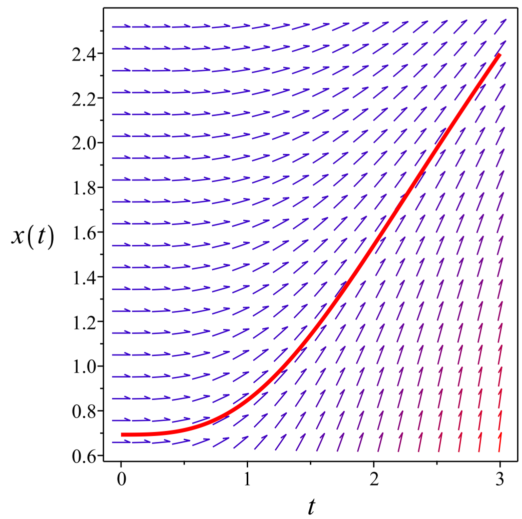
$$x = -\ln(3) + \ln(t^3 + 6)$$

Summary
The solution(s) found are the following

$$x = -\ln(3) + \ln(t^3 + 6) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -\ln\left(3\right)+\ln\left(t^{3}+6\right)$$

Verified OK.

4.18.6 Maple step by step solution

Let's solve

$\left[x'-t^{2}\mathrm{e}^{-x}=0, x(0)=\ln\left(2\right)\right]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{\mathrm{e}^{-x}}=t^{2}$

- Integrate both sides with respect to t

  $\int \frac{x'}{\mathrm{e}^{-x}}dt=\int t^{2}dt+c_{1}$

- Evaluate integral

  $\frac{1}{\mathrm{e}^{-x}}=\frac{t^{3}}{3}+c_{1}$

- Solve for x

  $x=-\ln\left(\frac{3}{t^{3}+3c_{1}}\right)$

- Use initial condition $x(0) = \ln(2)$

  $\ln(2) = -\ln\left(\frac{1}{c_1}\right)$

- Solve for c_1

  $c_1 = 2$

- Substitute $c_1 = 2$ into general solution and simplify

  $x = -\ln(3) - \ln\left(\frac{1}{t^3+6}\right)$

- Solution to the IVP

  $x = -\ln(3) - \ln\left(\frac{1}{t^3+6}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 15

```
dsolve([diff(x(t),t)=t^2*exp(-x(t)),x(0) = ln(2)],x(t), singsol=all)
```

$$x(t) = -\ln(3) + \ln(t^3 + 6)$$

✓ Solution by Mathematica

Time used: 0.474 (sec). Leaf size: 15

```
DSolve[{x'[t]==t^2*Exp[-x[t]],{x[0]==Log[2]}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \log\left(\frac{1}{3}(t^3 + 6)\right)$$

4.19 problem 9

4.19.1 Existence and uniqueness analysis 325
4.19.2 Solving as quadrature ode 326
4.19.3 Maple step by step solution 327

Internal problem ID [11386]
Internal file name [`OUTPUT/10368_Wednesday_May_17_2023_07_50_07_PM_22720490/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' - x(4 + x) = 0}$$

With initial conditions

$$[x(0) = 1]$$

4.19.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t, x) \\ &= x(x + 4) \end{aligned}$$

The x domain of $f(t, x)$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x(x+4)) \\
&= 2x + 4
\end{aligned}
$$

The x domain of $\frac{\partial f}{\partial x}$ when $t = 0$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 1$ is inside this domain. Therefore solution exists and is unique.

4.19.2 Solving as quadrature ode

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{x(x+4)} dx &= \int dt \\
-\frac{\ln(x+4)}{4} + \frac{\ln(x)}{4} &= t + c_1
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
\left(-\frac{1}{4}\right)(\ln(x+4) - \ln(x)) &= t + c_1 \\
\ln(x+4) - \ln(x) &= (-4)(t + c_1) \\
&= -4t - 4c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{\ln(x+4) - \ln(x)} = -4c_1 e^{-4t}
$$

Which simplifies to

$$
\frac{x+4}{x} = c_2 e^{-4t}
$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = \frac{4}{-1 + c_2}
$$

$$
c_2 = 5
$$

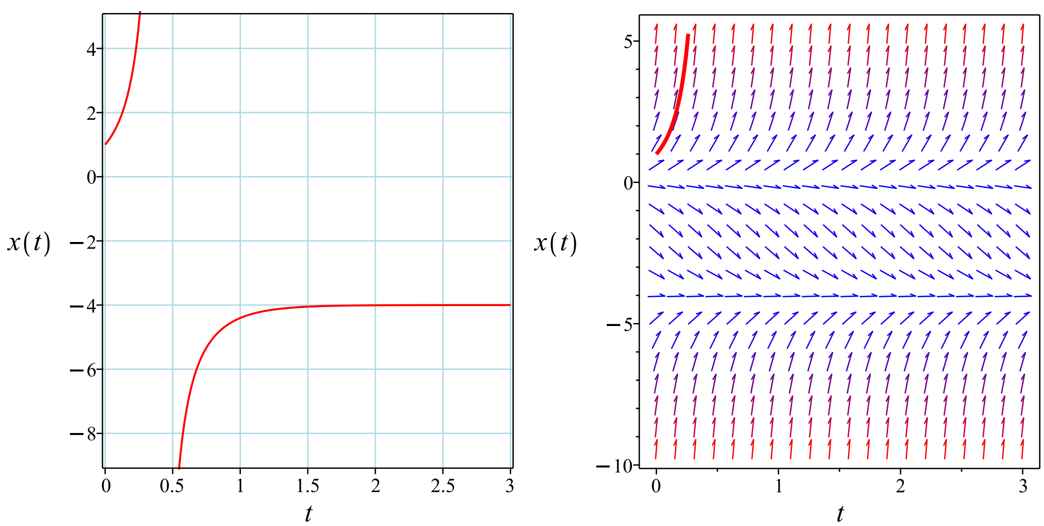
Substituting c_2 found above in the general solution gives

$$x = \frac{4}{-1+5\,e^{-4t}}$$

Summary

The solution(s) found are the following

$$x = \frac{4}{-1+5\,e^{-4t}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \frac{4}{-1+5\,e^{-4t}}$$

Verified OK.

4.19.3 Maple step by step solution

Let's solve

$[x' - x(4+x) = 0, x(0) = 1]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

$\frac{x'}{x(4+x)} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{x(4+x)} dt = \int 1 dt + c_1$
- Evaluate integral

  $-\frac{\ln(4+x)}{4} + \frac{\ln(x)}{4} = t + c_1$
- Solve for x

  $x = -\frac{4\,\mathrm{e}^{4t+4c_1}}{\mathrm{e}^{4t+4c_1}-1}$
- Use initial condition $x(0) = 1$

  $1 = -\frac{4\,\mathrm{e}^{4c_1}}{\mathrm{e}^{4c_1}-1}$
- Solve for c_1

  $c_1 = -\frac{\ln(5)}{4}$
- Substitute $c_1 = -\frac{\ln(5)}{4}$ into general solution and simplify

  $x = -\frac{4\,\mathrm{e}^{4t}}{\mathrm{e}^{4t}-5}$
- Solution to the IVP

  $x = -\frac{4\,\mathrm{e}^{4t}}{\mathrm{e}^{4t}-5}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(x(t),t)=x(t)*(4+x(t)),x(0) = 1],x(t), singsol=all)
```

$$x(t) = \frac{4}{-1 + 5\,\mathrm{e}^{-4t}}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 21

```
DSolve[{x'[t]==x[t]*(4+x[t]),{x[0]==1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{4e^{4t}}{e^{4t} - 5}$$

4.20 problem 10(a)

4.20.1 Existence and uniqueness analysis 330
4.20.2 Solving as separable ode 331
4.20.3 Solving as first order special form ID 1 ode 333
4.20.4 Solving as first order ode lie symmetry lookup ode 334
4.20.5 Solving as exact ode 338
4.20.6 Maple step by step solution 342

Internal problem ID [11387]
Internal file name [`OUTPUT/10369_Wednesday_May_17_2023_07_50_08_PM_43417451/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 10(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' - e^{t+x} = 0}$$

With initial conditions

$$[x(0) = 0]$$

4.20.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t, x) \\ &= e^{t+x} \end{aligned}$$

The t domain of $f(t,x)$ when $x = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The x domain of $f(t, x)$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(\mathrm{e}^{t+x}\right)\\ &= \mathrm{e}^{t+x}\end{aligned}$$

The t domain of $\frac{\partial f}{\partial x}$ when $x = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The x domain of $\frac{\partial f}{\partial x}$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. Therefore solution exists and is unique.

4.20.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}x' &= F(t, x)\\ &= f(t)g(x)\\ &= \mathrm{e}^t \mathrm{e}^x\end{aligned}$$

Where $f(t) = \mathrm{e}^t$ and $g(x) = \mathrm{e}^x$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\mathrm{e}^x}\, dx &= \mathrm{e}^t\, dt\\ \int \frac{1}{\mathrm{e}^x}\, dx &= \int \mathrm{e}^t\, dt\\ -\mathrm{e}^{-x} &= \mathrm{e}^t + c_1\end{aligned}$$

Which results in

$$x = \ln\left(-\frac{1}{\mathrm{e}^t + c_1}\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln\left(-\frac{1}{1 + c_1}\right)$$

$$c_1 = -2$$

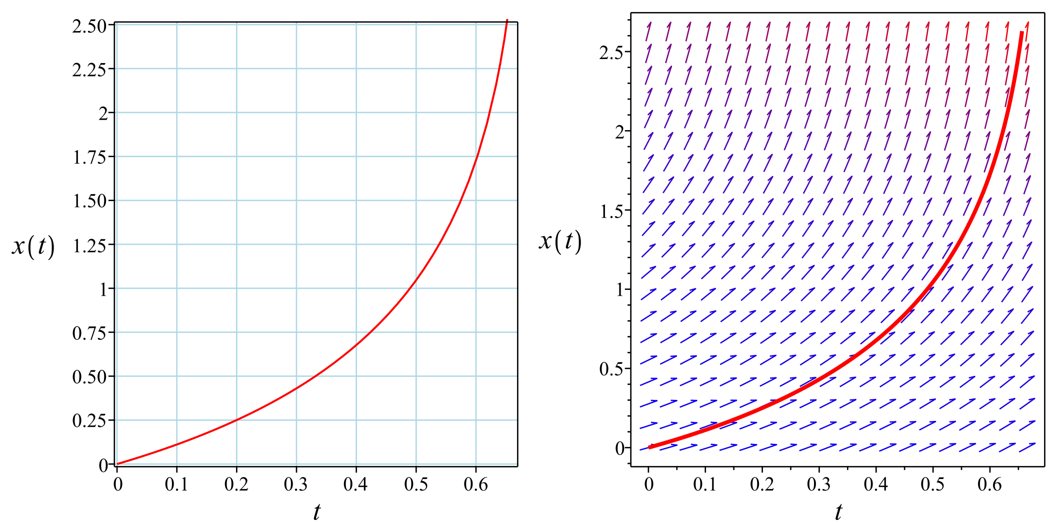
Substituting c_1 found above in the general solution gives

$$x = \ln\left(-\frac{1}{e^t - 2}\right)$$

Summary

The solution(s) found are the following

$$x = \ln\left(-\frac{1}{e^t - 2}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \ln\left(-\frac{1}{e^t - 2}\right)$$

Verified OK.

4.20.3 Solving as first order special form ID 1 ode

Writing the ode as

$$x' = e^{t+x} \tag{1}$$

And using the substitution $u = e^{-x}$ then

$$u' = -x' e^{-x}$$

The above shows that

$$\begin{aligned} x' &= -u'(t)\, e^{x} \\ &= -\frac{u'(t)}{u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(t)}{u} = \frac{e^{t}}{u}$$

The above simplifies to

$$u'(t) = -e^{t} \tag{2}$$

Now ode (2) is solved for $u(t)$ Integrating both sides gives

$$\begin{aligned} u(t) &= \int -e^{t}\, dt \\ &= -e^{t} + c_1 \end{aligned}$$

Substituting the solution found for $u(t)$ in $u = e^{-x}$ gives

$$\begin{aligned} x &= -\ln (u(t)) \\ &= -\ln \left(-e^{t} + c_1\right) \\ &= -\ln \left(-e^{t} + c_1\right) \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\ln (-1 + c_1)$$

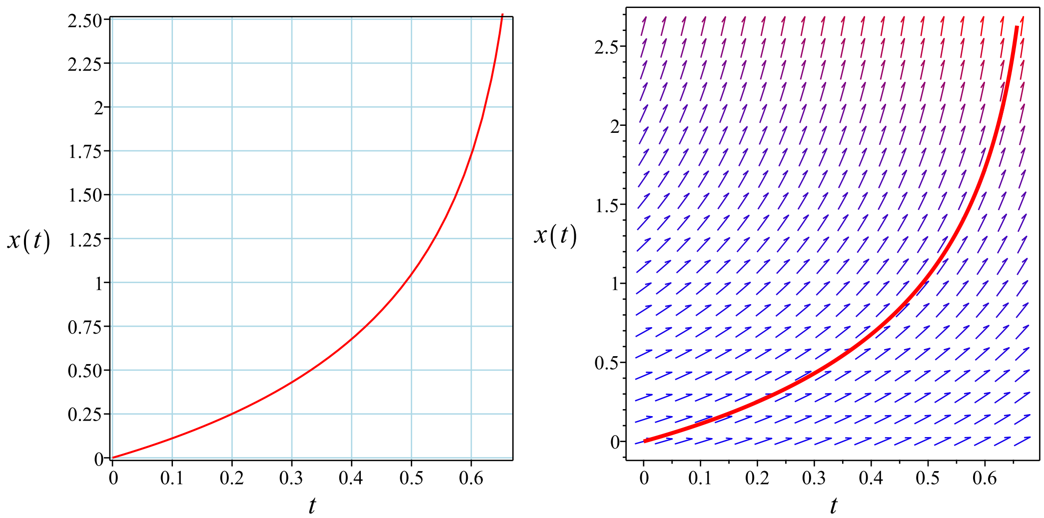
$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$x = -\ln\left(-e^t + 2\right)$$

Summary
The solution(s) found are the following

$$x = -\ln\left(-e^t + 2\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = -\ln\left(-e^t + 2\right)$$

Verified OK.

4.20.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = e^{t+x}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 65: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= \mathrm{e}^{-t} \\
\eta(t,x) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\mathrm{e}^{-t}} dt \end{aligned}$$

Which results in

$$S = \mathrm{e}^{t}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = \mathrm{e}^{t+x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_x &= 1 \\ S_t &= \mathrm{e}^{t} \\ S_x &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -e^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$e^t = -e^{-x} + c_1$$

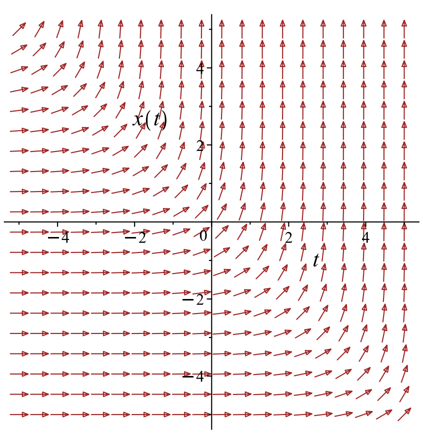
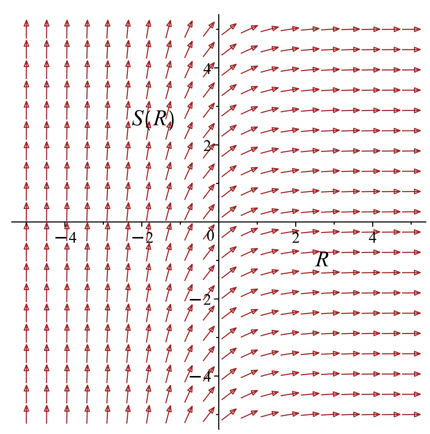
Which simplifies to

$$e^t = -e^{-x} + c_1$$

Which gives

$$x = -\ln\left(-e^t + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = e^{t+x}$ 	$R = x$ $S = e^t$	$\frac{dS}{dR} = e^{-R}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\ln(-1 + c_1)$$

$$c_1 = 2$$

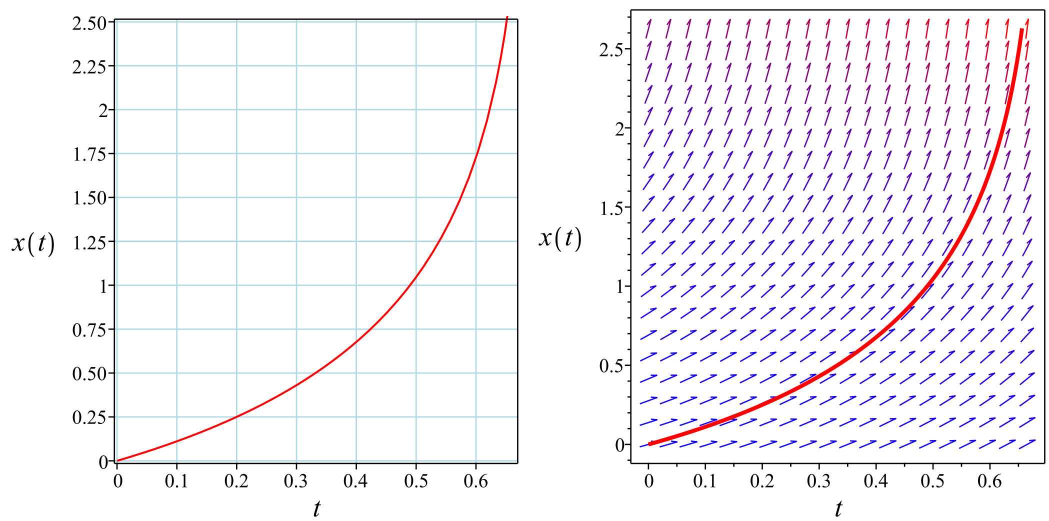
Substituting c_1 found above in the general solution gives

$$x = -\ln\left(-e^t + 2\right)$$

Summary

The solution(s) found are the following

$$x = -\ln\left(-e^t + 2\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = -\ln\left(-e^t + 2\right)$$

Verified OK.

4.20.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\, \mathrm{d}t + N(t, x)\, \mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\mathrm{e}^{-x}\right) \mathrm{d}x &= \left(\mathrm{e}^{t}\right) \mathrm{d}t \\
\left(-\mathrm{e}^{t}\right) \mathrm{d}t + \left(\mathrm{e}^{-x}\right) \mathrm{d}x &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(t, x) &= -\mathrm{e}^{t} \\
N(t, x) &= \mathrm{e}^{-x}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\mathrm{e}^{t}\right) \\
&= 0
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\mathrm{e}^{-x}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int M \,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int -\mathrm{e}^{t} \,\mathrm{d}t
\end{aligned}
$$

$$
\phi = -\mathrm{e}^{t} + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$
\mathrm{e}^{-x} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = \mathrm{e}^{-x}
$$

Integrating the above w.r.t x gives

$$
\begin{aligned}
\int f'(x) \,\mathrm{d}x &= \int \left(\mathrm{e}^{-x}\right) \mathrm{d}x \\
f(x) &= -\mathrm{e}^{-x} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -e^{t} - e^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -e^{t} - e^{-x}$$

The solution becomes

$$x = -\ln\left(-e^{t} - c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\ln\left(-1 - c_1\right)$$

$$c_1 = -2$$

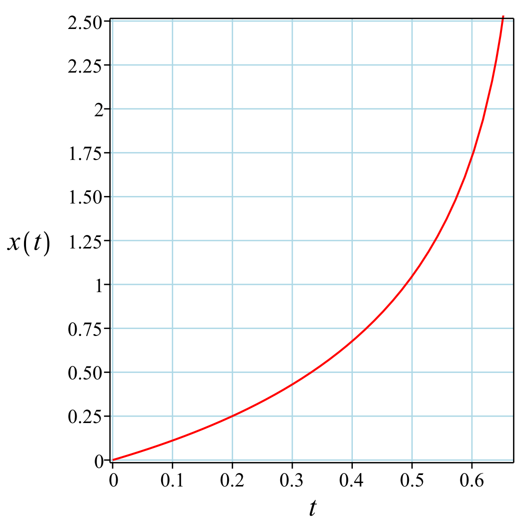
Substituting c_1 found above in the general solution gives

$$x = -\ln\left(-e^{t} + 2\right)$$

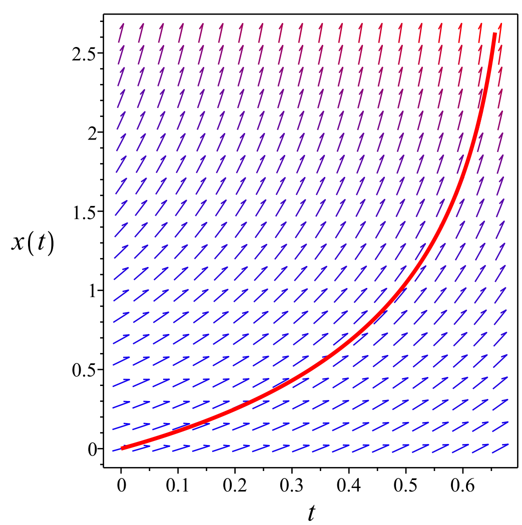
Summary

The solution(s) found are the following

$$x = -\ln\left(-e^{t} + 2\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -\ln\left(-e^{t} + 2\right)$$

Verified OK.

4.20.6 Maple step by step solution

Let's solve

$[x' - e^{t+x} = 0, x(0) = 0]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{e^{x}} = e^{t}$

- Integrate both sides with respect to t

  $\int \frac{x'}{e^{x}} dt = \int e^{t} dt + c_1$

- Evaluate integral

  $-\frac{1}{e^{x}} = e^{t} + c_1$

- Solve for x

  $x = \ln\left(-\frac{1}{e^{t}+c_1}\right)$

- Use initial condition $x(0) = 0$

  $0 = \ln\left(-\frac{1}{1+c_1}\right)$

- Solve for c_1

  $c_1 = -2$

- Substitute $c_1 = -2$ into general solution and simplify

  $x = \ln\left(-\frac{1}{e^{t}-2}\right)$

- Solution to the IVP

  $x = \ln\left(-\frac{1}{e^{t}-2}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 13

```
dsolve([diff(x(t),t)=exp(t+x(t)),x(0) = 0],x(t), singsol=all)
```

$$x(t) = -\ln\left(-e^{t} + 2\right)$$

✓ Solution by Mathematica

Time used: 1.309 (sec). Leaf size: 15

```
DSolve[{x'[t]==Exp[t+x[t]],{x[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\log\left(2 - e^{t}\right)$$

4.21 problem 10(b)

4.21.1 Existence and uniqueness analysis 345
4.21.2 Solving as separable ode 345
4.21.3 Solving as first order ode lie symmetry lookup ode 347
4.21.4 Solving as exact ode 351
4.21.5 Solving as riccati ode 354
4.21.6 Maple step by step solution 356

Internal problem ID [11388]
Internal file name [`OUTPUT/10370_Wednesday_May_17_2023_07_50_10_PM_12919399/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 10(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{T' - 2at(T^2 - a^2) = 0}$$

With initial conditions

$$[T(0) = 0]$$

4.21.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned}
T' &= f(t,T) \\
&= 2at(T^2 - a^2)
\end{aligned}$$

The t domain of $f(t,T)$ when $T=0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The T domain of $f(t,T)$ when $t=0$ is

$$\{-\infty < T < \infty\}$$

And the point $T_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial T} &= \frac{\partial}{\partial T}\left(2at(T^2 - a^2)\right) \\
&= 4atT
\end{aligned}$$

The t domain of $\frac{\partial f}{\partial T}$ when $T=0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The T domain of $\frac{\partial f}{\partial T}$ when $t=0$ is

$$\{-\infty < T < \infty\}$$

And the point $T_0 = 0$ is inside this domain. Therefore solution exists and is unique.

4.21.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
T' &= F(t,T) \\
&= f(t)g(T) \\
&= 2at(T^2 - a^2)
\end{aligned}$$

Where $f(t) = 2ta$ and $g(T) = T^2 - a^2$. Integrating both sides gives

$$\frac{1}{T^2 - a^2}\, dT = 2ta\, dt$$

$$\int \frac{1}{T^2 - a^2}\, dT = \int 2ta\, dt$$

$$\frac{\ln(T-a)}{2a} - \frac{\ln(T+a)}{2a} = t^2 a + c_1$$

The above can be written as

$$\left(\frac{1}{2a}\right)(\ln(T-a) - \ln(T+a)) = t^2 a + 2c_1$$

$$\ln(T-a) - \ln(T+a) = (2a)\left(t^2 a + 2c_1\right)$$
$$= 2a\left(t^2 a + 2c_1\right)$$

Raising both side to exponential gives

$$e^{\ln(T-a) - \ln(T+a)} = e^{2a\left(t^2 a + c_1\right)}$$

Which simplifies to

$$\frac{T-a}{T+a} = 2c_1 a\, e^{2a^2 t^2}$$
$$= c_2 e^{2a^2 t^2}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $T = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-ac_2 - a}{-1 + c_2}$$

$$c_2 = -1$$

Substituting c_2 found above in the general solution gives

$$T = \frac{-a\, e^{2a^2 t^2} + a}{e^{2a^2 t^2} + 1}$$

Summary

The solution(s) found are the following

$$T = \frac{-a\, e^{2a^2 t^2} + a}{e^{2a^2 t^2} + 1} \tag{1}$$

Verification of solutions

$$
T = \frac{-a\,\mathrm{e}^{2a^2t^2}+a}{\mathrm{e}^{2a^2t^2}+1}
$$

Verified OK.

4.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
T' &= 2at\left(T^2-a^2\right)\\
T' &= \omega(t,T)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t+\omega(\eta_T-\xi_t)-\omega^2\xi_T-\omega_t\xi-\omega_T\eta=0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 68: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, T) &= \frac{1}{2ta} \\
\eta(t, T) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, T) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dT}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial T}\right) S(t, T) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = T$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{2ta}} dt \end{aligned}$$

Which results in

$$S = t^2 a$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,T) S_T}{R_t + \omega(t,T) R_T} \tag{2}$$

Where in the above R_t, R_T, S_t, S_T are all partial derivatives and $\omega(t,T)$ is the right hand side of the original ode given by

$$\omega(t,T) = 2at\left(T^2 - a^2\right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_T &= 1 \\ S_t &= 2ta \\ S_T &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{T^2 - a^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, T in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 - a^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R-a)}{2a} - \frac{\ln(R+a)}{2a} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, T coordinates. This results in

$$t^2 a = \frac{\ln(T-a)}{2a} - \frac{\ln(T+a)}{2a} + c_1$$

Which simplifies to

$$t^2 a = \frac{\ln(T-a)}{2a} - \frac{\ln(T+a)}{2a} + c_1$$

Which gives

$$T = \frac{a\left(1 + \mathrm{e}^{-2a^2t^2+2ac_1}\right)}{\mathrm{e}^{-2a^2t^2+2ac_1} - 1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $T = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{a\,\mathrm{e}^{2ac_1} + a}{\mathrm{e}^{2ac_1} - 1}$$

$$c_1 = \frac{i\pi}{2a}$$

Substituting c_1 found above in the general solution gives

$$T = \frac{a\,\mathrm{e}^{-2a^2t^2} - a}{\mathrm{e}^{-2a^2t^2} + 1}$$

Summary

The solution(s) found are the following

$$T = \frac{a\,\mathrm{e}^{-2a^2t^2} - a}{\mathrm{e}^{-2a^2t^2} + 1} \tag{1}$$

Verification of solutions

$$T = \frac{a\,\mathrm{e}^{-2a^2t^2} - a}{\mathrm{e}^{-2a^2t^2} + 1}$$

Verified OK.

4.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,T)\,\mathrm{d}t+N(t,T)\,\mathrm{d}T=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{2a\left(T^2-a^2\right)}\right)\mathrm{d}T&=(t)\,\mathrm{d}t\\ (-t)\,\mathrm{d}t+\left(\frac{1}{2a\left(T^2-a^2\right)}\right)\mathrm{d}T&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, T) &= -t \\
N(t, T) &= \frac{1}{2a\left(T^{2}-a^{2}\right)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial T} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial T} &= \frac{\partial}{\partial T}(-t) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{2a\left(T^{2}-a^{2}\right)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial T} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, T)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} &= M \qquad (1) \\
\frac{\partial \phi}{\partial T} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -t \, \mathrm{d}t \\
\phi &= -\frac{t^{2}}{2} + f(T) \qquad (3)
\end{aligned}
$$

Where $f(T)$ is used for the constant of integration since ϕ is a function of both t and T. Taking derivative of equation (3) w.r.t T gives

$$
\frac{\partial \phi}{\partial T} = 0 + f'(T) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial T} = \frac{1}{2a(T^2-a^2)}$. Therefore equation (4) becomes

$$\frac{1}{2a\left(T^2-a^2\right)} = 0 + f'(T) \tag{5}$$

Solving equation (5) for $f'(T)$ gives

$$\begin{aligned} f'(T) &= \frac{1}{2a\left(T^2-a^2\right)} \\ &= \frac{1}{2T^2a-2a^3} \end{aligned}$$

Integrating the above w.r.t T results in

$$\begin{aligned} \int f'(T)\,\mathrm{d}T &= \int \left(\frac{1}{2T^2a-2a^3}\right) \mathrm{d}T \\ f(T) &= \frac{\ln\left(T-a\right)}{4a^2} - \frac{\ln\left(T+a\right)}{4a^2} + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(T)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} + \frac{\ln\left(T-a\right)}{4a^2} - \frac{\ln\left(T+a\right)}{4a^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} + \frac{\ln\left(T-a\right)}{4a^2} - \frac{\ln\left(T+a\right)}{4a^2}$$

The solution becomes

$$T = -\frac{a\left(\mathrm{e}^{2a^2t^2+4a^2c_1}+1\right)}{-1+\mathrm{e}^{2a^2t^2+4a^2c_1}}$$

Initial conditions are used to solve for c_1. Substituting $t=0$ and $T=0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-a\,\mathrm{e}^{4a^2c_1}-a}{-1+\mathrm{e}^{4a^2c_1}}$$

$$c_1 = \frac{i\pi}{4a^2}$$

Substituting c_1 found above in the general solution gives

$$T = \frac{-a\,\mathrm{e}^{2a^2t^2} + a}{\mathrm{e}^{2a^2t^2} + 1}$$

Summary

The solution(s) found are the following

$$T = \frac{-a\,\mathrm{e}^{2a^2t^2} + a}{\mathrm{e}^{2a^2t^2} + 1} \tag{1}$$

Verification of solutions

$$T = \frac{-a\,\mathrm{e}^{2a^2t^2} + a}{\mathrm{e}^{2a^2t^2} + 1}$$

Verified OK.

4.21.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} T' &= F(t, T) \\ &= 2at\left(T^2 - a^2\right) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$T' = 2T^2at - 2a^3t$$

With Riccati ODE standard form

$$T' = f_0(t) + f_1(t)T + f_2(t)T^2$$

Shows that $f_0(t) = -2a^3t$, $f_1(t) = 0$ and $f_2(t) = 2ta$. Let

$$\begin{aligned} T &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{2tau} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$
\begin{aligned}
f_2' &= 2a \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= -8t^3 a^5
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
2tau''(t) - 2au'(t) - 8t^3 a^5 u(t) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(t) = c_1 \sinh\left(a^2 t^2\right) + c_2 \cosh\left(a^2 t^2\right)
$$

The above shows that

$$
u'(t) = 2a^2 t\left(c_1 \cosh\left(a^2 t^2\right) + c_2 \sinh\left(a^2 t^2\right)\right)
$$

Using the above in (1) gives the solution

$$
T = -\frac{a(c_1 \cosh(a^2 t^2) + c_2 \sinh(a^2 t^2))}{c_1 \sinh(a^2 t^2) + c_2 \cosh(a^2 t^2)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
T = -\frac{a(c_3 \cosh(a^2 t^2) + \sinh(a^2 t^2))}{c_3 \sinh(a^2 t^2) + \cosh(a^2 t^2)}
$$

Initial conditions are used to solve for c_3. Substituting $t = 0$ and $T = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = -c_3 a
$$

$$
c_3 = 0
$$

Substituting c_3 found above in the general solution gives

$$
T = -\frac{\sinh(a^2 t^2)\, a}{\cosh(a^2 t^2)}
$$

Summary
The solution(s) found are the following

$$T = -\frac{\sinh(a^2t^2)\,a}{\cosh(a^2t^2)} \tag{1}$$

Verification of solutions

$$T = -\frac{\sinh(a^2t^2)\,a}{\cosh(a^2t^2)}$$

Verified OK.

4.21.6 Maple step by step solution

Let's solve

$[T' - 2at(T^2 - a^2) = 0, T(0) = 0]$

- Highest derivative means the order of the ODE is 1

  T'

- Separate variables

  $\frac{T'}{T^2-a^2} = 2ta$

- Integrate both sides with respect to t

  $\int \frac{T'}{T^2-a^2}dt = \int 2tadt + c_1$

- Evaluate integral

  $\frac{\ln(T-a)}{2a} - \frac{\ln(T+a)}{2a} = t^2a + c_1$

- Solve for T

  $T = -\frac{a\left(e^{2a^2t^2+2c_1a}+1\right)}{-1+e^{2a^2t^2+2c_1a}}$

- Use initial condition $T(0) = 0$

  $0 = -\frac{(1+e^{2c_1a})a}{e^{2c_1a}-1}$

- Solve for c_1

  $c_1 = \frac{\frac{I}{2}\pi}{a}$

- Substitute $c_1 = \frac{\frac{I}{2}\pi}{a}$ into general solution and simplify

  $T = -\frac{\left(e^{2a^2t^2}-1\right)a}{e^{2a^2t^2}+1}$

- Solution to the IVP

$$T = -\frac{\left(\mathrm{e}^{2a^2t^2}-1\right)a}{\mathrm{e}^{2a^2t^2}+1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.281 (sec). Leaf size: 31

```
dsolve([diff(T(t),t)=2*a*t*(T(t)^2-a^2),T(0) = 0],T(t), singsol=all)
```

$$T(t) = -\frac{a\left(\mathrm{e}^{2t^2a^2}-1\right)}{\mathrm{e}^{2t^2a^2}+1}$$

✓ Solution by Mathematica

Time used: 3.308 (sec). Leaf size: 16

```
DSolve[{T'[t]==2*a*t*(T[t]^2-a^2),{T[0]==0}},T[t],t,IncludeSingularSolutions -> True]
```

$$T(t) \to -a\tanh\left(a^2t^2\right)$$

4.22 problem 10(c)

4.22.1 Existence and uniqueness analysis 358
4.22.2 Solving as separable ode 359
4.22.3 Solving as first order ode lie symmetry lookup ode 360
4.22.4 Solving as exact ode 364
4.22.5 Maple step by step solution 367

Internal problem ID [11389]
Internal file name [`OUTPUT/10371_Wednesday_May_17_2023_07_50_11_PM_32960068/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 10(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - t^2 \tan(y) = 0}$$

With initial conditions

$$[y(0) = 0]$$

4.22.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= t^2 \tan(y) \end{aligned}$$

The t domain of $f(t, y)$ when $y = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(t^2 \tan(y)\right) \\ &= t^2\left(1 + \tan(y)^2\right)\end{aligned}$$

The t domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

4.22.2 Solving as separable ode

In canonical form the ODE is

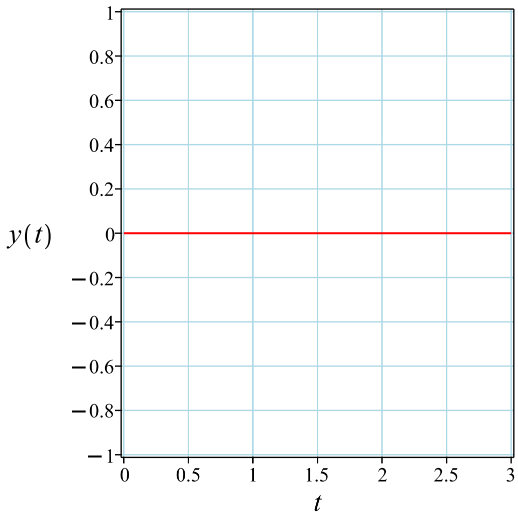
$$\begin{aligned}y' &= F(t, y) \\ &= f(t)g(y) \\ &= t^2 \tan(y)\end{aligned}$$

Where $f(t) = t^2$ and $g(y) = \tan(y)$. Since unique solution exists and $g(y)$ evaluated at $y_0 = 0$ is zero, then the solution is

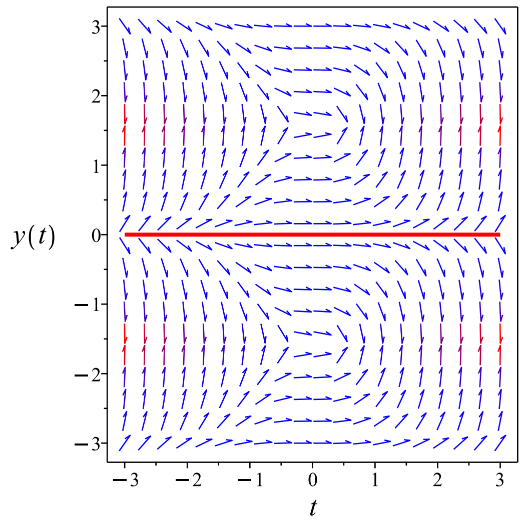
$$\begin{aligned}y &= y_0 \\ &= 0\end{aligned}$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 0$$

Verified OK.

4.22.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = t^2 \tan(y)$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 71: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= \frac{1}{t^2} \\
\eta(t, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{t^2}} dt \end{aligned}$$

Which results in

$$S = \frac{t^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = t^2 \tan(y)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= t^2 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cot(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cot(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(\sin(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{t^3}{3} = \ln(\sin(y)) + c_1$$

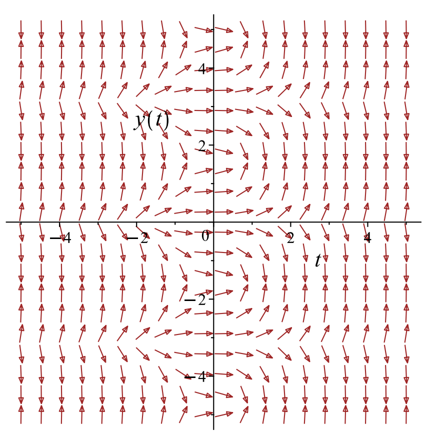
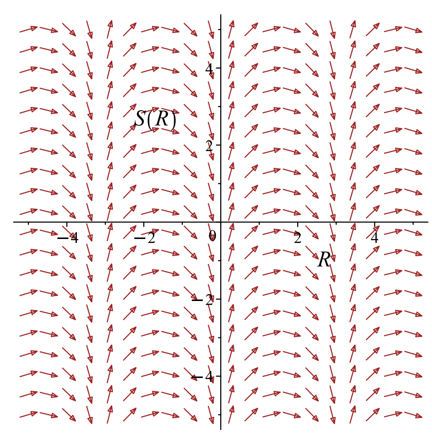
Which simplifies to

$$\frac{t^3}{3} = \ln(\sin(y)) + c_1$$

Which gives

$$y = \arcsin\left(e^{\frac{t^3}{3} - c_1}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = t^2 \tan(y)$ 	$R = y$ $S = \frac{t^3}{3}$	$\frac{dS}{dR} = \cot(R)$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

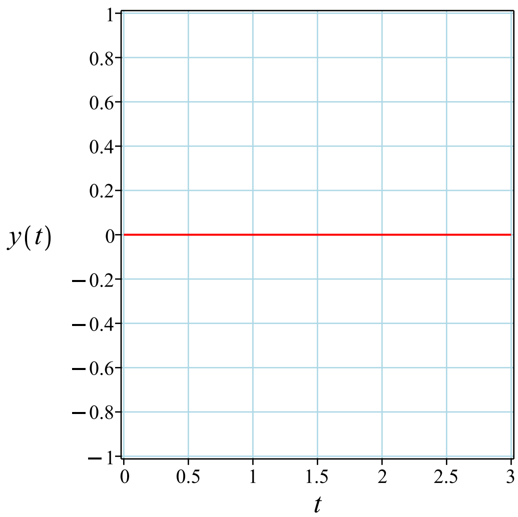
$$0 = \arcsin\left(e^{-c_1}\right)$$

Unable to solve for constant of integration. Since $\lim_{c_1 \to \infty}$ gives $y = \arcsin\left(e^{\frac{t^3}{3}-c_1}\right) =$ $y = 0$ and this result satisfies the given initial condition.

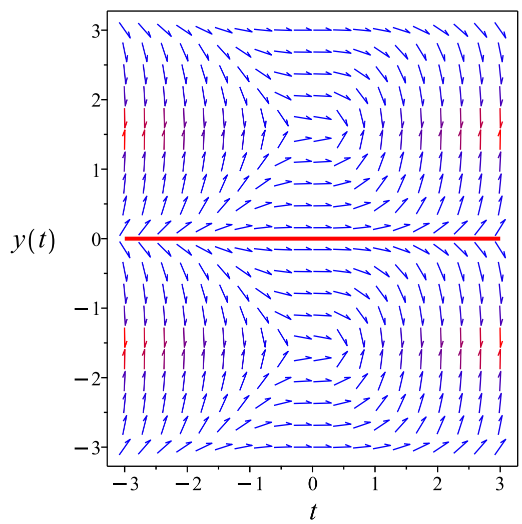
Summary

The solution(s) found are the following

$$y = 0$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 0$$

Verified OK.

4.22.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{\tan(y)}\right)\mathrm{d}y &= \left(t^2\right)\mathrm{d}t \\
\left(-t^2\right)\mathrm{d}t + \left(\frac{1}{\tan(y)}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t,y) &= -t^2 \\
N(t,y) &= \frac{1}{\tan(y)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-t^2\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{\tan(y)}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -t^2\,\mathrm{d}t \\
\phi &= -\frac{t^3}{3} + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\tan(y)}$. Therefore equation (4) becomes

$$\frac{1}{\tan(y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}
f'(y) &= \frac{1}{\tan(y)} \\
&= \cot(y)
\end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int (\cot(y))\,\mathrm{d}y$$
$$f(y) = \ln(\sin(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^3}{3} + \ln(\sin(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^3}{3} + \ln(\sin(y))$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\infty = c_1$$

Unable to solve for constant of integration. Warning: Unable to solve for constant of integration.

Verification of solutions N/A

4.22.5 Maple step by step solution

Let's solve

$[y' - t^2\tan(y) = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\tan(y)} = t^2$

- Integrate both sides with respect to t

  $\int \frac{y'}{\tan(y)}dt = \int t^2 dt + c_1$

- Evaluate integral

  $\ln(\sin(y)) = \frac{t^3}{3} + c_1$

- Solve for y

  $y = \arcsin\left(\mathrm{e}^{\frac{t^3}{3}+c_1}\right)$

- Use initial condition $y(0) = 0$

  $0 = \arcsin(\mathrm{e}^{c_1})$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(t),t)=t^2*tan(y(t)),y(0) = 0],y(t), singsol=all)
```

$$y(t) = 0$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 6

```
DSolve[{y'[t]==t^2*Tan[y[t]],{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 0$$

4.23 problem 11

4.23.1 Existence and uniqueness analysis 370
4.23.2 Solving as separable ode 370
4.23.3 Solving as first order ode lie symmetry lookup ode 372
4.23.4 Solving as exact ode 377
4.23.5 Maple step by step solution 380

Internal problem ID [11390]
Internal file name [`OUTPUT/10372_Wednesday_May_17_2023_07_50_12_PM_14623495/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' - \frac{(4+2t)\,x}{\ln(x)} = 0}$$

With initial conditions

$$[x(0) = \mathrm{e}]$$

4.23.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t,x) \\ &= \frac{2x(t+2)}{\ln(x)} \end{aligned}$$

The t domain of $f(t,x)$ when $x = \mathrm{e}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The x domain of $f(t,x)$ when $t = 0$ is

$$\{0 < x < 1, 1 < x \leq \infty\}$$

And the point $x_0 = \mathrm{e}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2x(t+2)}{\ln(x)}\right) \\ &= \frac{4+2t}{\ln(x)} - \frac{2(t+2)}{\ln(x)^2} \end{aligned}$$

The t domain of $\frac{\partial f}{\partial x}$ when $x = \mathrm{e}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The x domain of $\frac{\partial f}{\partial x}$ when $t = 0$ is

$$\{0 < x < 1, 1 < x \leq \infty\}$$

And the point $x_0 = \mathrm{e}$ is inside this domain. Therefore solution exists and is unique.

4.23.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= f(t)g(x) \\ &= \frac{(4+2t)\,x}{\ln(x)} \end{aligned}$$

Where $f(t) = 4 + 2t$ and $g(x) = \frac{x}{\ln(x)}$. Integrating both sides gives

$$\frac{1}{\frac{x}{\ln(x)}} dx = 4 + 2t \, dt$$

$$\int \frac{1}{\frac{x}{\ln(x)}} dx = \int 4 + 2t \, dt$$

$$\frac{\ln(x)^2}{2} = t^2 + c_1 + 4t$$

Raising both side to exponential gives

$$e^{\frac{\ln(x)^2}{2}} = e^{t^2 + c_1 + 4t}$$

Which simplifies to

$$e^{\frac{\ln(x)^2}{2}} = c_2 e^{t^2 + 4t}$$

The solution is

$$e^{\frac{\ln(x)^2}{2}} = c_2 e^{t^2 + c_1 + 4t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = e$ in the above solution gives an equation to solve for the constant of integration.

$$e^{\frac{1}{2}} = e^{c_1} c_2$$

$$c_1 = -\ln(c_2) + \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$e^{\frac{\ln(x)^2}{2}} = e^{t^2 + \frac{1}{2} + 4t}$$

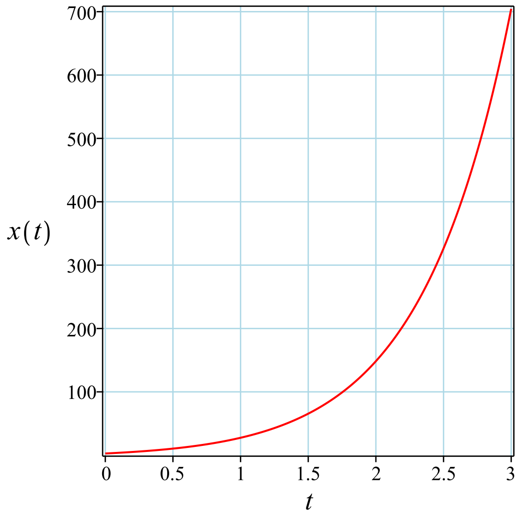
Solving for x from the above gives

$$x = e^{\sqrt{2t^2 + 8t + 1}}$$

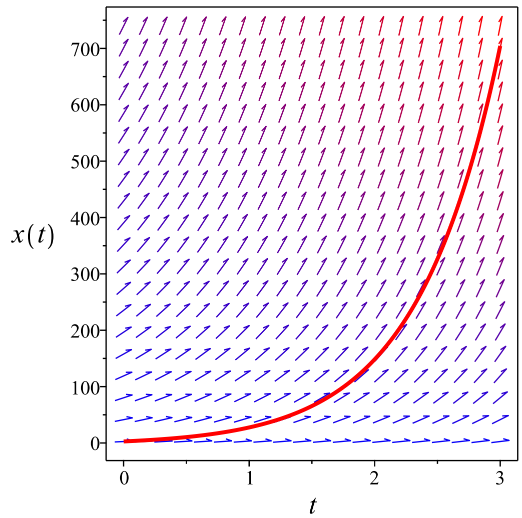
Summary

The solution(s) found are the following

$$x = e^{\sqrt{2t^2 + 8t + 1}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{\sqrt{2t^2+8t+1}}$$

Verified OK.

4.23.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = \frac{2x(t+2)}{\ln(x)}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 74: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= \frac{1}{4+2t} \\
\eta(t,x) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{4+2t}} dt \end{aligned}$$

Which results in

$$S = t^2 + 4t$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = \frac{2x(t+2)}{\ln(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_x &= 1 \\ S_t &= 4 + 2t \\ S_x &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\ln(x)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\ln(R)}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)^2}{2} + c_1 \tag{4}$$

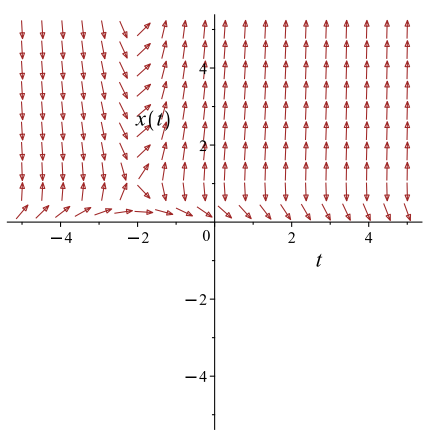
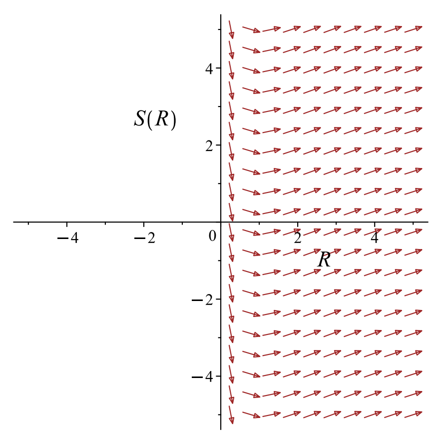
To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$t^2 + 4t = \frac{\ln(x)^2}{2} + c_1$$

Which simplifies to

$$t^2 + 4t = \frac{\ln(x)^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \frac{2x(t+2)}{\ln(x)}$ 	$R = x$ $S = t^2 + 4t$	$\frac{dS}{dR} = \frac{\ln(R)}{R}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = \mathrm{e}$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{2} + c_1$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$t^2 + 4t = \frac{\ln(x)^2}{2} - \frac{1}{2}$$

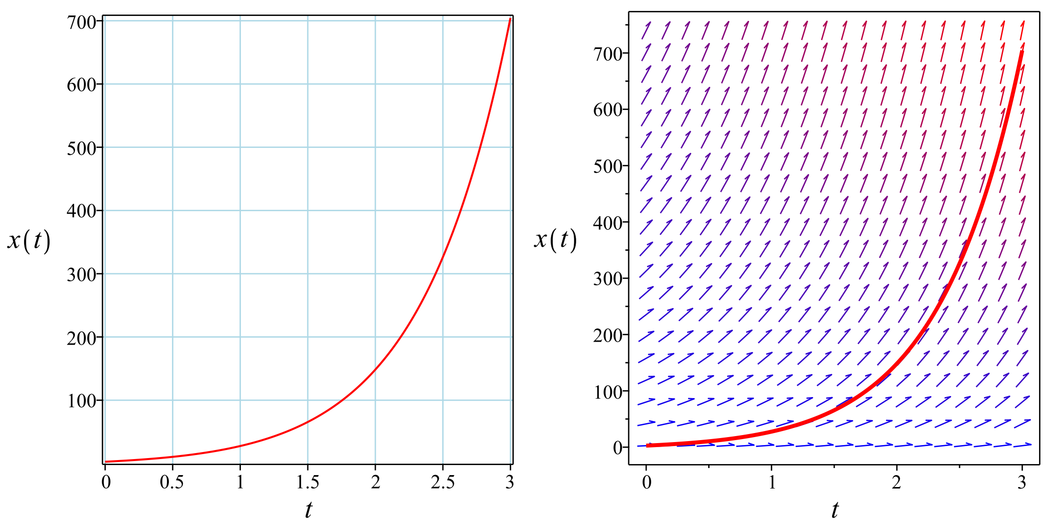
Solving for x from the above gives

$$x = e^{\sqrt{2t^2+8t+1}}$$

Summary

The solution(s) found are the following

$$x = e^{\sqrt{2t^2+8t+1}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = e^{\sqrt{2t^2+8t+1}}$$

Verified OK.

4.23.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \left(\frac{\ln(x)}{2x}\right)\mathrm{d}x &= (t+2)\,\mathrm{d}t \\ (-t-2)\,\mathrm{d}t + \left(\frac{\ln(x)}{2x}\right)\mathrm{d}x &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t,x) &= -t-2 \\
N(t,x) &= \frac{\ln(x)}{2x}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-t-2) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\ln(x)}{2x}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,x)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} &= M &\quad (1) \\
\frac{\partial \phi}{\partial x} &= N &\quad (2)
\end{aligned}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -t-2\,\mathrm{d}t \\
\phi &= -\frac{1}{2}t^2 - 2t + f(x) &\quad (3)
\end{aligned}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{\ln(x)}{2x}$. Therefore equation (4) becomes

$$\frac{\ln (x)}{2x} = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = \frac{\ln (x)}{2x}$$

Integrating the above w.r.t x gives

$$\int f'(x) \, \mathrm{d}x = \int \left(\frac{\ln (x)}{2x}\right) \mathrm{d}x$$

$$f(x) = \frac{\ln (x)^2}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} - 2t + \frac{\ln (x)^2}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} - 2t + \frac{\ln (x)^2}{4}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = \mathrm{e}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{4} = c_1$$

$$c_1 = \frac{1}{4}$$

Substituting c_1 found above in the general solution gives

$$-\frac{t^2}{2} - 2t + \frac{\ln (x)^2}{4} = \frac{1}{4}$$

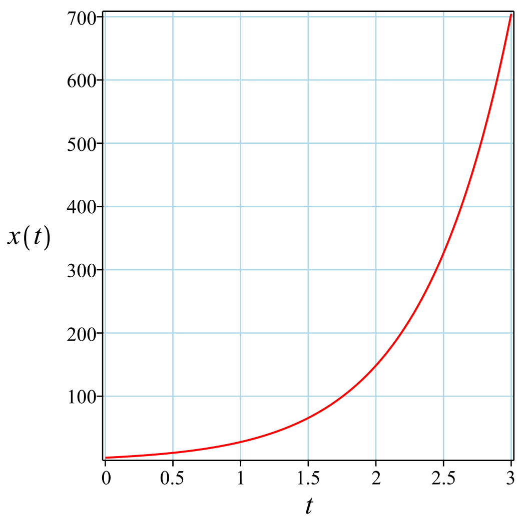
Solving for x from the above gives

$$x = e^{\sqrt{2t^2+8t+1}}$$

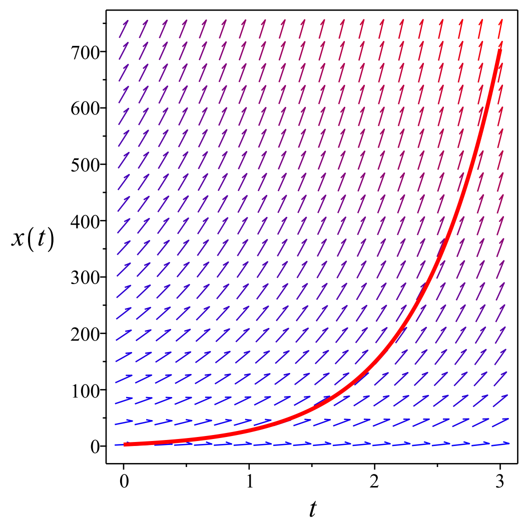
Summary

The solution(s) found are the following

$$x = e^{\sqrt{2t^2+8t+1}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{\sqrt{2t^2+8t+1}}$$

Verified OK.

4.23.5 Maple step by step solution

Let's solve

$\left[x' - \frac{(4+2t)x}{\ln(x)} = 0, x(0) = e\right]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

$\frac{x' \ln(x)}{x} = 4 + 2t$

- Integrate both sides with respect to t

  $\int \frac{x' \ln(x)}{x} dt = \int (4 + 2t)\, dt + c_1$

- Evaluate integral

  $\frac{\ln(x)^2}{2} = t^2 + c_1 + 4t$

- Solve for x

  $\left\{x = e^{\sqrt{2t^2+2c_1+8t}}, x = e^{-\sqrt{2t^2+2c_1+8t}}\right\}$

- Use initial condition $x(0) = e$

  $e = e^{\sqrt{c_1}\,\sqrt{2}}$

- Solve for c_1

  $c_1 = \frac{1}{2}$

- Substitute $c_1 = \frac{1}{2}$ into general solution and simplify

  $x = e^{\sqrt{2t^2+8t+1}}$

- Use initial condition $x(0) = e$

  $e = e^{-\sqrt{c_1}\,\sqrt{2}}$

- Solution does not satisfy initial condition

- Solution to the IVP

  $x = e^{\sqrt{2t^2+8t+1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 17

```
dsolve([diff(x(t),t)=(4+2*t)*x(t)/ln(x(t)),x(0) = exp(1)],x(t), singsol=all)
```

$$x(t) = \mathrm{e}^{\sqrt{2t^2+8t+1}}$$

✓ Solution by Mathematica

Time used: 1.447 (sec). Leaf size: 21

```
DSolve[{x'[t]==(4+2*t)*x[t]/Log[x[t]],{x[0]==Exp[1]}},x[t],t,IncludeSingularSolutions -> Tru
```

$$x(t) \to e^{\sqrt{2t^2+8t+1}}$$

4.24 problem 12

4.24.1 Existence and uniqueness analysis 384
4.24.2 Solving as separable ode 384
4.24.3 Solving as first order ode lie symmetry lookup ode 385
4.24.4 Solving as exact ode 389
4.24.5 Solving as riccati ode 393
4.24.6 Maple step by step solution 395

Internal problem ID [11391]
Internal file name [`OUTPUT/10373_Wednesday_May_17_2023_07_50_14_PM_39861667/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{2ty^2}{t^2+1} = 0}$$

With initial conditions

$$[y(0) = 0]$$

4.24.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
y' &= f(t, y) \\
&= \frac{2t\,y^2}{t^2+1}
\end{aligned}
$$

The t domain of $f(t, y)$ when $y = 0$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t, y)$ when $t = 0$ is

$$
\{-\infty < y < \infty\}
$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2t\,y^2}{t^2+1}\right) \\
&= \frac{4ty}{t^2+1}
\end{aligned}
$$

The t domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$
\{-\infty < y < \infty\}
$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

4.24.2 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(t, y) \\
&= f(t)g(y) \\
&= \frac{2t\,y^2}{t^2+1}
\end{aligned}
$$

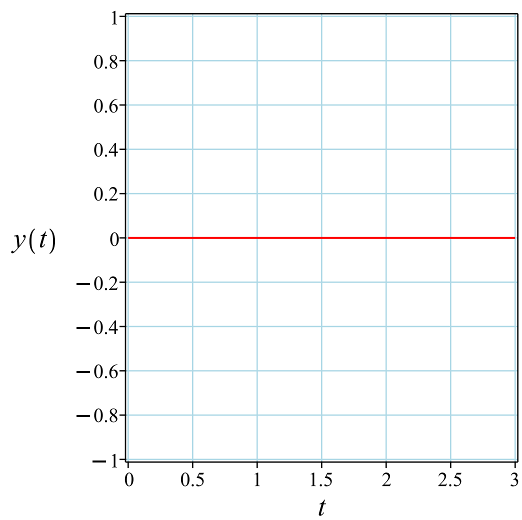
Where $f(t) = \frac{2t}{t^2+1}$ and $g(y) = y^2$. Since unique solution exists and $g(y)$ evaluated at $y_0 = 0$ is zero, then the solution is

$$
\begin{aligned}
y &= y_0 \\
&= 0
\end{aligned}
$$

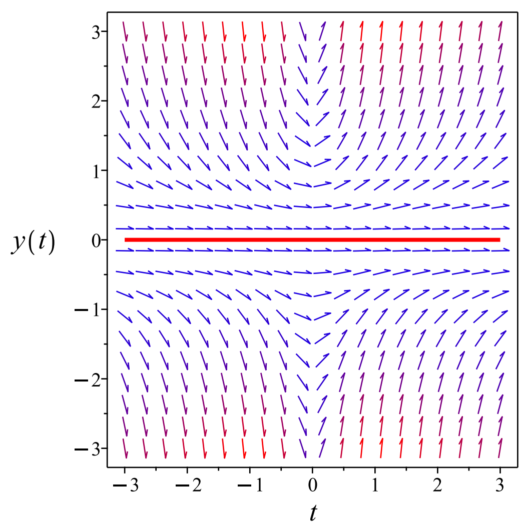
Summary

The solution(s) found are the following

$$
y = 0 \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = 0
$$

Verified OK.

4.24.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{2t\,y^2}{t^2+1} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 77: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= \frac{t^2 + 1}{2t} \\
\eta(t, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{t^2+1}{2t}} dt \end{aligned}$$

Which results in

$$S = \ln\left(t^2 + 1\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{2t\, y^2}{t^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= \frac{2t}{t^2 + 1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\ln\left(t^2+1\right) = -\frac{1}{y} + c_1$$

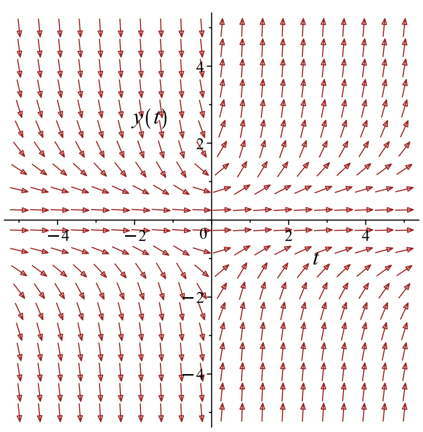
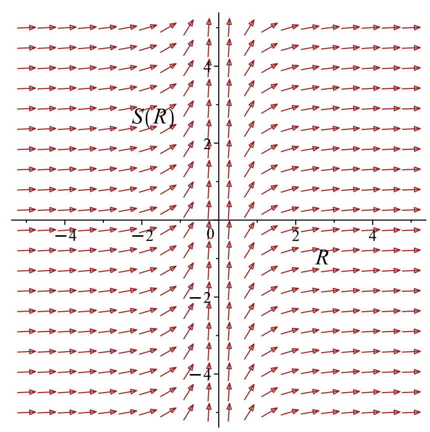
Which simplifies to

$$\ln\left(t^2+1\right) = -\frac{1}{y} + c_1$$

Which gives

$$y = -\frac{1}{\ln\left(t^2+1\right) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{2t\,y^2}{t^2+1}$ 	$R = y$ $S = \ln\left(t^2+1\right)$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

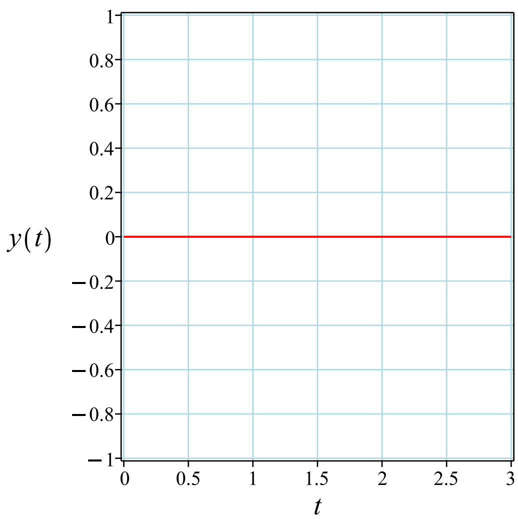
$$0 = \frac{1}{c_1}$$

Unable to solve for constant of integration. Since $\lim_{c_1 \to \infty}$ gives $y = -\frac{1}{\ln(t^2+1)-c_1} = y = 0$ and this result satisfies the given initial condition.

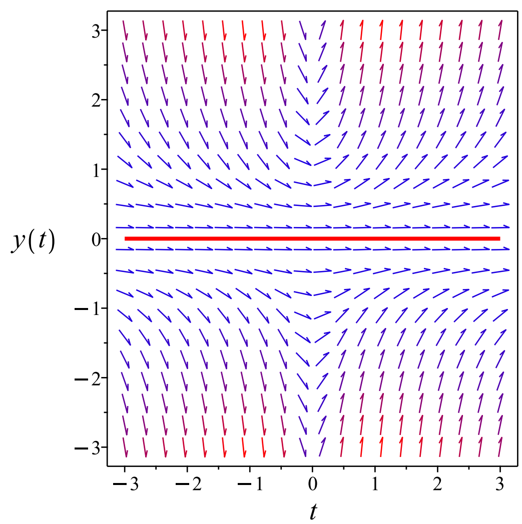
Summary

The solution(s) found are the following

$$y = 0$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 0$$

Verified OK.

4.24.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{2y^2}\right)\mathrm{d}y&=\left(\frac{t}{t^2+1}\right)\mathrm{d}t\\ \left(-\frac{t}{t^2+1}\right)\mathrm{d}t+\left(\frac{1}{2y^2}\right)\mathrm{d}y&=0 \qquad\qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-\frac{t}{t^2+1}\\ N(t,y)&=\frac{1}{2y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{t}{t^2+1}\right)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{2y^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t &= \int M\, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t &= \int -\frac{t}{t^2+1}\, \mathrm{d}t \\
\phi &= -\frac{\ln(t^2+1)}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y^2}$. Therefore equation (4) becomes

$$
\frac{1}{2y^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{2y^2}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{2y^2}\right)\mathrm{d}y$$

$$f(y) = -\frac{1}{2y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(t^2+1)}{2} - \frac{1}{2y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(t^2+1)}{2} - \frac{1}{2y}$$

The solution becomes

$$y = -\frac{1}{\ln(t^2+1) + 2c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

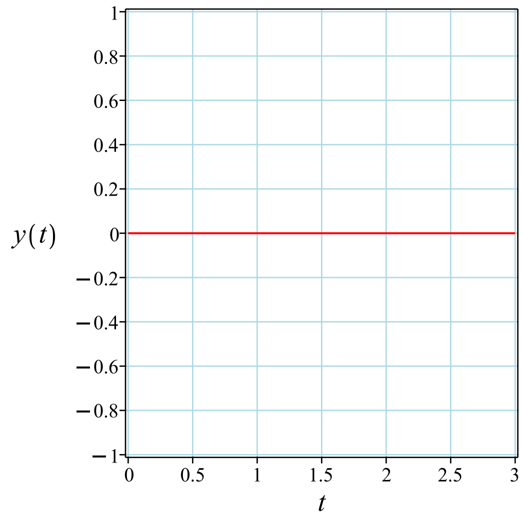
$$0 = -\frac{1}{2c_1}$$

Unable to solve for constant of integration. Since $\lim_{c_1 \to \infty}$ gives $y = -\frac{1}{\ln(t^2+1)+2c_1} = y = 0$ and this result satisfies the given initial condition.

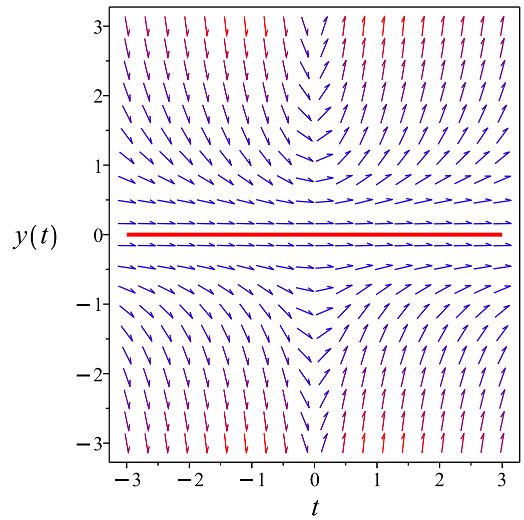
Summary

The solution(s) found are the following

$$y = 0$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 0$$

Verified OK.

4.24.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= \frac{2t\, y^2}{t^2 + 1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{2t\, y^2}{t^2 + 1}$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 0$, $f_1(t) = 0$ and $f_2(t) = \frac{2t}{t^2+1}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{2tu}{t^2+1}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{2}{t^2+1} - \frac{4t^2}{\left(t^2+1\right)^2} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= 0
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{2tu''(t)}{t^2+1} - \left(\frac{2}{t^2+1} - \frac{4t^2}{\left(t^2+1\right)^2}\right) u'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 + \ln\left(t^2+1\right) c_2$$

The above shows that

$$u'(t) = \frac{2tc_2}{t^2+1}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{c_1 + \ln\left(t^2+1\right) c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{c_3 + \ln\left(t^2+1\right)}$$

Initial conditions are used to solve for c_3. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

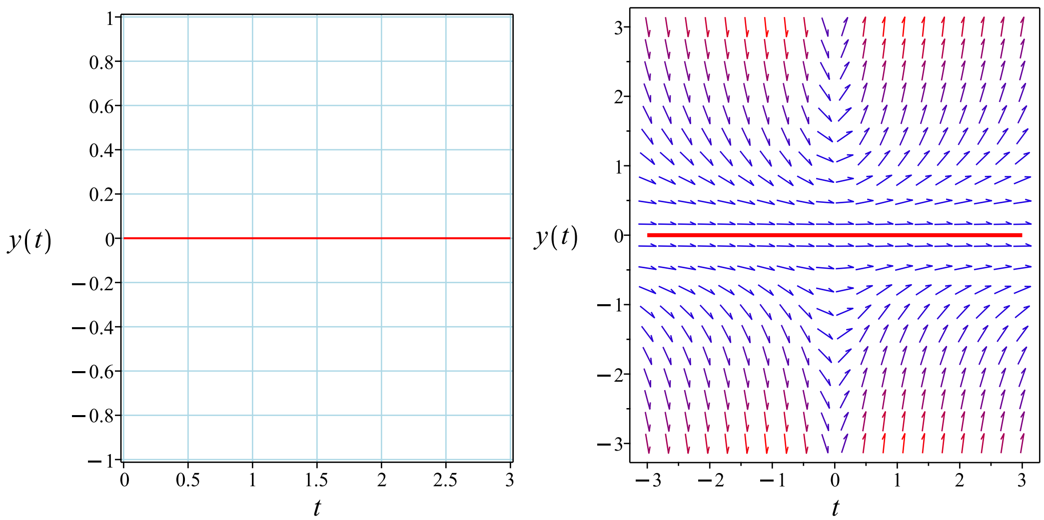
$$0 = -\frac{1}{c_3}$$

Unable to solve for constant of integration. Since $\lim_{c_1 \to \infty}$ gives $y = -\frac{1}{c_3 + \ln(t^2+1)} = y = 0$ and this result satisfies the given initial condition.

Summary

The solution(s) found are the following

$$y = 0$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 0$$

Verified OK.

4.24.6 Maple step by step solution

Let's solve

$\left[y' - \frac{2ty^2}{t^2+1} = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = \frac{2t}{t^2+1}$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2} dt = \int \frac{2t}{t^2+1} dt + c_1$

- Evaluate integral

  $-\frac{1}{y} = \ln(t^2+1) + c_1$

- Solve for y

  $y = -\frac{1}{\ln(t^2+1)+c_1}$

- Use initial condition $y(0) = 0$

  $0 = -\frac{1}{c_1}$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(t),t)=2*t*y(t)^2/(1+t^2),y(0) = 0],y(t), singsol=all)
```

$$y(t) = 0$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 6

```
DSolve[{y'[t]==2*t*y[t]^2/(1+t^2),{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 0$$

4.25 problem 13

4.25.1 Existence and uniqueness analysis 398
4.25.2 Solving as separable ode 398
4.25.3 Solving as differentialType ode 401
4.25.4 Solving as first order ode lie symmetry lookup ode 403
4.25.5 Solving as exact ode 407
4.25.6 Maple step by step solution 410

Internal problem ID [11392]
Internal file name [`OUTPUT/10374_Wednesday_May_17_2023_08_10_08_PM_27558722/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' - \frac{t^2}{1 - x^2} = 0}$$

With initial conditions

$$[x(1) = 1]$$

4.25.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
x' &= f(t,x) \\
&= -\frac{t^2}{x^2-1}
\end{aligned}
$$

$f(t,x)$ is not defined at $x=1$ therefore existence and uniqueness theorem do not apply.

4.25.2 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
x' &= F(t,x) \\
&= f(t)g(x) \\
&= -\frac{t^2}{x^2-1}
\end{aligned}
$$

Where $f(t)=-t^2$ and $g(x)=\frac{1}{x^2-1}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{x^2-1}}\,dx &= -t^2\,dt \\
\int \frac{1}{\frac{1}{x^2-1}}\,dx &= \int -t^2\,dt \\
\frac{1}{3}x^3 - x &= -\frac{t^3}{3} + c_1
\end{aligned}
$$

Which results in

$$
\begin{aligned}
x = &\frac{\left(-4t^3+12c_1+4\sqrt{t^6-6c_1t^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} \\
&+ \frac{2}{\left(-4t^3+12c_1+4\sqrt{t^6-6c_1t^3+9c_1^2-4}\right)^{\frac{1}{3}}}
\end{aligned}
$$

$$x = -\frac{\left(-4t^3 + 12c_1 + 4\sqrt{t^6 - 6c_1t^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4t^3 + 12c_1 + 4\sqrt{t^6 - 6c_1t^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-4t^3+12c_1+4\sqrt{t^6-6c_1t^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-4t^3+12c_1+4\sqrt{t^6-6c_1t^3+9c_1^2-4}\right)^{\frac{1}{3}}}\right)}{2}$$

$$x = -\frac{\left(-4t^3 + 12c_1 + 4\sqrt{t^6 - 6c_1t^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4t^3 + 12c_1 + 4\sqrt{t^6 - 6c_1t^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-4t^3+12c_1+4\sqrt{t^6-6c_1t^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-4t^3+12c_1+4\sqrt{t^6-6c_1t^3+9c_1^2-4}\right)^{\frac{1}{3}}}\right)}{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-i\sqrt{3}\left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{2}{3}} - \left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{2}{3}} + 4i\sqrt{3} - 4}{4\left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{1}{3}}}$$

$$c_1 = -\frac{1}{3}$$

Substituting c_1 found above in the general solution gives

$$x = \frac{-i\sqrt{3}\left(-4t^3 - 4 + 4\sqrt{t^6 + 2t^3 - 3}\right)^{\frac{2}{3}} + 4i\sqrt{3} - \left(-4t^3 - 4 + 4\sqrt{t^6 + 2t^3 - 3}\right)^{\frac{2}{3}} - 4}{4\left(-4t^3 - 4 + 4\sqrt{t^6 + 2t^3 - 3}\right)^{\frac{1}{3}}}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{i\sqrt{3}\left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{2}{3}} - 4i\sqrt{3} - \left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{2}{3}} - 4}{4\left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{1}{3}}}$$

Unable to solve for constant of integration. Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{4 + \left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{2}{3}}}{2\left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{1}{3}}}$$

$$c_1 = -\frac{1}{3}$$

Substituting c_1 found above in the general solution gives

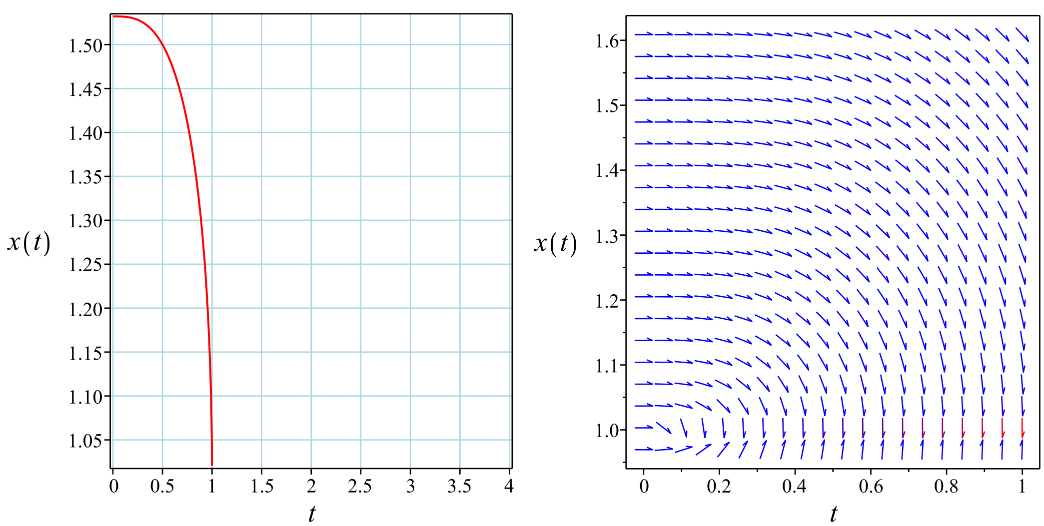
$$x = \frac{4 + \left(-4t^3 - 4 + 4\sqrt{t^6 + 2t^3 - 3}\right)^{\frac{2}{3}}}{2\left(-4t^3 - 4 + 4\sqrt{t^6 + 2t^3 - 3}\right)^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$x = \frac{4 + \left(-4t^3 - 4 + 4\sqrt{t^6 + 2t^3 - 3}\right)^{\frac{2}{3}}}{2\left(-4t^3 - 4 + 4\sqrt{t^6 + 2t^3 - 3}\right)^{\frac{1}{3}}} \tag{1}$$

$$x = \frac{-i\sqrt{3}\left(-4t^3 - 4 + 4\sqrt{t^6 + 2t^3 - 3}\right)^{\frac{2}{3}} + 4i\sqrt{3} - \left(-4t^3 - 4 + 4\sqrt{t^6 + 2t^3 - 3}\right)^{\frac{2}{3}} - 4}{4\left(-4t^3 - 4 + 4\sqrt{t^6 + 2t^3 - 3}\right)^{\frac{1}{3}}} \tag{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \frac{4+\left(-4t^3-4+4\sqrt{t^6+2t^3-3}\right)^{\frac{2}{3}}}{2\left(-4t^3-4+4\sqrt{t^6+2t^3-3}\right)^{\frac{1}{3}}}$$

Verified OK.

$$x = \frac{-i\sqrt{3}\left(-4t^3-4+4\sqrt{t^6+2t^3-3}\right)^{\frac{2}{3}}+4i\sqrt{3}-\left(-4t^3-4+4\sqrt{t^6+2t^3-3}\right)^{\frac{2}{3}}-4}{4\left(-4t^3-4+4\sqrt{t^6+2t^3-3}\right)^{\frac{1}{3}}}$$

Verified OK.

4.25.3 Solving as differentialType ode

Writing the ode as

$$x' = \frac{t^2}{1-x^2} \tag{1}$$

Which becomes

$$\left(x^2-1\right)dx = \left(-t^2\right)dt \tag{2}$$

But the RHS is complete differential because

$$\left(-t^2\right)dt = d\left(-\frac{t^3}{3}\right)$$

Hence (2) becomes

$$\left(x^2-1\right)dx = d\left(-\frac{t^3}{3}\right)$$

Integrating both sides gives gives these solutions

$$x = \frac{\left(-4t^3 + 12c_1 + 4\sqrt{t^6 - 6c_1t^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(-4t^3 + 12c_1 + 4\sqrt{t^6 - 6c_1t^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}} + c_1$$

$$x = -\frac{\left(-4t^3 + 12c_1 + 4\sqrt{t^6 - 6c_1t^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4t^3 + 12c_1 + 4\sqrt{t^6 - 6c_1t^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\,\Big(}{}$$

$$x = -\frac{\left(-4t^3 + 12c_1 + 4\sqrt{t^6 - 6c_1t^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4t^3 + 12c_1 + 4\sqrt{t^6 - 6c_1t^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\,\Big(}{}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-i\sqrt{3}\left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{2}{3}} + 4i\sqrt{3} - \left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{2}{3}} + 4c_1\big(-4 + 12}{4\left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{i\sqrt{3}\left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{2}{3}} - 4i\sqrt{3} - \left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{2}{3}} + 4c_1\big(-4 + 12c}{4\left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{2}{3}} + 2c_1\left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{1}{3}} + 4}{2\left(-4 + 12c_1 + 4\sqrt{9c_1^2 - 6c_1 - 3}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration.

Verification of solutions N/A

4.25.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= -\frac{t^2}{x^2-1} \\
x' &= \omega(t,x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 80: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= -\frac{1}{t^2} \\
\eta(t, x) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dt \\
&= \int \frac{1}{-\frac{1}{t^2}} dt
\end{aligned}
$$

Which results in

$$
S = -\frac{t^3}{3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}
$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$
\omega(t, x) = -\frac{t^2}{x^2 - 1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_x &= 1 \\
S_t &= -t^2 \\
S_x &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = x^2 - 1 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = R^2 - 1
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{1}{3}R^3 - R + c_1 \tag{4}
$$

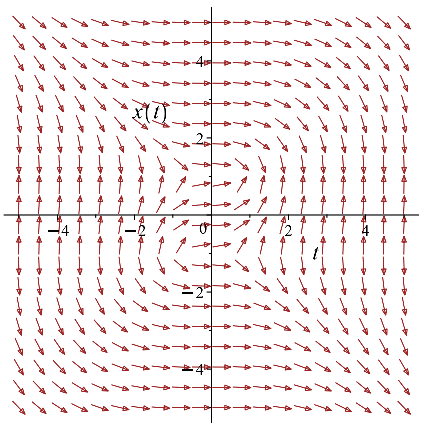
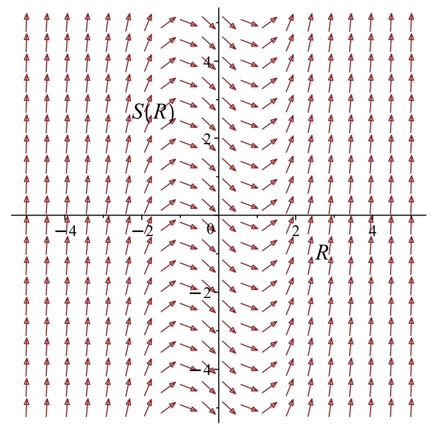
To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$
-\frac{t^3}{3} = \frac{x^3}{3} - x + c_1
$$

Which simplifies to

$$
-\frac{t^3}{3} = \frac{x^3}{3} - x + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\frac{t^2}{x^2-1}$ 	$R = x$ $S = -\frac{t^3}{3}$	$\frac{dS}{dR} = R^2 - 1$ 

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{3} = c_1 - \frac{2}{3}$$

$$c_1 = \frac{1}{3}$$

Substituting c_1 found above in the general solution gives

$$-\frac{t^3}{3} = \frac{1}{3} + \frac{1}{3}x^3 - x$$

Summary

The solution(s) found are the following

$$-\frac{t^3}{3} = \frac{1}{3} + \frac{x^3}{3} - x \tag{1}$$

Verification of solutions

$$-\frac{t^3}{3} = \frac{1}{3} + \frac{x^3}{3} - x$$

Verified OK.

4.25.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-x^2+1\right)\mathrm{d}x&=\left(t^2\right)\mathrm{d}t\\ \left(-t^2\right)\mathrm{d}t+\left(-x^2+1\right)\mathrm{d}x&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x)&=-t^2\\ N(t,x)&=-x^2+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-t^2) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(-x^2 + 1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -t^2 \, \mathrm{d}t$$

$$\phi = -\frac{t^3}{3} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = -x^2 + 1$. Therefore equation (4) becomes

$$-x^2 + 1 = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -x^2 + 1$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int (-x^2 + 1)\,\mathrm{d}x$$

$$f(x) = -\frac{1}{3}x^3 + x + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{3}t^3 - \frac{1}{3}x^3 + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{3}t^3 - \frac{1}{3}x^3 + x$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{3} = c_1$$

$$c_1 = \frac{1}{3}$$

Substituting c_1 found above in the general solution gives

$$-\frac{1}{3}t^3 - \frac{1}{3}x^3 + x = \frac{1}{3}$$

Summary
The solution(s) found are the following

$$-\frac{t^3}{3} - \frac{x^3}{3} + x = \frac{1}{3} \tag{1}$$

Verification of solutions

$$-\frac{t^3}{3} - \frac{x^3}{3} + x = \frac{1}{3}$$

Verified OK.

4.25.6 Maple step by step solution

Let's solve

$\left[x' - \frac{t^2}{1-x^2} = 0, x(1) = 1\right]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $x'(1-x^2) = t^2$

- Integrate both sides with respect to t

  $\int x'(1-x^2)\, dt = \int t^2 dt + c_1$

- Evaluate integral

  $-\frac{x^3}{3} + x = \frac{t^3}{3} + c_1$

- Solve for x

  $x = \frac{\left(-4t^3-12c_1+4\sqrt{t^6+6t^3c_1+9c_1^2-4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(-4t^3-12c_1+4\sqrt{t^6+6t^3c_1+9c_1^2-4}\right)^{\frac{1}{3}}}$

- Use initial condition $x(1) = 1$

  $1 = \frac{\left(-4-12c_1+4\sqrt{9c_1^2+6c_1-3}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(-4-12c_1+4\sqrt{9c_1^2+6c_1-3}\right)^{\frac{1}{3}}}$

- Solve for c_1

  $c_1 = \frac{1}{3}$

- Substitute $c_1 = \frac{1}{3}$ into general solution and simplify

  $x = \frac{4+\left(-4t^3-4+4\sqrt{t^6+2t^3-3}\right)^{\frac{2}{3}}}{2\left(-4t^3-4+4\sqrt{t^6+2t^3-3}\right)^{\frac{1}{3}}}$

- Solution to the IVP

  $x = \frac{4+\left(-4t^3-4+4\sqrt{t^6+2t^3-3}\right)^{\frac{2}{3}}}{2\left(-4t^3-4+4\sqrt{t^6+2t^3-3}\right)^{\frac{1}{3}}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 122

```
dsolve([diff(x(t),t)=t^2/(1-x(t)^2),x(1) = 1],x(t), singsol=all)
```

$$x(t) = \frac{\left(-4-4t^3+4\sqrt{t^6+2t^3-3}\right)^{\frac{2}{3}}+4}{2\left(-4-4t^3+4\sqrt{t^6+2t^3-3}\right)^{\frac{1}{3}}}$$

$$x(t) = -\frac{\left(1+i\sqrt{3}\right)\left(-4-4t^3+4\sqrt{t^6+2t^3-3}\right)^{\frac{2}{3}}-4i\sqrt{3}+4}{4\left(-4-4t^3+4\sqrt{t^6+2t^3-3}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 4.187 (sec). Leaf size: 188

```
DSolve[{x'[t]==t^2/(1-x[t]^2),{x[1]==1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{\sqrt[3]{-t^3+\sqrt{t^6+2t^3-3}-1}}{\sqrt[3]{2}}+\frac{\sqrt[3]{2}}{\sqrt[3]{-t^3+\sqrt{t^6+2t^3-3}-1}}$$

$$x(t) \to \frac{-i\sqrt[3]{2}\sqrt{3}\left(-t^3+\sqrt{t^6+2t^3-3}-1\right)^{2/3}-\sqrt[3]{2}\left(-t^3+\sqrt{t^6+2t^3-3}-1\right)^{2/3}+2i\sqrt{3}-2}{2\,2^{2/3}\sqrt[3]{-t^3+\sqrt{t^6+2t^3-3}-1}}$$

4.26 problem 15

4.26.1 Solving as separable ode 412
4.26.2 Solving as first order ode lie symmetry lookup ode 414
4.26.3 Solving as exact ode 418
4.26.4 Maple step by step solution 422

Internal problem ID [11393]
Internal file name [`OUTPUT/10375_Wednesday_May_17_2023_08_10_09_PM_13701037/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' - 6t(x-1)^{\frac{2}{3}} = 0}$$

4.26.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= f(t)g(x) \\ &= 6t(x-1)^{\frac{2}{3}} \end{aligned}$$

Where $f(t) = 6t$ and $g(x) = (x-1)^{\frac{2}{3}}$. Integrating both sides gives

$$\frac{1}{(x-1)^{\frac{2}{3}}}\,dx = 6t\,dt$$

$$
\int \frac{1}{(x-1)^{\frac{2}{3}}}dx = \int 6t\,dt
$$

$$
3(x-1)^{\frac{1}{3}} = 3t^2 + c_1
$$

The solution is

$$
3(x-1)^{\frac{1}{3}} - 3t^2 - c_1 = 0
$$

Summary

The solution(s) found are the following

$$
3(x-1)^{\frac{1}{3}} - 3t^2 - c_1 = 0 \tag{1}
$$

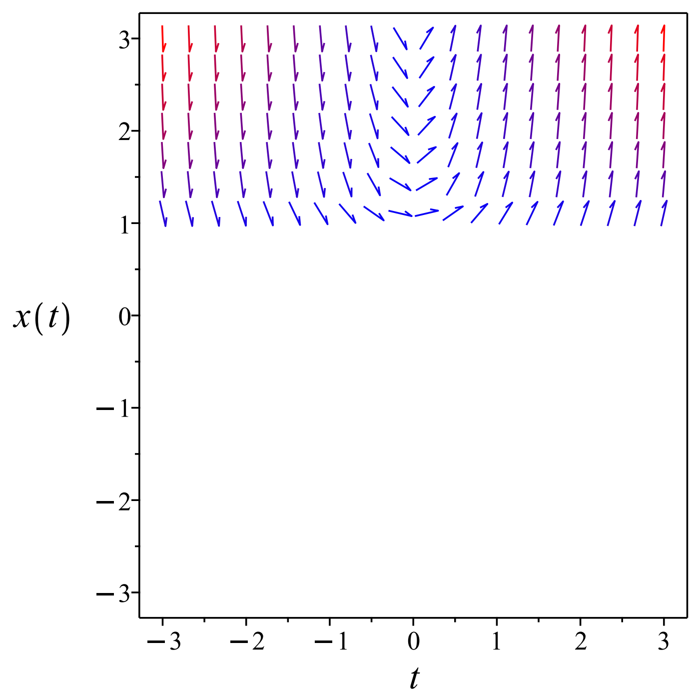


Figure 101: Slope field plot

Verification of solutions

$$
3(x-1)^{\frac{1}{3}} - 3t^2 - c_1 = 0
$$

Verified OK.

4.26.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= 6t(x-1)^{\frac{2}{3}} \\
x' &= \omega(t,x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 83: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t,x) &= \frac{1}{6t} \\ \eta(t,x) &= 0 \end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{6t}} dt\end{aligned}$$

Which results in

$$S = 3t^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = 6t(x-1)^{\frac{2}{3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_t &= 0 \\ R_x &= 1 \\ S_t &= 6t \\ S_x &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(x-1)^{\frac{2}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(R-1)^{\frac{2}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 3(R-1)^{\frac{1}{3}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$3t^2 = 3(x-1)^{\frac{1}{3}} + c_1$$

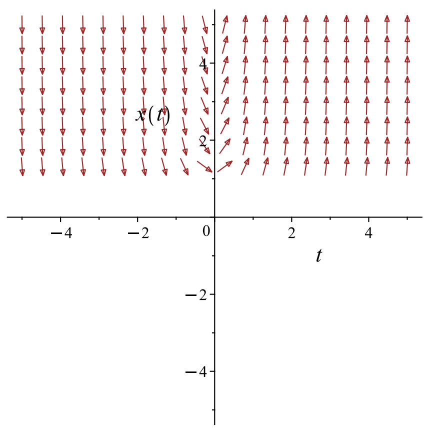
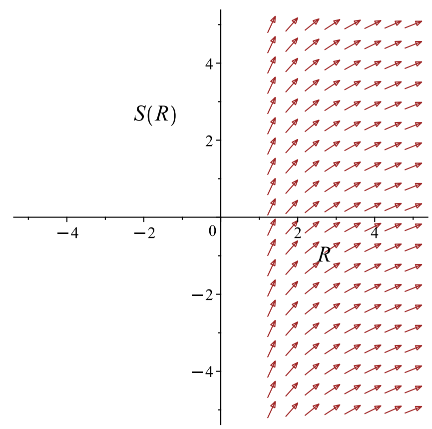
Which simplifies to

$$3t^2 = 3(x-1)^{\frac{1}{3}} + c_1$$

Which gives

$$x = t^6 - t^4 c_1 + \frac{1}{3} t^2 c_1^2 - \frac{1}{27} c_1^3 + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = 6t(x-1)^{\frac{2}{3}}$ 	$R = x$ $S = 3t^2$	$\frac{dS}{dR} = \frac{1}{(R-1)^{\frac{2}{3}}}$ 

Summary

The solution(s) found are the following

$$x = t^6 - t^4 c_1 + \frac{1}{3} t^2 c_1^2 - \frac{1}{27} c_1^3 + 1 \tag{1}$$

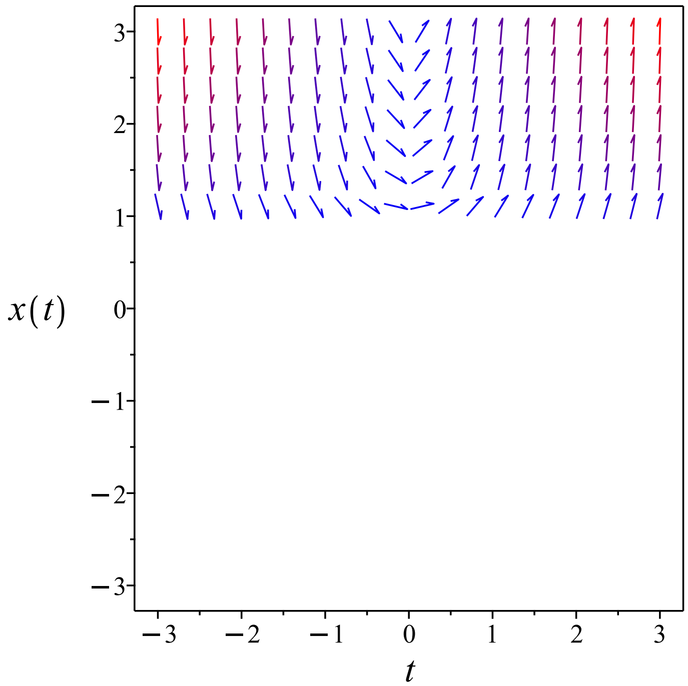


Figure 102: Slope field plot

<u>Verification of solutions</u>

$$x = t^6 - t^4 c_1 + \frac{1}{3} t^2 c_1^2 - \frac{1}{27} c_1^3 + 1$$

Verified OK.

4.26.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\, \mathrm{d}t + N(t, x)\, \mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{6\,(x-1)^{\frac{2}{3}}}\right) \mathrm{d}x &= (t)\, \mathrm{d}t\\ (-t)\, \mathrm{d}t + \left(\frac{1}{6\,(x-1)^{\frac{2}{3}}}\right) \mathrm{d}x &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, x) &= -t\\ N(t, x) &= \frac{1}{6\,(x-1)^{\frac{2}{3}}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-t)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{6\left(x-1\right)^{\frac{2}{3}}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x}=\frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,x)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} &= M \qquad (1) \\
\frac{\partial \phi}{\partial x} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -t\,\mathrm{d}t \\
\phi &= -\frac{t^2}{2}+f(x) \qquad (3)
\end{aligned}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0+f'(x) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial x}=\frac{1}{6(x-1)^{\frac{2}{3}}}$. Therefore equation (4) becomes

$$
\frac{1}{6\left(x-1\right)^{\frac{2}{3}}} = 0+f'(x) \qquad (5)
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = \frac{1}{6\left(x-1\right)^{\frac{2}{3}}}
$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(\frac{1}{6\,(x-1)^{\frac{2}{3}}}\right) \mathrm{d}x$$

$$f(x) = \frac{(x-1)^{\frac{1}{3}}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} + \frac{(x-1)^{\frac{1}{3}}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} + \frac{(x-1)^{\frac{1}{3}}}{2}$$

The solution becomes

$$x = t^6 + 6t^4 c_1 + 12t^2 c_1^2 + 8c_1^3 + 1$$

<u>Summary</u>

The solution(s) found are the following

$$x = t^6 + 6t^4 c_1 + 12t^2 c_1^2 + 8c_1^3 + 1 \tag{1}$$

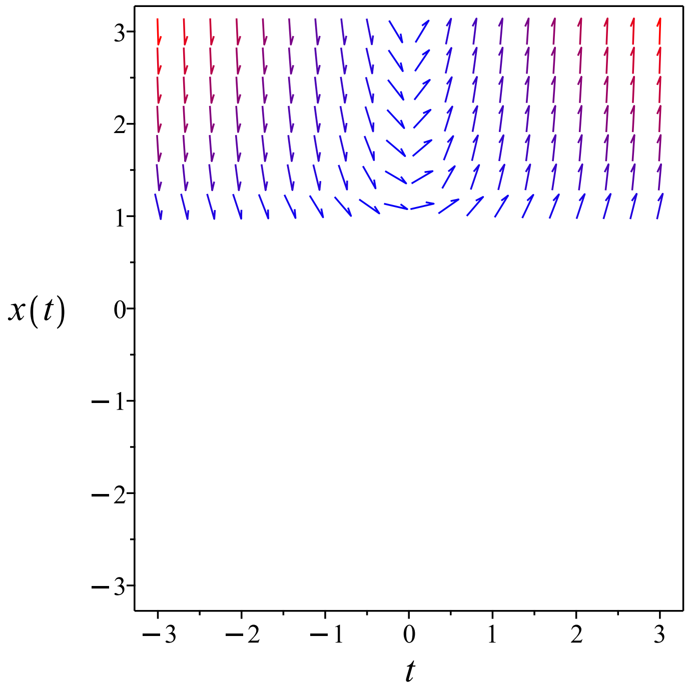


Figure 103: Slope field plot

Verification of solutions

$$x = t^6 + 6t^4c_1 + 12t^2c_1^2 + 8c_1^3 + 1$$

Verified OK.

4.26.4 Maple step by step solution

Let's solve

$x' - 6t(x-1)^{\frac{2}{3}} = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{(x-1)^{\frac{2}{3}}} = 6t$

- Integrate both sides with respect to t

  $\int \frac{x'}{(x-1)^{\frac{2}{3}}}dt = \int 6tdt + c_1$

- Evaluate integral

$$3(x-1)^{\frac{1}{3}} = 3t^2 + c_1$$

- Solve for x

$$x = t^6 + t^4 c_1 + \frac{1}{3} t^2 c_1^2 + \frac{1}{27} c_1^3 + 1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(x(t),t)=6*t*(x(t)-1)^(2/3),x(t), singsol=all)
```

$$c_1 + t^2 - (x(t)-1)^{\frac{1}{3}} = 0$$

✓ Solution by Mathematica

Time used: 0.298 (sec). Leaf size: 40

```
DSolve[x'[t]==6*t*(x[t]-1)^(2/3),x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to t^6 + c_1 t^4 + \frac{c_1{}^2 t^2}{3} + 1 + \frac{c_1{}^3}{27}$$
$$x(t) \to 1$$

4.27 problem 21

4.27.1 Solving as homogeneousTypeD2 ode 424
4.27.2 Solving as first order ode lie symmetry lookup ode 426
4.27.3 Solving as bernoulli ode 430
4.27.4 Solving as exact ode 433

Internal problem ID [11394]
Internal file name [`OUTPUT/10376_Wednesday_May_17_2023_08_10_11_PM_11615872/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{x' - \frac{4t^2 + 3x^2}{2xt} = 0}$$

4.27.1 Solving as homogeneousTypeD2 ode

Using the change of variables $x = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u'(t)\,t + u(t) - \frac{4t^2 + 3u(t)^2\,t^2}{2u\,(t)\,t^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= \frac{u^2+4}{2tu} \end{aligned}$$

Where $f(t) = \frac{1}{2t}$ and $g(u) = \frac{u^2+4}{u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^2+4}{u}}\,du &= \frac{1}{2t}\,dt \\
\int \frac{1}{\frac{u^2+4}{u}}\,du &= \int \frac{1}{2t}\,dt \\
\frac{\ln(u^2+4)}{2} &= \frac{\ln(t)}{2} + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\sqrt{u^2+4} = \mathrm{e}^{\frac{\ln(t)}{2}+c_2}
$$

Which simplifies to

$$
\sqrt{u^2+4} = c_3\sqrt{t}
$$

Which simplifies to

$$
\sqrt{u(t)^2+4} = c_3\sqrt{t}\,\mathrm{e}^{c_2}
$$

The solution is

$$
\sqrt{u(t)^2+4} = c_3\sqrt{t}\,\mathrm{e}^{c_2}
$$

Replacing $u(t)$ in the above solution by $\frac{x}{t}$ results in the solution for x in implicit form

$$
\begin{aligned}
\sqrt{\frac{x^2}{t^2}+4} &= c_3\sqrt{t}\,\mathrm{e}^{c_2} \\
\sqrt{\frac{x^2+4t^2}{t^2}} &= c_3\sqrt{t}\,\mathrm{e}^{c_2}
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\sqrt{\frac{x^2+4t^2}{t^2}} = c_3\sqrt{t}\,\mathrm{e}^{c_2} \tag{1}
$$

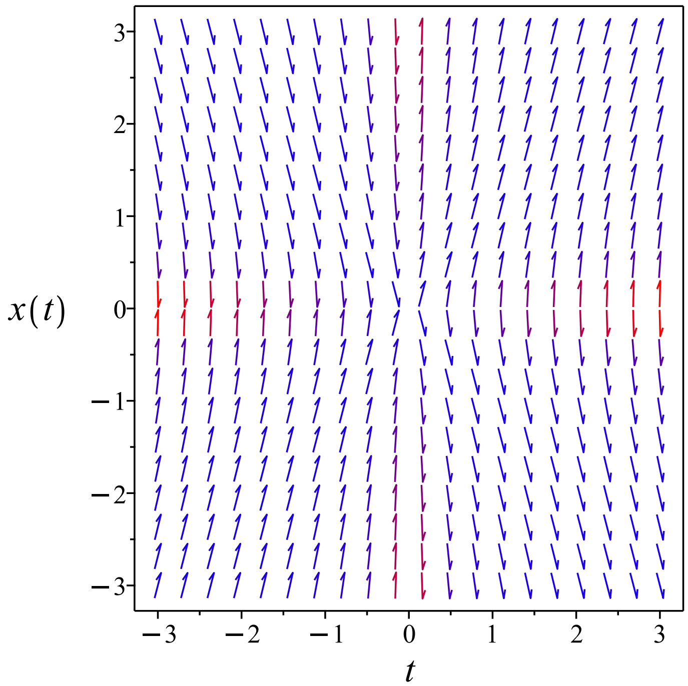


Figure 104: Slope field plot

Verification of solutions

$$\sqrt{\frac{x^2+4t^2}{t^2}} = c_3\sqrt{t}\,\mathrm{e}^{c_2}$$

Verified OK.

4.27.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = \frac{4t^2+3x^2}{2xt}$$
$$x' = \omega(t,x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 86: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \frac{t^3}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{t^3}{x}} dy \end{aligned}$$

Which results in

$$S = \frac{x^2}{2t^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = \frac{4t^2 + 3x^2}{2xt}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= -\frac{3x^2}{2t^4} \\ S_x &= \frac{x}{t^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2}{t^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{2}{R} + c_1 \tag{4}$$

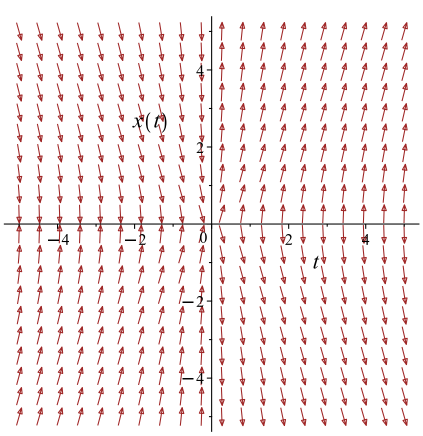
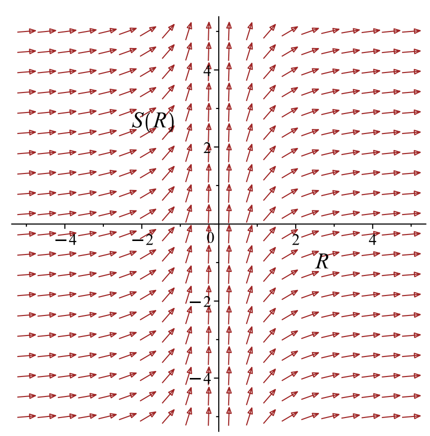
To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\frac{x^2}{2t^3} = -\frac{2}{t} + c_1$$

Which simplifies to

$$\frac{x^2}{2t^3} = -\frac{2}{t} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \frac{4t^2+3x^2}{2xt}$  	$R = t$ $S = \frac{x^2}{2t^3}$	$\frac{dS}{dR} = \frac{2}{R^2}$ 

Summary

The solution(s) found are the following

$$\frac{x^2}{2t^3} = -\frac{2}{t} + c_1 \tag{1}$$

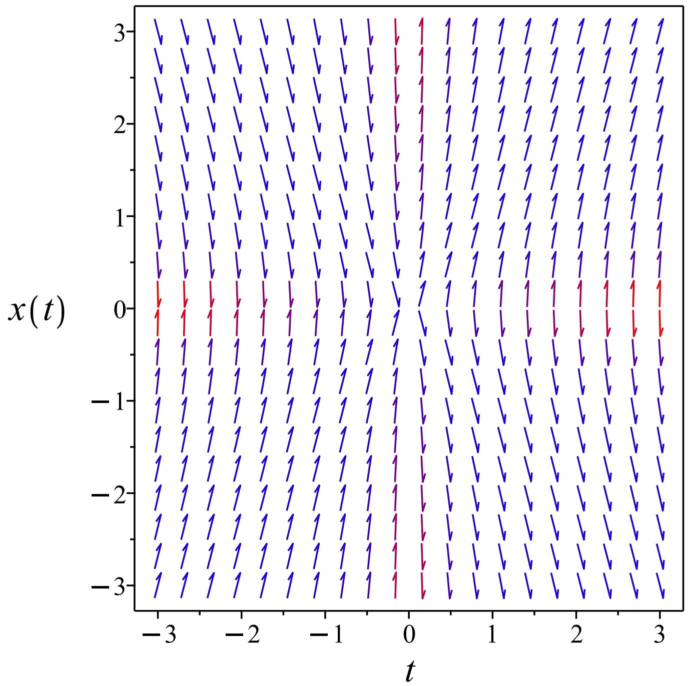


Figure 105: Slope field plot

Verification of solutions

$$\frac{x^2}{2t^3} = -\frac{2}{t} + c_1$$

Verified OK.

4.27.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= \frac{4t^2 + 3x^2}{2xt} \end{aligned}$$

This is a Bernoulli ODE.

$$x' = \frac{3}{2t}x + 2t\frac{1}{x} \tag{1}$$

The standard Bernoulli ODE has the form

$$x' = f_0(t)x + f_1(t)x^n \tag{2}$$

The first step is to divide the above equation by x^n which gives

$$\frac{x'}{x^n} = f_0(t)x^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $w = x^{1-n}$ in equation (3) which generates a new ODE in $w(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $x(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(t) &= \frac{3}{2t} \\ f_1(t) &= 2t \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $x^n = \frac{1}{x}$ gives

$$x'x = \frac{3x^2}{2t} + 2t \tag{4}$$

Let

$$\begin{aligned} w &= x^{1-n} \\ &= x^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t t gives

$$w' = 2xx' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(t)}{2} &= \frac{3w(t)}{2t} + 2t \\ w' &= \frac{3w}{t} + 4t \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(t) + p(t)w(t) = q(t)$$

Where here

$$\begin{aligned} p(t) &= -\frac{3}{t} \\ q(t) &= 4t \end{aligned}$$

Hence the ode is

$$w'(t) - \frac{3w(t)}{t} = 4t$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{3}{t}dt} \\
&= \frac{1}{t^3}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu w) &= (\mu)(4t) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{w}{t^3}\right) &= \left(\frac{1}{t^3}\right)(4t) \\
\mathrm{d}\left(\frac{w}{t^3}\right) &= \left(\frac{4}{t^2}\right)\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{t^3} &= \int \frac{4}{t^2}\,\mathrm{d}t \\
\frac{w}{t^3} &= -\frac{4}{t} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^3}$ results in

$$
w(t) = c_1 t^3 - 4t^2
$$

Replacing w in the above by x^2 using equation (5) gives the final solution.

$$
x^2 = c_1 t^3 - 4t^2
$$

Solving for x gives

$$
\begin{aligned}
x(t) &= \sqrt{c_1 t - 4}\,t \\
x(t) &= -\sqrt{c_1 t - 4}\,t
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = \sqrt{c_1 t - 4}\,t \tag{1}
$$

$$
x = -\sqrt{c_1 t - 4}\,t \tag{2}
$$

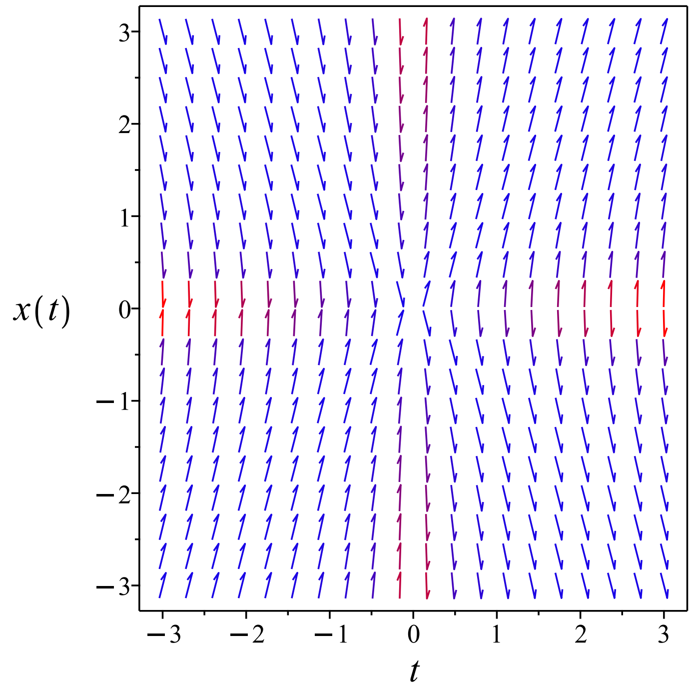


Figure 106: Slope field plot

Verification of solutions

$$x = \sqrt{c_1 t - 4}\, t$$

Verified OK.

$$x = -\sqrt{c_1 t - 4}\, t$$

Verified OK.

4.27.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2tx)\,\mathrm{d}x&=\left(4t^2+3x^2\right)\mathrm{d}t\\ \left(-4t^2-3x^2\right)\mathrm{d}t+(2tx)\,\mathrm{d}x&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x)&=-4t^2-3x^2\\ N(t,x)&=2tx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x}&=\frac{\partial}{\partial x}\left(-4t^2-3x^2\right)\\ &=-6x\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(2tx) \\
&= 2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= \frac{1}{2xt}((-6x) - (2x)) \\
&= -\frac{4}{t}
\end{aligned}
$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -\frac{4}{t} \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-4\ln(t)} \\
&= \frac{1}{t^4}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{t^4}\left(-4t^2 - 3x^2\right) \\
&= \frac{-4t^2 - 3x^2}{t^4}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{t^4}(2tx) \\
&= \frac{2x}{t^3}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\
\left(\frac{-4t^2 - 3x^2}{t^4}\right) + \left(\frac{2x}{t^3}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \mathrm{d}t &= \int \overline{M} \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \mathrm{d}t &= \int \frac{-4t^2 - 3x^2}{t^4} \mathrm{d}t
\end{aligned}
$$

$$
\phi = \frac{x^2}{t^3} + \frac{4}{t} + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = \frac{2x}{t^3} + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{2x}{t^3}$. Therefore equation (4) becomes

$$
\frac{2x}{t^3} = \frac{2x}{t^3} + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = 0
$$

Therefore

$$
f(x) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{x^2}{t^3} + \frac{4}{t} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2}{t^3} + \frac{4}{t}$$

Summary
The solution(s) found are the following

$$\frac{x^2}{t^3} + \frac{4}{t} = c_1 \tag{1}$$

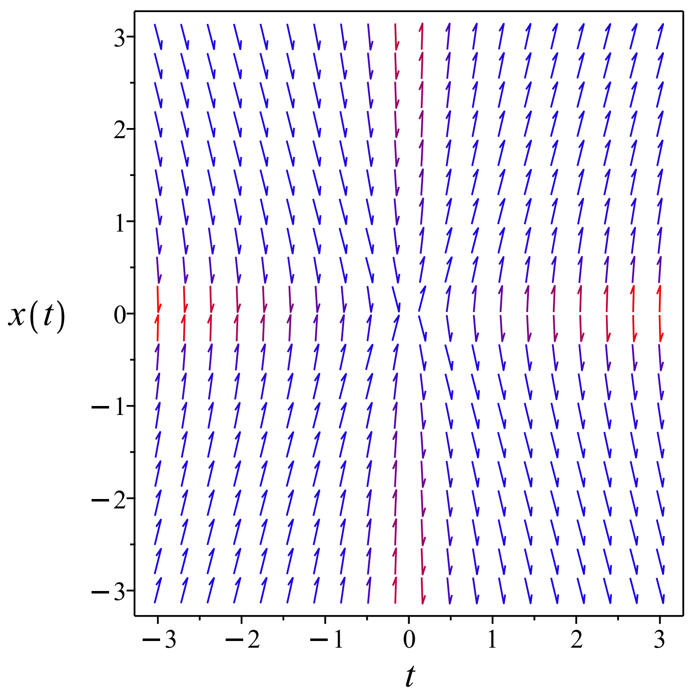


Figure 107: Slope field plot

Verification of solutions

$$\frac{x^2}{t^3} + \frac{4}{t} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(x(t),t)=(4*t^2+3*x(t)^2)/(2*t*x(t)),x(t), singsol=all)
```

$$x(t) = \sqrt{c_1 t - 4}\, t$$
$$x(t) = -\sqrt{c_1 t - 4}\, t$$

✓ Solution by Mathematica

Time used: 0.434 (sec). Leaf size: 34

```
DSolve[x'[t]==(4*t^2+3*x[t]^2)/(2*t*x[t]),x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -t\sqrt{-4 + c_1 t}$$
$$x(t) \to t\sqrt{-4 + c_1 t}$$

4.28 problem 23

4.28.1 Existence and uniqueness analysis 439
4.28.2 Solving as linear ode 440
4.28.3 Solving as first order ode lie symmetry lookup ode 442
4.28.4 Solving as exact ode 446
4.28.5 Maple step by step solution 449

Internal problem ID [11395]
Internal file name [`OUTPUT/10377_Wednesday_May_17_2023_08_10_12_PM_86939990/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{x' e^{2t} + 2x\, e^{2t} = e^{-t}}$$

With initial conditions

$$[x(0) = 3]$$

4.28.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = 2$$
$$q(t) = e^{-3t}$$

Hence the ode is

$$x' + 2x = e^{-3t}$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = e^{-3t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.28.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 2dt}\\ &= e^{2t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu x) &= (\mu)\left(e^{-3t}\right)\\ \frac{d}{dt}\left(e^{2t}x\right) &= \left(e^{2t}\right)\left(e^{-3t}\right)\\ d\left(e^{2t}x\right) &= e^{-t}\,dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{2t}x &= \int e^{-t}\,dt\\ e^{2t}x &= -e^{-t} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2t}$ results in

$$x = -e^{-2t}e^{-t} + c_1 e^{-2t}$$

which simplifies to

$$x = -e^{-3t} + c_1 e^{-2t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = -1 + c_1$$

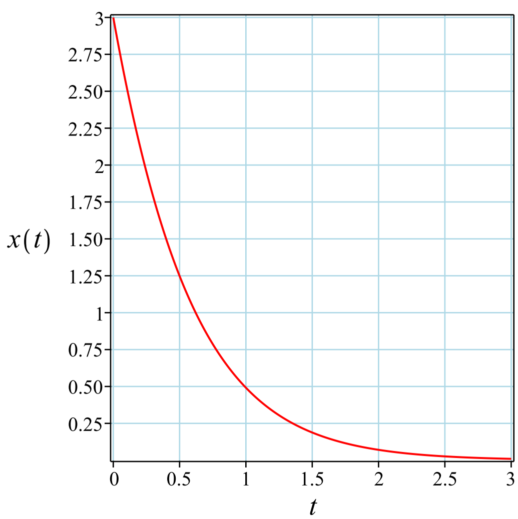
$$c_1 = 4$$

Substituting c_1 found above in the general solution gives

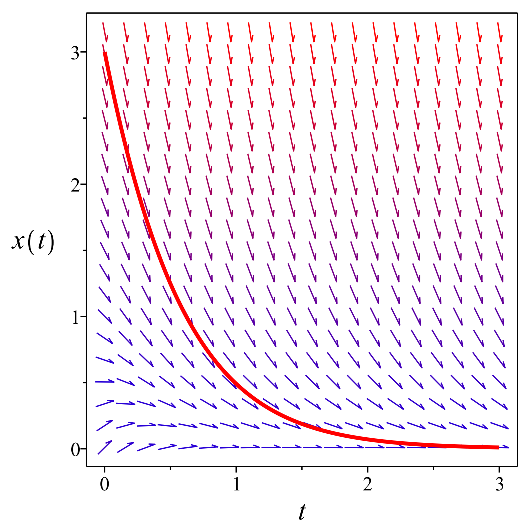
$$x = 4\,\mathrm{e}^{-2t} - \mathrm{e}^{-3t}$$

Summary
The solution(s) found are the following

$$x = 4\,\mathrm{e}^{-2t} - \mathrm{e}^{-3t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 4\,\mathrm{e}^{-2t} - \mathrm{e}^{-3t}$$

Verified OK.

4.28.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= -\left(2\,\mathrm{e}^{2t}x - \mathrm{e}^{-t}\right)\mathrm{e}^{-2t} \\
x' &= \omega(t,x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 88: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= 0 \\
\eta(t, x) &= \mathrm{e}^{-2t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{-2t}} dy
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{2t} x
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}
$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$
\omega(t, x) = -\left(2\,\mathrm{e}^{2t} x - \mathrm{e}^{-t}\right) \mathrm{e}^{-2t}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_x &= 0 \\
S_t &= 2\,\mathrm{e}^{2t} x \\
S_x &= \mathrm{e}^{2t}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{-t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$x\,\mathrm{e}^{2t} = -\mathrm{e}^{-t} + c_1$$

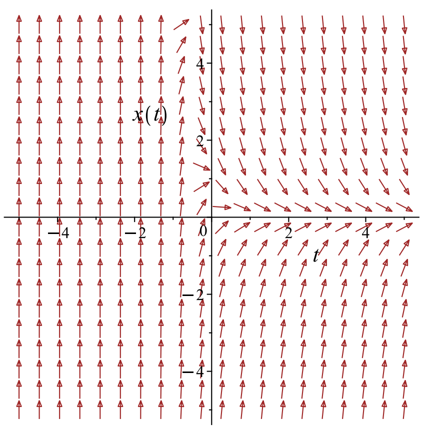
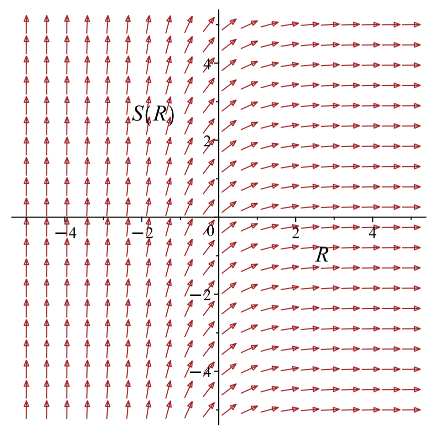
Which simplifies to

$$x\,\mathrm{e}^{2t} = -\mathrm{e}^{-t} + c_1$$

Which gives

$$x = -\left(\mathrm{e}^{-t} - c_1\right) \mathrm{e}^{-2t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\left(2\,\mathrm{e}^{2t} x - \mathrm{e}^{-t}\right) \mathrm{e}^{-2t}$ 	$R = t$ $S = \mathrm{e}^{2t} x$	$\frac{dS}{dR} = \mathrm{e}^{-R}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = -1 + c_1$$

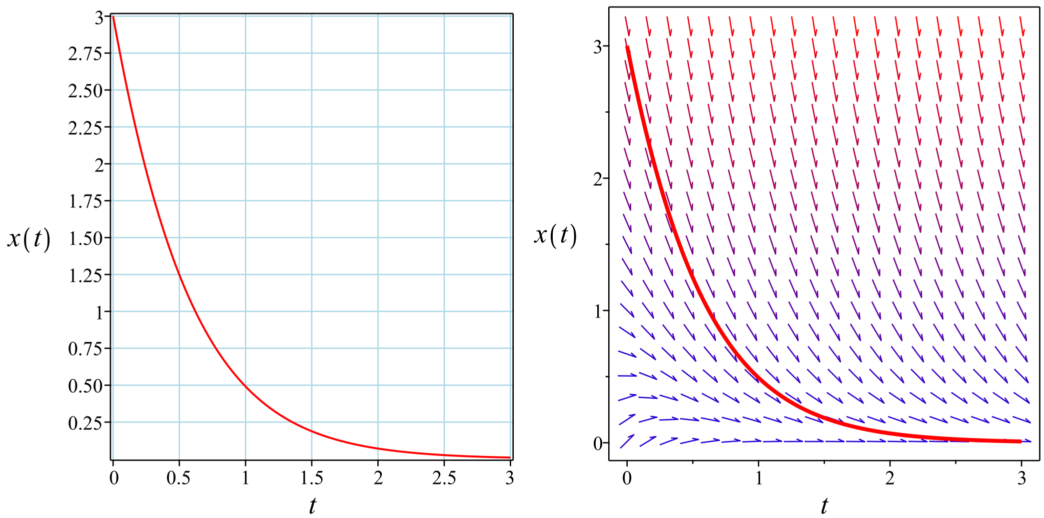
$$c_1 = 4$$

Substituting c_1 found above in the general solution gives

$$x = 4\,\mathrm{e}^{-2t} - \mathrm{e}^{-3t}$$

<u>Summary</u>
The solution(s) found are the following

$$x = 4\,\mathrm{e}^{-2t} - \mathrm{e}^{-3t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$x = 4\,\mathrm{e}^{-2t} - \mathrm{e}^{-3t}$$

Verified OK.

4.28.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\mathrm{e}^{2t}\right)\mathrm{d}x &= \left(-2\,\mathrm{e}^{2t}x + \mathrm{e}^{-t}\right)\mathrm{d}t \\
\left(2\,\mathrm{e}^{2t}x - \mathrm{e}^{-t}\right)\mathrm{d}t + \left(\mathrm{e}^{2t}\right)\mathrm{d}x &= 0 && \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(t, x) &= 2\,\mathrm{e}^{2t}x - \mathrm{e}^{-t} \\
N(t, x) &= \mathrm{e}^{2t}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(2\,\mathrm{e}^{2t}x - \mathrm{e}^{-t}\right) \\
&= 2\,\mathrm{e}^{2t}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\mathrm{e}^{2t}\right) \\
&= 2\,\mathrm{e}^{2t}
\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int M\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int 2\,\mathrm{e}^{2t}x - \mathrm{e}^{-t}\,\mathrm{d}t$$
$$\phi = \mathrm{e}^{2t}x + \mathrm{e}^{-t} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \mathrm{e}^{2t} + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \mathrm{e}^{2t}$. Therefore equation (4) becomes

$$\mathrm{e}^{2t} = \mathrm{e}^{2t} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{2t}x + \mathrm{e}^{-t} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{2t}x + \mathrm{e}^{-t}$$

The solution becomes

$$x = -\left(\mathrm{e}^{-t} - c_1\right)\mathrm{e}^{-2t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = -1 + c_1$$

$$c_1 = 4$$

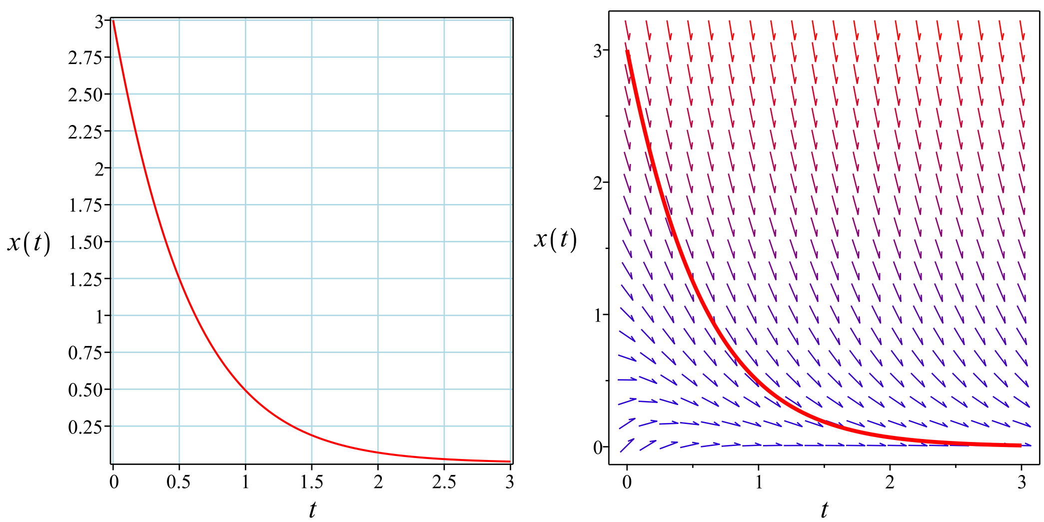
Substituting c_1 found above in the general solution gives

$$x = 4\,\mathrm{e}^{-2t} - \mathrm{e}^{-3t}$$

Summary

The solution(s) found are the following

$$x = 4\,\mathrm{e}^{-2t} - \mathrm{e}^{-3t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = 4\,\mathrm{e}^{-2t} - \mathrm{e}^{-3t}$$

Verified OK.

4.28.5 Maple step by step solution

Let's solve

$[x'\mathrm{e}^{2t} + 2x\,\mathrm{e}^{2t} = \mathrm{e}^{-t}, x(0) = 3]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

$x' = -2x + \frac{e^{-t}}{e^{2t}}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

$x' + 2x = \frac{e^{-t}}{e^{2t}}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)\left(x' + 2x\right) = \frac{\mu(t)e^{-t}}{e^{2t}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

$\mu(t)\left(x' + 2x\right) = \mu'(t)\,x + \mu(t)\,x'$

- Isolate $\mu'(t)$

$\mu'(t) = 2\mu(t)$

- Solve to find the integrating factor

$\mu(t) = \left(e^{2t}\right)^2 e^{-2t}$

- Integrate both sides with respect to t

$\int \left(\frac{d}{dt}(\mu(t)\,x)\right) dt = \int \frac{\mu(t)e^{-t}}{e^{2t}} dt + c_1$

- Evaluate the integral on the lhs

$\mu(t)\,x = \int \frac{\mu(t)e^{-t}}{e^{2t}} dt + c_1$

- Solve for x

$x = \frac{\int \frac{\mu(t)e^{-t}}{e^{2t}} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \left(e^{2t}\right)^2 e^{-2t}$

$x = \frac{\int e^{2t}e^{-2t}e^{-t} dt + c_1}{\left(e^{2t}\right)^2 e^{-2t}}$

- Evaluate the integrals on the rhs

$x = \frac{-e^{-t} + c_1}{\left(e^{2t}\right)^2 e^{-2t}}$

- Simplify

$x = e^{-2t}\left(-e^{-t} + c_1\right)$

- Use initial condition $x(0) = 3$

$3 = -1 + c_1$

- Solve for c_1

$c_1 = 4$

- Substitute $c_1 = 4$ into general solution and simplify

$$x = -(\mathrm{e}^{-t} - 4)\,\mathrm{e}^{-2t}$$

- Solution to the IVP

$$x = -(\mathrm{e}^{-t} - 4)\,\mathrm{e}^{-2t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(x(t)*exp(2*t),t)=exp(-t),x(0) = 3],x(t), singsol=all)
```

$$x(t) = -\left(\mathrm{e}^{-t} - 4\right)\mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.093 (sec). Leaf size: 18

```
DSolve[{D[x[t]*Exp[2*t],t]==Exp[-t],{x[0]==3}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-3t}\left(4e^t - 1\right)$$

4.29 problem 24

4.29.1 Solving as second order ode missing y ode 452
4.29.2 Solving as second order ode non constant coeff transformation on B ode 454
4.29.3 Solving using Kovacic algorithm 458
4.29.4 Maple step by step solution 465

Internal problem ID [11396]
Internal file name [`OUTPUT/10378_Wednesday_May_17_2023_08_10_14_PM_33607129/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{\frac{x' + t x''}{t} = -2}$$

4.29.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable x. Let

$$p(t) = x'$$

Then

$$p'(t) = x''$$

Hence the ode becomes

$$p'(t) + \frac{p(t)}{t} + 2 = 0$$

Which is now solve for $p(t)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(t) + p(t)p(t) = q(t)$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{t} \\ q(t) &= -2 \end{aligned}$$

Hence the ode is

$$p'(t) + \frac{p(t)}{t} = -2$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{t} dt} \\ &= t \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu p) &= (\mu)(-2) \\ \frac{\mathrm{d}}{\mathrm{d}t}(tp) &= (t)(-2) \\ \mathrm{d}(tp) &= (-2t)\ \mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} tp &= \int -2t \, \mathrm{d}t \\ tp &= -t^2 + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$p(t) = -t + \frac{c_1}{t}$$

Since $p = x'$ then the new first order ode to solve is

$$x' = -t + \frac{c_1}{t}$$

Integrating both sides gives

$$\begin{aligned} x &= \int \frac{-t^2 + c_1}{t} \, \mathrm{d}t \\ &= -\frac{t^2}{2} + c_1 \ln(t) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$x = -\frac{t^2}{2} + c_1 \ln(t) + c_2 \tag{1}$$

Verification of solutions

$$x = -\frac{t^2}{2} + c_1 \ln(t) + c_2$$

Verified OK.

4.29.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ax'' + Bx' + Cx = F(t)$$

This method reduces the order ode the ODE by one by applying the transformation

$$x = Bv$$

This results in

$$\begin{aligned} x' &= B'v + v'B \\ x'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + (2AB' + B^2)\, v' + (AB'' + BB' + CB)\, v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)\, v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)\, u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $x = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The ODE is now normalized to

$$x' + tx'' = -2$$

Where now

$$\begin{aligned} A &= t \\ B &= 1 \\ C &= 0 \\ F &= -2 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (t)\,(0) + (1)\,(0) + (0)\,(1) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$tv'' + (1)\,v' = 0$$

Now by applying $v' = u$ the above becomes

$$tu'(t) + u(t) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= -\frac{u}{t} \end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{1}{t}\,dt \\ \int \frac{1}{u}\,du &= \int -\frac{1}{t}\,dt \\ \ln(u) &= -\ln(t) + c_1 \\ u &= e^{-\ln(t)+c_1} \\ &= \frac{c_1}{t} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{t} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(t) &= \int \frac{c_1}{t}\, \mathrm{d}t \\ &= c_1 \ln(t) + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} x_h(t) &= Bv \\ &= (1)\,(c_1 \ln(t) + c_2) \\ &= c_1 \ln(t) + c_2 \end{aligned}$$

And now the particular solution $x_p(t)$ will be found. The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} x_1 &= 1 \\ x_2 &= \ln(t) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(t) \\ \frac{d}{dt}(1) & \frac{d}{dt}(\ln(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(t) \\ 0 & \frac{1}{t} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{t}\right) - (\ln(t))(0)$$

Which simplifies to

$$W = \frac{1}{t}$$

Which simplifies to

$$W = \frac{1}{t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2\ln(t)}{1}\, dt$$

Which simplifies to

$$u_1 = -\int -2\ln(t)\, dt$$

Hence

$$u_1 = 2t\ln(t) - 2t$$

And Eq. (3) becomes

$$u_2 = \int \frac{-2}{1}\, dt$$

Which simplifies to

$$u_2 = \int (-2)\, dt$$

Hence

$$u_2 = -2t$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = -2t$$

Hence the complete solution is

$$\begin{aligned}
x(t) &= x_h + x_p \\
&= (c_1 \ln(t) + c_2) + (-2t) \\
&= c_1 \ln(t) + c_2 - 2t
\end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 \ln(t) + c_2 - 2t \tag{1}$$

Verification of solutions

$$x = c_1 \ln(t) + c_2 - 2t$$

Verified OK.

4.29.3 Solving using Kovacic algorithm

Writing the ode as

$$x' + tx'' = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= t \\
B &= 1 \\
C &= 0
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 91: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4t^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2t} + (-)\,(0) \\
&= \frac{1}{2t} \\
&= \frac{1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2t}\right)^2 - \left(-\frac{1}{4t^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= pe^{\int \omega \, dt} \\ &= e^{\int \frac{1}{2t} dt} \\ &= \sqrt{t} \end{aligned}$$

The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{t} \, dt} \\ &= z_1 e^{-\frac{\ln(t)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{t}}\right) \end{aligned}$$

Which simplifies to

$$x_1 = 1$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{x_1^2} \, dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{1}{t} \, dt}}{(x_1)^2} \, dt \\ &= x_1 \int \frac{e^{-\ln(t)}}{(x_1)^2} \, dt \\ &= x_1(\ln(t)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1x_1 + c_2x_2 \\
&= c_1(1) + c_2(1(\ln(t)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x' + tx'' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 + c_2 \ln(t)
$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
x_p(t) = u_1x_1 + u_2x_2 \tag{1}
$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
x_1 &= 1 \\
x_2 &= \ln(t)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}
$$

$$
u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}
$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(t) \\ \frac{d}{dt}(1) & \frac{d}{dt}(\ln(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(t) \\ 0 & \frac{1}{t} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{t}\right) - (\ln(t))(0)$$

Which simplifies to

$$W = \frac{1}{t}$$

Which simplifies to

$$W = \frac{1}{t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2\ln(t)}{1}\, dt$$

Which simplifies to

$$u_1 = -\int -2\ln(t)\, dt$$

Hence

$$u_1 = 2t\ln(t) - 2t$$

And Eq. (3) becomes

$$u_2 = \int \frac{-2}{1}\, dt$$

Which simplifies to

$$u_2 = \int (-2)\, dt$$

Hence

$$u_2 = -2t$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = -2t$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1 + c_2 \ln(t)) + (-2t) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$x = c_1 + c_2 \ln(t) - 2t \tag{1}$$

<u>Verification of solutions</u>

$$x = c_1 + c_2 \ln(t) - 2t$$

Verified OK.

4.29.4 Maple step by step solution

Let's solve

$x' + tx'' = -2$

- Highest derivative means the order of the ODE is 2

  x''

- Make substitution $u = x'$ to reduce order of ODE

  $u(t) + tu'(t) = -2$

- Integrate both sides with respect to t

  $\int (u(t) + tu'(t))\, dt = \int (-2)\, dt + c_1$

- Evaluate integral

$$tu(t) = -2t + c_1$$

- Solve for $u(t)$

$$u(t) = \frac{-2t+c_1}{t}$$

- Solve 1st ODE for $u(t)$

$$u(t) = \frac{-2t+c_1}{t}$$

- Make substitution $u = x'$

$$x' = \frac{-2t+c_1}{t}$$

- Integrate both sides to solve for x

$$\int x' dt = \int \frac{-2t+c_1}{t} dt + c_2$$

- Compute integrals

$$x = c_1 \ln(t) + c_2 - 2t$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(_b(_a)+2*_a)/_a, _b(_a)`   *** Sublev
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(1/t*diff(t*diff(x(t),t),t)=-2,x(t), singsol=all)
```

$$x(t) = -\frac{t^2}{2} + c_1 \ln(t) + c_2$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 20

```
DSolve[1/t*D[t*x'[t],t]==-2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{t^2}{2} + c_1 \log(t) + c_2$$

4.30 problem 26

4.30.1 Solving as homogeneousTypeD2 ode 468
4.30.2 Solving as first order ode lie symmetry lookup ode 470
4.30.3 Solving as bernoulli ode 474
4.30.4 Solving as exact ode 478
4.30.5 Solving as riccati ode 482

Internal problem ID [11397]
Internal file name [`OUTPUT/10379_Wednesday_May_17_2023_08_10_16_PM_92028239/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y' - \frac{y^2 + 2ty}{t^2} = 0}$$

4.30.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\, t$ on the above ode results in new ode in $u(t)$

$$u'(t)\, t + u(t) - \frac{u(t)^2\, t^2 + 2t^2 u(t)}{t^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t) g(u) \\ &= \frac{u(u+1)}{t} \end{aligned}$$

Where $f(t) = \frac{1}{t}$ and $g(u) = u(u+1)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u\left(u+1\right)}\,du &= \frac{1}{t}\,dt \\
\int \frac{1}{u\left(u+1\right)}\,du &= \int \frac{1}{t}\,dt \\
-\ln\left(u+1\right)+\ln\left(u\right) &= \ln\left(t\right)+c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{-\ln(u+1)+\ln(u)} = \mathrm{e}^{\ln(t)+c_2}
$$

Which simplifies to

$$
\frac{u}{u+1} = c_3 t
$$

Therefore the solution y is

$$
\begin{aligned}
y &= tu \\
&= -\frac{t^2 c_3}{c_3 t - 1}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{t^2 c_3}{c_3 t - 1} \tag{1}
$$

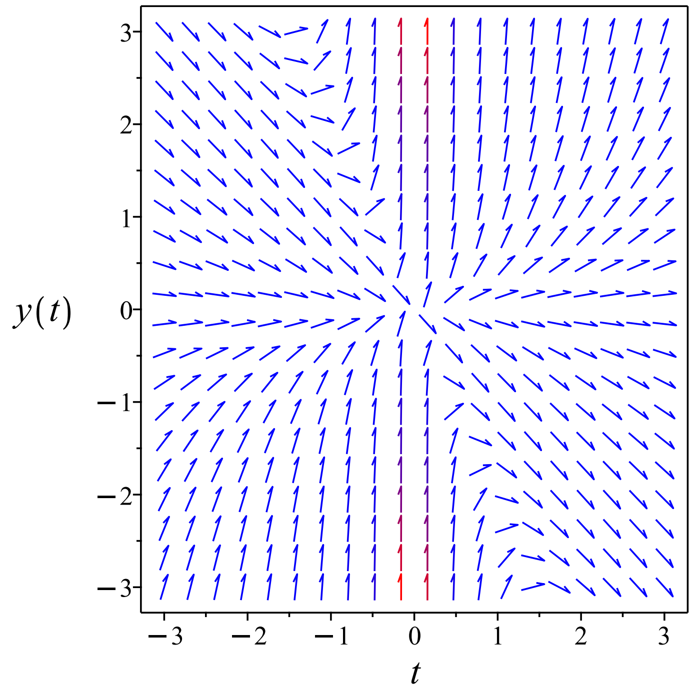


Figure 111: Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{t^2 c_3}{c_3 t - 1}$$

Verified OK.

4.30.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(2t + y)}{t^2}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 93: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \frac{y^2}{t^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{t^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{t^2}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{y(2t + y)}{t^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{2t}{y} \\ S_y &= \frac{t^2}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$-\frac{t^2}{y} = t + c_1$$

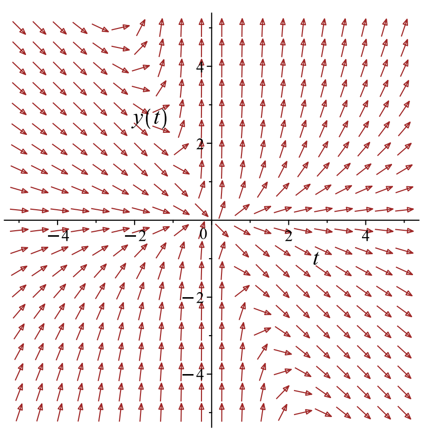
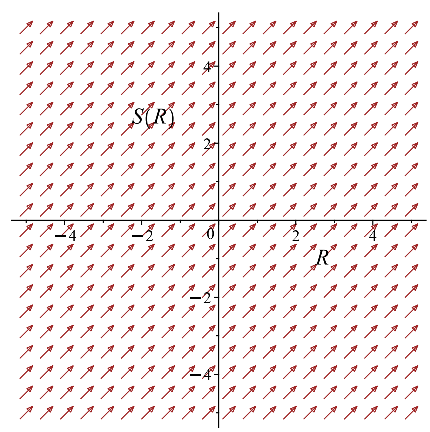
Which simplifies to

$$-\frac{t^2}{y} = t + c_1$$

Which gives

$$y = -\frac{t^2}{t + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{y(2t+y)}{t^2}$ 	$R = t$ $S = -\frac{t^2}{y}$	$\frac{dS}{dR} = 1$ 

Summary
The solution(s) found are the following

$$y = -\frac{t^2}{t + c_1} \tag{1}$$

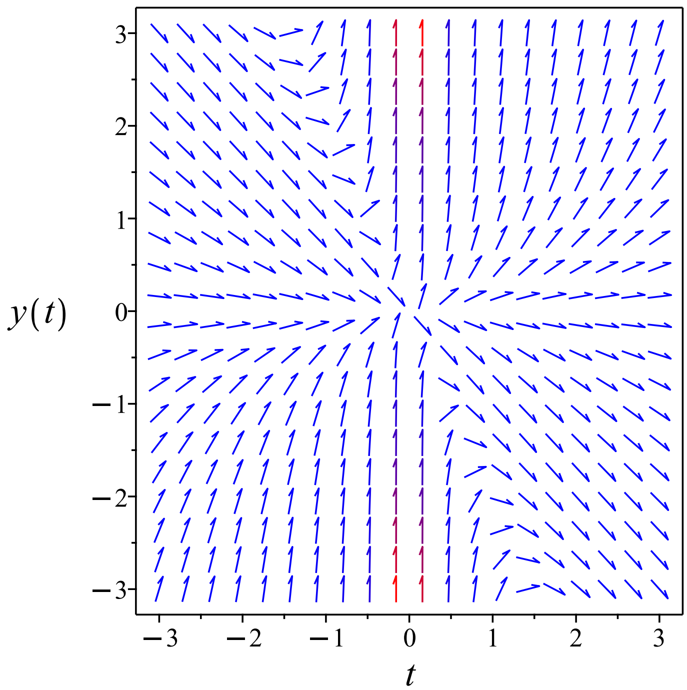


Figure 112: Slope field plot

Verification of solutions

$$y = -\frac{t^2}{t + c_1}$$

Verified OK.

4.30.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= \frac{y(2t + y)}{t^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{2}{t}y + \frac{1}{t^2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(t)y + f_1(t)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(t)y^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(t) &= \frac{2}{t} \\ f_1(t) &= \frac{1}{t^2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{2}{ty} + \frac{1}{t^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t t gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(t) &= \frac{2w(t)}{t} + \frac{1}{t^2} \\ w' &= -\frac{2w}{t} - \frac{1}{t^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(t) + p(t)w(t) = q(t)$$

Where here

$$\begin{aligned} p(t) &= \frac{2}{t} \\ q(t) &= -\frac{1}{t^2} \end{aligned}$$

Hence the ode is

$$w'(t) + \frac{2w(t)}{t} = -\frac{1}{t^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{2}{t} dt} \\ &= t^2 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dt}(\mu w) &= (\mu)\left(-\frac{1}{t^2}\right) \\ \frac{d}{dt}(t^2 w) &= (t^2)\left(-\frac{1}{t^2}\right) \\ d(t^2 w) &= -1\, dt \end{aligned}$$

Integrating gives

$$\begin{aligned} t^2 w &= \int -1\, dt \\ t^2 w &= -t + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$w(t) = -\frac{1}{t} + \frac{c_1}{t^2}$$

which simplifies to

$$w(t) = \frac{-t + c_1}{t^2}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{-t + c_1}{t^2}$$

Or

$$y = \frac{t^2}{-t + c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{t^2}{-t + c_1} \tag{1}$$

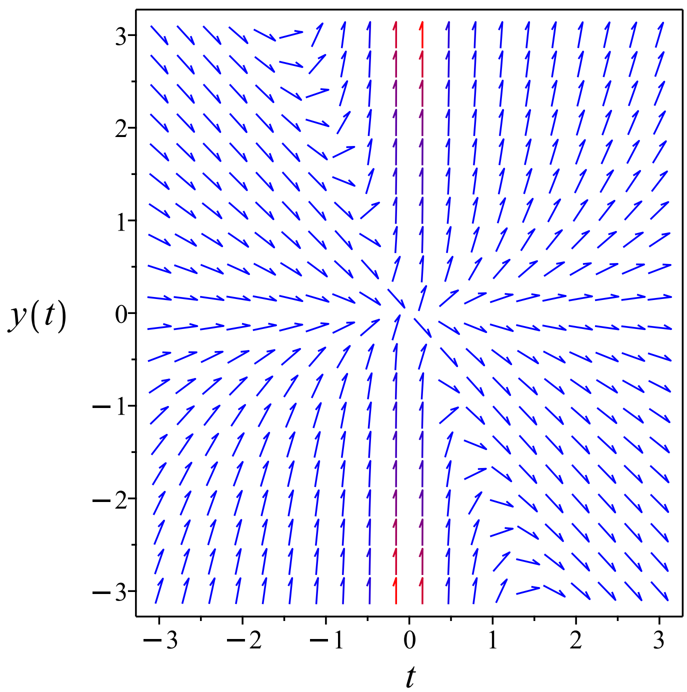


Figure 113: Slope field plot

Verification of solutions

$$y = \frac{t^2}{-t + c_1}$$

Verified OK.

4.30.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(t^2\right)\mathrm{d}y&=(y(2t+y))\,\mathrm{d}t\\ (-y(2t+y))\,\mathrm{d}t+\left(t^2\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-y(2t+y)\\ N(t,y)&=t^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y(2t+y)) \\ &= -2t - 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t^2) \\ &= 2t\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= \frac{1}{t^2}((-2t-2y) - (2t)) \\ &= \frac{-4t-2y}{t^2}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial t} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{y(2t+y)}((2t) - (-2t-2y)) \\ &= -\frac{2}{y}\end{aligned}$$

Since B does not depend on t, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \,\mathrm{d}y} \\ &= e^{\int -\frac{2}{y} \,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(y)}\\ &= \frac{1}{y^2}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{y^2}(-y(2t+y))\\ &= \frac{-2t-y}{y}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{y^2}(t^2)\\ &= \frac{t^2}{y^2}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\\ \left(\frac{-2t-y}{y}\right) + \left(\frac{t^2}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \frac{-2t-y}{y}\,\mathrm{d}t\end{aligned}$$

$$\phi = -\frac{(t+y)\,t}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= -\frac{t}{y} + \frac{(t+y)\,t}{y^2} + f'(y) \qquad (4)\\
&= \frac{t^2}{y^2} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{t^2}{y^2}$. Therefore equation (4) becomes

$$
\frac{t^2}{y^2} = \frac{t^2}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{(t+y)\,t}{y} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{(t+y)\,t}{y}
$$

The solution becomes

$$
y = -\frac{t^2}{t + c_1}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = -\frac{t^2}{t + c_1} \tag{1}
$$

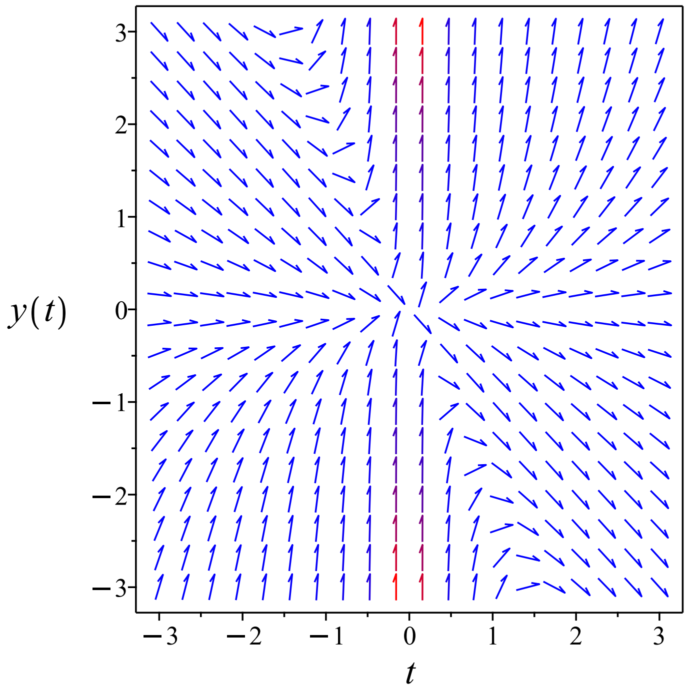


Figure 114: Slope field plot

Verification of solutions

$$y = -\frac{t^2}{t + c_1}$$

Verified OK.

4.30.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= \frac{y(2t + y)}{t^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{2y}{t} + \frac{y^2}{t^2}$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 0$, $f_1(t) = \frac{2}{t}$ and $f_2(t) = \frac{1}{t^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{t^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2) u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{t^3} \\ f_1 f_2 &= \frac{2}{t^3} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(t)}{t^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 t + c_2$$

The above shows that

$$u'(t) = c_1$$

Using the above in (1) gives the solution

$$y = -\frac{c_1 t^2}{c_1 t + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{c_3 t^2}{c_3 t + 1}$$

Summary
The solution(s) found are the following

$$y = -\frac{c_3 t^2}{c_3 t + 1} \tag{1}$$

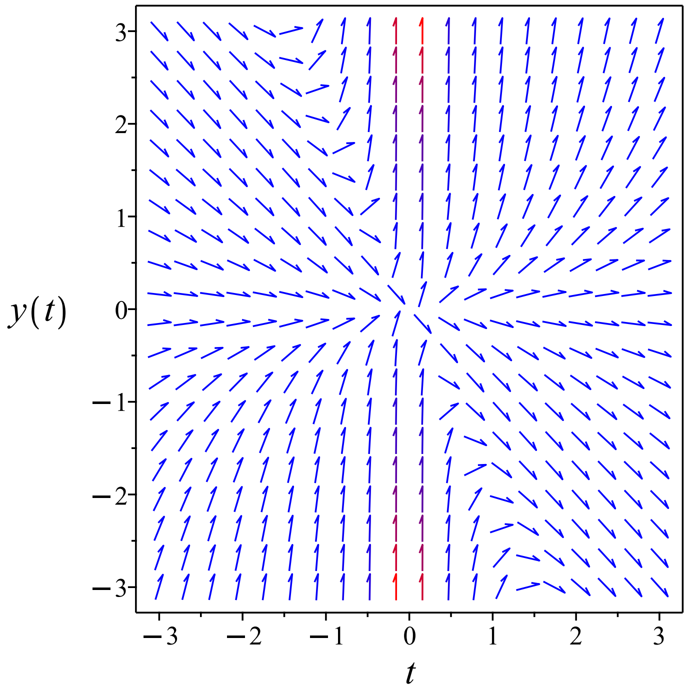


Figure 115: Slope field plot

Verification of solutions

$$y = -\frac{c_3 t^2}{c_3 t + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(t),t)=(y(t)^2+2*t*y(t))/t^2,y(t), singsol=all)
```

$$y(t) = \frac{t^2}{-t + c_1}$$

✓ Solution by Mathematica

Time used: 0.248 (sec). Leaf size: 23

```
DSolve[y'[t]==(y[t]^2+2*t*y[t])/t^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{t^2}{t - c_1}$$
$$y(t) \to 0$$

4.31 problem 28

4.31.1 Existence and uniqueness analysis 486
4.31.2 Solving as separable ode 487
4.31.3 Solving as first order ode lie symmetry lookup ode 489
4.31.4 Solving as exact ode 493
4.31.5 Solving as riccati ode 497

Internal problem ID [11398]
Internal file name [`OUTPUT/10380_Wednesday_May_17_2023_08_10_17_PM_50563810/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.3.1 Separable equations. Exercises page 26
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + y^2 e^{-t^2} = 0}$$

With initial conditions

$$\left[y(0) = \frac{1}{2}\right]$$

4.31.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t,y) \\ &= -y^2 e^{-t^2} \end{aligned}$$

The t domain of $f(t,y)$ when $y = \frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-y^2 \mathrm{e}^{-t^2}\right)\\ &= -2y\,\mathrm{e}^{-t^2}\end{aligned}$$

The t domain of $\frac{\partial f}{\partial y}$ when $y = \frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

4.31.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(t, y)\\ &= f(t)g(y)\\ &= -y^2 \mathrm{e}^{-t^2}\end{aligned}$$

Where $f(t) = -\mathrm{e}^{-t^2}$ and $g(y) = y^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^2}\,dy &= -\mathrm{e}^{-t^2}\,dt\\ \int \frac{1}{y^2}\,dy &= \int -\mathrm{e}^{-t^2}\,dt\\ -\frac{1}{y} &= -\frac{\sqrt{\pi}\,\mathrm{erf}\,(t)}{2} + c_1\end{aligned}$$

Which results in

$$y = \frac{2}{\sqrt{\pi}\,\mathrm{erf}\,(t) - 2c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -\frac{1}{c_1}$$

$$c_1 = -2$$

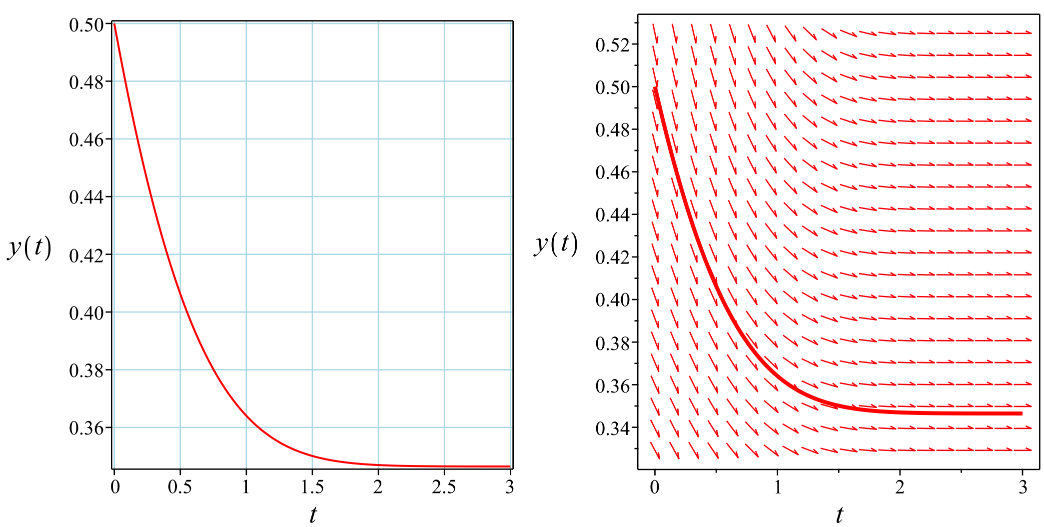
Substituting c_1 found above in the general solution gives

$$y = \frac{2}{\sqrt{\pi}\,\operatorname{erf}(t) + 4}$$

Summary

The solution(s) found are the following

$$y = \frac{2}{\sqrt{\pi}\,\operatorname{erf}(t) + 4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{2}{\sqrt{\pi}\,\operatorname{erf}(t) + 4}$$

Verified OK.

4.31.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -y^2 e^{-t^2} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 95: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= -\mathrm{e}^{t^2} \\
\eta(t,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dt \\
&= \int \frac{1}{-\mathrm{e}^{t^2}} dt
\end{aligned}
$$

Which results in

$$
S = -\frac{\sqrt{\pi}\,\mathrm{erf}\,(t)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$
\omega(t,y) = -y^2\mathrm{e}^{-t^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_y &= 1 \\
S_t &= -\mathrm{e}^{-t^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$-\frac{\sqrt{\pi}\, \mathrm{erf}\,(t)}{2} = -\frac{1}{y} + c_1$$

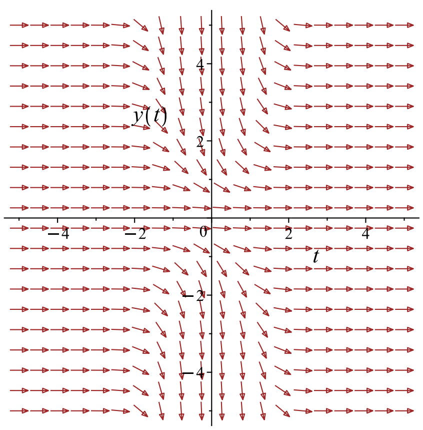
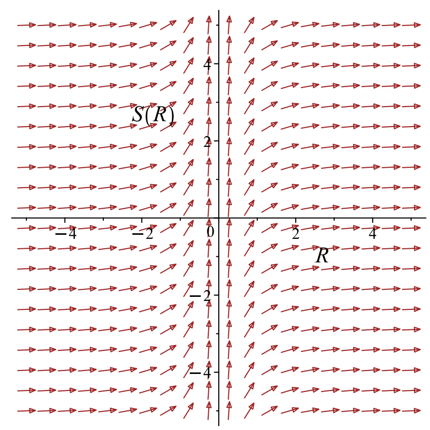
Which simplifies to

$$-\frac{\sqrt{\pi}\, \mathrm{erf}\,(t)}{2} = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{2}{\sqrt{\pi}\, \mathrm{erf}\,(t) + 2c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -y^2 e^{-t^2}$ 	$R = y$ $S = -\frac{\sqrt{\pi}\,\operatorname{erf}(t)}{2}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{1}{c_1}$$

$$c_1 = 2$$

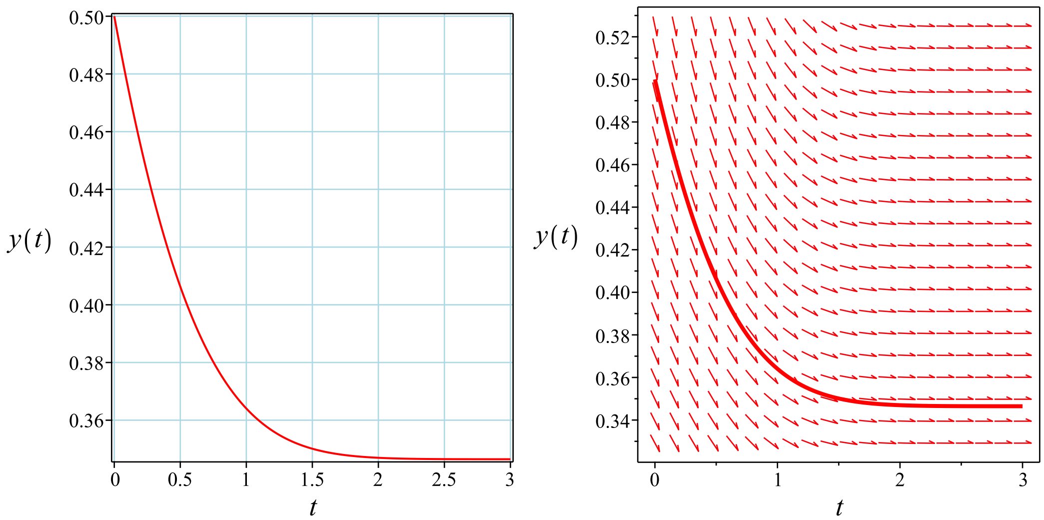
Substituting c_1 found above in the general solution gives

$$y = \frac{2}{\sqrt{\pi}\,\operatorname{erf}(t) + 4}$$

Summary

The solution(s) found are the following

$$y = \frac{2}{\sqrt{\pi}\,\operatorname{erf}(t) + 4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{2}{\sqrt{\pi}\ \mathrm{erf}\,(t) + 4}$$

Verified OK.

4.31.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y^2}\right)\mathrm{d}y &= \left(\mathrm{e}^{-t^2}\right)\mathrm{d}t\\ \left(-\mathrm{e}^{-t^2}\right)\mathrm{d}t + \left(-\frac{1}{y^2}\right)\mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= -\mathrm{e}^{-t^2}\\ N(t,y) &= -\frac{1}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\mathrm{e}^{-t^2}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{1}{y^2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -\mathrm{e}^{-t^2} \, \mathrm{d}t$$

$$\phi = -\frac{\sqrt{\pi}\, \mathrm{erf}\,(t)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y^2}$. Therefore equation (4) becomes

$$-\frac{1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{1}{y^2}\right) \mathrm{d}y$$

$$f(y) = \frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\sqrt{\pi}\,\operatorname{erf}(t)}{2} + \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\sqrt{\pi}\,\operatorname{erf}(t)}{2} + \frac{1}{y}$$

The solution becomes

$$y = \frac{2}{\sqrt{\pi}\,\operatorname{erf}(t) + 2c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{1}{c_1}$$

$$c_1 = 2$$

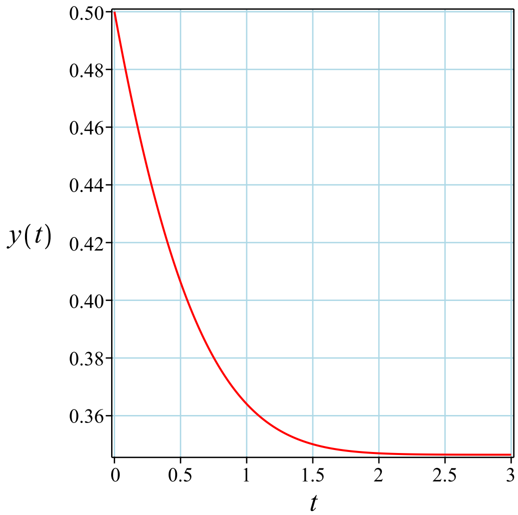
Substituting c_1 found above in the general solution gives

$$y = \frac{2}{\sqrt{\pi}\,\operatorname{erf}(t) + 4}$$

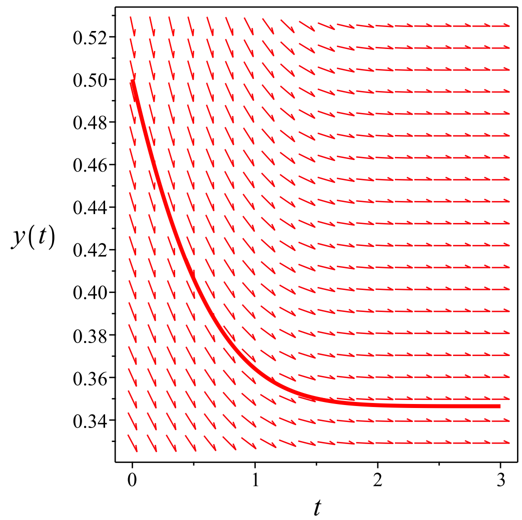
Summary

The solution(s) found are the following

$$y = \frac{2}{\sqrt{\pi}\,\operatorname{erf}(t) + 4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2}{\sqrt{\pi}\, \operatorname{erf}(t) + 4}$$

Verified OK.

4.31.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= -y^2 \mathrm{e}^{-t^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y^2 \mathrm{e}^{-t^2}$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 0$, $f_1(t) = 0$ and $f_2(t) = -\mathrm{e}^{-t^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\mathrm{e}^{-t^2} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(t) - (f_2' + f_1f_2)\,u'(t) + f_2^2f_0u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2t\,\mathrm{e}^{-t^2} \\ f_1f_2 &= 0 \\ f_2^2f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\mathrm{e}^{-t^2}u''(t) - 2t\,\mathrm{e}^{-t^2}u'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 + \operatorname{erf}(t)\,c_2$$

The above shows that

$$u'(t) = \frac{2\,\mathrm{e}^{-t^2}c_2}{\sqrt{\pi}}$$

Using the above in (1) gives the solution

$$y = \frac{2c_2}{\sqrt{\pi}\,(c_1 + \operatorname{erf}(t)\,c_2)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2}{\sqrt{\pi}\,(c_3 + \operatorname{erf}(t))}$$

Initial conditions are used to solve for c_3. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{2}{c_3\sqrt{\pi}}$$

$$c_3 = \frac{4}{\sqrt{\pi}}$$

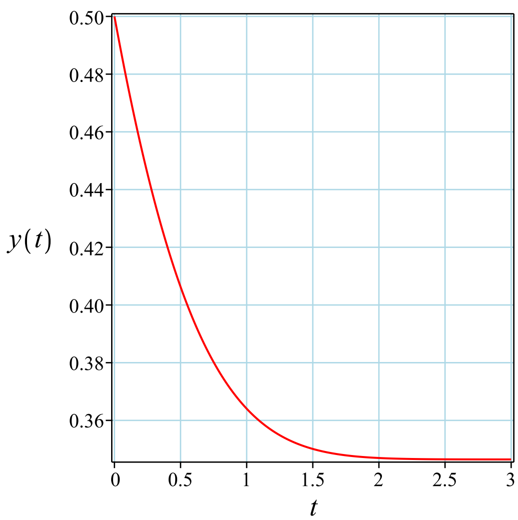
Substituting c_3 found above in the general solution gives

$$y = \frac{2}{\sqrt{\pi}\,\operatorname{erf}(t) + 4}$$

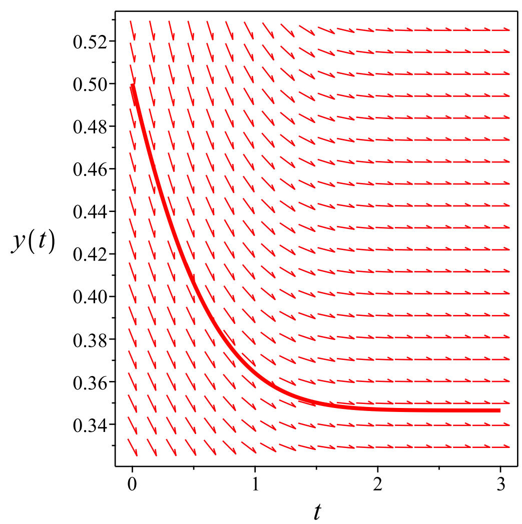
Summary

The solution(s) found are the following

$$y = \frac{2}{\sqrt{\pi}\,\operatorname{erf}(t) + 4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2}{\sqrt{\pi}\,\operatorname{erf}(t) + 4}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 16

```
dsolve([diff(y(t),t)=-y(t)^2*exp(-t^2),y(0) = 1/2],y(t), singsol=all)
```

$$y(t) = \frac{2}{4 + \sqrt{\pi}\,\mathrm{erf}\,(t)}$$

✓ Solution by Mathematica

Time used: 0.365 (sec). Leaf size: 19

```
DSolve[{y'[t]==-y[t]^2*Exp[-t^2],{y[0]==1/2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{2}{\sqrt{\pi}\mathrm{erf}(t) + 4}$$

5 Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41

5.1 problem 1(a) 503
5.2 problem 1(b) 517
5.3 problem 1(c) 529
5.4 problem 1(d) 533
5.5 problem 1(e) 547
5.6 problem 1(f) 549
5.7 problem 2(a) 552
5.8 problem 2(b) 565
5.9 problem 2(c) 578
5.10 problem 2(d) 591
5.11 problem 2(e) 605
5.12 problem 2(f) 616
5.13 problem 3(a) 630
5.14 problem 3(b) 644
5.15 problem 3(c) 657
5.16 problem 3(d) 671
5.17 problem 3(e) 684
5.18 problem 3(f) 696
5.19 problem 4 712
5.20 problem 5 723
5.21 problem 6 738
5.22 problem 7 752
5.23 problem 8 772
5.24 problem 9 782
5.25 problem 12 785
5.26 problem 15(a) 795
5.27 problem 15(b) 810
5.28 problem 15(c) 821
5.29 problem 15(d) 837
5.30 problem 15(e) 850
5.31 problem 15(f) 853
5.32 problem 16-b(i) 862
5.33 problem 16-b(ii) 877
5.34 problem 16-b(iii) 885

5.35 problem 16-b(iv) 890
5.36 problem 16-b(v) 902
5.37 problem 16-b(vi) 917

5.1 problem 1(a)

5.1.1 Solving as linear ode 503
5.1.2 Solving as first order ode lie symmetry lookup ode 505
5.1.3 Solving as exact ode 509
5.1.4 Maple step by step solution 514

Internal problem ID [11399]
Internal file name [`OUTPUT/10381_Wednesday_May_17_2023_08_10_18_PM_3687755/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 1(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x' - 2t^3 x = -6}$$

5.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= -2t^3 \\ q(t) &= -6 \end{aligned}$$

Hence the ode is

$$x' - 2t^3 x = -6$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -2t^3 dt} \\
&= \mathrm{e}^{-\frac{t^4}{2}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)(-6) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\frac{t^4}{2}} x\right) &= \left(\mathrm{e}^{-\frac{t^4}{2}}\right)(-6) \\
\mathrm{d}\left(\mathrm{e}^{-\frac{t^4}{2}} x\right) &= \left(-6\,\mathrm{e}^{-\frac{t^4}{2}}\right)\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-\frac{t^4}{2}} x &= \int -6\,\mathrm{e}^{-\frac{t^4}{2}}\,\mathrm{d}t \\
\mathrm{e}^{-\frac{t^4}{2}} x &= -\frac{3\,2^{\frac{1}{4}}\left(\frac{8t2^{\frac{7}{8}}\,\mathrm{e}^{-\frac{t^4}{4}}\,\mathrm{WhittakerM}\left(\frac{1}{8},\frac{5}{8},\frac{t^4}{2}\right)}{5(t^4)^{\frac{1}{8}}} + \frac{4\,2^{\frac{7}{8}}\,\mathrm{e}^{-\frac{t^4}{4}}\,\mathrm{WhittakerM}\left(\frac{9}{8},\frac{5}{8},\frac{t^4}{2}\right)}{t^3(t^4)^{\frac{1}{8}}}\right)}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{t^4}{2}}$ results in

$$
x = -\frac{3\,\mathrm{e}^{\frac{t^4}{2}}\,2^{\frac{1}{4}}\left(\frac{8t2^{\frac{7}{8}}\,\mathrm{e}^{-\frac{t^4}{4}}\,\mathrm{WhittakerM}\left(\frac{1}{8},\frac{5}{8},\frac{t^4}{2}\right)}{5(t^4)^{\frac{1}{8}}} + \frac{4\,2^{\frac{7}{8}}\,\mathrm{e}^{-\frac{t^4}{4}}\,\mathrm{WhittakerM}\left(\frac{9}{8},\frac{5}{8},\frac{t^4}{2}\right)}{t^3(t^4)^{\frac{1}{8}}}\right)}{2} + c_1\mathrm{e}^{\frac{t^4}{2}}
$$

which simplifies to

$$
x = \frac{-\frac{24\,\mathrm{WhittakerM}\left(\frac{1}{8},\frac{5}{8},\frac{t^4}{2}\right)2^{\frac{1}{8}}\,t\,\mathrm{e}^{\frac{t^4}{4}}}{5} + \left(c_1\mathrm{e}^{\frac{t^4}{2}} - 6t\right)(t^4)^{\frac{1}{8}}}{(t^4)^{\frac{1}{8}}}
$$

<u>Summary</u>

The solution(s) found are the following

$$
x = \frac{-\frac{24\,\mathrm{WhittakerM}\left(\frac{1}{8},\frac{5}{8},\frac{t^4}{2}\right)2^{\frac{1}{8}}\,t\,\mathrm{e}^{\frac{t^4}{4}}}{5} + \left(c_1\mathrm{e}^{\frac{t^4}{2}} - 6t\right)(t^4)^{\frac{1}{8}}}{(t^4)^{\frac{1}{8}}} \tag{1}
$$

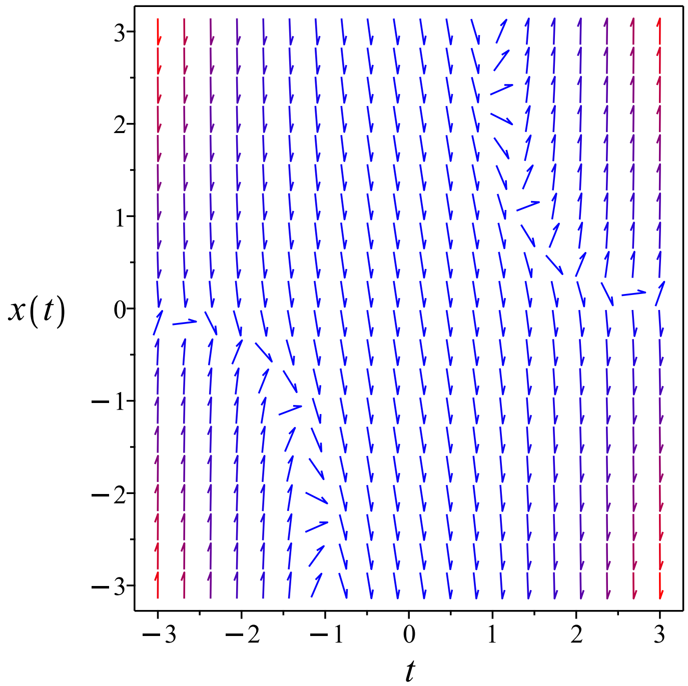


Figure 120: Slope field plot

Verification of solutions

$$x = \frac{-\frac{24\,\mathrm{WhittakerM}\left(\frac{1}{8},\frac{5}{8},\frac{t^4}{2}\right)2^{\frac{1}{8}}t\,\mathrm{e}^{\frac{t^4}{4}}}{5} + \left(c_1\mathrm{e}^{\frac{t^4}{2}} - 6t\right)\left(t^4\right)^{\frac{1}{8}}}{\left(t^4\right)^{\frac{1}{8}}}$$

Verified OK.

5.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = 2t^3x - 6$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 97: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \mathrm{e}^{\frac{t^4}{2}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\frac{t^4}{2}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-\frac{t^4}{2}} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = 2t^3 x - 6$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= -2t^3 \mathrm{e}^{-\frac{t^4}{2}} x \\ S_x &= \mathrm{e}^{-\frac{t^4}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -6\,\mathrm{e}^{-\frac{t^4}{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -6\,\mathrm{e}^{-\frac{R^4}{2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{3R\,\mathrm{e}^{-\frac{R^4}{4}}\,\mathrm{WhittakerM}\left(\frac{1}{8}, \frac{5}{8}, \frac{R^4}{2}\right) 128^{\frac{7}{8}}}{40\left(R^4\right)^{\frac{1}{8}}} - \frac{3\,\mathrm{e}^{-\frac{R^4}{4}}\,\mathrm{WhittakerM}\left(\frac{9}{8}, \frac{5}{8}, \frac{R^4}{2}\right) 128^{\frac{7}{8}}}{16R^3\left(R^4\right)^{\frac{1}{8}}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\mathrm{e}^{-\frac{t^4}{2}} x = -\frac{3t\,\mathrm{e}^{-\frac{t^4}{4}}\,\mathrm{WhittakerM}\left(\frac{1}{8}, \frac{5}{8}, \frac{t^4}{2}\right) 128^{\frac{7}{8}}}{40\left(t^4\right)^{\frac{1}{8}}} - \frac{3\,\mathrm{e}^{-\frac{t^4}{4}}\,\mathrm{WhittakerM}\left(\frac{9}{8}, \frac{5}{8}, \frac{t^4}{2}\right) 128^{\frac{7}{8}}}{16t^3\left(t^4\right)^{\frac{1}{8}}} + c_1$$

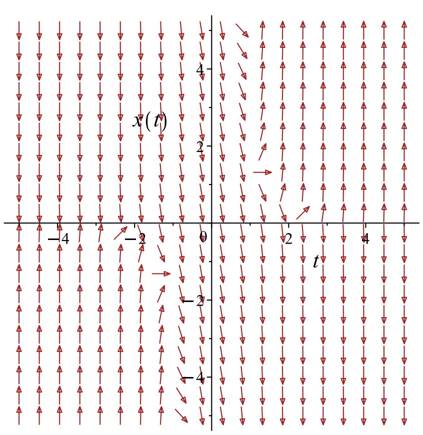
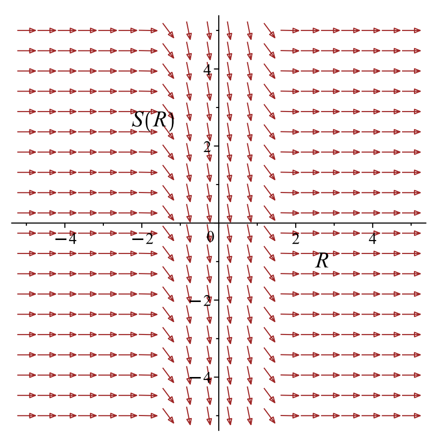
Which simplifies to

$$\frac{\left(24\sqrt{t}\,\mathrm{WhittakerM}\left(\frac{1}{8}, \frac{5}{8}, \frac{t^4}{2}\right) 2^{\frac{1}{8}}\mathrm{e}^{\frac{t^4}{4}} - 5c_1\mathrm{e}^{\frac{t^4}{2}} + 30t + 5x\right)\mathrm{e}^{-\frac{t^4}{2}}}{5} = 0$$

Which gives

$$x = -\frac{24\sqrt{t}\,\mathrm{WhittakerM}\left(\frac{1}{8}, \frac{5}{8}, \frac{t^4}{2}\right) 2^{\frac{1}{8}}\mathrm{e}^{\frac{t^4}{4}}}{5} + c_1\mathrm{e}^{\frac{t^4}{2}} - 6t$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = 2t^3x - 6$ 	$R = t$ $S = \mathrm{e}^{-\frac{t^4}{2}} x$	$\frac{dS}{dR} = -6\,\mathrm{e}^{-\frac{R^4}{2}}$ 

Summary
The solution(s) found are the following

$$x = -\frac{24\sqrt{t}\ \mathrm{WhittakerM}\left(\frac{1}{8}, \frac{5}{8}, \frac{t^4}{2}\right) 2^{\frac{1}{8}} \mathrm{e}^{\frac{t^4}{4}}}{5} + c_1 \mathrm{e}^{\frac{t^4}{2}} - 6t \tag{1}$$

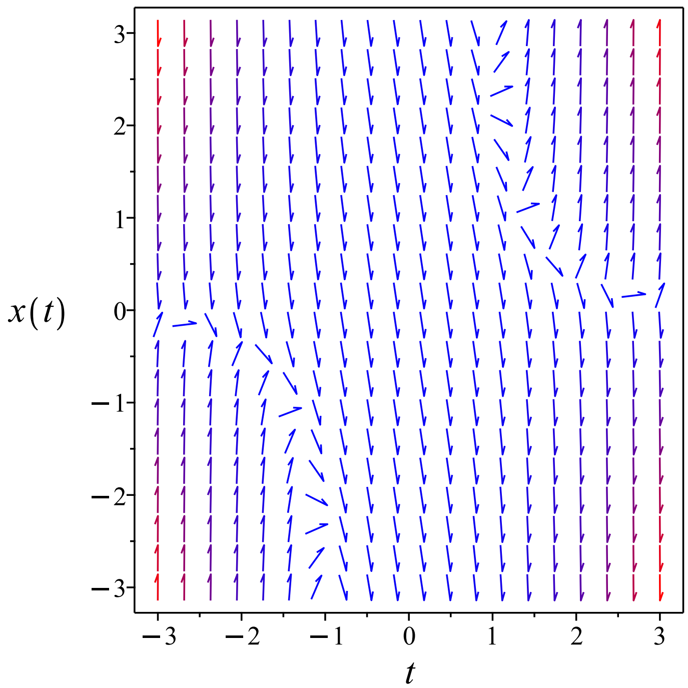


Figure 121: Slope field plot

Verification of solutions

$$x = -\frac{24\sqrt{t}\ \mathrm{WhittakerM}\left(\frac{1}{8}, \frac{5}{8}, \frac{t^4}{2}\right) 2^{\frac{1}{8}} \mathrm{e}^{\frac{t^4}{4}}}{5} + c_1 \mathrm{e}^{\frac{t^4}{2}} - 6t$$

Verified OK.

5.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}x&=\left(2t^3x-6\right)\mathrm{d}t\\ \left(-2t^3x+6\right)\mathrm{d}t+\mathrm{d}x&=0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x)&=-2t^3x+6\\ N(t,x)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-2t^3x + 6\right)\\ &= -2t^3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right)\\ &= 1\left(\left(-2t^3\right) - (0)\right)\\ &= -2t^3\end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}t}\\ &= e^{\int -2t^3 \,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{t^4}{2}}\\ &= \mathrm{e}^{-\frac{t^4}{2}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-\frac{t^4}{2}}\left(-2t^3x + 6\right)\\ &= -2\left(t^3x - 3\right)\mathrm{e}^{-\frac{t^4}{2}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-\frac{t^4}{2}}(1)\\ &= \mathrm{e}^{-\frac{t^4}{2}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}x}{\mathrm{d}t}&=0\\
\left(-2\left(t^{3}x-3\right)\mathrm{e}^{-\frac{t^{4}}{2}}\right)+\left(\mathrm{e}^{-\frac{t^{4}}{2}}\right)\frac{\mathrm{d}x}{\mathrm{d}t}&=0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$
\frac{\partial\phi}{\partial t}=\overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial x}=\overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t&=\int\overline{M}\,\mathrm{d}t\\
\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t&=\int-2\left(t^{3}x-3\right)\mathrm{e}^{-\frac{t^{4}}{2}}\,\mathrm{d}t
\end{aligned}
$$

$$
\phi=\frac{\frac{24\,\mathrm{e}^{-\frac{t^{4}}{4}}\,\mathrm{WhittakerM}\left(\frac{1}{8},\frac{5}{8},\frac{t^{4}}{2}\right)2^{\frac{1}{8}}t}{5}+6\left(\left(t+\frac{x}{6}\right)\mathrm{e}^{-\frac{t^{4}}{2}}-\frac{x}{6}\right)\left(t^{4}\right)^{\frac{1}{8}}}{\left(t^{4}\right)^{\frac{1}{8}}}+f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial\phi}{\partial x}=-1+\mathrm{e}^{-\frac{t^{4}}{2}}+f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial x}=\mathrm{e}^{-\frac{t^{4}}{2}}$. Therefore equation (4) becomes

$$
\mathrm{e}^{-\frac{t^{4}}{2}}=-1+\mathrm{e}^{-\frac{t^{4}}{2}}+f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x)=1
$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int (1)\,\mathrm{d}x$$
$$f(x) = x + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{\frac{24\,\mathrm{e}^{-\frac{t^4}{4}}\,\mathrm{WhittakerM}\left(\frac{1}{8},\frac{5}{8},\frac{t^4}{2}\right)2^{\frac{1}{8}}t}{5} + 6\left(\left(t+\frac{x}{6}\right)\mathrm{e}^{-\frac{t^4}{2}} - \frac{x}{6}\right)\left(t^4\right)^{\frac{1}{8}}}{\left(t^4\right)^{\frac{1}{8}}} + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\frac{24\,\mathrm{e}^{-\frac{t^4}{4}}\,\mathrm{WhittakerM}\left(\frac{1}{8},\frac{5}{8},\frac{t^4}{2}\right)2^{\frac{1}{8}}t}{5} + 6\left(\left(t+\frac{x}{6}\right)\mathrm{e}^{-\frac{t^4}{2}} - \frac{x}{6}\right)\left(t^4\right)^{\frac{1}{8}}}{\left(t^4\right)^{\frac{1}{8}}} + x$$

The solution becomes

$$x = -\frac{\mathrm{e}^{\frac{t^4}{2}}\left(24\,\mathrm{e}^{-\frac{t^4}{4}}\,\mathrm{WhittakerM}\left(\frac{1}{8},\frac{5}{8},\frac{t^4}{2}\right)2^{\frac{1}{8}}t + 30\,\mathrm{e}^{-\frac{t^4}{2}}\left(t^4\right)^{\frac{1}{8}}t - 5c_1\left(t^4\right)^{\frac{1}{8}}\right)}{5\left(t^4\right)^{\frac{1}{8}}}$$

Summary

The solution(s) found are the following

$$x = -\frac{\mathrm{e}^{\frac{t^4}{2}}\left(24\,\mathrm{e}^{-\frac{t^4}{4}}\,\mathrm{WhittakerM}\left(\frac{1}{8},\frac{5}{8},\frac{t^4}{2}\right)2^{\frac{1}{8}}t + 30\,\mathrm{e}^{-\frac{t^4}{2}}\left(t^4\right)^{\frac{1}{8}}t - 5c_1\left(t^4\right)^{\frac{1}{8}}\right)}{5\left(t^4\right)^{\frac{1}{8}}} \tag{1}$$

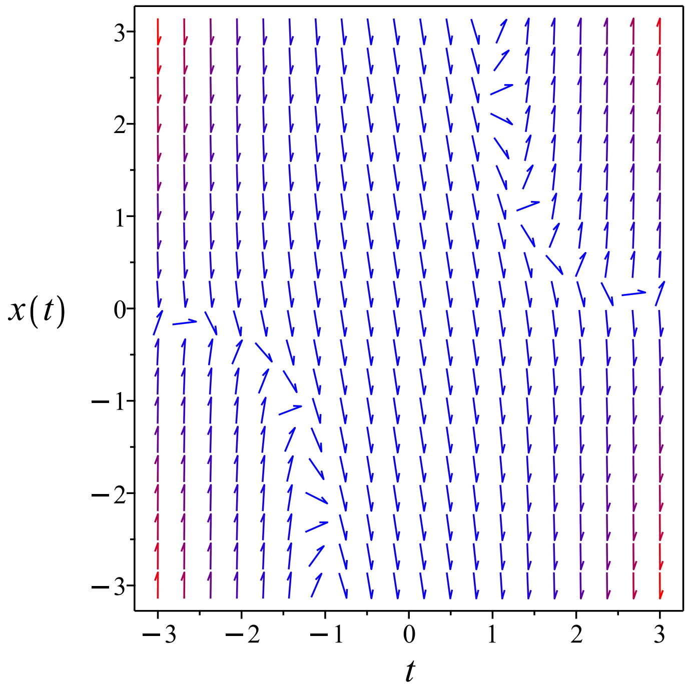


Figure 122: Slope field plot

Verification of solutions

$$x = -\frac{e^{\frac{t^4}{2}}\left(24\,e^{-\frac{t^4}{4}}\,\text{WhittakerM}\left(\frac{1}{8},\frac{5}{8},\frac{t^4}{2}\right)2^{\frac{1}{8}}t+30\,e^{-\frac{t^4}{2}}\left(t^4\right)^{\frac{1}{8}}t-5c_1\left(t^4\right)^{\frac{1}{8}}\right)}{5\left(t^4\right)^{\frac{1}{8}}}$$

Verified OK.

5.1.4 Maple step by step solution

Let's solve

$x' - 2t^3x = -6$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = 2t^3x - 6$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' - 2t^3x = -6$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $$\mu(t)\left(x' - 2t^3 x\right) = -6\mu(t)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, x)$

  $$\mu(t)\left(x' - 2t^3 x\right) = \mu'(t)\, x + \mu(t)\, x'$$

- Isolate $\mu'(t)$

  $$\mu'(t) = -2\mu(t)\, t^3$$

- Solve to find the integrating factor

  $$\mu(t) = \mathrm{e}^{-\frac{t^4}{2}}$$

- Integrate both sides with respect to t

  $$\int \left(\frac{d}{dt}(\mu(t)\, x)\right) dt = \int -6\mu(t)\, dt + c_1$$

- Evaluate the integral on the lhs

  $$\mu(t)\, x = \int -6\mu(t)\, dt + c_1$$

- Solve for x

  $$x = \frac{\int -6\mu(t) dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = \mathrm{e}^{-\frac{t^4}{2}}$

  $$x = \frac{\int -6\,\mathrm{e}^{-\frac{t^4}{2}}\, dt + c_1}{\mathrm{e}^{-\frac{t^4}{2}}}$$

- Evaluate the integrals on the rhs

  $$x = \frac{-\frac{3\, 2^{\frac{1}{4}} \left(\frac{8t 2^{\frac{7}{8}}\,\mathrm{e}^{-\frac{t^4}{4}}\, \mathit{WhittakerM}\left(\frac{1}{8}, \frac{5}{8}, \frac{t^4}{2}\right)}{5\left(t^4\right)^{\frac{1}{8}}} + \frac{4\, 2^{\frac{7}{8}}\,\mathrm{e}^{-\frac{t^4}{4}}\, \mathit{WhittakerM}\left(\frac{9}{8}, \frac{5}{8}, \frac{t^4}{2}\right)}{t^3\left(t^4\right)^{\frac{1}{8}}}\right)}{2} + c_1}{\mathrm{e}^{-\frac{t^4}{2}}}$$

- Simplify

  $$x = \frac{-\frac{24\, \mathit{WhittakerM}\left(\frac{1}{8}, \frac{5}{8}, \frac{t^4}{2}\right) 2^{\frac{1}{8}}\, t\, \mathrm{e}^{\frac{t^4}{4}}}{5} + \left(c_1 \mathrm{e}^{\frac{t^4}{2}} - 6t\right)\left(t^4\right)^{\frac{1}{8}}}{\left(t^4\right)^{\frac{1}{8}}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
dsolve(diff(x(t),t)=2*t^3*x(t)-6,x(t), singsol=all)
```

$$x(t) = \frac{-\frac{24\,\mathrm{WhittakerM}\left(\frac{1}{8},\frac{5}{8},\frac{t^4}{2}\right)\mathrm{e}^{\frac{t^4}{4}}\,2^{\frac{1}{8}}\,t}{5} + (t^4)^{\frac{1}{8}}\left(\mathrm{e}^{\frac{t^4}{2}}c_1 - 6t\right)}{(t^4)^{\frac{1}{8}}}$$

✓ Solution by Mathematica

Time used: 0.152 (sec). Leaf size: 49

```
DSolve[x'[t]==2*t^3*x[t]-6,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2}e^{\frac{t^4}{2}}\left(\frac{3\sqrt[4]{2}t\Gamma\left(\frac{1}{4},\frac{t^4}{2}\right)}{\sqrt[4]{t^4}} + 2c_1\right)$$

5.2 problem 1(b)

5.2.1 Solving as separable ode 517
5.2.2 Solving as first order ode lie symmetry lookup ode 519
5.2.3 Solving as exact ode 523
5.2.4 Maple step by step solution 527

Internal problem ID [11400]
Internal file name [`OUTPUT/10382_Wednesday_May_17_2023_08_10_20_PM_46076406/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 1(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\cos(t)\, x' - 2x \sin(x) = 0}$$

5.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= f(t)g(x) \\ &= \frac{2x \sin(x)}{\cos(t)} \end{aligned}$$

Where $f(t) = \frac{2}{\cos(t)}$ and $g(x) = x\sin(x)$. Integrating both sides gives

$$\frac{1}{x \sin(x)}\, dx = \frac{2}{\cos(t)}\, dt$$

$$\int \frac{1}{x \sin(x)}\, dx = \int \frac{2}{\cos(t)}\, dt$$

$$\int^{x} \frac{1}{\_a \sin(\_a)} d\_a = 2 \ln(\sec(t) + \tan(t)) + c_1$$

Which results in

$$\int^{x} \frac{1}{\_a \sin(\_a)} d\_a = 2 \ln(\sec(t) + \tan(t)) + c_1$$

The solution is

$$\int^{x} \frac{1}{\_a \sin(\_a)} d\_a - 2 \ln(\sec(t) + \tan(t)) - c_1 = 0$$

Summary

The solution(s) found are the following

$$\int^{x} \frac{1}{\_a \sin(\_a)} d\_a - 2 \ln(\sec(t) + \tan(t)) - c_1 = 0 \tag{1}$$

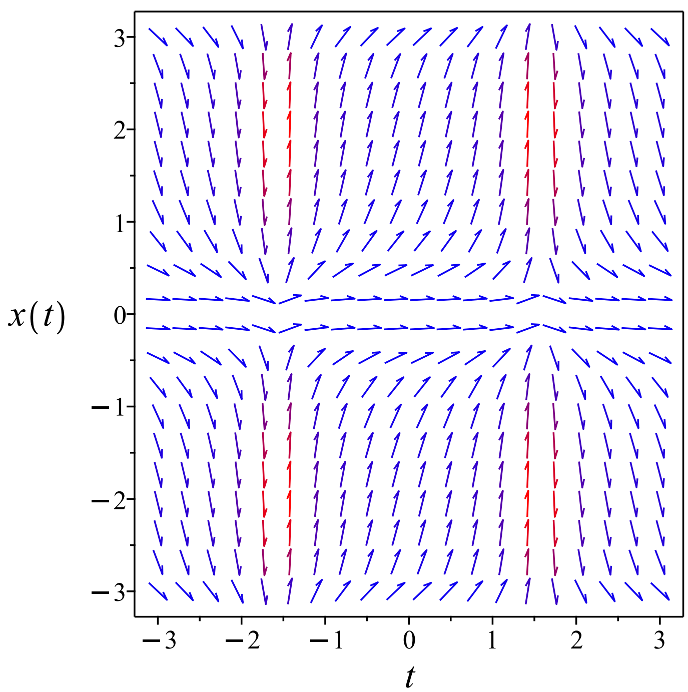


Figure 123: Slope field plot

Verification of solutions

$$\int^{x} \frac{1}{\_a \sin (\_a)}d\_a - 2 \ln (\sec (t) + \tan (t)) - c_1 = 0$$

Verified OK.

5.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = \frac{2x \sin (x)}{\cos (t)}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 100: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t,x) &= \frac{\cos(t)}{2} \\ \eta(t,x) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{\cos(t)}{2}} dt \end{aligned}$$

Which results in

$$S = 2 \ln (\sec (t) + \tan (t))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = \frac{2x \sin (x)}{\cos (t)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_x &= 1 \\ S_t &= 2 \sec (t) \\ S_x &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\csc (x)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\csc (R)}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{\csc(R)}{R} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$2\ln(\sec(t) + \tan(t)) = \int^x \frac{\csc(\_a)}{\_a} d\_a + c_1$$

Which simplifies to

$$2\ln(\sec(t) + \tan(t)) = \int^x \frac{\csc(\_a)}{\_a} d\_a + c_1$$

This results in

$$2\ln(\sec(t) + \tan(t)) = \int^x \frac{\csc(\_a)}{\_a} d\_a + c_1$$

<u>Summary</u>
The solution(s) found are the following

$$2\ln(\sec(t) + \tan(t)) = \int^x \frac{\csc(\_a)}{\_a} d\_a + c_1 \tag{1}$$

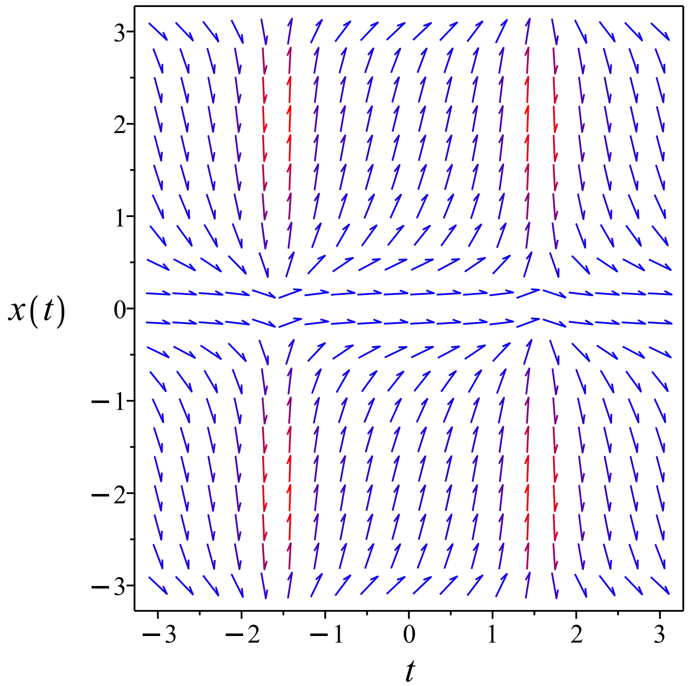


Figure 124: Slope field plot

Verification of solutions

$$2\ln\left(\sec\left(t\right)+\tan\left(t\right)\right)=\int^{x}\frac{\csc\left(\_a\right)}{\_a}d\_a+c_1$$

Verified OK.

5.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, x)\, \mathrm{d}t + N(t, x)\, \mathrm{d}x = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{2x \sin(x)}\right) \mathrm{d}x &= \left(\frac{1}{\cos(t)}\right) \mathrm{d}t \\
\left(-\frac{1}{\cos(t)}\right) \mathrm{d}t + \left(\frac{1}{2x \sin(x)}\right) \mathrm{d}x &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, x) &= -\frac{1}{\cos(t)} \\
N(t, x) &= \frac{1}{2x \sin(x)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{\cos(t)}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{2x\sin(x)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, x)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} &= M & (1) \\
\frac{\partial \phi}{\partial x} &= N & (2)
\end{aligned}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -\frac{1}{\cos(t)} \, \mathrm{d}t \\
\phi &= -\ln(\sec(t) + \tan(t)) + f(x) \qquad (3)
\end{aligned}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{2x\sin(x)}$. Therefore equation (4) becomes

$$
\frac{1}{2x\sin(x)} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
\begin{aligned}
f'(x) &= \frac{1}{2x\sin(x)} \\
&= \frac{\csc(x)}{2x}
\end{aligned}
$$

Integrating the above w.r.t x results in

$$\int f'(x)\,\mathrm{d}x = \int \left(\frac{\csc(x)}{2x}\right)\mathrm{d}x$$

$$f(x) = \int_0^x \frac{\csc(\_a)}{2\_a} d\_a + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\ln(\sec(t) + \tan(t)) + \int_0^x \frac{\csc(\_a)}{2\_a} d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\sec(t) + \tan(t)) + \int_0^x \frac{\csc(\_a)}{2\_a} d\_a$$

Summary

The solution(s) found are the following

$$-\ln(\sec(t) + \tan(t)) + \int_0^x \frac{\csc(\_a)}{2\_a} d\_a = c_1 \tag{1}$$

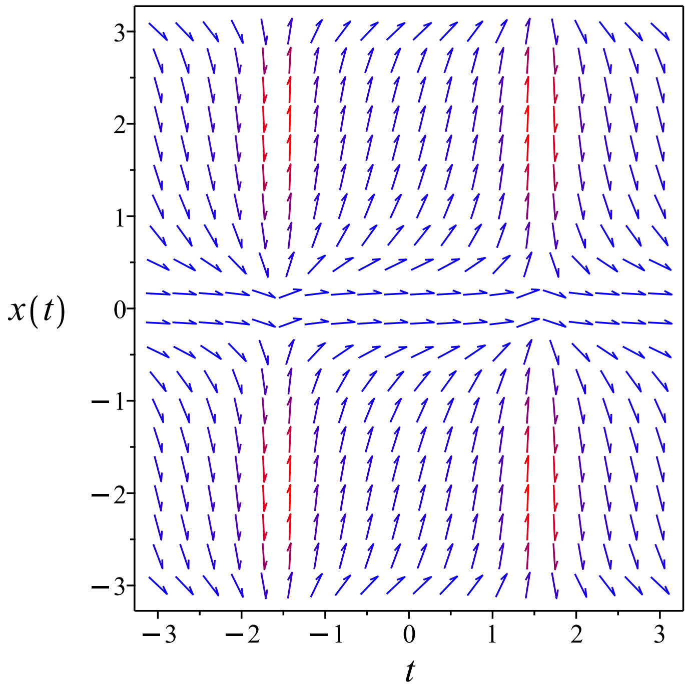


Figure 125: Slope field plot

Verification of solutions

$$-\ln\left(\sec\left(t\right)+\tan\left(t\right)\right)+\int_{0}^{x}\frac{\csc\left(\_a\right)}{2\_a}d\_a=c_1$$

Verified OK.

5.2.4 Maple step by step solution

Let's solve

$\cos\left(t\right)x'-2x\sin\left(x\right)=0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x\sin(x)}=\frac{2}{\cos(t)}$

- Integrate both sides with respect to t

  $\int\frac{x'}{x\sin(x)}dt=\int\frac{2}{\cos(t)}dt+c_1$

- Cannot compute integral

$$\int \frac{x'}{x\sin(x)}dt = 2\ln(\sec(t)+\tan(t)) + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(cos(t)*diff(x(t),t)-2*x(t)*sin(x(t))=0,x(t), singsol=all)
```

$$\ln(\sec(t)+\tan(t)) - \frac{\left(\int^{x(t)} \frac{\csc(\_a)}{\_a}d\_a\right)}{2} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 10.596 (sec). Leaf size: 40

```
DSolve[Cos[t]*x'[t]-2*x[t]*Sin[x[t]]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{\csc(K[1])}{K[1]}dK[1]\&\right]\left[4\text{arctanh}\left(\tan\left(\frac{t}{2}\right)\right) + c_1\right]$$
$$x(t) \to 0$$

5.3 problem 1(c)

5.3.1 Solving as riccati ode 529

Internal problem ID [11401]
Internal file name [`OUTPUT/10383_Wednesday_May_17_2023_08_10_22_PM_96590617/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 1(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_Riccati, _special]]
```

$$\boxed{x' + x^2 = t}$$

5.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= -x^2 + t \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$x' = -x^2 + t$$

With Riccati ODE standard form

$$x' = f_0(t) + f_1(t)x + f_2(t)x^2$$

Shows that $f_0(t) = t$, $f_1(t) = 0$ and $f_2(t) = -1$. Let

$$\begin{aligned} x &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(t) - (f_2' + f_1 f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$
\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= t
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$-u''(t) + tu(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 \operatorname{AiryAi}(t) + c_2 \operatorname{AiryBi}(t)$$

The above shows that

$$u'(t) = c_1 \operatorname{AiryAi}(1, t) + c_2 \operatorname{AiryBi}(1, t)$$

Using the above in (1) gives the solution

$$x = \frac{c_1 \operatorname{AiryAi}(1, t) + c_2 \operatorname{AiryBi}(1, t)}{c_1 \operatorname{AiryAi}(t) + c_2 \operatorname{AiryBi}(t)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$x = \frac{c_3 \operatorname{AiryAi}(1, t) + \operatorname{AiryBi}(1, t)}{c_3 \operatorname{AiryAi}(t) + \operatorname{AiryBi}(t)}$$

Summary

The solution(s) found are the following

$$x = \frac{c_3 \operatorname{AiryAi}(1, t) + \operatorname{AiryBi}(1, t)}{c_3 \operatorname{AiryAi}(t) + \operatorname{AiryBi}(t)} \tag{1}$$

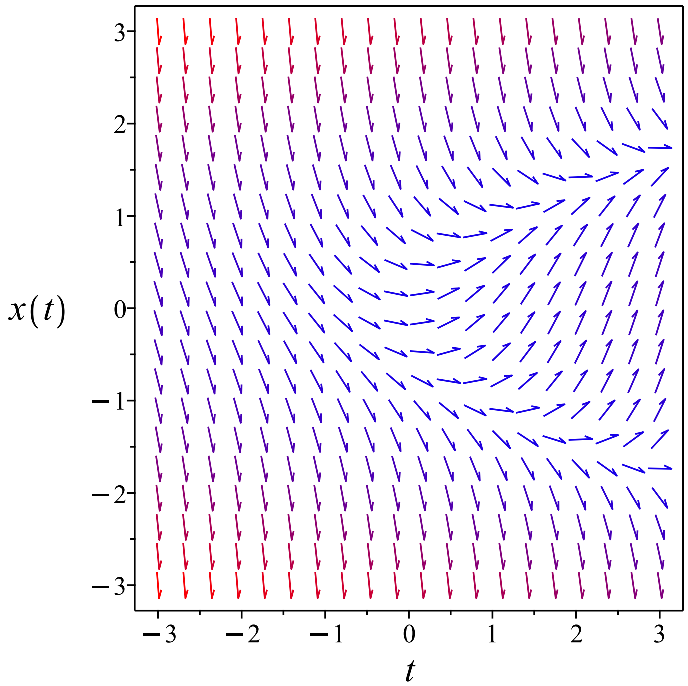


Figure 126: Slope field plot

Verification of solutions

$$x = \frac{c_3 \operatorname{AiryAi}(1, t) + \operatorname{AiryBi}(1, t)}{c_3 \operatorname{AiryAi}(t) + \operatorname{AiryBi}(t)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(x(t),t)=t-x(t)^2,x(t), singsol=all)
```

$$x(t) = \frac{c_1 \operatorname{AiryAi}(1,t) + \operatorname{AiryBi}(1,t)}{c_1 \operatorname{AiryAi}(t) + \operatorname{AiryBi}(t)}$$

✓ Solution by Mathematica

Time used: 0.221 (sec). Leaf size: 223

```
DSolve[x'[t]==t-x[t]^2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{-it^{3/2}\left(2 \operatorname{BesselJ}\left(-\frac{2}{3}, \frac{2}{3}it^{3/2}\right) + c_1\left(\operatorname{BesselJ}\left(-\frac{4}{3}, \frac{2}{3}it^{3/2}\right) - \operatorname{BesselJ}\left(\frac{2}{3}, \frac{2}{3}it^{3/2}\right)\right)\right) - c_1 \operatorname{BesselJ}\left(-\frac{1}{3}, \frac{2}{3}it^{3/}\right.}{2t\left(\operatorname{BesselJ}\left(\frac{1}{3}, \frac{2}{3}it^{3/2}\right) + c_1 \operatorname{BesselJ}\left(-\frac{1}{3}, \frac{2}{3}it^{3/2}\right)\right)}$$

$$x(t) \to \frac{it^{3/2} \operatorname{BesselJ}\left(-\frac{4}{3}, \frac{2}{3}it^{3/2}\right) - it^{3/2} \operatorname{BesselJ}\left(\frac{2}{3}, \frac{2}{3}it^{3/2}\right) + \operatorname{BesselJ}\left(-\frac{1}{3}, \frac{2}{3}it^{3/2}\right)}{2t \operatorname{BesselJ}\left(-\frac{1}{3}, \frac{2}{3}it^{3/2}\right)}$$

5.4 problem 1(d)

5.4.1 Solving as linear ode 533
5.4.2 Solving as first order ode lie symmetry lookup ode 535
5.4.3 Solving as exact ode 539
5.4.4 Maple step by step solution 544

Internal problem ID [11402]
Internal file name [`OUTPUT/10384_Wednesday_May_17_2023_08_10_23_PM_300748/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 1(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{7t^2x' - 3x = -2t}$$

5.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = -\frac{3}{7t^2}$$
$$q(t) = -\frac{2}{7t}$$

Hence the ode is

$$x' - \frac{3x}{7t^2} = -\frac{2}{7t}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{3}{7t^2}dt} \\
&= \mathrm{e}^{\frac{3}{7t}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(-\frac{2}{7t}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\frac{3}{7t}} x\right) &= \left(\mathrm{e}^{\frac{3}{7t}}\right)\left(-\frac{2}{7t}\right) \\
\mathrm{d}\left(\mathrm{e}^{\frac{3}{7t}} x\right) &= \left(-\frac{2\,\mathrm{e}^{\frac{3}{7t}}}{7t}\right)\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\frac{3}{7t}} x &= \int -\frac{2\,\mathrm{e}^{\frac{3}{7t}}}{7t}\,\mathrm{d}t \\
\mathrm{e}^{\frac{3}{7t}} x &= -\frac{2\ \mathrm{expIntegral}_1\left(-\frac{3}{7t}\right)}{7} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{3}{7t}}$ results in

$$
x = -\frac{2\,\mathrm{e}^{-\frac{3}{7t}}\ \mathrm{expIntegral}_1\left(-\frac{3}{7t}\right)}{7} + c_1\mathrm{e}^{-\frac{3}{7t}}
$$

which simplifies to

$$
x = \mathrm{e}^{-\frac{3}{7t}}\left(-\frac{2\ \mathrm{expIntegral}_1\left(-\frac{3}{7t}\right)}{7} + c_1\right)
$$

Summary

The solution(s) found are the following

$$
x = \mathrm{e}^{-\frac{3}{7t}}\left(-\frac{2\ \mathrm{expIntegral}_1\left(-\frac{3}{7t}\right)}{7} + c_1\right) \tag{1}
$$

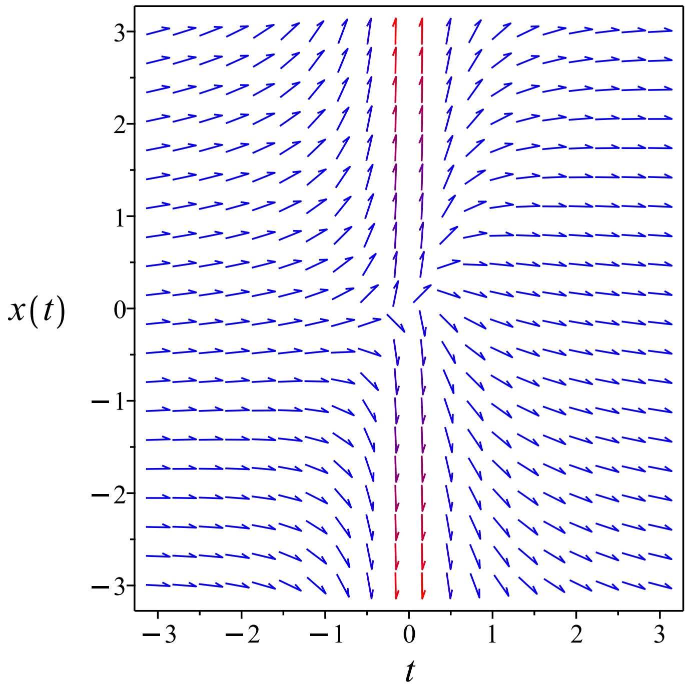


Figure 127: Slope field plot

Verification of solutions

$$x = e^{-\frac{3}{7t}} \left(-\frac{2 \operatorname{expIntegral}_1\left(-\frac{3}{7t}\right)}{7} + c_1 \right)$$

Verified OK.

5.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = \frac{3x - 2t}{7t^2}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 103: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \mathrm{e}^{-\frac{3}{7t}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{3}{7t}}} dy \end{aligned}$$

Which results in

$$S = e^{\frac{3}{7t}} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = \frac{3x - 2t}{7t^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= -\frac{3 e^{\frac{3}{7t}} x}{7t^2} \\ S_x &= e^{\frac{3}{7t}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2 e^{\frac{3}{7t}}}{7t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2 e^{\frac{3}{7R}}}{7R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{2\,\mathrm{expIntegral}_1\left(-\frac{3}{7R}\right)}{7} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\mathrm{e}^{\frac{3}{7t}} x = -\frac{2\,\mathrm{expIntegral}_1\left(-\frac{3}{7t}\right)}{7} + c_1$$

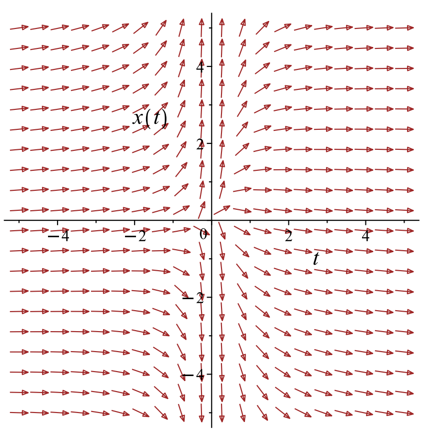
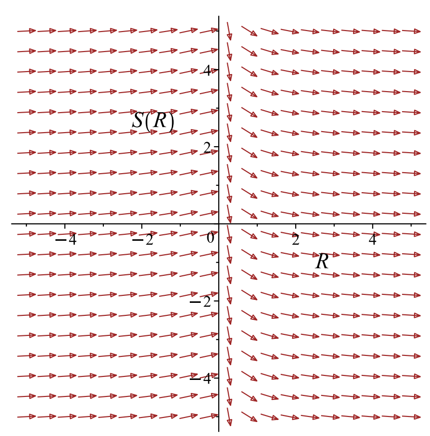
Which simplifies to

$$\mathrm{e}^{\frac{3}{7t}} x = -\frac{2\,\mathrm{expIntegral}_1\left(-\frac{3}{7t}\right)}{7} + c_1$$

Which gives

$$x = -\frac{\mathrm{e}^{-\frac{3}{7t}}\left(2\,\mathrm{expIntegral}_1\left(-\frac{3}{7t}\right) - 7c_1\right)}{7}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, x coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dx}{dt} = \frac{3x-2t}{7t^2}$

</td><td>$R = t$
$S = \mathrm{e}^{\frac{3}{7t}} x$</td><td>$\frac{dS}{dR} = -\frac{2\,\mathrm{e}^{\frac{3}{7R}}}{7R}$

</td></tr>
</table>

Summary
The solution(s) found are the following

$$x = -\frac{e^{-\frac{3}{7t}} \left(2 \operatorname{expIntegral}_1\left(-\frac{3}{7t}\right) - 7c_1\right)}{7} \tag{1}$$

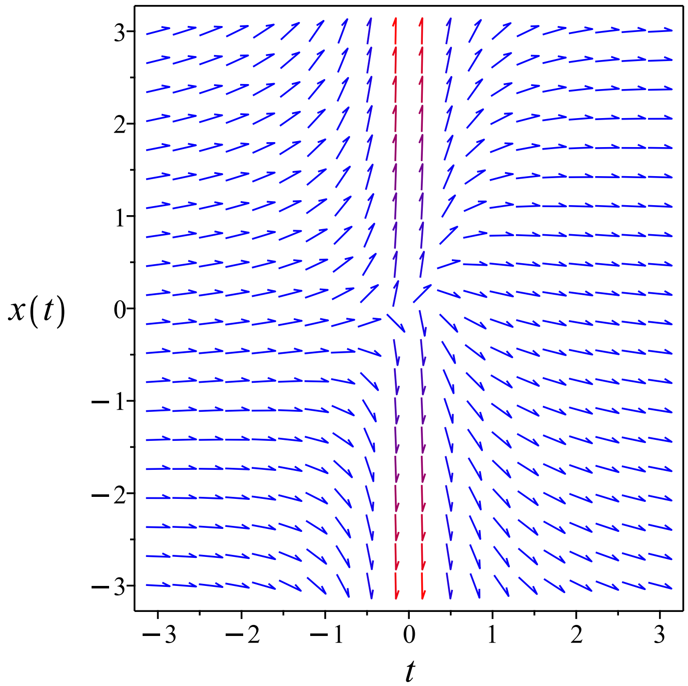


Figure 128: Slope field plot

Verification of solutions

$$x = -\frac{e^{-\frac{3}{7t}} \left(2 \operatorname{expIntegral}_1\left(-\frac{3}{7t}\right) - 7c_1\right)}{7}$$

Verified OK.

5.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(7t^2)\,\mathrm{d}x &= (3x - 2t)\,\mathrm{d}t\\ (-3x + 2t)\,\mathrm{d}t + (7t^2)\,\mathrm{d}x &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x) &= -3x + 2t\\ N(t,x) &= 7t^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-3x+2t) \\
&= -3
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(7t^2) \\
&= 14t
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= \frac{1}{7t^2}((-3) - (14t)) \\
&= \frac{-3-14t}{7t^2}
\end{aligned}
$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}t} \\
&= e^{\int \frac{-3-14t}{7t^2} \,\mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\frac{3}{7t} - 2\ln(t)} \\
&= \frac{\mathrm{e}^{\frac{3}{7t}}}{t^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\mathrm{e}^{\frac{3}{7t}}}{t^2}(-3x+2t) \\
&= \frac{(-3x+2t)\,\mathrm{e}^{\frac{3}{7t}}}{t^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\mathrm{e}^{\frac{3}{7t}}}{t^2}\left(7t^2\right) \\
&= 7\,\mathrm{e}^{\frac{3}{7t}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\
\left(\frac{(-3x+2t)\,\mathrm{e}^{\frac{3}{7t}}}{t^2}\right) + \left(7\,\mathrm{e}^{\frac{3}{7t}}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \frac{(-3x+2t)\,\mathrm{e}^{\frac{3}{7t}}}{t^2}\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = 2\,\mathrm{expIntegral}_1\left(-\frac{3}{7t}\right) + 7\,\mathrm{e}^{\frac{3}{7t}}x + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 7\,\mathrm{e}^{\frac{3}{7t}} + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = 7\,\mathrm{e}^{\frac{3}{7t}}$. Therefore equation (4) becomes

$$
7\,\mathrm{e}^{\frac{3}{7t}} = 7\,\mathrm{e}^{\frac{3}{7t}} + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = 2\,\text{expIntegral}_1\left(-\frac{3}{7t}\right) + 7\,\mathrm{e}^{\frac{3}{7t}} x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2\,\text{expIntegral}_1\left(-\frac{3}{7t}\right) + 7\,\mathrm{e}^{\frac{3}{7t}} x$$

The solution becomes

$$x = -\frac{\left(2\,\text{expIntegral}_1\left(-\frac{3}{7t}\right) - c_1\right)\mathrm{e}^{-\frac{3}{7t}}}{7}$$

<u>Summary</u>
The solution(s) found are the following

$$x = -\frac{\left(2\,\text{expIntegral}_1\left(-\frac{3}{7t}\right) - c_1\right)\mathrm{e}^{-\frac{3}{7t}}}{7} \tag{1}$$

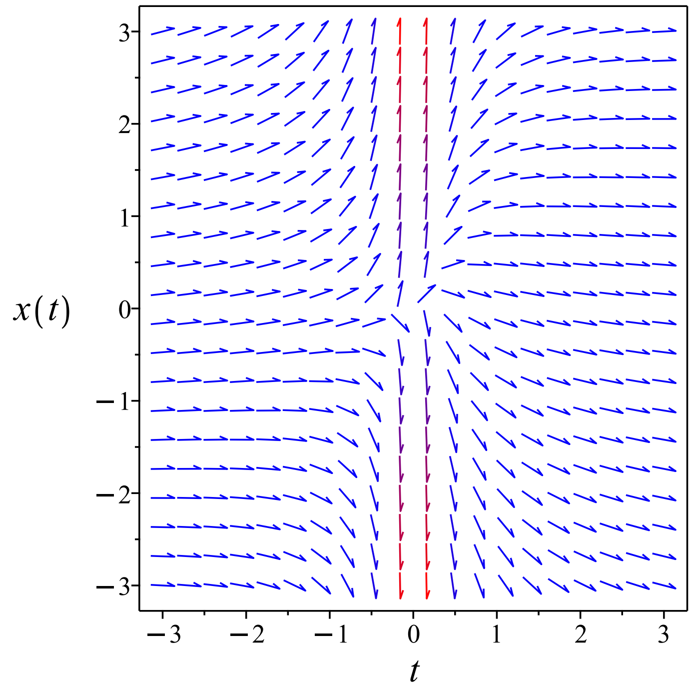


Figure 129: Slope field plot

Verification of solutions

$$x = -\frac{\left(2 \operatorname{expIntegral}_1\left(-\frac{3}{7t}\right) - c_1\right) e^{-\frac{3}{7t}}}{7}$$

Verified OK.

5.4.4 Maple step by step solution

Let's solve

$7t^2 x' - 3x = -2t$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = \frac{3x}{7t^2} - \frac{2}{7t}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' - \frac{3x}{7t^2} = -\frac{2}{7t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$$\mu(t)\left(x' - \frac{3x}{7t^2}\right) = -\frac{2\mu(t)}{7t}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

$$\mu(t)\left(x' - \frac{3x}{7t^2}\right) = \mu'(t)\,x + \mu(t)\,x'$$

- Isolate $\mu'(t)$

$$\mu'(t) = -\frac{3\mu(t)}{7t^2}$$

- Solve to find the integrating factor

$$\mu(t) = e^{\frac{3}{7t}}$$

- Integrate both sides with respect to t

$$\int \left(\frac{d}{dt}(\mu(t)\,x)\right) dt = \int -\frac{2\mu(t)}{7t} dt + c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\,x = \int -\frac{2\mu(t)}{7t} dt + c_1$$

- Solve for x

$$x = \frac{\int -\frac{2\mu(t)}{7t} dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = e^{\frac{3}{7t}}$

$$x = \frac{\int -\frac{2\,e^{\frac{3}{7t}}}{7t} dt + c_1}{e^{\frac{3}{7t}}}$$

- Evaluate the integrals on the rhs

$$x = \frac{-\frac{2\,\mathrm{Ei}_1\left(-\frac{3}{7t}\right)}{7} + c_1}{e^{\frac{3}{7t}}}$$

- Simplify

$$x = -\frac{e^{-\frac{3}{7t}}\left(2\,\mathrm{Ei}_1\left(-\frac{3}{7t}\right) - 7c_1\right)}{7}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(7*t^2*diff(x(t),t)=3*x(t)-2*t,x(t), singsol=all)
```

$$x(t) = -\frac{\left(2 \operatorname{expIntegral}_1\left(-\frac{3}{7t}\right) - 7c_1\right) \mathrm{e}^{-\frac{3}{7t}}}{7}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 33

```
DSolve[7*t^2*x'[t]==3*x[t]-2*t,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{7} e^{-\frac{3}{7}/t} \left(2 \operatorname{ExpIntegralEi}\left(\frac{3}{7t}\right) + 7c_1\right)$$

5.5 problem 1(e)

Internal problem ID [11403]
Internal file name [`OUTPUT/10385_Wednesday_May_17_2023_08_10_24_PM_21188209/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 1(e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{x'x + xt = 1}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(x(t)*diff(x(t),t)=1-t*x(t),x(t), singsol=all)
```

$$x(t) = -\frac{\left(2^{\frac{2}{3}}t^2 - 4\,\mathrm{RootOf}\left(\mathrm{AiryBi}\left(\_Z\right)2^{\frac{1}{3}}c_1t + 2^{\frac{1}{3}}t\,\mathrm{AiryAi}\left(\_Z\right) - 2\,\mathrm{AiryBi}\left(1, \_Z\right)c_1 - 2\,\mathrm{AiryAi}\left(1, \_Z\right)\right)\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.399 (sec). Leaf size: 121

```
DSolve[x[t]*x'[t]==1-t*x[t],x[t],t,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\frac{(-1)^{2/3}\sqrt[3]{2}t\,\mathrm{AiryAi}\left(-\frac{1}{2}\sqrt[3]{-\frac{1}{2}}(t^2+2x(t))\right) - 2\,\mathrm{AiryAiPrime}\left(-\frac{1}{2}\sqrt[3]{-\frac{1}{2}}(t^2+2x(t))\right)}{(-1)^{2/3}\sqrt[3]{2}t\,\mathrm{AiryBi}\left(-\frac{1}{2}\sqrt[3]{-\frac{1}{2}}(t^2+2x(t))\right) - 2\,\mathrm{AiryBiPrime}\left(-\frac{1}{2}\sqrt[3]{-\frac{1}{2}}(t^2+2x(t))\right)} + c_1 = 0, x(t)\right]$$

5.6 problem 1(f)

Internal problem ID [11404]
Internal file name [`OUTPUT/10386_Wednesday_May_17_2023_08_10_25_PM_15906252/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 1(f).
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{x'^2 + xt = \sqrt{1+t}}$$

Solving the given ode for x' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$x' = \sqrt{-xt + \sqrt{1+t}} \tag{1}$$

$$x' = -\sqrt{-xt + \sqrt{1+t}} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Unable to determine ODE type.

Unable to determine ODE type.

Solving equation (2)

Unable to determine ODE type.

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
trying simple symmetries for implicit equations
Successful isolation of dx/dt: 2 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 2
-> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
-> Calling odsolve with the ODE`, diff(y(x), x) = -2*y(x)^2*x^2*(1+(-4*y(x)*x^2+4*y(x)^2+1)^
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Calling odsolve with the ODE`, diff(y(x), x) = -4*y(x)*(y(x)+1)^(1/2)*x/(2*x*(y(x)+1)^(1/
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   trying inverse linear
   trying homogeneous types:
   trying Chini
   trying exact
   Looking for potential symmetries
```

✗ Solution by Maple

```
dsolve(diff(x(t),t)^2+t*x(t)=sqrt(1+t),x(t), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x'[t]^2+t*x[t]==Sqrt[1+t],x[t],t,IncludeSingularSolutions -> True]
```

Not solved

5.7 problem 2(a)

5.7.1 Solving as linear ode 552
5.7.2 Solving as first order ode lie symmetry lookup ode 554
5.7.3 Solving as exact ode 558
5.7.4 Maple step by step solution 563

Internal problem ID [11405]
Internal file name [`OUTPUT/10387_Wednesday_May_17_2023_08_10_26_PM_78299328/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 2(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x' + \frac{2x}{t} = t}$$

5.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = \frac{2}{t}$$
$$q(t) = t$$

Hence the ode is

$$x' + \frac{2x}{t} = t$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{t} dt} \\ &= t^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu x) &= (\mu)(t) \\ \frac{d}{dt}(t^2 x) &= (t^2)(t) \\ d(t^2 x) &= t^3 \, dt\end{aligned}$$

Integrating gives

$$\begin{aligned}t^2 x &= \int t^3 \, dt \\ t^2 x &= \frac{t^4}{4} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$x = \frac{t^2}{4} + \frac{c_1}{t^2}$$

Summary

The solution(s) found are the following

$$x = \frac{t^2}{4} + \frac{c_1}{t^2} \tag{1}$$

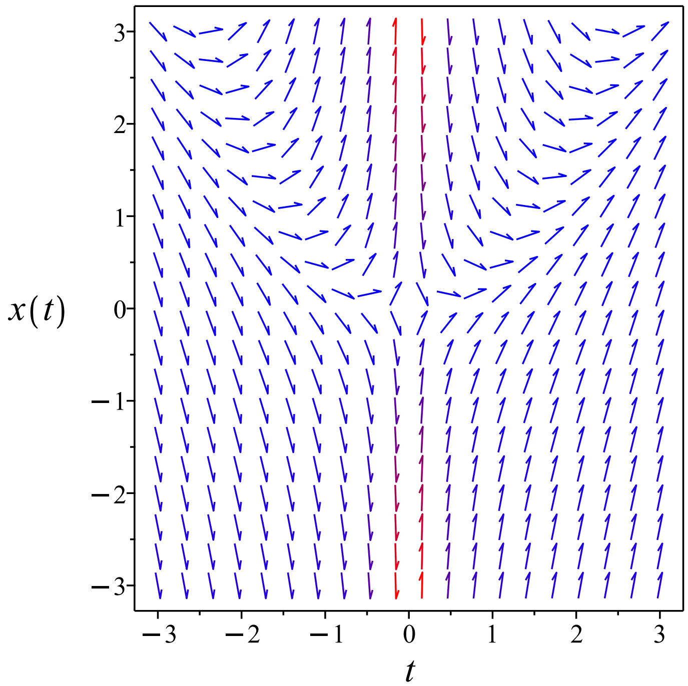


Figure 130: Slope field plot

Verification of solutions

$$x = \frac{t^2}{4} + \frac{c_1}{t^2}$$

Verified OK.

5.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = -\frac{-t^2 + 2x}{t}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 106: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \frac{1}{t^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{t^2}} dy \end{aligned}$$

Which results in

$$S = t^2 x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -\frac{-t^2 + 2x}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= 2tx \\ S_x &= t^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^4}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$xt^2 = \frac{t^4}{4} + c_1$$

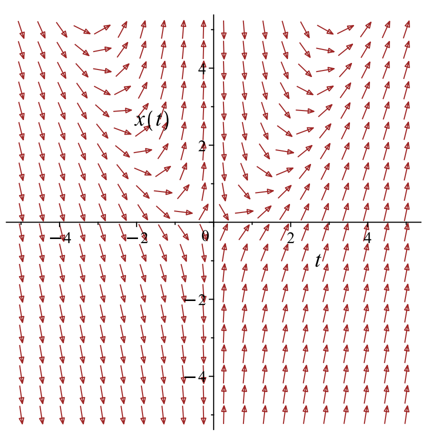
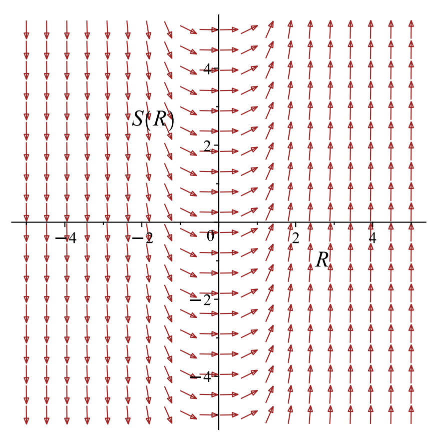
Which simplifies to

$$xt^2 = \frac{t^4}{4} + c_1$$

Which gives

$$x = \frac{t^4 + 4c_1}{4t^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\frac{-t^2+2x}{t}$ 	$R = t$ $S = t^2 x$	$\frac{dS}{dR} = R^3$ 

Summary

The solution(s) found are the following

$$x = \frac{t^4 + 4c_1}{4t^2} \tag{1}$$

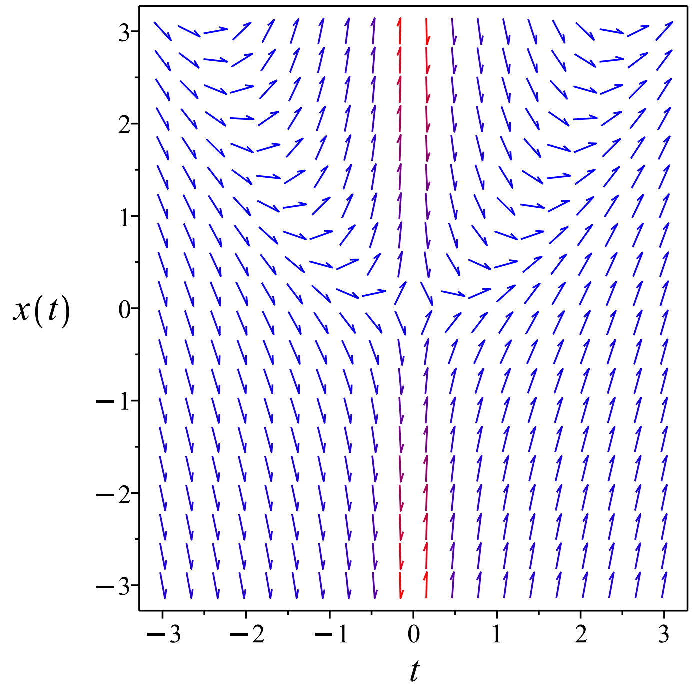


Figure 131: Slope field plot

Verification of solutions

$$x = \frac{t^4 + 4c_1}{4t^2}$$

Verified OK.

5.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\mathrm{d}x = \left(-\frac{2x}{t} + t\right)\mathrm{d}t$$

$$\left(-t + \frac{2x}{t}\right)\mathrm{d}t + \mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, x) &= -t + \frac{2x}{t} \\ N(t, x) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-t + \frac{2x}{t}\right) \\ &= \frac{2}{t}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= 1\left(\left(\frac{2}{t}\right) - (0)\right) \\
&= \frac{2}{t}
\end{aligned}
$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int \frac{2}{t} \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2\ln(t)} \\
&= t^2
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= t^2\left(-t + \frac{2x}{t}\right) \\
&= -t(t^2 - 2x)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= t^2(1) \\
&= t^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$
$$\left(-t\left(t^2 - 2x\right)\right) + \left(t^2\right)\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int -t\left(t^2 - 2x\right)\mathrm{d}t$$
$$\phi = -\frac{\left(t^2 - 2x\right)^2}{4} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = t^2 - 2x + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = t^2$. Therefore equation (4) becomes

$$t^2 = t^2 - 2x + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 2x$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int (2x)\,\mathrm{d}x$$
$$f(x) = x^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{(t^2 - 2x)^2}{4} + x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(t^2 - 2x)^2}{4} + x^2$$

The solution becomes

$$x = \frac{t^4 + 4c_1}{4t^2}$$

Summary

The solution(s) found are the following

$$x = \frac{t^4 + 4c_1}{4t^2} \tag{1}$$

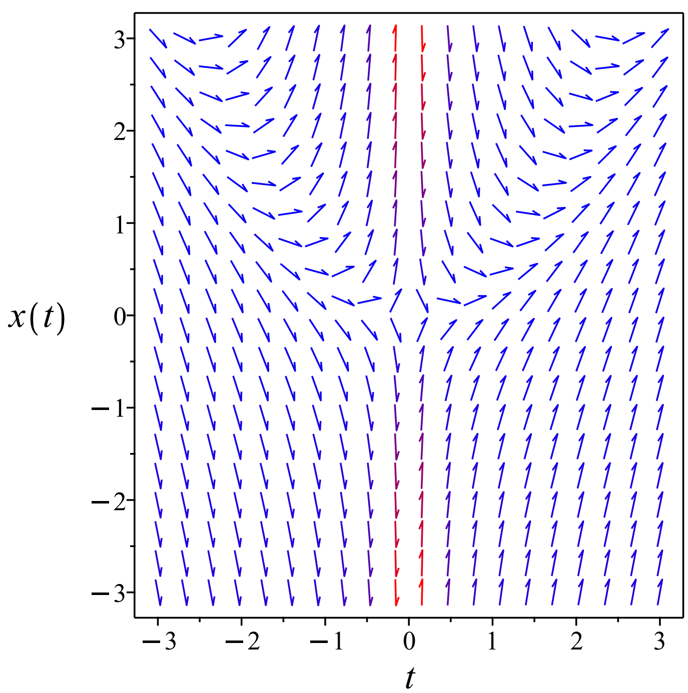


Figure 132: Slope field plot

Verification of solutions

$$x = \frac{t^4 + 4c_1}{4t^2}$$

Verified OK.

5.7.4 Maple step by step solution

Let's solve

$x' + \frac{2x}{t} = t$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -\frac{2x}{t} + t$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + \frac{2x}{t} = t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(x' + \frac{2x}{t}\right) = \mu(t)\, t$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, x)$

  $\mu(t)\left(x' + \frac{2x}{t}\right) = \mu'(t)\, x + \mu(t)\, x'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{2\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = t^2$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, x)\right) dt = \int \mu(t)\, t dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, x = \int \mu(t)\, t dt + c_1$

- Solve for x

  $x = \frac{\int \mu(t) t dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = t^2$

$x = \frac{\int t^3 dt + c_1}{t^2}$

- Evaluate the integrals on the rhs

$x = \frac{\frac{t^4}{4} + c_1}{t^2}$

- Simplify

$x = \frac{t^4 + 4c_1}{4t^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(x(t),t)=-(2/t)*x(t)+t,x(t), singsol=all)
```

$$x(t) = \frac{t^4 + 4c_1}{4t^2}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 19

```
DSolve[x'[t]==-(2/t)*x[t]+t,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{t^2}{4} + \frac{c_1}{t^2}$$

5.8 problem 2(b)

5.8.1 Solving as linear ode 565
5.8.2 Solving as first order ode lie symmetry lookup ode 567
5.8.3 Solving as exact ode 571
5.8.4 Maple step by step solution 575

Internal problem ID [11406]
Internal file name [`OUTPUT/10388_Wednesday_May_17_2023_08_10_28_PM_90718951/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 2(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = e^t}$$

5.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 1$$
$$q(t) = e^t$$

Hence the ode is

$$y' + y = e^t$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 1dt} \\ &= \mathrm{e}^{t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\mathrm{e}^{t}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{t} y\right) &= \left(\mathrm{e}^{t}\right)\left(\mathrm{e}^{t}\right) \\ \mathrm{d}\left(\mathrm{e}^{t} y\right) &= \mathrm{e}^{2t}\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{t} y &= \int \mathrm{e}^{2t}\,\mathrm{d}t \\ \mathrm{e}^{t} y &= \frac{\mathrm{e}^{2t}}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{t}$ results in

$$y = \frac{\mathrm{e}^{-t}\mathrm{e}^{2t}}{2} + c_1 \mathrm{e}^{-t}$$

which simplifies to

$$y = \frac{\mathrm{e}^{t}}{2} + c_1 \mathrm{e}^{-t}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{t}}{2} + c_1 \mathrm{e}^{-t} \tag{1}$$

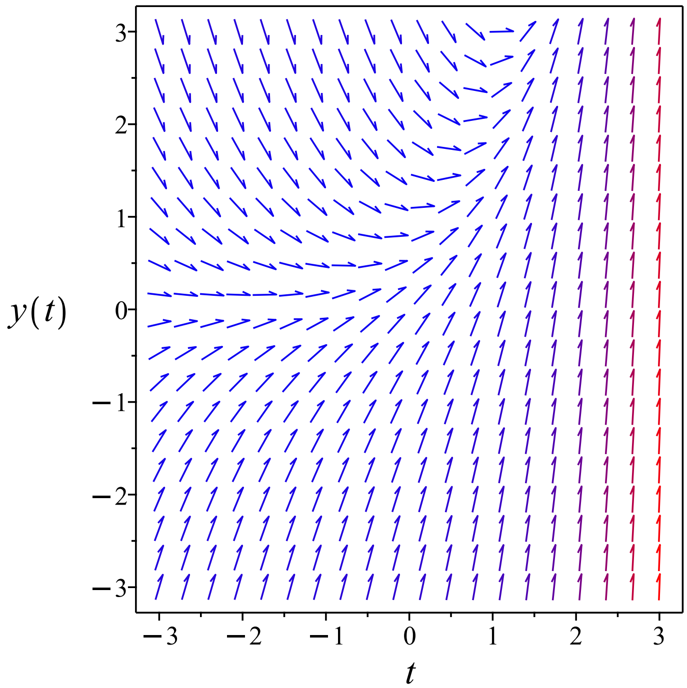


Figure 133: Slope field plot

Verification of solutions

$$y = \frac{e^t}{2} + c_1 e^{-t}$$

Verified OK.

5.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -y + e^t \\ y' &= \omega(t, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 109: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= 0 \\
\eta(t,y) &= \mathrm{e}^{-t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-t}} dy \end{aligned}$$

Which results in

$$S = e^t y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -y + e^t$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= e^t y \\ S_y &= e^t \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{2t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{2R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^t y = \frac{e^{2t}}{2} + c_1$$

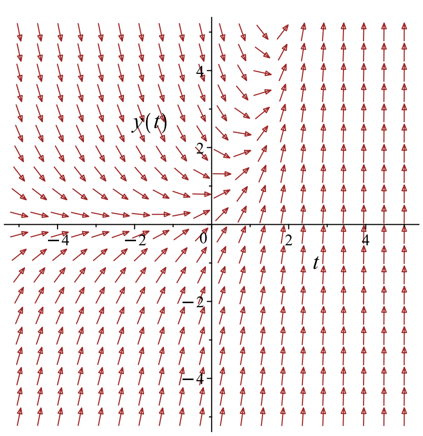
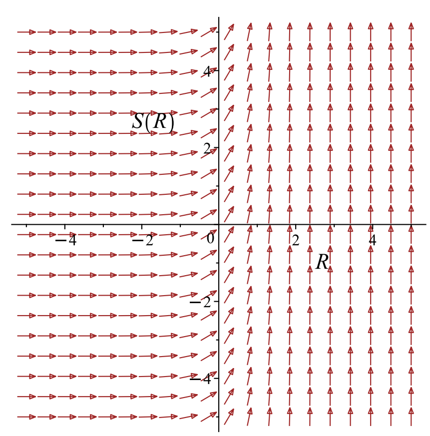
Which simplifies to

$$e^t y = \frac{e^{2t}}{2} + c_1$$

Which gives

$$y = \frac{(e^{2t} + 2c_1) e^{-t}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -y + e^t$ 	$R = t$ $S = e^t y$	$\frac{dS}{dR} = e^{2R}$ 

Summary

The solution(s) found are the following

$$y = \frac{(e^{2t} + 2c_1) e^{-t}}{2} \tag{1}$$

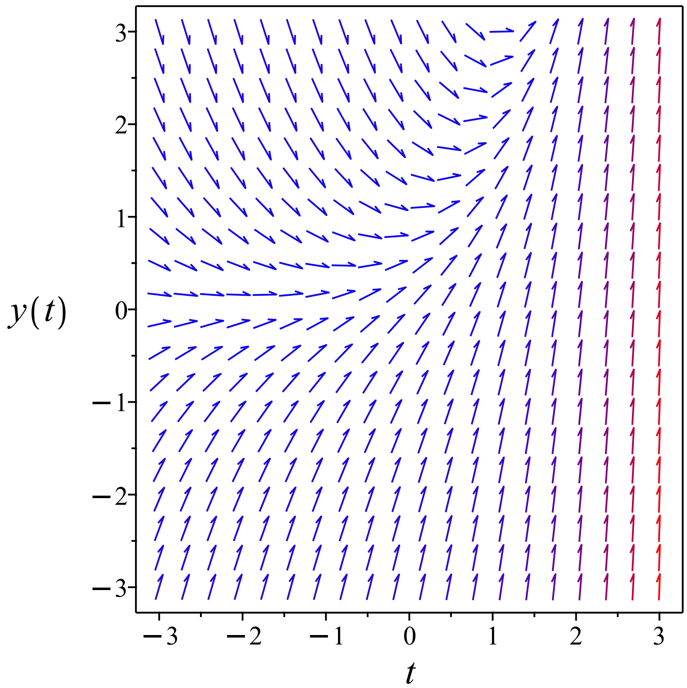


Figure 134: Slope field plot

Verification of solutions

$$y = \frac{(e^{2t} + 2c_1) e^{-t}}{2}$$

Verified OK.

5.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y)\, \mathrm{d}t + N(t, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(-y + \mathrm{e}^t\right) \mathrm{d}t \\
\left(y - \mathrm{e}^t\right) \mathrm{d}t + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= y - \mathrm{e}^t \\
N(t, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y - \mathrm{e}^t\right) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((1) - (0)) \\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int 1 \, \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{t} \\ &= \mathrm{e}^{t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{t}\left(y - \mathrm{e}^{t}\right) \\ &= \left(y - \mathrm{e}^{t}\right)\mathrm{e}^{t}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{t}(1) \\ &= \mathrm{e}^{t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(\left(y - \mathrm{e}^{t}\right)\mathrm{e}^{t}\right) + \left(\mathrm{e}^{t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \left(y - \mathrm{e}^{t}\right) \mathrm{e}^{t}\,\mathrm{d}t$$

$$\phi = \mathrm{e}^{t} y - \frac{\mathrm{e}^{2t}}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{t}$. Therefore equation (4) becomes

$$\mathrm{e}^{t} = \mathrm{e}^{t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{t} y - \frac{\mathrm{e}^{2t}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{t} y - \frac{\mathrm{e}^{2t}}{2}$$

The solution becomes

$$y = \frac{\left(\mathrm{e}^{2t} + 2c_1\right) \mathrm{e}^{-t}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{(e^{2t} + 2c_1) e^{-t}}{2} \tag{1}$$

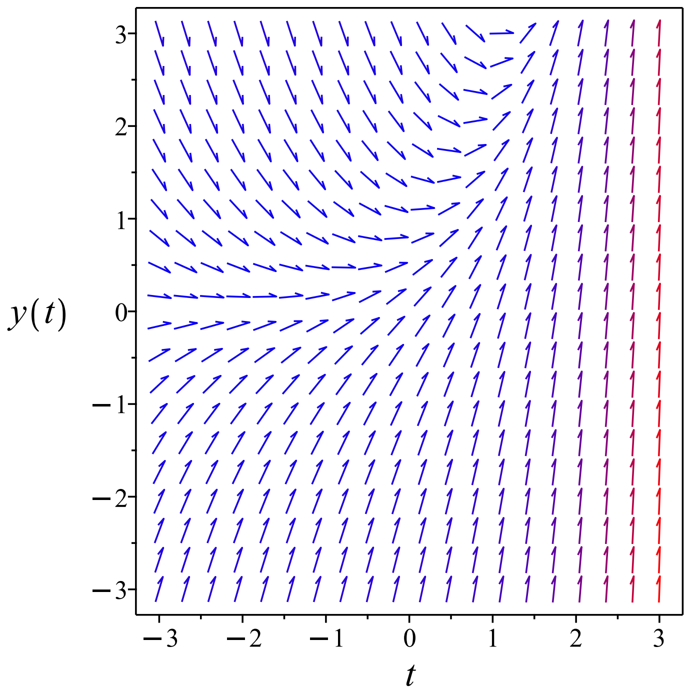


Figure 135: Slope field plot

Verification of solutions

$$y = \frac{(e^{2t} + 2c_1) e^{-t}}{2}$$

Verified OK.

5.8.4 Maple step by step solution

Let's solve

$y' + y = e^t$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -y + e^t$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE
  $y' + y = e^t$
- The ODE is linear; multiply by an integrating factor $\mu(t)$
  $\mu(t)(y' + y) = \mu(t) e^t$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t) y)$
  $\mu(t)(y' + y) = \mu'(t) y + \mu(t) y'$
- Isolate $\mu'(t)$
  $\mu'(t) = \mu(t)$
- Solve to find the integrating factor
  $\mu(t) = e^t$
- Integrate both sides with respect to t
  $\int \left(\frac{d}{dt}(\mu(t) y)\right) dt = \int \mu(t) e^t dt + c_1$
- Evaluate the integral on the lhs
  $\mu(t) y = \int \mu(t) e^t dt + c_1$
- Solve for y
  $y = \frac{\int \mu(t) e^t dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = e^t$
  $y = \frac{\int (e^t)^2 dt + c_1}{e^t}$
- Evaluate the integrals on the rhs
  $y = \frac{\frac{(e^t)^2}{2} + c_1}{e^t}$
- Simplify
  $y = \frac{e^t}{2} + c_1 e^{-t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(t),t)+y(t)=exp(t),y(t), singsol=all)
```

$$y(t) = \frac{e^{t}}{2} + e^{-t} c_1$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 21

```
DSolve[y'[t]+y[t]==Exp[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{e^t}{2} + c_1 e^{-t}$$

5.9 problem 2(c)

5.9.1 Solving as linear ode 578
5.9.2 Solving as first order ode lie symmetry lookup ode 580
5.9.3 Solving as exact ode 584
5.9.4 Maple step by step solution 589

Internal problem ID [11407]
Internal file name [`OUTPUT/10389_Wednesday_May_17_2023_08_10_29_PM_3027235/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 2(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x' + 2xt = e^{-t^2}}$$

5.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = 2t$$
$$q(t) = e^{-t^2}$$

Hence the ode is

$$x' + 2xt = e^{-t^2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 2t dt} \\
&= \mathrm{e}^{t^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(\mathrm{e}^{-t^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{t^2} x\right) &= \left(\mathrm{e}^{t^2}\right)\left(\mathrm{e}^{-t^2}\right) \\
\mathrm{d}\left(\mathrm{e}^{t^2} x\right) &= \mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{t^2} x &= \int \mathrm{d}t \\
\mathrm{e}^{t^2} x &= t + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{t^2}$ results in

$$
x = \mathrm{e}^{-t^2} t + c_1 \mathrm{e}^{-t^2}
$$

which simplifies to

$$
x = \mathrm{e}^{-t^2}(t + c_1)
$$

Summary

The solution(s) found are the following

$$
x = \mathrm{e}^{-t^2}(t + c_1) \tag{1}
$$

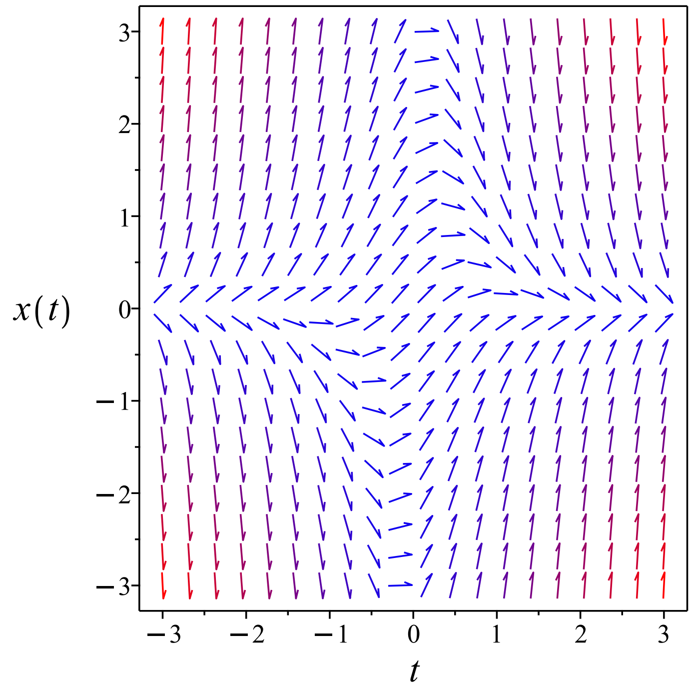


Figure 136: Slope field plot

Verification of solutions

$$x = e^{-t^2}(t + c_1)$$

Verified OK.

5.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = -2tx + e^{-t^2}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 112: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \mathrm{e}^{-t^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-t^2}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{t^2} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = -2tx + \mathrm{e}^{-t^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= 2t\,\mathrm{e}^{t^2} x \\ S_x &= \mathrm{e}^{t^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$x\,\mathrm{e}^{t^2} = t + c_1$$

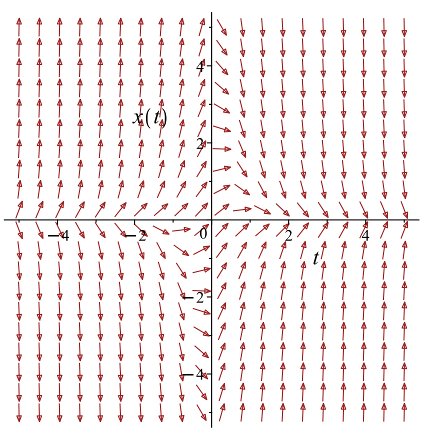
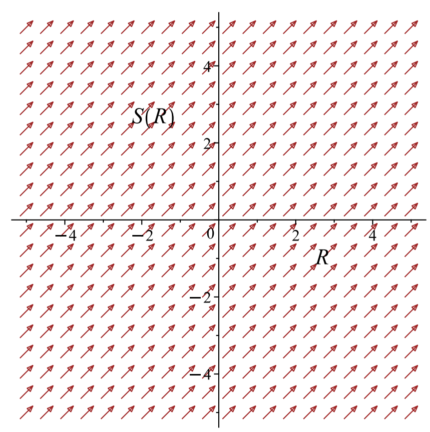
Which simplifies to

$$x\,\mathrm{e}^{t^2} = t + c_1$$

Which gives

$$x = \mathrm{e}^{-t^2}(t + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -2tx + \mathrm{e}^{-t^2}$ 	$R = t$ $S = \mathrm{e}^{t^2} x$	$\frac{dS}{dR} = 1$ 

<u>Summary</u>
The solution(s) found are the following

$$x = \mathrm{e}^{-t^2}(t + c_1) \tag{1}$$

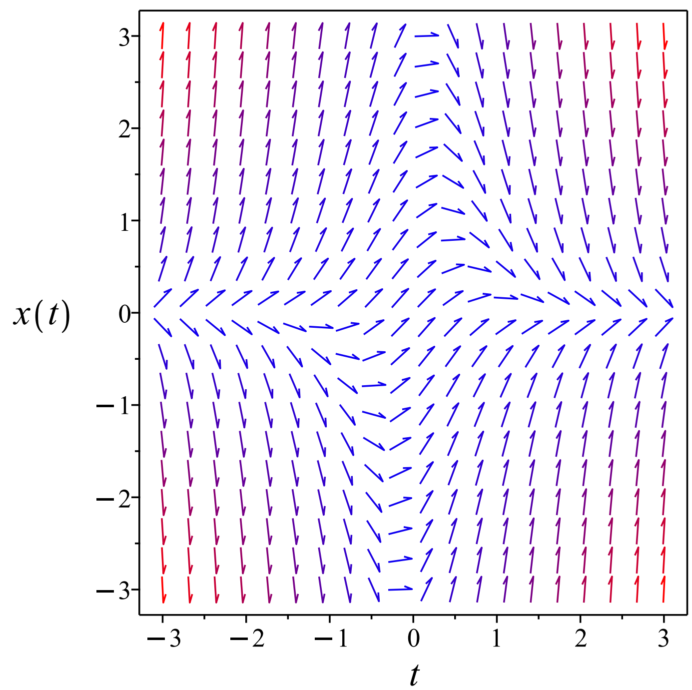


Figure 137: Slope field plot

Verification of solutions

$$x = e^{-t^2}(t + c_1)$$

Verified OK.

5.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}x &= \left(-2tx + \mathrm{e}^{-t^2}\right)\mathrm{d}t \\ \left(2tx - \mathrm{e}^{-t^2}\right)\mathrm{d}t + \mathrm{d}x &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x) &= 2tx - \mathrm{e}^{-t^2} \\ N(t,x) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(2tx - \mathrm{e}^{-t^2}\right) \\ &= 2t\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= 1((2t) - (0)) \\
&= 2t
\end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int 2t \, \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{t^2} \\
&= \mathrm{e}^{t^2}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{t^2}\left(2tx - \mathrm{e}^{-t^2}\right) \\
&= 2t\,\mathrm{e}^{t^2} x - 1
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{t^2}(1) \\
&= \mathrm{e}^{t^2}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\
\left(2t\,\mathrm{e}^{t^2} x - 1\right) + \left(\mathrm{e}^{t^2}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$
$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int 2t\,\mathrm{e}^{t^2}x - 1\,\mathrm{d}t$$
$$\phi = -t + \mathrm{e}^{t^2}x + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial\phi}{\partial x} = \mathrm{e}^{t^2} + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial x} = \mathrm{e}^{t^2}$. Therefore equation (4) becomes

$$\mathrm{e}^{t^2} = \mathrm{e}^{t^2} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = -t + \mathrm{e}^{t^2}x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -t + \mathrm{e}^{t^2}x$$

The solution becomes

$$x = e^{-t^2}(t + c_1)$$

Summary

The solution(s) found are the following

$$x = e^{-t^2}(t + c_1) \tag{1}$$

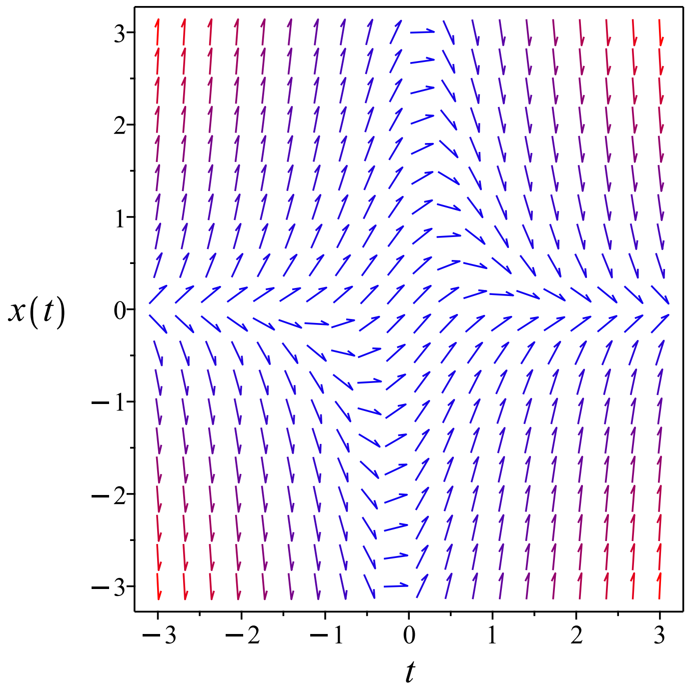


Figure 138: Slope field plot

Verification of solutions

$$x = e^{-t^2}(t + c_1)$$

Verified OK.

5.9.4 Maple step by step solution

Let's solve

$x' + 2xt = \mathrm{e}^{-t^2}$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -2xt + \mathrm{e}^{-t^2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + 2xt = \mathrm{e}^{-t^2}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(x' + 2xt) = \mu(t)\,\mathrm{e}^{-t^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

  $\mu(t)\,(x' + 2xt) = \mu'(t)\,x + \mu(t)\,x'$

- Isolate $\mu'(t)$

  $\mu'(t) = 2\mu(t)\,t$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{t^2}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,x)\right) dt = \int \mu(t)\,\mathrm{e}^{-t^2} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,x = \int \mu(t)\,\mathrm{e}^{-t^2} dt + c_1$

- Solve for x

  $x = \frac{\int \mu(t)\mathrm{e}^{-t^2} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{t^2}$

  $x = \frac{\int \mathrm{e}^{-t^2}\mathrm{e}^{t^2} dt + c_1}{\mathrm{e}^{t^2}}$

- Evaluate the integrals on the rhs

  $x = \frac{t + c_1}{\mathrm{e}^{t^2}}$

- Simplify

$$x = e^{-t^2}(t + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(x(t),t)+2*t*x(t)=exp(-t^2),x(t), singsol=all)
```

$$x(t) = (t + c_1)\,e^{-t^2}$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 17

```
DSolve[x'[t]+2*t*x[t]==Exp[-t^2],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-t^2}(t + c_1)$$

5.10 problem 2(d)

5.10.1 Solving as linear ode 591
5.10.2 Solving as differentialType ode 593
5.10.3 Solving as first order ode lie symmetry lookup ode 595
5.10.4 Solving as exact ode 599
5.10.5 Maple step by step solution 603

Internal problem ID [11408]
Internal file name [`OUTPUT/10390_Wednesday_May_17_2023_08_10_30_PM_3227727/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 2(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{tx' + x = t^2}$$

5.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = \frac{1}{t}$$
$$q(t) = t$$

Hence the ode is

$$x' + \frac{x}{t} = t$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{t}dt} \\ &= t\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu x) &= (\mu)(t) \\ \frac{d}{dt}(tx) &= (t)(t) \\ d(tx) &= t^2\, dt\end{aligned}$$

Integrating gives

$$\begin{aligned}tx &= \int t^2\, dt \\ tx &= \frac{t^3}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$x = \frac{t^2}{3} + \frac{c_1}{t}$$

Summary

The solution(s) found are the following

$$x = \frac{t^2}{3} + \frac{c_1}{t} \tag{1}$$

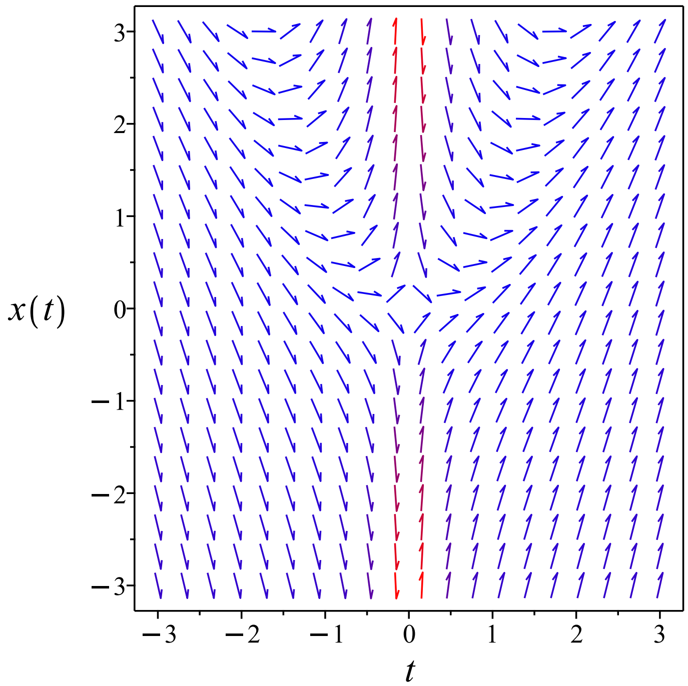


Figure 139: Slope field plot

Verification of solutions

$$x = \frac{t^2}{3} + \frac{c_1}{t}$$

Verified OK.

5.10.2 Solving as differentialType ode

Writing the ode as

$$x' = \frac{-x + t^2}{t} \tag{1}$$

Which becomes

$$0 = (-t)\, dx + (t^2 - x)\, dt \tag{2}$$

But the RHS is complete differential because

$$(-t)\, dx + (t^2 - x)\, dt = d\left(\frac{1}{3}t^3 - tx\right)$$

Hence (2) becomes

$$0 = d\left(\frac{1}{3}t^3 - tx\right)$$

Integrating both sides gives gives these solutions

$$x = \frac{t^3 + 3c_1}{3t} + c_1$$

Summary

The solution(s) found are the following

$$x = \frac{t^3 + 3c_1}{3t} + c_1 \tag{1}$$

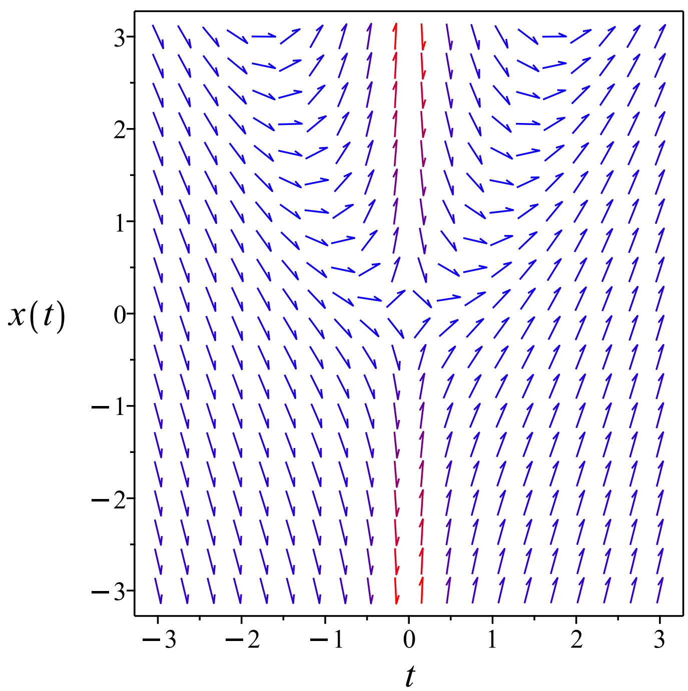


Figure 140: Slope field plot

Verification of solutions

$$x = \frac{t^3 + 3c_1}{3t} + c_1$$

Verified OK.

5.10.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= -\frac{-t^2 + x}{t} \\
x' &= \omega(t, x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 115: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \frac{1}{t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{t}} dy
\end{aligned}
$$

Which results in

$$
S = tx
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}
$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$
\omega(t,x) = -\frac{-t^2 + x}{t}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_x &= 0 \\
S_t &= x \\
S_x &= t
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^3}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$xt = \frac{t^3}{3} + c_1$$

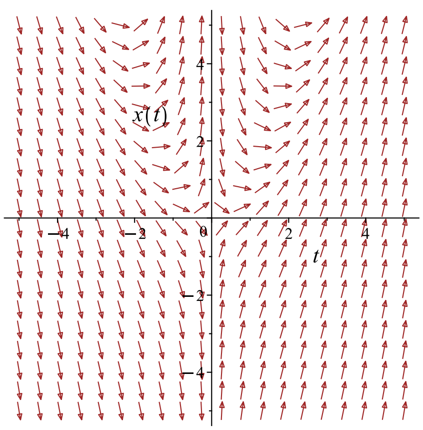
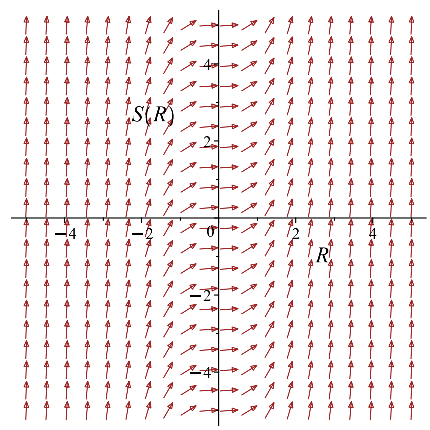
Which simplifies to

$$xt = \frac{t^3}{3} + c_1$$

Which gives

$$x = \frac{t^3 + 3c_1}{3t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\frac{-t^2+x}{t}$ 	$R = t$ $S = tx$	$\frac{dS}{dR} = R^2$ 

Summary

The solution(s) found are the following

$$x = \frac{t^3 + 3c_1}{3t} \tag{1}$$

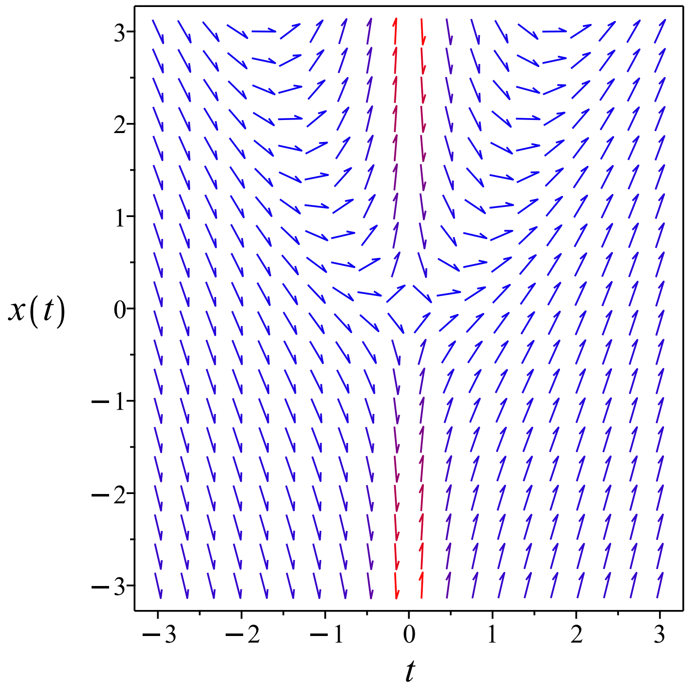


Figure 141: Slope field plot

Verification of solutions

$$x = \frac{t^3 + 3c_1}{3t}$$

Verified OK.

5.10.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(t)\,\mathrm{d}x &= \left(t^2 - x\right)\mathrm{d}t \\
\left(-t^2 + x\right)\mathrm{d}t + (t)\,\mathrm{d}x &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, x) &= -t^2 + x \\
N(t, x) &= t
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-t^2 + x\right) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -t^2 + x \, \mathrm{d}t$$
$$\phi = -\frac{1}{3}t^3 + tx + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = t + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = t$. Therefore equation (4) becomes

$$t = t + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{3}t^3 + tx + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{3}t^3 + tx$$

The solution becomes

$$x = \frac{t^3 + 3c_1}{3t}$$

Summary

The solution(s) found are the following

$$x = \frac{t^3 + 3c_1}{3t} \tag{1}$$

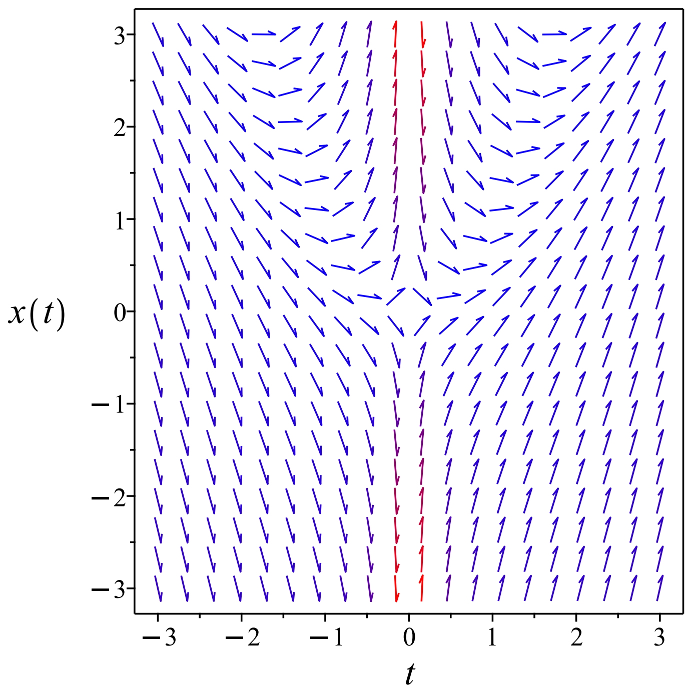


Figure 142: Slope field plot

Verification of solutions

$$x = \frac{t^3 + 3c_1}{3t}$$

Verified OK.

5.10.5 Maple step by step solution

Let's solve

$tx' + x = t^2$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -\frac{x}{t} + t$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + \frac{x}{t} = t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(x' + \frac{x}{t}\right) = \mu(t)\, t$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, x)$

  $\mu(t)\left(x' + \frac{x}{t}\right) = \mu'(t)\, x + \mu(t)\, x'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = t$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, x)\right) dt = \int \mu(t)\, t dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, x = \int \mu(t)\, t dt + c_1$

- Solve for x

  $x = \frac{\int \mu(t) t dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = t$

  $x = \frac{\int t^2 dt + c_1}{t}$

- Evaluate the integrals on the rhs

  $x = \frac{\frac{t^3}{3} + c_1}{t}$

- Simplify

$$x = \frac{t^3+3c_1}{3t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(t*diff(x(t),t)=-x(t)+t^2,x(t), singsol=all)
```

$$x(t) = \frac{t^3+3c_1}{3t}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 19

```
DSolve[t*x'[t]==-x[t]+t^2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{t^2}{3} + \frac{c_1}{t}$$

5.11 problem 2(e)

5.11.1 Solving as linear ode 605
5.11.2 Solving as first order ode lie symmetry lookup ode 607
5.11.3 Solving as exact ode 610
5.11.4 Maple step by step solution 613

Internal problem ID [11409]
Internal file name [`OUTPUT/10391_Wednesday_May_17_2023_08_10_31_PM_76859408/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 2(e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{\theta' + a\theta = e^{bt}}$$

5.11.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\theta' + p(t)\theta = q(t)$$

Where here

$$p(t) = a$$
$$q(t) = e^{bt}$$

Hence the ode is

$$\theta' + a\theta = e^{bt}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int a dt} \\ &= e^{ta}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu\theta) &= (\mu)\left(e^{bt}\right) \\ \frac{d}{dt}\left(e^{ta}\theta\right) &= \left(e^{ta}\right)\left(e^{bt}\right) \\ d\left(e^{ta}\theta\right) &= e^{t(a+b)}\, dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{ta}\theta &= \int e^{t(a+b)}\, dt \\ e^{ta}\theta &= \frac{e^{t(a+b)}}{a+b} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{ta}$ results in

$$\theta = \frac{e^{-ta}e^{t(a+b)}}{a+b} + c_1 e^{-ta}$$

which simplifies to

$$\theta = \frac{c_1(a+b)\, e^{-ta} + e^{bt}}{a+b}$$

Summary

The solution(s) found are the following

$$\theta = \frac{c_1(a+b)\, e^{-ta} + e^{bt}}{a+b} \tag{1}$$

Verification of solutions

$$\theta = \frac{c_1(a+b)\, e^{-ta} + e^{bt}}{a+b}$$

Verified OK.

5.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
\theta' &= -a\theta + \mathrm{e}^{bt} \\
\theta' &= \omega(t, \theta)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_\theta - \xi_t) - \omega^2 \xi_\theta - \omega_t \xi - \omega_\theta \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 118: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, \theta) &= 0 \\
\eta(t, \theta) &= \mathrm{e}^{-ta}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, \theta) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{d\theta}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial \theta}\right) S(t, \theta) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{-ta}} dy
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{ta} \theta
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t, \theta) S_\theta}{R_t + \omega(t, \theta) R_\theta} \tag{2}
$$

Where in the above $R_t, R_\theta, S_t, S_\theta$ are all partial derivatives and $\omega(t, \theta)$ is the right hand side of the original ode given by

$$
\omega(t, \theta) = -a\theta + \mathrm{e}^{bt}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_\theta &= 0 \\
S_t &= a\, \mathrm{e}^{ta} \theta \\
S_\theta &= \mathrm{e}^{ta}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{t(a+b)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, θ in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{R(a+b)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{R(a+b)}}{a+b} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, θ coordinates. This results in

$$e^{ta}\theta = \frac{e^{t(a+b)}}{a+b} + c_1$$

Which simplifies to

$$e^{ta}\theta = \frac{e^{t(a+b)}}{a+b} + c_1$$

Which gives

$$\theta = \frac{\left(c_1 a + c_1 b + e^{t(a+b)}\right) e^{-ta}}{a+b}$$

Summary

The solution(s) found are the following

$$\theta = \frac{\left(c_1 a + c_1 b + e^{t(a+b)}\right) e^{-ta}}{a+b} \tag{1}$$

Verification of solutions

$$\theta = \frac{\left(c_1 a + c_1 b + e^{t(a+b)}\right) e^{-ta}}{a+b}$$

Verified OK.

5.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,\theta)\,\mathrm{d}t+N(t,\theta)\,\mathrm{d}\theta=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}\theta&=\left(-a\theta+\mathrm{e}^{bt}\right)\mathrm{d}t\\ \left(a\theta-\mathrm{e}^{bt}\right)\mathrm{d}t+\mathrm{d}\theta&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,\theta)&=a\theta-\mathrm{e}^{bt}\\ N(t,\theta)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial \theta} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(a\theta - \mathrm{e}^{bt}\right) \\ &= a\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial \theta} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial \theta} - \frac{\partial N}{\partial t}\right) \\ &= 1((a) - (0)) \\ &= a\end{aligned}$$

Since A does not depend on θ, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int a \, \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{ta} \\ &= \mathrm{e}^{ta}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{ta}\left(a\theta - \mathrm{e}^{bt}\right) \\ &= \left(a\theta - \mathrm{e}^{bt}\right)\mathrm{e}^{ta}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{ta}(1) \\
&= \mathrm{e}^{ta}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= 0 \\
\left(\left(a\theta - \mathrm{e}^{bt}\right)\mathrm{e}^{ta}\right) + \left(\mathrm{e}^{ta}\right)\frac{\mathrm{d}\theta}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, \theta)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial \theta} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \left(a\theta - \mathrm{e}^{bt}\right)\mathrm{e}^{ta}\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = \frac{-\mathrm{e}^{t(a+b)} + \theta(a+b)\,\mathrm{e}^{ta}}{a+b} + f(\theta) \tag{3}
$$

Where $f(\theta)$ is used for the constant of integration since ϕ is a function of both t and θ. Taking derivative of equation (3) w.r.t θ gives

$$
\frac{\partial \phi}{\partial \theta} = \mathrm{e}^{ta} + f'(\theta) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial \theta} = \mathrm{e}^{ta}$. Therefore equation (4) becomes

$$
\mathrm{e}^{ta} = \mathrm{e}^{ta} + f'(\theta) \tag{5}
$$

Solving equation (5) for $f'(\theta)$ gives

$$
f'(\theta) = 0
$$

Therefore

$$f(\theta) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(\theta)$ into equation (3) gives ϕ

$$\phi = \frac{-\mathrm{e}^{t(a+b)} + \theta(a+b)\,\mathrm{e}^{ta}}{a+b} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-\mathrm{e}^{t(a+b)} + \theta(a+b)\,\mathrm{e}^{ta}}{a+b}$$

The solution becomes

$$\theta = \frac{\left(c_1 a + c_1 b + \mathrm{e}^{t(a+b)}\right) \mathrm{e}^{-ta}}{a+b}$$

<u>Summary</u>
The solution(s) found are the following

$$\theta = \frac{\left(c_1 a + c_1 b + \mathrm{e}^{t(a+b)}\right) \mathrm{e}^{-ta}}{a+b} \tag{1}$$

<u>Verification of solutions</u>

$$\theta = \frac{\left(c_1 a + c_1 b + \mathrm{e}^{t(a+b)}\right) \mathrm{e}^{-ta}}{a+b}$$

Verified OK.

5.11.4 Maple step by step solution

Let's solve

$\theta' + a\theta = \mathrm{e}^{bt}$

- Highest derivative means the order of the ODE is 1

  θ'

- Isolate the derivative

  $\theta' = -a\theta + \mathrm{e}^{bt}$

- Group terms with θ on the lhs of the ODE and the rest on the rhs of the ODE

$\theta' + a\theta = e^{bt}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(\theta' + a\theta) = \mu(t)\, e^{bt}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,\theta)$

  $\mu(t)\,(\theta' + a\theta) = \mu'(t)\,\theta + \mu(t)\,\theta'$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)\, a$

- Solve to find the integrating factor

  $\mu(t) = e^{ta}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,\theta)\right) dt = \int \mu(t)\, e^{bt} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,\theta = \int \mu(t)\, e^{bt} dt + c_1$

- Solve for θ

  $\theta = \frac{\int \mu(t) e^{bt} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = e^{ta}$

  $\theta = \frac{\int e^{bt} e^{ta} dt + c_1}{e^{ta}}$

- Evaluate the integrals on the rhs

  $\theta = \frac{\frac{e^{ta+bt}}{a+b} + c_1}{e^{ta}}$

- Simplify

  $\theta = \frac{(e^{t(a+b)} + (a+b)c_1)\, e^{-ta}}{a+b}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(theta(t),t)=-a*theta(t)+exp(b*t),theta(t), singsol=all)
```

$$\theta(t) = \frac{\left(e^{t(a+b)} + c_1(a+b)\right) e^{-at}}{a+b}$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 31

```
DSolve[theta'[t]==-a*theta[t]+Exp[b*t],theta[t],t,IncludeSingularSolutions -> True]
```

$$\theta(t) \to \frac{e^{-at}\left(e^{t(a+b)} + c_1(a+b)\right)}{a+b}$$

5.12 problem 2(f)

5.12.1 Solving as separable ode 616
5.12.2 Solving as linear ode 618
5.12.3 Solving as first order ode lie symmetry lookup ode 620
5.12.4 Solving as exact ode 624
5.12.5 Maple step by step solution 628

Internal problem ID [11410]
Internal file name [`OUTPUT/10392_Wednesday_May_17_2023_08_10_32_PM_29904459/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 2(f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(t^2+1)\, x' + 3xt = 6t}$$

5.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= f(t)g(x) \\ &= \frac{t(-3x+6)}{t^2+1} \end{aligned}$$

Where $f(t) = \frac{t}{t^2+1}$ and $g(x) = -3x+6$. Integrating both sides gives

$$\frac{1}{-3x+6}\, dx = \frac{t}{t^2+1}\, dt$$

$$\int \frac{1}{-3x+6}\,dx = \int \frac{t}{t^2+1}\,dt$$
$$-\frac{\ln(x-2)}{3} = \frac{\ln(t^2+1)}{2} + c_1$$

Raising both side to exponential gives

$$\frac{1}{(x-2)^{\frac{1}{3}}} = \mathrm{e}^{\frac{\ln(t^2+1)}{2}+c_1}$$

Which simplifies to

$$\frac{1}{(x-2)^{\frac{1}{3}}} = c_2\sqrt{t^2+1}$$

Which simplifies to

$$x = \frac{\left(2c_2^3\mathrm{e}^{3c_1}(t^2+1)^{\frac{3}{2}}+1\right)\mathrm{e}^{-3c_1}}{c_2^3(t^2+1)^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$x = \frac{\left(2c_2^3\mathrm{e}^{3c_1}(t^2+1)^{\frac{3}{2}}+1\right)\mathrm{e}^{-3c_1}}{c_2^3(t^2+1)^{\frac{3}{2}}} \tag{1}$$

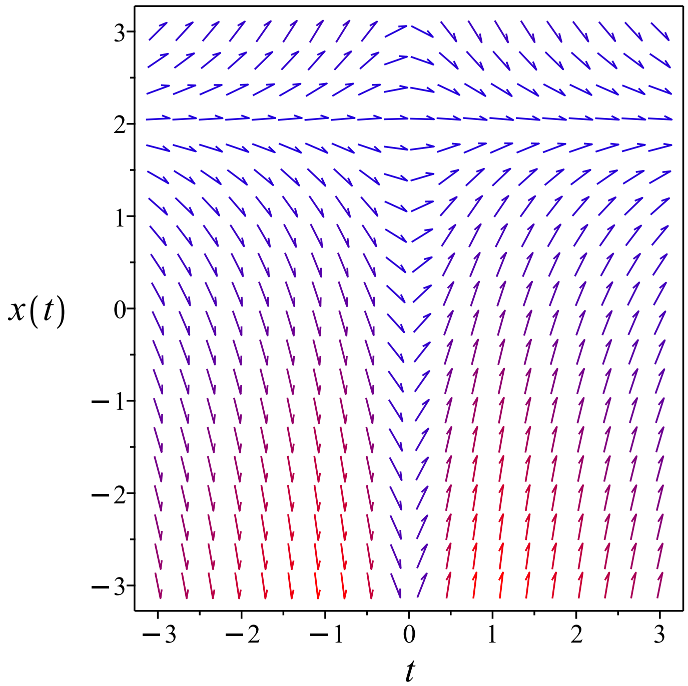


Figure 143: Slope field plot

Verification of solutions

$$x = \frac{\left(2c_2^3 \mathrm{e}^{3c_1}(t^2+1)^{\frac{3}{2}} + 1\right) \mathrm{e}^{-3c_1}}{c_2^3 (t^2+1)^{\frac{3}{2}}}$$

Verified OK.

5.12.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = \frac{3t}{t^2+1}$$
$$q(t) = \frac{6t}{t^2+1}$$

Hence the ode is

$$x' + \frac{3tx}{t^2+1} = \frac{6t}{t^2+1}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{3t}{t^2+1}dt} \\
&= \left(t^2+1\right)^{\frac{3}{2}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(\frac{6t}{t^2+1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(t^2+1\right)^{\frac{3}{2}} x\right) &= \left(\left(t^2+1\right)^{\frac{3}{2}}\right)\left(\frac{6t}{t^2+1}\right) \\
\mathrm{d}\left(\left(t^2+1\right)^{\frac{3}{2}} x\right) &= \left(6t\sqrt{t^2+1}\right)\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\left(t^2+1\right)^{\frac{3}{2}} x &= \int 6t\sqrt{t^2+1}\,\mathrm{d}t \\
\left(t^2+1\right)^{\frac{3}{2}} x &= 2\left(t^2+1\right)^{\frac{3}{2}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \left(t^2+1\right)^{\frac{3}{2}}$ results in

$$
x = 2 + \frac{c_1}{\left(t^2+1\right)^{\frac{3}{2}}}
$$

Summary

The solution(s) found are the following

$$
x = 2 + \frac{c_1}{\left(t^2+1\right)^{\frac{3}{2}}} \tag{1}
$$

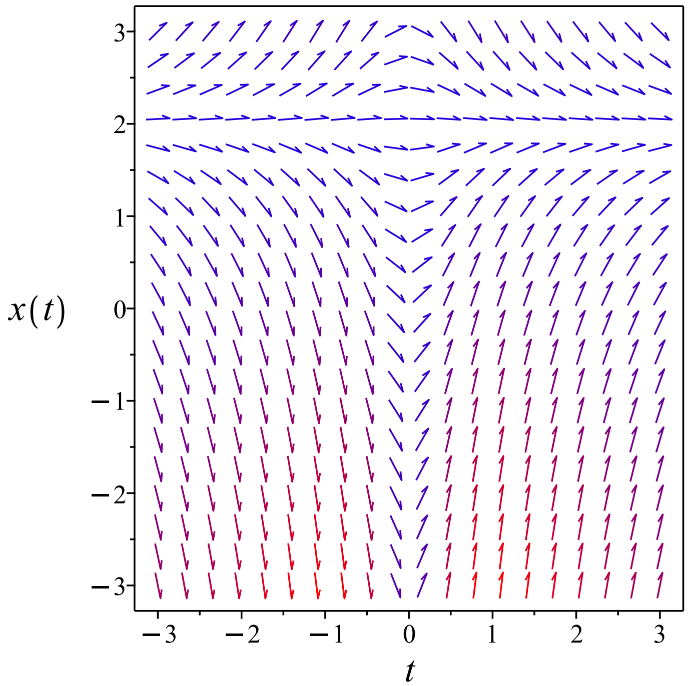


Figure 144: Slope field plot

Verification of solutions

$$x = 2 + \frac{c_1}{(t^2+1)^{\frac{3}{2}}}$$

Verified OK.

5.12.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} x' &= -\frac{3t(x-2)}{t^2+1} \\ x' &= \omega(t,x) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 121: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \frac{1}{(t^2+1)^{\frac{3}{2}}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(t^2+1)^{\frac{3}{2}}}} dy \end{aligned}$$

Which results in

$$S = \left(t^2 + 1\right)^{\frac{3}{2}} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -\frac{3t(x-2)}{t^2+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= 3\sqrt{t^2+1}\, xt \\ S_x &= \left(t^2+1\right)^{\frac{3}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 6t\sqrt{t^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 6R\sqrt{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\left(R^2+1\right)^{\frac{3}{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\left(t^2+1\right)^{\frac{3}{2}} x = 2\left(t^2+1\right)^{\frac{3}{2}} + c_1$$

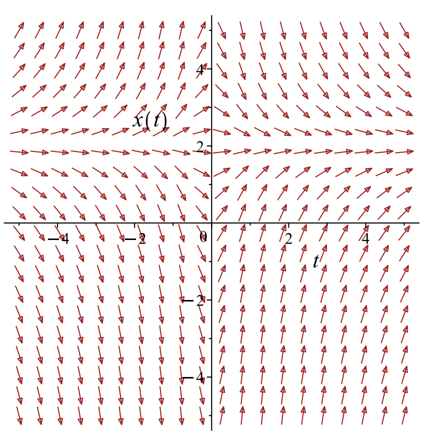
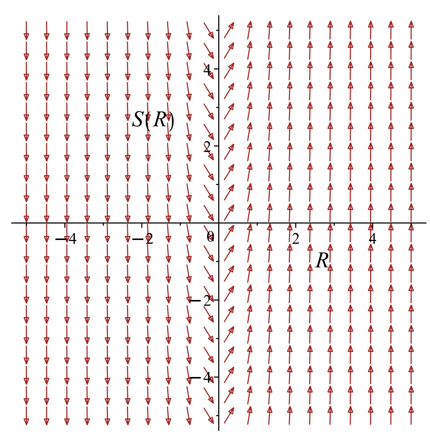
Which simplifies to

$$\left(t^2+1\right)^{\frac{3}{2}} (x-2) - c_1 = 0$$

Which gives

$$x = \frac{2(t^2+1)^{\frac{3}{2}} + c_1}{(t^2+1)^{\frac{3}{2}}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\frac{3t(x-2)}{t^2+1}$ 	$R = t$ $S = \left(t^2+1\right)^{\frac{3}{2}} x$	$\frac{dS}{dR} = 6R\sqrt{R^2+1}$ 

Summary

The solution(s) found are the following

$$x = \frac{2(t^2+1)^{\frac{3}{2}} + c_1}{(t^2+1)^{\frac{3}{2}}} \tag{1}$$

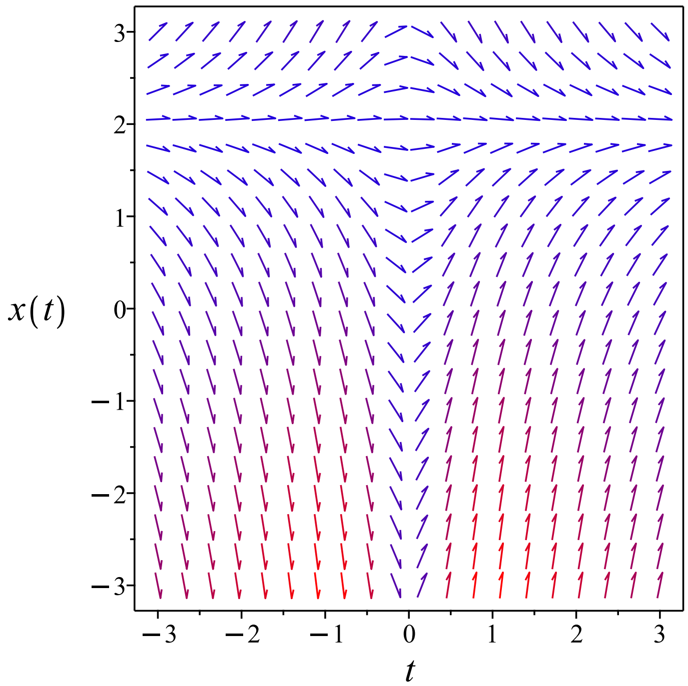


Figure 145: Slope field plot

Verification of solutions

$$x = \frac{2(t^2+1)^{\frac{3}{2}} + c_1}{(t^2+1)^{\frac{3}{2}}}$$

Verified OK.

5.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{-3x+6}\right)\mathrm{d}x&=\left(\frac{t}{t^2+1}\right)\mathrm{d}t\\ \left(-\frac{t}{t^2+1}\right)\mathrm{d}t+\left(\frac{1}{-3x+6}\right)\mathrm{d}x&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x)&=-\frac{t}{t^2+1}\\ N(t,x)&=\frac{1}{-3x+6}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{t}{t^2+1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{-3x+6}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -\frac{t}{t^2+1} \, \mathrm{d}t \\
\phi &= -\frac{\ln(t^2+1)}{2} + f(x) \qquad (3)
\end{aligned}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{-3x+6}$. Therefore equation (4) becomes

$$
\frac{1}{-3x+6} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = -\frac{1}{3(x-2)}
$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-\frac{1}{3x-6}\right) \mathrm{d}x$$

$$f(x) = -\frac{\ln(x-2)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(t^2+1)}{2} - \frac{\ln(x-2)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(t^2+1)}{2} - \frac{\ln(x-2)}{3}$$

The solution becomes

$$x = \mathrm{e}^{-\frac{3\ln(t^2+1)}{2} - 3c_1} + 2$$

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{-\frac{3\ln(t^2+1)}{2} - 3c_1} + 2 \tag{1}$$

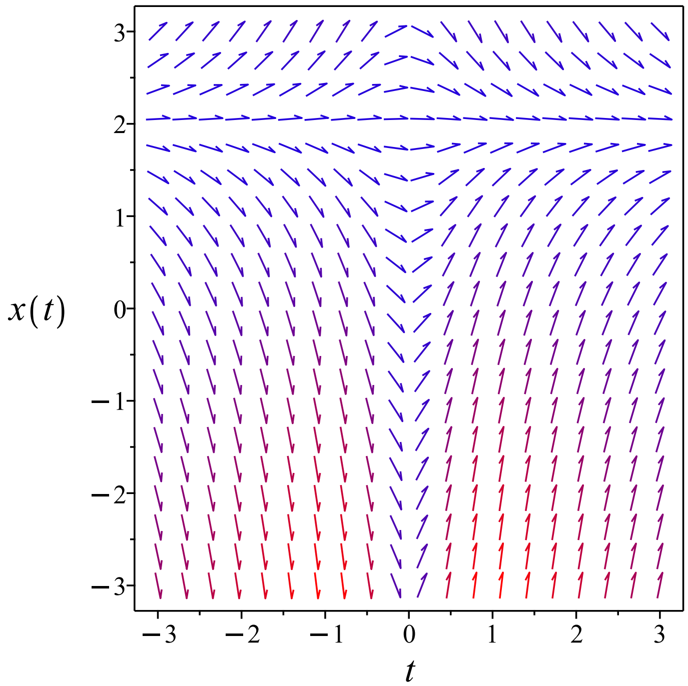


Figure 146: Slope field plot

Verification of solutions

$$x = \mathrm{e}^{-\frac{3\ln\left(t^2+1\right)}{2}-3c_1} + 2$$

Verified OK.

5.12.5 Maple step by step solution

Let's solve

$(t^2+1)\,x' + 3xt = 6t$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x-2} = -\frac{3t}{t^2+1}$

- Integrate both sides with respect to t

  $\int \frac{x'}{x-2}dt = \int -\frac{3t}{t^2+1}dt + c_1$

- Evaluate integral

$$\ln(x-2) = -\frac{3\ln(t^2+1)}{2} + c_1$$

- Solve for x

$$x = e^{-\frac{3\ln(t^2+1)}{2}+c_1} + 2$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((t^2+1)*diff(x(t),t)=-3*t*x(t)+6*t,x(t), singsol=all)
```

$$x(t) = 2 + \frac{c_1}{(t^2+1)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 24

```
DSolve[(t^2+1)*x'[t]==-3*t*x[t]+6*t,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 2 + \frac{c_1}{(t^2+1)^{3/2}}$$
$$x(t) \to 2$$

5.13 problem 3(a)

5.13.1 Existence and uniqueness analysis 630
5.13.2 Solving as linear ode 631
5.13.3 Solving as first order ode lie symmetry lookup ode 633
5.13.4 Solving as exact ode 637
5.13.5 Maple step by step solution 642

Internal problem ID [11411]
Internal file name [`OUTPUT/10393_Wednesday_May_17_2023_08_10_34_PM_59685877/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 3(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x' + \frac{5x}{t} = 1 + t}$$

With initial conditions

$$[x(1) = 1]$$

5.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = \frac{5}{t}$$
$$q(t) = 1 + t$$

Hence the ode is

$$x' + \frac{5x}{t} = 1 + t$$

The domain of $p(t) = \frac{5}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = 1 + t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

5.13.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{5}{t}dt} \\ &= t^5\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu x) &= (\mu)(1+t) \\ \frac{d}{dt}(t^5 x) &= (t^5)(1+t) \\ d(t^5 x) &= ((1+t)\,t^5)\,dt\end{aligned}$$

Integrating gives

$$\begin{aligned}t^5 x &= \int (1+t)\,t^5\,dt \\ t^5 x &= \frac{1}{7}t^7 + \frac{1}{6}t^6 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^5$ results in

$$x = \frac{\frac{1}{7}t^7 + \frac{1}{6}t^6}{t^5} + \frac{c_1}{t^5}$$

which simplifies to

$$x = \frac{6t^7 + 7t^6 + 42c_1}{42t^5}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 + \frac{13}{42}$$

$$c_1 = \frac{29}{42}$$

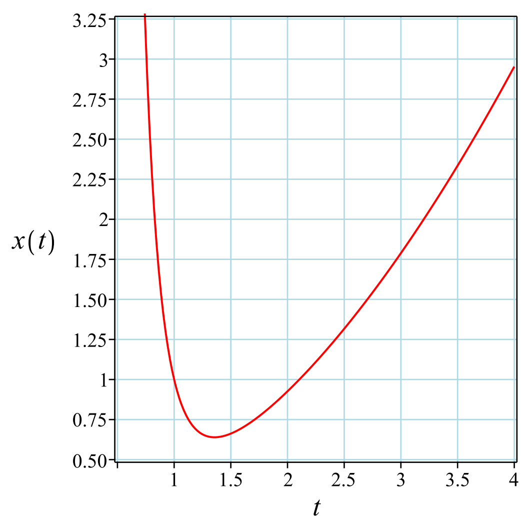
Substituting c_1 found above in the general solution gives

$$x = \frac{6t^7 + 7t^6 + 29}{42t^5}$$

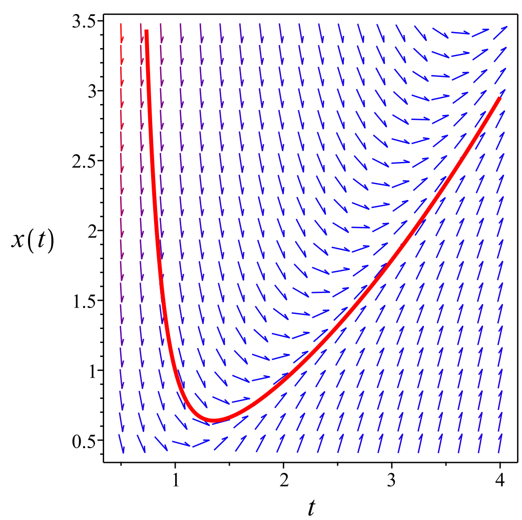
Summary

The solution(s) found are the following

$$x = \frac{6t^7 + 7t^6 + 29}{42t^5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{6t^7 + 7t^6 + 29}{42t^5}$$

Verified OK.

5.13.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= -\frac{-t^2 - t + 5x}{t} \\
x' &= \omega(t, x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 124: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \frac{1}{t^5}
\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{t^5}} dy
\end{aligned}$$

Which results in

$$S = t^5 x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -\frac{-t^2 - t + 5x}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= 5t^4 x \\ S_x &= t^5 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (1+t)\, t^5 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (1+R)\, R^5$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{7}R^7 + \frac{1}{6}R^6 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$t^5 x = \frac{1}{7}t^7 + \frac{1}{6}t^6 + c_1$$

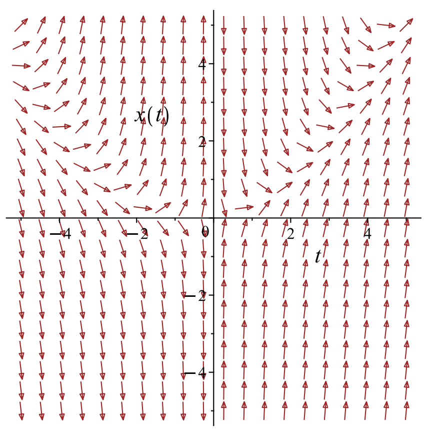
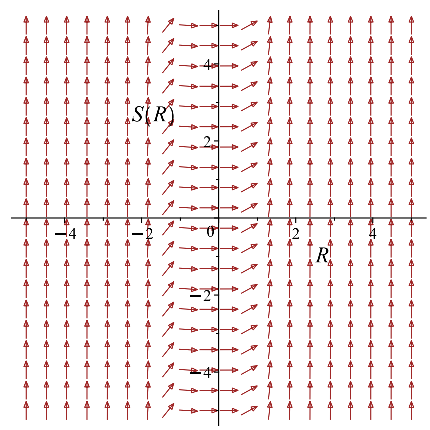
Which simplifies to

$$t^5 x = \frac{1}{7}t^7 + \frac{1}{6}t^6 + c_1$$

Which gives

$$x = \frac{6t^7 + 7t^6 + 42c_1}{42t^5}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\frac{-t^2-t+5x}{t}$ 	$R = t$ $S = t^5 x$	$\frac{dS}{dR} = (1+R) R^5$ 

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 + \frac{13}{42}$$

$$c_1 = \frac{29}{42}$$

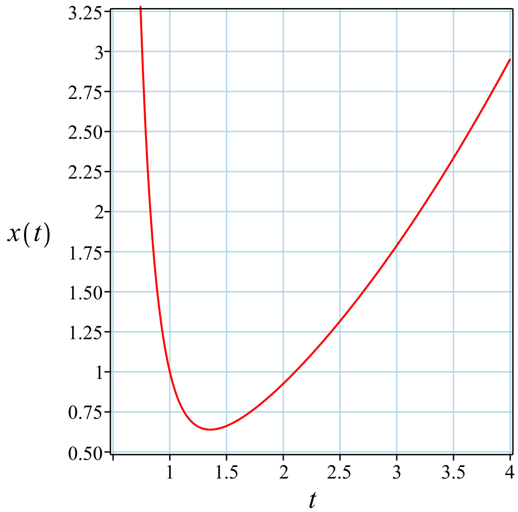
Substituting c_1 found above in the general solution gives

$$x = \frac{6t^7 + 7t^6 + 29}{42t^5}$$

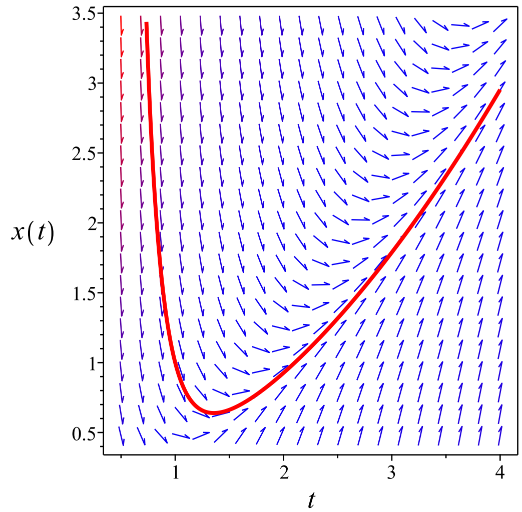
Summary

The solution(s) found are the following

$$x = \frac{6t^7 + 7t^6 + 29}{42t^5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$x = \frac{6t^7 + 7t^6 + 29}{42t^5}$$

Verified OK.

5.13.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\mathrm{d}x = \left(-\frac{5x}{t} + 1 + t\right)\mathrm{d}t$$

$$\left(-t - 1 + \frac{5x}{t}\right)\mathrm{d}t + \mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t, x) &= -t - 1 + \frac{5x}{t} \\ N(t, x) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-t - 1 + \frac{5x}{t}\right) \\ &= \frac{5}{t} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= 1\left(\left(\frac{5}{t}\right) - (0)\right) \\
&= \frac{5}{t}
\end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int \frac{5}{t} \, \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{5\ln(t)} \\
&= t^5
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= t^5\left(-t - 1 + \frac{5x}{t}\right) \\
&= -\left(t^2 + t - 5x\right) t^4
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= t^5(1) \\
&= t^5
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\
\left(-\left(t^2 + t - 5x\right) t^4\right) + \left(t^5\right) \frac{\mathrm{d}x}{\mathrm{d}t} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -(t^2 + t - 5x) \, t^4 \, \mathrm{d}t$$
$$\phi = -\frac{\left(t^2 + \frac{7}{6}t - 7x\right) t^5}{7} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = t^5 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = t^5$. Therefore equation (4) becomes

$$t^5 = t^5 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{\left(t^2 + \frac{7}{6}t - 7x\right) t^5}{7} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\left(t^2 + \frac{7}{6}t - 7x\right) t^5}{7}$$

The solution becomes

$$x = \frac{6t^7 + 7t^6 + 42c_1}{42t^5}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 + \frac{13}{42}$$

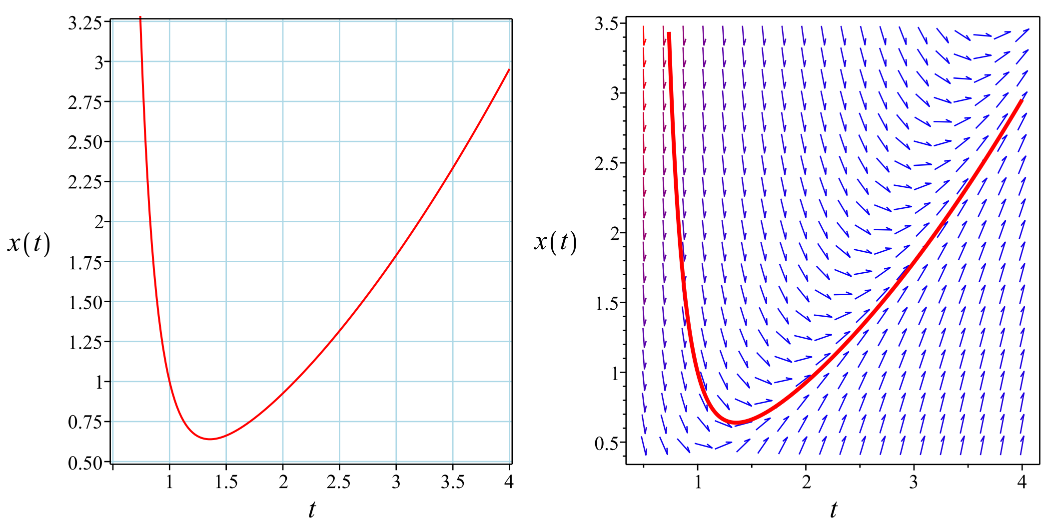
$$c_1 = \frac{29}{42}$$

Substituting c_1 found above in the general solution gives

$$x = \frac{6t^7 + 7t^6 + 29}{42t^5}$$

<u>Summary</u>
The solution(s) found are the following

$$x = \frac{6t^7 + 7t^6 + 29}{42t^5} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$x = \frac{6t^7 + 7t^6 + 29}{42t^5}$$

Verified OK.

5.13.5 Maple step by step solution

Let's solve

$\left[x' + \frac{5x}{t} = 1 + t, x(1) = 1\right]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -\frac{5x}{t} + 1 + t$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + \frac{5x}{t} = 1 + t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(x' + \frac{5x}{t}\right) = \mu(t)(1 + t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, x)$

  $\mu(t)\left(x' + \frac{5x}{t}\right) = \mu'(t)\, x + \mu(t)\, x'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{5\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = t^5$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, x)\right) dt = \int \mu(t)(1 + t)\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, x = \int \mu(t)(1 + t)\, dt + c_1$

- Solve for x

  $x = \frac{\int \mu(t)(1+t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = t^5$

  $x = \frac{\int (1+t)t^5 dt + c_1}{t^5}$

- Evaluate the integrals on the rhs

  $x = \frac{\frac{1}{7}t^7 + \frac{1}{6}t^6 + c_1}{t^5}$

- Use initial condition $x(1) = 1$

$1 = c_1 + \frac{13}{42}$

- Solve for c_1

  $c_1 = \frac{29}{42}$

- Substitute $c_1 = \frac{29}{42}$ into general solution and simplify

  $x = \frac{6t^7+7t^6+29}{42t^5}$

- Solution to the IVP

  $x = \frac{6t^7+7t^6+29}{42t^5}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve([diff(x(t),t)+(5/t)*x(t)=1+t,x(1) = 1],x(t), singsol=all)
```

$$x(t) = \frac{t^2}{7} + \frac{t}{6} + \frac{29}{42t^5}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 24

```
DSolve[{x'[t]+(5/t)*x[t]==1+t,{x[1]==1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{6t^7 + 7t^6 + 29}{42t^5}$$

5.14 problem 3(b)

5.14.1 Existence and uniqueness analysis 645
5.14.2 Solving as separable ode 645
5.14.3 Solving as linear ode 646
5.14.4 Solving as homogeneousTypeD2 ode 647
5.14.5 Solving as first order ode lie symmetry lookup ode 648
5.14.6 Solving as exact ode 652
5.14.7 Maple step by step solution 655

Internal problem ID [11412]
Internal file name [`OUTPUT/10394_Wednesday_May_17_2023_08_10_35_PM_52232036/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 3(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' - \left(a + \frac{b}{t}\right) x = 0}$$

With initial conditions

$$[x(1) = 1]$$

5.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= -\frac{ta+b}{t} \\ q(t) &= 0 \end{aligned}$$

Hence the ode is

$$x' - \frac{x(ta+b)}{t} = 0$$

The domain of $p(t) = -\frac{ta+b}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. Hence solution exists and is unique.

5.14.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= f(t)g(x) \\ &= \frac{x(ta+b)}{t} \end{aligned}$$

Where $f(t) = \frac{ta+b}{t}$ and $g(x) = x$. Integrating both sides gives

$$\begin{aligned} \frac{1}{x}\,dx &= \frac{ta+b}{t}\,dt \\ \int \frac{1}{x}\,dx &= \int \frac{ta+b}{t}\,dt \\ \ln(x) &= ta + b\ln(t) + c_1 \\ x &= \mathrm{e}^{ta+b\ln(t)+c_1} \\ &= c_1\mathrm{e}^{ta+b\ln(t)} \end{aligned}$$

Which can be simplified to become

$$x = c_1\mathrm{e}^{ta}t^b$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 \mathrm{e}^{a}$$

$$c_1 = \mathrm{e}^{-a}$$

Substituting c_1 found above in the general solution gives

$$x = t^{b} \mathrm{e}^{a(t-1)}$$

Summary

The solution(s) found are the following

$$x = t^{b} \mathrm{e}^{a(t-1)} \tag{1}$$

Verification of solutions

$$x = t^{b} \mathrm{e}^{a(t-1)}$$

Verified OK.

5.14.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{ta+b}{t}dt}\\ &= \mathrm{e}^{-ta-b\ln(t)}\end{aligned}$$

Which simplifies to

$$\mu = t^{-b} \mathrm{e}^{-ta}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu x &= 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(t^{-b} \mathrm{e}^{-ta} x\right) &= 0\end{aligned}$$

Integrating gives

$$t^{-b} \mathrm{e}^{-ta} x = c_1$$

Dividing both sides by the integrating factor $\mu = t^{-b} e^{-ta}$ results in

$$x = c_1 e^{ta} t^b$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 e^a$$

$$c_1 = e^{-a}$$

Substituting c_1 found above in the general solution gives

$$x = t^b e^{a(t-1)}$$

Summary

The solution(s) found are the following

$$x = t^b e^{a(t-1)} \tag{1}$$

Verification of solutions

$$x = t^b e^{a(t-1)}$$

Verified OK.

5.14.4 Solving as homogeneousTypeD2 ode

Using the change of variables $x = u(t)\, t$ on the above ode results in new ode in $u(t)$

$$u'(t)\, t + u(t) - \left(a + \frac{b}{t}\right) u(t)\, t = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t) g(u) \\ &= \frac{u(ta + b - 1)}{t} \end{aligned}$$

Where $f(t) = \frac{ta+b-1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{ta + b - 1}{t}\, dt \\ \int \frac{1}{u}\, du &= \int \frac{ta + b - 1}{t}\, dt \\ \ln(u) &= ta + (b-1)\ln(t) + c_2 \\ u &= e^{ta + (b-1)\ln(t) + c_2} \\ &= c_2 e^{ta + (b-1)\ln(t)} \end{aligned}$$

Which simplifies to

$$u(t) = \frac{c_2 \mathrm{e}^{ta} t^b}{t}$$

Therefore the solution x is

$$\begin{aligned} x &= ut \\ &= c_2 \mathrm{e}^{ta} t^b \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_2 \mathrm{e}^a$$

$$c_2 = \mathrm{e}^{-a}$$

Substituting c_2 found above in the general solution gives

$$x = t^b \mathrm{e}^{a(t-1)}$$

Summary
The solution(s) found are the following

$$x = t^b \mathrm{e}^{a(t-1)} \tag{1}$$

Verification of solutions

$$x = t^b \mathrm{e}^{a(t-1)}$$

Verified OK.

5.14.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} x' &= \frac{x(ta+b)}{t} \\ x' &= \omega(t, x) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 127: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= 0 \\
\eta(t, x) &= \mathrm{e}^{ta+b\ln(t)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{e^{ta+b\ln(t)}} dy
\end{aligned}$$

Which results in

$$S = e^{-ta-b\ln(t)} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = \frac{x(ta+b)}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_t &= 1 \\
R_x &= 0 \\
S_t &= -x\, t^{-1-b}(ta+b)\, e^{-ta} \\
S_x &= t^{-b} e^{-ta}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$t^{-b} \mathrm{e}^{-ta} x = c_1$$

Which simplifies to

$$t^{-b} \mathrm{e}^{-ta} x = c_1$$

Which gives

$$x = c_1 \mathrm{e}^{ta} t^b$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 \mathrm{e}^a$$

$$c_1 = \mathrm{e}^{-a}$$

Substituting c_1 found above in the general solution gives

$$x = t^b \mathrm{e}^{a(t-1)}$$

Summary
The solution(s) found are the following

$$x = t^b \mathrm{e}^{a(t-1)} \tag{1}$$

Verification of solutions

$$x = t^b \mathrm{e}^{a(t-1)}$$

Verified OK.

5.14.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{x}\right)\mathrm{d}x&=\left(\frac{ta+b}{t}\right)\mathrm{d}t\\ \left(-\frac{ta+b}{t}\right)\mathrm{d}t+\left(\frac{1}{x}\right)\mathrm{d}x&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x) &= -\frac{ta+b}{t}\\ N(t,x) &= \frac{1}{x}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{ta+b}{t}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{x}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{ta+b}{t}\,\mathrm{d}t\end{aligned}$$

$$\phi = -ta - b\ln(t) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = \frac{1}{x}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(\frac{1}{x}\right)\mathrm{d}x$$
$$f(x) = \ln(x) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -ta - b\ln(t) + \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -ta - b\ln(t) + \ln(x)$$

The solution becomes

$$x = \mathrm{e}^{ta + b\ln(t) + c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \mathrm{e}^{a + c_1}$$

$$c_1 = -a$$

Substituting c_1 found above in the general solution gives

$$x = t^b \mathrm{e}^{a(t-1)}$$

Summary
The solution(s) found are the following

$$x = t^b e^{a(t-1)} \tag{1}$$

Verification of solutions

$$x = t^b e^{a(t-1)}$$

Verified OK.

5.14.7 Maple step by step solution

Let's solve

$\left[x' - \left(a + \frac{b}{t}\right) x = 0, x(1) = 1\right]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x} = a + \frac{b}{t}$

- Integrate both sides with respect to t

  $\int \frac{x'}{x} dt = \int \left(a + \frac{b}{t}\right) dt + c_1$

- Evaluate integral

  $\ln(x) = ta + b\ln(t) + c_1$

- Solve for x

  $x = e^{ta + b\ln(t) + c_1}$

- Use initial condition $x(1) = 1$

  $1 = e^{a + c_1}$

- Solve for c_1

  $c_1 = -a$

- Substitute $c_1 = -a$ into general solution and simplify

  $x = t^b e^{a(t-1)}$

- Solution to the IVP

  $x = t^b e^{a(t-1)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(x(t),t)=(a+b/t)*x(t),x(1) = 1],x(t), singsol=all)
```

$$x(t) = t^b e^{a(t-1)}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 16

```
DSolve[{x'[t]==(a+b/t)*x[t],{x[1]==1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{a(t-1)} t^b$$

5.15 problem 3(c)

5.15.1 Existence and uniqueness analysis 657
5.15.2 Solving as linear ode 658
5.15.3 Solving as first order ode lie symmetry lookup ode 660
5.15.4 Solving as exact ode 664
5.15.5 Maple step by step solution 669

Internal problem ID [11413]
Internal file name [`OUTPUT/10395_Wednesday_May_17_2023_08_10_36_PM_84307721/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 3(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{R' + \frac{R}{t} = \frac{2}{t^2+1}}$$

With initial conditions

$$[R(1) = 3\ln(2)]$$

5.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$R' + p(t)R = q(t)$$

Where here

$$p(t) = \frac{1}{t}$$
$$q(t) = \frac{2}{t^2+1}$$

Hence the ode is

$$R' + \frac{R}{t} = \frac{2}{t^2 + 1}$$

The domain of $p(t) = \frac{1}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = \frac{2}{t^2+1}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

5.15.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{t} dt} \\ &= t\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu R) &= (\mu)\left(\frac{2}{t^2 + 1}\right) \\ \frac{d}{dt}(tR) &= (t)\left(\frac{2}{t^2 + 1}\right) \\ d(tR) &= \left(\frac{2t}{t^2 + 1}\right) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}tR &= \int \frac{2t}{t^2 + 1} dt \\ tR &= \ln\left(t^2 + 1\right) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$R = \frac{\ln\left(t^2 + 1\right)}{t} + \frac{c_1}{t}$$

which simplifies to

$$R = \frac{\ln\left(t^2 + 1\right) + c_1}{t}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $R = 3\ln(2)$ in the above solution gives an equation to solve for the constant of integration.

$$3\ln(2) = \ln(2) + c_1$$

$$c_1 = 2\ln(2)$$

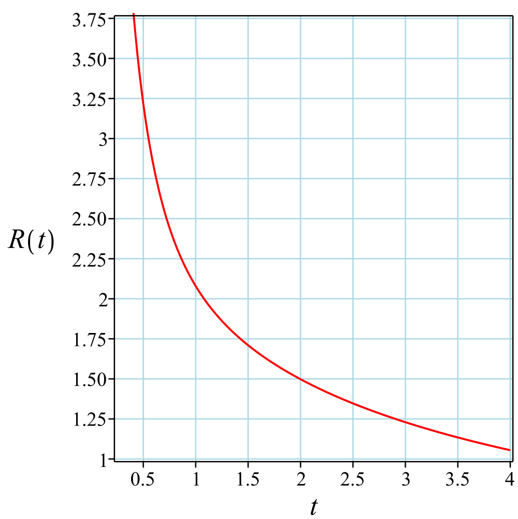
Substituting c_1 found above in the general solution gives

$$R = \frac{\ln(t^2+1) + 2\ln(2)}{t}$$

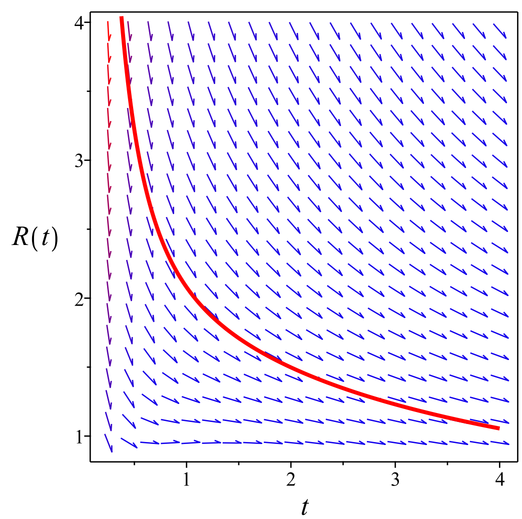
Summary

The solution(s) found are the following

$$R = \frac{\ln(t^2+1) + 2\ln(2)}{t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$R = \frac{\ln(t^2+1) + 2\ln(2)}{t}$$

Verified OK.

5.15.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
R' &= -\frac{R\,t^2 + R - 2t}{t\,(t^2+1)} \\
R' &= \omega(t, R)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_R - \xi_t) - \omega^2 \xi_R - \omega_t \xi - \omega_R \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 130: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, R) &= 0 \\
\eta(t, R) &= \frac{1}{t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, R) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dR}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial R}\right) S(t, R) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{t}} dy
\end{aligned}
$$

Which results in

$$
S = tR
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t, R) S_R}{R_t + \omega(t, R) R_R} \tag{2}
$$

Where in the above R_t, R_R, S_t, S_R are all partial derivatives and $\omega(t, R)$ is the right hand side of the original ode given by

$$
\omega(t, R) = -\frac{R t^2 + R - 2t}{t\,(t^2 + 1)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_R &= 0 \\
S_t &= R \\
S_R &= t
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{2t}{t^2 + 1} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for t, R in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{2R}{R^2 + 1}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln\left(R^2 + 1\right) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to t, R coordinates. This results in

$$
tR = \ln\left(t^2 + 1\right) + c_1
$$

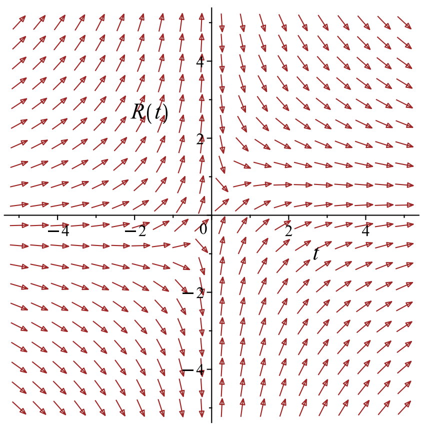
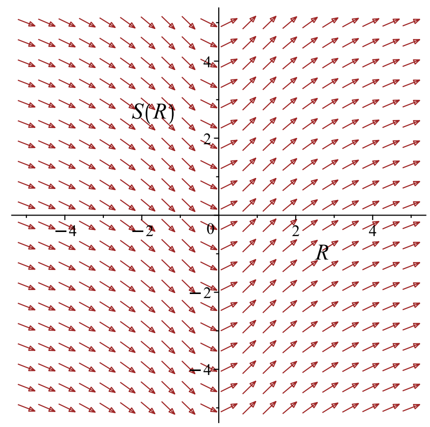
Which simplifies to

$$
tR = \ln\left(t^2 + 1\right) + c_1
$$

Which gives

$$
R = \frac{\ln(t^2 + 1) + c_1}{t}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, R coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dR}{dt} = -\frac{R\,t^2+R-2t}{t(t^2+1)}$ 	$R = t$ $S = tR$	$\frac{dS}{dR} = \frac{2R}{R^2+1}$ 

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $R = 3\ln(2)$ in the above solution gives an equation to solve for the constant of integration.

$$3\ln(2) = \ln(2) + c_1$$

$$c_1 = 2\ln(2)$$

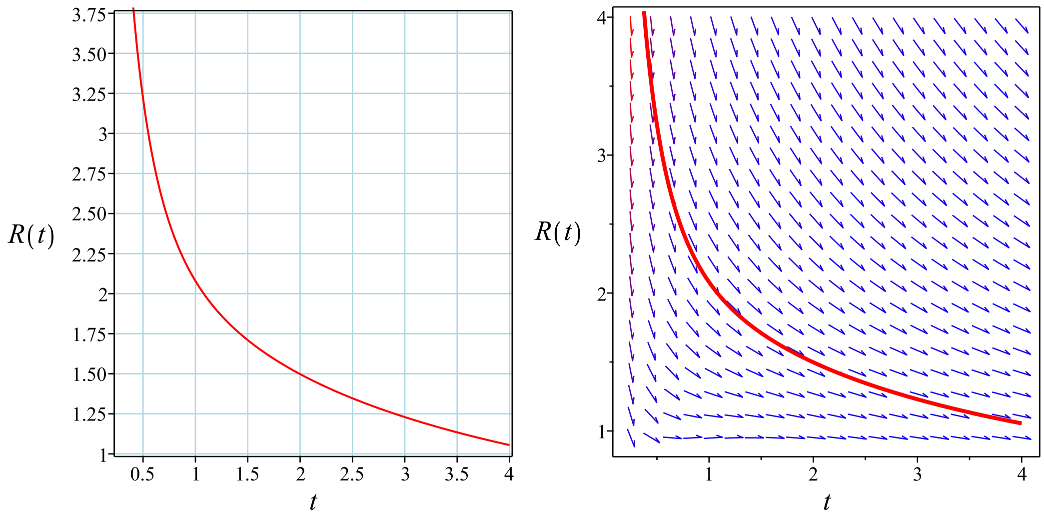
Substituting c_1 found above in the general solution gives

$$R = \frac{\ln(t^2+1) + 2\ln(2)}{t}$$

Summary

The solution(s) found are the following

$$R = \frac{\ln(t^2+1) + 2\ln(2)}{t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$R = \frac{\ln(t^2+1) + 2\ln(2)}{t}$$

Verified OK.

5.15.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,R)\,\mathrm{d}t+N(t,R)\,\mathrm{d}R=0 \tag{1A}$$

Therefore

$$\mathrm{d}R=\left(-\frac{R}{t}+\frac{2}{t^2+1}\right)\mathrm{d}t$$

$$\left(\frac{R}{t}-\frac{2}{t^2+1}\right)\mathrm{d}t+\mathrm{d}R=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,R)&=\frac{R}{t}-\frac{2}{t^2+1}\\ N(t,R)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial R}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial R}&=\frac{\partial}{\partial R}\left(\frac{R}{t}-\frac{2}{t^2+1}\right)\\ &=\frac{1}{t}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t}&=\frac{\partial}{\partial t}(1)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial R} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial R} - \frac{\partial N}{\partial t}\right) \\
&= 1\left(\left(\frac{1}{t}\right) - (0)\right) \\
&= \frac{1}{t}
\end{aligned}$$

Since A does not depend on R, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int \frac{1}{t} \, \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{\ln(t)} \\
&= t
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= t\left(\frac{R}{t} - \frac{2}{t^2+1}\right) \\
&= \frac{R t^2 + R - 2t}{t^2+1}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= t(1) \\
&= t
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}R}{\mathrm{d}t} &= 0 \\
\left(\frac{R t^2 + R - 2t}{t^2+1}\right) + (t)\frac{\mathrm{d}R}{\mathrm{d}t} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,R)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial R} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \frac{Rt^2 + R - 2t}{t^2+1}\,\mathrm{d}t$$
$$\phi = tR - \ln\left(t^2+1\right) + f(R) \tag{3}$$

Where $f(R)$ is used for the constant of integration since ϕ is a function of both t and R. Taking derivative of equation (3) w.r.t R gives

$$\frac{\partial \phi}{\partial R} = t + f'(R) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial R} = t$. Therefore equation (4) becomes

$$t = t + f'(R) \tag{5}$$

Solving equation (5) for $f'(R)$ gives

$$f'(R) = 0$$

Therefore

$$f(R) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(R)$ into equation (3) gives ϕ

$$\phi = tR - \ln\left(t^2+1\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = tR - \ln\left(t^2+1\right)$$

The solution becomes

$$R = \frac{\ln(t^2+1)+c_1}{t}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $R = 3\ln(2)$ in the above solution gives an equation to solve for the constant of integration.

$$3\ln(2) = \ln(2) + c_1$$

$$c_1 = 2\ln(2)$$

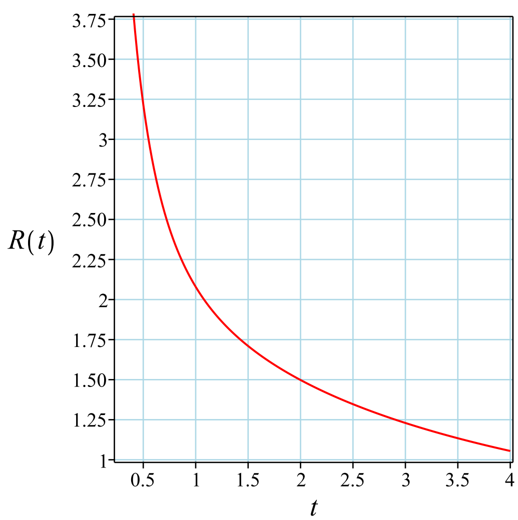
Substituting c_1 found above in the general solution gives

$$R = \frac{\ln(t^2+1)+2\ln(2)}{t}$$

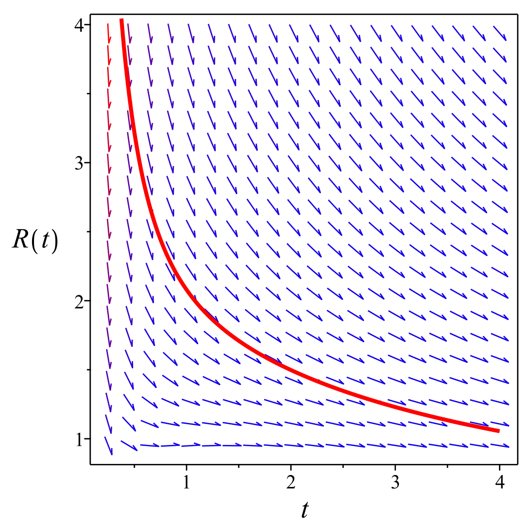
Summary

The solution(s) found are the following

$$R = \frac{\ln(t^2+1)+2\ln(2)}{t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$R = \frac{\ln(t^2+1)+2\ln(2)}{t}$$

Verified OK.

5.15.5 Maple step by step solution

Let's solve

$\left[R' + \frac{R}{t} = \frac{2}{t^2+1}, R(1) = 3\ln(2)\right]$

- Highest derivative means the order of the ODE is 1

  R'

- Isolate the derivative

  $R' = -\frac{R}{t} + \frac{2}{t^2+1}$

- Group terms with R on the lhs of the ODE and the rest on the rhs of the ODE

  $R' + \frac{R}{t} = \frac{2}{t^2+1}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(R' + \frac{R}{t}\right) = \frac{2\mu(t)}{t^2+1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,R)$

  $\mu(t)\left(R' + \frac{R}{t}\right) = \mu'(t)\,R + \mu(t)\,R'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = t$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,R)\right) dt = \int \frac{2\mu(t)}{t^2+1} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,R = \int \frac{2\mu(t)}{t^2+1} dt + c_1$

- Solve for R

  $R = \frac{\int \frac{2\mu(t)}{t^2+1} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = t$

  $R = \frac{\int \frac{2t}{t^2+1} dt + c_1}{t}$

- Evaluate the integrals on the rhs

  $R = \frac{\ln(t^2+1) + c_1}{t}$

- Use initial condition $R(1) = 3\ln(2)$

$$3\ln(2) = \ln(2) + c_1$$

- Solve for c_1

$$c_1 = 2\ln(2)$$

- Substitute $c_1 = 2\ln(2)$ into general solution and simplify

$$R = \frac{\ln(t^2+1)+2\ln(2)}{t}$$

- Solution to the IVP

$$R = \frac{\ln(t^2+1)+2\ln(2)}{t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve([diff(R(t),t)+R(t)/t=2/(1+t^2),R(1) = ln(8)],R(t), singsol=all)
```

$$R(t) = \frac{\ln(t^2+1)+2\ln(2)}{t}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 17

```
DSolve[{R'[t]+R[t]/t==2/(1+t^2),{R[1]==Log[8]}},R[t],t,IncludeSingularSolutions -> True]
```

$$R(t) \to \frac{\log(4t^2+4)}{t}$$

5.16 problem 3(d)

5.16.1 Solving as linear ode 671
5.16.2 Solving as first order ode lie symmetry lookup ode 673
5.16.3 Solving as exact ode 677
5.16.4 Maple step by step solution 681

Internal problem ID [11414]
Internal file name [`OUTPUT/10396_Wednesday_May_17_2023_08_10_37_PM_86140142/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 3(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[[_linear, `class A`]]`

$$\boxed{N' - N = -9\,\mathrm{e}^{-t}}$$

5.16.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$N' + p(t)N = q(t)$$

Where here

$$p(t) = -1$$
$$q(t) = -9\,\mathrm{e}^{-t}$$

Hence the ode is

$$N' - N = -9\,\mathrm{e}^{-t}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-1)dt} \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu N) &= (\mu)\left(-9\,\mathrm{e}^{-t}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-t}N\right) &= \left(\mathrm{e}^{-t}\right)\left(-9\,\mathrm{e}^{-t}\right) \\
\mathrm{d}\left(\mathrm{e}^{-t}N\right) &= \left(-9\,\mathrm{e}^{-2t}\right)\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-t}N &= \int -9\,\mathrm{e}^{-2t}\,\mathrm{d}t \\
\mathrm{e}^{-t}N &= \frac{9\,\mathrm{e}^{-2t}}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-t}$ results in

$$
N = \frac{9\,\mathrm{e}^{t}\mathrm{e}^{-2t}}{2} + c_1\mathrm{e}^{t}
$$

which simplifies to

$$
N = \frac{9\,\mathrm{e}^{-t}}{2} + c_1\mathrm{e}^{t}
$$

Summary

The solution(s) found are the following

$$
N = \frac{9\,\mathrm{e}^{-t}}{2} + c_1\mathrm{e}^{t} \tag{1}
$$

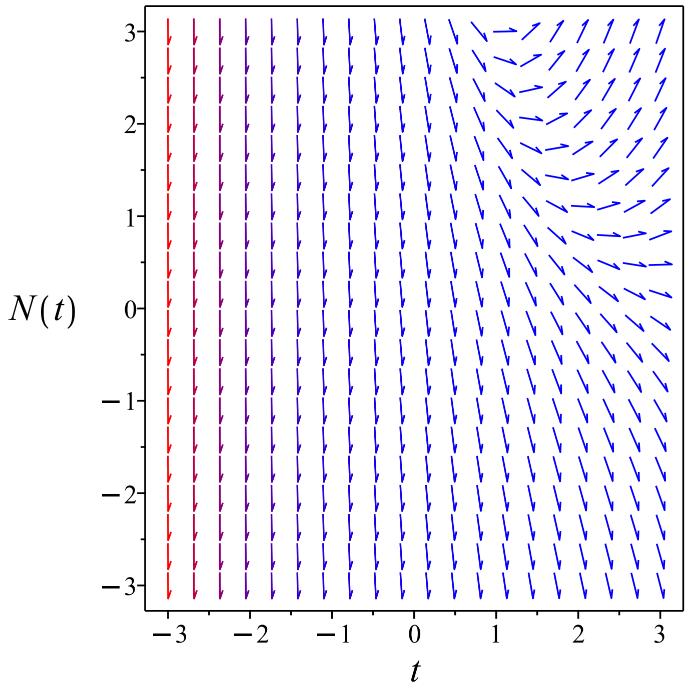


Figure 153: Slope field plot

Verification of solutions

$$N = \frac{9\,e^{-t}}{2} + c_1 e^{t}$$

Verified OK.

5.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
N' &= N - 9\,e^{-t} \\
N' &= \omega(t, N)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_N - \xi_t) - \omega^2 \xi_N - \omega_t \xi - \omega_N \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 133: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, N) &= 0 \\ \eta(t, N) &= \mathrm{e}^t \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, N) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dN}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial N}\right) S(t, N) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^t} dy \end{aligned}$$

Which results in

$$S = e^{-t} N$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, N) S_N}{R_t + \omega(t, N) R_N} \tag{2}$$

Where in the above R_t, R_N, S_t, S_N are all partial derivatives and $\omega(t, N)$ is the right hand side of the original ode given by

$$\omega(t, N) = N - 9\,e^{-t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_N &= 0 \\ S_t &= -e^{-t} N \\ S_N &= e^{-t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -9\,e^{-2t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, N in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -9\,e^{-2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{9\,\mathrm{e}^{-2R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, N coordinates. This results in

$$\mathrm{e}^{-t} N = \frac{9\,\mathrm{e}^{-2t}}{2} + c_1$$

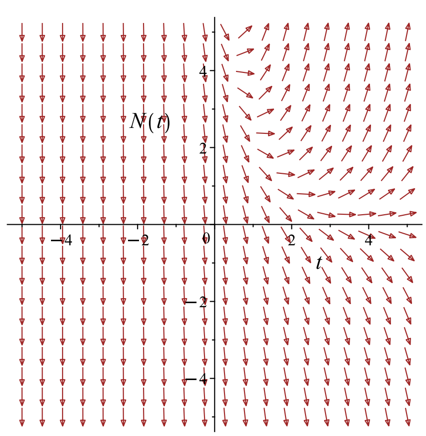
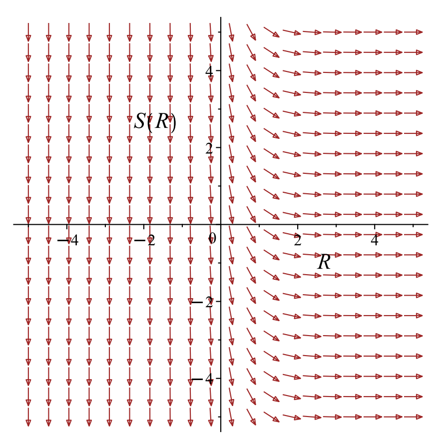
Which simplifies to

$$\mathrm{e}^{-t} N = \frac{9\,\mathrm{e}^{-2t}}{2} + c_1$$

Which gives

$$N = \frac{(9\,\mathrm{e}^{-2t} + 2c_1)\,\mathrm{e}^{t}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, N coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dN}{dt} = N - 9\,\mathrm{e}^{-t}$ 	$R = t$ $S = \mathrm{e}^{-t} N$	$\frac{dS}{dR} = -9\,\mathrm{e}^{-2R}$ 

Summary

The solution(s) found are the following

$$N = \frac{(9\,\mathrm{e}^{-2t} + 2c_1)\,\mathrm{e}^{t}}{2} \tag{1}$$

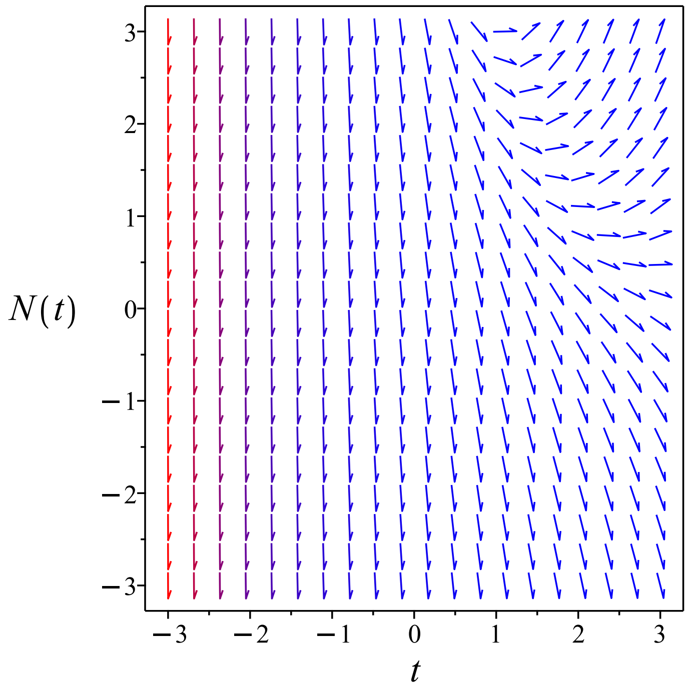


Figure 154: Slope field plot

Verification of solutions

$$N = \frac{(9\,\mathrm{e}^{-2t} + 2c_1)\,\mathrm{e}^{t}}{2}$$

Verified OK.

5.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, N)\,\mathrm{d}t + N(t, N)\,\mathrm{d}N = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}N &= \left(N - 9\,\mathrm{e}^{-t}\right)\mathrm{d}t \\
\left(-N + 9\,\mathrm{e}^{-t}\right)\mathrm{d}t + \mathrm{d}N &= 0 \qquad (2\mathrm{A})
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, N) &= -N + 9\,\mathrm{e}^{-t} \\
N(t, N) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial N} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial N} &= \frac{\partial}{\partial N}\left(-N + 9\,\mathrm{e}^{-t}\right) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial N} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial N} - \frac{\partial N}{\partial t}\right) \\
&= 1((-1) - (0)) \\
&= -1
\end{aligned}
$$

Since A does not depend on N, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -1 \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-t} \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-t}\left(-N + 9\,\mathrm{e}^{-t}\right) \\
&= -\mathrm{e}^{-t}\left(N - 9\,\mathrm{e}^{-t}\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-t}(1) \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}N}{\mathrm{d}t} &= 0 \\
\left(-\mathrm{e}^{-t}\left(N - 9\,\mathrm{e}^{-t}\right)\right) + \left(\mathrm{e}^{-t}\right)\frac{\mathrm{d}N}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, N)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial N} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int \overline{M}\, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int -\mathrm{e}^{-t}\left(N - 9\,\mathrm{e}^{-t}\right) \mathrm{d}t$$

$$\phi = \mathrm{e}^{-t} N - \frac{9\,\mathrm{e}^{-2t}}{2} + f(N) \tag{3}$$

Where $f(N)$ is used for the constant of integration since ϕ is a function of both t and N. Taking derivative of equation (3) w.r.t N gives

$$\frac{\partial \phi}{\partial N} = \mathrm{e}^{-t} + f'(N) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial N} = \mathrm{e}^{-t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-t} = \mathrm{e}^{-t} + f'(N) \tag{5}$$

Solving equation (5) for $f'(N)$ gives

$$f'(N) = 0$$

Therefore

$$f(N) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(N)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{-t} N - \frac{9\,\mathrm{e}^{-2t}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{-t} N - \frac{9\,\mathrm{e}^{-2t}}{2}$$

The solution becomes

$$N = \frac{\left(9\,\mathrm{e}^{-2t} + 2c_1\right) \mathrm{e}^{t}}{2}$$

Summary

The solution(s) found are the following

$$N = \frac{(9\,\mathrm{e}^{-2t} + 2c_1)\,\mathrm{e}^{t}}{2} \tag{1}$$

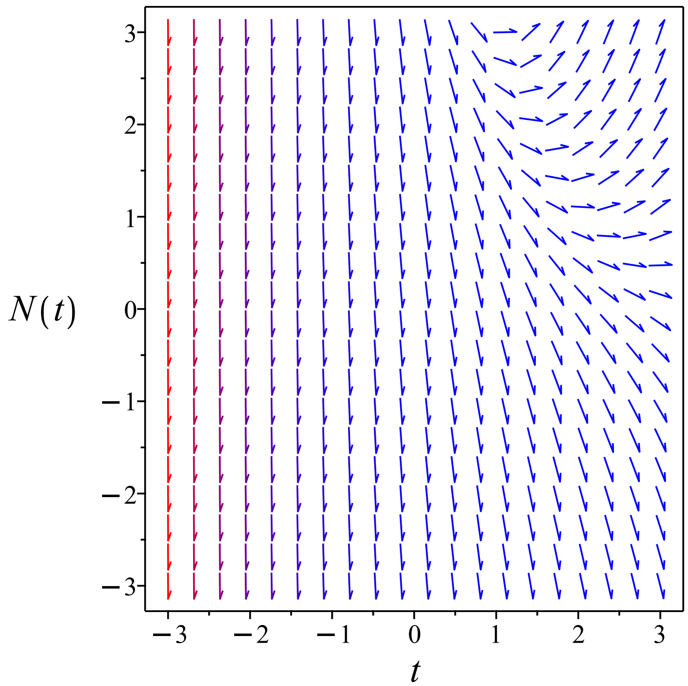


Figure 155: Slope field plot

Verification of solutions

$$N = \frac{(9\,\mathrm{e}^{-2t} + 2c_1)\,\mathrm{e}^{t}}{2}$$

Verified OK.

5.16.4 Maple step by step solution

Let's solve

$N' - N = -9\,\mathrm{e}^{-t}$

- Highest derivative means the order of the ODE is 1

  N'

- Isolate the derivative

$N' = N - 9\,\mathrm{e}^{-t}$

- Group terms with N on the lhs of the ODE and the rest on the rhs of the ODE

  $N' - N = -9\,\mathrm{e}^{-t}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(N' - N) = -9\mu(t)\,\mathrm{e}^{-t}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,N)$

  $\mu(t)\,(N' - N) = \mu'(t)\,N + \mu(t)\,N'$
- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,N)\right) dt = \int -9\mu(t)\,\mathrm{e}^{-t} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,N = \int -9\mu(t)\,\mathrm{e}^{-t} dt + c_1$
- Solve for N

  $N = \frac{\int -9\mu(t)\mathrm{e}^{-t} dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{-t}$

  $N = \frac{\int -9(\mathrm{e}^{-t})^2 dt + c_1}{\mathrm{e}^{-t}}$
- Evaluate the integrals on the rhs

  $N = \frac{\frac{9(\mathrm{e}^{-t})^2}{2} + c_1}{\mathrm{e}^{-t}}$
- Simplify

  $N = \frac{9\,\mathrm{e}^{-t}}{2} + c_1 \mathrm{e}^{t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(N(t),t)=N(t)-9*exp(-t),N(t), singsol=all)
```

$$N(t) = \frac{9\,\mathrm{e}^{-t}}{2} + c_1\mathrm{e}^{t}$$

✓ Solution by Mathematica

Time used: 0.112 (sec). Leaf size: 32

```
DSolve[n'[t]==n[t]-9*exp[-t],n[t],t,IncludeSingularSolutions -> True]
```

$$n(t) \to e^t\left(\int_1^t -9e^{-K[1]}\exp(-K[1])dK[1] + c_1\right)$$

5.17 problem 3(e)

5.17.1 Existence and uniqueness analysis 685
5.17.2 Solving as separable ode 685
5.17.3 Solving as linear ode 686
5.17.4 Solving as first order ode lie symmetry lookup ode 687
5.17.5 Solving as exact ode 691
5.17.6 Maple step by step solution 694

Internal problem ID [11415]
Internal file name [`OUTPUT/10397_Wednesday_May_17_2023_08_10_39_PM_16924072/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 3(e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\cos(\theta)\, v' + v = 3}$$

With initial conditions

$$\left[v\left(\frac{\pi}{2}\right) = 1\right]$$

5.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$v' + p(\theta)v = q(\theta)$$

Where here

$$\begin{aligned} p(\theta) &= \sec(\theta) \\ q(\theta) &= 3\sec(\theta) \end{aligned}$$

Hence the ode is

$$v' + \sec(\theta)\, v = 3\sec(\theta)$$

The domain of $p(\theta) = \sec(\theta)$ is

$$\left\{\theta < \frac{1}{2}\pi + \pi\_Z87 \vee \frac{1}{2}\pi + \pi\_Z87 < \theta\right\}$$

But the point $\theta_0 = \frac{\pi}{2}$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

5.17.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} v' &= F(\theta, v) \\ &= f(\theta)g(v) \\ &= \frac{-v+3}{\cos(\theta)} \end{aligned}$$

Where $f(\theta) = \frac{1}{\cos(\theta)}$ and $g(v) = -v + 3$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-v+3}\, dv &= \frac{1}{\cos(\theta)}\, d\theta \\ \int \frac{1}{-v+3}\, dv &= \int \frac{1}{\cos(\theta)}\, d\theta \\ -\ln(v-3) &= \ln(\sec(\theta) + \tan(\theta)) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{v-3} = e^{\ln(\sec(\theta)+\tan(\theta))+c_1}$$

Which simplifies to

$$\frac{1}{v-3} = c_2(\sec(\theta) + \tan(\theta))$$

Initial conditions are used to solve for c_1. Substituting $\theta = \frac{\pi}{2}$ and $v = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 3$$

This solution is valid for any c_1. Hence there are infinite number of solutions.

Summary

The solution(s) found are the following

$$v = \frac{\left(3c_2 e^{\ln(\sec(\theta)+\tan(\theta))+c_1} + 1\right) e^{-c_1}}{c_2 (\sec(\theta) + \tan(\theta))} \tag{1}$$

Verification of solutions

$$v = \frac{\left(3c_2 e^{\ln(\sec(\theta)+\tan(\theta))+c_1} + 1\right) e^{-c_1}}{c_2 (\sec(\theta) + \tan(\theta))}$$

Warning, solution could not be verified

5.17.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \sec(\theta)d\theta} \\ &= \sec(\theta) + \tan(\theta)\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{d\theta}(\mu v) &= (\mu)(3\sec(\theta)) \\ \frac{d}{d\theta}((\sec(\theta) + \tan(\theta))v) &= (\sec(\theta) + \tan(\theta))(3\sec(\theta)) \\ d((\sec(\theta) + \tan(\theta))v) &= \left(-\frac{3}{\sin(\theta) - 1}\right) d\theta\end{aligned}$$

Integrating gives

$$\left(\sec\left(\theta\right)+\tan\left(\theta\right)\right) v = \int -\frac{3}{\sin\left(\theta\right)-1}\mathrm{d}\theta$$
$$\left(\sec\left(\theta\right)+\tan\left(\theta\right)\right) v = -\frac{6}{\tan\left(\frac{\theta}{2}\right)-1}+c_1$$

Dividing both sides by the integrating factor $\mu = \sec\left(\theta\right)+\tan\left(\theta\right)$ results in

$$v = -\frac{6}{\left(\sec\left(\theta\right)+\tan\left(\theta\right)\right)\left(\tan\left(\frac{\theta}{2}\right)-1\right)}+\frac{c_1}{\sec\left(\theta\right)+\tan\left(\theta\right)}$$

Initial conditions are used to solve for c_1. Substituting $\theta = \frac{\pi}{2}$ and $v = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 3$$

This solution is valid for any c_1. Hence there are infinite number of solutions.

Summary

The solution(s) found are the following

$$v = -\frac{6}{\left(\sec\left(\theta\right)+\tan\left(\theta\right)\right)\left(\tan\left(\frac{\theta}{2}\right)-1\right)}+\frac{c_1}{\sec\left(\theta\right)+\tan\left(\theta\right)} \tag{1}$$

Verification of solutions

$$v = -\frac{6}{\left(\sec\left(\theta\right)+\tan\left(\theta\right)\right)\left(\tan\left(\frac{\theta}{2}\right)-1\right)}+\frac{c_1}{\sec\left(\theta\right)+\tan\left(\theta\right)}$$

Warning, solution could not be verified

5.17.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$v' = -\frac{v-3}{\cos\left(\theta\right)}$$
$$v' = \omega(\theta, v)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_\theta + \omega(\eta_v - \xi_\theta) - \omega^2\xi_v - \omega_\theta\xi - \omega_v\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 136: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(\theta, v) &= 0 \\
\eta(\theta, v) &= \frac{1}{\sec(\theta) + \tan(\theta)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(\theta, v) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{d\theta}{\xi} = \frac{dv}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial v}\right) S(\theta, v) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = \theta$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sec(\theta)+\tan(\theta)}} dy\end{aligned}$$

Which results in

$$S = (\sec(\theta) + \tan(\theta))\, v$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_\theta + \omega(\theta, v) S_v}{R_\theta + \omega(\theta, v) R_v} \tag{2}$$

Where in the above $R_\theta, R_v, S_\theta, S_v$ are all partial derivatives and $\omega(\theta, v)$ is the right hand side of the original ode given by

$$\omega(\theta, v) = -\frac{v-3}{\cos(\theta)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_\theta &= 1 \\ R_v &= 0 \\ S_\theta &= -\frac{v}{\sin(\theta) - 1} \\ S_v &= \sec(\theta) + \tan(\theta)\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{3}{\sin(\theta) - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for θ, v in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{3}{\sin(R) - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{6}{\tan\left(\frac{R}{2}\right) - 1} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to θ, v coordinates. This results in

$$(\sec(\theta) + \tan(\theta))\, v = -\frac{6}{\tan\left(\frac{\theta}{2}\right) - 1} + c_1$$

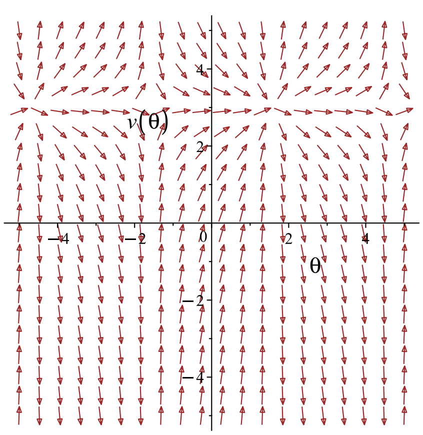
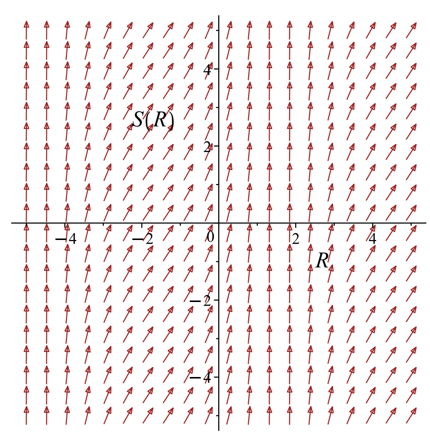
Which simplifies to

$$(\sec(\theta) + \tan(\theta))\, v = -\frac{6}{\tan\left(\frac{\theta}{2}\right) - 1} + c_1$$

Which gives

$$v = \frac{c_1 \tan\left(\frac{\theta}{2}\right) - c_1 - 6}{\tan(\theta)\tan\left(\frac{\theta}{2}\right) + \sec(\theta)\tan\left(\frac{\theta}{2}\right) - \tan(\theta) - \sec(\theta)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in θ, v coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dv}{d\theta} = -\frac{v-3}{\cos(\theta)}$
</td><td>$R = \theta$
$S = (\sec(\theta) + \tan(\theta))\, v$</td><td>$\frac{dS}{dR} = -\frac{3}{\sin(R)-1}$
</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $\theta = \frac{\pi}{2}$ and $v = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 3$$

This solution is valid for any c_1. Hence there are infinite number of solutions.

Summary

The solution(s) found are the following

$$v = \frac{c_1 \tan\left(\frac{\theta}{2}\right) - c_1 - 6}{\tan(\theta)\tan\left(\frac{\theta}{2}\right) + \sec(\theta)\tan\left(\frac{\theta}{2}\right) - \tan(\theta) - \sec(\theta)} \tag{1}$$

Verification of solutions

$$v = \frac{c_1 \tan\left(\frac{\theta}{2}\right) - c_1 - 6}{\tan(\theta)\tan\left(\frac{\theta}{2}\right) + \sec(\theta)\tan\left(\frac{\theta}{2}\right) - \tan(\theta) - \sec(\theta)}$$

Warning, solution could not be verified

5.17.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(\theta, v)\,\mathrm{d}\theta + N(\theta, v)\,\mathrm{d}v = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{-v+3}\right)\mathrm{d}v &= \left(\frac{1}{\cos(\theta)}\right)\mathrm{d}\theta \\ \left(-\frac{1}{\cos(\theta)}\right)\mathrm{d}\theta + \left(\frac{1}{-v+3}\right)\mathrm{d}v &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(\theta, v) &= -\frac{1}{\cos(\theta)} \\ N(\theta, v) &= \frac{1}{-v+3} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial v} = \frac{\partial N}{\partial \theta}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial v} &= \frac{\partial}{\partial v}\left(-\frac{1}{\cos(\theta)}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(\frac{1}{-v+3}\right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial v} = \frac{\partial N}{\partial \theta}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(\theta, v)$

$$\frac{\partial \phi}{\partial \theta} = M \tag{1}$$

$$\frac{\partial \phi}{\partial v} = N \tag{2}$$

Integrating (1) w.r.t. θ gives

$$\int \frac{\partial \phi}{\partial \theta} \mathrm{d}\theta = \int M \mathrm{d}\theta$$

$$\int \frac{\partial \phi}{\partial \theta} \mathrm{d}\theta = \int -\frac{1}{\cos(\theta)} \mathrm{d}\theta$$

$$\phi = -\ln(\sec(\theta) + \tan(\theta)) + f(v) \tag{3}$$

Where $f(v)$ is used for the constant of integration since ϕ is a function of both θ and v. Taking derivative of equation (3) w.r.t v gives

$$\frac{\partial \phi}{\partial v} = 0 + f'(v) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial v} = \frac{1}{-v+3}$. Therefore equation (4) becomes

$$\frac{1}{-v+3} = 0 + f'(v) \tag{5}$$

Solving equation (5) for $f'(v)$ gives

$$f'(v) = -\frac{1}{v-3}$$

Integrating the above w.r.t v gives

$$\int f'(v) \mathrm{d}v = \int \left(-\frac{1}{v-3}\right) \mathrm{d}v$$

$$f(v) = -\ln(v-3) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(v)$ into equation (3) gives ϕ

$$\phi = -\ln(\sec(\theta) + \tan(\theta)) - \ln(v-3) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\sec(\theta) + \tan(\theta)) - \ln(v-3)$$

The solution becomes

$$v = \frac{\left(3\,\mathrm{e}^{c_1} \sin\left(\theta\right)+3\,\mathrm{e}^{c_1}+\cos\left(\theta\right)\right) \mathrm{e}^{-c_1}}{1+\sin\left(\theta\right)}$$

Initial conditions are used to solve for c_1. Substituting $\theta = \frac{\pi}{2}$ and $v = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 3$$

This solution is valid for any c_1. Hence there are infinite number of solutions.

Summary

The solution(s) found are the following

$$v = \frac{\left(3\,\mathrm{e}^{c_1} \sin\left(\theta\right)+3\,\mathrm{e}^{c_1}+\cos\left(\theta\right)\right) \mathrm{e}^{-c_1}}{1+\sin\left(\theta\right)} \tag{1}$$

Verification of solutions

$$v = \frac{\left(3\,\mathrm{e}^{c_1} \sin\left(\theta\right)+3\,\mathrm{e}^{c_1}+\cos\left(\theta\right)\right) \mathrm{e}^{-c_1}}{1+\sin\left(\theta\right)}$$

Warning, solution could not be verified

5.17.6 Maple step by step solution

Let's solve

$\left[\cos\left(\theta\right) v'+v=3, v\left(\frac{\pi}{2}\right)=1\right]$

- Highest derivative means the order of the ODE is 1

  v'

- Separate variables

  $\frac{v'}{-v+3}=\frac{1}{\cos(\theta)}$

- Integrate both sides with respect to θ

  $\int \frac{v'}{-v+3}d\theta=\int \frac{1}{\cos(\theta)}d\theta+c_1$

- Evaluate integral

  $-\ln\left(-v+3\right)=\ln\left(\sec\left(\theta\right)+\tan\left(\theta\right)\right)+c_1$

- Solve for v

  $v=\frac{3\,\mathrm{e}^{c_1}\sin(\theta)+3\,\mathrm{e}^{c_1}-\cos(\theta)}{\mathrm{e}^{c_1}(1+\sin(\theta))}$

- Use initial condition $v\left(\frac{\pi}{2}\right) = 1$

  $$1 = 3$$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✗ Solution by Maple

```
dsolve([cos(theta)*diff(v(theta),theta)+v(theta)=3,v(1/2*Pi) = 1],v(theta), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{Cos[theta]*v'[theta]+v[theta]==3,{v[Pi/2]==1}},v[theta],theta,IncludeSingularSolutio
```

$$\{\}$$

5.18 problem 3(f)

5.18.1 Existence and uniqueness analysis 697
5.18.2 Solving as linear ode 697
5.18.3 Solving as homogeneousTypeD2 ode 699
5.18.4 Solving as first order ode lie symmetry lookup ode 700
5.18.5 Solving as exact ode 704
5.18.6 Maple step by step solution 709

Internal problem ID [11416]
Internal file name [`OUTPUT/10398_Wednesday_May_17_2023_08_10_40_PM_14770727/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 3(f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{R' - \frac{R}{t} = t\,\mathrm{e}^{-t}}$$

With initial conditions

$$[R(1) = 1]$$

5.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$R' + p(t)R = q(t)$$

Where here

$$\begin{aligned} p(t) &= -\frac{1}{t} \\ q(t) &= t\,\mathrm{e}^{-t} \end{aligned}$$

Hence the ode is

$$R' - \frac{R}{t} = t\,\mathrm{e}^{-t}$$

The domain of $p(t) = -\frac{1}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = t\,\mathrm{e}^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

5.18.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{t}dt} \\ &= \frac{1}{t} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu R) &= (\mu)\left(t\,\mathrm{e}^{-t}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R}{t}\right) &= \left(\frac{1}{t}\right)\left(t\,\mathrm{e}^{-t}\right) \\ \mathrm{d}\left(\frac{R}{t}\right) &= \mathrm{e}^{-t}\,\mathrm{d}t \end{aligned}$$

Integrating gives

$$\frac{R}{t} = \int \mathrm{e}^{-t}\, \mathrm{d}t$$

$$\frac{R}{t} = -\mathrm{e}^{-t} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t}$ results in

$$R = -t\,\mathrm{e}^{-t} + c_1 t$$

which simplifies to

$$R = t\left(-\mathrm{e}^{-t} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $R = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\mathrm{e}^{-1} + c_1$$

$$c_1 = \mathrm{e}^{-1} + 1$$

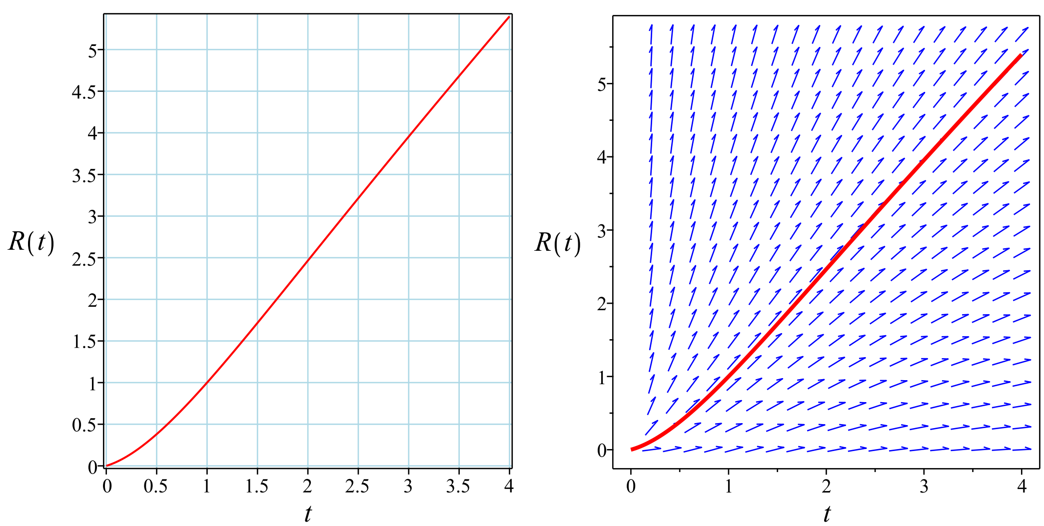
Substituting c_1 found above in the general solution gives

$$R = -t\left(\mathrm{e}^{-t} - 1 - \mathrm{e}^{-1}\right)$$

Summary

The solution(s) found are the following

$$R = -t\left(\mathrm{e}^{-t} - 1 - \mathrm{e}^{-1}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
R = -t\left(\mathrm{e}^{-t} - 1 - \mathrm{e}^{-1}\right)
$$

Verified OK.

5.18.3 Solving as homogeneousTypeD2 ode

Using the change of variables $R = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$
u'(t)\,t = t\,\mathrm{e}^{-t}
$$

Integrating both sides gives

$$
\begin{aligned}
u(t) &= \int \mathrm{e}^{-t}\,\mathrm{d}t \\
&= -\mathrm{e}^{-t} + c_2
\end{aligned}
$$

Therefore the solution R is

$$
\begin{aligned}
R &= tu \\
&= t\left(-\mathrm{e}^{-t} + c_2\right)
\end{aligned}
$$

Initial conditions are used to solve for c_2. Substituting $t = 1$ and $R = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = -\mathrm{e}^{-1} + c_2
$$

$$
c_2 = \mathrm{e}^{-1} + 1
$$

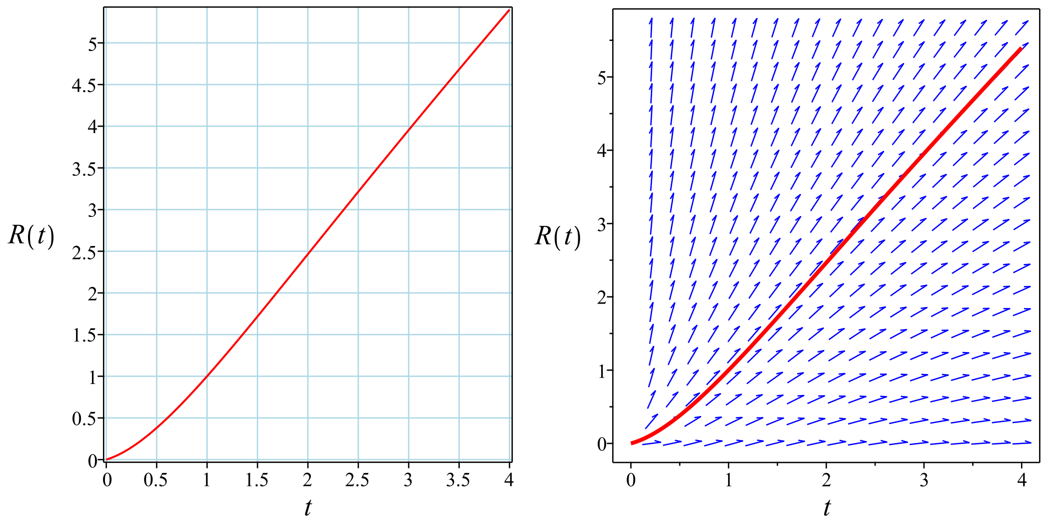
Substituting c_2 found above in the general solution gives

$$
R = -t\left(\mathrm{e}^{-t} - 1 - \mathrm{e}^{-1}\right)
$$

Summary

The solution(s) found are the following

$$
R = -t\left(\mathrm{e}^{-t} - 1 - \mathrm{e}^{-1}\right) \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$R = -t\left(\mathrm{e}^{-t} - 1 - \mathrm{e}^{-1}\right)$$

Verified OK.

5.18.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$R' = \frac{t^2\mathrm{e}^{-t} + R}{t}$$
$$R' = \omega(t, R)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_R - \xi_t) - \omega^2\xi_R - \omega_t\xi - \omega_R\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 139: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, R) &= 0 \\
\eta(t, R) &= t
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, R) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dR}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial R}\right) S(t, R) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{t} dy \end{aligned}$$

Which results in

$$S = \frac{R}{t}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, R) S_R}{R_t + \omega(t, R) R_R} \tag{2}$$

Where in the above R_t, R_R, S_t, S_R are all partial derivatives and $\omega(t, R)$ is the right hand side of the original ode given by

$$\omega(t, R) = \frac{t^2 \mathrm{e}^{-t} + R}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_R &= 0 \\ S_t &= -\frac{R}{t^2} \\ S_R &= \frac{1}{t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{-t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, R in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, R coordinates. This results in

$$\frac{R}{t} = -\mathrm{e}^{-t} + c_1$$

Which simplifies to

$$\frac{R}{t} = -\mathrm{e}^{-t} + c_1$$

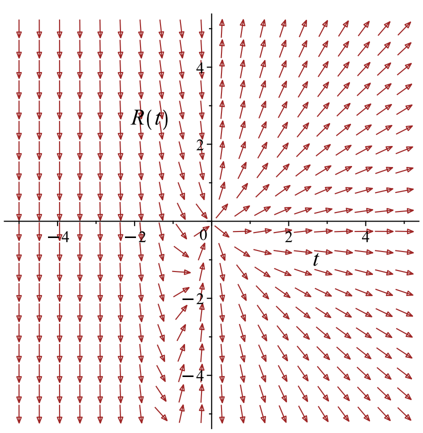
Which gives

$$R = -t\left(\mathrm{e}^{-t} - c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, R coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dR}{dt} = \frac{t^2\mathrm{e}^{-t}+R}{t}$



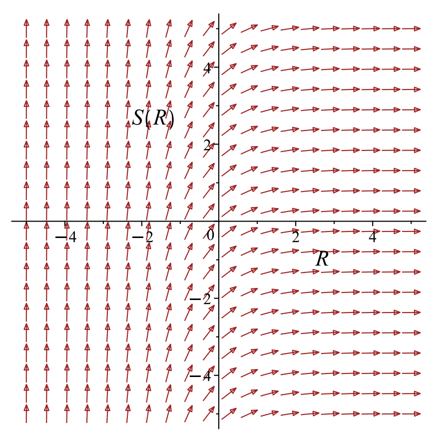
</td><td>

$R = t$

$S = \frac{R}{t}$

</td><td>

$\frac{dS}{dR} = \mathrm{e}^{-R}$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $R = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\mathrm{e}^{-1} + c_1$$

$$c_1 = e^{-1} + 1$$

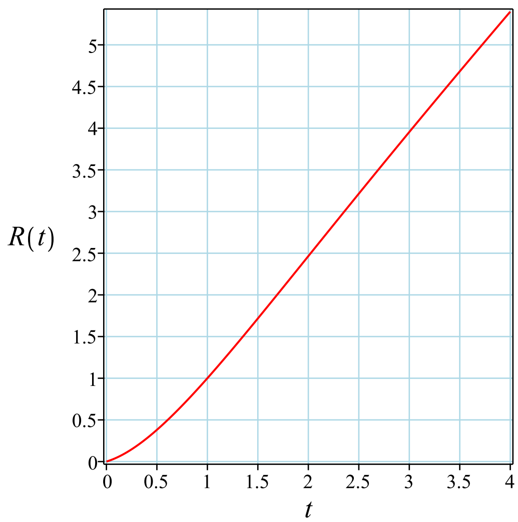
Substituting c_1 found above in the general solution gives

$$R = -t\left(e^{-t} - 1 - e^{-1}\right)$$

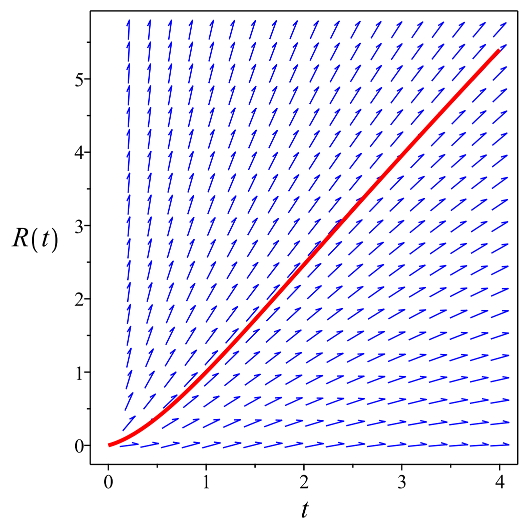
Summary

The solution(s) found are the following

$$R = -t\left(e^{-t} - 1 - e^{-1}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$R = -t\left(e^{-t} - 1 - e^{-1}\right)$$

Verified OK.

5.18.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,R)\,\mathrm{d}t + N(t,R)\,\mathrm{d}R = 0 \tag{1A}$$

Therefore

$$\mathrm{d}R = \left(\frac{R}{t} + t\,\mathrm{e}^{-t}\right)\mathrm{d}t$$

$$\left(-\frac{R}{t} - t\,\mathrm{e}^{-t}\right)\mathrm{d}t + \mathrm{d}R = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,R) &= -\frac{R}{t} - t\,\mathrm{e}^{-t} \\ N(t,R) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial R} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial R} &= \frac{\partial}{\partial R}\left(-\frac{R}{t} - t\,\mathrm{e}^{-t}\right) \\ &= -\frac{1}{t}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial R} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial R} - \frac{\partial N}{\partial t}\right) \\ &= 1\left(\left(-\frac{1}{t}\right) - (0)\right) \\ &= -\frac{1}{t}\end{aligned}$$

Since A does not depend on R, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int -\frac{1}{t} \, \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(t)} \\ &= \frac{1}{t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{t}\left(-\frac{R}{t} - t\,\mathrm{e}^{-t}\right) \\ &= \frac{-t^2\mathrm{e}^{-t} - R}{t^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{t}(1) \\ &= \frac{1}{t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}R}{\mathrm{d}t} = 0$$

$$\left(\frac{-t^2\mathrm{e}^{-t} - R}{t^2}\right) + \left(\frac{1}{t}\right)\frac{\mathrm{d}R}{\mathrm{d}t} = 0$$

The following equations are now set up to solve for the function $\phi(t, R)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial R} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \frac{-t^2\mathrm{e}^{-t} - R}{t^2}\,\mathrm{d}t$$

$$\phi = \frac{t\,\mathrm{e}^{-t} + R}{t} + f(R) \tag{3}$$

Where $f(R)$ is used for the constant of integration since ϕ is a function of both t and R. Taking derivative of equation (3) w.r.t R gives

$$\frac{\partial \phi}{\partial R} = \frac{1}{t} + f'(R) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial R} = \frac{1}{t}$. Therefore equation (4) becomes

$$\frac{1}{t} = \frac{1}{t} + f'(R) \tag{5}$$

Solving equation (5) for $f'(R)$ gives

$$f'(R) = 0$$

Therefore

$$f(R) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(R)$ into equation (3) gives ϕ

$$\phi = \frac{t\,\mathrm{e}^{-t} + R}{t} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{t\,\mathrm{e}^{-t} + R}{t}$$

The solution becomes

$$R = -t\left(\mathrm{e}^{-t} - c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $R = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\mathrm{e}^{-1} + c_1$$

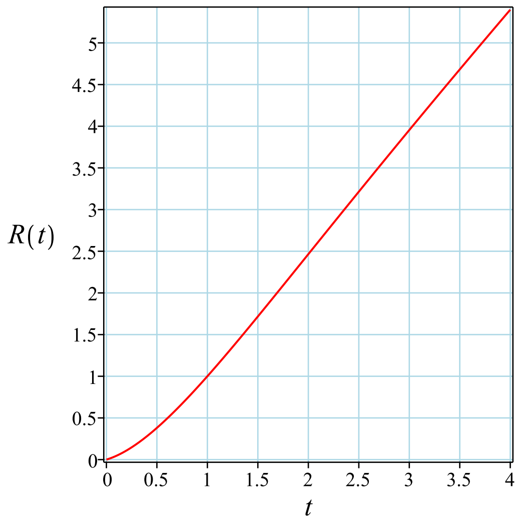
$$c_1 = \mathrm{e}^{-1} + 1$$

Substituting c_1 found above in the general solution gives

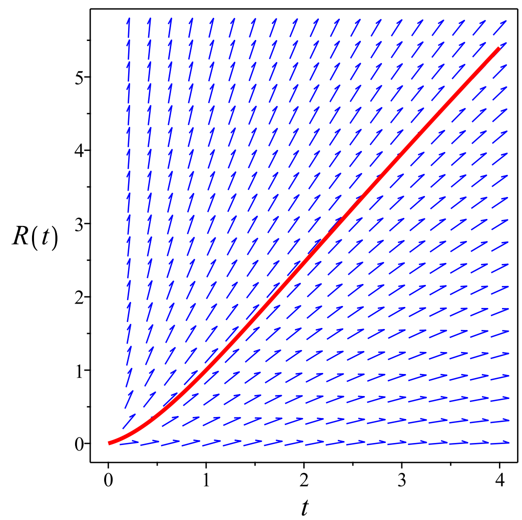
$$R = -t\left(\mathrm{e}^{-t} - 1 - \mathrm{e}^{-1}\right)$$

Summary
The solution(s) found are the following

$$R = -t\left(\mathrm{e}^{-t} - 1 - \mathrm{e}^{-1}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$R = -t\left(e^{-t} - 1 - e^{-1}\right)$$

Verified OK.

5.18.6 Maple step by step solution

Let's solve

$\left[R' - \frac{R}{t} = t\,e^{-t}, R(1) = 1\right]$

- Highest derivative means the order of the ODE is 1

  R'

- Isolate the derivative

  $R' = \frac{R}{t} + t\,e^{-t}$

- Group terms with R on the lhs of the ODE and the rest on the rhs of the ODE

  $R' - \frac{R}{t} = t\,e^{-t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(R' - \frac{R}{t}\right) = \mu(t)\,t\,e^{-t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,R)$

  $\mu(t)\left(R' - \frac{R}{t}\right) = \mu'(t)\,R + \mu(t)\,R'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{\mu(t)}{t}$
- Solve to find the integrating factor

  $\mu(t) = \frac{1}{t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, R)\right) dt = \int \mu(t)\, t\, \mathrm{e}^{-t} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, R = \int \mu(t)\, t\, \mathrm{e}^{-t} dt + c_1$
- Solve for R

  $R = \frac{\int \mu(t) t\, \mathrm{e}^{-t} dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \frac{1}{t}$

  $R = t\left(\int \mathrm{e}^{-t} dt + c_1\right)$
- Evaluate the integrals on the rhs

  $R = t(-\mathrm{e}^{-t} + c_1)$
- Use initial condition $R(1) = 1$

  $1 = -\mathrm{e}^{-1} + c_1$
- Solve for c_1

  $c_1 = \mathrm{e}^{-1} + 1$
- Substitute $c_1 = \mathrm{e}^{-1} + 1$ into general solution and simplify

  $R = t(-\mathrm{e}^{-t} + \mathrm{e}^{-1} + 1)$
- Solution to the IVP

  $R = t(-\mathrm{e}^{-t} + \mathrm{e}^{-1} + 1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(R(t),t)=R(t)/t+t*exp(-t),R(1) = 1],R(t), singsol=all)
```

$$R(t) = \left(-\mathrm{e}^{-t} + 1 + \mathrm{e}^{-1}\right) t$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 19

```
DSolve[{R'[t]==R[t]/t+t*Exp[-t],{R[1]==1}},R[t],t,IncludeSingularSolutions -> True]
```

$$R(t) \rightarrow \left(-e^{-t} + 1 + \frac{1}{e}\right) t$$

5.19 problem 4

5.19.1 Solving as linear ode 712
5.19.2 Solving as first order ode lie symmetry lookup ode 714
5.19.3 Solving as exact ode 717
5.19.4 Maple step by step solution 721

Internal problem ID [11417]
Internal file name [`OUTPUT/10399_Wednesday_May_17_2023_08_10_41_PM_57528556/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + ay = \sqrt{1+t}}$$

5.19.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= a \\ q(t) &= \sqrt{1+t} \end{aligned}$$

Hence the ode is

$$y' + ay = \sqrt{1+t}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int a dt} \\ &= \mathrm{e}^{ta}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\sqrt{1+t}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{ta} y\right) &= \left(\mathrm{e}^{ta}\right)\left(\sqrt{1+t}\right) \\ \mathrm{d}\left(\mathrm{e}^{ta} y\right) &= \left(\sqrt{1+t}\,\mathrm{e}^{ta}\right)\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{ta} y &= \int \sqrt{1+t}\,\mathrm{e}^{ta}\,\mathrm{d}t \\ \mathrm{e}^{ta} y &= 2\,\mathrm{e}^{-a}\left(\frac{\sqrt{1+t}\,\mathrm{e}^{(1+t)a}}{2a} - \frac{\sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)}{4a\sqrt{-a}}\right) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{ta}$ results in

$$y = 2\,\mathrm{e}^{-ta}\mathrm{e}^{-a}\left(\frac{\sqrt{1+t}\,\mathrm{e}^{(1+t)a}}{2a} - \frac{\sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)}{4a\sqrt{-a}}\right) + c_1\mathrm{e}^{-ta}$$

which simplifies to

$$y = \frac{2c_1\mathrm{e}^{-ta}(-a)^{\frac{3}{2}} + \sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)\mathrm{e}^{-(1+t)a} - 2\sqrt{-a}\,\sqrt{1+t}}{2\,(-a)^{\frac{3}{2}}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{2c_1\mathrm{e}^{-ta}(-a)^{\frac{3}{2}} + \sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)\mathrm{e}^{-(1+t)a} - 2\sqrt{-a}\,\sqrt{1+t}}{2\,(-a)^{\frac{3}{2}}} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{2c_1\mathrm{e}^{-ta}(-a)^{\frac{3}{2}} + \sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)\mathrm{e}^{-(1+t)a} - 2\sqrt{-a}\,\sqrt{1+t}}{2\,(-a)^{\frac{3}{2}}}$$

Verified OK.

5.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -ay + \sqrt{1+t} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 142: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= 0 \\
\eta(t,y) &= \mathrm{e}^{-ta}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{-ta}} dy
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{ta} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$
\omega(t,y) = -ay + \sqrt{1+t}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= a\,\mathrm{e}^{ta} y \\
S_y &= \mathrm{e}^{ta}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sqrt{1+t}\,\mathrm{e}^{ta} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sqrt{1+R}\,\mathrm{e}^{Ra}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\,\mathrm{e}^{-a}\left(\frac{\sqrt{1+R}\,\mathrm{e}^{(1+R)a}}{2a} - \frac{\sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+R}\right)}{4a\sqrt{-a}}\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{ta}y = 2\,\mathrm{e}^{-a}\left(\frac{\sqrt{1+t}\,\mathrm{e}^{(1+t)a}}{2a} - \frac{\sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)}{4a\sqrt{-a}}\right) + c_1$$

Which simplifies to

$$\mathrm{e}^{ta}y = 2\,\mathrm{e}^{-a}\left(\frac{\sqrt{1+t}\,\mathrm{e}^{(1+t)a}}{2a} - \frac{\sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)}{4a\sqrt{-a}}\right) + c_1$$

Which gives

$$y = \frac{\left(2c_1\mathrm{e}^{a}a\sqrt{-a} + 2\sqrt{1+t}\,\mathrm{e}^{(1+t)a}\sqrt{-a} - \sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)\right)\mathrm{e}^{-ta}\mathrm{e}^{-a}}{2a\sqrt{-a}}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(2c_1\mathrm{e}^{a}a\sqrt{-a} + 2\sqrt{1+t}\,\mathrm{e}^{(1+t)a}\sqrt{-a} - \sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)\right)\mathrm{e}^{-ta}\mathrm{e}^{-a}}{2a\sqrt{-a}} \tag{1}$$

Verification of solutions

$$y = \frac{\left(2c_1\mathrm{e}^{a}a\sqrt{-a} + 2\sqrt{1+t}\,\mathrm{e}^{(1+t)a}\sqrt{-a} - \sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)\right)\mathrm{e}^{-ta}\mathrm{e}^{-a}}{2a\sqrt{-a}}$$

Verified OK.

5.19.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(-ay+\sqrt{1+t}\right)\mathrm{d}t$$

$$\left(ay-\sqrt{1+t}\right)\mathrm{d}t+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=ay-\sqrt{1+t}\\ N(t,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(ay - \sqrt{1+t}\right)\\ &= a\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= 1((a) - (0))\\ &= a\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}t}\\ &= e^{\int a \, \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{ta}\\ &= \mathrm{e}^{ta}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{ta}\left(ay - \sqrt{1+t}\right)\\ &= \left(ay - \sqrt{1+t}\right)\mathrm{e}^{ta}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{ta}(1) \\
&= \mathrm{e}^{ta}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(\left(ay - \sqrt{1+t}\right)\mathrm{e}^{ta}\right) + \left(\mathrm{e}^{ta}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \left(ay - \sqrt{1+t}\right)\mathrm{e}^{ta}\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = \frac{\mathrm{e}^{-a}\left(2y\,\mathrm{e}^{(1+t)a}(-a)^{\frac{3}{2}} + 2\sqrt{1+t}\,\mathrm{e}^{(1+t)a}\sqrt{-a} - \sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)\right)}{2\,(-a)^{\frac{3}{2}}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \mathrm{e}^{(1+t)a}\mathrm{e}^{-a} + f'(y) \qquad (4) \\
&= \mathrm{e}^{ta} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{ta}$. Therefore equation (4) becomes

$$
\mathrm{e}^{ta} = \mathrm{e}^{ta} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\mathrm{e}^{-a}\left(2y\,\mathrm{e}^{(1+t)a}(-a)^{\frac{3}{2}} + 2\sqrt{1+t}\,\mathrm{e}^{(1+t)a}\sqrt{-a} - \sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)\right)}{2\,(-a)^{\frac{3}{2}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\mathrm{e}^{-a}\left(2y\,\mathrm{e}^{(1+t)a}(-a)^{\frac{3}{2}} + 2\sqrt{1+t}\,\mathrm{e}^{(1+t)a}\sqrt{-a} - \sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)\right)}{2\,(-a)^{\frac{3}{2}}}$$

The solution becomes

$$y = -\frac{\left(-2(-a)^{\frac{3}{2}}\,c_1 + 2\sqrt{1+t}\,\mathrm{e}^{(1+t)a}\sqrt{-a}\,\mathrm{e}^{-a} - \sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)\mathrm{e}^{-a}\right)\mathrm{e}^{-(1+t)a}\mathrm{e}^{a}}{2\,(-a)^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(-2(-a)^{\frac{3}{2}}\,c_1 + 2\sqrt{1+t}\,\mathrm{e}^{(1+t)a}\sqrt{-a}\,\mathrm{e}^{-a} - \sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)\mathrm{e}^{-a}\right)\mathrm{e}^{-(1+t)a}\mathrm{e}^{a}}{2\,(-a)^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(-2(-a)^{\frac{3}{2}}\,c_1 + 2\sqrt{1+t}\,\mathrm{e}^{(1+t)a}\sqrt{-a}\,\mathrm{e}^{-a} - \sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)\mathrm{e}^{-a}\right)\mathrm{e}^{-(1+t)a}\mathrm{e}^{a}}{2\,(-a)^{\frac{3}{2}}}$$

Verified OK.

5.19.4 Maple step by step solution

Let's solve

$y' + ay = \sqrt{1+t}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -ay + \sqrt{1+t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + ay = \sqrt{1+t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' + ay) = \mu(t)\,\sqrt{1+t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' + ay) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)\,a$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{ta}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\,\sqrt{1+t}dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \mu(t)\,\sqrt{1+t}dt + c_1$

- Solve for y

  $y = \frac{\int \mu(t)\sqrt{1+t}dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{ta}$

  $y = \frac{\int \sqrt{1+t}\,\mathrm{e}^{ta}dt + c_1}{\mathrm{e}^{ta}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{2\left(\frac{\sqrt{1+t}\,\mathrm{e}^{(1+t)a}}{2a} - \frac{\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)}{4a\sqrt{-a}}\right)}{\mathrm{e}^{a}} + c_1}{\mathrm{e}^{ta}}$

- Simplify

$$y = \frac{2c_1 \mathrm{e}^{-ta}(-a)^{\frac{3}{2}} + \sqrt{\pi}\,\mathrm{erf}\left(\sqrt{-a}\,\sqrt{1+t}\right)\mathrm{e}^{-(1+t)a} - 2\sqrt{-a}\,\sqrt{1+t}}{2(-a)^{\frac{3}{2}}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 60

```
dsolve(diff(y(t),t)+a*y(t)=sqrt(1+t),y(t), singsol=all)
```

$$y(t) = \frac{2c_1 \mathrm{e}^{-at}(-a)^{\frac{3}{2}} - 2\sqrt{-a}\,\sqrt{t+1} + \sqrt{\pi}\ \mathrm{erf}\left(\sqrt{-a}\,\sqrt{t+1}\right)\mathrm{e}^{-(t+1)a}}{2\,(-a)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.461 (sec). Leaf size: 49

```
DSolve[y'[t]+a*y[t]==Sqrt[1+t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-at}\left(\frac{ae^{-a}(t+1)^{5/2}\Gamma\left(\frac{3}{2}, -a(t+1)\right)}{(-a(t+1))^{5/2}} + c_1\right)$$

5.20 problem 5

5.20.1 Solving as separable ode 723
5.20.2 Solving as linear ode 725
5.20.3 Solving as homogeneousTypeD2 ode 726
5.20.4 Solving as first order ode lie symmetry lookup ode 728
5.20.5 Solving as exact ode 732
5.20.6 Maple step by step solution 736

Internal problem ID [11418]
Internal file name [`OUTPUT/10400_Wednesday_May_17_2023_08_10_42_PM_27763354/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{x' - 2xt = 0}$$

5.20.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
x' &= F(t, x) \\
&= f(t)g(x) \\
&= 2tx
\end{aligned}$$

Where $f(t) = 2t$ and $g(x) = x$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{x}\,dx &= 2t\,dt \\
\int \frac{1}{x}\,dx &= \int 2t\,dt \\
\ln(x) &= t^2 + c_1 \\
x &= \mathrm{e}^{t^2+c_1} \\
&= c_1 \mathrm{e}^{t^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1 \mathrm{e}^{t^2} \tag{1}
$$

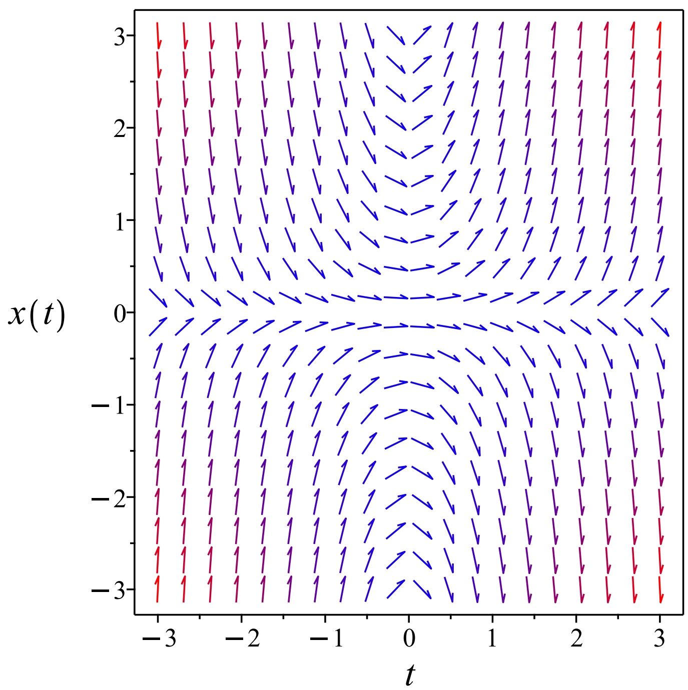


Figure 160: Slope field plot

Verification of solutions

$$
x = c_1 \mathrm{e}^{t^2}
$$

Verified OK.

5.20.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned}p(t) &= -2t\\ q(t) &= 0\end{aligned}$$

Hence the ode is

$$x' - 2xt = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -2t dt}\\ &= \mathrm{e}^{-t^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu x &= 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-t^2}x\right) &= 0\end{aligned}$$

Integrating gives

$$\mathrm{e}^{-t^2}x = c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-t^2}$ results in

$$x = c_1\mathrm{e}^{t^2}$$

Summary
The solution(s) found are the following

$$x = c_1\mathrm{e}^{t^2} \tag{1}$$

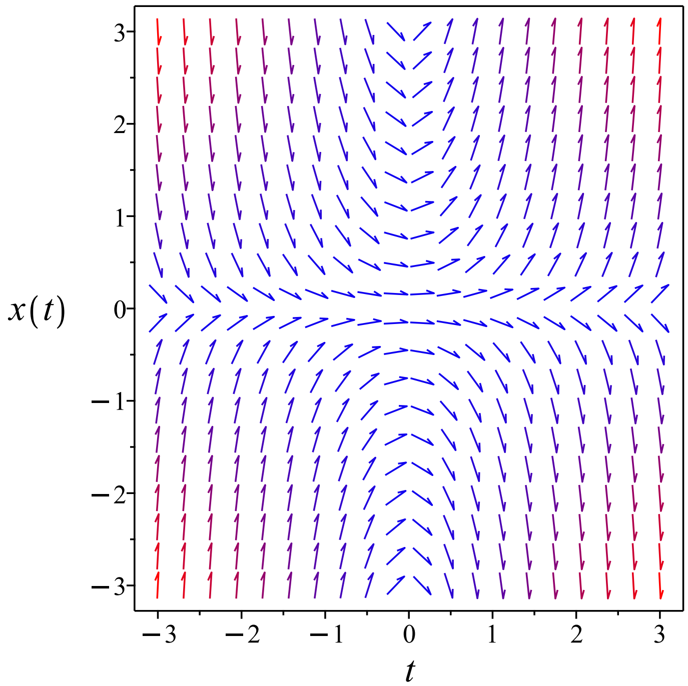


Figure 161: Slope field plot

Verification of solutions

$$x = c_1 e^{t^2}$$

Verified OK.

5.20.3 Solving as homogeneousTypeD2 ode

Using the change of variables $x = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u'(t)\,t + u(t) - 2u(t)\,t^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= \frac{u(2t^2 - 1)}{t} \end{aligned}$$

Where $f(t) = \frac{2t^2-1}{t}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{2t^2-1}{t}\,dt \\
\int \frac{1}{u}\,du &= \int \frac{2t^2-1}{t}\,dt \\
\ln(u) &= t^2 - \ln(t) + c_2 \\
u &= e^{t^2-\ln(t)+c_2} \\
&= c_2 e^{t^2-\ln(t)}
\end{aligned}
$$

Which simplifies to

$$
u(t) = \frac{c_2 e^{t^2}}{t}
$$

Therefore the solution x is

$$
\begin{aligned}
x &= tu \\
&= c_2 e^{t^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_2 e^{t^2} \tag{1}
$$

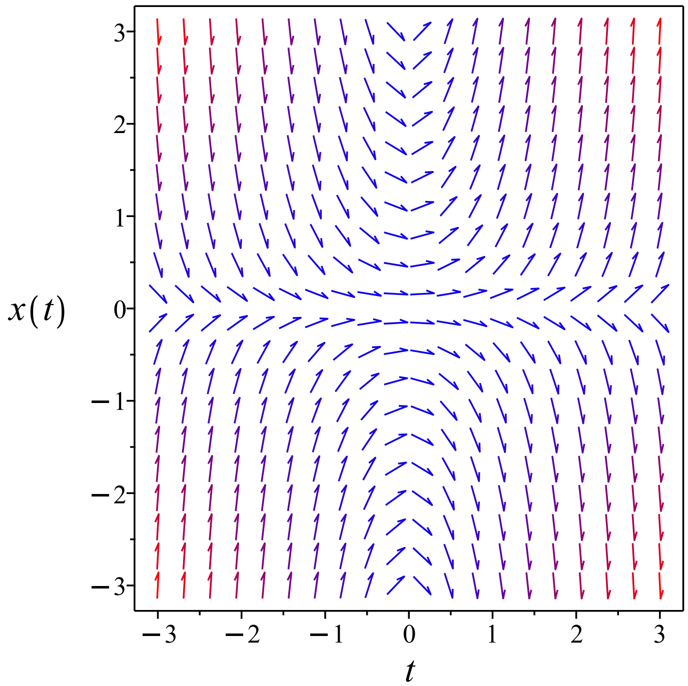


Figure 162: Slope field plot

<u>Verification of solutions</u>

$$x = c_2 e^{t^2}$$

Verified OK.

5.20.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} x' &= 2tx \\ x' &= \omega(t, x) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 145: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \mathrm{e}^{t^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{t^2}} dy \end{aligned}$$

Which results in

$$S = e^{-t^2} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = 2tx$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= -2\, e^{-t^2} tx \\ S_x &= e^{-t^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$e^{-t^2} x = c_1$$

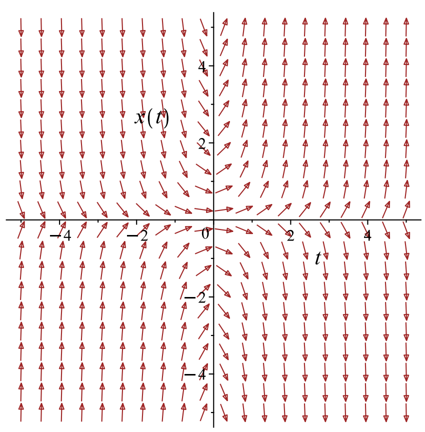
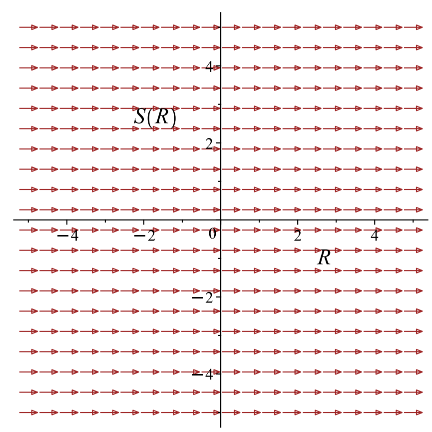
Which simplifies to

$$e^{-t^2} x = c_1$$

Which gives

$$x = c_1 e^{t^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, x coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dx}{dt} = 2tx$

</td><td>$R = t$
$S = e^{-t^2} x$</td><td>$\frac{dS}{dR} = 0$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$x = c_1 e^{t^2} \tag{1}$$

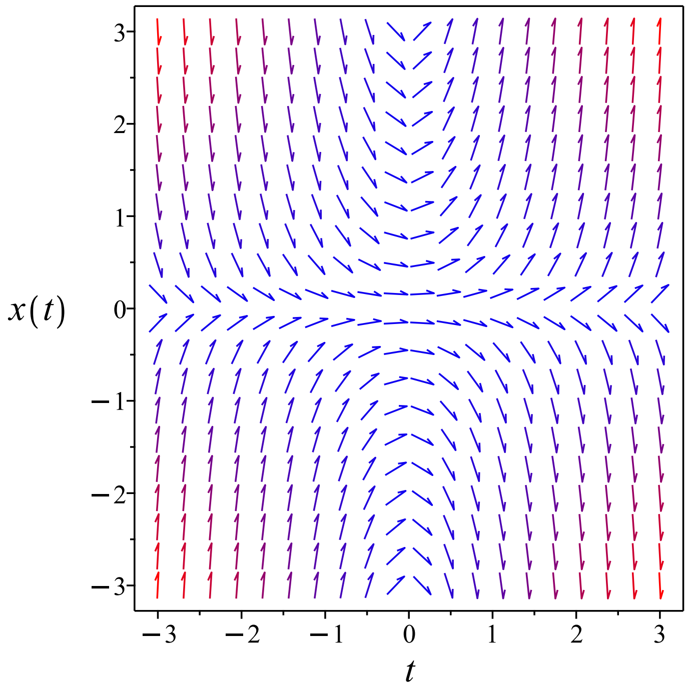


Figure 163: Slope field plot

<u>Verification of solutions</u>

$$x = c_1 e^{t^2}$$

Verified OK.

5.20.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, x)\, \mathrm{d}t + N(t, x)\, \mathrm{d}x = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{2x}\right) \mathrm{d}x &= (t)\, \mathrm{d}t \\
(-t)\, \mathrm{d}t + \left(\frac{1}{2x}\right) \mathrm{d}x &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, x) &= -t \\
N(t, x) &= \frac{1}{2x}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-t) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{2x}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -t \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{t^2}{2} + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{2x}$. Therefore equation (4) becomes

$$
\frac{1}{2x} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = \frac{1}{2x}
$$

Integrating the above w.r.t x gives

$$
\begin{aligned}
\int f'(x) \, \mathrm{d}x &= \int \left(\frac{1}{2x}\right) \mathrm{d}x \\
f(x) &= \frac{\ln (x)}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} + \frac{\ln(x)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} + \frac{\ln(x)}{2}$$

The solution becomes

$$x = e^{t^2+2c_1}$$

Summary

The solution(s) found are the following

$$x = e^{t^2+2c_1} \tag{1}$$

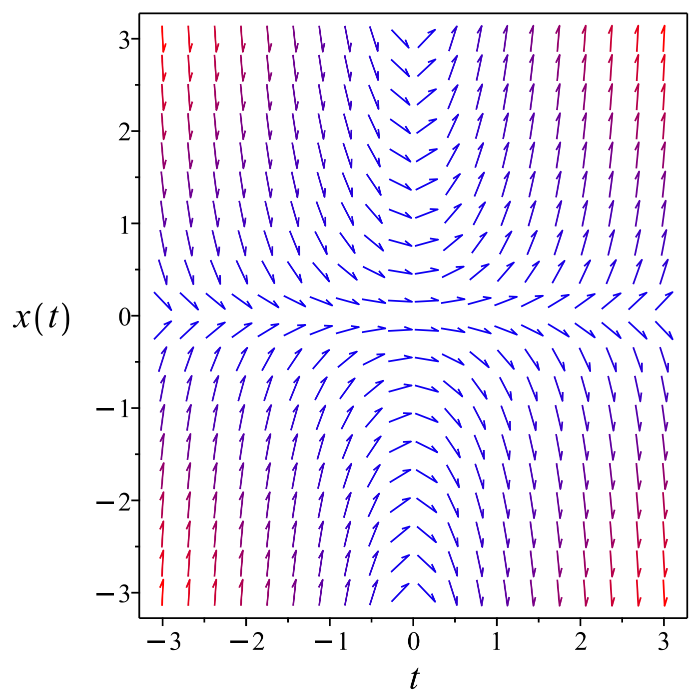


Figure 164: Slope field plot

Verification of solutions

$$x = e^{t^2+2c_1}$$

Verified OK.

5.20.6 Maple step by step solution

Let's solve

$x' - 2xt = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x} = 2t$

- Integrate both sides with respect to t

  $\int \frac{x'}{x} dt = \int 2t dt + c_1$

- Evaluate integral

  $\ln(x) = t^2 + c_1$

- Solve for x

  $x = e^{t^2+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(x(t),t)=2*t*x(t),x(t), singsol=all)
```

$$x(t) = c_1 e^{t^2}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 18

```
DSolve[x'[t]==2*t*x[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^{t^2}$$
$$x(t) \to 0$$

5.21 problem 6

5.21.1 Existence and uniqueness analysis 738
5.21.2 Solving as linear ode 739
5.21.3 Solving as first order ode lie symmetry lookup ode 740
5.21.4 Solving as exact ode 744
5.21.5 Maple step by step solution 749

Internal problem ID [11419]
Internal file name [`OUTPUT/10401_Wednesday_May_17_2023_08_10_44_PM_53964256/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x' + \frac{e^{-t}x}{t} = t}$$

With initial conditions

$$[x(1) = 0]$$

5.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = \frac{e^{-t}}{t}$$
$$q(t) = t$$

Hence the ode is

$$x' + \frac{e^{-t} x}{t} = t$$

The domain of $p(t) = \frac{e^{-t}}{t}$ is

$$\{t < 0 \vee 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

5.21.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int_0^t \frac{e^{-\_a}}{\_a} d\_a}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dt}(\mu x) &= (\mu)(t) \\
\frac{d}{dt}\left(e^{\int_0^t \frac{e^{-_a}}{_a} d_a} x\right) &= \left(e^{\int_0^t \frac{e^{-_a}}{_a} d_a}\right)(t) \\
d\left(e^{\int_0^t \frac{e^{-_a}}{_a} d_a} x\right) &= \left(t\, e^{\int_0^t \frac{e^{-_a}}{_a} d_a}\right) dt
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{\int_0^t \frac{e^{-_a}}{_a} d_a} x &= \int t\, e^{\int_0^t \frac{e^{-_a}}{_a} d_a} dt \\
e^{\int_0^t \frac{e^{-_a}}{_a} d_a} x &= \int_0^t _a\, e^{\int_0^{_a} \frac{e^{-_a}}{_a} d_a} d_a + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\int_0^t \frac{e^{-\_a}}{\_a} d\_a}$ results in

$$x = e^{-\left(\int_0^t \frac{e^{-\_a}}{\_a} d\_a\right)} \left(\int_0^t \_a\, e^{\int_0^{\_a} \frac{e^{-\_a}}{\_a} d\_a} d\_a\right) + c_1 e^{-\left(\int_0^t \frac{e^{-\_a}}{\_a} d\_a\right)}$$

which simplifies to

$$x = e^{-\left(\int_0^t \frac{e^{-\_a}}{\_a} d\_a\right)} \left(\int_0^t \_a\, e^{\int_0^{\_a} \frac{e^{-\_a}}{\_a} d\_a} d\_a + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \mathrm{e}^{-\left(\int_0^1 \frac{\mathrm{e}^{-\_a}}{\_a} d\_a\right)} \left(\int_0^1 \_a\,\mathrm{e}^{\int_0^{\_a} \frac{\mathrm{e}^{-\_a}}{\_a} d\_a} d\_a + c_1\right)$$

$$c_1 = -\left(\int_0^1 \_a\,\mathrm{e}^{\int_0^{\_a} \frac{\mathrm{e}^{-\_a}}{\_a} d\_a} d\_a\right)$$

Substituting c_1 found above in the general solution gives

$$x = -\mathrm{e}^{-\left(\int_0^t \frac{\mathrm{e}^{-\_a}}{\_a} d\_a\right)} \left(\int_0^1 \_a\,\mathrm{e}^{\int_0^{\_a} \frac{\mathrm{e}^{-\_a}}{\_a} d\_a} d\_a\right) + \mathrm{e}^{-\left(\int_0^t \frac{\mathrm{e}^{-\_a}}{\_a} d\_a\right)} \left(\int_0^t \_a\,\mathrm{e}^{\int_0^{\_a} \frac{\mathrm{e}^{-\_a}}{\_a} d\_a} d\_a\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} x = &-\mathrm{e}^{-\left(\int_0^t \frac{\mathrm{e}^{-\_a}}{\_a} d\_a\right)} \left(\int_0^1 \_a\,\mathrm{e}^{\int_0^{\_a} \frac{\mathrm{e}^{-\_a}}{\_a} d\_a} d\_a\right) \\ &+ \mathrm{e}^{-\left(\int_0^t \frac{\mathrm{e}^{-\_a}}{\_a} d\_a\right)} \left(\int_0^t \_a\,\mathrm{e}^{\int_0^{\_a} \frac{\mathrm{e}^{-\_a}}{\_a} d\_a} d\_a\right) \end{aligned} \tag{1}$$

Verification of solutions

$$x = -\mathrm{e}^{-\left(\int_0^t \frac{\mathrm{e}^{-\_a}}{\_a} d\_a\right)} \left(\int_0^1 \_a\,\mathrm{e}^{\int_0^{\_a} \frac{\mathrm{e}^{-\_a}}{\_a} d\_a} d\_a\right) + \mathrm{e}^{-\left(\int_0^t \frac{\mathrm{e}^{-\_a}}{\_a} d\_a\right)} \left(\int_0^t \_a\,\mathrm{e}^{\int_0^{\_a} \frac{\mathrm{e}^{-\_a}}{\_a} d\_a} d\_a\right)$$

Verified OK.

5.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} x' &= -\frac{\mathrm{e}^{-t}x - t^2}{t} \\ x' &= \omega(t, x) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 148: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= 0 \\
\eta(t, x) &= \mathrm{e}^{\text{expIntegral}_1(t)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\mathrm{expIntegral}_1(t)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-\mathrm{expIntegral}_1(t)} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -\frac{\mathrm{e}^{-t} x - t^2}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= \frac{x\, \mathrm{e}^{-t-\mathrm{expIntegral}_1(t)}}{t} \\ S_x &= \mathrm{e}^{-\mathrm{expIntegral}_1(t)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{-\mathrm{expIntegral}_1(t)} t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{-\mathrm{expIntegral}_1(R)} R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \mathrm{e}^{-\operatorname{expIntegral}_1(R)} R dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\mathrm{e}^{-\operatorname{expIntegral}_1(t)} x = \int \mathrm{e}^{-\operatorname{expIntegral}_1(t)} t dt + c_1$$

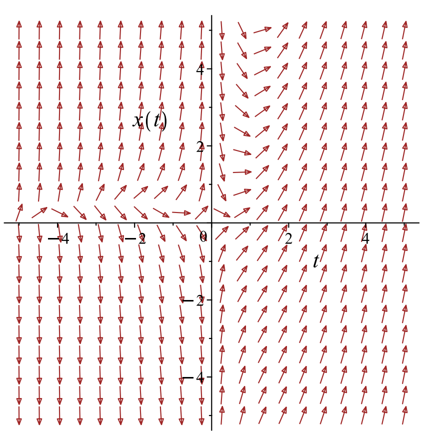
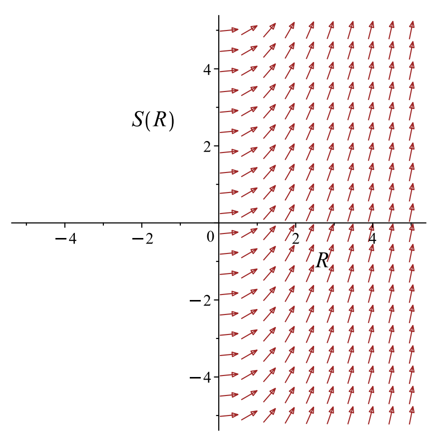
Which simplifies to

$$\mathrm{e}^{-\operatorname{expIntegral}_1(t)} x = \int \mathrm{e}^{-\operatorname{expIntegral}_1(t)} t dt + c_1$$

Which gives

$$x = \left(\int \mathrm{e}^{-\operatorname{expIntegral}_1(t)} t dt + c_1\right) \mathrm{e}^{\operatorname{expIntegral}_1(t)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\frac{\mathrm{e}^{-t} x - t^2}{t}$ 	$R = t$ $S = \mathrm{e}^{-\operatorname{expIntegral}_1(t)} x$	$\frac{dS}{dR} = \mathrm{e}^{-\operatorname{expIntegral}_1(R)} R$ 

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \left(\int^{1} \mathrm{e}^{-\operatorname{expIntegral}_1(\_a)} \_ad\_a + c_1\right) \mathrm{e}^{\operatorname{expIntegral}_1(1)}$$

$$c_1 = -\left(\int^{1} \mathrm{e}^{-\operatorname{expIntegral}_1(\_a)} \_ad\_a\right)$$

Substituting c_1 found above in the general solution gives

$$x = \left(\int \mathrm{e}^{-\operatorname{expIntegral}_1(t)} t dt - \left(\int^{1} \mathrm{e}^{-\operatorname{expIntegral}_1(\_a)} \_ad\_a\right)\right) \mathrm{e}^{\operatorname{expIntegral}_1(t)}$$

Summary
The solution(s) found are the following

$$x = \left(\int \mathrm{e}^{-\operatorname{expIntegral}_1(t)} t dt - \left(\int^{1} \mathrm{e}^{-\operatorname{expIntegral}_1(\_a)} \_ad\_a\right)\right) \mathrm{e}^{\operatorname{expIntegral}_1(t)} \tag{1}$$

Verification of solutions

$$x = \left(\int \mathrm{e}^{-\operatorname{expIntegral}_1(t)} t dt - \left(\int^{1} \mathrm{e}^{-\operatorname{expIntegral}_1(\_a)} \_ad\_a\right)\right) \mathrm{e}^{\operatorname{expIntegral}_1(t)}$$

Verified OK.

5.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\, \mathrm{d}t + N(t, x)\, \mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}x &= \left(-\frac{\mathrm{e}^{-t} x}{t} + t\right) \mathrm{d}t \\ \left(\frac{\mathrm{e}^{-t} x}{t} - t\right) \mathrm{d}t + \mathrm{d}x &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, x) &= \frac{\mathrm{e}^{-t} x}{t} - t \\ N(t, x) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\mathrm{e}^{-t} x}{t} - t\right) \\ &= \frac{\mathrm{e}^{-t}}{t}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right)\\ &= 1\left(\left(\frac{\mathrm{e}^{-t}}{t}\right) - (0)\right)\\ &= \frac{\mathrm{e}^{-t}}{t}\end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\operatorname{expIntegral}_1(t)}\\ &= \mathrm{e}^{-\operatorname{expIntegral}_1(t)}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-\operatorname{expIntegral}_1(t)}\left(\frac{\mathrm{e}^{-t}x}{t} - t\right)\\ &= -\frac{\mathrm{e}^{-\operatorname{expIntegral}_1(t)}(-\mathrm{e}^{-t}x + t^2)}{t}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-\operatorname{expIntegral}_1(t)}(1)\\ &= \mathrm{e}^{-\operatorname{expIntegral}_1(t)}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

$$\left(-\frac{\mathrm{e}^{-\operatorname{expIntegral}_1(t)}(-\mathrm{e}^{-t}x + t^2)}{t}\right) + \left(\mathrm{e}^{-\operatorname{expIntegral}_1(t)}\right)\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$

$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int -\frac{\mathrm{e}^{-\operatorname{expIntegral}_1(t)}(-\mathrm{e}^{-t}x + t^2)}{t}\,\mathrm{d}t$$

$$\phi = \int_1^t -\frac{\mathrm{e}^{-\operatorname{expIntegral}_1(\_a)}(-\mathrm{e}^{-\_a}x + \_a^2)}{\_a} d\_a + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial\phi}{\partial x} = \int_1^t \frac{\mathrm{e}^{-\operatorname{expIntegral}_1(\_a)}\mathrm{e}^{-\_a}}{\_a} d\_a + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial x} = \mathrm{e}^{-\operatorname{expIntegral}_1(t)}$. Therefore equation (4) becomes

$$\mathrm{e}^{-\operatorname{expIntegral}_1(t)} = \int_1^t \frac{\mathrm{e}^{-\operatorname{expIntegral}_1(\_a)-\_a}}{\_a} d\_a + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\left(\int_1^t \frac{\mathrm{e}^{-\operatorname{expIntegral}_1(\_a)-\_a}}{\_a} d\_a\right) + \mathrm{e}^{-\operatorname{expIntegral}_1(t)}$$

Integrating the above w.r.t x gives

$$
\begin{aligned}
\int f'(x)\,\mathrm{d}x &= \int \left(-\left(\int_1^t \frac{\mathrm{e}^{-\operatorname{expIntegral}_1(\_a)-\_a}}{\_a} d\_a\right) + \mathrm{e}^{-\operatorname{expIntegral}_1(t)}\right) \mathrm{d}x \\
f(x) &= \left(-\left(\int_1^t \frac{\mathrm{e}^{-\operatorname{expIntegral}_1(\_a)-\_a}}{\_a} d\_a\right) + \mathrm{e}^{-\operatorname{expIntegral}_1(t)}\right) x + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$
\begin{aligned}
\phi = \int_1^t -\frac{\mathrm{e}^{-\operatorname{expIntegral}_1(\_a)}\left(-\mathrm{e}^{-\_a}x + \_a^2\right)}{\_a} d\_a \\
+ \left(-\left(\int_1^t \frac{\mathrm{e}^{-\operatorname{expIntegral}_1(\_a)-\_a}}{\_a} d\_a\right) + \mathrm{e}^{-\operatorname{expIntegral}_1(t)}\right) x + c_1
\end{aligned}
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
\begin{aligned}
c_1 = \int_1^t -\frac{\mathrm{e}^{-\operatorname{expIntegral}_1(\_a)}\left(-\mathrm{e}^{-\_a}x + \_a^2\right)}{\_a} d\_a \\
+ \left(-\left(\int_1^t \frac{\mathrm{e}^{-\operatorname{expIntegral}_1(\_a)-\_a}}{\_a} d\_a\right) + \mathrm{e}^{-\operatorname{expIntegral}_1(t)}\right) x
\end{aligned}
$$

The solution becomes

$$
x = \left(\int_1^t \mathrm{e}^{-\operatorname{expIntegral}_1(\_a)}\_a d\_a + c_1\right) \mathrm{e}^{\operatorname{expIntegral}_1(t)}
$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = c_1 \mathrm{e}^{\operatorname{expIntegral}_1(1)}
$$

$$
c_1 = 0
$$

Substituting c_1 found above in the general solution gives

$$
x = \left(\int_1^t \mathrm{e}^{-\operatorname{expIntegral}_1(\_a)}\_a d\_a\right) \mathrm{e}^{\operatorname{expIntegral}_1(t)}
$$

Summary
The solution(s) found are the following

$$x = \left(\int_{1}^{t} \mathrm{e}^{-\operatorname{expIntegral}_{1}(\_a)} \_a d\_a\right) \mathrm{e}^{\operatorname{expIntegral}_{1}(t)} \tag{1}$$

Verification of solutions

$$x = \left(\int_{1}^{t} \mathrm{e}^{-\operatorname{expIntegral}_{1}(\_a)} \_a d\_a\right) \mathrm{e}^{\operatorname{expIntegral}_{1}(t)}$$

Verified OK.

5.21.5 Maple step by step solution

Let's solve
$\left[x' + \frac{\mathrm{e}^{-t} x}{t} = t, x(1) = 0\right]$

- Highest derivative means the order of the ODE is 1
  x'
- Isolate the derivative
  $x' = -\frac{\mathrm{e}^{-t} x}{t} + t$
- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE
  $x' + \frac{\mathrm{e}^{-t} x}{t} = t$
- The ODE is linear; multiply by an integrating factor $\mu(t)$
  $\mu(t) \left(x' + \frac{\mathrm{e}^{-t} x}{t}\right) = \mu(t)\, t$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, x)$
  $\mu(t) \left(x' + \frac{\mathrm{e}^{-t} x}{t}\right) = \mu'(t)\, x + \mu(t)\, x'$
- Isolate $\mu'(t)$
  $\mu'(t) = \frac{\mu(t) \mathrm{e}^{-t}}{t}$
- Solve to find the integrating factor
  $\mu(t) = \mathrm{e}^{-\mathrm{Ei}_1(t)}$
- Integrate both sides with respect to t
  $\int \left(\frac{d}{dt}(\mu(t)\, x)\right) dt = \int \mu(t)\, t dt + c_1$
- Evaluate the integral on the lhs

$$\mu(t)\,x = \int \mu(t)\,t dt + c_1$$

- Solve for x

$$x = \frac{\int \mu(t) t dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = \mathrm{e}^{-\mathrm{Ei}_1(t)}$

$$x = \frac{\int \mathrm{e}^{-\mathrm{Ei}_1(t)} t dt + c_1}{\mathrm{e}^{-\mathrm{Ei}_1(t)}}$$

- Simplify

$$x = \left(\int \mathrm{e}^{-\mathrm{Ei}_1(t)} t dt + c_1\right) \mathrm{e}^{\mathrm{Ei}_1(t)}$$

- Use initial condition $x(1) = 0$

$$0 = \left(\int^1 \mathrm{e}^{-\mathrm{Ei}_1(\_a)} \_a d\_a + c_1\right) \mathrm{e}^{\mathrm{Ei}_1(1)}$$

- Solve for c_1

$$c_1 = -\left(\int^1 \mathrm{e}^{-\mathrm{Ei}_1(\_a)} \_a d\_a\right)$$

- Substitute $c_1 = -\left(\int^1 \mathrm{e}^{-\mathrm{Ei}_1(\_a)} \_a d\_a\right)$ into general solution and simplify

$$x = \left(\int \mathrm{e}^{-\mathrm{Ei}_1(t)} t dt - \left(\int^1 \mathrm{e}^{-\mathrm{Ei}_1(\_a)} \_a d\_a\right)\right) \mathrm{e}^{\mathrm{Ei}_1(t)}$$

- Solution to the IVP

$$x = \left(\int \mathrm{e}^{-\mathrm{Ei}_1(t)} t dt - \left(\int^1 \mathrm{e}^{-\mathrm{Ei}_1(\_a)} \_a d\_a\right)\right) \mathrm{e}^{\mathrm{Ei}_1(t)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 23

```
dsolve([diff(x(t),t)+exp(-t)/t*x(t)=t,x(1) = 0],x(t), singsol=all)
```

$$x(t) = \left(\int_1^t \_z1\, \mathrm{e}^{-\operatorname{expIntegral}_1(\_z1)} d\_z1\right) \mathrm{e}^{\operatorname{expIntegral}_1(t)}$$

✓ Solution by Mathematica

Time used: 0.169 (sec). Leaf size: 31

```
DSolve[{x'[t]+Exp[-t]/t*x[t]==t,{x[1]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-\text{ExpIntegralEi}(-t)} \int_1^t e^{\text{ExpIntegralEi}(-K[1])} K[1] dK[1]$$

5.22 problem 7

5.22.1 Solving as second order linear constant coeff ode 752
5.22.2 Solving as second order integrable as is ode 756
5.22.3 Solving as second order ode missing y ode 758
5.22.4 Solving as type second_order_integrable_as_is (not using ABC version) 760
5.22.5 Solving using Kovacic algorithm 762
5.22.6 Solving as exact linear second order ode ode 767
5.22.7 Maple step by step solution 769

Internal problem ID [11420]
Internal file name [`OUTPUT/10402_Thursday_May_18_2023_04_18_28_AM_75656588/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x'' + x' = 3t}$$

5.22.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = 0, f(t) = 3t$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''+x'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t)+Bx'(t)+Cx(t)=0$$

Where in the above $A=1, B=1, C=0$. Let the solution be $x=e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t}+\lambda \mathrm{e}^{\lambda t}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2+\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=1, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-1}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{1^2-(4)\,(1)\,(0)}\\&=-\frac{1}{2}\pm\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-\frac{1}{2}+\frac{1}{2}\\\lambda_2&=-\frac{1}{2}-\frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=0\\\lambda_2&=-1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x&=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}\\x&=c_1e^{(0)t}+c_2e^{(-1)t}\end{aligned}$$

Or

$$x = c_1 + c_2 \mathrm{e}^{-t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 + c_2 \mathrm{e}^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-t}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t, t^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_2 t^2 + A_1 t$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2tA_2 + A_1 + 2A_2 = 3t$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -3, A_2 = \frac{3}{2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{3}{2}t^2 - 3t$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(c_1 + c_2 \mathrm{e}^{-t}\right) + \left(\frac{3}{2}t^2 - 3t\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
x = c_1 + c_2 \mathrm{e}^{-t} + \frac{3t^2}{2} - 3t \tag{1}
$$

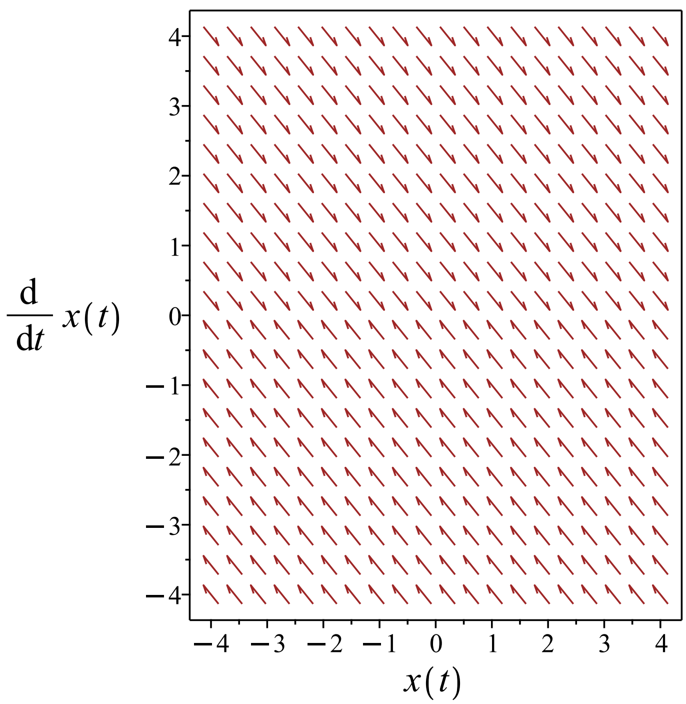


Figure 165: Slope field plot

<u>Verification of solutions</u>

$$
x = c_1 + c_2 \mathrm{e}^{-t} + \frac{3t^2}{2} - 3t
$$

Verified OK.

5.22.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$
\begin{aligned}
\int (x'' + x')\, dt &= \int 3t dt \\
x' + x &= \frac{3t^2}{2} + c_1
\end{aligned}
$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
x' + p(t)x = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= 1 \\
q(t) &= \frac{3t^2}{2} + c_1
\end{aligned}
$$

Hence the ode is

$$
x' + x = \frac{3t^2}{2} + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1 dt} \\
&= \mathrm{e}^t
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(\frac{3t^2}{2} + c_1\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^t x) &= (\mathrm{e}^t)\left(\frac{3t^2}{2} + c_1\right) \\
\mathrm{d}(\mathrm{e}^t x) &= \left(\frac{(3t^2 + 2c_1)\,\mathrm{e}^t}{2}\right) \mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^t x &= \int \frac{(3t^2 + 2c_1)\,\mathrm{e}^t}{2}\, \mathrm{d}t \\
\mathrm{e}^t x &= \frac{(3t^2 + 2c_1 - 6t + 6)\,\mathrm{e}^t}{2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^t$ results in

$$x = \frac{e^{-t}(3t^2 + 2c_1 - 6t + 6)\,e^t}{2} + c_2 e^{-t}$$

which simplifies to

$$x = \frac{3t^2}{2} + c_1 - 3t + 3 + c_2 e^{-t}$$

Summary

The solution(s) found are the following

$$x = \frac{3t^2}{2} + c_1 - 3t + 3 + c_2 e^{-t} \tag{1}$$

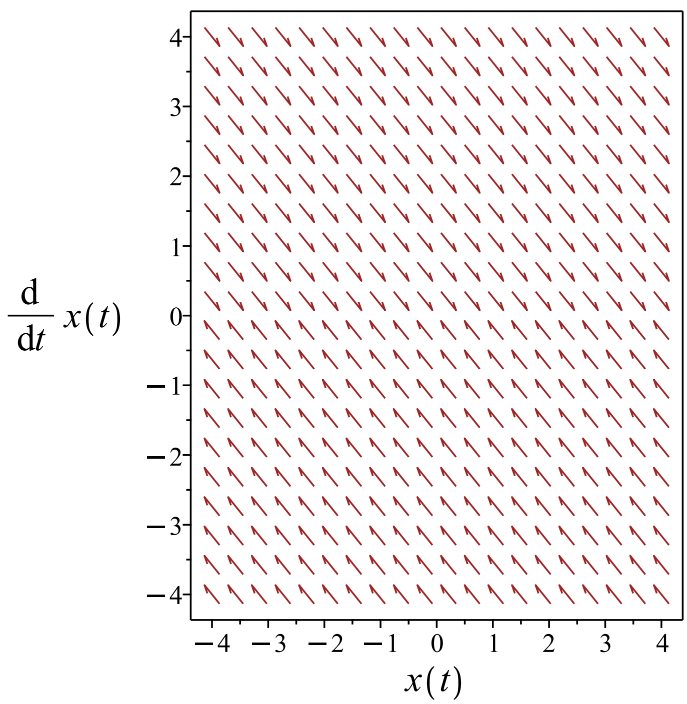


Figure 166: Slope field plot

Verification of solutions

$$x = \frac{3t^2}{2} + c_1 - 3t + 3 + c_2 e^{-t}$$

Verified OK.

5.22.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable x. Let

$$p(t) = x'$$

Then

$$p'(t) = x''$$

Hence the ode becomes

$$p'(t) + p(t) - 3t = 0$$

Which is now solve for $p(t)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(t) + p(t)p(t) = q(t)$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= 3t \end{aligned}$$

Hence the ode is

$$p'(t) + p(t) = 3t$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dt} \\ &= e^{t} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dt}(\mu p) &= (\mu)(3t) \\ \frac{d}{dt}(e^{t} p) &= (e^{t})(3t) \\ d(e^{t} p) &= (3t\,e^{t})\; dt \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{t} p &= \int 3t\,e^{t}\, dt \\ e^{t} p &= 3(t-1)\,e^{t} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{t}$ results in

$$p(t) = 3\,\mathrm{e}^{-t}(t-1)\,\mathrm{e}^{t} + c_1 \mathrm{e}^{-t}$$

which simplifies to

$$p(t) = 3t - 3 + c_1 \mathrm{e}^{-t}$$

Since $p = x'$ then the new first order ode to solve is

$$x' = 3t - 3 + c_1 \mathrm{e}^{-t}$$

Integrating both sides gives

$$\begin{aligned} x &= \int 3t - 3 + c_1 \mathrm{e}^{-t}\, \mathrm{d}t \\ &= -3t - c_1 \mathrm{e}^{-t} + \frac{3t^2}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$x = -3t - c_1 \mathrm{e}^{-t} + \frac{3t^2}{2} + c_2 \tag{1}$$

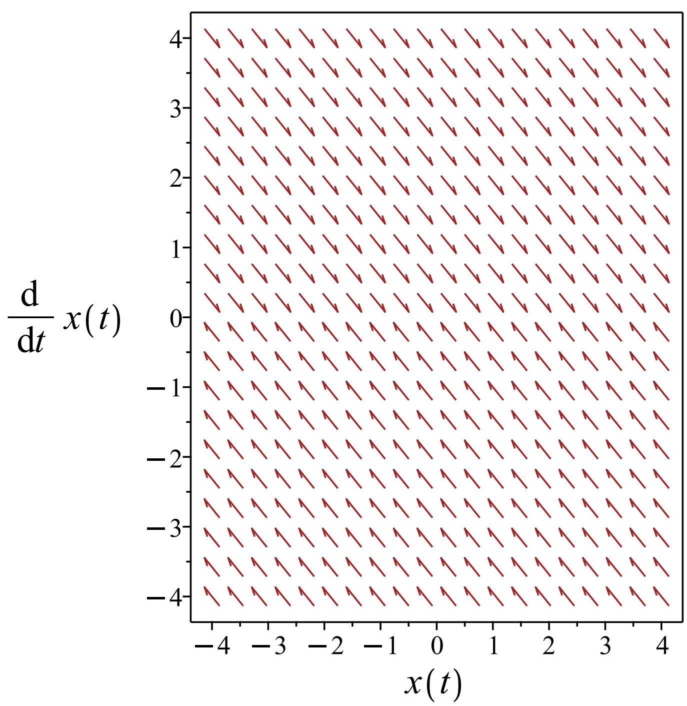


Figure 167: Slope field plot

Verification of solutions

$$x = -3t - c_1 e^{-t} + \frac{3t^2}{2} + c_2$$

Verified OK.

5.22.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x'' + x' = 3t$$

Integrating both sides of the ODE w.r.t t gives

$$\int (x'' + x')\,dt = \int 3t dt$$
$$x' + x = \frac{3t^2}{2} + c_1$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = 1$$
$$q(t) = \frac{3t^2}{2} + c_1$$

Hence the ode is

$$x' + x = \frac{3t^2}{2} + c_1$$

The integrating factor μ is

$$\mu = e^{\int 1dt}$$
$$= e^t$$

The ode becomes

$$\frac{d}{dt}(\mu x) = (\mu)\left(\frac{3t^2}{2} + c_1\right)$$
$$\frac{d}{dt}(e^t x) = (e^t)\left(\frac{3t^2}{2} + c_1\right)$$
$$d(e^t x) = \left(\frac{(3t^2 + 2c_1)\,e^t}{2}\right) dt$$

Integrating gives

$$
\begin{aligned}
e^{t} x &= \int \frac{(3t^{2}+2c_1)\,e^{t}}{2}\,dt \\
e^{t} x &= \frac{(3t^{2}+2c_1-6t+6)\,e^{t}}{2}+c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{t}$ results in

$$
x = \frac{e^{-t}(3t^{2}+2c_1-6t+6)\,e^{t}}{2}+c_2 e^{-t}
$$

which simplifies to

$$
x = \frac{3t^{2}}{2}+c_1-3t+3+c_2 e^{-t}
$$

Summary

The solution(s) found are the following

$$
x = \frac{3t^{2}}{2}+c_1-3t+3+c_2 e^{-t} \tag{1}
$$

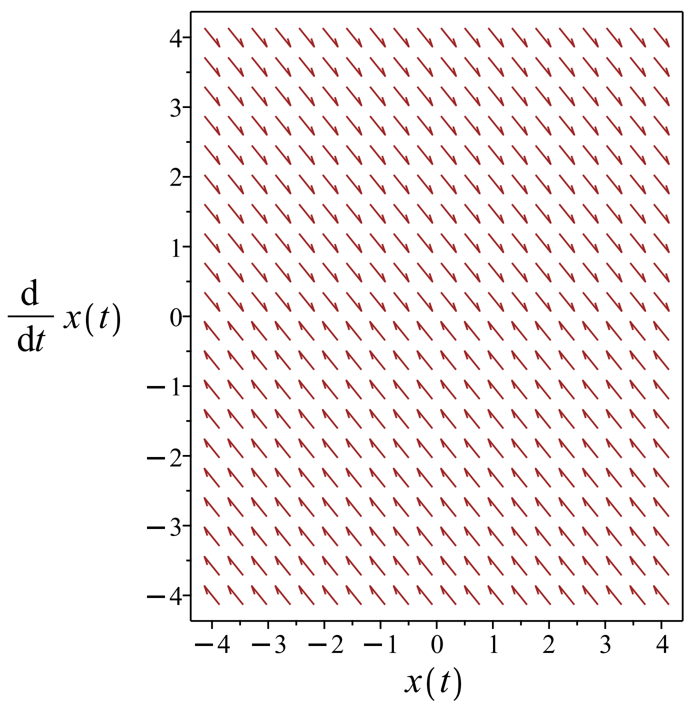


Figure 168: Slope field plot

Verification of solutions

$$x = \frac{3t^2}{2} + c_1 - 3t + 3 + c_2 e^{-t}$$

Verified OK.

5.22.5 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 151: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\,dt \\
&= x_1 (\mathrm{e}^{t})
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 (\mathrm{e}^{-t}) + c_2 (\mathrm{e}^{-t} (\mathrm{e}^{t}))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''+x'=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 e^{-t} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1,t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-t}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t,t^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_2 t^2 + A_1 t$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2tA_2 + A_1 + 2A_2 = 3t$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -3, A_2 = \frac{3}{2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{3}{2}t^2 - 3t$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1 e^{-t} + c_2) + \left(\frac{3}{2}t^2 - 3t\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{-t} + c_2 + \frac{3t^2}{2} - 3t \tag{1}$$

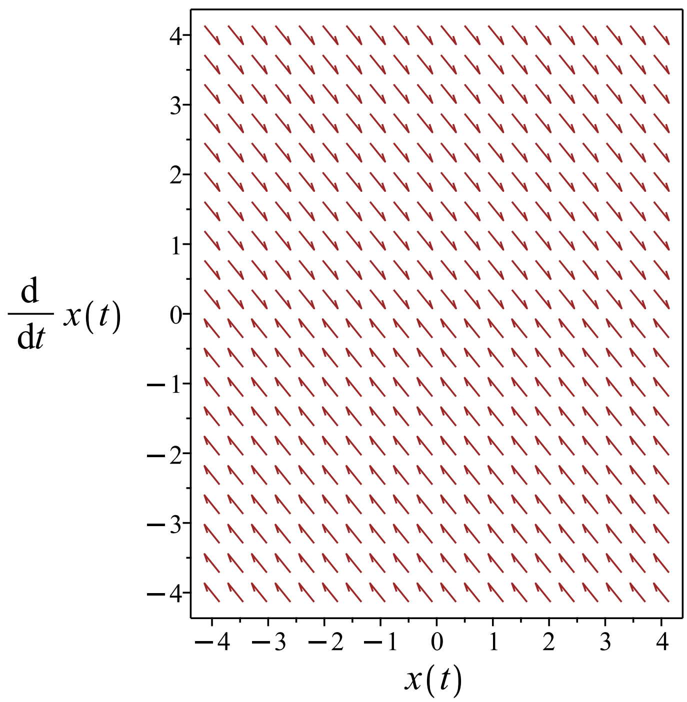


Figure 169: Slope field plot

Verification of solutions

$$x = c_1 e^{-t} + c_2 + \frac{3t^2}{2} - 3t$$

Verified OK.

5.22.6 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\, x'' + q(t)\, x' + r(t)\, x = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= 3t \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(t)\, x' + (q(t) - p'(t))\, x\right)' = s(x)$$

Integrating gives

$$p(t)\, x' + (q(t) - p'(t))\, x = \int s(t)\, dt$$

Substituting the above values for p, q, r, s gives

$$x' + x = \int 3t\, dt$$

We now have a first order ode to solve which is

$$x' + x = \frac{3t^2}{2} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= \frac{3t^2}{2} + c_1 \end{aligned}$$

Hence the ode is

$$x' + x = \frac{3t^2}{2} + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 1 dt} \\ &= \mathrm{e}^t \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(\frac{3t^2}{2} + c_1\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^t x) &= (\mathrm{e}^t)\left(\frac{3t^2}{2} + c_1\right) \\ \mathrm{d}(\mathrm{e}^t x) &= \left(\frac{(3t^2 + 2c_1)\,\mathrm{e}^t}{2}\right) \mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^t x &= \int \frac{(3t^2 + 2c_1)\,\mathrm{e}^t}{2}\, \mathrm{d}t \\ \mathrm{e}^t x &= \frac{(3t^2 + 2c_1 - 6t + 6)\,\mathrm{e}^t}{2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^t$ results in

$$x = \frac{\mathrm{e}^{-t}(3t^2 + 2c_1 - 6t + 6)\,\mathrm{e}^t}{2} + c_2 \mathrm{e}^{-t}$$

which simplifies to

$$x = \frac{3t^2}{2} + c_1 - 3t + 3 + c_2 \mathrm{e}^{-t}$$

Summary
The solution(s) found are the following

$$x = \frac{3t^2}{2} + c_1 - 3t + 3 + c_2 \mathrm{e}^{-t} \tag{1}$$

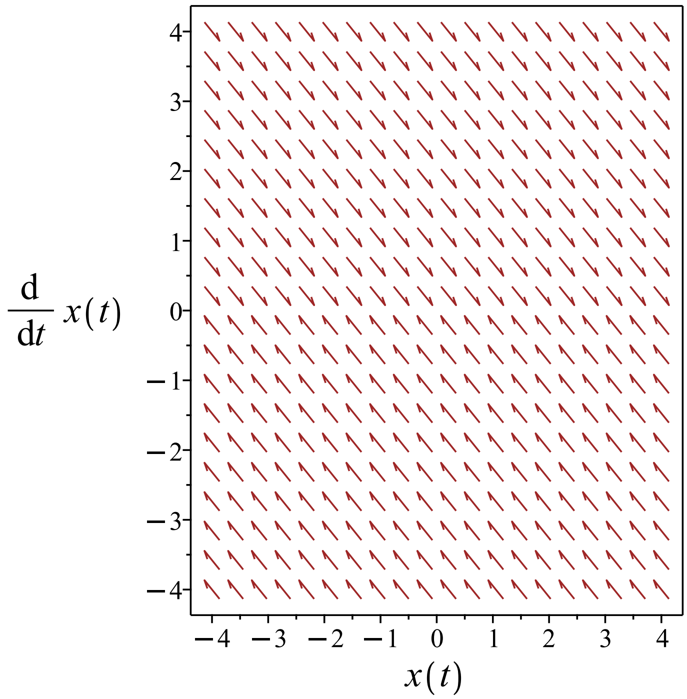


Figure 170: Slope field plot

Verification of solutions

$$x = \frac{3t^2}{2} + c_1 - 3t + 3 + c_2 \mathrm{e}^{-t}$$

Verified OK.

5.22.7 Maple step by step solution

Let's solve

$x'' + x' = 3t$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

$r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = 1$

- General solution of the ODE

  $x = c_1x_1(t) + c_2x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1\mathrm{e}^{-t} + c_2 + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = 3t\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} & 1 \\ -\mathrm{e}^{-t} & 0 \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = \mathrm{e}^{-t}$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -3\,\mathrm{e}^{-t}\left(\int t\,\mathrm{e}^{t}dt\right) + 3\left(\int t dt\right)$

  - Compute integrals

    $x_p(t) = -3t + 3 + \frac{3}{2}t^2$

- Substitute particular solution into general solution to ODE

  $x = c_1\mathrm{e}^{-t} + c_2 - 3t + 3 + \frac{3t^2}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)+3*_a, _b(_a)`   *** Sublevel 2
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(x(t),t$2)+diff(x(t),t)=3*t,x(t), singsol=all)
```

$$x(t) = -\mathrm{e}^{-t}c_1 + \frac{3t^2}{2} - 3t + c_2$$

✓ Solution by Mathematica
Time used: 0.071 (sec). Leaf size: 27

```
DSolve[x''[t]+x'[t]==3*t,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{3t^2}{2} - 3t - c_1 e^{-t} + c_2$$

5.23 problem 8

5.23.1 Solving as homogeneousTypeC ode 772
5.23.2 Solving as first order ode lie symmetry lookup ode 774
5.23.3 Solving as riccati ode 778

Internal problem ID [11421]
Internal file name [`OUTPUT/10403_Thursday_May_18_2023_04_18_30_AM_64152595/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _Riccati]
```

$$\boxed{x' - (t + x)^2 = 0}$$

5.23.1 Solving as homogeneousTypeC ode

Let

$$z = t + x \tag{1}$$

Then

$$z'(t) = 1 + x'$$

Therefore

$$x' = z'(t) - 1$$

Hence the given ode can now be written as

$$z'(t) - 1 = z^2$$

This is separable first order ode. Integrating

$$\int dt = \int \frac{1}{z^2+1} dz$$
$$t + c_1 = \arctan(z)$$

Replacing z back by its value from (1) then the above gives the solution as

$$x = -t + \tan(t + c_1)$$

$$x = -t + \tan(t + c_1)$$

Summary
The solution(s) found are the following

$$x = -t + \tan(t + c_1) \tag{1}$$

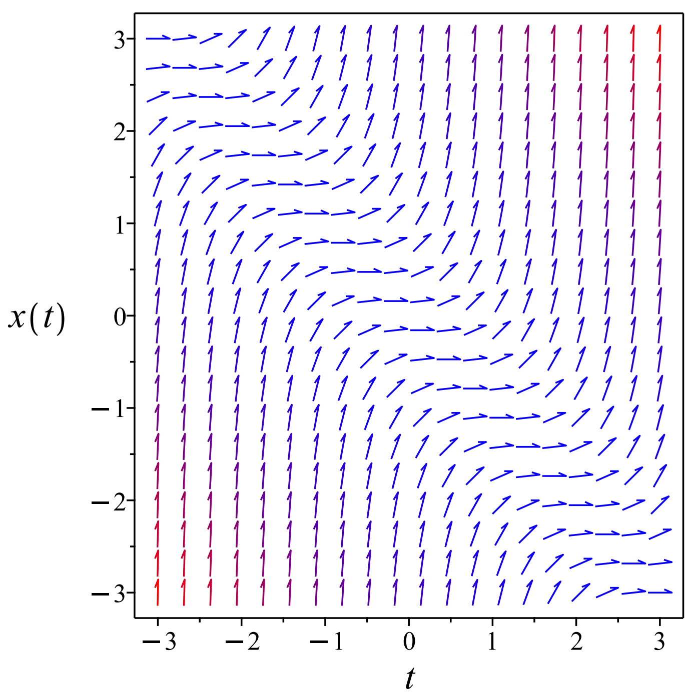


Figure 171: Slope field plot

Verification of solutions

$$x = -t + \tan(t + c_1)$$

Verified OK.

5.23.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= (t+x)^2 \\
x' &= \omega(t,x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 153: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= 1 \\
\eta(t, x) &= -1
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{\eta}{\xi} \\
&= \frac{-1}{1} \\
&= -1
\end{aligned}
$$

This is easily solved to give

$$
x = -t + c_1
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = t + x
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dt}{\xi} \\
&= \frac{dt}{1}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dt}{T} \\
&= t
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = (t+x)^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 1 \\ S_t &= 1 \\ S_x &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{1+(t+x)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$t = \arctan(t+x) + c_1$$

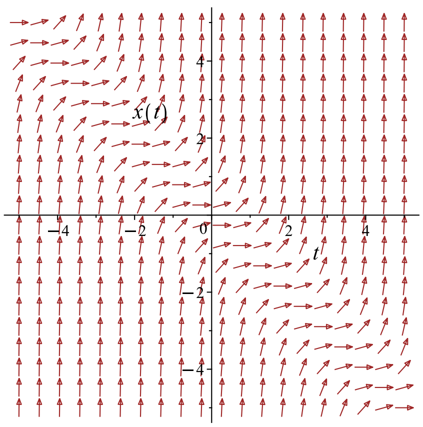
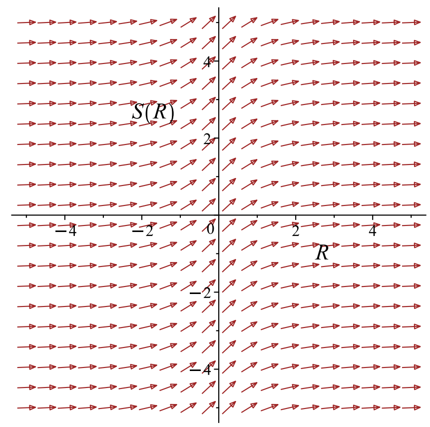
Which simplifies to

$$t = \arctan(t+x) + c_1$$

Which gives

$$x = -t - \tan(-t + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = (t + x)^2$ 	$R = t + x$ $S = t$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

Summary

The solution(s) found are the following

$$x = -t - \tan(-t + c_1) \tag{1}$$

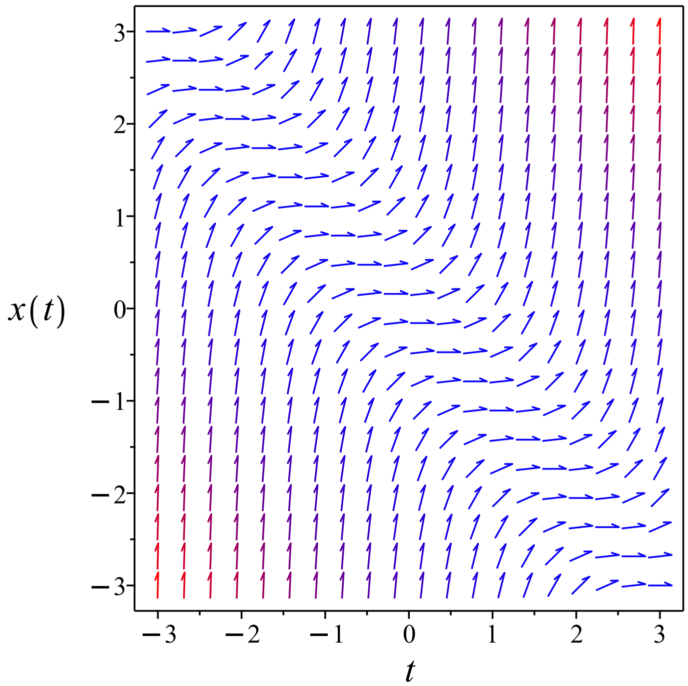


Figure 172: Slope field plot

Verification of solutions

$$x = -t - \tan(-t + c_1)$$

Verified OK.

5.23.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= (t + x)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$x' = t^2 + 2tx + x^2$$

With Riccati ODE standard form

$$x' = f_0(t) + f_1(t)x + f_2(t)x^2$$

Shows that $f_0(t) = t^2$, $f_1(t) = 2t$ and $f_2(t) = 1$. Let

$$\begin{aligned} x &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 2t \\ f_2^2 f_0 &= t^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(t) - 2tu'(t) + t^2 u(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = \mathrm{e}^{\frac{t^2}{2}} \left(c_1 \cos(t) + c_2 \sin(t)\right)$$

The above shows that

$$u'(t) = \mathrm{e}^{\frac{t^2}{2}} \left((c_1 t + c_2) \cos(t) + \sin(t)\,(c_2 t - c_1)\right)$$

Using the above in (1) gives the solution

$$x = -\frac{(c_1 t + c_2) \cos(t) + \sin(t)\,(c_2 t - c_1)}{c_1 \cos(t) + c_2 \sin(t)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$x = \frac{(-c_3 t - 1) \cos(t) - \sin(t)\,(-c_3 + t)}{c_3 \cos(t) + \sin(t)}$$

Summary

The solution(s) found are the following

$$x = \frac{(-c_3 t - 1)\cos(t) - \sin(t)(-c_3 + t)}{c_3 \cos(t) + \sin(t)} \tag{1}$$

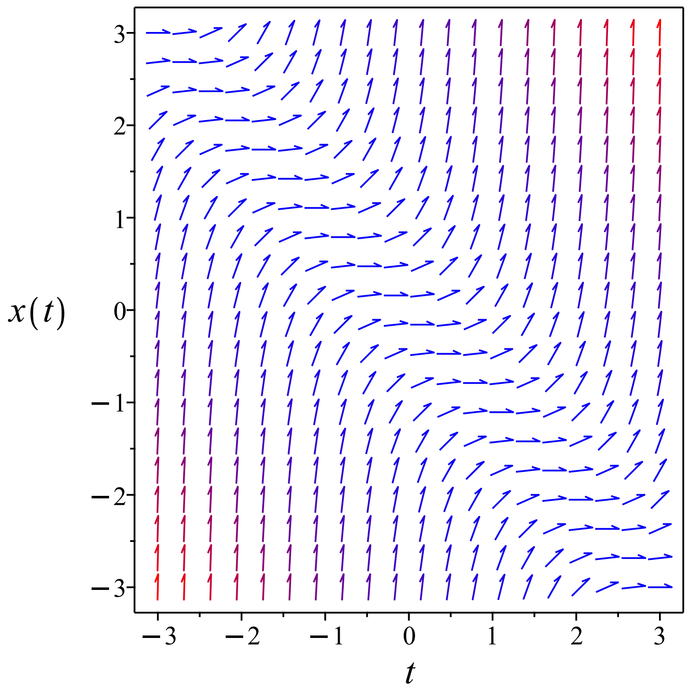


Figure 173: Slope field plot

Verification of solutions

$$x = \frac{(-c_3 t - 1)\cos(t) - \sin(t)(-c_3 + t)}{c_3 \cos(t) + \sin(t)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(x(t),t)=(t+x(t))^2,x(t), singsol=all)
```

$$x(t) = -t - \tan(-t + c_1)$$

✓ Solution by Mathematica

Time used: 0.681 (sec). Leaf size: 14

```
DSolve[x'[t]==(t+x[t])^2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -t + \tan(t + c_1)$$

5.24 problem 9

5.24.1 Solving as quadrature ode 782
5.24.2 Maple step by step solution 783

Internal problem ID [11422]
Internal file name [`OUTPUT/10404_Thursday_May_18_2023_04_18_30_AM_37563748/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' - ax = b}$$

5.24.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{ax+b}dx = \int dt$$
$$\frac{\ln (ax+b)}{a} = t + c_1$$

Raising both side to exponential gives

$$e^{\frac{\ln(ax+b)}{a}} = e^{t+c_1}$$

Which simplifies to

$$(ax+b)^{\frac{1}{a}} = c_2 e^t$$

Summary
The solution(s) found are the following

$$x = \frac{(c_2 e^t)^a - b}{a} \tag{1}$$

Verification of solutions

$$x = \frac{(c_2 e^t)^a - b}{a}$$

Verified OK.

5.24.2 Maple step by step solution

Let's solve

$x' - ax = b$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{ax+b} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{ax+b} dt = \int 1 dt + c_1$

- Evaluate integral

  $\frac{\ln(ax+b)}{a} = t + c_1$

- Solve for x

  $x = \frac{e^{c_1 a + ta} - b}{a}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(x(t),t)=a*x(t)+b,x(t), singsol=all)
```

$$x(t) = \frac{e^{at} c_1 a - b}{a}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 30

```
DSolve[x'[t]==a*x[t]+b,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{b}{a} + c_1 e^{at}$$
$$x(t) \to -\frac{b}{a}$$

5.25 problem 12

5.25.1 Solving as separable ode 785
5.25.2 Solving as linear ode 786
5.25.3 Solving as homogeneousTypeD2 ode 787
5.25.4 Solving as first order ode lie symmetry lookup ode 788
5.25.5 Solving as exact ode 790
5.25.6 Maple step by step solution 793

Internal problem ID [11423]
Internal file name [`OUTPUT/10405_Thursday_May_18_2023_04_18_31_AM_21883334/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{x' + p(t)\, x = 0}$$

5.25.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= f(t)g(x) \\ &= -p(t)\, x \end{aligned}$$

Where $f(t) = -p(t)$ and $g(x) = x$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{x}\, dx &= -p(t)\ dt \\
\int \frac{1}{x}\, dx &= \int -p(t)\ dt \\
\ln(x) &= \int -p(t)\, dt + c_1 \\
x &= \mathrm{e}^{\int -p(t)dt+c_1} \\
&= c_1 \mathrm{e}^{\int -p(t)dt}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1 \mathrm{e}^{\int -p(t)dt} \tag{1}
$$

Verification of solutions

$$
x = c_1 \mathrm{e}^{\int -p(t)dt}
$$

Verified OK.

5.25.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
x' + p(t)x = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= p(t) \\
q(t) &= 0
\end{aligned}
$$

Hence the ode is

$$
x' + p(t)\, x = 0
$$

The integrating factor μ is

$$
\mu = \mathrm{e}^{\int p(t)dt}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mu x &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\int p(t)dt} x\right) &= 0
\end{aligned}
$$

Integrating gives

$$e^{\int p(t)dt}x = c_1$$

Dividing both sides by the integrating factor $\mu = e^{\int p(t)dt}$ results in

$$x = c_1 e^{-\left(\int p(t)dt\right)}$$

<u>Summary</u>
The solution(s) found are the following

$$x = c_1 e^{-\left(\int p(t)dt\right)} \tag{1}$$

<u>Verification of solutions</u>

$$x = c_1 e^{-\left(\int p(t)dt\right)}$$

Verified OK.

5.25.3 Solving as homogeneousTypeD2 ode

Using the change of variables $x = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u'(t)\,t + u(t) + p(t)\,u(t)\,t = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= -\frac{u(p(t)\,t+1)}{t} \end{aligned}$$

Where $f(t) = -\frac{p(t)t+1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{p(t)\,t+1}{t}\,dt \\ \int \frac{1}{u}\,du &= \int -\frac{p(t)\,t+1}{t}\,dt \\ \ln(u) &= \int -\frac{p(t)\,t+1}{t}dt + c_2 \\ u &= e^{\int -\frac{p(t)t+1}{t}dt+c_2} \\ &= c_2 e^{\int -\frac{p(t)t+1}{t}dt} \end{aligned}$$

Therefore the solution x is

$$
\begin{aligned}
x &= tu \\
&= t c_2 \mathrm{e}^{\int -\frac{p(t)t+1}{t}dt}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = t c_2 \mathrm{e}^{\int -\frac{p(t)t+1}{t}dt} \tag{1}
$$

Verification of solutions

$$
x = t c_2 \mathrm{e}^{\int -\frac{p(t)t+1}{t}dt}
$$

Verified OK.

5.25.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= -p(t)\, x \\
x' &= \omega(t, x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 156: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \mathrm{e}^{\int -p(t)dt}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\int -p(t)dt}} dy \end{aligned}$$

5.25.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{x}\right)\mathrm{d}x &= (p(t))\,\mathrm{d}t\\ (-p(t))\,\mathrm{d}t + \left(-\frac{1}{x}\right)\mathrm{d}x &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x) &= -p(t)\\ N(t,x) &= -\frac{1}{x}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-p(t))\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -p(t) \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = \int^{t} -p(\_a) \, d\_a + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = -\frac{1}{x}$. Therefore equation (4) becomes

$$
-\frac{1}{x} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = -\frac{1}{x}
$$

Integrating the above w.r.t x gives

$$
\begin{aligned}
\int f'(x) \, \mathrm{d}x &= \int \left(-\frac{1}{x}\right) \mathrm{d}x \\
f(x) &= -\ln(x) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$
\phi = \int^{t} -p(\_a) \, d\_a - \ln(x) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \int^{t} -p(\_a) \, d\_a - \ln(x)
$$

The solution becomes

$$x = e^{\int^t -p(\_a)d\_a - c_1}$$

Summary
The solution(s) found are the following

$$x = e^{\int^t -p(\_a)d\_a - c_1} \tag{1}$$

Verification of solutions

$$x = e^{\int^t -p(\_a)d\_a - c_1}$$

Verified OK.

5.25.6 Maple step by step solution

Let's solve

$x' + p(t)\, x = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x} = -p(t)$

- Integrate both sides with respect to t

  $\int \frac{x'}{x} dt = \int -p(t)\, dt + c_1$

- Evaluate integral

  $\ln(x) = \int -p(t)\, dt + c_1$

- Solve for x

  $x = e^{-\left(\int p(t)dt\right) + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(x(t),t)+p(t)*x(t)=0,x(t), singsol=all)
```

$$x(t) = c_1 e^{-\left(\int p(t)dt\right)}$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 27

```
DSolve[x'[t]+p[t]*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 \exp\left(\int_1^t -p(K[1])dK[1]\right)$$
$$x(t) \to 0$$

5.26 problem 15(a)

5.26.1 Solving as homogeneousTypeD2 ode 795
5.26.2 Solving as first order ode lie symmetry lookup ode 797
5.26.3 Solving as bernoulli ode 801
5.26.4 Solving as exact ode 804

Internal problem ID [11424]
Internal file name [`OUTPUT/10406_Thursday_May_18_2023_04_18_33_AM_4787677/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 15(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{x' - \frac{2x}{3t} - \frac{2t}{x} = 0}$$

5.26.1 Solving as homogeneousTypeD2 ode

Using the change of variables $x = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u'(t)\,t + \frac{u(t)}{3} - \frac{2}{u\,(t)} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{u^2 - 6}{3tu} \end{aligned}$$

Where $f(t) = -\frac{1}{3t}$ and $g(u) = \frac{u^2-6}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-6}{u}}\,du = -\frac{1}{3t}\,dt$$

$$\int \frac{1}{\frac{u^2-6}{u}}\,du = \int -\frac{1}{3t}\,dt$$

$$\frac{\ln(u^2-6)}{2} = -\frac{\ln(t)}{3} + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2-6} = \mathrm{e}^{-\frac{\ln(t)}{3}+c_2}$$

Which simplifies to

$$\sqrt{u^2-6} = \frac{c_3}{t^{\frac{1}{3}}}$$

Which simplifies to

$$\sqrt{u(t)^2-6} = \frac{c_3\mathrm{e}^{c_2}}{t^{\frac{1}{3}}}$$

The solution is

$$\sqrt{u(t)^2-6} = \frac{c_3\mathrm{e}^{c_2}}{t^{\frac{1}{3}}}$$

Replacing $u(t)$ in the above solution by $\frac{x}{t}$ results in the solution for x in implicit form

$$\sqrt{\frac{x^2}{t^2}-6} = \frac{c_3\mathrm{e}^{c_2}}{t^{\frac{1}{3}}}$$

$$\sqrt{\frac{x^2-6t^2}{t^2}} = \frac{c_3\mathrm{e}^{c_2}}{t^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{x^2-6t^2}{t^2}} = \frac{c_3\mathrm{e}^{c_2}}{t^{\frac{1}{3}}} \tag{1}$$

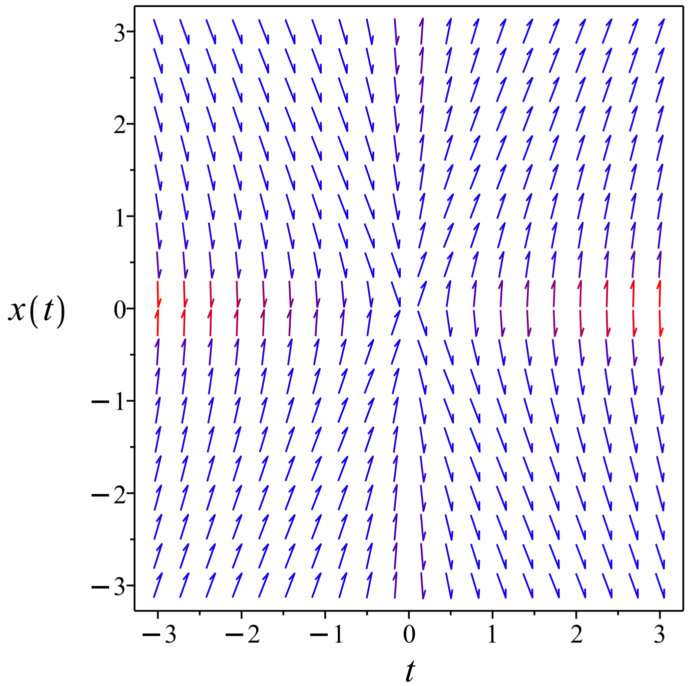


Figure 174: Slope field plot

Verification of solutions

$$\sqrt{\frac{x^2 - 6t^2}{t^2}} = \frac{c_3 e^{c_2}}{t^{\frac{1}{3}}}$$

Verified OK.

5.26.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = \frac{\frac{2x^2}{3} + 2t^2}{tx}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 159: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= 0 \\
\eta(t, x) &= \frac{t^{\frac{4}{3}}}{x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{t^{\frac{4}{3}}}{x}} dy \end{aligned}$$

Which results in

$$S = \frac{x^2}{2t^{\frac{4}{3}}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = \frac{\frac{2x^2}{3} + 2t^2}{tx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= -\frac{2x^2}{3t^{\frac{7}{3}}} \\ S_x &= \frac{x}{t^{\frac{4}{3}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2}{t^{\frac{1}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2}{R^{\frac{1}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 3R^{\frac{2}{3}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\frac{x^2}{2t^{\frac{4}{3}}} = 3t^{\frac{2}{3}} + c_1$$

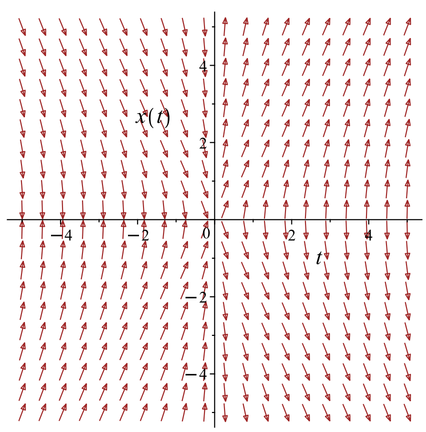
Which simplifies to

$$\frac{x^2}{2t^{\frac{4}{3}}} = 3t^{\frac{2}{3}} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, x coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dx}{dt} = \frac{\frac{2x^2}{3}+2t^2}{tx}$



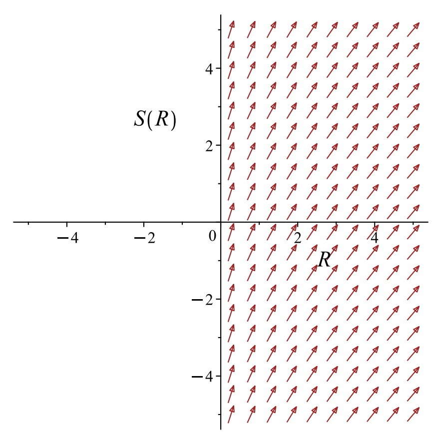
</td><td>

$R = t$

$S = \frac{x^2}{2t^{\frac{4}{3}}}$

</td><td>

$\frac{dS}{dR} = \frac{2}{R^{\frac{1}{3}}}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{x^2}{2t^{\frac{4}{3}}} = 3t^{\frac{2}{3}} + c_1 \tag{1}$$

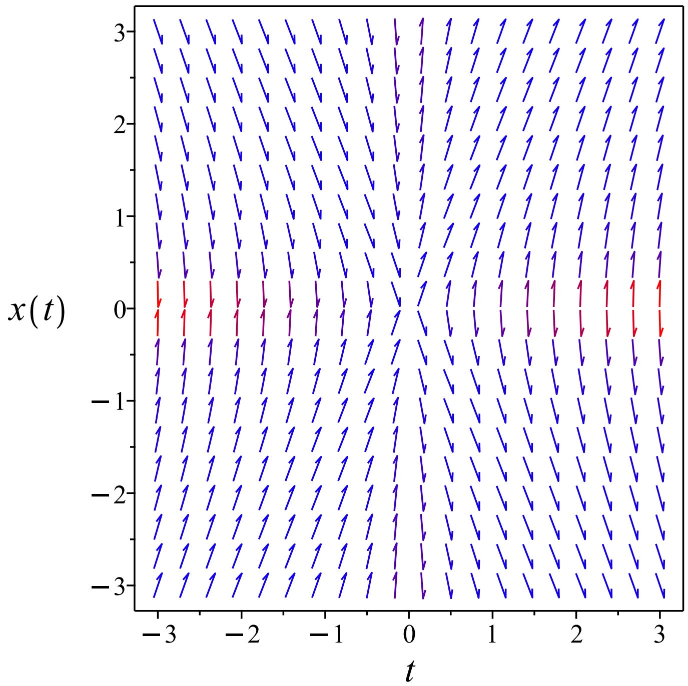


Figure 175: Slope field plot

Verification of solutions

$$\frac{x^2}{2t^{\frac{4}{3}}} = 3t^{\frac{2}{3}} + c_1$$

Verified OK.

5.26.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= \frac{\frac{2x^2}{3} + 2t^2}{tx} \end{aligned}$$

This is a Bernoulli ODE.

$$x' = \frac{2}{3t}x + 2t\frac{1}{x} \tag{1}$$

The standard Bernoulli ODE has the form

$$x' = f_0(t)x + f_1(t)x^n \tag{2}$$

The first step is to divide the above equation by x^n which gives

$$\frac{x'}{x^n} = f_0(t)x^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $w = x^{1-n}$ in equation (3) which generates a new ODE in $w(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $x(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(t) &= \frac{2}{3t} \\
f_1(t) &= 2t \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $x^n = \frac{1}{x}$ gives

$$
x'x = \frac{2x^2}{3t} + 2t \tag{4}
$$

Let

$$
\begin{aligned}
w &= x^{1-n} \\
&= x^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t t gives

$$
w' = 2xx' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(t)}{2} &= \frac{2w(t)}{3t} + 2t \\
w' &= \frac{4w}{3t} + 4t
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(t) + p(t)w(t) = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= -\frac{4}{3t} \\
q(t) &= 4t
\end{aligned}
$$

Hence the ode is

$$
w'(t) - \frac{4w(t)}{3t} = 4t
$$

The integrating factor μ is

$$\mu = e^{\int -\frac{4}{3t}dt}$$
$$= \frac{1}{t^{\frac{4}{3}}}$$

The ode becomes

$$\frac{d}{dt}(\mu w) = (\mu)(4t)$$
$$\frac{d}{dt}\left(\frac{w}{t^{\frac{4}{3}}}\right) = \left(\frac{1}{t^{\frac{4}{3}}}\right)(4t)$$
$$d\left(\frac{w}{t^{\frac{4}{3}}}\right) = \left(\frac{4}{t^{\frac{1}{3}}}\right) dt$$

Integrating gives

$$\frac{w}{t^{\frac{4}{3}}} = \int \frac{4}{t^{\frac{1}{3}}} dt$$
$$\frac{w}{t^{\frac{4}{3}}} = 6t^{\frac{2}{3}} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^{\frac{4}{3}}}$ results in

$$w(t) = 6t^2 + c_1 t^{\frac{4}{3}}$$

Replacing w in the above by x^2 using equation (5) gives the final solution.

$$x^2 = 6t^2 + c_1 t^{\frac{4}{3}}$$

Solving for x gives

$$x(t) = \sqrt{t^{\frac{4}{3}}\left(6t^{\frac{2}{3}} + c_1\right)}$$
$$x(t) = -\sqrt{t^{\frac{4}{3}}\left(6t^{\frac{2}{3}} + c_1\right)}$$

Summary

The solution(s) found are the following

$$x = \sqrt{t^{\frac{4}{3}}\left(6t^{\frac{2}{3}} + c_1\right)} \tag{1}$$
$$x = -\sqrt{t^{\frac{4}{3}}\left(6t^{\frac{2}{3}} + c_1\right)} \tag{2}$$

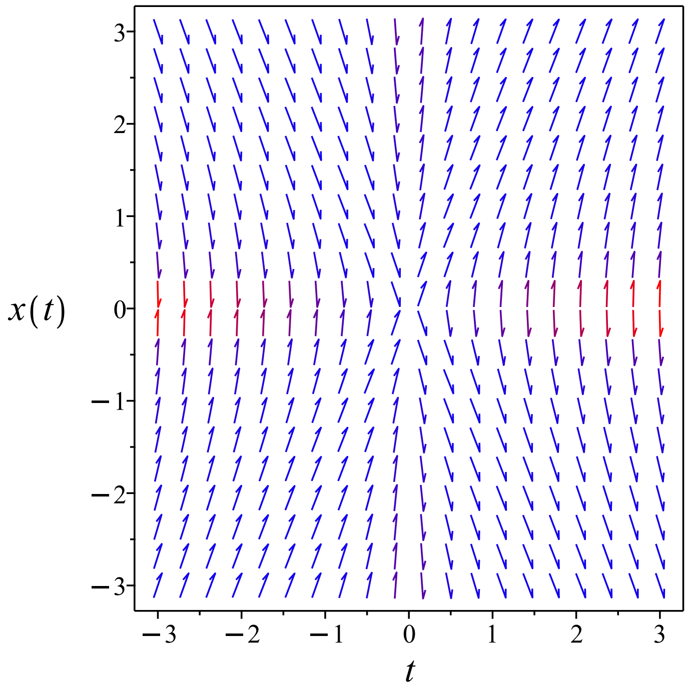


Figure 176: Slope field plot

Verification of solutions

$$x = \sqrt{t^{\frac{4}{3}} \left(6t^{\frac{2}{3}} + c_1\right)}$$

Verified OK.

$$x = -\sqrt{t^{\frac{4}{3}} \left(6t^{\frac{2}{3}} + c_1\right)}$$

Verified OK.

5.26.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3tx)\,\mathrm{d}x &= \left(6t^2 + 2x^2\right)\mathrm{d}t\\ \left(-6t^2 - 2x^2\right)\mathrm{d}t + (3tx)\,\mathrm{d}x &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x) &= -6t^2 - 2x^2\\ N(t,x) &= 3tx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-6t^2 - 2x^2\right)\\ &= -4x\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(3tx) \\
&= 3x
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= \frac{1}{3tx}((-4x) - (3x)) \\
&= -\frac{7}{3t}
\end{aligned}
$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -\frac{7}{3t} \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{7\ln(t)}{3}} \\
&= \frac{1}{t^{\frac{7}{3}}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{t^{\frac{7}{3}}}\left(-6t^2 - 2x^2\right) \\
&= \frac{-6t^2 - 2x^2}{t^{\frac{7}{3}}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{t^{\frac{7}{3}}}(3tx) \\
&= \frac{3x}{t^{\frac{4}{3}}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$
$$\left(\frac{-6t^2 - 2x^2}{t^{\frac{7}{3}}}\right) + \left(\frac{3x}{t^{\frac{4}{3}}}\right)\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \frac{-6t^2 - 2x^2}{t^{\frac{7}{3}}}\,\mathrm{d}t$$
$$\phi = \frac{-9t^2 + \frac{3x^2}{2}}{t^{\frac{4}{3}}} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \frac{3x}{t^{\frac{4}{3}}} + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{3x}{t^{\frac{4}{3}}}$. Therefore equation (4) becomes

$$\frac{3x}{t^{\frac{4}{3}}} = \frac{3x}{t^{\frac{4}{3}}} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{-9t^2 + \frac{3x^2}{2}}{t^{\frac{4}{3}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-9t^2 + \frac{3x^2}{2}}{t^{\frac{4}{3}}}$$

Summary

The solution(s) found are the following

$$\frac{\frac{3x^2}{2} - 9t^2}{t^{\frac{4}{3}}} = c_1 \tag{1}$$

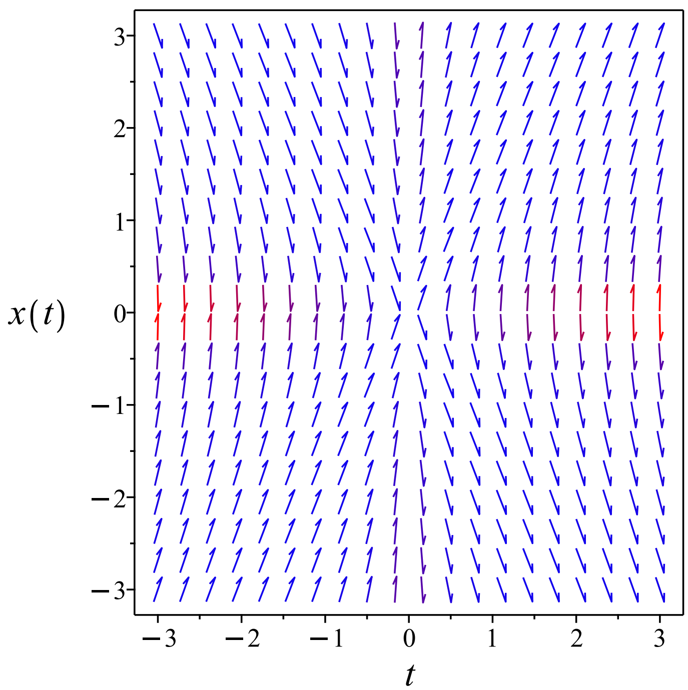


Figure 177: Slope field plot

Verification of solutions

$$\frac{\frac{3x^2}{2} - 9t^2}{t^{\frac{4}{3}}} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(x(t),t)=2/(3*t)*x(t)+2*t/x(t),x(t), singsol=all)
```

$$x(t) = \sqrt{\left(6t^{\frac{2}{3}} + c_1\right) t^{\frac{4}{3}}}$$
$$x(t) = -\sqrt{\left(6t^{\frac{2}{3}} + c_1\right) t^{\frac{4}{3}}}$$

✓ Solution by Mathematica

Time used: 5.087 (sec). Leaf size: 47

```
DSolve[x'[t]==2/(3*t)*x[t]+2*t/x[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\sqrt{6t^2 + c_1 t^{4/3}}$$
$$x(t) \to \sqrt{6t^2 + c_1 t^{4/3}}$$

5.27 problem 15(b)

5.27.1 Solving as first order ode lie symmetry lookup ode 810
5.27.2 Solving as bernoulli ode 814
5.27.3 Solving as riccati ode 818

Internal problem ID [11425]
Internal file name [`OUTPUT/10407_Thursday_May_18_2023_04_18_34_AM_84307621/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 15(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Bernoulli]
```

$$\boxed{x' - x(1 + x\,\mathrm{e}^{t}) = 0}$$

5.27.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = x(1 + \mathrm{e}^{t} x)$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 161: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= 0 \\
\eta(t, x) &= x^2 \mathrm{e}^{-t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{x^2 \mathrm{e}^{-t}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\mathrm{e}^t}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = x(1 + \mathrm{e}^t x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_t &= 1 \\
R_x &= 0 \\
S_t &= -\frac{\mathrm{e}^t}{x} \\
S_x &= \frac{\mathrm{e}^t}{x^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{2t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{2R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$-\frac{e^t}{x} = \frac{e^{2t}}{2} + c_1$$

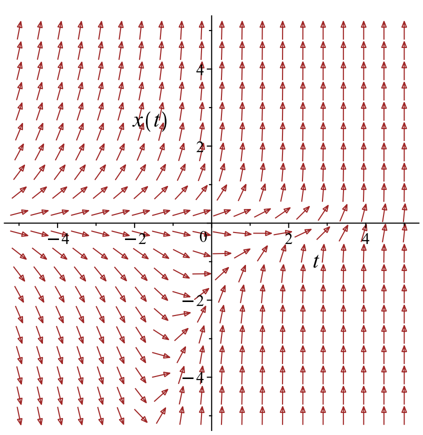
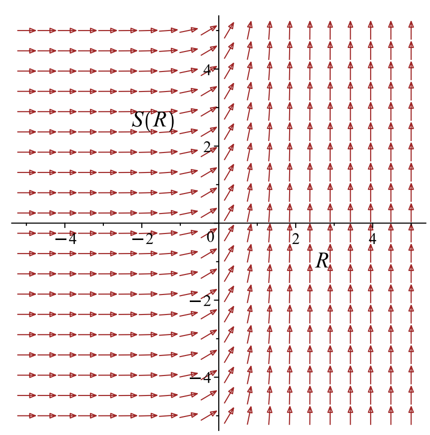
Which simplifies to

$$-\frac{e^t}{x} = \frac{e^{2t}}{2} + c_1$$

Which gives

$$x = -\frac{2\,e^t}{e^{2t} + 2c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = x(1 + e^t x)$ 	$R = t$ $S = -\frac{e^t}{x}$	$\frac{dS}{dR} = e^{2R}$ 

Summary

The solution(s) found are the following

$$x = -\frac{2\,\mathrm{e}^{t}}{\mathrm{e}^{2t}+2c_1} \tag{1}$$

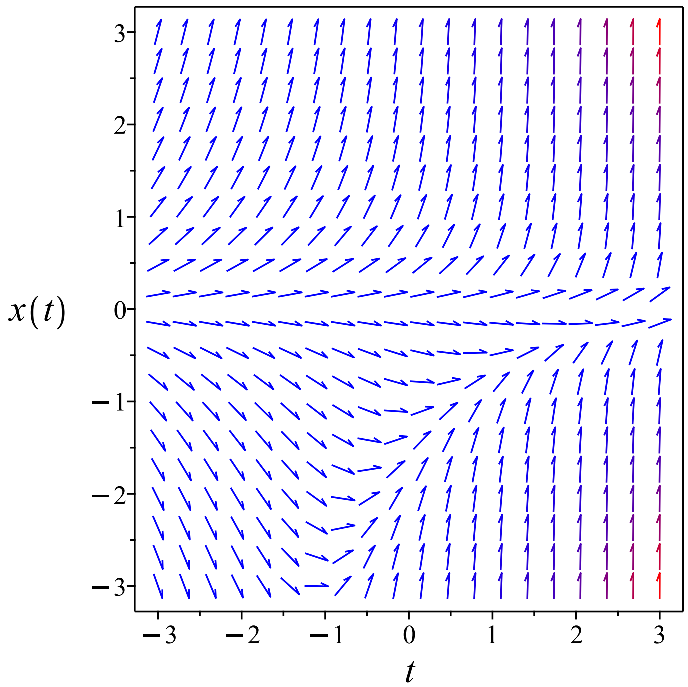


Figure 178: Slope field plot

Verification of solutions

$$x = -\frac{2\,\mathrm{e}^{t}}{\mathrm{e}^{2t}+2c_1}$$

Verified OK.

5.27.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= x\left(1+\mathrm{e}^{t}x\right) \end{aligned}$$

This is a Bernoulli ODE.

$$x' = x + \mathrm{e}^{t}x^{2} \tag{1}$$

The standard Bernoulli ODE has the form

$$x' = f_0(t)x + f_1(t)x^n \tag{2}$$

The first step is to divide the above equation by x^n which gives

$$\frac{x'}{x^n} = f_0(t)x^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $w = x^{1-n}$ in equation (3) which generates a new ODE in $w(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $x(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(t) &= 1 \\ f_1(t) &= \mathrm{e}^t \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $x^n = x^2$ gives

$$x'\frac{1}{x^2} = \frac{1}{x} + \mathrm{e}^t \tag{4}$$

Let

$$\begin{aligned} w &= x^{1-n} \\ &= \frac{1}{x} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t t gives

$$w' = -\frac{1}{x^2}x' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(t) &= w(t) + \mathrm{e}^t \\ w' &= -w - \mathrm{e}^t \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(t) + p(t)w(t) = q(t)$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= -\mathrm{e}^t \end{aligned}$$

Hence the ode is

$$w'(t) + w(t) = -\mathrm{e}^t$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 1 dt} \\ &= \mathrm{e}^t \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu w) &= (\mu)\left(-\mathrm{e}^t\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^t w\right) &= \left(\mathrm{e}^t\right)\left(-\mathrm{e}^t\right) \\ \mathrm{d}\left(\mathrm{e}^t w\right) &= \left(-\mathrm{e}^{2t}\right)\,\mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^t w &= \int -\mathrm{e}^{2t}\,\mathrm{d}t \\ \mathrm{e}^t w &= -\frac{\mathrm{e}^{2t}}{2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^t$ results in

$$w(t) = -\frac{\mathrm{e}^{-t}\mathrm{e}^{2t}}{2} + c_1\mathrm{e}^{-t}$$

which simplifies to

$$w(t) = -\frac{\mathrm{e}^t}{2} + c_1\mathrm{e}^{-t}$$

Replacing w in the above by $\frac{1}{x}$ using equation (5) gives the final solution.

$$\frac{1}{x} = -\frac{\mathrm{e}^t}{2} + c_1\mathrm{e}^{-t}$$

Or

$$x = \frac{1}{-\frac{\mathrm{e}^t}{2} + c_1\mathrm{e}^{-t}}$$

Which is simplified to

$$x = -\frac{2\,\mathrm{e}^{t}}{\mathrm{e}^{2t} - 2c_1}$$

Summary

The solution(s) found are the following

$$x = -\frac{2\,\mathrm{e}^{t}}{\mathrm{e}^{2t} - 2c_1} \tag{1}$$

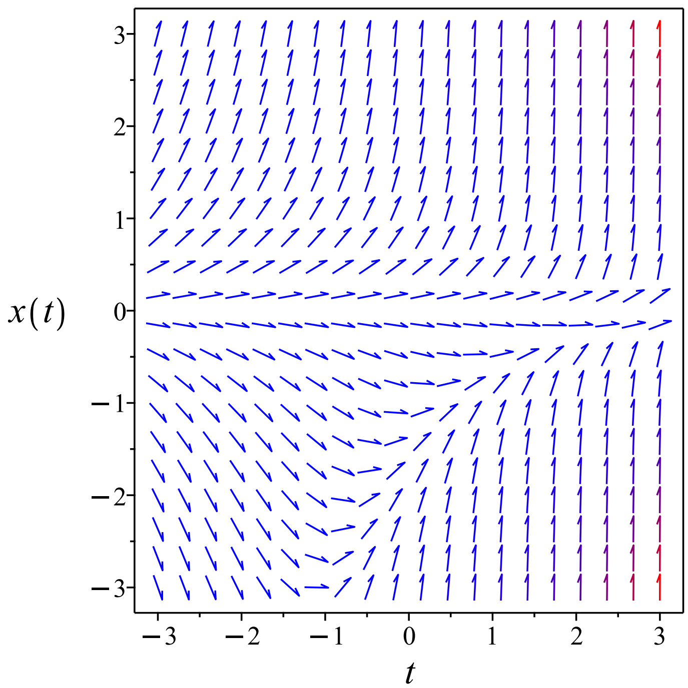


Figure 179: Slope field plot

Verification of solutions

$$x = -\frac{2\,\mathrm{e}^{t}}{\mathrm{e}^{2t} - 2c_1}$$

Verified OK.

5.27.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}
x' &= F(t, x) \\
&= x(1 + e^t x)
\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$x' = e^t x^2 + x$$

With Riccati ODE standard form

$$x' = f_0(t) + f_1(t)x + f_2(t)x^2$$

Shows that $f_0(t) = 0$, $f_1(t) = 1$ and $f_2(t) = e^t$. Let

$$\begin{aligned}
x &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{e^t u}
\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2) u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= e^t \\
f_1 f_2 &= e^t \\
f_2^2 f_0 &= 0
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$e^t u''(t) - 2 e^t u'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 + c_2 e^{2t}$$

The above shows that

$$u'(t) = 2c_2 e^{2t}$$

Using the above in (1) gives the solution

$$x = -\frac{2c_2 e^{2t} e^{-t}}{c_1 + c_2 e^{2t}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$x = -\frac{2\,e^{t}}{c_3 + e^{2t}}$$

Summary

The solution(s) found are the following

$$x = -\frac{2\,e^{t}}{c_3 + e^{2t}} \tag{1}$$

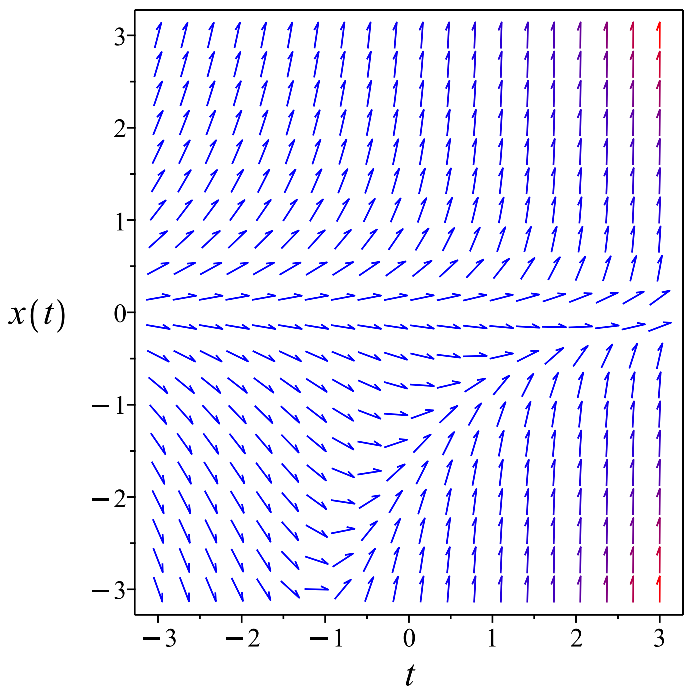


Figure 180: Slope field plot

Verification of solutions

$$x = -\frac{2\,e^{t}}{c_3 + e^{2t}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(x(t),t)=x(t)*(1+x(t)*exp(t)),x(t), singsol=all)
```

$$x(t) = -\frac{2\,\mathrm{e}^{t}}{\mathrm{e}^{2t} - 2c_1}$$

✓ Solution by Mathematica

Time used: 0.323 (sec). Leaf size: 27

```
DSolve[x'[t]==x[t]*(1+x[t]*Exp[t]),x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{2e^{t}}{e^{2t} - 2c_1}$$
$$x(t) \to 0$$

5.28 problem 15(c)

5.28.1 Solving as separable ode 821
5.28.2 Solving as first order ode lie symmetry lookup ode 823
5.28.3 Solving as bernoulli ode 827
5.28.4 Solving as exact ode 831
5.28.5 Maple step by step solution 834

Internal problem ID [11426]
Internal file name [`OUTPUT/10408_Thursday_May_18_2023_04_18_36_AM_52077065/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 15(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' + \frac{x}{t} - \frac{1}{t x^2} = 0}$$

5.28.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= f(t)g(x) \\ &= -\frac{x^3-1}{t\,x^2} \end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(x) = \frac{x^3-1}{x^2}$. Integrating both sides gives

$$\frac{1}{\frac{x^3-1}{x^2}}\,dx = -\frac{1}{t}\,dt$$

$$\int \frac{1}{\frac{x^3-1}{x^2}}\,dx = \int -\frac{1}{t}\,dt$$

$$\frac{\ln\left(x^3-1\right)}{3} = -\ln\left(t\right) + c_1$$

Raising both side to exponential gives

$$\left(x^3-1\right)^{\frac{1}{3}} = \mathrm{e}^{-\ln(t)+c_1}$$

Which simplifies to

$$\left(x^3-1\right)^{\frac{1}{3}} = \frac{c_2}{t}$$

Which simplifies to

$$\left(x^3-1\right)^{\frac{1}{3}} = \frac{c_2 \mathrm{e}^{c_1}}{t}$$

The solution is

$$\left(x^3-1\right)^{\frac{1}{3}} = \frac{c_2 \mathrm{e}^{c_1}}{t}$$

<u>Summary</u>
The solution(s) found are the following

$$\left(x^3-1\right)^{\frac{1}{3}} = \frac{c_2 \mathrm{e}^{c_1}}{t} \tag{1}$$

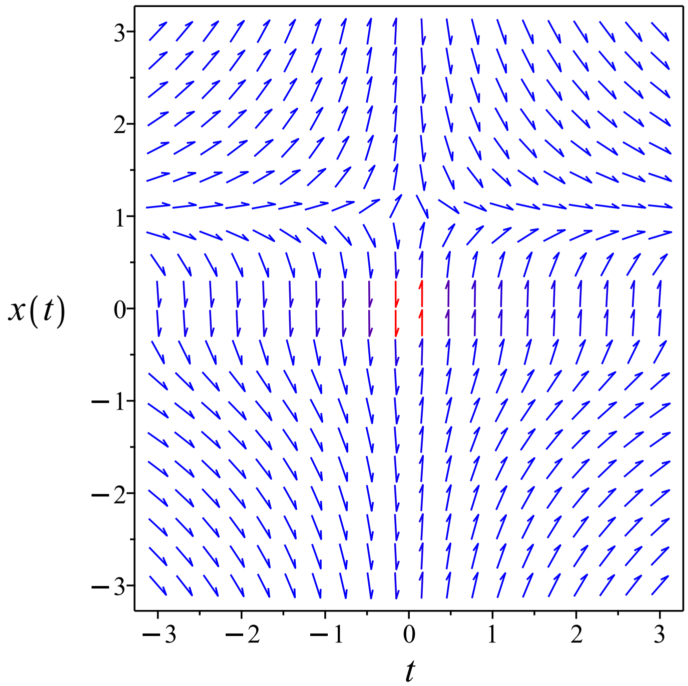


Figure 181: Slope field plot

Verification of solutions

$$\left(x^3-1\right)^{\frac{1}{3}} = \frac{c_2 \mathrm{e}^{c_1}}{t}$$

Verified OK.

5.28.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} x' &= -\frac{x^3-1}{t\,x^2} \\ x' &= \omega(t,x) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 163: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= -t \\
\eta(t,x) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{-t} dt \end{aligned}$$

Which results in

$$S = -\ln(t)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -\frac{x^3 - 1}{t\,x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_x &= 1 \\ S_t &= -\frac{1}{t} \\ S_x &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x^2}{x^3 - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^2}{R^3 - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln\left((R-1)\left(R^2+R+1\right)\right)}{3} + c_1 \tag{4}$$

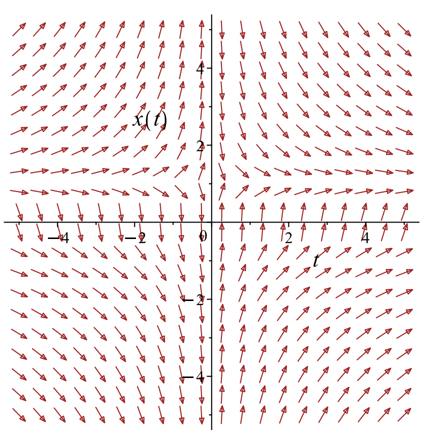
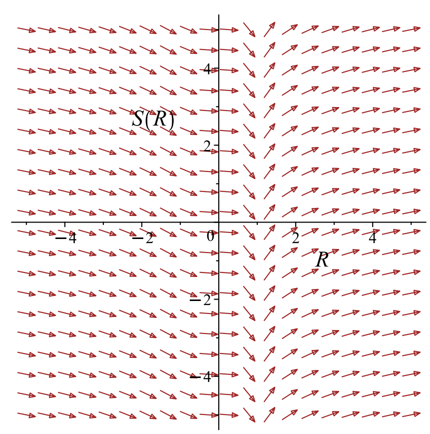
To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$-\ln(t) = \frac{\ln\left((x-1)\left(x^2+x+1\right)\right)}{3} + c_1$$

Which simplifies to

$$-\ln(t) = \frac{\ln\left((x-1)\left(x^2+x+1\right)\right)}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\frac{x^3-1}{t\,x^2}$ 	$R = x$ $S = -\ln(t)$	$\frac{dS}{dR} = \frac{R^2}{R^3-1}$ 

Summary

The solution(s) found are the following

$$-\ln(t) = \frac{\ln\left((x-1)\left(x^2+x+1\right)\right)}{3} + c_1 \tag{1}$$

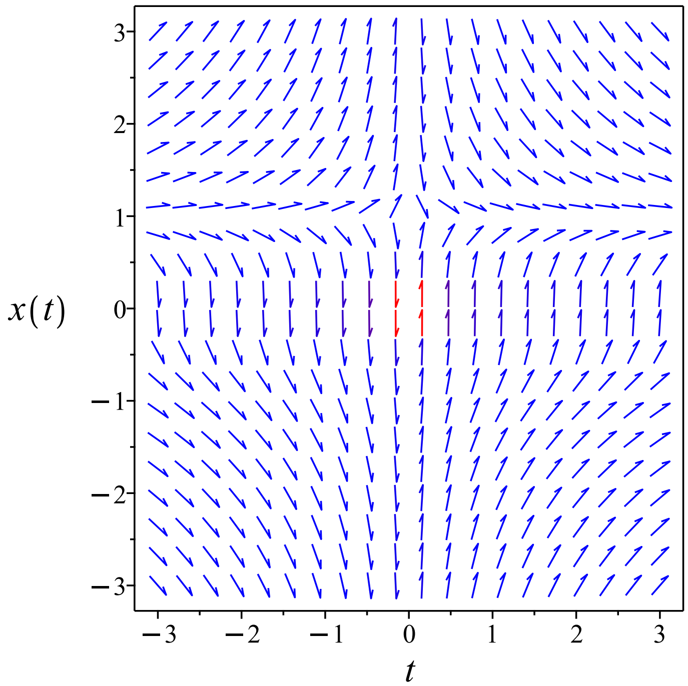


Figure 182: Slope field plot

Verification of solutions

$$-\ln(t) = \frac{\ln((x-1)(x^2+x+1))}{3} + c_1$$

Verified OK.

5.28.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= -\frac{x^3 - 1}{t\,x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$x' = -\frac{1}{t}x + \frac{1}{t}\frac{1}{x^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$x' = f_0(t)x + f_1(t)x^n \tag{2}$$

The first step is to divide the above equation by x^n which gives

$$\frac{x'}{x^n} = f_0(t)x^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $w = x^{1-n}$ in equation (3) which generates a new ODE in $w(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $x(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(t) &= -\frac{1}{t} \\ f_1(t) &= \frac{1}{t} \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $x^n = \frac{1}{x^2}$ gives

$$x'x^2 = -\frac{x^3}{t} + \frac{1}{t} \tag{4}$$

Let

$$\begin{aligned} w &= x^{1-n} \\ &= x^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t t gives

$$w' = 3x^2x' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(t)}{3} &= -\frac{w(t)}{t} + \frac{1}{t} \\ w' &= -\frac{3w}{t} + \frac{3}{t} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(t) + p(t)w(t) = q(t)$$

Where here

$$\begin{aligned} p(t) &= \frac{3}{t} \\ q(t) &= \frac{3}{t} \end{aligned}$$

Hence the ode is

$$w'(t) + \frac{3w(t)}{t} = \frac{3}{t}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{3}{t}dt} \\
&= t^3
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dt}(\mu w) &= (\mu)\left(\frac{3}{t}\right) \\
\frac{d}{dt}(t^3 w) &= (t^3)\left(\frac{3}{t}\right) \\
d(t^3 w) &= (3t^2)\ dt
\end{aligned}$$

Integrating gives

$$\begin{aligned}
t^3 w &= \int 3t^2\, dt \\
t^3 w &= t^3 + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^3$ results in

$$w(t) = 1 + \frac{c_1}{t^3}$$

Replacing w in the above by x^3 using equation (5) gives the final solution.

$$x^3 = 1 + \frac{c_1}{t^3}$$

Solving for x gives

$$\begin{aligned}
x(t) &= \frac{(t^3 + c_1)^{\frac{1}{3}}}{t} \\
x(t) &= \frac{(t^3 + c_1)^{\frac{1}{3}}\,(i\sqrt{3} - 1)}{2t} \\
x(t) &= -\frac{(t^3 + c_1)^{\frac{1}{3}}\,(1 + i\sqrt{3})}{2t}
\end{aligned}$$

Summary

The solution(s) found are the following

$$x = \frac{(t^3 + c_1)^{\frac{1}{3}}}{t} \tag{1}$$

$$x = \frac{(t^3 + c_1)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2t} \tag{2}$$

$$x = -\frac{(t^3 + c_1)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2t} \tag{3}$$

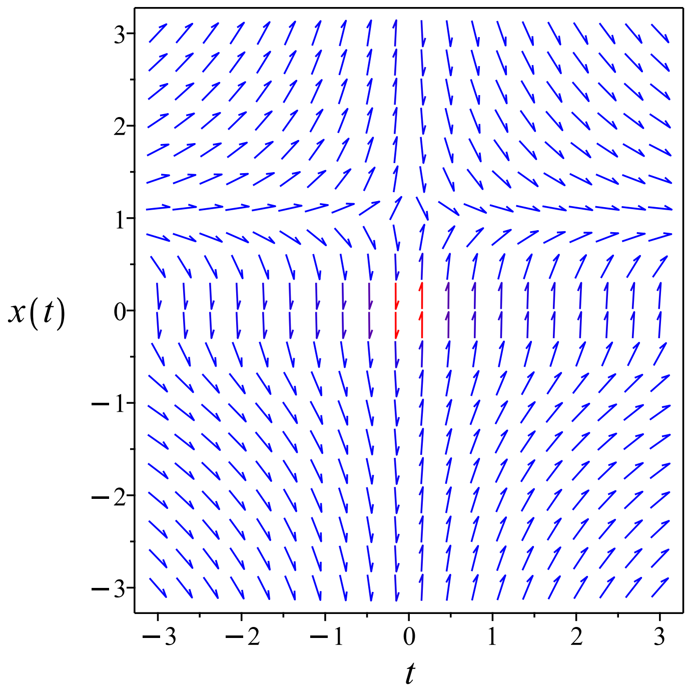


Figure 183: Slope field plot

Verification of solutions

$$x = \frac{(t^3 + c_1)^{\frac{1}{3}}}{t}$$

Verified OK.

$$x = \frac{(t^3 + c_1)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2t}$$

Verified OK.

$$x = -\frac{(t^3 + c_1)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2t}$$

Verified OK.

5.28.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\left(-\frac{x^2}{x^3-1}\right)\mathrm{d}x = \left(\frac{1}{t}\right)\mathrm{d}t$$
$$\left(-\frac{1}{t}\right)\mathrm{d}t + \left(-\frac{x^2}{x^3-1}\right)\mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t,x) = -\frac{1}{t}$$
$$N(t,x) = -\frac{x^2}{x^3-1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{t}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{x^2}{x^3-1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{1}{t}\,\mathrm{d}t \\
\phi &= -\ln(t) + f(x) \qquad (3)
\end{aligned}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = -\frac{x^2}{x^3-1}$. Therefore equation (4) becomes

$$
-\frac{x^2}{x^3-1} = 0 + f'(x) \qquad (5)
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = -\frac{x^2}{x^3-1}
$$

Integrating the above w.r.t x gives

$$
\begin{aligned}
\int f'(x)\,\mathrm{d}x &= \int \left(-\frac{x^2}{x^3-1}\right)\mathrm{d}x \\
f(x) &= -\frac{\ln(x^3-1)}{3} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$
\phi = -\ln(t) - \frac{\ln(x^3-1)}{3} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\ln(t) - \frac{\ln(x^3-1)}{3}
$$

Summary
The solution(s) found are the following

$$
-\ln(t) - \frac{\ln(x^3 - 1)}{3} = c_1 \tag{1}
$$

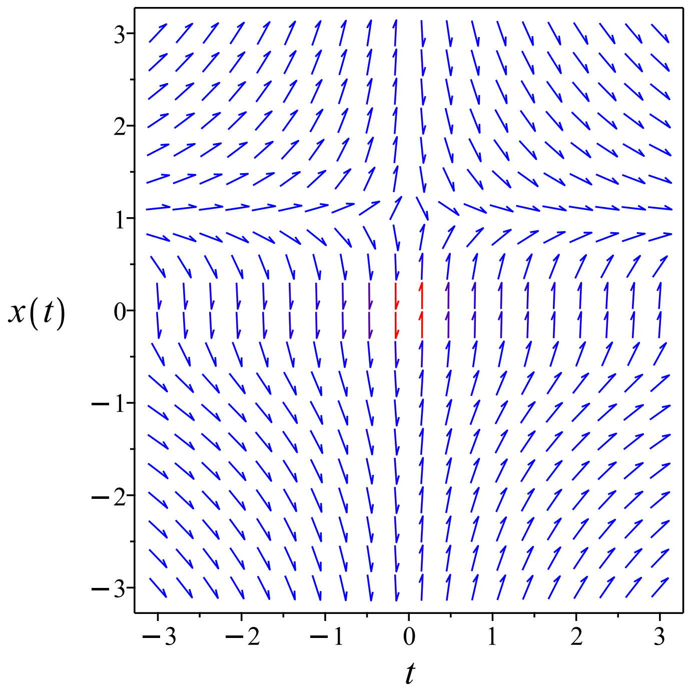


Figure 184: Slope field plot

Verification of solutions

$$
-\ln(t) - \frac{\ln(x^3 - 1)}{3} = c_1
$$

Verified OK.

5.28.5 Maple step by step solution

Let's solve

$x' + \frac{x}{t} - \frac{1}{t x^2} = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

$$\frac{x'x^2}{(x-1)(x^2+x+1)} = -\frac{1}{t}$$

- Integrate both sides with respect to t

$$\int \frac{x'x^2}{(x-1)(x^2+x+1)}dt = \int -\frac{1}{t}dt + c_1$$

- Evaluate integral

$$\frac{\ln((x-1)(x^2+x+1))}{3} = -\ln(t) + c_1$$

- Solve for x

$$x = \frac{\left((\mathrm{e}^{c_1})^3 + t^3\right)^{\frac{1}{3}}}{t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 61

```
dsolve(diff(x(t),t)=-1/t*x(t)+1/(t*x(t)^2),x(t), singsol=all)
```

$$x(t) = \frac{(t^3 + c_1)^{\frac{1}{3}}}{t}$$
$$x(t) = -\frac{(t^3 + c_1)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2t}$$
$$x(t) = \frac{(t^3 + c_1)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2t}$$

✓ Solution by Mathematica

Time used: 0.456 (sec). Leaf size: 159

```
DSolve[x'[t]==-1/t*x[t]+1/(t*x[t]^2),x[t],t,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
x(t) &\to \frac{\sqrt[3]{t^3+e^{3c_1}}}{t} \\
x(t) &\to -\frac{\sqrt[3]{-1}\sqrt[3]{t^3+e^{3c_1}}}{t} \\
x(t) &\to \frac{(-1)^{2/3}\sqrt[3]{t^3+e^{3c_1}}}{t} \\
x(t) &\to 1 \\
x(t) &\to -\sqrt[3]{-1} \\
x(t) &\to (-1)^{2/3} \\
x(t) &\to \frac{\sqrt[3]{t^3}}{t} \\
x(t) &\to -\frac{\sqrt[3]{-1}\sqrt[3]{t^3}}{t} \\
x(t) &\to \frac{(-1)^{2/3}\sqrt[3]{t^3}}{t}
\end{aligned}
$$

5.29 problem 15(d)

5.29.1 Solving as homogeneousTypeD2 ode 837
5.29.2 Solving as first order ode lie symmetry lookup ode 839
5.29.3 Solving as bernoulli ode 843
5.29.4 Solving as riccati ode 847

Internal problem ID [11427]
Internal file name [`OUTPUT/10409_Thursday_May_18_2023_04_18_38_AM_80034732/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 15(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{t^2y' + 2ty - y^2 = 0}$$

5.29.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$t^2(u'(t)\,t + u(t)) + 2t^2u(t) - u(t)^2\,t^2 = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(t,u) \\
&= f(t)g(u) \\
&= \frac{u(u-3)}{t}
\end{aligned}$$

Where $f(t) = \frac{1}{t}$ and $g(u) = u(u-3)$. Integrating both sides gives

$$\frac{1}{u(u-3)}\, du = \frac{1}{t}\, dt$$
$$\int \frac{1}{u(u-3)}\, du = \int \frac{1}{t}\, dt$$
$$\frac{\ln (u-3)}{3} - \frac{\ln (u)}{3} = \ln (t) + c_2$$

The above can be written as

$$\left(\frac{1}{3}\right)(\ln (u-3) - \ln (u)) = \ln (t) + 2c_2$$
$$\ln (u-3) - \ln (u) = (3)(\ln (t) + 2c_2)$$
$$= 3\ln (t) + 6c_2$$

Raising both side to exponential gives

$$e^{\ln(u-3)-\ln(u)} = e^{3\ln(t)+3c_2}$$

Which simplifies to

$$\frac{u-3}{u} = 3c_2t^3$$
$$= c_3t^3$$

Therefore the solution y is

$$y = tu$$
$$= -\frac{3t}{c_3t^3-1}$$

Summary

The solution(s) found are the following

$$y = -\frac{3t}{c_3t^3-1} \tag{1}$$

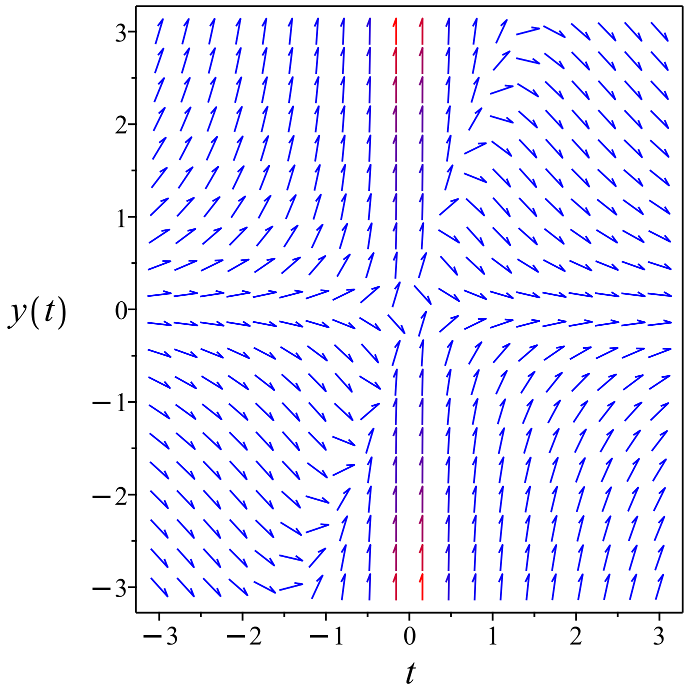


Figure 185: Slope field plot

Verification of solutions

$$y = -\frac{3t}{c_3 t^3 - 1}$$

Verified OK.

5.29.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(-2t + y)}{t^2}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 166: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= t^2 y^2 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{t^2 y^2} dy
\end{aligned}$$

Which results in

$$S = -\frac{1}{t^2 y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{y(-2t + y)}{t^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= \frac{2}{t^3 y} \\
S_y &= \frac{1}{t^2 y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{t^4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{3R^3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$-\frac{1}{t^2 y} = -\frac{1}{3t^3} + c_1$$

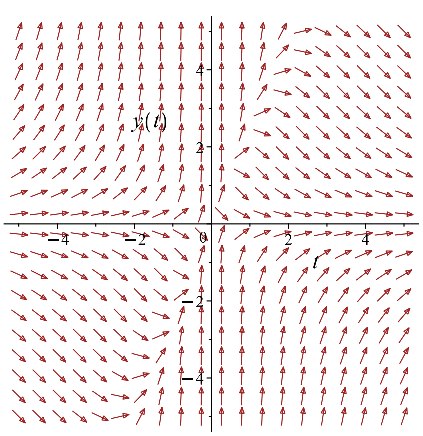
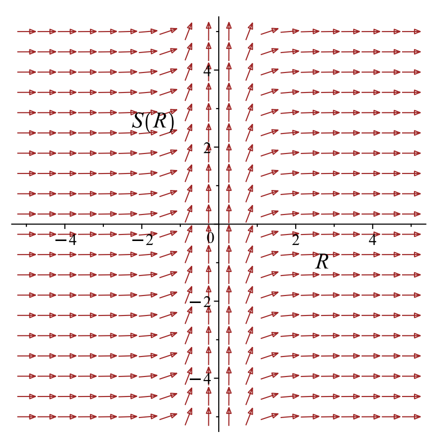
Which simplifies to

$$-\frac{1}{t^2 y} = -\frac{1}{3t^3} + c_1$$

Which gives

$$y = -\frac{3t}{3c_1 t^3 - 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{y(-2t+y)}{t^2}$ 	$R = t$ $S = -\frac{1}{t^2 y}$	$\frac{dS}{dR} = \frac{1}{R^4}$ 

Summary
The solution(s) found are the following

$$y = -\frac{3t}{3c_1 t^3 - 1} \tag{1}$$

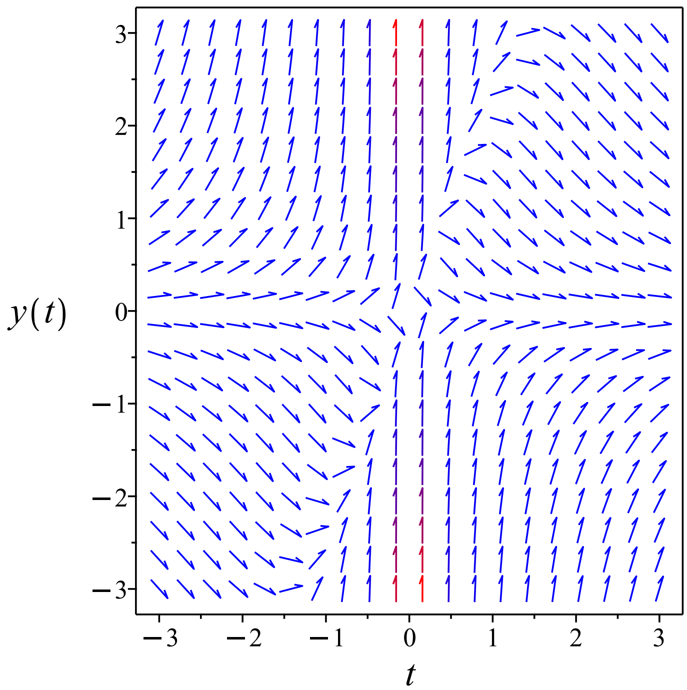


Figure 186: Slope field plot

Verification of solutions

$$y = -\frac{3t}{3c_1 t^3 - 1}$$

Verified OK.

5.29.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= \frac{y(-2t + y)}{t^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2}{t}y + \frac{1}{t^2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(t)y + f_1(t)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(t)y^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(t) &= -\frac{2}{t} \\ f_1(t) &= \frac{1}{t^2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{2}{ty} + \frac{1}{t^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t t gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(t) &= -\frac{2w(t)}{t} + \frac{1}{t^2} \\ w' &= \frac{2w}{t} - \frac{1}{t^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(t) + p(t)w(t) = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= -\frac{2}{t} \\
q(t) &= -\frac{1}{t^2}
\end{aligned}
$$

Hence the ode is

$$
w'(t) - \frac{2w(t)}{t} = -\frac{1}{t^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{2}{t}dt} \\
&= \frac{1}{t^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu w) &= (\mu)\left(-\frac{1}{t^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{w}{t^2}\right) &= \left(\frac{1}{t^2}\right)\left(-\frac{1}{t^2}\right) \\
\mathrm{d}\left(\frac{w}{t^2}\right) &= \left(-\frac{1}{t^4}\right)\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{t^2} &= \int -\frac{1}{t^4}\,\mathrm{d}t \\
\frac{w}{t^2} &= \frac{1}{3t^3} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$
w(t) = \frac{1}{3t} + t^2 c_1
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = \frac{1}{3t} + t^2 c_1
$$

Or

$$y = \frac{1}{\frac{1}{3t} + t^2 c_1}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{\frac{1}{3t} + t^2 c_1} \tag{1}$$

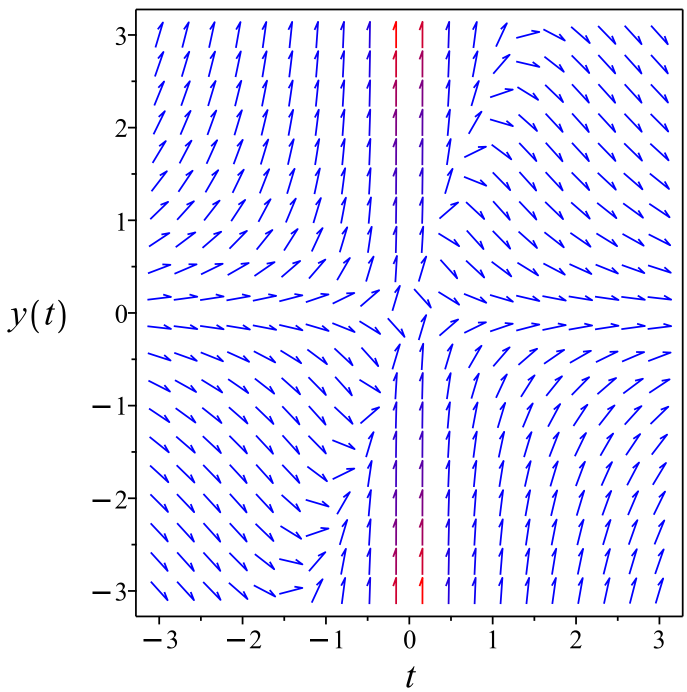


Figure 187: Slope field plot

Verification of solutions

$$y = \frac{1}{\frac{1}{3t} + t^2 c_1}$$

Verified OK.

5.29.4 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(t, y) \\
&= \frac{y(-2t + y)}{t^2}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = -\frac{2y}{t} + \frac{y^2}{t^2}
$$

With Riccati ODE standard form

$$
y' = f_0(t) + f_1(t)y + f_2(t)y^2
$$

Shows that $f_0(t) = 0$, $f_1(t) = -\frac{2}{t}$ and $f_2(t) = \frac{1}{t^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{t^2}} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(t) - (f_2' + f_1 f_2) u'(t) + f_2^2 f_0 u(t) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= -\frac{2}{t^3} \\
f_1 f_2 &= -\frac{2}{t^3} \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(t)}{t^2} + \frac{4u'(t)}{t^3} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(t) = c_1 + \frac{c_2}{t^3}
$$

The above shows that

$$u'(t) = -\frac{3c_2}{t^4}$$

Using the above in (1) gives the solution

$$y = \frac{3c_2}{t^2 \left(c_1 + \frac{c_2}{t^3}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{3t}{c_3 t^3 + 1}$$

Summary
The solution(s) found are the following

$$y = \frac{3t}{c_3 t^3 + 1} \tag{1}$$

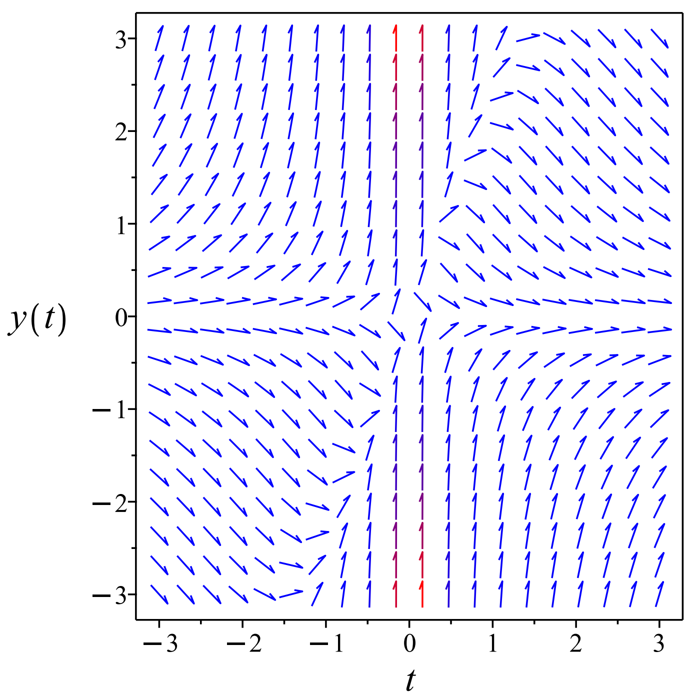


Figure 188: Slope field plot

Verification of solutions

$$y = \frac{3t}{c_3 t^3 + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(t^2*diff(y(t),t)+2*t*y(t)-y(t)^2=0,y(t), singsol=all)
```

$$y(t) = \frac{3t}{3c_1 t^3 + 1}$$

✓ Solution by Mathematica

Time used: 0.246 (sec). Leaf size: 24

```
DSolve[t^2*y'[t]+2*t*y[t]-y[t]^2==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{3t}{1 + 3c_1 t^3}$$
$$y(t) \to 0$$

5.30 problem 15(e)

5.30.1 Solving as quadrature ode 850
5.30.2 Maple step by step solution 851

Internal problem ID [11428]
Internal file name [`OUTPUT/10410_Thursday_May_18_2023_04_18_39_AM_39873401/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 15(e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' - ax - bx^3 = 0}$$

5.30.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{x^3 b + ax} dx = \int dt$$
$$-\frac{\ln (b\,x^2 + a)}{2a} + \frac{\ln (x)}{a} = t + c_1$$

Raising both side to exponential gives

$$e^{-\frac{\ln \left(b\,x^2+a\right)}{2a}+\frac{\ln(x)}{a}} = e^{t+c_1}$$

Which simplifies to

$$\left(b\,x^2 + a\right)^{-\frac{1}{2a}} x^{\frac{1}{a}} = c_2 e^t$$

<u>Summary</u>
The solution(s) found are the following

$$x = \operatorname{RootOf}\left(-\left(\_Z^2 b + a\right)^{-\frac{1}{2a}} \_Z^{\frac{1}{a}} + c_2 e^t\right) \tag{1}$$

Verification of solutions

$$x = \operatorname{RootOf}\left(-\left(\_Z^2 b + a\right)^{-\frac{1}{2a}} \_Z^{\frac{1}{a}} + c_2 \mathrm{e}^{t}\right)$$

Verified OK.

5.30.2 Maple step by step solution

Let's solve

$x' - ax - bx^3 = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{ax+bx^3} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{ax+bx^3} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\frac{\ln(bx^2+a)}{2a} + \frac{\ln(x)}{a} = t + c_1$

- Solve for x

  $\left\{x = \frac{\sqrt{-(b\,\mathrm{e}^{2c_1a+2ta}-1)a\,\mathrm{e}^{2c_1a+2ta}}}{b\,\mathrm{e}^{2c_1a+2ta}-1}, x = -\frac{\sqrt{-(b\,\mathrm{e}^{2c_1a+2ta}-1)a\,\mathrm{e}^{2c_1a+2ta}}}{b\,\mathrm{e}^{2c_1a+2ta}-1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 70

```
dsolve(diff(x(t),t)=a*x(t)+b*x(t)^3,x(t), singsol=all)
```

$$x(t) = \frac{\sqrt{(c_1 a \,\mathrm{e}^{-2at} - b)\, a}}{c_1 a \,\mathrm{e}^{-2at} - b}$$
$$x(t) = -\frac{\sqrt{(c_1 a \,\mathrm{e}^{-2at} - b)\, a}}{c_1 a \,\mathrm{e}^{-2at} - b}$$

✓ Solution by Mathematica

Time used: 2.841 (sec). Leaf size: 118

```
DSolve[x'[t]==a*x[t]+b*x[t]^3,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{i\sqrt{a}e^{a(t+c_1)}}{\sqrt{-1+be^{2a(t+c_1)}}}$$
$$x(t) \to \frac{i\sqrt{a}e^{a(t+c_1)}}{\sqrt{-1+be^{2a(t+c_1)}}}$$
$$x(t) \to 0$$
$$x(t) \to -\frac{i\sqrt{a}}{\sqrt{b}}$$
$$x(t) \to \frac{i\sqrt{a}}{\sqrt{b}}$$

5.31 problem 15(f)

5.31.1 Solving as first order ode lie symmetry lookup ode 853
5.31.2 Solving as bernoulli ode 857

Internal problem ID [11429]
Internal file name [`OUTPUT/10411_Thursday_May_18_2023_04_18_45_AM_14858344/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 15(f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{w' - wt - t^3w^3 = 0}$$

5.31.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} w' &= t^3w^3 + wt \\ w' &= \omega(t, w) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_w - \xi_t) - \omega^2\xi_w - \omega_t\xi - \omega_w\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 169: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t, w) &= 0 \\ \eta(t, w) &= w^3 \mathrm{e}^{-t^2}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, w) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dw}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial w}\right) S(t, w) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{w^3 \mathrm{e}^{-t^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\mathrm{e}^{t^2}}{2w^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,w) S_w}{R_t + \omega(t,w) R_w} \tag{2}$$

Where in the above R_t, R_w, S_t, S_w are all partial derivatives and $\omega(t,w)$ is the right hand side of the original ode given by

$$\omega(t,w) = t^3 w^3 + wt$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_w &= 0 \\ S_t &= -\frac{t\,\mathrm{e}^{t^2}}{w^2} \\ S_w &= \frac{\mathrm{e}^{t^2}}{w^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t^3 \mathrm{e}^{t^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, w in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3 \mathrm{e}^{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{(R^2 - 1)\,\mathrm{e}^{R^2}}{2} + c_1 \tag{4}$$

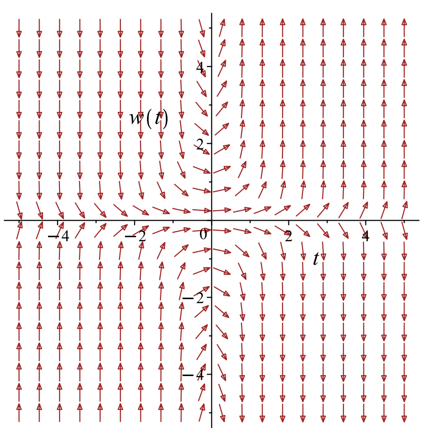
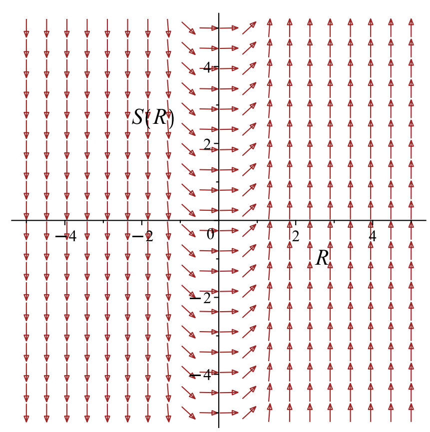
To complete the solution, we just need to transform (4) back to t, w coordinates. This results in

$$-\frac{\mathrm{e}^{t^2}}{2w^2} = \frac{(t^2 - 1)\,\mathrm{e}^{t^2}}{2} + c_1$$

Which simplifies to

$$-\frac{\mathrm{e}^{t^2}}{2w^2} = \frac{(t^2 - 1)\,\mathrm{e}^{t^2}}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, w coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dw}{dt} = t^3 w^3 + wt$ 	$R = t$ $S = -\frac{\mathrm{e}^{t^2}}{2w^2}$	$\frac{dS}{dR} = R^3 \mathrm{e}^{R^2}$ 

Summary
The solution(s) found are the following

$$-\frac{\mathrm{e}^{t^2}}{2w^2} = \frac{(t^2 - 1)\,\mathrm{e}^{t^2}}{2} + c_1 \tag{1}$$

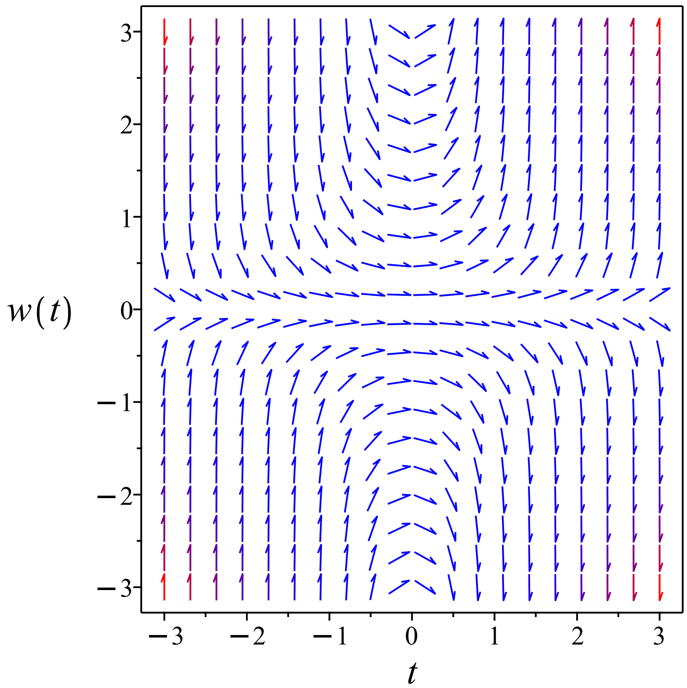


Figure 189: Slope field plot

<u>Verification of solutions</u>

$$-\frac{e^{t^2}}{2w^2} = \frac{(t^2-1)\,e^{t^2}}{2} + c_1$$

Verified OK.

5.31.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} w' &= F(t,w) \\ &= t^3w^3 + wt \end{aligned}$$

This is a Bernoulli ODE.

$$w' = tw + t^3w^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$w' = f_0(t)w + f_1(t)w^n \tag{2}$$

The first step is to divide the above equation by w^n which gives

$$\frac{w'}{w^n} = f_0(t)w^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $v = w^{1-n}$ in equation (3) which generates a new ODE in $v(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $w(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(t) &= t \\
f_1(t) &= t^3 \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $w^n = w^3$ gives

$$
w' \frac{1}{w^3} = \frac{t}{w^2} + t^3 \tag{4}
$$

Let

$$
\begin{aligned}
v &= w^{1-n} \\
&= \frac{1}{w^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t t gives

$$
v' = -\frac{2}{w^3} w' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{v'(t)}{2} &= v(t)\, t + t^3 \\
v' &= -2t^3 - 2tv
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $v(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
v'(t) + p(t)v(t) = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= 2t \\
q(t) &= -2t^3
\end{aligned}
$$

Hence the ode is

$$
v'(t) + 2v(t)\, t = -2t^3
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 2t dt} \\
&= \mathrm{e}^{t^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu v) &= (\mu)\left(-2t^3\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{t^2} v\right) &= \left(\mathrm{e}^{t^2}\right)\left(-2t^3\right) \\
\mathrm{d}\left(\mathrm{e}^{t^2} v\right) &= \left(-2t^3 \mathrm{e}^{t^2}\right)\, \mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{t^2} v &= \int -2t^3 \mathrm{e}^{t^2}\, \mathrm{d}t \\
\mathrm{e}^{t^2} v &= -\left(t^2-1\right) \mathrm{e}^{t^2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{t^2}$ results in

$$
v(t) = -\mathrm{e}^{-t^2}\left(t^2-1\right) \mathrm{e}^{t^2} + c_1 \mathrm{e}^{-t^2}
$$

which simplifies to

$$
v(t) = -t^2 + 1 + c_1 \mathrm{e}^{-t^2}
$$

Replacing v in the above by $\frac{1}{w^2}$ using equation (5) gives the final solution.

$$
\frac{1}{w^2} = -t^2 + 1 + c_1 \mathrm{e}^{-t^2}
$$

Solving for w gives

$$
\begin{aligned}
w(t) &= \frac{1}{\sqrt{-t^2+1+c_1 \mathrm{e}^{-t^2}}} \\
w(t) &= -\frac{1}{\sqrt{-t^2+1+c_1 \mathrm{e}^{-t^2}}}
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
w = \frac{1}{\sqrt{-t^2+1+c_1 \mathrm{e}^{-t^2}}} \tag{1}
$$

$$
w = -\frac{1}{\sqrt{-t^2+1+c_1 \mathrm{e}^{-t^2}}} \tag{2}
$$

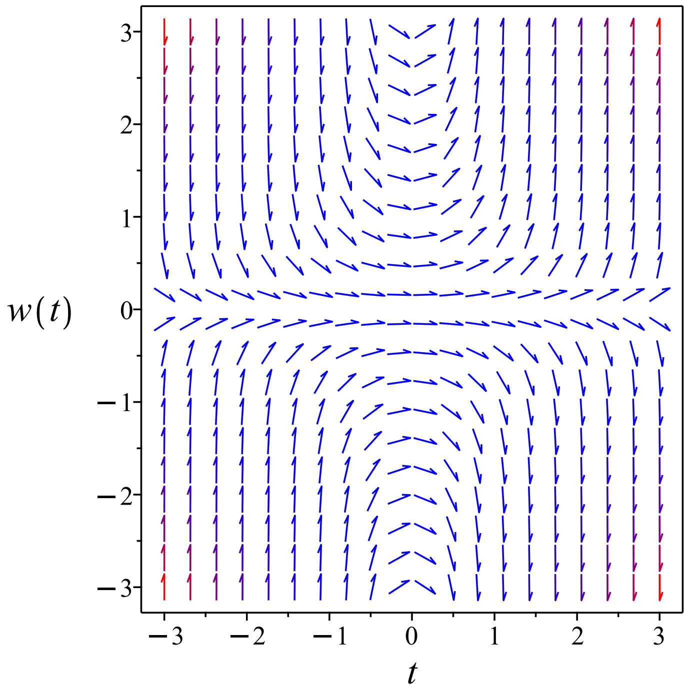


Figure 190: Slope field plot

Verification of solutions

$$w = \frac{1}{\sqrt{-t^2 + 1 + c_1 e^{-t^2}}}$$

Verified OK.

$$w = -\frac{1}{\sqrt{-t^2 + 1 + c_1 e^{-t^2}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(diff(w(t),t)=t*w(t)+t^3*w(t)^3,w(t), singsol=all)
```

$$w(t) = \frac{1}{\sqrt{e^{-t^2} c_1 - t^2 + 1}}$$
$$w(t) = -\frac{1}{\sqrt{e^{-t^2} c_1 - t^2 + 1}}$$

✓ Solution by Mathematica

Time used: 2.07 (sec). Leaf size: 80

```
DSolve[w'[t]==t*w[t]+t^3*w[t]^3,w[t],t,IncludeSingularSolutions -> True]
```

$$w(t) \to -\frac{ie^{\frac{t^2}{2}}}{\sqrt{e^{t^2} (t^2 - 1) - c_1}}$$
$$w(t) \to \frac{ie^{\frac{t^2}{2}}}{\sqrt{e^{t^2} (t^2 - 1) - c_1}}$$
$$w(t) \to 0$$

5.32 problem 16-b(i)

5.32.1 Solving as separable ode 862
5.32.2 Solving as linear ode 864
5.32.3 Solving as homogeneousTypeD2 ode 865
5.32.4 Solving as first order ode lie symmetry lookup ode 867
5.32.5 Solving as exact ode 871
5.32.6 Maple step by step solution 875

Internal problem ID [11430]
Internal file name [`OUTPUT/10412_Thursday_May_18_2023_04_18_47_AM_35323218/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 16-b(i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{x^3 + 3x'tx^2 = 0}$$

5.32.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= f(t)g(x) \\ &= -\frac{x}{3t} \end{aligned}$$

Where $f(t) = -\frac{1}{3t}$ and $g(x) = x$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{x}\,dx &= -\frac{1}{3t}\,dt \\
\int \frac{1}{x}\,dx &= \int -\frac{1}{3t}\,dt \\
\ln(x) &= -\frac{\ln(t)}{3} + c_1 \\
x &= \mathrm{e}^{-\frac{\ln(t)}{3}+c_1} \\
&= \frac{c_1}{t^{\frac{1}{3}}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$x = \frac{c_1}{t^{\frac{1}{3}}} \tag{1}$$

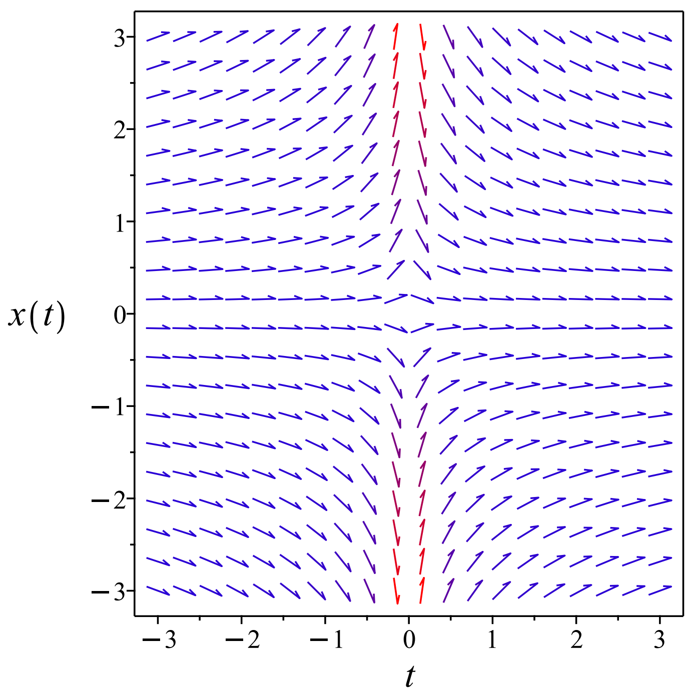


Figure 191: Slope field plot

Verification of solutions

$$x = \frac{c_1}{t^{\frac{1}{3}}}$$

Verified OK.

5.32.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned}p(t) &= \frac{1}{3t} \\ q(t) &= 0\end{aligned}$$

Hence the ode is

$$x' + \frac{x}{3t} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{1}{3t}dt} \\ &= t^{\frac{1}{3}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu x &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(t^{\frac{1}{3}}x\right) &= 0\end{aligned}$$

Integrating gives

$$t^{\frac{1}{3}}x = c_1$$

Dividing both sides by the integrating factor $\mu = t^{\frac{1}{3}}$ results in

$$x = \frac{c_1}{t^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$x = \frac{c_1}{t^{\frac{1}{3}}} \tag{1}$$

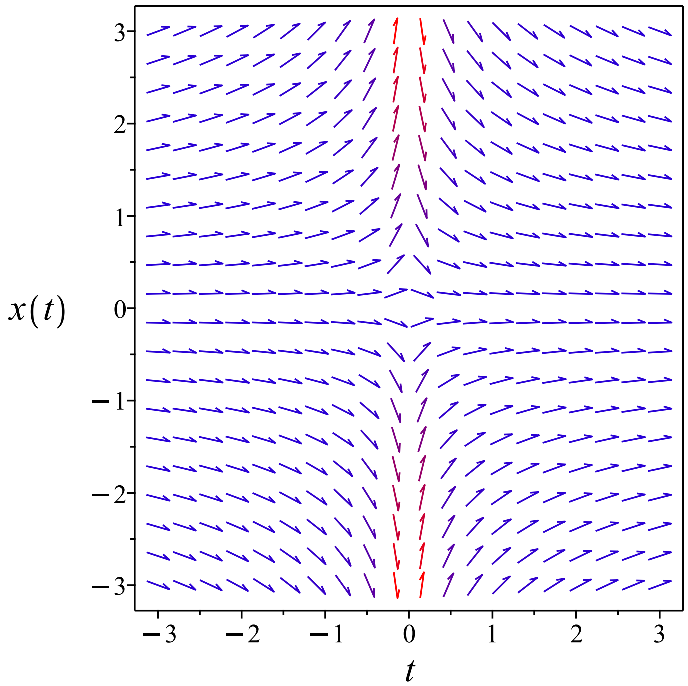


Figure 192: Slope field plot

Verification of solutions

$$x = \frac{c_1}{t^{\frac{1}{3}}}$$

Verified OK.

5.32.3 Solving as homogeneousTypeD2 ode

Using the change of variables $x = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u(t)^3\,t^3 + 3(u'(t)\,t + u(t))\,t^3 u(t)^2 = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(t, u) \\
&= f(t)g(u) \\
&= -\frac{4u}{3t}
\end{aligned}$$

Where $f(t) = -\frac{4}{3t}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{4}{3t}\,dt \\
\int \frac{1}{u}\,du &= \int -\frac{4}{3t}\,dt \\
\ln(u) &= -\frac{4\ln(t)}{3} + c_2 \\
u &= \mathrm{e}^{-\frac{4\ln(t)}{3}+c_2} \\
&= \frac{c_2}{t^{\frac{4}{3}}}
\end{aligned}
$$

Therefore the solution x is

$$
\begin{aligned}
x &= tu \\
&= \frac{c_2}{t^{\frac{1}{3}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = \frac{c_2}{t^{\frac{1}{3}}} \tag{1}
$$

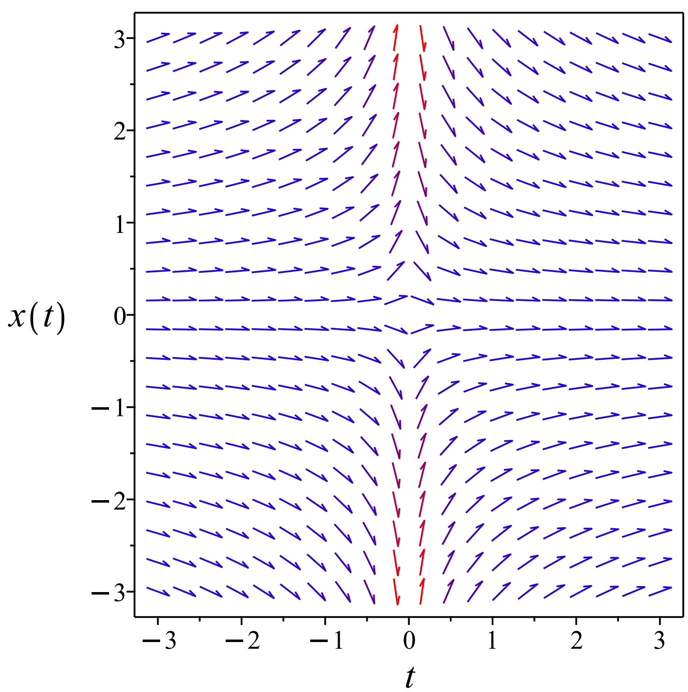


Figure 193: Slope field plot

Verification of solutions

$$x = \frac{c_2}{t^{\frac{1}{3}}}$$

Verified OK.

5.32.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = -\frac{x}{3t}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 171: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \frac{1}{t^{\frac{1}{3}}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{t^{\frac{1}{3}}}} dy
\end{aligned}$$

Which results in

$$S = t^{\frac{1}{3}} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -\frac{x}{3t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_t &= 1 \\
R_x &= 0 \\
S_t &= \frac{x}{3t^{\frac{2}{3}}} \\
S_x &= t^{\frac{1}{3}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$x t^{\frac{1}{3}} = c_1$$

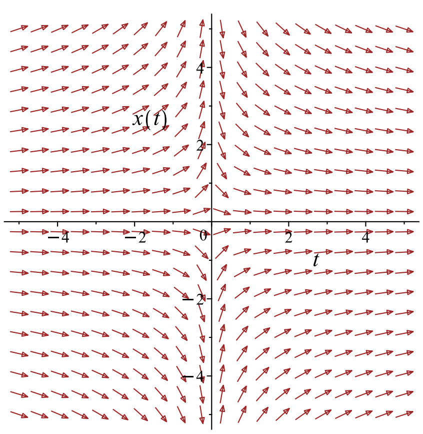
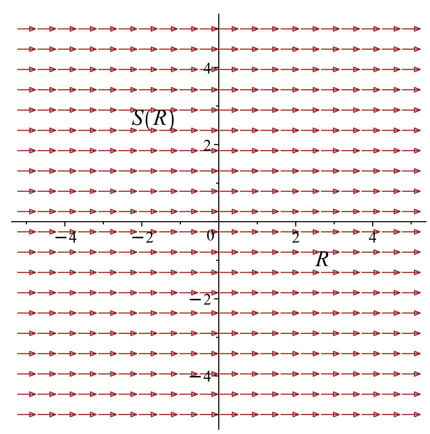
Which simplifies to

$$x t^{\frac{1}{3}} = c_1$$

Which gives

$$x = \frac{c_1}{t^{\frac{1}{3}}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\frac{x}{3t}$ 	$R = t$ $S = t^{\frac{1}{3}} x$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$x = \frac{c_1}{t^{\frac{1}{3}}} \tag{1}$$

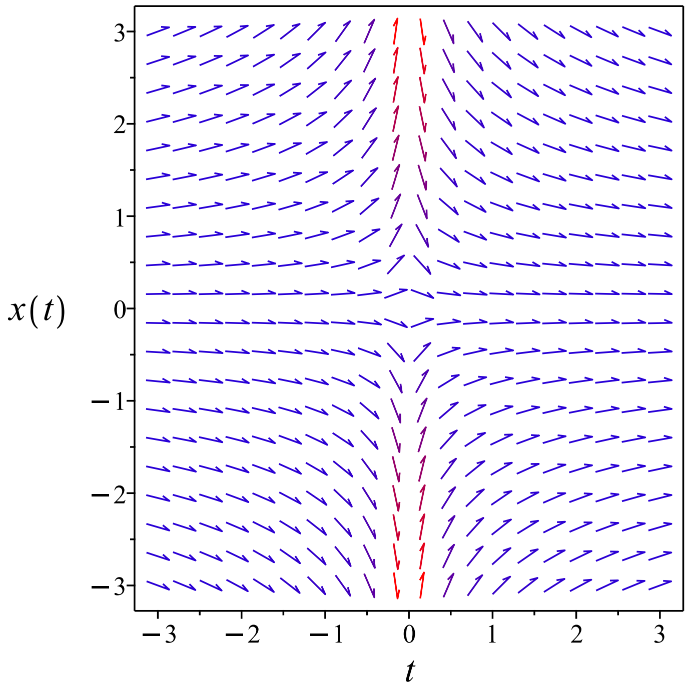


Figure 194: Slope field plot

<u>Verification of solutions</u>

$$x = \frac{c_1}{t^{\frac{1}{3}}}$$

Verified OK.

5.32.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, x) \,\mathrm{d}t + N(t, x) \,\mathrm{d}x = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{3}{x}\right) \mathrm{d}x &= \left(\frac{1}{t}\right) \mathrm{d}t \\
\left(-\frac{1}{t}\right) \mathrm{d}t + \left(-\frac{3}{x}\right) \mathrm{d}x &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, x) &= -\frac{1}{t} \\
N(t, x) &= -\frac{3}{x}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{t}\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{3}{x}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{1}{t}\,\mathrm{d}t\end{aligned}$$

$$\phi = -\ln(t) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = -\frac{3}{x}$. Therefore equation (4) becomes

$$-\frac{3}{x} = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\frac{3}{x}$$

Integrating the above w.r.t x gives

$$\begin{aligned}\int f'(x)\,\mathrm{d}x &= \int \left(-\frac{3}{x}\right)\mathrm{d}x\\ f(x) &= -3\ln(x) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\ln(t) - 3\ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(t) - 3\ln(x)$$

The solution becomes

$$x = \mathrm{e}^{-\frac{\ln(t)}{3}-\frac{c_1}{3}}$$

<u>Summary</u>
The solution(s) found are the following

$$x = \mathrm{e}^{-\frac{\ln(t)}{3}-\frac{c_1}{3}} \tag{1}$$

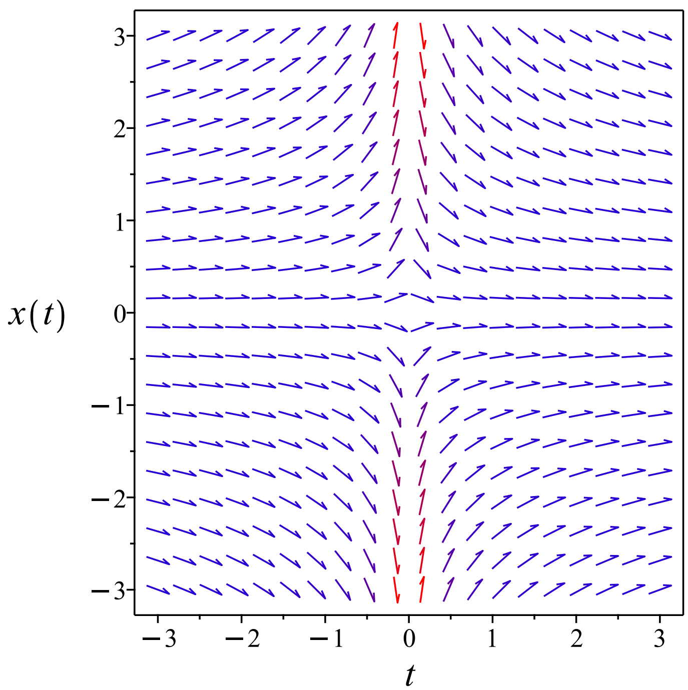


Figure 195: Slope field plot

<u>Verification of solutions</u>

$$x = \mathrm{e}^{-\frac{\ln(t)}{3}-\frac{c_1}{3}}$$

Verified OK.

5.32.6 Maple step by step solution

Let's solve

$x^3 + 3x'tx^2 = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Integrate both sides with respect to t

  $\int (x^3 + 3x'tx^2)\, dt = \int 0dt + c_1$

- Evaluate integral

  $x^3t = c_1$

- Solve for x

  $x = \frac{(t^2c_1)^{\frac{1}{3}}}{t}$

Maple trace

```
`Classification methods on request
Methods to be used are: [exact]
----------------------------
* Tackling ODE using method: exact
--- Trying classification methods ---
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 68

```
dsolve(x(t)^3+3*t*x(t)^2*diff(x(t),t)=0,x(t), singsol=all)
```

$$x(t) = 0$$
$$x(t) = \frac{(-c_1t^2)^{\frac{1}{3}}}{t}$$
$$x(t) = -\frac{(-c_1t^2)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2t}$$
$$x(t) = \frac{(-c_1t^2)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2t}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 23

```
DSolve[x[t]^3+3*t*x[t]^2*x'[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 0$$
$$x(t) \to \frac{c_1}{\sqrt[3]{t}}$$
$$x(t) \to 0$$

5.33 problem 16-b(ii)

5.33.1 Solving as exact ode 877
5.33.2 Maple step by step solution 881

Internal problem ID [11431]
Internal file name [`OUTPUT/10413_Thursday_May_18_2023_04_18_48_AM_20918723/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 16-b(ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{\frac{x}{t} + \left(x^2 + \ln(t)\right) x' = -t^3}$$

5.33.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, x)\, \mathrm{d}t + N(t, x)\, \mathrm{d}x = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(x^2 + \ln(t)\right) \mathrm{d}x &= \left(-t^3 - \frac{x}{t}\right) \mathrm{d}t \\
\left(t^3 + \frac{x}{t}\right) \mathrm{d}t + \left(x^2 + \ln(t)\right) \mathrm{d}x &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, x) &= t^3 + \frac{x}{t} \\
N(t, x) &= x^2 + \ln(t)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(t^3 + \frac{x}{t}\right) \\
&= \frac{1}{t}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(x^2 + \ln(t)\right) \\
&= \frac{1}{t}
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int t^3 + \frac{x}{t} \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = \frac{t^4}{4} + \ln(t)\, x + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = \ln(t) + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = x^2 + \ln(t)$. Therefore equation (4) becomes

$$
x^2 + \ln(t) = \ln(t) + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = x^2
$$

Integrating the above w.r.t x gives

$$
\int f'(x) \, \mathrm{d}x = \int \left(x^2\right) \mathrm{d}x
$$

$$
f(x) = \frac{x^3}{3} + c_1
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{t^4}{4} + \ln(t)\, x + \frac{x^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{t^4}{4} + \ln(t)\, x + \frac{x^3}{3}$$

Summary

The solution(s) found are the following

$$\frac{t^4}{4} + \ln(t)\, x + \frac{x^3}{3} = c_1 \tag{1}$$

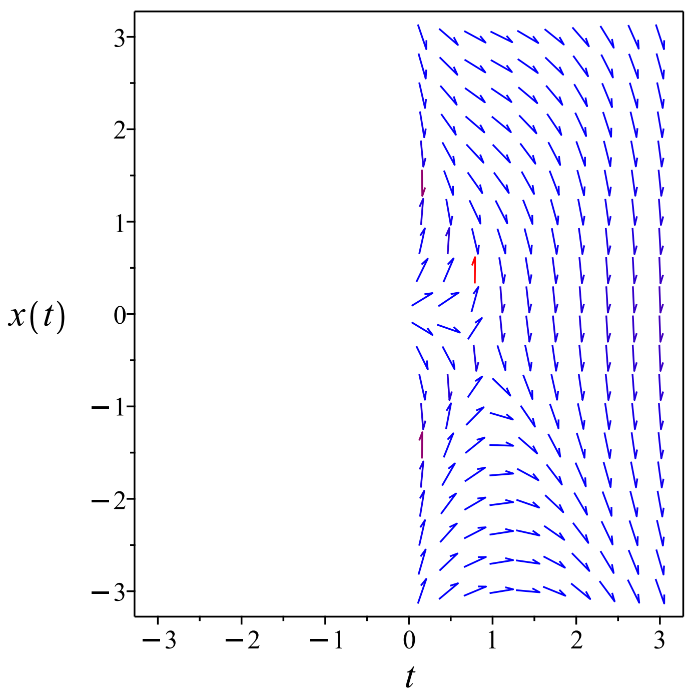


Figure 196: Slope field plot

Verification of solutions

$$\frac{t^4}{4} + \ln(t)\, x + \frac{x^3}{3} = c_1$$

Verified OK.

5.33.2 Maple step by step solution

Let's solve

$\frac{x}{t} + (x^2 + \ln(t))\, x' = -t^3$

- Highest derivative means the order of the ODE is 1

  x'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(t, x) = 0$
  - Compute derivative of lhs

    $F'(t, x) + \left(\frac{\partial}{\partial x} F(t, x)\right) x' = 0$
  - Evaluate derivatives

    $\frac{1}{t} = \frac{1}{t}$
  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(t, x) = c_1, M(t, x) = F'(t, x), N(t, x) = \frac{\partial}{\partial x} F(t, x)\right]$

- Solve for $F(t, x)$ by integrating $M(t, x)$ with respect to t

  $F(t, x) = \int \left(t^3 + \frac{x}{t}\right) dt + f_1(x)$

- Evaluate integral

  $F(t, x) = \frac{t^4}{4} + \ln(t)\, x + f_1(x)$

- Take derivative of $F(t, x)$ with respect to x

  $N(t, x) = \frac{\partial}{\partial x} F(t, x)$

- Compute derivative

  $x^2 + \ln(t) = \ln(t) + \frac{d}{dx} f_1(x)$

- Isolate for $\frac{d}{dx} f_1(x)$

  $\frac{d}{dx} f_1(x) = x^2$

- Solve for $f_1(x)$

  $f_1(x) = \frac{x^3}{3}$

- Substitute $f_1(x)$ into equation for $F(t, x)$

  $F(t, x) = \frac{t^4}{4} + \ln(t)\, x + \frac{x^3}{3}$

- Substitute $F(t,x)$ into the solution of the ODE

  $\frac{t^4}{4}+\ln(t)\,x+\frac{x^3}{3}=c_1$
- Solve for x

$$\left\{x=\frac{\left(-3t^4+12c_1+\sqrt{64\ln(t)^3+9t^8-72t^4c_1+144c_1^2}\right)^{\frac{1}{3}}}{2}-\frac{2\ln(t)}{\left(-3t^4+12c_1+\sqrt{64\ln(t)^3+9t^8-72t^4c_1+144c_1^2}\right)^{\frac{1}{3}}}, x=-\frac{\left(-3t^4+\right.}{}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 307

```
dsolve(t^3+x(t)/t+(x(t)^2+ln(t))*diff(x(t),t)=0,x(t), singsol=all)
```

$$x(t) = \frac{\left(-3t^4 - 12c_1 + \sqrt{64\ln(t)^3 + 9(t^4 + 4c_1)^2}\right)^{\frac{2}{3}} - 4\ln(t)}{2\left(-3t^4 - 12c_1 + \sqrt{64\ln(t)^3 + 9(t^4 + 4c_1)^2}\right)^{\frac{1}{3}}}$$

$$x(t) = \frac{i\left(-\left(-3t^4 - 12c_1 + \sqrt{64\ln(t)^3 + 9(t^4 + 4c_1)^2}\right)^{\frac{2}{3}} - 4\ln(t)\right)\sqrt{3} - \left(-3t^4 - 12c_1 + \sqrt{64\ln(t)^3 + 9(t^4}\right.}{4\left(-3t^4 - 12c_1 + \sqrt{64\ln(t)^3 + 9(t^4 + 4c_1)^2}\right)^{\frac{1}{3}}}$$

$$x(t) = \frac{i\left(\left(-3t^4 - 12c_1 + \sqrt{64\ln(t)^3 + 9(t^4 + 4c_1)^2}\right)^{\frac{2}{3}} + 4\ln(t)\right)\sqrt{3} - \left(-3t^4 - 12c_1 + \sqrt{64\ln(t)^3 + 9(t^4 -}\right.}{4\left(-3t^4 - 12c_1 + \sqrt{64\ln(t)^3 + 9(t^4 + 4c_1)^2}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 2.922 (sec). Leaf size: 307

```
DSolve[t^3+x[t]/t+(x[t]^2+Log[t])*x'[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{-4\log(t) + \left(-3t^4 + \sqrt{64\log^3(t) + 9\left(t^4 - 4c_1\right)^2} + 12c_1\right)^{2/3}}{2\sqrt[3]{-3t^4 + \sqrt{64\log^3(t) + 9\left(t^4 - 4c_1\right)^2} + 12c_1}}$$

$$x(t) \to \frac{i(\sqrt{3} + i)\left(-3t^4 + \sqrt{64\log^3(t) + 9\left(t^4 - 4c_1\right)^2} + 12c_1\right)^{2/3} + \left(4 + 4i\sqrt{3}\right)\log(t)}{4\sqrt[3]{-3t^4 + \sqrt{64\log^3(t) + 9\left(t^4 - 4c_1\right)^2} + 12c_1}}$$

$$x(t) \to \frac{\left(-1 - i\sqrt{3}\right)\left(-3t^4 + \sqrt{64\log^3(t) + 9\left(t^4 - 4c_1\right)^2} + 12c_1\right)^{2/3} + \left(4 - 4i\sqrt{3}\right)\log(t)}{4\sqrt[3]{-3t^4 + \sqrt{64\log^3(t) + 9\left(t^4 - 4c_1\right)^2} + 12c_1}}$$

5.34 problem 16-b(iii)

5.34.1 Solving as exact ode 885

Internal problem ID [11432]
Internal file name [`OUTPUT/10414_Thursday_May_18_2023_04_18_50_AM_14832249/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 16-b(iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[NONE]`

$$\boxed{x' + \frac{\sin(x) - x\sin(t)}{t\cos(x) + \cos(t)} = 0}$$

5.34.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, x) \, \mathrm{d}t + N(t, x) \, \mathrm{d}x = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(t \cos(x) + \cos(t)) \, \mathrm{d}x &= (x \sin(t) - \sin(x)) \, \mathrm{d}t \\
(\sin(x) - x \sin(t)) \, \mathrm{d}t + (t \cos(x) + \cos(t)) \, \mathrm{d}x &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, x) &= \sin(x) - x \sin(t) \\
N(t, x) &= t \cos(x) + \cos(t)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(\sin(x) - x \sin(t)) \\
&= \cos(x) - \sin(t)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t \cos(x) + \cos(t)) \\
&= \cos(x) - \sin(t)
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int M\,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \sin(x) - x\sin(t)\,\mathrm{d}t$$

$$\phi = x\cos(t) + t\sin(x) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = t\cos(x) + \cos(t) + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = t\cos(x) + \cos(t)$. Therefore equation (4) becomes

$$t\cos(x) + \cos(t) = t\cos(x) + \cos(t) + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = x\cos(t) + t\sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\cos(t) + t\sin(x)$$

Summary

The solution(s) found are the following

$$x \cos(t) + \sin(x) t = c_1 \tag{1}$$

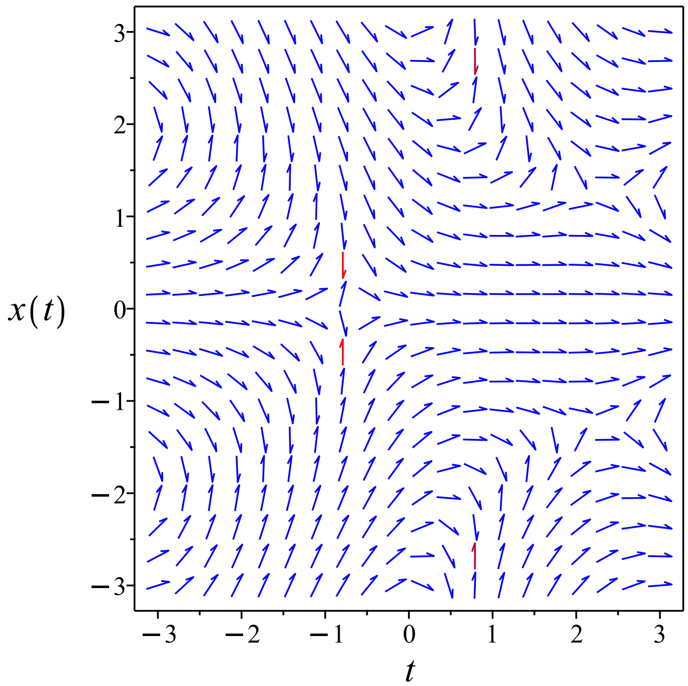


Figure 197: Slope field plot

Verification of solutions

$$x \cos(t) + \sin(x) t = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 15

```
dsolve(diff(x(t),t)=- (sin(x(t))-x(t)*sin(t))/(t*cos(x(t))+cos(t)),x(t), singsol=all)
```

$$\cos(t)\,x(t)+t\sin(x(t))+c_1=0$$

✓ Solution by Mathematica

Time used: 0.27 (sec). Leaf size: 17

```
DSolve[x'[t]==- (Sin[x[t]]-x[t]*Sin[t])/(t*Cos[x[t]]+Cos[t]),x[t],t,IncludeSingularSolutions
```

$$\text{Solve}[t\sin(x(t))+x(t)\cos(t)=c_1,x(t)]$$

5.35 problem 16-b(iv)

5.35.1 Solving as separable ode 890
5.35.2 Solving as first order ode lie symmetry lookup ode 892
5.35.3 Solving as exact ode 896
5.35.4 Maple step by step solution 900

Internal problem ID [11433]
Internal file name [`OUTPUT/10415_Thursday_May_18_2023_04_18_54_AM_25927860/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 16-b(iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x + 3x' t x^2 = 0}$$

5.35.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= f(t)g(x) \\ &= -\frac{1}{3xt} \end{aligned}$$

Where $f(t) = -\frac{1}{3t}$ and $g(x) = \frac{1}{x}$. Integrating both sides gives

$$\frac{1}{\frac{1}{x}}\, dx = -\frac{1}{3t}\, dt$$

$$\int \frac{1}{\frac{1}{x}}\, dx = \int -\frac{1}{3t}\, dt$$

$$\frac{x^2}{2} = -\frac{\ln(t)}{3} + c_1$$

Which results in

$$x = \frac{\sqrt{-6\ln(t) + 18c_1}}{3}$$

$$x = -\frac{\sqrt{-6\ln(t) + 18c_1}}{3}$$

Summary

The solution(s) found are the following

$$x = \frac{\sqrt{-6\ln(t) + 18c_1}}{3} \quad (1)$$

$$x = -\frac{\sqrt{-6\ln(t) + 18c_1}}{3} \quad (2)$$

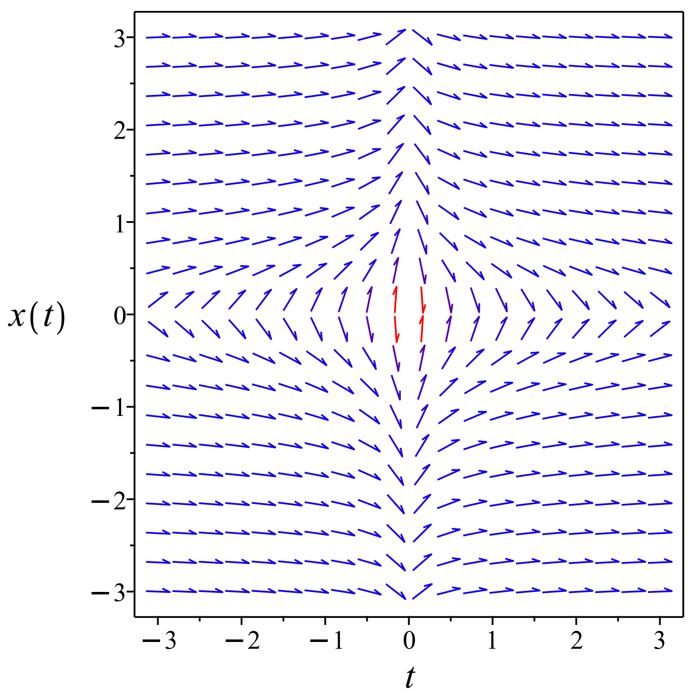


Figure 198: Slope field plot

Verification of solutions

$$x = \frac{\sqrt{-6\ln\left(t\right)+18c_1}}{3}$$

Verified OK.

$$x = -\frac{\sqrt{-6\ln\left(t\right)+18c_1}}{3}$$

Verified OK.

5.35.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} x' &= -\frac{1}{3xt} \\ x' &= \omega(t,x) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 175: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= -3t \\
\eta(t, x) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{-3t} dt \end{aligned}$$

Which results in

$$S = -\frac{\ln(t)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -\frac{1}{3xt}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_x &= 1 \\ S_t &= -\frac{1}{3t} \\ S_x &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

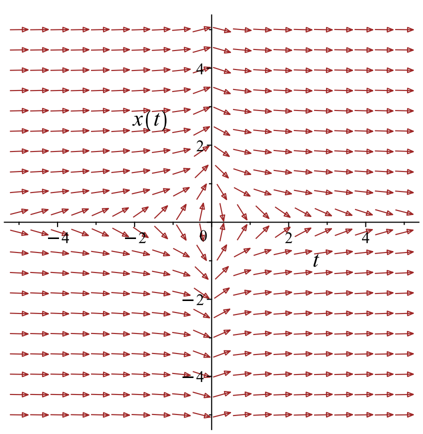
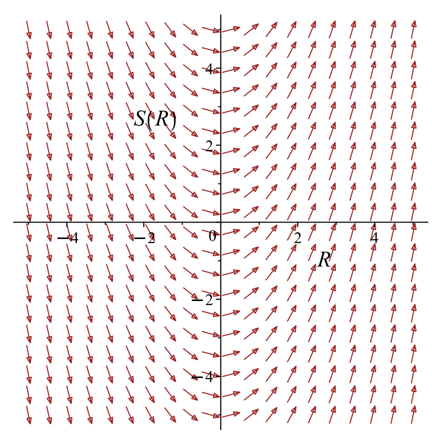
To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$-\frac{\ln(t)}{3} = \frac{x^2}{2} + c_1$$

Which simplifies to

$$-\frac{\ln(t)}{3} = \frac{x^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\frac{1}{3xt}$ 	$R = x$ $S = -\frac{\ln(t)}{3}$	$\frac{dS}{dR} = R$ 

Summary
The solution(s) found are the following

$$-\frac{\ln(t)}{3} = \frac{x^2}{2} + c_1 \tag{1}$$

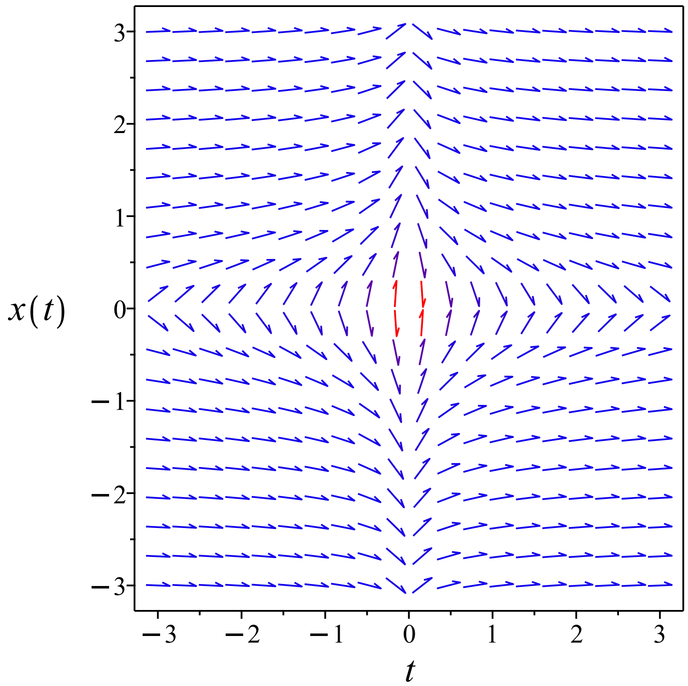


Figure 199: Slope field plot

Verification of solutions

$$
-\frac{\ln(t)}{3} = \frac{x^2}{2} + c_1
$$

Verified OK.

5.35.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x, y) = 0
$$

Hence

$$
\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-3x)\,\mathrm{d}x&=\left(\frac{1}{t}\right)\mathrm{d}t\\ \left(-\frac{1}{t}\right)\mathrm{d}t+(-3x)\,\mathrm{d}x&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x)&=-\frac{1}{t}\\ N(t,x)&=-3x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x}&=\frac{\partial}{\partial x}\left(-\frac{1}{t}\right)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(-3x)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{1}{t}\,\mathrm{d}t\\ \phi &= -\ln(t) + f(x)\end{aligned} \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = -3x$. Therefore equation (4) becomes

$$-3x = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -3x$$

Integrating the above w.r.t x gives

$$\begin{aligned}\int f'(x)\,\mathrm{d}x &= \int (-3x)\,\mathrm{d}x\\ f(x) &= -\frac{3x^2}{2} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\ln(t) - \frac{3x^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(t) - \frac{3x^2}{2}$$

Summary

The solution(s) found are the following

$$-\ln(t) - \frac{3x^2}{2} = c_1 \tag{1}$$

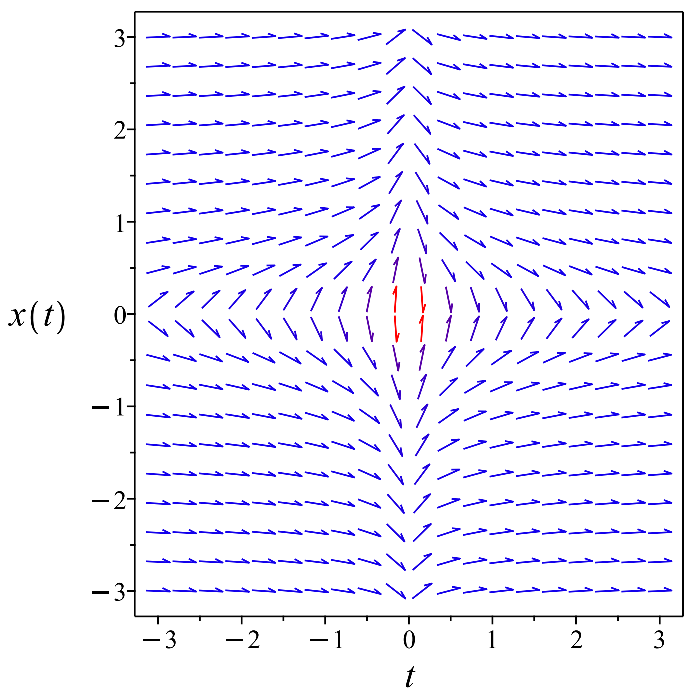


Figure 200: Slope field plot

Verification of solutions

$$-\ln(t) - \frac{3x^2}{2} = c_1$$

Verified OK.

5.35.4 Maple step by step solution

Let's solve

$x + 3x'tx^2 = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $x'x = -\frac{1}{3t}$

- Integrate both sides with respect to t

  $\int x'xdt = \int -\frac{1}{3t}dt + c_1$

- Evaluate integral

  $\frac{x^2}{2} = -\frac{\ln(t)}{3} + c_1$

- Solve for x

  $\left\{x = -\frac{\sqrt{-6\ln(t)+18c_1}}{3}, x = \frac{\sqrt{-6\ln(t)+18c_1}}{3}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
dsolve(x(t)+3*t*x(t)^2*diff(x(t),t)=0,x(t), singsol=all)
```

$$x(t) = 0$$
$$x(t) = -\frac{\sqrt{-6\ln(t) + 9c_1}}{3}$$
$$x(t) = \frac{\sqrt{-6\ln(t) + 9c_1}}{3}$$

✓ Solution by Mathematica

Time used: 0.113 (sec). Leaf size: 51

```
DSolve[x[t]+3*t*x[t]^2*x'[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 0$$
$$x(t) \to -\sqrt{-\frac{2\log(t)}{3} + 2c_1}$$
$$x(t) \to \sqrt{-\frac{2\log(t)}{3} + 2c_1}$$
$$x(t) \to 0$$

5.36 problem 16-b(v)

5.36.1 Solving as separable ode 902
5.36.2 Solving as homogeneousTypeD2 ode 904
5.36.3 Solving as first order ode lie symmetry lookup ode 905
5.36.4 Solving as exact ode 909
5.36.5 Solving as riccati ode 913
5.36.6 Maple step by step solution 915

Internal problem ID [11434]
Internal file name [`OUTPUT/10416_Thursday_May_18_2023_04_18_56_AM_25923686/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 16-b(v).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{x^2 - t^2 x' = 0}$$

5.36.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= f(t)g(x) \\ &= \frac{x^2}{t^2} \end{aligned}$$

Where $f(t) = \frac{1}{t^2}$ and $g(x) = x^2$. Integrating both sides gives

$$\frac{1}{x^2}\, dx = \frac{1}{t^2}\, dt$$
$$\int \frac{1}{x^2}\, dx = \int \frac{1}{t^2}\, dt$$
$$-\frac{1}{x} = -\frac{1}{t} + c_1$$

Which results in

$$x = -\frac{t}{c_1 t - 1}$$

<u>Summary</u>
The solution(s) found are the following

$$x = -\frac{t}{c_1 t - 1} \tag{1}$$

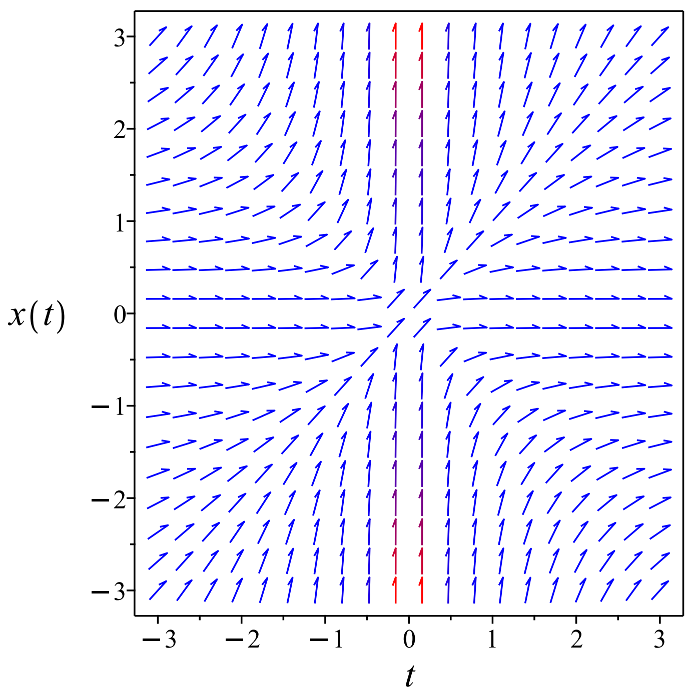


Figure 201: Slope field plot

<u>Verification of solutions</u>

$$x = -\frac{t}{c_1 t - 1}$$

Verified OK.

5.36.2 Solving as homogeneousTypeD2 ode

Using the change of variables $x = u(t)\, t$ on the above ode results in new ode in $u(t)$

$$u(t)^2 t^2 - t^2 (u'(t)\, t + u(t)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t) g(u) \\ &= \frac{u(u-1)}{t} \end{aligned}$$

Where $f(t) = \frac{1}{t}$ and $g(u) = u(u-1)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u\,(u-1)}\, du &= \frac{1}{t}\, dt \\ \int \frac{1}{u\,(u-1)}\, du &= \int \frac{1}{t}\, dt \\ \ln(u-1) - \ln(u) &= \ln(t) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(u-1)-\ln(u)} = e^{\ln(t)+c_2}$$

Which simplifies to

$$\frac{u-1}{u} = c_3 t$$

Therefore the solution x is

$$\begin{aligned} x &= tu \\ &= -\frac{t}{c_3 t - 1} \end{aligned}$$

Summary
The solution(s) found are the following

$$x = -\frac{t}{c_3 t - 1} \tag{1}$$

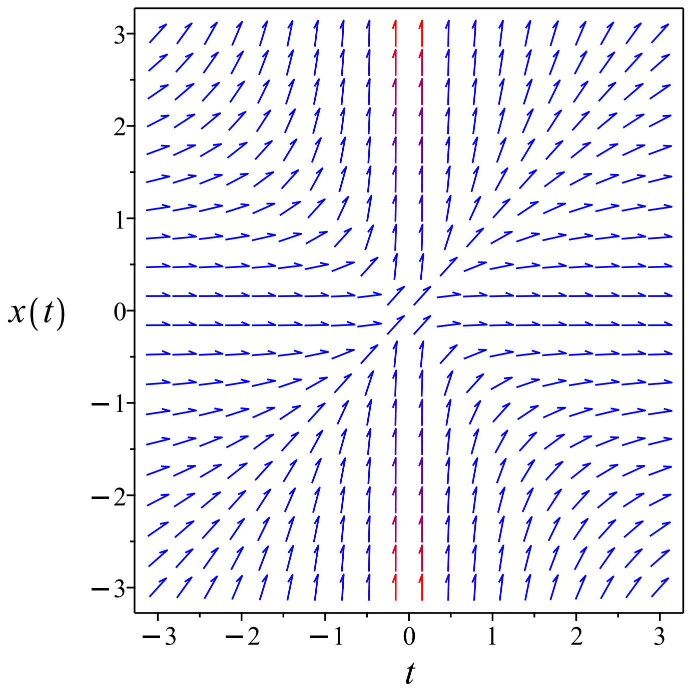


Figure 202: Slope field plot

Verification of solutions

$$x = -\frac{t}{c_3 t - 1}$$

Verified OK.

5.36.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = \frac{x^2}{t^2}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 178: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= t^2 \\
\eta(t, x) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{t^2} dt \end{aligned}$$

Which results in

$$S = -\frac{1}{t}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = \frac{x^2}{t^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_x &= 1 \\ S_t &= \frac{1}{t^2} \\ S_x &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$-\frac{1}{t} = -\frac{1}{x} + c_1$$

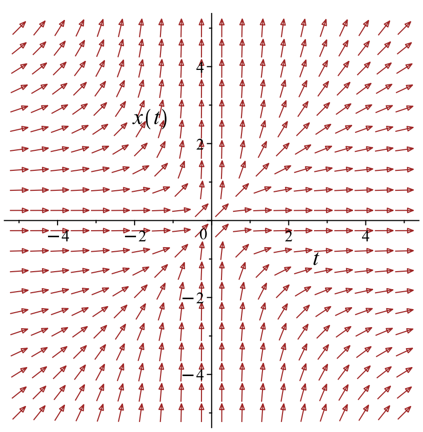
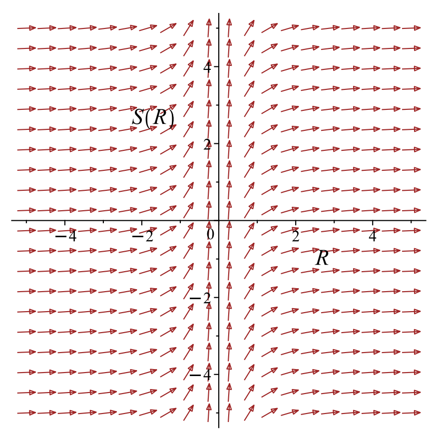
Which simplifies to

$$-\frac{1}{t} = -\frac{1}{x} + c_1$$

Which gives

$$x = \frac{t}{c_1 t + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \frac{x^2}{t^2}$ 	$R = x$ $S = -\frac{1}{t}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$x = \frac{t}{c_1 t + 1} \tag{1}$$

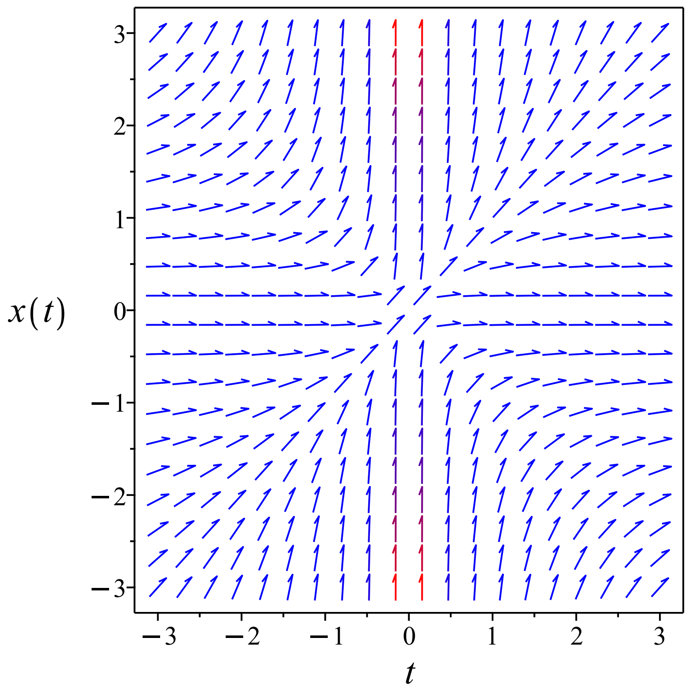


Figure 203: Slope field plot

Verification of solutions

$$x = \frac{t}{c_1 t + 1}$$

Verified OK.

5.36.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{x^2}\right)\mathrm{d}x&=\left(\frac{1}{t^2}\right)\mathrm{d}t\\ \left(-\frac{1}{t^2}\right)\mathrm{d}t+\left(\frac{1}{x^2}\right)\mathrm{d}x&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x)&=-\frac{1}{t^2}\\ N(t,x)&=\frac{1}{x^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x}&=\frac{\partial}{\partial x}\left(-\frac{1}{t^2}\right)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{x^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -\frac{1}{t^2} \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = \frac{1}{t} + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{x^2}$. Therefore equation (4) becomes

$$
\frac{1}{x^2} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = \frac{1}{x^2}
$$

Integrating the above w.r.t x gives

$$
\begin{aligned}
\int f'(x) \, \mathrm{d}x &= \int \left(\frac{1}{x^2}\right) \mathrm{d}x \\
f(x) &= -\frac{1}{x} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{1}{t} - \frac{1}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{t} - \frac{1}{x}$$

The solution becomes

$$x = -\frac{t}{c_1 t - 1}$$

Summary

The solution(s) found are the following

$$x = -\frac{t}{c_1 t - 1} \tag{1}$$

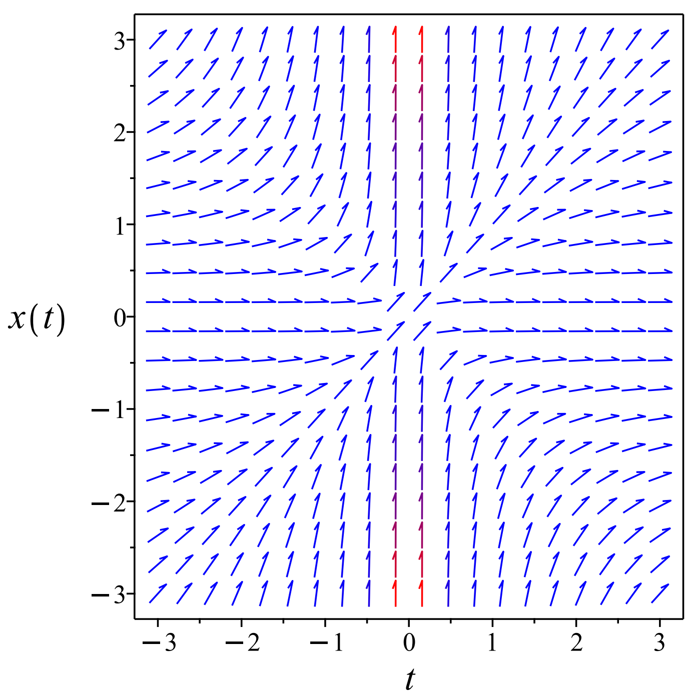


Figure 204: Slope field plot

Verification of solutions

$$x = -\frac{t}{c_1 t - 1}$$

Verified OK.

5.36.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= \frac{x^2}{t^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$x' = \frac{x^2}{t^2}$$

With Riccati ODE standard form

$$x' = f_0(t) + f_1(t)x + f_2(t)x^2$$

Shows that $f_0(t) = 0$, $f_1(t) = 0$ and $f_2(t) = \frac{1}{t^2}$. Let

$$\begin{aligned} x &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{t^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2) u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{t^3} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(t)}{t^2} + \frac{2u'(t)}{t^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 + \frac{c_2}{t}$$

The above shows that

$$u'(t) = -\frac{c_2}{t^2}$$

Using the above in (1) gives the solution

$$x = \frac{c_2}{c_1 + \frac{c_2}{t}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$x = \frac{1}{c_3 + \frac{1}{t}}$$

Summary

The solution(s) found are the following

$$x = \frac{1}{c_3 + \frac{1}{t}} \tag{1}$$

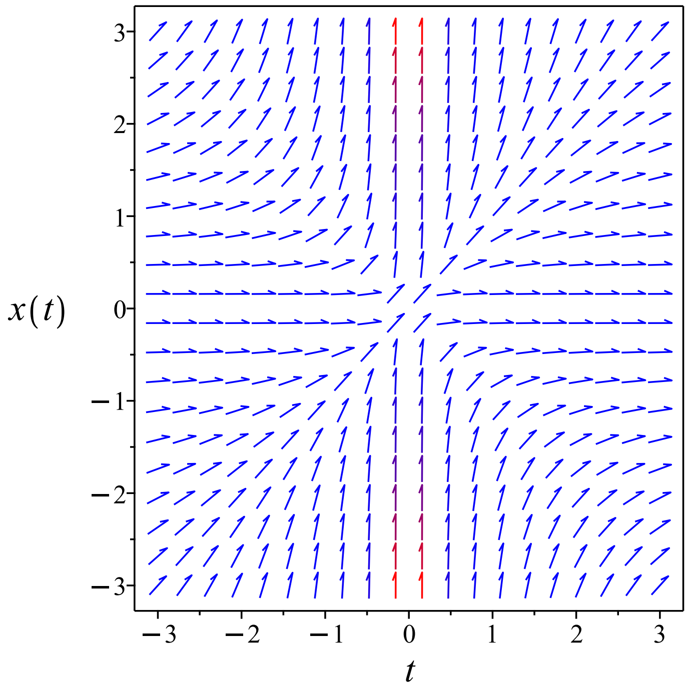


Figure 205: Slope field plot

Verification of solutions

$$x = \frac{1}{c_3 + \frac{1}{t}}$$

Verified OK.

5.36.6 Maple step by step solution

Let's solve

$x^2 - t^2 x' = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x^2} = \frac{1}{t^2}$

- Integrate both sides with respect to t

  $\int \frac{x'}{x^2} dt = \int \frac{1}{t^2} dt + c_1$

- Evaluate integral

$$-\frac{1}{x} = -\frac{1}{t} + c_1$$

- Solve for x

$$x = -\frac{t}{c_1 t - 1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(x(t)^2-t^2*diff(x(t),t)=0,x(t), singsol=all)
```

$$x(t) = \frac{t}{c_1 t + 1}$$

✓ Solution by Mathematica

Time used: 0.192 (sec). Leaf size: 21

```
DSolve[x[t]^2-t^2*x'[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{t}{1 - c_1 t}$$
$$x(t) \to 0$$

5.37 problem 16-b(vi)

5.37.1 Solving as separable ode 917
5.37.2 Solving as first order ode lie symmetry lookup ode 919
5.37.3 Solving as exact ode 923
5.37.4 Maple step by step solution 927

Internal problem ID [11435]
Internal file name [`OUTPUT/10417_Thursday_May_18_2023_04_18_57_AM_54387768/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 1, First order differential equations. Section 1.4.1. Integrating factors. Exercises page 41
Problem number: 16-b(vi).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{t \cot(x) x' = -2}$$

5.37.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= f(t)g(x) \\ &= -\frac{2\tan(x)}{t} \end{aligned}$$

Where $f(t) = -\frac{2}{t}$ and $g(x) = \tan(x)$. Integrating both sides gives

$$\frac{1}{\tan(x)} \, dx = -\frac{2}{t} \, dt$$

$$\int \frac{1}{\tan(x)} \, dx = \int -\frac{2}{t} \, dt$$

$$\ln(\sin(x)) = -2\ln(t) + c_1$$

Raising both side to exponential gives

$$\sin(x) = e^{-2\ln(t)+c_1}$$

Which simplifies to

$$\sin(x) = \frac{c_2}{t^2}$$

Summary

The solution(s) found are the following

$$x = \arcsin\left(\frac{c_2 e^{c_1}}{t^2}\right) \tag{1}$$

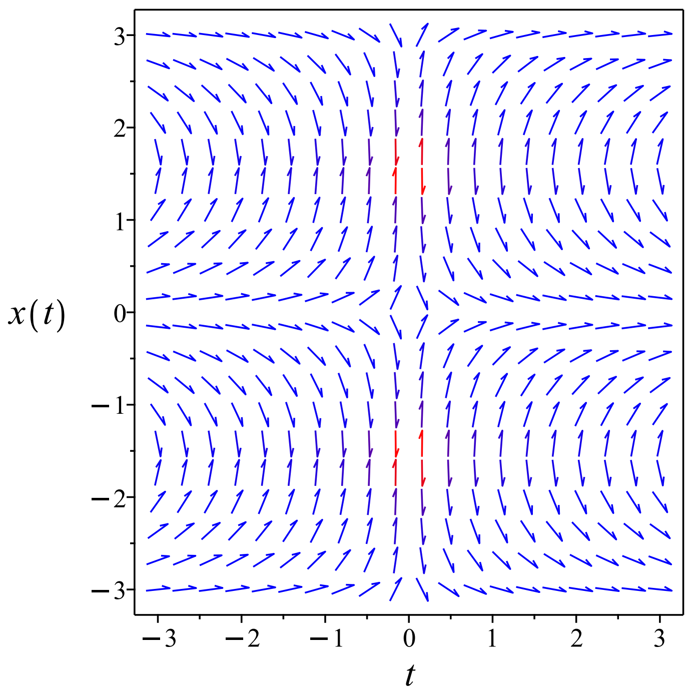


Figure 206: Slope field plot

Verification of solutions

$$x = \arcsin\left(\frac{c_2 e^{c_1}}{t^2}\right)$$

Verified OK.

5.37.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= -\frac{2}{t\cot(x)} \\
x' &= \omega(t,x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 181: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= -\frac{t}{2} \\
\eta(t,x) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dt \\
&= \int \frac{1}{-\frac{t}{2}} dt
\end{aligned}
$$

Which results in

$$
S = -2\ln(t)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}
$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$
\omega(t,x) = -\frac{2}{t\cot(x)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_x &= 1 \\
S_t &= -\frac{2}{t} \\
S_x &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \cot(x) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \cot(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(\sin(R)) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$
-2\ln(t) = \ln(\sin(x)) + c_1
$$

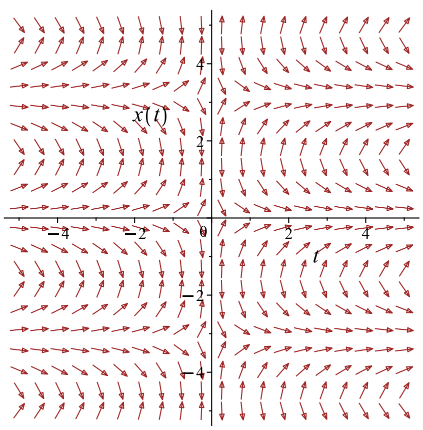
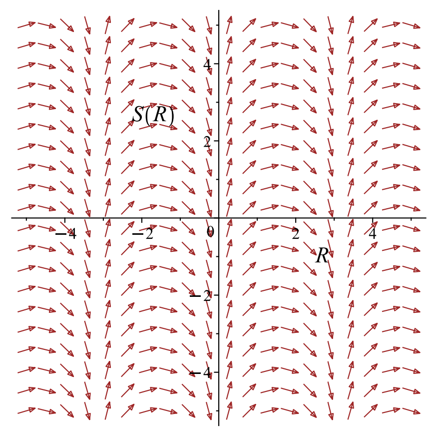
Which simplifies to

$$
-2\ln(t) = \ln(\sin(x)) + c_1
$$

Which gives

$$
x = \arcsin\left(\frac{e^{-c_1}}{t^2}\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\frac{2}{t\cot(x)}$ 	$R = x$ $S = -2\ln(t)$	$\frac{dS}{dR} = \cot(R)$ 

Summary

The solution(s) found are the following

$$x = \arcsin\left(\frac{e^{-c_1}}{t^2}\right) \tag{1}$$

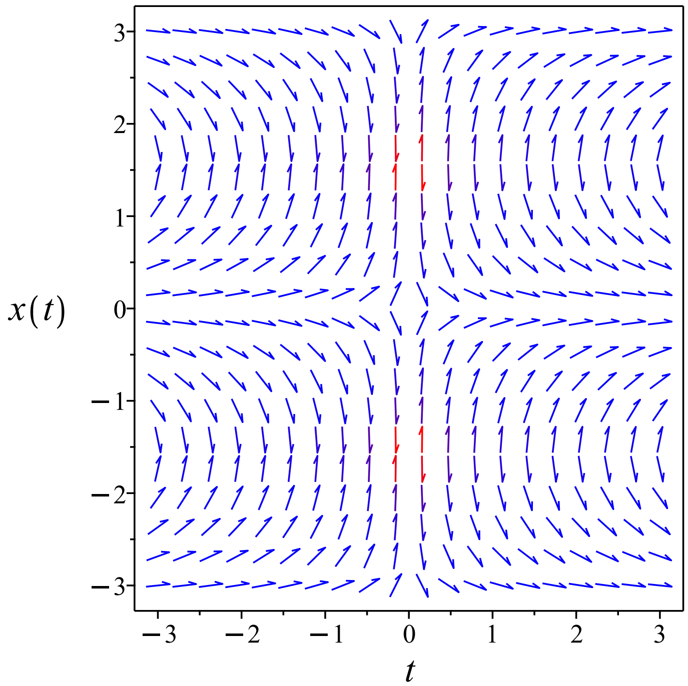


Figure 207: Slope field plot

Verification of solutions

$$x = \arcsin\left(\frac{\mathrm{e}^{-c_1}}{t^2}\right)$$

Verified OK.

5.37.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\left(-\frac{\cot(x)}{2}\right)\mathrm{d}x = \left(\frac{1}{t}\right)\mathrm{d}t$$
$$\left(-\frac{1}{t}\right)\mathrm{d}t + \left(-\frac{\cot(x)}{2}\right)\mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t,x) = -\frac{1}{t}$$
$$N(t,x) = -\frac{\cot(x)}{2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{t}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{\cot(x)}{2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t &= \int M\, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t &= \int -\frac{1}{t}\, \mathrm{d}t
\end{aligned}
$$

$$
\phi = -\ln(t) + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = -\frac{\cot(x)}{2}$. Therefore equation (4) becomes

$$
-\frac{\cot(x)}{2} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = -\frac{\cot(x)}{2}
$$

Integrating the above w.r.t x gives

$$
\begin{aligned}
\int f'(x)\, \mathrm{d}x &= \int \left(-\frac{\cot(x)}{2}\right) \mathrm{d}x \\
f(x) &= -\frac{\ln(\sin(x))}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\ln(t) - \frac{\ln(\sin(x))}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(t) - \frac{\ln(\sin(x))}{2}$$

Summary

The solution(s) found are the following

$$-\ln(t) - \frac{\ln(\sin(x))}{2} = c_1 \tag{1}$$

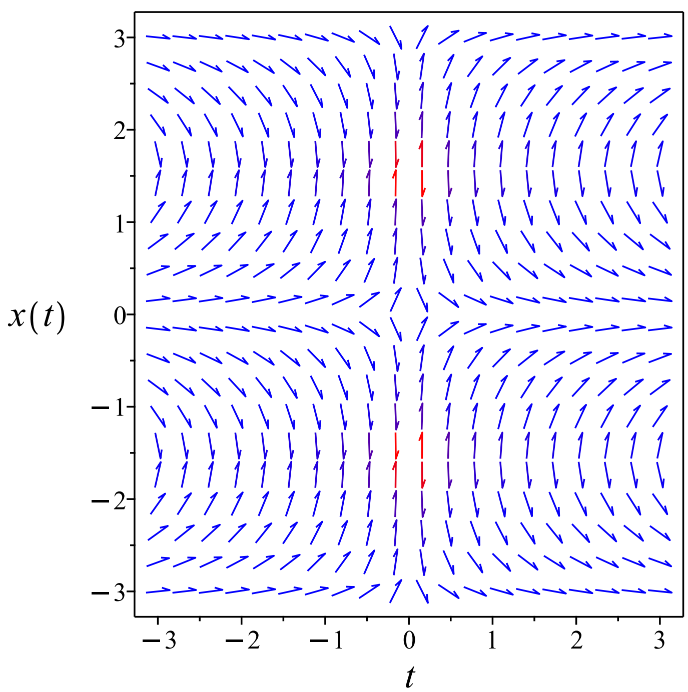


Figure 208: Slope field plot

Verification of solutions

$$-\ln(t) - \frac{\ln(\sin(x))}{2} = c_1$$

Verified OK.

5.37.4 Maple step by step solution

Let's solve

$t \cot(x) x' = -2$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $x' \cot(x) = -\frac{2}{t}$

- Integrate both sides with respect to t

  $\int x' \cot(x) dt = \int -\frac{2}{t} dt + c_1$

- Evaluate integral

  $\ln(\sin(x)) = -2\ln(t) + c_1$

- Solve for x

  $x = \arcsin\left(\frac{e^{c_1}}{t^2}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 10

```
dsolve(t*cot(x(t))*diff(x(t),t)=-2,x(t), singsol=all)
```

$$x(t) = \arcsin\left(\frac{c_1}{t^2}\right)$$

✓ Solution by Mathematica

Time used: 0.122 (sec). Leaf size: 14

```
DSolve[t*Cot[x[t]]*x'[t]==-2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \arcsin\left(\frac{e^{c_1}}{t^2}\right)$$

6 Chapter 2, Second order linear equations. Section 2.2.2 Real eigenvalues. Exercises page 90

6.1 problem 1(a) 930
6.2 problem 1(b) 942
6.3 problem 1(c) 960
6.4 problem 1(d) 972
6.5 problem 3(a) 982
6.6 problem 3(b) 994
6.7 problem 3(c) 1010
6.8 problem 3(d) 1022

6.1 problem 1(a)

6.1.1 Existence and uniqueness analysis 931
6.1.2 Solving as second order linear constant coeff ode 931
6.1.3 Solving as linear second order ode solved by an integrating factor ode 933
6.1.4 Solving using Kovacic algorithm 935
6.1.5 Maple step by step solution 939

Internal problem ID [11436]
Internal file name [`OUTPUT/10418_Thursday_May_18_2023_04_18_59_AM_89672287/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.2 Real eigenvalues. Exercises page 90
Problem number: 1(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' - 4x' + 4x = 0}$$

With initial conditions

$$[x(0) = 1, x'(0) = 0]$$

6.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' - 4x' + 4x = 0$$

The domain of $p(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.1.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - 4\lambda\, \mathrm{e}^{\lambda t} + 4\, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-4)^2 - (4)(1)(4)}\\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$x = c_1 e^{2t} + c_2 e^{2t} t \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{2t} + c_2 t\, e^{2t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t\, e^{2t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = 2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}c_1 &= 1\\ c_2 &= -2\end{aligned}$$

Substituting these values back in above solution results in

$$x = -2\, e^{2t} t + e^{2t}$$

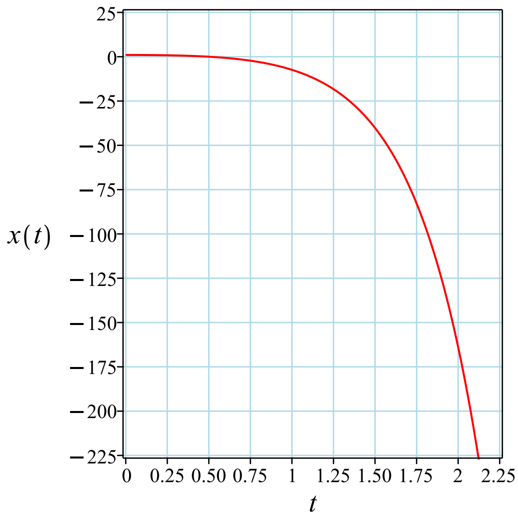
Which simplifies to

$$x = e^{2t}(1 - 2t)$$

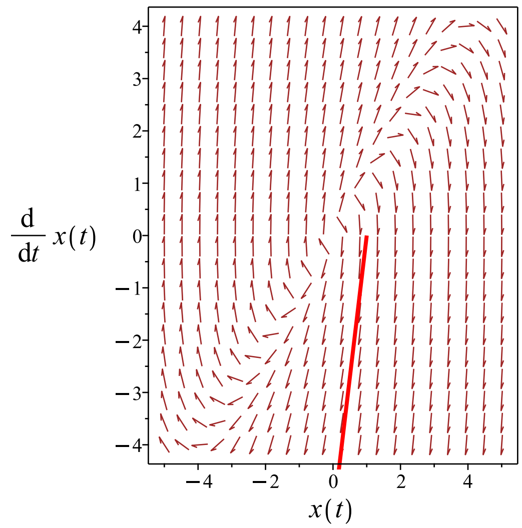
Summary

The solution(s) found are the following

$$x = e^{2t}(1 - 2t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$x = e^{2t}(1 - 2t)$$

Verified OK.

6.1.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$x'' + p(t)\,x' + \frac{\left(p(t)^2 + p'(t)\right)x}{2} = f(t)$$

Where $p(t) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -4\,dx} \\ &= e^{-2t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)x)'' &= 0 \\ \left(e^{-2t}x\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-2t}x\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-2t}x\right) = c_1 t + c_2$$

Hence the solution is

$$x = \frac{c_1 t + c_2}{\mathrm{e}^{-2t}}$$

Or

$$x = c_1 t\,\mathrm{e}^{2t} + c_2 \mathrm{e}^{2t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 t\,\mathrm{e}^{2t} + c_2 \mathrm{e}^{2t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1 \mathrm{e}^{2t} + 2c_1 t\,\mathrm{e}^{2t} + 2c_2 \mathrm{e}^{2t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -2 \\ c_2 &= 1 \end{aligned}$$

Substituting these values back in above solution results in

$$x = -2\,\mathrm{e}^{2t} t + \mathrm{e}^{2t}$$

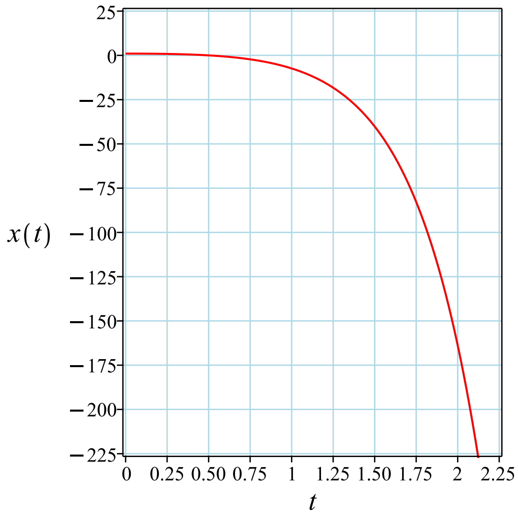
Which simplifies to

$$x = \mathrm{e}^{2t}(1 - 2t)$$

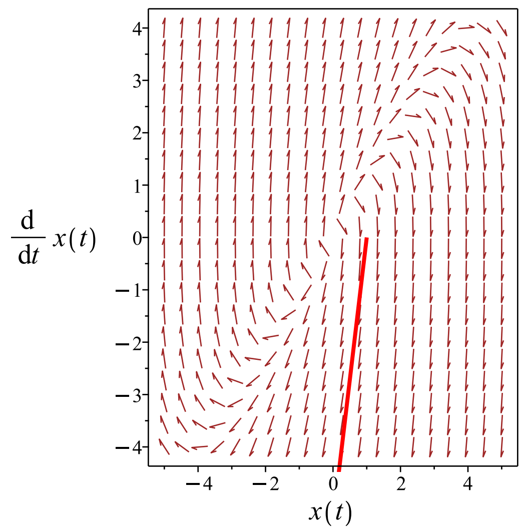
Summary

The solution(s) found are the following

$$x = e^{2t}(1 - 2t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{2t}(1 - 2t)$$

Verified OK.

6.1.4 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 4x' + 4x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 184: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dt}\\ &= z_1 e^{2t}\\ &= z_1\left(e^{2t}\right)\end{aligned}$$

Which simplifies to

$$x_1 = e^{2t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}x_2 &= x_1 \int \frac{e^{\int -\frac{-4}{1}\,dt}}{(x_1)^2}\,dt\\ &= x_1 \int \frac{e^{4t}}{(x_1)^2}\,dt\\ &= x_1(t)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1x_1 + c_2x_2 \\
&= c_1\left(\mathrm{e}^{2t}\right) + c_2\left(\mathrm{e}^{2t}(t)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1\mathrm{e}^{2t} + c_2 t\,\mathrm{e}^{2t} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = 2c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{2t} + 2c_2 t\,\mathrm{e}^{2t}
$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$
0 = 2c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= -2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
x = -2\,\mathrm{e}^{2t}t + \mathrm{e}^{2t}
$$

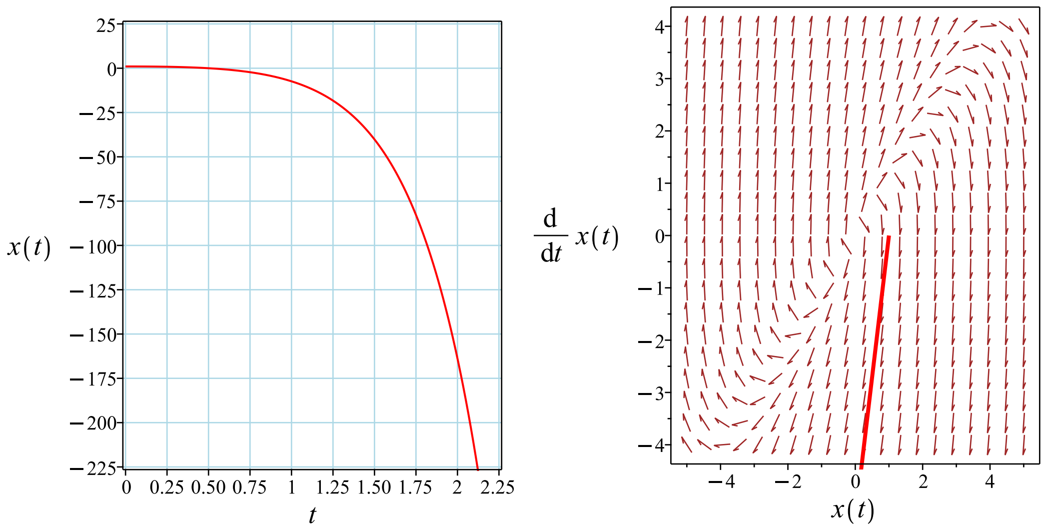
Which simplifies to

$$
x = \mathrm{e}^{2t}(1 - 2t)
$$

Summary

The solution(s) found are the following

$$
x = \mathrm{e}^{2t}(1 - 2t) \tag{1}
$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$x = e^{2t}(1 - 2t)$$

Verified OK.

6.1.5 Maple step by step solution

Let's solve

$$\left[x'' - 4x' + 4x = 0, x(0) = 1, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r - 2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$

- 1st solution of the ODE

$x_1(t) = \mathrm{e}^{2t}$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t) = \mathrm{e}^{2t} t$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 \mathrm{e}^{2t} + c_2 t\,\mathrm{e}^{2t}$

□ Check validity of solution $x = c_1 \mathrm{e}^{2t} + c_2 t \mathrm{e}^{2t}$

  - Use initial condition $x(0) = 1$

    $1 = c_1$

  - Compute derivative of the solution

    $x' = 2c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{2t} + 2c_2 t\,\mathrm{e}^{2t}$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 0$

    $0 = 2c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = -2\}$

  - Substitute constant values into general solution and simplify

    $x = \mathrm{e}^{2t}(1 - 2t)$

- Solution to the IVP

  $x = \mathrm{e}^{2t}(1 - 2t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(x(t),t$2)-4*diff(x(t),t)+4*x(t)=0,x(0) = 1, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = e^{2t}(-2t + 1)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 16

```
DSolve[{x''[t]-4*x'[t]+4*x[t]==0,{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True
```

$$x(t) \to e^{2t}(1 - 2t)$$

6.2 problem 1(b)

6.2.1 Existence and uniqueness analysis 943
6.2.2 Solving as second order linear constant coeff ode 943
6.2.3 Solving as second order integrable as is ode 945
6.2.4 Solving as second order ode missing y ode 947
6.2.5 Solving as type second_order_integrable_as_is (not using ABC version) 949
6.2.6 Solving using Kovacic algorithm 951
6.2.7 Solving as exact linear second order ode ode 955
6.2.8 Maple step by step solution 958

Internal problem ID [11437]
Internal file name [`OUTPUT/10419_Thursday_May_18_2023_04_19_01_AM_61070416/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.2 Real eigenvalues. Exercises page 90
Problem number: 1(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' - 2x' = 0}$$

With initial conditions

$$[x(0) = 1, x'(0) = 0]$$

6.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 0 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' - 2x' = 0$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

6.2.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -2, C = 0$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - 2\lambda \, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 2\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 0$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(0)} \\ &= 1 \pm 1 \end{aligned}$$

Hence

$$\lambda_1 = 1 + 1$$
$$\lambda_2 = 1 - 1$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = 0$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$x = c_1 e^{(2)t} + c_2 e^{(0)t}$$

Or

$$x = c_1 \mathrm{e}^{2t} + c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{2t} + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = 2c_1 \mathrm{e}^{2t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = 2c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 1$$

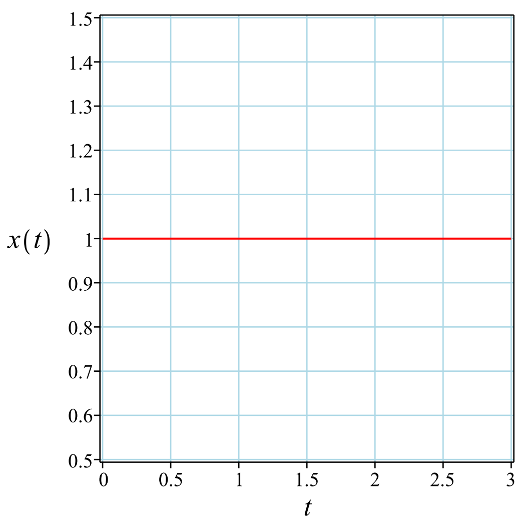
Substituting these values back in above solution results in

$$x = 1$$

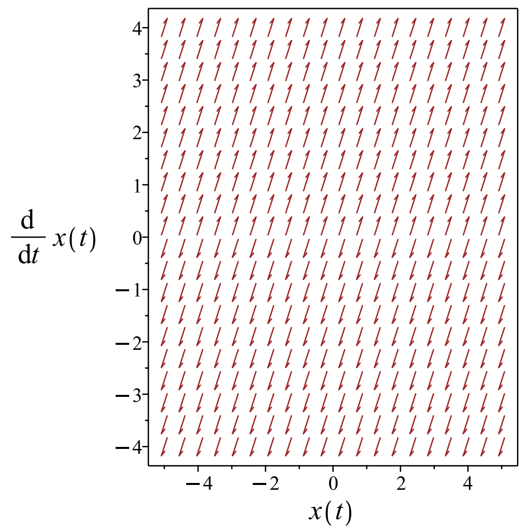
Summary

The solution(s) found are the following

$$x = 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 1$$

Verified OK.

6.2.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int (x'' - 2x')\, dt = 0$$
$$-2x + x' = c_1$$

Which is now solved for x. Integrating both sides gives

$$\int \frac{1}{2x + c_1} dx = \int dt$$
$$\frac{\ln (2x + c_1)}{2} = t + c_2$$

Raising both side to exponential gives

$$\sqrt{2x + c_1} = \mathrm{e}^{t+c_2}$$

Which simplifies to

$$\sqrt{2x + c_1} = c_3 \mathrm{e}^{t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \frac{c_3^2 \mathrm{e}^{2t}}{2} - \frac{c_1}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = \frac{c_3^2}{2} - \frac{c_1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_3^2 \mathrm{e}^{2t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = c_3^2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -2 \\ c_3 &= 0 \end{aligned}$$

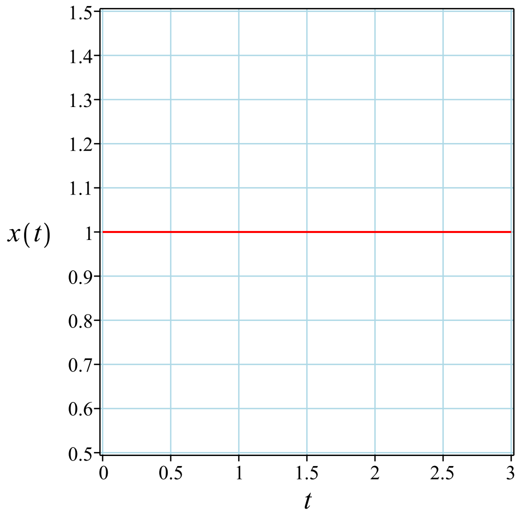
Substituting these values back in above solution results in

$$x = 1$$

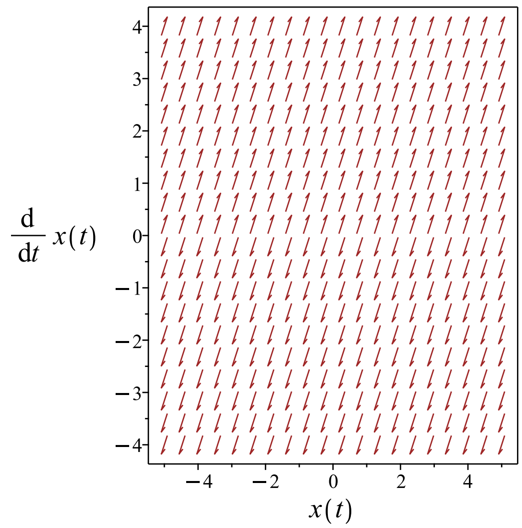
Summary

The solution(s) found are the following

$$x = 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 1$$

Verified OK.

6.2.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable x. Let

$$p(t) = x'$$

Then

$$p'(t) = x''$$

Hence the ode becomes

$$p'(t) - 2p(t) = 0$$

Which is now solve for $p(t)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{2p} dp = \int dt$$
$$\frac{\ln(p)}{2} = t + c_1$$

Raising both side to exponential gives

$$\sqrt{p} = e^{t+c_1}$$

Which simplifies to

$$\sqrt{p} = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $p = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_2^2$$

$$c_2 = 0$$

Substituting c_2 found above in the general solution gives

$$p(t) = 0$$

Since $p = x'$ then the new first order ode to solve is

$$x' = 0$$

Integrating both sides gives

$$\begin{aligned} x &= \int 0 \,\mathrm{d}t \\ &= c_3 \end{aligned}$$

Initial conditions are used to solve for c_3. Substituting $t = 0$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_3$$

$$c_3 = 1$$

Substituting c_3 found above in the general solution gives

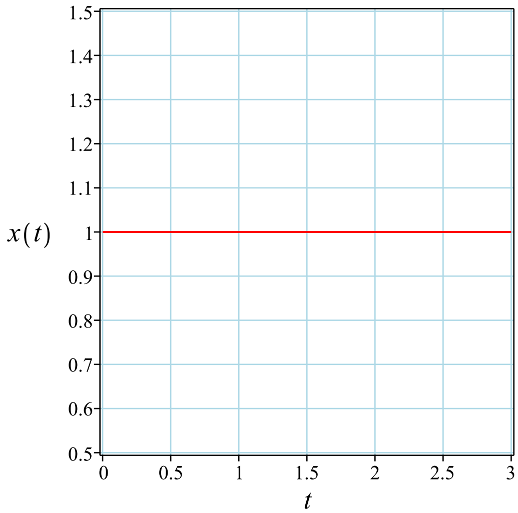
$$x = 1$$

Initial conditions are used to solve for the constants of integration.

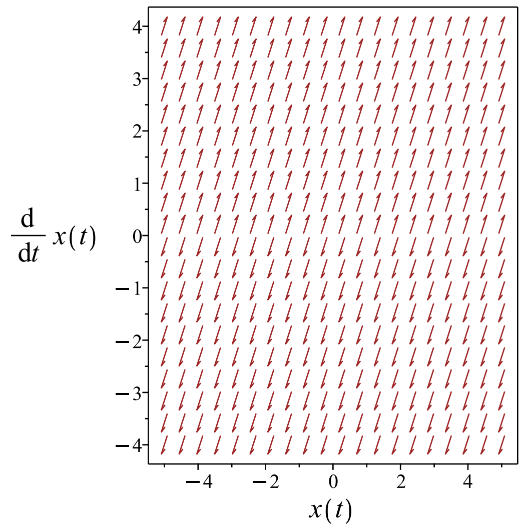
Summary

The solution(s) found are the following

$$x = 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 1$$

Verified OK.

6.2.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x'' - 2x' = 0$$

Integrating both sides of the ODE w.r.t t gives

$$\int (x'' - 2x')\,dt = 0$$
$$-2x + x' = c_1$$

Which is now solved for x. Integrating both sides gives

$$\int \frac{1}{2x + c_1} dx = \int dt$$
$$\frac{\ln(2x + c_1)}{2} = t + c_2$$

Raising both side to exponential gives

$$\sqrt{2x + c_1} = e^{t + c_2}$$

Which simplifies to

$$\sqrt{2x + c_1} = c_3 \mathrm{e}^t$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \frac{c_3^2 \mathrm{e}^{2t}}{2} - \frac{c_1}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = \frac{c_3^2}{2} - \frac{c_1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_3^2 \mathrm{e}^{2t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = c_3^2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -2 \\ c_3 &= 0 \end{aligned}$$

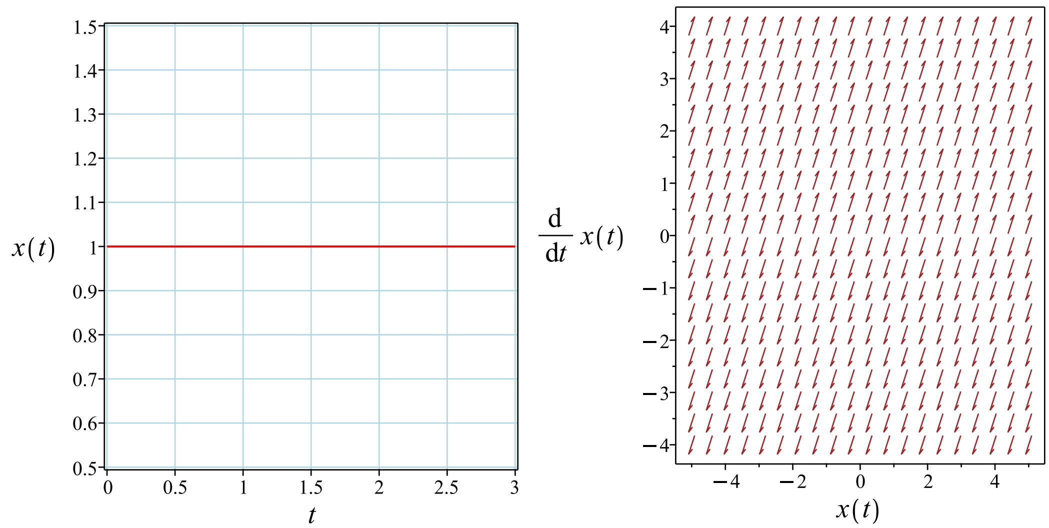
Substituting these values back in above solution results in

$$x = 1$$

Summary

The solution(s) found are the following

$$x = 1 \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$x = 1$$

Verified OK.

6.2.6 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 2x' = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 186: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dt} \\
&= z_1 e^{t} \\
&= z_1 \left(\mathrm{e}^{t}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = 1
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-2}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{2t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(1) + c_2\left(1\left(\frac{e^{2t}}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 + \frac{c_2 e^{2t}}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$
1 = c_1 + \frac{c_2}{2} \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = c_2 e^{2t}
$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$
0 = c_2 \tag{2A}
$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= 0
\end{aligned}
$$

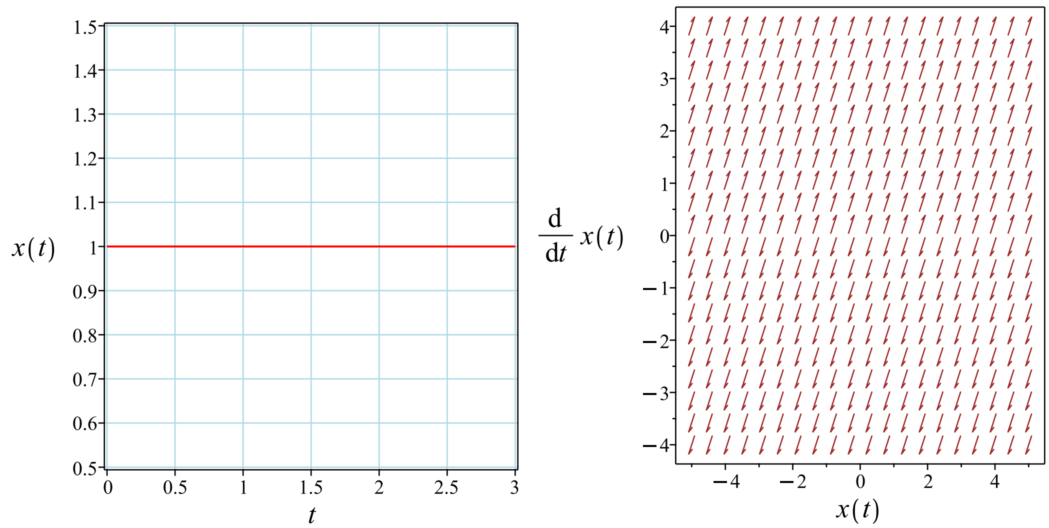
Substituting these values back in above solution results in

$$
x = 1
$$

Summary

The solution(s) found are the following

$$
x = 1 \tag{1}
$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$x = 1$$

Verified OK.

6.2.7 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\, x'' + q(t)\, x' + r(t)\, x = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -2 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\, x' + (q(t) - p'(t))\, x)' = s(x)$$

Integrating gives

$$p(t)\, x' + (q(t) - p'(t))\, x = \int s(t)\, dt$$

Substituting the above values for p, q, r, s gives

$$-2x + x' = c_1$$

We now have a first order ode to solve which is

$$-2x + x' = c_1$$

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{2x + c_1} dx &= \int dt \\ \frac{\ln (2x + c_1)}{2} &= t + c_2\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{2x + c_1} = \mathrm{e}^{t + c_2}$$

Which simplifies to

$$\sqrt{2x + c_1} = c_3 \mathrm{e}^{t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \frac{c_3^2 \mathrm{e}^{2t}}{2} - \frac{c_1}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = \frac{c_3^2}{2} - \frac{c_1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_3^2 e^{2t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = c_3^2 \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_3 = 0$$

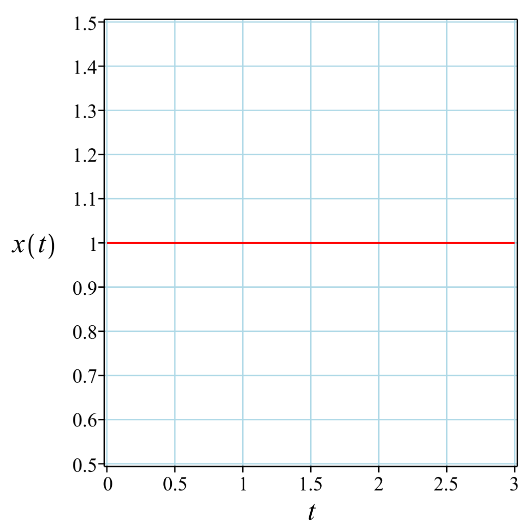
Substituting these values back in above solution results in

$$x = 1$$

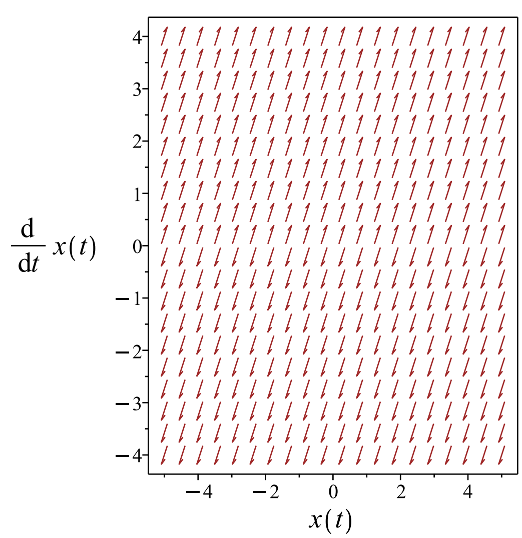
Summary

The solution(s) found are the following

$$x = 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 1$$

Verified OK.

6.2.8 Maple step by step solution

Let's solve

$\left[x'' - 2x' = 0, x(0) = 1, x'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 - 2r = 0$

- Factor the characteristic polynomial

  $r(r - 2) = 0$

- Roots of the characteristic polynomial

  $r = (0, 2)$

- 1st solution of the ODE

  $x_1(t) = 1$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{2t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 + c_2 \mathrm{e}^{2t}$

☐ Check validity of solution $x = c_1 + c_2 \mathrm{e}^{2t}$

  - Use initial condition $x(0) = 1$

    $1 = c_1 + c_2$

  - Compute derivative of the solution

    $x' = 2c_2 \mathrm{e}^{2t}$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 0$

    $0 = 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $x = 1$

- Solution to the IVP

  $x = 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(x(t),t$2)-2*diff(x(t),t)=0,x(0) = 1, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = 1$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 6

```
DSolve[{x''[t]-2*x'[t]==0,{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 1$$

6.3 problem 1(c)

6.3.1 Existence and uniqueness analysis 961
6.3.2 Solving as second order linear constant coeff ode 961
6.3.3 Solving as linear second order ode solved by an integrating factor ode 963
6.3.4 Solving using Kovacic algorithm 965
6.3.5 Maple step by step solution 970

Internal problem ID [11438]
Internal file name [`OUTPUT/10420_Thursday_May_18_2023_04_19_02_AM_14820731/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.2 Real eigenvalues. Exercises page 90
Problem number: 1(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{\frac{x''}{2}+x'+\frac{x}{2}=0}$$

With initial conditions

$$[x(0)=1, x'(0)=0]$$

6.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + 2x' + x = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.3.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = \frac{1}{2}, B = 1, C = \frac{1}{2}$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\frac{\lambda^2 \mathrm{e}^{\lambda t}}{2} + \lambda \,\mathrm{e}^{\lambda t} + \frac{\mathrm{e}^{\lambda t}}{2} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\frac{1}{2}\lambda^2 + \lambda + \frac{1}{2} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = \frac{1}{2}, B = 1, C = \frac{1}{2}$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\left(\frac{1}{2}\right)} \pm \frac{1}{(2)\left(\frac{1}{2}\right)}\sqrt{(1)^2 - (4)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} \\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$x = c_1 e^{-t} + c_2 t\, e^{-t} \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{-t} + c_2 t\, e^{-t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -c_1 e^{-t} + c_2 e^{-t} - c_2 t\, e^{-t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 + c_2 \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}c_1 &= 1 \\ c_2 &= 1\end{aligned}$$

Substituting these values back in above solution results in

$$x = t\, e^{-t} + e^{-t}$$

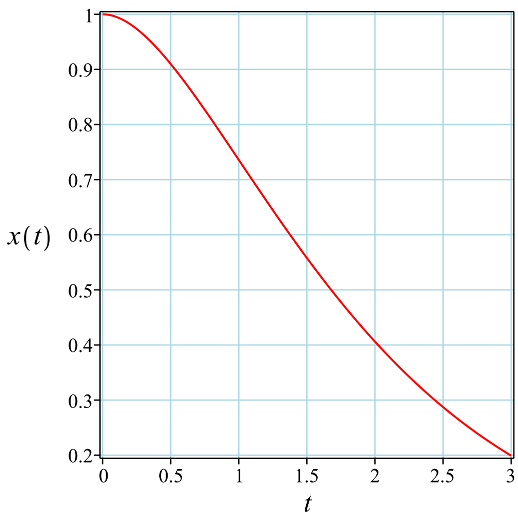
Which simplifies to

$$x = (1 + t)\, e^{-t}$$

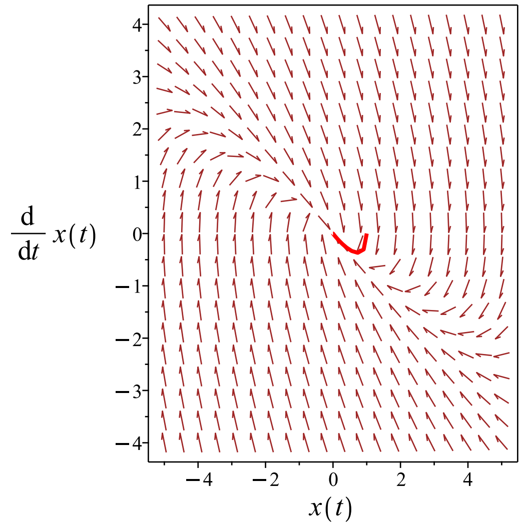
Summary

The solution(s) found are the following

$$x = (1 + t)\, e^{-t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = (1 + t)\,\mathrm{e}^{-t}$$

Verified OK.

6.3.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$x'' + p(t)\,x' + \frac{\left(p(t)^2 + p'(t)\right)x}{2} = f(t)$$

Where $p(t) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= \mathrm{e}^{t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)x)'' &= 0 \\ \left(\mathrm{e}^{t}x\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{t} x\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{t} x\right) = c_1 t + c_2$$

Hence the solution is

$$x = \frac{c_1 t + c_2}{\mathrm{e}^{t}}$$

Or

$$x = c_1 t\,\mathrm{e}^{-t} + c_2 \mathrm{e}^{-t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 t\,\mathrm{e}^{-t} + c_2 \mathrm{e}^{-t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1 \mathrm{e}^{-t} - c_1 t\,\mathrm{e}^{-t} - c_2 \mathrm{e}^{-t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 1$$

Substituting these values back in above solution results in

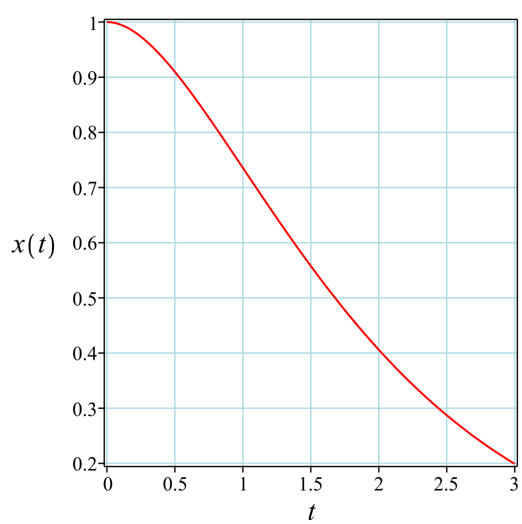
$$x = t\,\mathrm{e}^{-t} + \mathrm{e}^{-t}$$

Which simplifies to

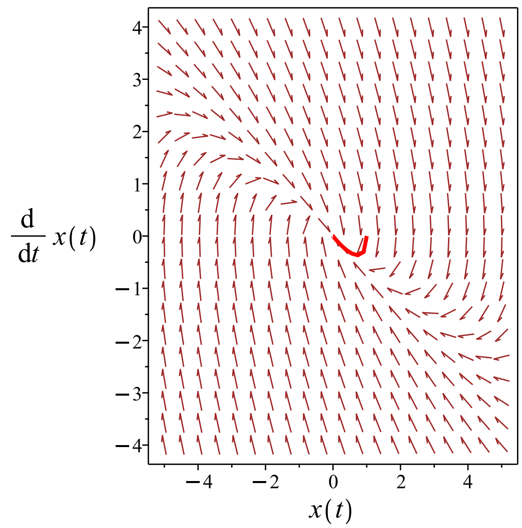
$$x = (1 + t)\,\mathrm{e}^{-t}$$

Summary
The solution(s) found are the following

$$x = (1+t)\,\mathrm{e}^{-t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = (1+t)\,\mathrm{e}^{-t}$$

Verified OK.

6.3.4 Solving using Kovacic algorithm

Writing the ode as

$$\frac{x''}{2} + x' + \frac{x}{2} = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= \frac{1}{2} \\ B &= 1 \\ C &= \frac{1}{2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 188: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{\frac{1}{2}}\,dt} \\ &= z_1 e^{-t} \\ &= z_1\left(\mathrm{e}^{-t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{\frac{1}{2}}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{-2t}}{(x_1)^2}\, dt \\
&= x_1(t)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\mathrm{e}^{-t}\right) + c_2\left(\mathrm{e}^{-t}(t)\right)
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-t} + c_2 t\, \mathrm{e}^{-t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-t} - c_2 t\, \mathrm{e}^{-t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$x = t\,e^{-t} + e^{-t}$$

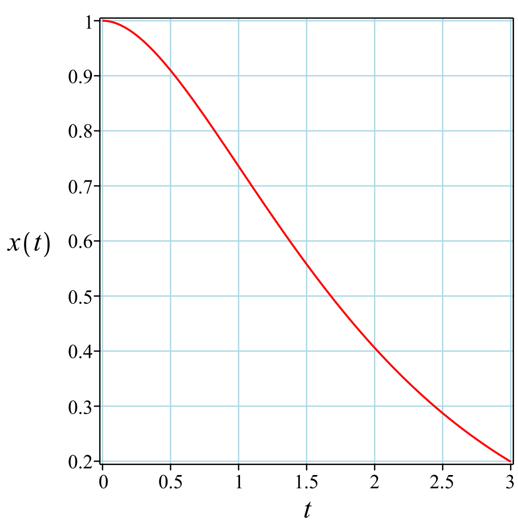
Which simplifies to

$$x = (1+t)\,e^{-t}$$

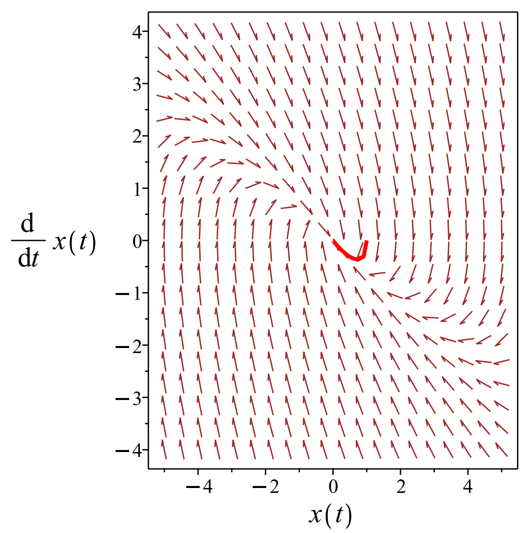
Summary

The solution(s) found are the following

$$x = (1+t)\,e^{-t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = (1+t)\,e^{-t}$$

Verified OK.

6.3.5 Maple step by step solution

Let's solve

$\left[\frac{x''}{2}+x'+\frac{x}{2}=0, x(0)=1, x'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x''=-2x'-x$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x''+2x'+x=0$

- Characteristic polynomial of ODE

  $r^2+2r+1=0$

- Factor the characteristic polynomial

  $(r+1)^2=0$

- Root of the characteristic polynomial

  $r=-1$

- 1st solution of the ODE

  $x_1(t)=\mathrm{e}^{-t}$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t)=t\,\mathrm{e}^{-t}$

- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)$

- Substitute in solutions

  $x=c_1\mathrm{e}^{-t}+c_2t\,\mathrm{e}^{-t}$

□ Check validity of solution $x=c_1\mathrm{e}^{-t}+c_2t\mathrm{e}^{-t}$

  - Use initial condition $x(0)=1$

    $1=c_1$

  - Compute derivative of the solution

    $x'=-c_1\mathrm{e}^{-t}+c_2\mathrm{e}^{-t}-c_2t\,\mathrm{e}^{-t}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = -c_1 + c_2$
- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 1\}$
- Substitute constant values into general solution and simplify

  $x = (1 + t)\,\mathrm{e}^{-t}$

- Solution to the IVP

  $x = (1 + t)\,\mathrm{e}^{-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([1/2*diff(x(t),t$2)+diff(x(t),t)+1/2*x(t)=0,x(0) = 1, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \mathrm{e}^{-t}(t + 1)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 14

```
DSolve[{1/2*x''[t]+x'[t]+1/2*x[t]==0,{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolutions ->
```

$$x(t) \to e^{-t}(t + 1)$$

6.4 problem 1(d)

6.4.1 Existence and uniqueness analysis 972
6.4.2 Solving as second order linear constant coeff ode 973
6.4.3 Solving using Kovacic algorithm 975
6.4.4 Maple step by step solution 980

Internal problem ID [11439]
Internal file name [`OUTPUT/10421_Thursday_May_18_2023_04_19_03_AM_78349429/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.2 Real eigenvalues. Exercises page 90
Problem number: 1(d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{x'' + 4x' + 3x = 0}$$

With initial conditions

$$[x(0) = 1, x'(0) = 0]$$

6.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 3 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + 4x' + 3x = 0$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.4.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 4, C = 3$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda\, e^{\lambda t} + 3\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(3)} \\ &= -2 \pm 1\end{aligned}$$

Hence

$$\lambda_1 = -2 + 1$$
$$\lambda_2 = -2 - 1$$

Which simplifies to

$$\lambda_1 = -1$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$x = c_1 e^{(-1)t} + c_2 e^{(-3)t}$$

Or

$$x = c_1 e^{-t} + c_2 e^{-3t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{-t} + c_2 e^{-3t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -c_1 e^{-t} - 3c_2 e^{-3t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{3}{2}$$
$$c_2 = -\frac{1}{2}$$

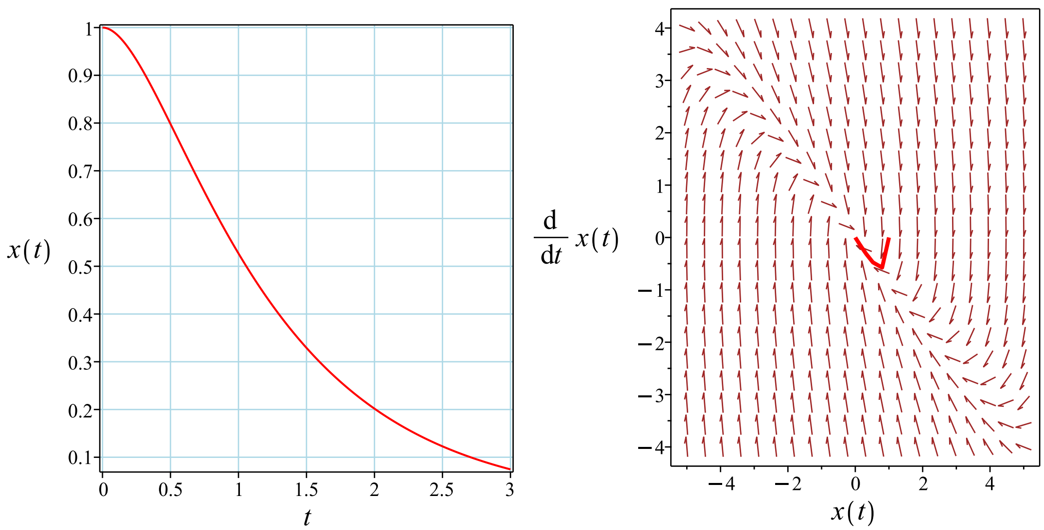
Substituting these values back in above solution results in

$$x = \frac{3\,e^{-t}}{2} - \frac{e^{-3t}}{2}$$

Summary

The solution(s) found are the following

$$x = \frac{3\,\mathrm{e}^{-t}}{2} - \frac{\mathrm{e}^{-3t}}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \frac{3\,\mathrm{e}^{-t}}{2} - \frac{\mathrm{e}^{-3t}}{2}$$

Verified OK.

6.4.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 4x' + 3x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 190: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\ &= z_1 e^{-2t} \\ &= z_1\left(\mathrm{e}^{-2t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-3t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{4}{1}\, dt}}{(x_1)^2}\, dt \\ &= x_1 \int \frac{e^{-4t}}{(x_1)^2}\, dt \\ &= x_1 \left(\frac{\mathrm{e}^{2t}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1 \left(\mathrm{e}^{-3t}\right) + c_2 \left(\mathrm{e}^{-3t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-3t} + \frac{c_2 \mathrm{e}^{-t}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 + \frac{c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -3c_1 \mathrm{e}^{-3t} - \frac{c_2 \mathrm{e}^{-t}}{2}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -3c_1 - \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{2}$$
$$c_2 = 3$$

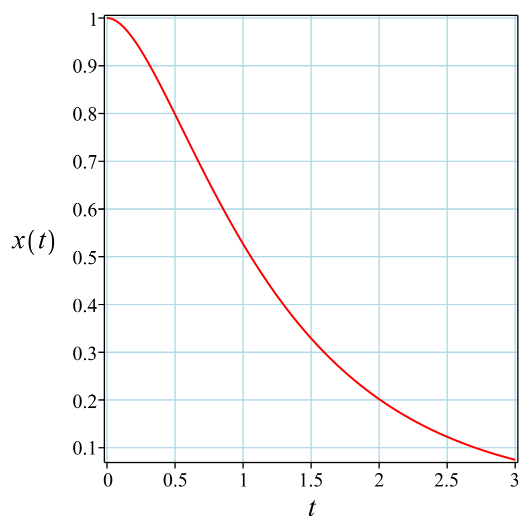
Substituting these values back in above solution results in

$$x = \frac{3\,\mathrm{e}^{-t}}{2} - \frac{\mathrm{e}^{-3t}}{2}$$

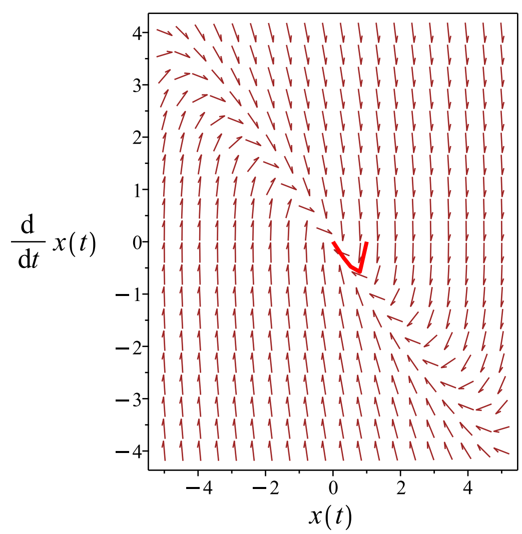
Summary

The solution(s) found are the following

$$x = \frac{3\,\mathrm{e}^{-t}}{2} - \frac{\mathrm{e}^{-3t}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{3\,\mathrm{e}^{-t}}{2} - \frac{\mathrm{e}^{-3t}}{2}$$

Verified OK.

6.4.4 Maple step by step solution

Let's solve

$\left[x''+4x'+3x=0, x(0)=1, x'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2+4r+3=0$

- Factor the characteristic polynomial

  $(r+3)(r+1)=0$

- Roots of the characteristic polynomial

  $r=(-3,-1)$

- 1st solution of the ODE

  $x_1(t)=\mathrm{e}^{-3t}$

- 2nd solution of the ODE

  $x_2(t)=\mathrm{e}^{-t}$

- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)$

- Substitute in solutions

  $x=c_1\mathrm{e}^{-3t}+c_2\mathrm{e}^{-t}$

□ Check validity of solution $x=c_1\mathrm{e}^{-3t}+c_2\mathrm{e}^{-t}$

  - Use initial condition $x(0)=1$

    $1=c_1+c_2$

  - Compute derivative of the solution

    $x'=-3c_1\mathrm{e}^{-3t}-c_2\mathrm{e}^{-t}$

  - Use the initial condition $x'\Big|_{\{t=0\}}=0$

    $0=-3c_1-c_2$

  - Solve for c_1 and c_2

    $\left\{c_1=-\frac{1}{2}, c_2=\frac{3}{2}\right\}$

- Substitute constant values into general solution and simplify

  $x = \frac{3\,\mathrm{e}^{-t}}{2} - \frac{\mathrm{e}^{-3t}}{2}$

- Solution to the IVP

  $x = \frac{3\,\mathrm{e}^{-t}}{2} - \frac{\mathrm{e}^{-3t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve([diff(x(t),t$2)+4*diff(x(t),t)+3*x(t)=0,x(0) = 1, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = -\frac{\mathrm{e}^{-3t}}{2} + \frac{3\,\mathrm{e}^{-t}}{2}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 23

```
DSolve[{x''[t]+4*x'[t]+3*x[t]==0,{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True
```

$$x(t) \to \frac{1}{2}e^{-3t}\left(3e^{2t} - 1\right)$$

6.5 problem 3(a)

6.5.1 Existence and uniqueness analysis 983
6.5.2 Solving as second order linear constant coeff ode 983
6.5.3 Solving as linear second order ode solved by an integrating factor ode 985
6.5.4 Solving using Kovacic algorithm 987
6.5.5 Maple step by step solution 991

Internal problem ID [11440]
Internal file name [`OUTPUT/10422_Thursday_May_18_2023_04_19_05_AM_44364650/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.2 Real eigenvalues. Exercises page 90
Problem number: 3(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' - 4x' + 4x = 0}$$

With initial conditions

$$[x(0) = -1, x'(0) = 2]$$

6.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' - 4x' + 4x = 0$$

The domain of $p(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.5.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - 4\lambda\, e^{\lambda t} + 4\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{(-4)^2 - (4)(1)(4)} \\
&= 2
\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$x = c_1 e^{2t} + c_2 e^{2t} t \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{2t} + c_2 t\, e^{2t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 0$ in the above gives

$$-1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t\, e^{2t}$$

substituting $x' = 2$ and $t = 0$ in the above gives

$$2 = 2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}
c_1 &= -1 \\
c_2 &= 4
\end{aligned}$$

Substituting these values back in above solution results in

$$x = 4\, e^{2t} t - e^{2t}$$

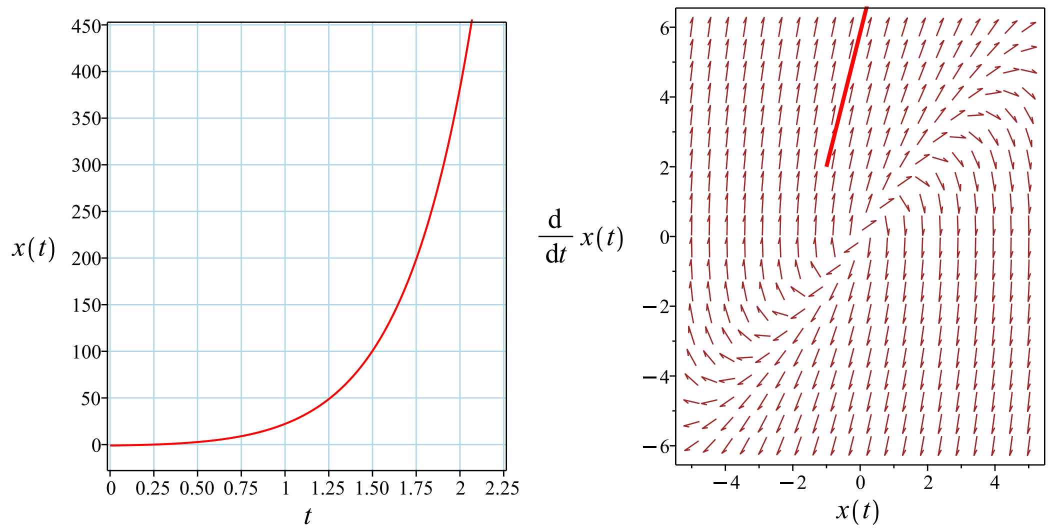
Which simplifies to

$$x = e^{2t}(-1 + 4t)$$

Summary

The solution(s) found are the following

$$x = e^{2t}(-1 + 4t) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \mathrm{e}^{2t}(-1+4t)$$

Verified OK.

6.5.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$x'' + p(t)\,x' + \frac{\left(p(t)^2 + p'(t)\right)x}{2} = f(t)$$

Where $p(t) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -4\,dx} \\ &= \mathrm{e}^{-2t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)x)'' &= 0 \\ \left(\mathrm{e}^{-2t}x\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(e^{-2t}x\right)' = c_1$$

Integrating again gives

$$\left(e^{-2t}x\right) = c_1 t + c_2$$

Hence the solution is

$$x = \frac{c_1 t + c_2}{e^{-2t}}$$

Or

$$x = c_1 t\, e^{2t} + c_2 e^{2t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 t\, e^{2t} + c_2 e^{2t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 0$ in the above gives

$$-1 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1 e^{2t} + 2c_1 t\, e^{2t} + 2c_2 e^{2t}$$

substituting $x' = 2$ and $t = 0$ in the above gives

$$2 = c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 4 \\ c_2 &= -1 \end{aligned}$$

Substituting these values back in above solution results in

$$x = 4\, e^{2t} t - e^{2t}$$

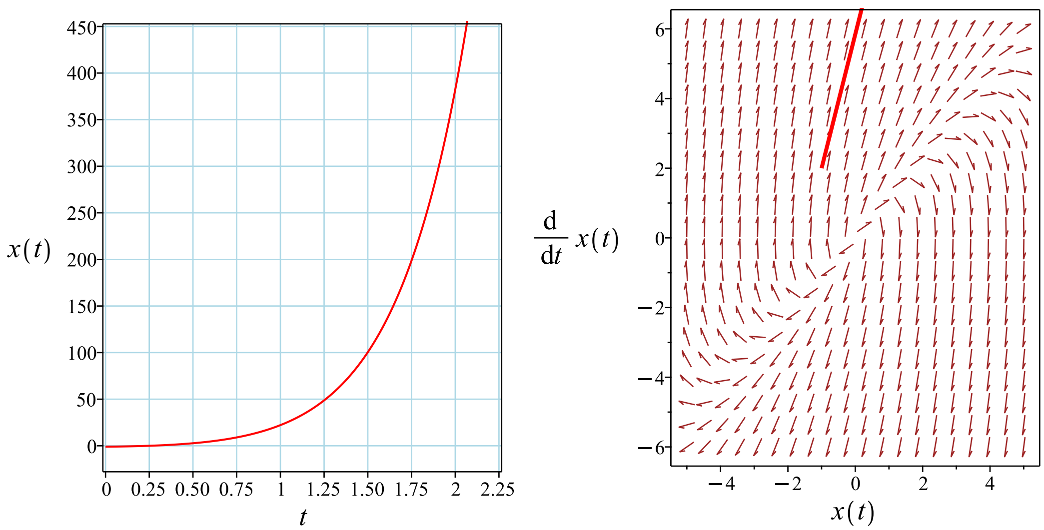
Which simplifies to

$$x = e^{2t}(-1 + 4t)$$

Summary

The solution(s) found are the following

$$x = e^{2t}(-1 + 4t) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = e^{2t}(-1 + 4t)$$

Verified OK.

6.5.4 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 4x' + 4x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 192: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dt} \\ &= z_1 e^{2t} \\ &= z_1\left(\mathrm{e}^{2t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{2t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{-4}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{4t}}{(x_1)^2}\,dt \\ &= x_1(t) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{2t}\right) + c_2 \left(\mathrm{e}^{2t}(t)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 \mathrm{e}^{2t} + c_2 t\, \mathrm{e}^{2t} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 0$ in the above gives

$$
-1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = 2c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{2t} + 2c_2 t\, \mathrm{e}^{2t}
$$

substituting $x' = 2$ and $t = 0$ in the above gives

$$
2 = 2c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -1 \\
c_2 &= 4
\end{aligned}
$$

Substituting these values back in above solution results in

$$
x = 4\, \mathrm{e}^{2t} t - \mathrm{e}^{2t}
$$

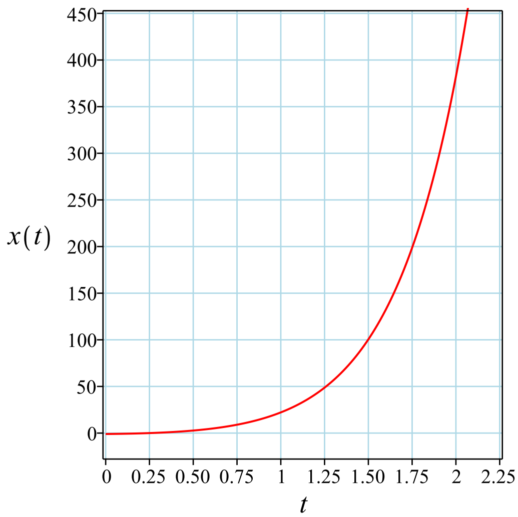
Which simplifies to

$$
x = \mathrm{e}^{2t}(-1 + 4t)
$$

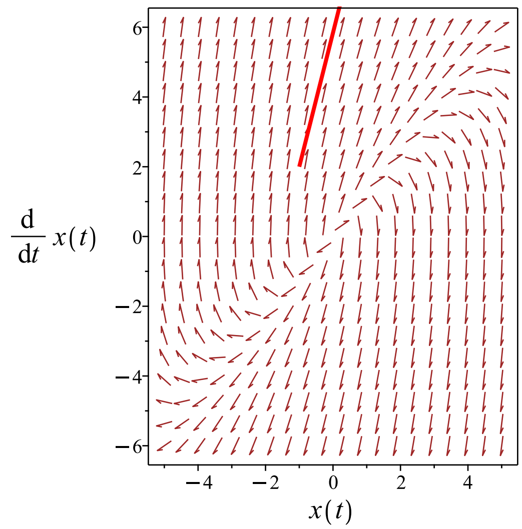
Summary

The solution(s) found are the following

$$
x = \mathrm{e}^{2t}(-1 + 4t) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{2t}(-1 + 4t)$$

Verified OK.

6.5.5 Maple step by step solution

Let's solve

$$\left[x'' - 4x' + 4x = 0, x(0) = -1, x'\Big|_{\{t=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  x''
- Characteristic polynomial of ODE

  $r^2 - 4r + 4 = 0$
- Factor the characteristic polynomial

  $(r-2)^2 = 0$
- Root of the characteristic polynomial

  $r = 2$
- 1st solution of the ODE

$x_1(t) = e^{2t}$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t) = e^{2t}t$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 e^{2t} + c_2 t\, e^{2t}$

- □ Check validity of solution $x = c_1 e^{2t} + c_2 t e^{2t}$

  - Use initial condition $x(0) = -1$

    $-1 = c_1$

  - Compute derivative of the solution

    $x' = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t\, e^{2t}$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 2$

    $2 = 2c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = -1, c_2 = 4\}$

  - Substitute constant values into general solution and simplify

    $x = e^{2t}(-1 + 4t)$

- Solution to the IVP

  $x = e^{2t}(-1 + 4t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(x(t),t$2)-4*diff(x(t),t)+4*x(t)=0,x(0) = -1, D(x)(0) = 2],x(t), singsol=all)
```

$$x(t) = e^{2t}(-1 + 4t)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 16

```
DSolve[{x''[t]-4*x'[t]+4*x[t]==0,{x[0]==-1,x'[0]==2}},x[t],t,IncludeSingularSolutions -> Tru
```

$$x(t) \to e^{2t}(4t - 1)$$

6.6 problem 3(b)

6.6.1 Existence and uniqueness analysis 995
6.6.2 Solving as second order linear constant coeff ode 995
6.6.3 Solving as second order integrable as is ode 997
6.6.4 Solving as second order ode missing y ode 998
6.6.5 Solving as type second_order_integrable_as_is (not using ABC version) 1000
6.6.6 Solving using Kovacic algorithm 1001
6.6.7 Solving as exact linear second order ode ode 1006
6.6.8 Maple step by step solution 1008

Internal problem ID [11441]
Internal file name [`OUTPUT/10423_Thursday_May_18_2023_04_19_06_AM_50170927/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.2 Real eigenvalues. Exercises page 90
Problem number: 3(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' - 2x' = 0}$$

With initial conditions

$$[x(0) = -1, x'(0) = 2]$$

6.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 0 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' - 2x' = 0$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

6.6.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -2, C = 0$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - 2\lambda\, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 2\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 0$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(0)} \\ &= 1 \pm 1 \end{aligned}$$

Hence

$$\lambda_1 = 1 + 1$$
$$\lambda_2 = 1 - 1$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = 0$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$x = c_1 e^{(2)t} + c_2 e^{(0)t}$$

Or

$$x = c_1 \mathrm{e}^{2t} + c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{2t} + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 0$ in the above gives

$$-1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = 2c_1 \mathrm{e}^{2t}$$

substituting $x' = 2$ and $t = 0$ in the above gives

$$2 = 2c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -2$$

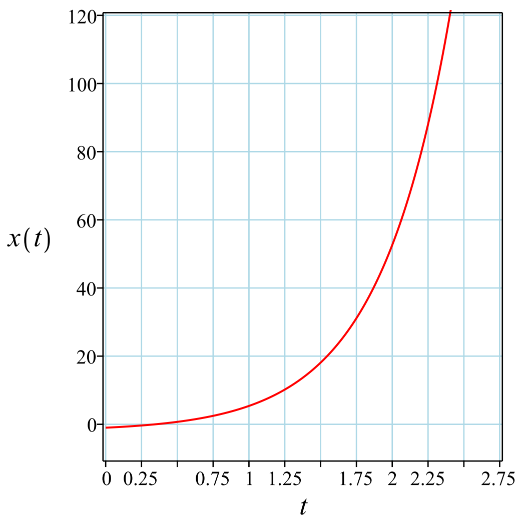
Substituting these values back in above solution results in

$$x = e^{2t} - 2$$

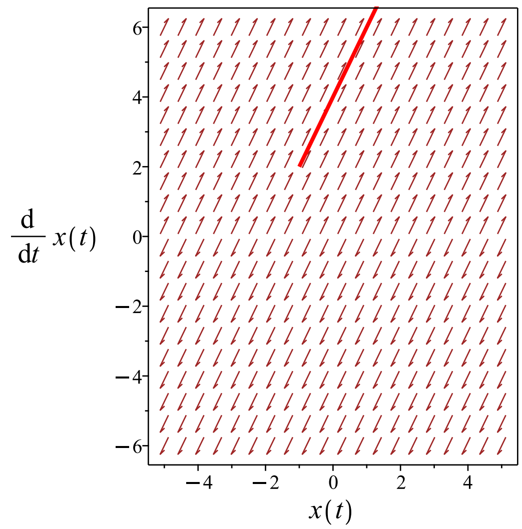
Summary

The solution(s) found are the following

$$x = e^{2t} - 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{2t} - 2$$

Verified OK.

6.6.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int (x'' - 2x')\, dt = 0$$
$$-2x + x' = c_1$$

Which is now solved for x. Integrating both sides gives

$$\int \frac{1}{2x + c_1} dx = \int dt$$
$$\frac{\ln(2x + c_1)}{2} = t + c_2$$

Raising both side to exponential gives

$$\sqrt{2x + c_1} = e^{t + c_2}$$

Which simplifies to

$$\sqrt{2x + c_1} = c_3 e^t$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \frac{c_3^2 e^{2t}}{2} - \frac{c_1}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 0$ in the above gives

$$-1 = \frac{c_3^2}{2} - \frac{c_1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_3^2 e^{2t}$$

substituting $x' = 2$ and $t = 0$ in the above gives

$$2 = c_3^2 \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

Verification of solutions N/A

6.6.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable x. Let

$$p(t) = x'$$

Then

$$p'(t) = x''$$

Hence the ode becomes

$$p'(t) - 2p(t) = 0$$

Which is now solve for $p(t)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{2p} dp = \int dt$$
$$\frac{\ln(p)}{2} = t + c_1$$

Raising both side to exponential gives

$$\sqrt{p} = e^{t+c_1}$$

Which simplifies to

$$\sqrt{p} = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $p = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_2^2$$

$$c_2 = -\sqrt{2}$$

Substituting c_2 found above in the general solution gives

$$p(t) = 2\,e^{2t}$$

Since $p = x'$ then the new first order ode to solve is

$$x' = 2\,e^{2t}$$

Integrating both sides gives

$$x = \int 2\,e^{2t}\,\mathrm{d}t$$
$$= c_3 + e^{2t}$$

Initial conditions are used to solve for c_3. Substituting $t = 0$ and $x = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = 1 + c_3$$

$$c_3 = -2$$

Substituting c_3 found above in the general solution gives

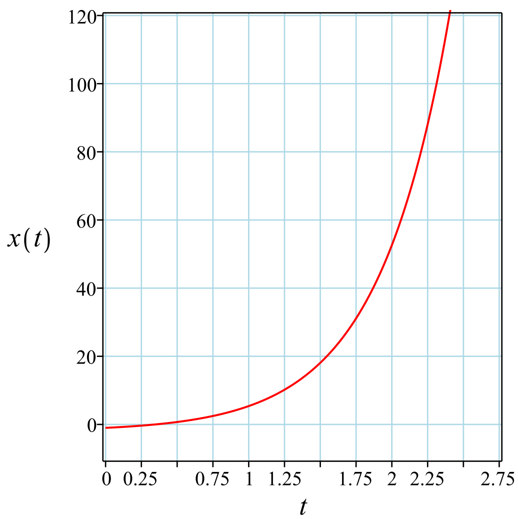
$$x = e^{2t} - 2$$

Initial conditions are used to solve for the constants of integration.

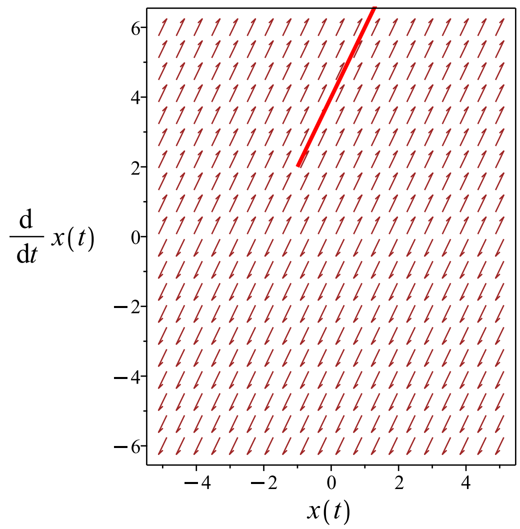
Summary

The solution(s) found are the following

$$x = e^{2t} - 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{2t} - 2$$

Verified OK.

6.6.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x'' - 2x' = 0$$

Integrating both sides of the ODE w.r.t t gives

$$\int (x'' - 2x')\, dt = 0$$
$$-2x + x' = c_1$$

Which is now solved for x. Integrating both sides gives

$$\int \frac{1}{2x + c_1} dx = \int dt$$
$$\frac{\ln(2x + c_1)}{2} = t + c_2$$

Raising both side to exponential gives

$$\sqrt{2x + c_1} = e^{t+c_2}$$

Which simplifies to

$$\sqrt{2x + c_1} = c_3 e^t$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \frac{c_3^2 e^{2t}}{2} - \frac{c_1}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 0$ in the above gives

$$-1 = \frac{c_3^2}{2} - \frac{c_1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_3^2 e^{2t}$$

substituting $x' = 2$ and $t = 0$ in the above gives

$$2 = c_3^2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

6.6.6 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 2x' = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -2 \\
C &= 0
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = xe^{\int \frac{B}{2A}\, dt}
$$

Then (2) becomes

$$
z''(t) = rz(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 194: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dt} \\
&= z_1 e^{t} \\
&= z_1 \left(\mathrm{e}^{t}\right)
\end{aligned}
$$

Which simplifies to

$$x_1 = 1$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-2}{1}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{2t}}{(x_1)^2}\, dt \\
&= x_1 \left(\frac{e^{2t}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(1) + c_2\left(1\left(\frac{e^{2t}}{2}\right)\right)
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 + \frac{c_2 e^{2t}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 0$ in the above gives

$$-1 = c_1 + \frac{c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_2 e^{2t}$$

substituting $x' = 2$ and $t = 0$ in the above gives

$$2 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -2 \\ c_2 &= 2 \end{aligned}$$

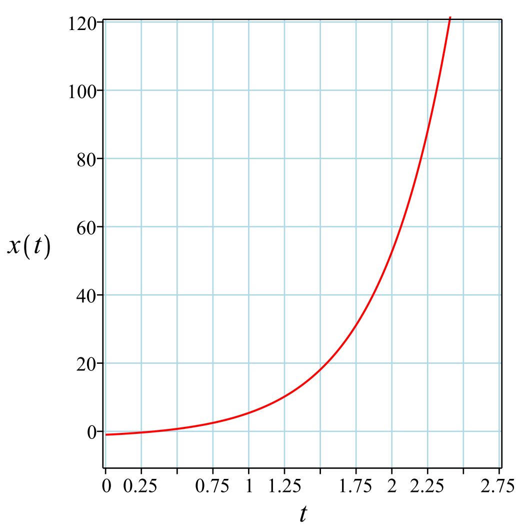
Substituting these values back in above solution results in

$$x = e^{2t} - 2$$

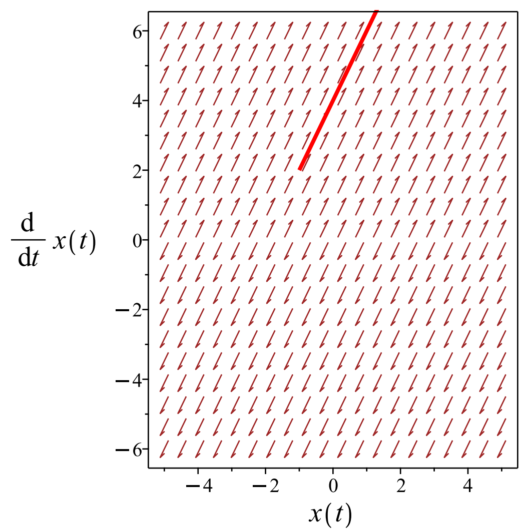
Summary

The solution(s) found are the following

$$x = e^{2t} - 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{2t} - 2$$

Verified OK.

6.6.7 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\,x'' + q(t)\,x' + r(t)\,x = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 1\\
q(x) &= -2\\
r(x) &= 0\\
s(x) &= 0
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 0\\
q'(x) &= 0
\end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,x' + (q(t) - p'(t))\,x)' = s(x)$$

Integrating gives

$$p(t)\,x' + (q(t) - p'(t))\,x = \int s(t)\;dt$$

Substituting the above values for p, q, r, s gives

$$-2x + x' = c_1$$

We now have a first order ode to solve which is

$$-2x + x' = c_1$$

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{2x + c_1} dx &= \int dt \\
\frac{\ln (2x + c_1)}{2} &= t + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\sqrt{2x + c_1} = \mathrm{e}^{t + c_2}
$$

Which simplifies to

$$
\sqrt{2x + c_1} = c_3 \mathrm{e}^{t}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = \frac{c_3^2 \mathrm{e}^{2t}}{2} - \frac{c_1}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 0$ in the above gives

$$
-1 = \frac{c_3^2}{2} - \frac{c_1}{2} \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = c_3^2 \mathrm{e}^{2t}
$$

substituting $x' = 2$ and $t = 0$ in the above gives

$$
2 = c_3^2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

Verification of solutions N/A

6.6.8 Maple step by step solution

Let's solve

$$\left[x'' - 2x' = 0, x(0) = -1, x'\Big|_{\{t=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 - 2r = 0$

- Factor the characteristic polynomial

  $r(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (0, 2)$

- 1st solution of the ODE

  $x_1(t) = 1$

- 2nd solution of the ODE

  $x_2(t) = e^{2t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 + c_2 e^{2t}$

☐ Check validity of solution $x = c_1 + c_2 e^{2t}$

  - Use initial condition $x(0) = -1$

    $-1 = c_1 + c_2$

  - Compute derivative of the solution

    $x' = 2c_2 e^{2t}$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 2$

    $2 = 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = -2, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $x = e^{2t} - 2$

- Solution to the IVP

  $x = e^{2t} - 2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve([diff(x(t),t$2)-2*diff(x(t),t)=0,x(0) = -1, D(x)(0) = 2],x(t), singsol=all)
```

$$x(t) = -2 + e^{2t}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 12

```
DSolve[{x''[t]-2*x'[t]==0,{x[0]==-1,x'[0]==2}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{2t} - 2$$

6.7 problem 3(c)

6.7.1 Existence and uniqueness analysis 1011
6.7.2 Solving as second order linear constant coeff ode 1011
6.7.3 Solving as linear second order ode solved by an integrating factor ode 1013
6.7.4 Solving using Kovacic algorithm 1015
6.7.5 Maple step by step solution 1020

Internal problem ID [11442]
Internal file name [`OUTPUT/10424_Thursday_May_18_2023_04_19_08_AM_41798993/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.2 Real eigenvalues. Exercises page 90
Problem number: 3(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{\frac{x''}{2} + x' + \frac{x}{2} = 0}$$

With initial conditions

$$[x(0) = -1, x'(0) = 2]$$

6.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + 2x' + x = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.7.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = \frac{1}{2}, B = 1, C = \frac{1}{2}$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\frac{\lambda^2 \mathrm{e}^{\lambda t}}{2} + \lambda\, \mathrm{e}^{\lambda t} + \frac{\mathrm{e}^{\lambda t}}{2} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\frac{1}{2}\lambda^2 + \lambda + \frac{1}{2} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = \frac{1}{2}, B = 1, C = \frac{1}{2}$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\left(\frac{1}{2}\right)} \pm \frac{1}{(2)\left(\frac{1}{2}\right)}\sqrt{(1)^2 - (4)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} \\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$x = c_1 \mathrm{e}^{-t} + c_2 t\,\mathrm{e}^{-t} \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-t} + c_2 t\,\mathrm{e}^{-t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 0$ in the above gives

$$-1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-t} - c_2 t\,\mathrm{e}^{-t}$$

substituting $x' = 2$ and $t = 0$ in the above gives

$$2 = -c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}c_1 &= -1 \\ c_2 &= 1\end{aligned}$$

Substituting these values back in above solution results in

$$x = t\,\mathrm{e}^{-t} - \mathrm{e}^{-t}$$

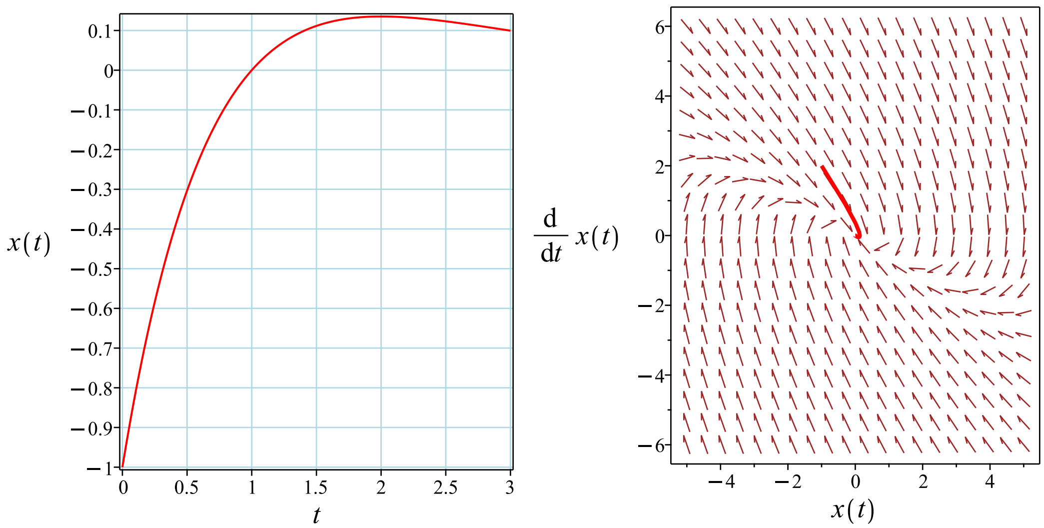
Which simplifies to

$$x = \mathrm{e}^{-t}(t - 1)$$

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{-t}(t - 1) \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$
x = \mathrm{e}^{-t}(t-1)
$$

Verified OK.

6.7.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$
x'' + p(t)\,x' + \frac{\left(p(t)^2 + p'(t)\right)x}{2} = f(t)
$$

Where $p(t) = 2$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int 2\,dx} \\
&= \mathrm{e}^{t}
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)x)'' &= 0 \\
\left(\mathrm{e}^{t}x\right)'' &= 0
\end{aligned}
$$

Integrating once gives

$$(e^t x)' = c_1$$

Integrating again gives

$$(e^t x) = c_1 t + c_2$$

Hence the solution is

$$x = \frac{c_1 t + c_2}{e^t}$$

Or

$$x = c_1 t\, e^{-t} + c_2 e^{-t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 t\, e^{-t} + c_2 e^{-t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 0$ in the above gives

$$-1 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1 e^{-t} - c_1 t\, e^{-t} - c_2 e^{-t}$$

substituting $x' = 2$ and $t = 0$ in the above gives

$$2 = c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$x = t\, e^{-t} - e^{-t}$$

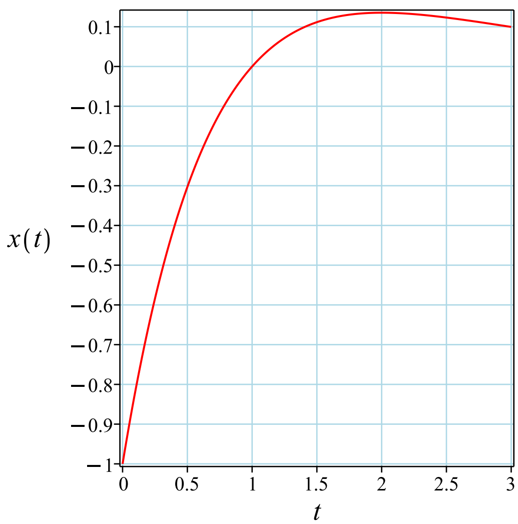
Which simplifies to

$$x = e^{-t}(t - 1)$$

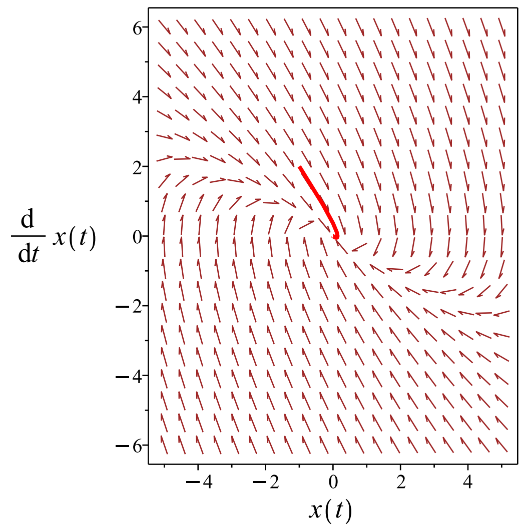
Summary

The solution(s) found are the following

$$x = e^{-t}(t-1) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{-t}(t-1)$$

Verified OK.

6.7.4 Solving using Kovacic algorithm

Writing the ode as

$$\frac{x''}{2} + x' + \frac{x}{2} = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= \frac{1}{2} \\ B &= 1 \\ C &= \frac{1}{2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 196: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{\frac{1}{2}}\, dt} \\
&= z_1 e^{-t} \\
&= z_1 \left(\mathrm{e}^{-t}\right)
\end{aligned}
$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{\frac{1}{2}}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{-2t}}{(x_1)^2}\, dt \\
&= x_1(t)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\mathrm{e}^{-t}\right) + c_2\left(\mathrm{e}^{-t}(t)\right)
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-t} + c_2 t\, \mathrm{e}^{-t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 0$ in the above gives

$$-1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-t} - c_2 t\, \mathrm{e}^{-t}$$

substituting $x' = 2$ and $t = 0$ in the above gives

$$2 = -c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -1 \\
c_2 &= 1
\end{aligned}
$$

Substituting these values back in above solution results in

$$
x = t\,\mathrm{e}^{-t} - \mathrm{e}^{-t}
$$

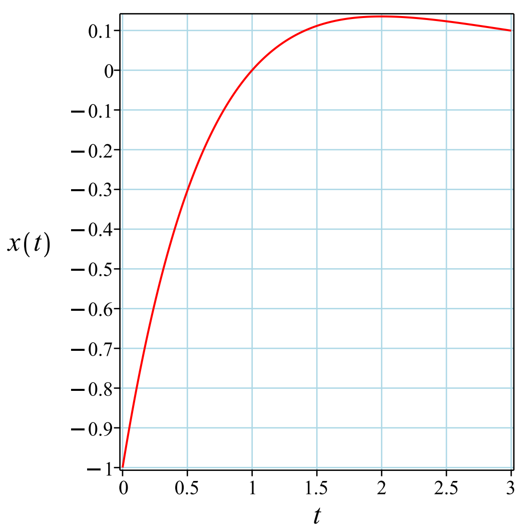
Which simplifies to

$$
x = \mathrm{e}^{-t}(t-1)
$$

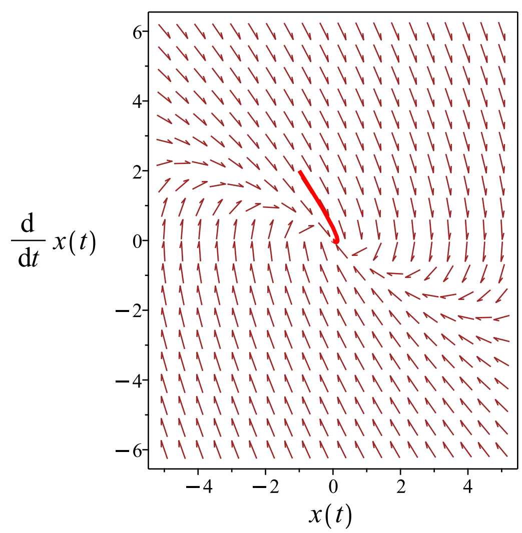
Summary

The solution(s) found are the following

$$
x = \mathrm{e}^{-t}(t-1) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
x = \mathrm{e}^{-t}(t-1)
$$

Verified OK.

6.7.5 Maple step by step solution

Let's solve

$\left[\frac{x''}{2}+x'+\frac{x}{2}=0, x(0)=-1, x'\Big|_{\{t=0\}}=2\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x''=-2x'-x$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x''+2x'+x=0$

- Characteristic polynomial of ODE

  $r^2+2r+1=0$

- Factor the characteristic polynomial

  $(r+1)^2=0$

- Root of the characteristic polynomial

  $r=-1$

- 1st solution of the ODE

  $x_1(t)=\mathrm{e}^{-t}$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t)=t\,\mathrm{e}^{-t}$

- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)$

- Substitute in solutions

  $x=c_1\mathrm{e}^{-t}+c_2t\,\mathrm{e}^{-t}$

☐ Check validity of solution $x=c_1\mathrm{e}^{-t}+c_2t\mathrm{e}^{-t}$

  - Use initial condition $x(0)=-1$

    $-1=c_1$

  - Compute derivative of the solution

    $x'=-c_1\mathrm{e}^{-t}+c_2\mathrm{e}^{-t}-c_2t\,\mathrm{e}^{-t}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 2$

  $2 = -c_1 + c_2$
- Solve for c_1 and c_2

  $\{c_1 = -1, c_2 = 1\}$
- Substitute constant values into general solution and simplify

  $x = e^{-t}(t-1)$

- Solution to the IVP

  $x = e^{-t}(t-1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve([1/2*diff(x(t),t$2)+diff(x(t),t)+1/2*x(t)=0,x(0) = -1, D(x)(0) = 2],x(t), singsol=all
```

$$x(t) = e^{-t}(t-1)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 14

```
DSolve[{1/2*x''[t]+x'[t]+1/2*x[t]==0,{x[0]==-1,x'[0]==2}},x[t],t,IncludeSingularSolutions ->
```

$$x(t) \to e^{-t}(t-1)$$

6.8 problem 3(d)

6.8.1 Existence and uniqueness analysis 1022
6.8.2 Solving as second order linear constant coeff ode 1023
6.8.3 Solving using Kovacic algorithm 1025
6.8.4 Maple step by step solution 1030

Internal problem ID [11443]
Internal file name [`OUTPUT/10425_Thursday_May_18_2023_04_19_09_AM_20700232/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.2 Real eigenvalues. Exercises page 90
Problem number: 3(d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{x'' + 4x' + 3x = 0}$$

With initial conditions

$$[x(0) = -1, x'(0) = 2]$$

6.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 3 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + 4x' + 3x = 0$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.8.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 4, C = 3$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda\, e^{\lambda t} + 3\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(3)}\\ &= -2 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 1\\ \lambda_2 &= -2 - 1\end{aligned}$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= -1 \\
\lambda_2 &= -3
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
x &= c_1 e^{(-1)t} + c_2 e^{(-3)t}
\end{aligned}
$$

Or

$$
x = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-3t}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-3t} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 0$ in the above gives

$$
-1 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = -c_1 \mathrm{e}^{-t} - 3c_2 \mathrm{e}^{-3t}
$$

substituting $x' = 2$ and $t = 0$ in the above gives

$$
2 = -c_1 - 3c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{1}{2} \\
c_2 &= -\frac{1}{2}
\end{aligned}
$$

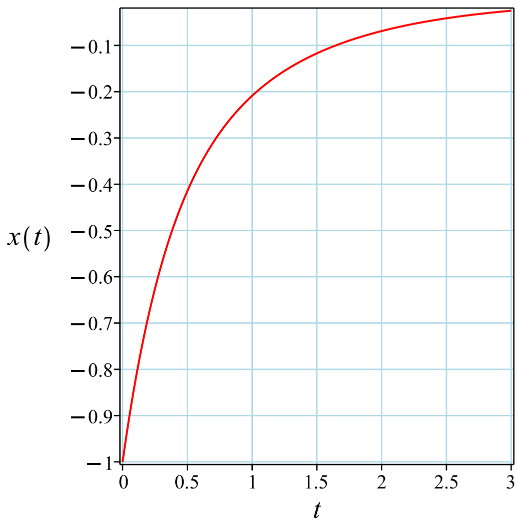
Substituting these values back in above solution results in

$$
x = -\frac{\mathrm{e}^{-t}}{2} - \frac{\mathrm{e}^{-3t}}{2}
$$

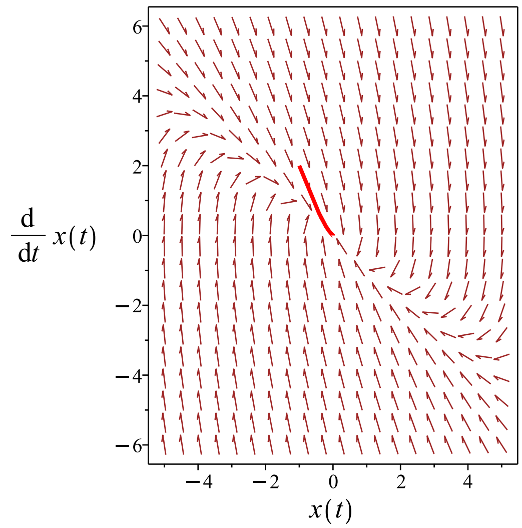
Summary

The solution(s) found are the following

$$x = -\frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -\frac{e^{-t}}{2} - \frac{e^{-3t}}{2}$$

Verified OK.

6.8.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 4x' + 3x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 198: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{4}{1} \, dt} \\
&= z_1 e^{-2t} \\
&= z_1 \left(\mathrm{e}^{-2t}\right)
\end{aligned}
$$

Which simplifies to

$$x_1 = \mathrm{e}^{-3t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{4}{1}\, dt}}{(x_1)^2}\, dt \\ &= x_1 \int \frac{e^{-4t}}{(x_1)^2}\, dt \\ &= x_1 \left(\frac{\mathrm{e}^{2t}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1 \left(\mathrm{e}^{-3t}\right) + c_2 \left(\mathrm{e}^{-3t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-3t} + \frac{c_2 \mathrm{e}^{-t}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 0$ in the above gives

$$-1 = c_1 + \frac{c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -3c_1 \mathrm{e}^{-3t} - \frac{c_2 \mathrm{e}^{-t}}{2}$$

substituting $x' = 2$ and $t = 0$ in the above gives

$$2 = -3c_1 - \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

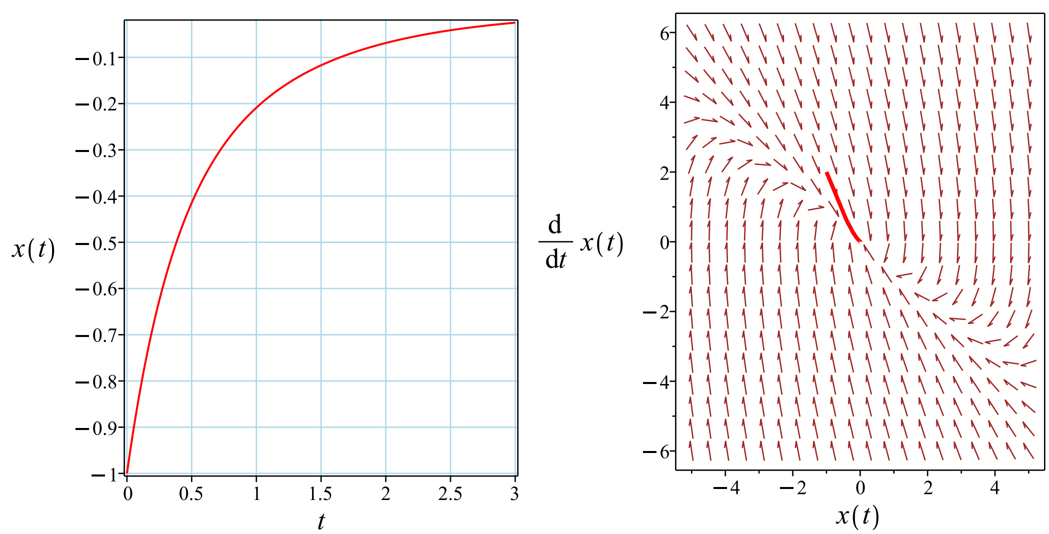
$$c_1 = -\frac{1}{2}$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$x = -\frac{e^{-t}}{2} - \frac{e^{-3t}}{2}$$

Summary
The solution(s) found are the following

$$x = -\frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = -\frac{e^{-t}}{2} - \frac{e^{-3t}}{2}$$

Verified OK.

6.8.4 Maple step by step solution

Let's solve

$$\left[x'' + 4x' + 3x = 0, x(0) = -1, x'\Big|_{\{t=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 + 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r+3)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-3, -1)$

- 1st solution of the ODE

  $x_1(t) = e^{-3t}$

- 2nd solution of the ODE

  $x_2(t) = e^{-t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 e^{-3t} + c_2 e^{-t}$

□ Check validity of solution $x = c_1 e^{-3t} + c_2 e^{-t}$

  - Use initial condition $x(0) = -1$

    $-1 = c_1 + c_2$

  - Compute derivative of the solution

    $x' = -3c_1 e^{-3t} - c_2 e^{-t}$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 2$

    $2 = -3c_1 - c_2$

  - Solve for c_1 and c_2

    $\left\{c_1 = -\frac{1}{2}, c_2 = -\frac{1}{2}\right\}$

- Substitute constant values into general solution and simplify

  $x = -\frac{e^{-t}}{2} - \frac{e^{-3t}}{2}$

- Solution to the IVP

  $x = -\frac{e^{-t}}{2} - \frac{e^{-3t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve([diff(x(t),t$2)+4*diff(x(t),t)+3*x(t)=0,x(0) = -1, D(x)(0) = 2],x(t), singsol=all)
```

$$x(t) = -\frac{e^{-3t}}{2} - \frac{e^{-t}}{2}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 21

```
DSolve[{x''[t]+4*x'[t]+3*x[t]==0,{x[0]==-1,x'[0]==2}},x[t],t,IncludeSingularSolutions -> Tru
```

$$x(t) \to -\frac{1}{2}e^{-3t}(e^{2t} + 1)$$

7 Chapter 2, Second order linear equations. Section 2.2.3 Complex eigenvalues. Exercises page 94

7.1 problem 1(a) 1033
7.2 problem 1(b) 1044
7.3 problem 1(c) 1054
7.4 problem 1(d) 1066
7.5 problem 1(e) 1079
7.6 problem 1(f) 1090

7.1 problem 1(a)

7.1.1 Existence and uniqueness analysis 1033
7.1.2 Solving as second order linear constant coeff ode 1034
7.1.3 Solving using Kovacic algorithm 1037
7.1.4 Maple step by step solution 1041

Internal problem ID [11444]
Internal file name [`OUTPUT/10426_Thursday_May_18_2023_04_19_11_AM_56841255/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.3 Complex eigenvalues. Exercises page 94
Problem number: 1(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{x'' + x' + 4x = 0}$$

With initial conditions

$$[x(0) = 1, x'(0) = 0]$$

7.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= 4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + x' + 4x = 0$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.1.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 1, C = 4$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \lambda \,\mathrm{e}^{\lambda t} + 4\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(4)}\\ &= -\frac{1}{2} \pm \frac{i\sqrt{15}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{15}}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{15}}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{15}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{15}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{15}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{15}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\, t}{2}\right)\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{15}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\, t}{2}\right)\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{\mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{15}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\, t}{2}\right)\right)}{2} + \mathrm{e}^{-\frac{t}{2}} \left(-\frac{c_1\sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{2}\right)}{2} + \frac{c_2\sqrt{15}\, \cos\left(\frac{\sqrt{15}\, t}{2}\right)}{2}\right)$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{c_1}{2} + \frac{\sqrt{15}\, c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = \frac{\sqrt{15}}{15}$$

Substituting these values back in above solution results in

$$x = \frac{e^{-\frac{t}{2}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{2}\right)}{15} + e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{15}\, t}{2}\right)$$

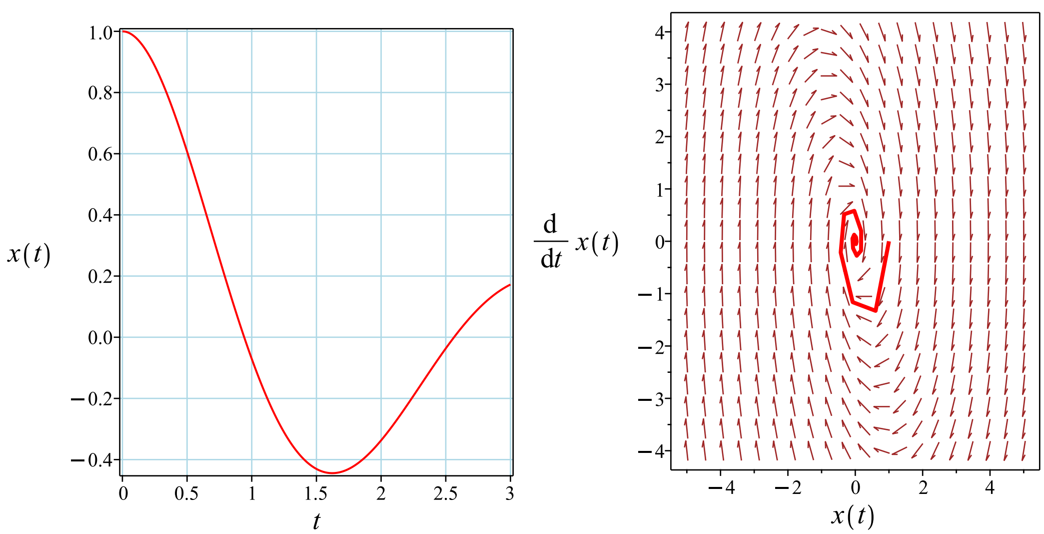
Which simplifies to

$$x = \frac{\left(\sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{2}\right) + 15 \cos\left(\frac{\sqrt{15}\, t}{2}\right)\right) e^{-\frac{t}{2}}}{15}$$

Summary

The solution(s) found are the following

$$x = \frac{\left(\sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{2}\right) + 15 \cos\left(\frac{\sqrt{15}\, t}{2}\right)\right) e^{-\frac{t}{2}}}{15} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \frac{\left(\sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{2}\right) + 15 \cos\left(\frac{\sqrt{15}\, t}{2}\right)\right) e^{-\frac{t}{2}}}{15}$$

Verified OK.

7.1.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' + 4x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-15}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -15 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{15z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\,e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 200: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{15}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos\left(\frac{\sqrt{15}\, t}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{15}\,t}{2}\right)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{2\sqrt{15}\,\tan\left(\frac{\sqrt{15}\,t}{2}\right)}{15}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{15}\,t}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{15}\,t}{2}\right) \left(\frac{2\sqrt{15}\,\tan\left(\frac{\sqrt{15}\,t}{2}\right)}{15}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{15}\, t}{2}\right)+\frac{2 c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{2}\right)}{15} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{15}\, t}{2}\right)}{2}-\frac{c_1 \mathrm{e}^{-\frac{t}{2}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{2}\right)}{2}-\frac{c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{2}\right)}{15}+c_2 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{15}\, t}{2}\right)$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{c_1}{2}+c_2 \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= \frac{1}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$x = \frac{\mathrm{e}^{-\frac{t}{2}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{2}\right)}{15}+\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{15}\, t}{2}\right)$$

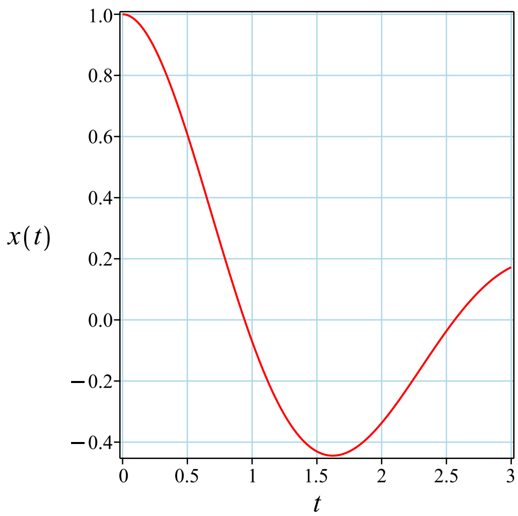
Which simplifies to

$$x = \frac{\left(\sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{2}\right)+15 \cos\left(\frac{\sqrt{15}\, t}{2}\right)\right) \mathrm{e}^{-\frac{t}{2}}}{15}$$

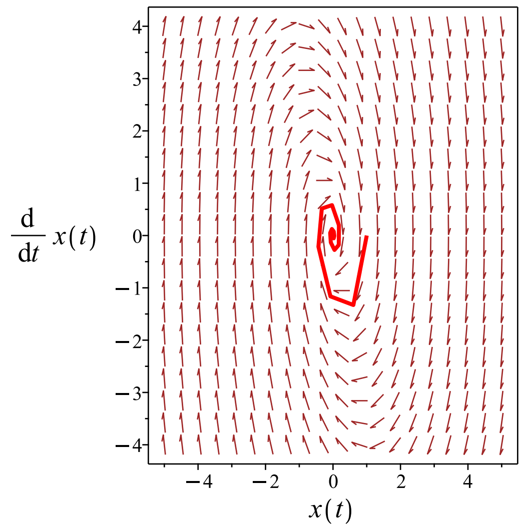
<u>Summary</u>

The solution(s) found are the following

$$x = \frac{\left(\sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{2}\right)+15 \cos\left(\frac{\sqrt{15}\, t}{2}\right)\right) \mathrm{e}^{-\frac{t}{2}}}{15} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right) + 15\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)\mathrm{e}^{-\frac{t}{2}}}{15}$$

Verified OK.

7.1.4 Maple step by step solution

Let's solve

$\left[x'' + x' + 4x = 0, x(0) = 1, x'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 + r + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-15})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{15}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{15}}{2}\right)$

- 1st solution of the ODE

  $x_1(t) = e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{15}\, t}{2}\right)$

- 2nd solution of the ODE

  $x_2(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{15}\, t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{15}\, t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{15}\, t}{2}\right) c_2$

☐ Check validity of solution $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{15}\, t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{15}\, t}{2}\right) c_2$

  - Use initial condition $x(0) = 1$

    $1 = c_1$

  - Compute derivative of the solution

    $x' = -\frac{c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{15}\, t}{2}\right)}{2} - \frac{c_1 e^{-\frac{t}{2}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{15}\, t}{2}\right) c_2}{2} + \frac{e^{-\frac{t}{2}} \sqrt{15}\, \cos\left(\frac{\sqrt{15}\, t}{2}\right) c_2}{2}$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 0$

    $0 = -\frac{c_1}{2} + \frac{\sqrt{15}\, c_2}{2}$

  - Solve for c_1 and c_2

    $\left\{c_1 = 1, c_2 = \frac{\sqrt{15}}{15}\right\}$

  - Substitute constant values into general solution and simplify

    $x = \frac{\left(\sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{2}\right) + 15 \cos\left(\frac{\sqrt{15}\, t}{2}\right)\right) e^{-\frac{t}{2}}}{15}$

- Solution to the IVP

  $x = \frac{\left(\sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{2}\right) + 15 \cos\left(\frac{\sqrt{15}\, t}{2}\right)\right) e^{-\frac{t}{2}}}{15}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 31

```
dsolve([diff(x(t),t$2)+diff(x(t),t)+4*x(t)=0,x(0) = 1, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \frac{\mathrm{e}^{-\frac{t}{2}} \left(\sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{2}\right) + 15 \cos\left(\frac{\sqrt{15}\, t}{2}\right)\right)}{15}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 47

```
DSolve[{x''[t]+x'[t]+4*x[t]==0,{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{15} e^{-t/2} \left(\sqrt{15} \sin\left(\frac{\sqrt{15}t}{2}\right) + 15 \cos\left(\frac{\sqrt{15}t}{2}\right)\right)$$

7.2 problem 1(b)

7.2.1 Existence and uniqueness analysis 1044
7.2.2 Solving as second order linear constant coeff ode 1045
7.2.3 Solving using Kovacic algorithm 1048
7.2.4 Maple step by step solution 1052

Internal problem ID [11445]
Internal file name [`OUTPUT/10427_Thursday_May_18_2023_04_19_12_AM_23116747/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.3 Complex eigenvalues. Exercises page 94
Problem number: 1(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' - 4x' + 6x = 0}$$

With initial conditions

$$[x(0) = 1, x'(0) = 0]$$

7.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 6 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' - 4x' + 6x = 0$$

The domain of $p(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.2.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -4, C = 6$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - 4\lambda\, e^{\lambda t} + 6\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 4\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(6)}\\ &= 2 \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 2 + i\sqrt{2}\\ \lambda_2 &= 2 - i\sqrt{2}\end{aligned}$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 2 + i\sqrt{2} \\
\lambda_2 &= 2 - i\sqrt{2}
\end{aligned}
$$

Since roots are complex conjugate of each others, then let the roots be

$$
\lambda_{1,2} = \alpha \pm i\beta
$$

Where $\alpha = 2$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$
x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))
$$

Which becomes

$$
x = e^{2t}\left(c_1 \cos\left(\sqrt{2}\, t\right) + c_2 \sin\left(\sqrt{2}\, t\right)\right)
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = \mathrm{e}^{2t}\left(c_1 \cos\left(\sqrt{2}\, t\right) + c_2 \sin\left(\sqrt{2}\, t\right)\right) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = 2\,\mathrm{e}^{2t}\left(c_1 \cos\left(\sqrt{2}\, t\right) + c_2 \sin\left(\sqrt{2}\, t\right)\right) + \mathrm{e}^{2t}\left(-c_1\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) + c_2\sqrt{2}\, \cos\left(\sqrt{2}\, t\right)\right)
$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$
0 = 2c_1 + \sqrt{2}\, c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= -\sqrt{2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$x = -\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) e^{2t} + e^{2t} \cos\left(\sqrt{2}\, t\right)$$

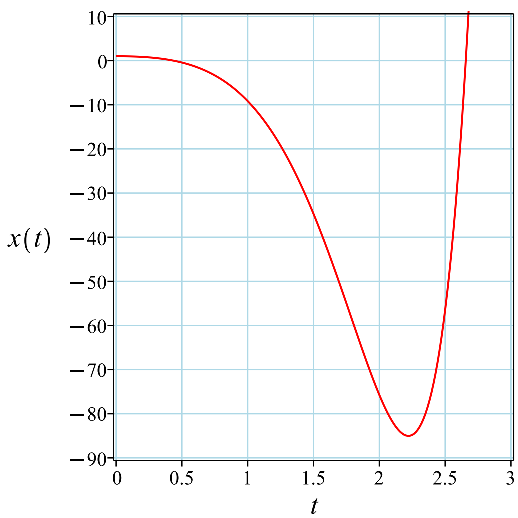
Which simplifies to

$$x = \left(-\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) + \cos\left(\sqrt{2}\, t\right)\right) e^{2t}$$

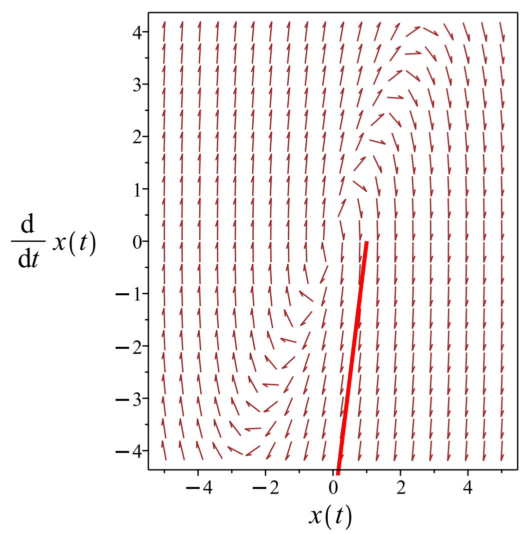
Summary

The solution(s) found are the following

$$x = \left(-\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) + \cos\left(\sqrt{2}\, t\right)\right) e^{2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \left(-\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) + \cos\left(\sqrt{2}\, t\right)\right) e^{2t}$$

Verified OK.

7.2.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 4x' + 6x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -2z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 202: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\sqrt{2}\, t\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

$$
\begin{aligned}
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dt} \\
&= z_1 e^{2t} \\
&= z_1\left(\mathrm{e}^{2t}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{2t} \cos\left(\sqrt{2}\, t\right)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-4}{1}\,dt}}{\left(x_1\right)^2}\,dt \\
&= x_1 \int \frac{e^{4t}}{\left(x_1\right)^2}\,dt \\
&= x_1 \left(\frac{\sqrt{2}\,\tan\left(\sqrt{2}\,t\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\mathrm{e}^{2t} \cos\left(\sqrt{2}\, t\right)\right) + c_2\left(\mathrm{e}^{2t} \cos\left(\sqrt{2}\, t\right)\left(\frac{\sqrt{2}\,\tan\left(\sqrt{2}\,t\right)}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 \mathrm{e}^{2t} \cos\left(\sqrt{2}\, t\right) + \frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)\mathrm{e}^{2t}}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$x' = 2c_1 e^{2t} \cos\left(\sqrt{2}\, t\right) - c_1 e^{2t} \sqrt{2}\, \sin\left(\sqrt{2}\, t\right) + c_2 \cos\left(\sqrt{2}\, t\right) e^{2t} + c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, t\right) e^{2t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = 2c_1 + c_2 \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -2$$

Substituting these values back in above solution results in

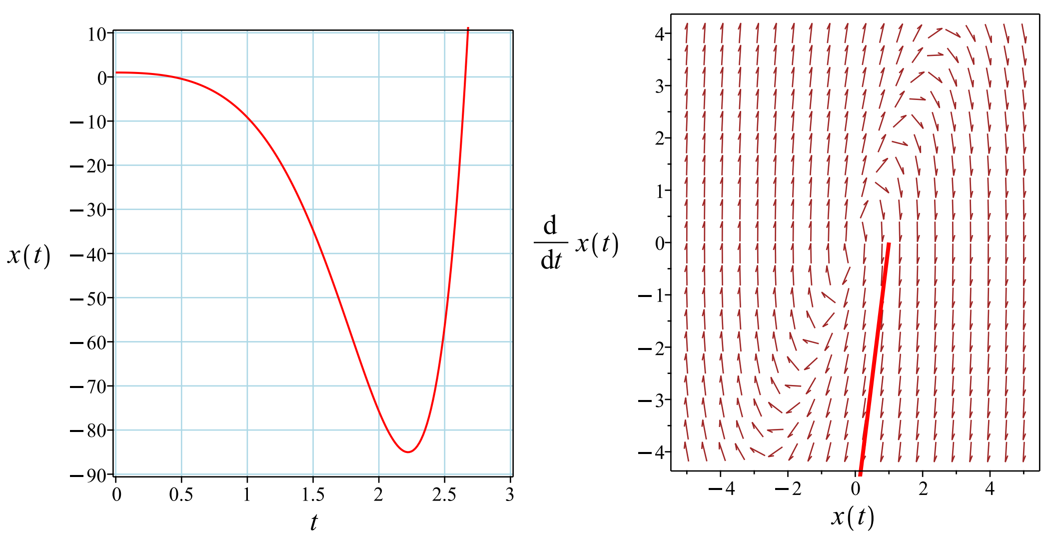
$$x = -\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) e^{2t} + e^{2t} \cos\left(\sqrt{2}\, t\right)$$

Which simplifies to

$$x = \left(-\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) + \cos\left(\sqrt{2}\, t\right)\right) e^{2t}$$

Summary
The solution(s) found are the following

$$x = \left(-\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) + \cos\left(\sqrt{2}\, t\right)\right) e^{2t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \left(-\sqrt{2}\,\sin\left(\sqrt{2}\,t\right) + \cos\left(\sqrt{2}\,t\right)\right) e^{2t}$$

Verified OK.

7.2.4 Maple step by step solution

Let's solve

$\left[x'' - 4x' + 6x = 0, x(0) = 1, x'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 - 4r + 6 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4 \pm (\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

  $r = \left(2 - \mathrm{I}\sqrt{2}, 2 + \mathrm{I}\sqrt{2}\right)$

- 1st solution of the ODE

  $x_1(t) = e^{2t}\cos\left(\sqrt{2}\,t\right)$

- 2nd solution of the ODE

  $x_2(t) = e^{2t}\sin\left(\sqrt{2}\,t\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 e^{2t}\cos\left(\sqrt{2}\,t\right) + \sin\left(\sqrt{2}\,t\right) e^{2t} c_2$

□ Check validity of solution $x = c_1 e^{2t}\cos\left(\sqrt{2}\,t\right) + \sin\left(\sqrt{2}\,t\right) e^{2t} c_2$

  - Use initial condition $x(0) = 1$

    $1 = c_1$

  - Compute derivative of the solution

    $x' = 2c_1 e^{2t}\cos\left(\sqrt{2}\,t\right) - c_1 e^{2t}\sqrt{2}\,\sin\left(\sqrt{2}\,t\right) + \sqrt{2}\,\cos\left(\sqrt{2}\,t\right) e^{2t} c_2 + 2\sin\left(\sqrt{2}\,t\right) e^{2t} c_2$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = 2c_1 + \sqrt{2}\, c_2$
- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = -\sqrt{2}\}$
- Substitute constant values into general solution and simplify

  $x = \left(-\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) + \cos\left(\sqrt{2}\, t\right)\right) \mathrm{e}^{2t}$

- Solution to the IVP

  $x = \left(-\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) + \cos\left(\sqrt{2}\, t\right)\right) \mathrm{e}^{2t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 27

```
dsolve([diff(x(t),t$2)-4*diff(x(t),t)+6*x(t)=0,x(0) = 1, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \mathrm{e}^{2t}\left(-\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) + \cos\left(\sqrt{2}\, t\right)\right)$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 35

```
DSolve[{x''[t]-4*x'[t]+6*x[t]==0,{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True
```

$$x(t) \to e^{2t}\left(\cos\left(\sqrt{2}t\right) - \sqrt{2}\sin\left(\sqrt{2}t\right)\right)$$

7.3 problem 1(c)

7.3.1 Existence and uniqueness analysis 1054
7.3.2 Solving as second order linear constant coeff ode 1055
7.3.3 Solving as second order ode can be made integrable ode 1057
7.3.4 Solving using Kovacic algorithm 1060
7.3.5 Maple step by step solution 1064

Internal problem ID [11446]
Internal file name [`OUTPUT/10428_Thursday_May_18_2023_04_19_14_AM_33206177/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.3 Complex eigenvalues. Exercises page 94
Problem number: 1(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{x'' + 9x = 0}$$

With initial conditions

$$[x(0) = 1, x'(0) = 0]$$

7.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 9 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + 9x = 0$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.3.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 9\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(9)} \\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i \\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\lambda_1 = 3i$$
$$\lambda_2 = -3i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{0}(c_1 \cos(3t) + c_2 \sin(3t))$$

Or

$$x = c_1 \cos(3t) + c_2 \sin(3t)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \cos(3t) + c_2 \sin(3t) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -3c_1 \sin(3t) + 3c_2 \cos(3t)$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 0$$

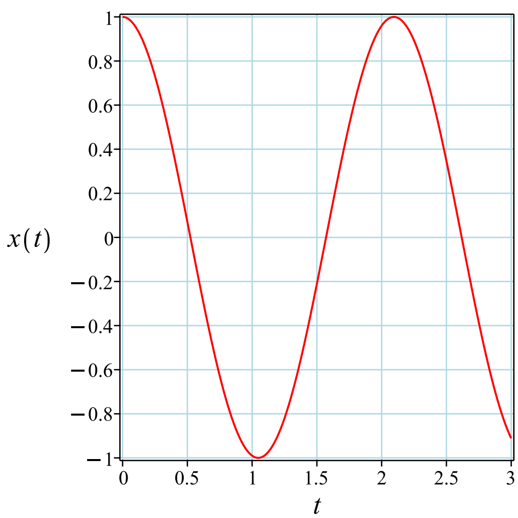
Substituting these values back in above solution results in

$$x = \cos(3t)$$

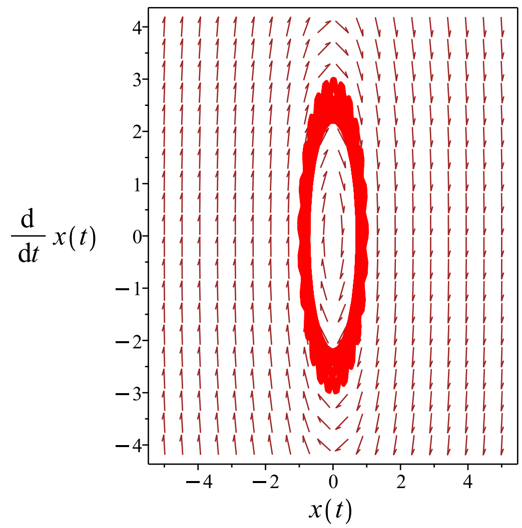
<u>Summary</u>

The solution(s) found are the following

$$x = \cos(3t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$x = \cos(3t)$$

Verified OK.

7.3.3 Solving as second order ode can be made integrable ode

Multiplying the ode by x' gives

$$x'x'' + 9x'x = 0$$

Integrating the above w.r.t t gives

$$\int (x'x'' + 9x'x)\, dt = 0$$

$$\frac{x'^2}{2} + \frac{9x^2}{2} = c_2$$

Which is now solved for x. Solving the given ode for x' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$x' = \sqrt{-9x^2 + 2c_1} \tag{1}$$
$$x' = -\sqrt{-9x^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-9x^2 + 2c_1}} dx = \int dt$$
$$\frac{\arctan\left(\frac{3x}{\sqrt{-9x^2+2c_1}}\right)}{3} = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-9x^2 + 2c_1}} dx = \int dt$$
$$-\frac{\arctan\left(\frac{3x}{\sqrt{-9x^2+2c_1}}\right)}{3} = t + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\arctan\left(\frac{3x}{\sqrt{-9x^2+2c_1}}\right)}{3} = t + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$\frac{\arctan\left(\frac{3}{\sqrt{-9+2c_1}}\right)}{3} = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{\left(3\tan\left(3t+3c_2\right)^2+3\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(3t+3c_2)^2+1}}}{3} - \frac{\tan\left(3t+3c_2\right)^2\sqrt{2}\,c_1\left(3\tan\left(3t+3c_2\right)^2+3\right)}{3\sqrt{\frac{c_1}{\tan(3t+3c_2)^2+1}}\,\left(\tan\left(3t+3c_2\right)^2+1\right)^2}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = \frac{\cos\left(3c_2\right)^2\sqrt{2}\,c_1}{\sqrt{\cos\left(3c_2\right)^2 c_1}} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$-\frac{\arctan\left(\frac{3x}{\sqrt{-9x^2+2c_1}}\right)}{3} = t + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$-\frac{\arctan\left(\frac{3}{\sqrt{-9+2c_1}}\right)}{3} = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{\left(3\tan\left(3t+3c_3\right)^2+3\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(3t+3c_3)^2+1}}}{3} + \frac{\tan\left(3t+3c_3\right)^2\sqrt{2}\,c_1\left(3\tan\left(3t+3c_3\right)^2+3\right)}{3\sqrt{\frac{c_1}{\tan(3t+3c_3)^2+1}}\,\left(\tan\left(3t+3c_3\right)^2+1\right)^2}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{\cos\left(3c_3\right)^2\sqrt{2}\,c_1}{\sqrt{\cos\left(3c_3\right)^2 c_1}} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

<u>Verification of solutions</u> N/A

7.3.4 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 9x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 204: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(3t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
x_1 &= z_1 \\
&= \cos(3t)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \cos(3t)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2}\,dt \\
&= \cos(3t) \int \frac{1}{\cos(3t)^2}\,dt \\
&= \cos(3t) \left(\frac{\tan(3t)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(\cos(3t)) + c_2\left(\cos(3t)\left(\frac{\tan(3t)}{3}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$x' = -3c_1 \sin(3t) + c_2 \cos(3t)$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

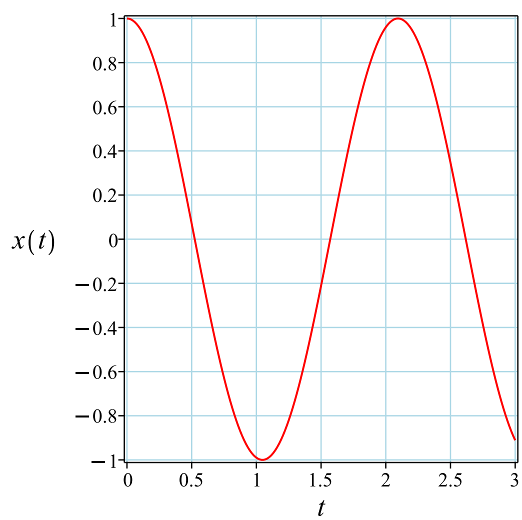
$$c_1 = 1$$
$$c_2 = 0$$

Substituting these values back in above solution results in

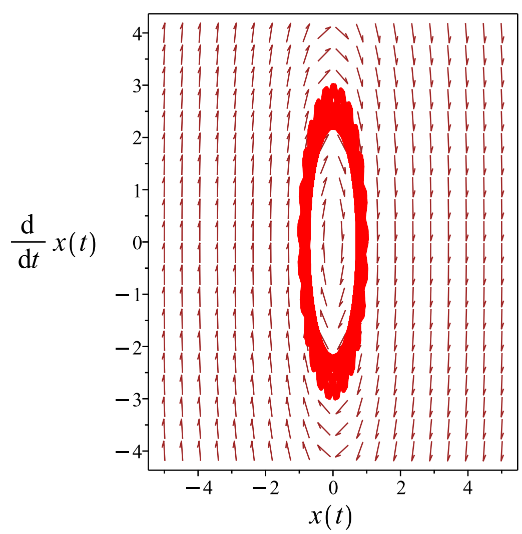
$$x = \cos(3t)$$

Summary
The solution(s) found are the following

$$x = \cos(3t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \cos(3t)$$

Verified OK.

7.3.5 Maple step by step solution

Let's solve

$$\left[x'' + 9x = 0, x(0) = 1, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the ODE

  $x_1(t) = \cos(3t)$

- 2nd solution of the ODE

  $x_2(t) = \sin(3t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 \cos(3t) + c_2 \sin(3t)$

□ Check validity of solution $x = c_1 \cos(3t) + c_2 \sin(3t)$

  - Use initial condition $x(0) = 1$

    $1 = c_1$

  - Compute derivative of the solution

    $x' = -3c_1 \sin(3t) + 3c_2 \cos(3t)$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 0$

    $0 = 3c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = 0\}$

- ◦ Substitute constant values into general solution and simplify

  $$x = \cos(3t)$$

- Solution to the IVP

  $$x = \cos(3t)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 8

```
dsolve([diff(x(t),t$2)+9*x(t)=0,x(0) = 1, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \cos(3t)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 9

```
DSolve[{x''[t]+9*x[t]==0,{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \cos(3t)$$

7.4 problem 1(d)

7.4.1 Existence and uniqueness analysis 1066
7.4.2 Solving as second order linear constant coeff ode 1067
7.4.3 Solving as second order ode can be made integrable ode 1069
7.4.4 Solving using Kovacic algorithm 1073
7.4.5 Maple step by step solution 1077

Internal problem ID [11447]
Internal file name [`OUTPUT/10429_Thursday_May_18_2023_04_19_15_AM_24766642/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.3 Complex eigenvalues. Exercises page 94
Problem number: 1(d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{x'' - 12x = 0}$$

With initial conditions

$$[x(0) = 1, x'(0) = 0]$$

7.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= -12 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' - 12x = 0$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -12$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.4.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = -12$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - 12\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 12 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -12$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-12)}\\ &= \pm 2\sqrt{3}\end{aligned}$$

Hence

$$\lambda_1 = +2\sqrt{3}$$
$$\lambda_2 = -2\sqrt{3}$$

Which simplifies to

$$\lambda_1 = 2\sqrt{3}$$
$$\lambda_2 = -2\sqrt{3}$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$x = c_1 e^{\left(2\sqrt{3}\right)t} + c_2 e^{\left(-2\sqrt{3}\right)t}$$

Or

$$x = c_1 \mathrm{e}^{2\sqrt{3}\,t} + c_2 \mathrm{e}^{-2\sqrt{3}\,t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{2\sqrt{3}\,t} + c_2 \mathrm{e}^{-2\sqrt{3}\,t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = 2c_1\sqrt{3}\,\mathrm{e}^{2\sqrt{3}\,t} - 2c_2\sqrt{3}\,\mathrm{e}^{-2\sqrt{3}\,t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = 2\sqrt{3}\,(c_1 - c_2) \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{2}$$
$$c_2 = \frac{1}{2}$$

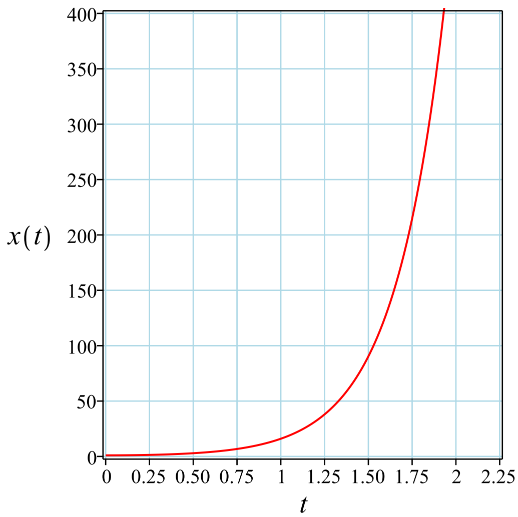
Substituting these values back in above solution results in

$$x = \frac{\mathrm{e}^{2\sqrt{3}\,t}}{2} + \frac{\mathrm{e}^{-2\sqrt{3}\,t}}{2}$$

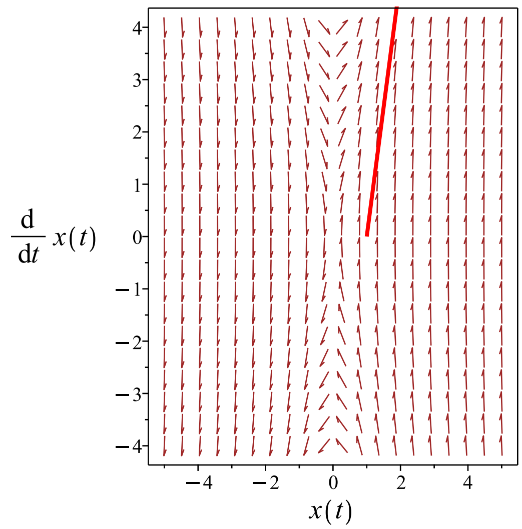
Summary

The solution(s) found are the following

$$x = \frac{e^{2\sqrt{3}\, t}}{2}+\frac{e^{-2\sqrt{3}\, t}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{e^{2\sqrt{3}\, t}}{2}+\frac{e^{-2\sqrt{3}\, t}}{2}$$

Verified OK.

7.4.3 Solving as second order ode can be made integrable ode

Multiplying the ode by x' gives

$$x'x'' - 12x'x = 0$$

Integrating the above w.r.t t gives

$$\int (x'x'' - 12x'x)\, dt = 0$$
$$\frac{x'^2}{2} - 6x^2 = c_2$$

Which is now solved for x. Solving the given ode for x' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$x' = \sqrt{12x^2 + 2c_1} \tag{1}$$
$$x' = -\sqrt{12x^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\int \frac{1}{\sqrt{12x^2 + 2c_1}} dx = \int dt$$
$$\frac{\ln\left(x\sqrt{12} + \sqrt{12x^2 + 2c_1}\right)\sqrt{12}}{12} = t + c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln\left(x\sqrt{12} + \sqrt{12x^2 + 2c_1}\right)\sqrt{12}}{12}} = e^{t + c_2}$$

Which simplifies to

$$\left(2\sqrt{3}\, x + \sqrt{12x^2 + 2c_1}\right)^{\frac{\sqrt{3}}{6}} = c_3 e^{t}$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\int -\frac{1}{\sqrt{12x^2 + 2c_1}} dx = \int dt$$
$$-\frac{\ln\left(x\sqrt{12} + \sqrt{12x^2 + 2c_1}\right)\sqrt{12}}{12} = t + c_4$$

Raising both side to exponential gives

$$e^{-\frac{\ln\left(x\sqrt{12} + \sqrt{12x^2 + 2c_1}\right)\sqrt{12}}{12}} = e^{t + c_4}$$

Which simplifies to

$$\left(2\sqrt{3}\, x + \sqrt{12x^2 + 2c_1}\right)^{-\frac{\sqrt{3}}{6}} = c_5 e^{t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$x = \frac{\left(\left(c_3 \mathrm{e}^{t}\right)^{4\sqrt{3}} - 2c_1\right)\sqrt{3}\left(c_3 \mathrm{e}^{t}\right)^{-2\sqrt{3}}}{12} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = \frac{\left(-2c_3^{-2\sqrt{3}}c_1 + c_3^{2\sqrt{3}}\right)\sqrt{3}}{12} \tag{1A}$$

Taking derivative of the solution gives

$$x' = \left(c_3 \mathrm{e}^{t}\right)^{2\sqrt{3}} - \frac{\left(\left(c_3 \mathrm{e}^{t}\right)^{4\sqrt{3}} - 2c_1\right)\left(c_3 \mathrm{e}^{t}\right)^{-2\sqrt{3}}}{2}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = \frac{c_3^{2\sqrt{3}}}{2} + c_3^{-2\sqrt{3}}c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -6 \\ c_3 &= 2^{\frac{\sqrt{3}}{6}} 3^{\frac{\sqrt{3}}{12}} \end{aligned}$$

Substituting these values back in above solution results in

$$x = \frac{\left(\mathrm{e}^{t}\right)^{2\sqrt{3}}}{2} + \frac{\left(\mathrm{e}^{t}\right)^{-2\sqrt{3}}}{2}$$

Looking at the Second solution

$$x = \frac{\left(\left(c_5 \mathrm{e}^{t}\right)^{-4\sqrt{3}} - 2c_1\right)\left(c_5 \mathrm{e}^{t}\right)^{2\sqrt{3}}\sqrt{3}}{12} \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = -\frac{\left(2c_5^{2\sqrt{3}}c_1 - c_5^{-2\sqrt{3}}\right)\sqrt{3}}{12} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\left(c_5 \mathrm{e}^t\right)^{-2\sqrt{3}} + \frac{\left(\left(c_5 \mathrm{e}^t\right)^{-4\sqrt{3}} - 2c_1\right)\left(c_5 \mathrm{e}^t\right)^{2\sqrt{3}}}{2}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{c_5^{-2\sqrt{3}}}{2} - c_5^{2\sqrt{3}} c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_5\}$. Solving for the constants gives

$$c_1 = -6$$
$$c_5 = 2^{-\frac{\sqrt{3}}{6}} 3^{-\frac{\sqrt{3}}{12}}$$

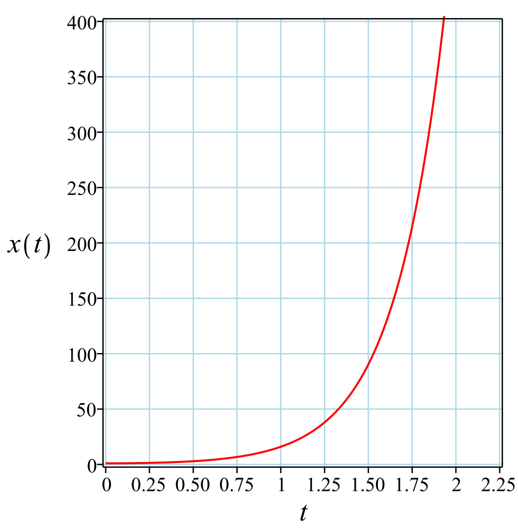
Substituting these values back in above solution results in

$$x = \frac{\left(\mathrm{e}^t\right)^{2\sqrt{3}}}{2} + \frac{\left(\mathrm{e}^t\right)^{-2\sqrt{3}}}{2}$$

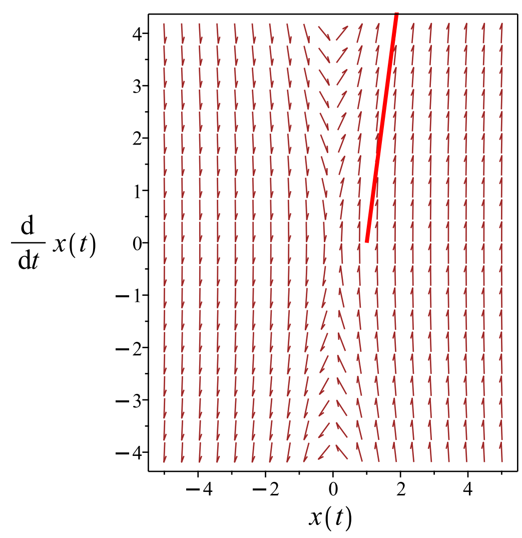
Summary

The solution(s) found are the following

$$x = \frac{\left(\mathrm{e}^t\right)^{2\sqrt{3}}}{2} + \frac{\left(\mathrm{e}^t\right)^{-2\sqrt{3}}}{2} \tag{1}$$
$$x = \frac{\left(\mathrm{e}^t\right)^{2\sqrt{3}}}{2} + \frac{\left(\mathrm{e}^t\right)^{-2\sqrt{3}}}{2} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{(e^t)^{2\sqrt{3}}}{2} + \frac{(e^t)^{-2\sqrt{3}}}{2}$$

Verified OK.

$$x = \frac{(e^t)^{2\sqrt{3}}}{2} + \frac{(e^t)^{-2\sqrt{3}}}{2}$$

Verified OK.

7.4.4 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 12x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{12}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 12 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 12z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 206: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 12$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-2\sqrt{3}\,t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}x_1 &= z_1 \\ &= \mathrm{e}^{-2\sqrt{3}\,t}\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-2\sqrt{3}\,t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}x_2 &= x_1 \int \frac{1}{x_1^2}\,dt \\ &= \mathrm{e}^{-2\sqrt{3}\,t} \int \frac{1}{\mathrm{e}^{-4\sqrt{3}\,t}}\,dt \\ &= \mathrm{e}^{-2\sqrt{3}\,t} \left(\frac{\mathrm{e}^{4\sqrt{3}\,t}\sqrt{3}}{12}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}x &= c_1 x_1 + c_2 x_2 \\ &= c_1\left(\mathrm{e}^{-2\sqrt{3}\,t}\right) + c_2\left(\mathrm{e}^{-2\sqrt{3}\,t}\left(\frac{\mathrm{e}^{4\sqrt{3}\,t}\sqrt{3}}{12}\right)\right)\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-2\sqrt{3}\,t} + \frac{c_2\sqrt{3}\,\mathrm{e}^{2\sqrt{3}\,t}}{12} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 + \frac{c_2\sqrt{3}}{12} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -2c_1\sqrt{3}\,\mathrm{e}^{-2\sqrt{3}\,t} + \frac{c_2 \mathrm{e}^{2\sqrt{3}\,t}}{2}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -2\sqrt{3}\,c_1 + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{2}$$
$$c_2 = 2\sqrt{3}$$

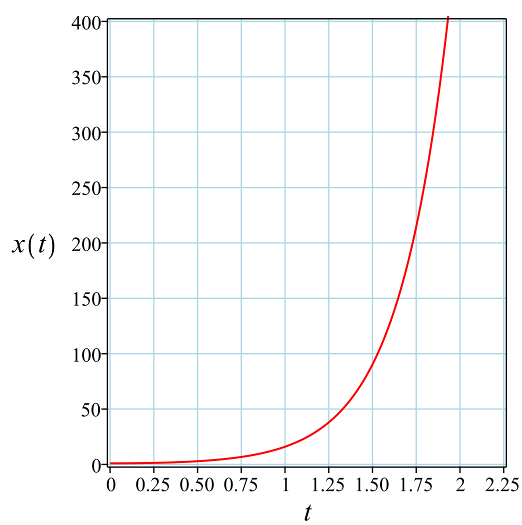
Substituting these values back in above solution results in

$$x = \frac{\mathrm{e}^{2\sqrt{3}\,t}}{2} + \frac{\mathrm{e}^{-2\sqrt{3}\,t}}{2}$$

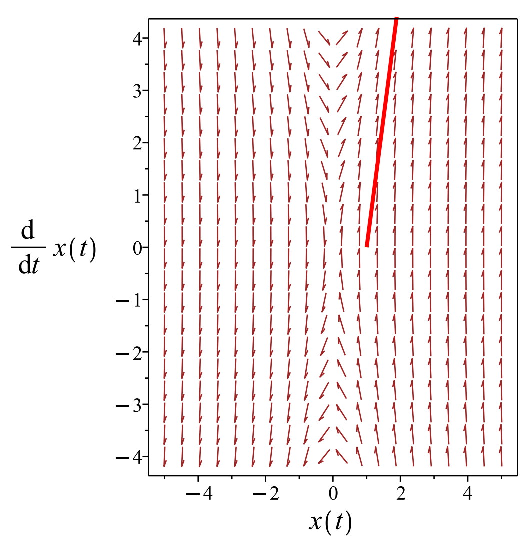
Summary

The solution(s) found are the following

$$x = \frac{\mathrm{e}^{2\sqrt{3}\,t}}{2} + \frac{\mathrm{e}^{-2\sqrt{3}\,t}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{e^{2\sqrt{3}\,t}}{2} + \frac{e^{-2\sqrt{3}\,t}}{2}$$

Verified OK.

7.4.5 Maple step by step solution

Let's solve

$\left[x'' - 12x = 0, x(0) = 1, x'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 - 12 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{48}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-2\sqrt{3}, 2\sqrt{3}\right)$

- 1st solution of the ODE

  $x_1(t) = e^{-2\sqrt{3}\,t}$

- 2nd solution of the ODE

  $x_2(t) = e^{2\sqrt{3}\,t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 e^{-2\sqrt{3}\,t} + c_2 e^{2\sqrt{3}\,t}$

□ Check validity of solution $x = c_1 e^{-2\sqrt{3}\,t} + c_2 e^{2\sqrt{3}\,t}$

  - Use initial condition $x(0) = 1$

    $1 = c_1 + c_2$

  - Compute derivative of the solution

    $x' = -2c_1\sqrt{3}\, e^{-2\sqrt{3}\,t} + 2c_2\sqrt{3}\, e^{2\sqrt{3}\,t}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = -2\sqrt{3}\, c_1 + 2c_2\sqrt{3}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{2}, c_2 = \frac{1}{2}\right\}$

- Substitute constant values into general solution and simplify

  $x = \frac{e^{2\sqrt{3}\, t}}{2} + \frac{e^{-2\sqrt{3}\, t}}{2}$

- Solution to the IVP

  $x = \frac{e^{2\sqrt{3}\, t}}{2} + \frac{e^{-2\sqrt{3}\, t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 23

```
dsolve([diff(x(t),t$2)-12*x(t)=0,x(0) = 1, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \frac{e^{2\sqrt{3}\, t}}{2} + \frac{e^{-2\sqrt{3}\, t}}{2}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 31

```
DSolve[{x''[t]-12*x[t]==0,{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2} e^{-2\sqrt{3}t}\left(e^{4\sqrt{3}t} + 1\right)$$

7.5 problem 1(e)

7.5.1 Existence and uniqueness analysis 1079
7.5.2 Solving as second order linear constant coeff ode 1080
7.5.3 Solving using Kovacic algorithm 1083
7.5.4 Maple step by step solution 1087

Internal problem ID [11448]
Internal file name [`OUTPUT/10430_Thursday_May_18_2023_04_19_17_AM_20027788/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.3 Complex eigenvalues. Exercises page 94
Problem number: 1(e).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{2x'' + 3x' + 3x = 0}$$

With initial conditions

$$[x(0) = 1, x'(0) = 0]$$

7.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= \frac{3}{2} \\ q(t) &= \frac{3}{2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + \frac{3x'}{2} + \frac{3x}{2} = 0$$

The domain of $p(t) = \frac{3}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{3}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.5.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 2, B = 3, C = 3$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$2\lambda^2 \mathrm{e}^{\lambda t} + 3\lambda \, \mathrm{e}^{\lambda t} + 3 \, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$2\lambda^2 + 3\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 3, C = 3$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-3}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{3^2 - (4)\,(2)\,(3)} \\
&= -\frac{3}{4} \pm \frac{i\sqrt{15}}{4}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{3}{4} + \frac{i\sqrt{15}}{4} \\
\lambda_2 &= -\frac{3}{4} - \frac{i\sqrt{15}}{4}
\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{3}{4} + \frac{i\sqrt{15}}{4}$$
$$\lambda_2 = -\frac{3}{4} - \frac{i\sqrt{15}}{4}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{3}{4}$ and $\beta = \frac{\sqrt{15}}{4}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{3t}{4}}\left(c_1 \cos\left(\frac{\sqrt{15}\,t}{4}\right) + c_2 \sin\left(\frac{\sqrt{15}\,t}{4}\right)\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \mathrm{e}^{-\frac{3t}{4}}\left(c_1 \cos\left(\frac{\sqrt{15}\,t}{4}\right) + c_2 \sin\left(\frac{\sqrt{15}\,t}{4}\right)\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{3\,\mathrm{e}^{-\frac{3t}{4}}\left(c_1 \cos\left(\frac{\sqrt{15}\,t}{4}\right) + c_2 \sin\left(\frac{\sqrt{15}\,t}{4}\right)\right)}{4} + \mathrm{e}^{-\frac{3t}{4}}\left(-\frac{c_1\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{4}\right)}{4} + \frac{c_2\sqrt{15}\,\cos\left(\frac{\sqrt{15}\,t}{4}\right)}{4}\right)$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{3c_1}{4} + \frac{\sqrt{15}\,c_2}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = \frac{\sqrt{15}}{5}$$

Substituting these values back in above solution results in

$$x = \frac{\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{4}\right) \mathrm{e}^{-\frac{3t}{4}}}{5} + \mathrm{e}^{-\frac{3t}{4}} \cos\left(\frac{\sqrt{15}\,t}{4}\right)$$

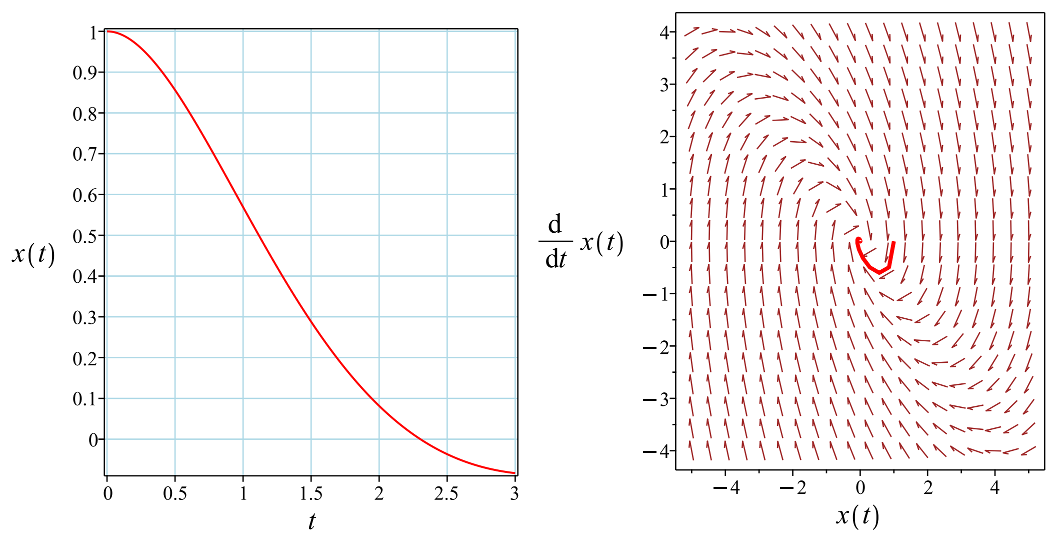
Which simplifies to

$$x = \frac{\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{4}\right) + 5\cos\left(\frac{\sqrt{15}\,t}{4}\right)\right) \mathrm{e}^{-\frac{3t}{4}}}{5}$$

Summary

The solution(s) found are the following

$$x = \frac{\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{4}\right) + 5\cos\left(\frac{\sqrt{15}\,t}{4}\right)\right) \mathrm{e}^{-\frac{3t}{4}}}{5} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \frac{\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{4}\right) + 5\cos\left(\frac{\sqrt{15}\,t}{4}\right)\right) \mathrm{e}^{-\frac{3t}{4}}}{5}$$

Verified OK.

7.5.3 Solving using Kovacic algorithm

Writing the ode as

$$2x'' + 3x' + 3x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 3 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-15}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -15 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{15z(t)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 208: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{15}{16}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos\left(\frac{\sqrt{15}\,t}{4}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{2}\,dt} \\
&= z_1 e^{-\frac{3t}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3t}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-\frac{3t}{4}} \cos\left(\frac{\sqrt{15}\,t}{4}\right)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{3}{2}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-\frac{3t}{2}}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{4\sqrt{15}\,\tan\left(\frac{\sqrt{15}\,t}{4}\right)}{15}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{3t}{4}} \cos\left(\frac{\sqrt{15}\,t}{4}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{3t}{4}} \cos\left(\frac{\sqrt{15}\,t}{4}\right) \left(\frac{4\sqrt{15}\,\tan\left(\frac{\sqrt{15}\,t}{4}\right)}{15}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{-\frac{3t}{4}} \cos\left(\frac{\sqrt{15}\, t}{4}\right) + \frac{4 c_2 \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{4}\right) e^{-\frac{3t}{4}}}{15} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{3 c_1 e^{-\frac{3t}{4}} \cos\left(\frac{\sqrt{15}\, t}{4}\right)}{4} - \frac{c_1 e^{-\frac{3t}{4}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{4}\right)}{4} + c_2 \cos\left(\frac{\sqrt{15}\, t}{4}\right) e^{-\frac{3t}{4}} - \frac{c_2 \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{4}\right) e^{-\frac{3t}{4}}}{5}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{3 c_1}{4} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= \frac{3}{4} \end{aligned}$$

Substituting these values back in above solution results in

$$x = \frac{\sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{4}\right) e^{-\frac{3t}{4}}}{5} + e^{-\frac{3t}{4}} \cos\left(\frac{\sqrt{15}\, t}{4}\right)$$

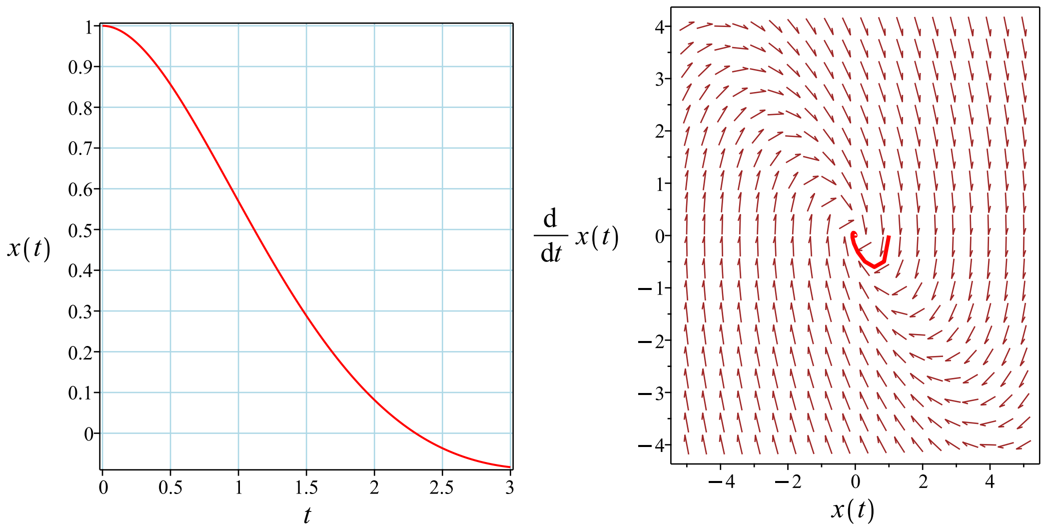
Which simplifies to

$$x = \frac{\left(\sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{4}\right) + 5 \cos\left(\frac{\sqrt{15}\, t}{4}\right)\right) e^{-\frac{3t}{4}}}{5}$$

Summary

The solution(s) found are the following

$$x = \frac{\left(\sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{4}\right) + 5 \cos\left(\frac{\sqrt{15}\, t}{4}\right)\right) e^{-\frac{3t}{4}}}{5} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$x = \frac{\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{4}\right) + 5\cos\left(\frac{\sqrt{15}\,t}{4}\right)\right)\mathrm{e}^{-\frac{3t}{4}}}{5}$$

Verified OK.

7.5.4 Maple step by step solution

Let's solve

$\left[2x'' + 3x' + 3x = 0, x(0) = 1, x'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = -\frac{3x'}{2} - \frac{3x}{2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + \frac{3x'}{2} + \frac{3x}{2} = 0$

- Characteristic polynomial of ODE

  $r^2 + \frac{3}{2}r + \frac{3}{2} = 0$

- Use quadratic formula to solve for r

  $r = \frac{\left(-\frac{3}{2}\right)\pm\left(\sqrt{-\frac{15}{4}}\right)}{2}$
- Roots of the characteristic polynomial

  $r = \left(-\frac{3}{4}-\frac{\mathrm{I}\sqrt{15}}{4}, -\frac{3}{4}+\frac{\mathrm{I}\sqrt{15}}{4}\right)$
- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^{-\frac{3t}{4}} \cos\left(\frac{\sqrt{15}\,t}{4}\right)$
- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{-\frac{3t}{4}} \sin\left(\frac{\sqrt{15}\,t}{4}\right)$
- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$
- Substitute in solutions

  $x = c_1 \mathrm{e}^{-\frac{3t}{4}} \cos\left(\frac{\sqrt{15}\,t}{4}\right) + \sin\left(\frac{\sqrt{15}\,t}{4}\right) \mathrm{e}^{-\frac{3t}{4}} c_2$

☐ Check validity of solution $x = c_1 \mathrm{e}^{-\frac{3t}{4}} \cos\left(\frac{\sqrt{15}\,t}{4}\right) + \sin\left(\frac{\sqrt{15}\,t}{4}\right) \mathrm{e}^{-\frac{3t}{4}} c_2$

  - Use initial condition $x(0) = 1$

    $1 = c_1$
  - Compute derivative of the solution

    $x' = -\frac{3c_1 \mathrm{e}^{-\frac{3t}{4}} \cos\left(\frac{\sqrt{15}\,t}{4}\right)}{4} - \frac{c_1 \mathrm{e}^{-\frac{3t}{4}} \sqrt{15} \sin\left(\frac{\sqrt{15}\,t}{4}\right)}{4} + \frac{\sqrt{15} \cos\left(\frac{\sqrt{15}\,t}{4}\right) \mathrm{e}^{-\frac{3t}{4}} c_2}{4} - \frac{3 \sin\left(\frac{\sqrt{15}\,t}{4}\right) \mathrm{e}^{-\frac{3t}{4}} c_2}{4}$
  - Use the initial condition $x'\Big|_{\{t=0\}} = 0$

    $0 = -\frac{3c_1}{4} + \frac{\sqrt{15}\,c_2}{4}$
  - Solve for c_1 and c_2

    $\left\{c_1 = 1, c_2 = \frac{\sqrt{15}}{5}\right\}$
  - Substitute constant values into general solution and simplify

    $x = \frac{\left(\sqrt{15} \sin\left(\frac{\sqrt{15}\,t}{4}\right) + 5 \cos\left(\frac{\sqrt{15}\,t}{4}\right)\right) \mathrm{e}^{-\frac{3t}{4}}}{5}$
- Solution to the IVP

  $x = \frac{\left(\sqrt{15} \sin\left(\frac{\sqrt{15}\,t}{4}\right) + 5 \cos\left(\frac{\sqrt{15}\,t}{4}\right)\right) \mathrm{e}^{-\frac{3t}{4}}}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 31

```
dsolve([2*diff(x(t),t$2)+3*diff(x(t),t)+3*x(t)=0,x(0) = 1, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \frac{e^{-\frac{3t}{4}}\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{4}\right)+5\cos\left(\frac{\sqrt{15}\,t}{4}\right)\right)}{5}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 42

```
DSolve[{x''[t]+3*x'[t]+3*x[t]==0,{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True
```

$$x(t)\to e^{-3t/2}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}t}{2}\right)+\cos\left(\frac{\sqrt{3}t}{2}\right)\right)$$

7.6 problem 1(f)

7.6.1 Existence and uniqueness analysis 1090
7.6.2 Solving as second order linear constant coeff ode 1091
7.6.3 Solving using Kovacic algorithm 1094
7.6.4 Maple step by step solution 1098

Internal problem ID [11449]
Internal file name [`OUTPUT/10431_Thursday_May_18_2023_04_19_18_AM_40239304/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.3 Complex eigenvalues. Exercises page 94
Problem number: 1(f).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{\frac{x''}{2}+\frac{5x'}{6}+\frac{2x}{9}=0}$$

With initial conditions

$$[x(0)=1, x'(0)=0]$$

7.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x''+p(t)x'+q(t)x=F$$

Where here

$$\begin{aligned}p(t)&=\frac{5}{3}\\ q(t)&=\frac{4}{9}\\ F&=0\end{aligned}$$

Hence the ode is

$$x'' + \frac{5x'}{3} + \frac{4x}{9} = 0$$

The domain of $p(t) = \frac{5}{3}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{4}{9}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.6.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = \frac{1}{2}, B = \frac{5}{6}, C = \frac{2}{9}$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\frac{\lambda^2 \mathrm{e}^{\lambda t}}{2} + \frac{5\lambda\, \mathrm{e}^{\lambda t}}{6} + \frac{2\,\mathrm{e}^{\lambda t}}{9} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\frac{1}{2}\lambda^2 + \frac{5}{6}\lambda + \frac{2}{9} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = \frac{1}{2}, B = \frac{5}{6}, C = \frac{2}{9}$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-\frac{5}{6}}{(2)\left(\frac{1}{2}\right)} \pm \frac{1}{(2)\left(\frac{1}{2}\right)}\sqrt{\frac{5}{6}^2 - (4)\left(\frac{1}{2}\right)\left(\frac{2}{9}\right)} \\ &= -\frac{5}{6} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{5}{6} + \frac{1}{2}$$
$$\lambda_2 = -\frac{5}{6} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{3}$$
$$\lambda_2 = -\frac{4}{3}$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$x = c_1 e^{\left(-\frac{1}{3}\right)t} + c_2 e^{\left(-\frac{4}{3}\right)t}$$

Or

$$x = c_1 \mathrm{e}^{-\frac{t}{3}} + c_2 \mathrm{e}^{-\frac{4t}{3}}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-\frac{t}{3}} + c_2 \mathrm{e}^{-\frac{4t}{3}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{c_1 \mathrm{e}^{-\frac{t}{3}}}{3} - \frac{4c_2 \mathrm{e}^{-\frac{4t}{3}}}{3}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{c_1}{3} - \frac{4c_2}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{4}{3}$$
$$c_2 = -\frac{1}{3}$$

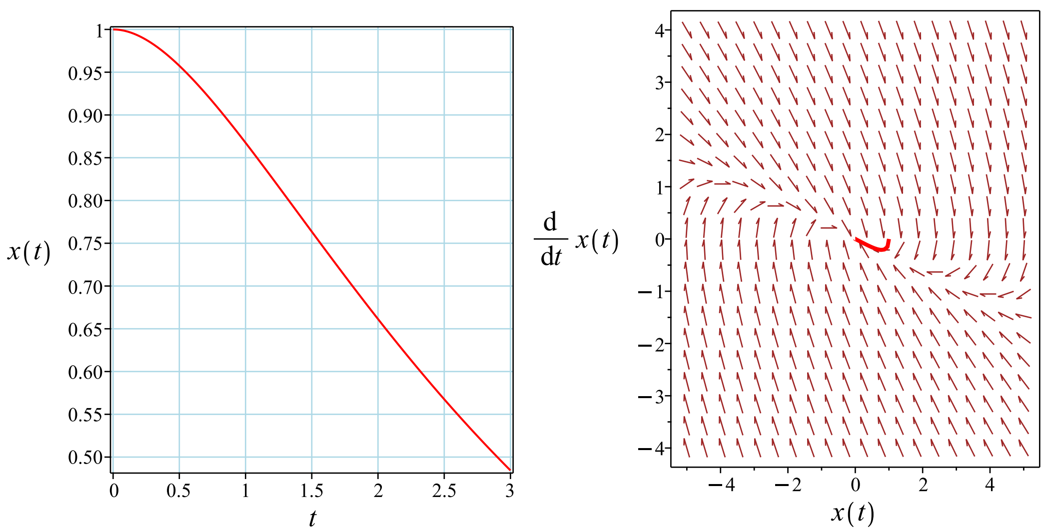
Substituting these values back in above solution results in

$$x = \frac{4\,\mathrm{e}^{-\frac{t}{3}}}{3} - \frac{\mathrm{e}^{-\frac{4t}{3}}}{3}$$

Summary

The solution(s) found are the following

$$x = \frac{4\,\mathrm{e}^{-\frac{t}{3}}}{3} - \frac{\mathrm{e}^{-\frac{4t}{3}}}{3} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \frac{4\,\mathrm{e}^{-\frac{t}{3}}}{3} - \frac{\mathrm{e}^{-\frac{4t}{3}}}{3}$$

Verified OK.

7.6.3 Solving using Kovacic algorithm

Writing the ode as

$$\frac{x''}{2} + \frac{5x'}{6} + \frac{2x}{9} = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= \frac{1}{2} \\ B &= \frac{5}{6} \\ C &= \frac{2}{9} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 210: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{5}{6}}{\frac{1}{2}}\,dt} \\
&= z_1 e^{-\frac{5t}{6}} \\
&= z_1\left(\mathrm{e}^{-\frac{5t}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-\frac{4t}{3}}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{\frac{5}{6}}{\frac{1}{2}}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-\frac{5t}{3}}}{(x_1)^2}\,dt \\
&= x_1\left(\mathrm{e}^{t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\mathrm{e}^{-\frac{4t}{3}}\right) + c_2\left(\mathrm{e}^{-\frac{4t}{3}}\left(\mathrm{e}^{t}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 \mathrm{e}^{-\frac{4t}{3}} + c_2 \mathrm{e}^{-\frac{t}{3}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{4c_1 e^{-\frac{4t}{3}}}{3} - \frac{c_2 e^{-\frac{t}{3}}}{3}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{4c_1}{3} - \frac{c_2}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{3}$$
$$c_2 = \frac{4}{3}$$

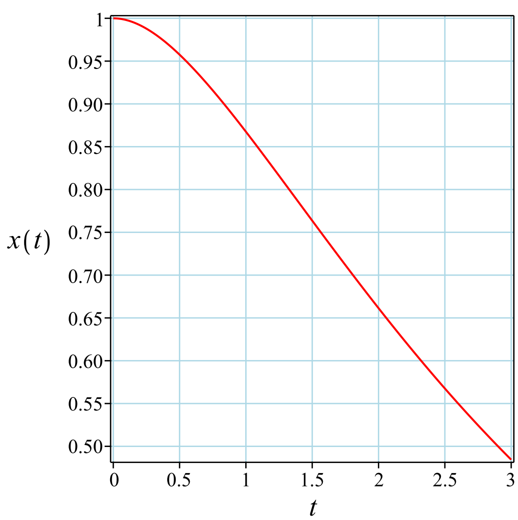
Substituting these values back in above solution results in

$$x = \frac{4\,e^{-\frac{t}{3}}}{3} - \frac{e^{-\frac{4t}{3}}}{3}$$

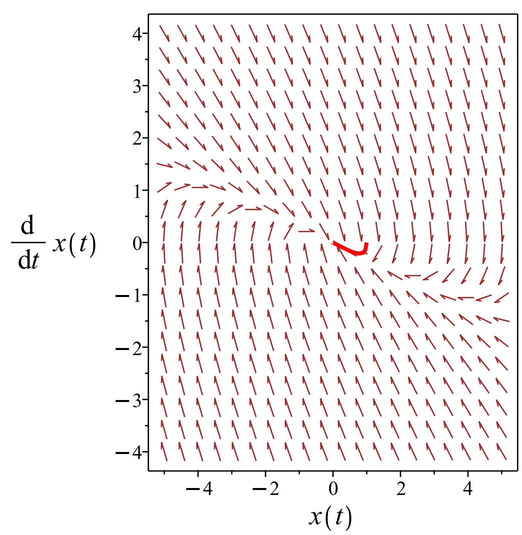
Summary

The solution(s) found are the following

$$x = \frac{4\,e^{-\frac{t}{3}}}{3} - \frac{e^{-\frac{4t}{3}}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{4\,\mathrm{e}^{-\frac{t}{3}}}{3} - \frac{\mathrm{e}^{-\frac{4t}{3}}}{3}$$

Verified OK.

7.6.4 Maple step by step solution

Let's solve

$\left[\frac{x''}{2} + \frac{5x'}{6} + \frac{2x}{9} = 0, x(0) = 1, x'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = -\frac{5x'}{3} - \frac{4x}{9}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + \frac{5x'}{3} + \frac{4x}{9} = 0$

- Characteristic polynomial of ODE

  $r^2 + \frac{5}{3}r + \frac{4}{9} = 0$

- Factor the characteristic polynomial

  $\frac{(3r+4)(3r+1)}{9} = 0$

- Roots of the characteristic polynomial

  $r = \left(-\frac{4}{3}, -\frac{1}{3}\right)$

- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^{-\frac{4t}{3}}$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{-\frac{t}{3}}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 \mathrm{e}^{-\frac{4t}{3}} + c_2 \mathrm{e}^{-\frac{t}{3}}$

□ Check validity of solution $x = c_1 \mathrm{e}^{-\frac{4t}{3}} + c_2 \mathrm{e}^{-\frac{t}{3}}$

- Use initial condition $x(0) = 1$

  $1 = c_1 + c_2$
- Compute derivative of the solution

  $x' = -\frac{4c_1 e^{-\frac{4t}{3}}}{3} - \frac{c_2 e^{-\frac{t}{3}}}{3}$
- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = -\frac{4c_1}{3} - \frac{c_2}{3}$
- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{3}, c_2 = \frac{4}{3}\right\}$
- Substitute constant values into general solution and simplify

  $x = \frac{4\,e^{-\frac{t}{3}}}{3} - \frac{e^{-\frac{4t}{3}}}{3}$

- Solution to the IVP

  $x = \frac{4\,e^{-\frac{t}{3}}}{3} - \frac{e^{-\frac{4t}{3}}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([1/2*diff(x(t),t$2)+5/6*diff(x(t),t)+2/9*x(t)=0,x(0) = 1, D(x)(0) = 0],x(t), singsol=
```

$$x(t) = \frac{4\,e^{-\frac{t}{3}}}{3} - \frac{e^{-\frac{4t}{3}}}{3}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 23

```
DSolve[{1/2*x''[t]+5/6*x'[t]+2/9*x[t]==0,{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolutions
```

$$x(t) \to \frac{1}{3} e^{-4t/3} \left(4e^t - 1\right)$$

8 Chapter 2, Second order linear equations. Section 2.2.4. Applications. Exercises page 99

8.1 problem 1 1102
8.2 problem 2 1113

8.1 problem 1

8.1.1 Existence and uniqueness analysis 1102
8.1.2 Solving as second order linear constant coeff ode 1103
8.1.3 Solving using Kovacic algorithm 1106
8.1.4 Maple step by step solution 1110

Internal problem ID [11450]
Internal file name [`OUTPUT/10432_Thursday_May_18_2023_04_19_20_AM_87188324/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.4. Applications. Exercises page 99
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' + x' + x = 0}$$

With initial conditions

$$[x(0) = 1, x'(0) = 1]$$

8.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + x' + x = 0$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.1.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \lambda \,\mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{\mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)}{2} + \mathrm{e}^{-\frac{t}{2}} \left(-\frac{c_1\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} + \frac{c_2\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2}\right)$$

substituting $x' = 1$ and $t = 0$ in the above gives

$$1 = -\frac{c_1}{2} + \frac{c_2\sqrt{3}}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = \sqrt{3}$$

Substituting these values back in above solution results in

$$x = e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right) + e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)$$

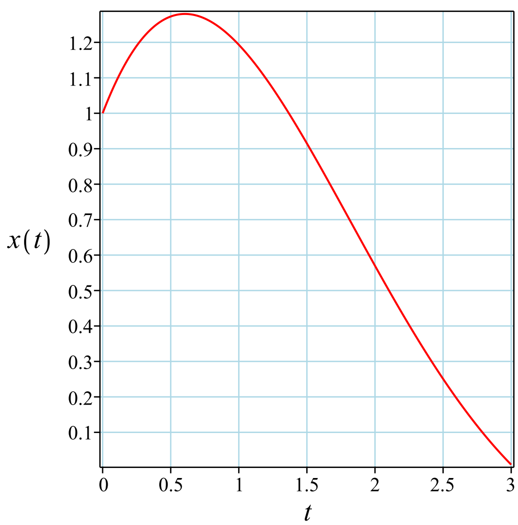
Which simplifies to

$$x = \left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right) + \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)e^{-\frac{t}{2}}$$

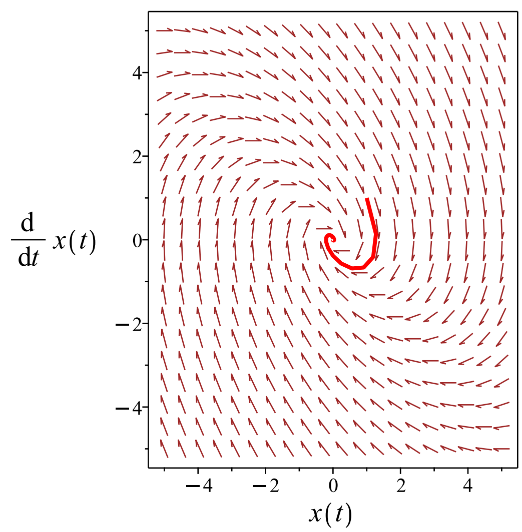
Summary

The solution(s) found are the following

$$x = \left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right) + \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)e^{-\frac{t}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right) + \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)e^{-\frac{t}{2}}$$

Verified OK.

8.1.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{3z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 212: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos\left(\frac{\sqrt{3}\,t}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{1}{1} \, dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}} \right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2} \right)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{x_1^2} \, dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1} \, dt}}{(x_1)^2} \, dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2} \, dt \\
&= x_1 \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, t}{2} \right)}{3} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2} \right) \right) + c_2 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2} \right) \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, t}{2} \right)}{3} \right) \right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)+\frac{2 c_2 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = -\frac{c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)}{2}-\frac{c_1 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{2}-\frac{c_2 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3}+c_2 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)
$$

substituting $x' = 1$ and $t = 0$ in the above gives

$$
1 = -\frac{c_1}{2}+c_2 \tag{2A}
$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= \frac{3}{2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
x = e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)+e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)
$$

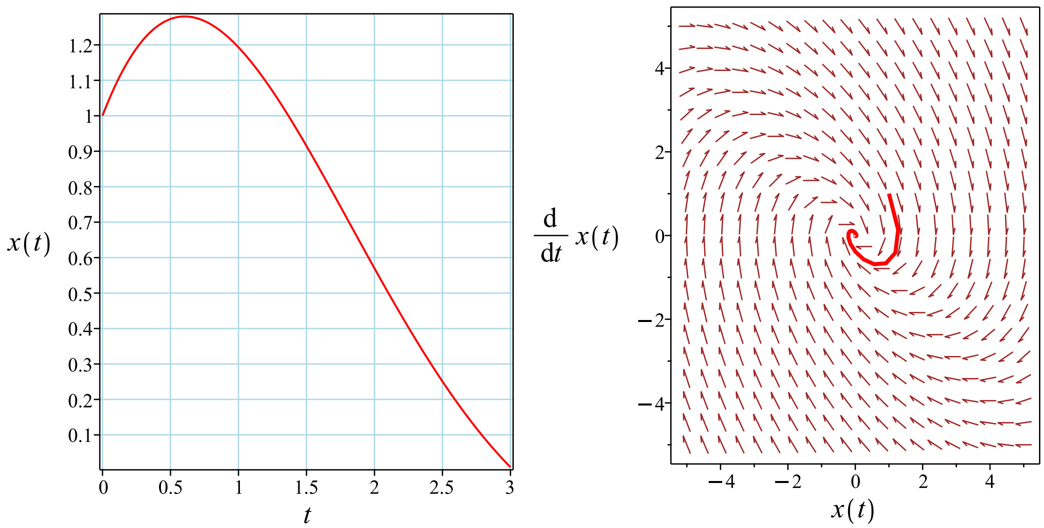
Which simplifies to

$$
x = \left(\sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)+\cos\left(\frac{\sqrt{3}\, t}{2}\right)\right) e^{-\frac{t}{2}}
$$

Summary

The solution(s) found are the following

$$
x = \left(\sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)+\cos\left(\frac{\sqrt{3}\, t}{2}\right)\right) e^{-\frac{t}{2}} \tag{1}
$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$x = \left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right) + \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) e^{-\frac{t}{2}}$$

Verified OK.

8.1.4 Maple step by step solution

Let's solve

$\left[x'' + x' + x = 0, x(0) = 1, x'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the ODE

  $x_1(t) = e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)$

- 2nd solution of the ODE

  $x_2(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) c_2$

☐ Check validity of solution $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) c_2$

  - Use initial condition $x(0) = 1$

    $1 = c_1$

  - Compute derivative of the solution

    $x' = -\frac{c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} - \frac{c_1 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) c_2}{2} + \frac{e^{-\frac{t}{2}} \sqrt{3}\, \cos\left(\frac{\sqrt{3}\,t}{2}\right) c_2}{2}$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 1$

    $1 = -\frac{c_1}{2} + \frac{c_2\sqrt{3}}{2}$

  - Solve for c_1 and c_2

    $\left\{c_1 = 1, c_2 = \sqrt{3}\right\}$

  - Substitute constant values into general solution and simplify

    $x = \left(\sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right) + \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) e^{-\frac{t}{2}}$

- Solution to the IVP

  $x = \left(\sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right) + \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) e^{-\frac{t}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve([diff(x(t),t$2)+diff(x(t),t)+x(t)=0,x(0) = 1, D(x)(0) = 1],x(t), singsol=all)
```

$$x(t) = e^{-\frac{t}{2}}\left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)+\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 42

```
DSolve[{x''[t]+x'[t]+x[t]==0,{x[0]==1,x'[0]==1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-t/2}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}t}{2}\right)+\cos\left(\frac{\sqrt{3}t}{2}\right)\right)$$

8.2 problem 2

8.2.1 Existence and uniqueness analysis 1113
8.2.2 Solving as second order linear constant coeff ode 1114
8.2.3 Solving using Kovacic algorithm 1117
8.2.4 Maple step by step solution 1121

Internal problem ID [11451]
Internal file name [`OUTPUT/10433_Thursday_May_18_2023_04_19_21_AM_94252793/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.2.4. Applications. Exercises page 99
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{x'' + \frac{x'}{8} + x = 0}$$

With initial conditions

$$[x(0) = 2, x'(0) = 0]$$

8.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{8} \\ q(t) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + \frac{x'}{8} + x = 0$$

The domain of $p(t) = \frac{1}{8}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.2.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = \frac{1}{8}, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \frac{\lambda e^{\lambda t}}{8} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \frac{1}{8}\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = \frac{1}{8}, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-\frac{1}{8}}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{\frac{1}{8}^2 - (4)(1)(1)} \\
&= -\frac{1}{16} \pm \frac{i\sqrt{255}}{16}
\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{16} + \frac{i\sqrt{255}}{16}$$
$$\lambda_2 = -\frac{1}{16} - \frac{i\sqrt{255}}{16}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{16} + \frac{i\sqrt{255}}{16}$$
$$\lambda_2 = -\frac{1}{16} - \frac{i\sqrt{255}}{16}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{16}$ and $\beta = \frac{\sqrt{255}}{16}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{t}{16}}\left(c_1 \cos\left(\frac{\sqrt{255}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{255}\, t}{16}\right)\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \mathrm{e}^{-\frac{t}{16}}\left(c_1 \cos\left(\frac{\sqrt{255}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{255}\, t}{16}\right)\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{\mathrm{e}^{-\frac{t}{16}}\left(c_1 \cos\left(\frac{\sqrt{255}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{255}\, t}{16}\right)\right)}{16} + \mathrm{e}^{-\frac{t}{16}}\left(-\frac{c_1\sqrt{255}\, \sin\left(\frac{\sqrt{255}\, t}{16}\right)}{16} + \frac{c_2\sqrt{255}\, \cos\left(\frac{\sqrt{255}\, t}{16}\right)}{16}\right)$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{c_1}{16} + \frac{\sqrt{255}\, c_2}{16} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 2 \\ c_2 &= \frac{2\sqrt{255}}{255} \end{aligned}$$

Substituting these values back in above solution results in

$$x = \frac{2 \sin\left(\frac{\sqrt{255}\, t}{16}\right) \sqrt{255}\, \mathrm{e}^{-\frac{t}{16}}}{255} + 2\,\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{255}\, t}{16}\right)$$

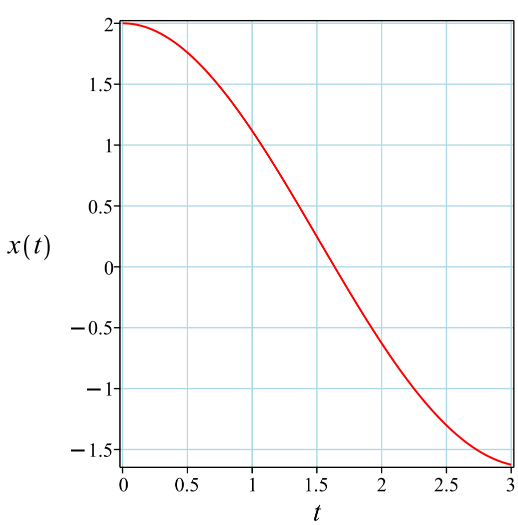
Which simplifies to

$$x = \frac{2\left(\sqrt{255}\, \sin\left(\frac{\sqrt{255}\, t}{16}\right) + 255 \cos\left(\frac{\sqrt{255}\, t}{16}\right)\right) \mathrm{e}^{-\frac{t}{16}}}{255}$$

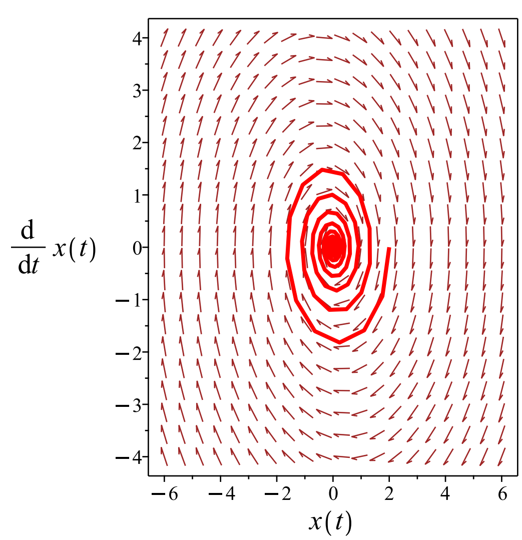
Summary

The solution(s) found are the following

$$x = \frac{2\left(\sqrt{255}\, \sin\left(\frac{\sqrt{255}\, t}{16}\right) + 255 \cos\left(\frac{\sqrt{255}\, t}{16}\right)\right) \mathrm{e}^{-\frac{t}{16}}}{255} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{2\left(\sqrt{255}\,\sin\left(\frac{\sqrt{255}\,t}{16}\right)+255\cos\left(\frac{\sqrt{255}\,t}{16}\right)\right)\mathrm{e}^{-\frac{t}{16}}}{255}$$

Verified OK.

8.2.3 Solving using Kovacic algorithm

Writing the ode as

$$x''+\frac{x'}{8}+x=0 \tag{1}$$

$$Ax''+Bx'+Cx=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{1}{8} \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-255}{256} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -255 \\ t &= 256 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{255z(t)}{256} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 214: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{255}{256}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\frac{\sqrt{255}\, t}{16}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{1}{8}}{1}\, dt} \\
&= z_1 e^{-\frac{t}{16}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{16}}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{255}\, t}{16}\right)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{\frac{1}{8}}{1}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{-\frac{t}{8}}}{(x_1)^2}\, dt \\
&= x_1 \left(\frac{16\sqrt{255}\, \tan\left(\frac{\sqrt{255}\, t}{16}\right)}{255}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{255}\, t}{16}\right)\right) + c_2\left(\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{255}\, t}{16}\right)\left(\frac{16\sqrt{255}\, \tan\left(\frac{\sqrt{255}\, t}{16}\right)}{255}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{255}\, t}{16}\right) + \frac{16 c_2 \sin\left(\frac{\sqrt{255}\, t}{16}\right) \sqrt{255}\, \mathrm{e}^{-\frac{t}{16}}}{255} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 0$ in the above gives

$$
2 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = -\frac{c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{255}\, t}{16}\right)}{16} - \frac{c_1 \mathrm{e}^{-\frac{t}{16}} \sqrt{255}\, \sin\left(\frac{\sqrt{255}\, t}{16}\right)}{16} + c_2 \cos\left(\frac{\sqrt{255}\, t}{16}\right) \mathrm{e}^{-\frac{t}{16}} - \frac{c_2 \sin\left(\frac{\sqrt{255}\, t}{16}\right) \sqrt{255}\, \mathrm{e}^{-\frac{t}{16}}}{255}
$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$
0 = -\frac{c_1}{16} + c_2 \tag{2A}
$$

Equations $\{1\text{A}, 2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= \frac{1}{8}
\end{aligned}
$$

Substituting these values back in above solution results in

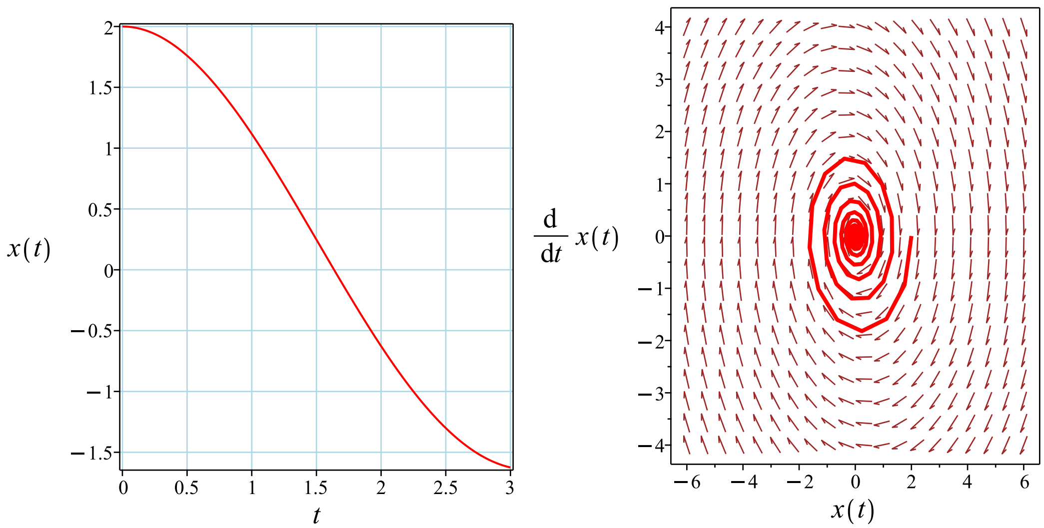
$$
x = \frac{2 \sin\left(\frac{\sqrt{255}\, t}{16}\right) \sqrt{255}\, \mathrm{e}^{-\frac{t}{16}}}{255} + 2\,\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{255}\, t}{16}\right)
$$

Which simplifies to

$$
x = \frac{2\left(\sqrt{255}\, \sin\left(\frac{\sqrt{255}\, t}{16}\right) + 255 \cos\left(\frac{\sqrt{255}\, t}{16}\right)\right) \mathrm{e}^{-\frac{t}{16}}}{255}
$$

Summary
The solution(s) found are the following

$$x = \frac{2\left(\sqrt{255}\, \sin\left(\frac{\sqrt{255}\, t}{16}\right)+255 \cos\left(\frac{\sqrt{255}\, t}{16}\right)\right) \mathrm{e}^{-\frac{t}{16}}}{255} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \frac{2\left(\sqrt{255}\, \sin\left(\frac{\sqrt{255}\, t}{16}\right)+255 \cos\left(\frac{\sqrt{255}\, t}{16}\right)\right) \mathrm{e}^{-\frac{t}{16}}}{255}$$

Verified OK.

8.2.4 Maple step by step solution

Let's solve

$\left[x''+\frac{x'}{8}+x=0, x(0)=2, x'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2+\frac{1}{8}r+1=0$

- Use quadratic formula to solve for r

  $r = \frac{\left(-\frac{1}{8}\right)\pm\left(\sqrt{-\frac{255}{64}}\right)}{2}$
- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{16}-\frac{\mathrm{I}\sqrt{255}}{16}, -\frac{1}{16}+\frac{\mathrm{I}\sqrt{255}}{16}\right)$
- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{255}\, t}{16}\right)$
- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{255}\, t}{16}\right)$
- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$
- Substitute in solutions

  $x = c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{255}\, t}{16}\right) + \sin\left(\frac{\sqrt{255}\, t}{16}\right) \mathrm{e}^{-\frac{t}{16}} c_2$

☐ Check validity of solution $x = c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{255}\, t}{16}\right) + \sin\left(\frac{\sqrt{255}\, t}{16}\right) \mathrm{e}^{-\frac{t}{16}} c_2$

  - Use initial condition $x(0) = 2$

    $2 = c_1$
  - Compute derivative of the solution

    $x' = -\frac{c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{255}\, t}{16}\right)}{16} - \frac{c_1 \mathrm{e}^{-\frac{t}{16}} \sqrt{255} \sin\left(\frac{\sqrt{255}\, t}{16}\right)}{16} + \frac{\sqrt{255} \cos\left(\frac{\sqrt{255}\, t}{16}\right) \mathrm{e}^{-\frac{t}{16}} c_2}{16} - \frac{\sin\left(\frac{\sqrt{255}\, t}{16}\right) \mathrm{e}^{-\frac{t}{16}} c_2}{16}$
  - Use the initial condition $x'\Big|_{\{t=0\}} = 0$

    $0 = -\frac{c_1}{16} + \frac{\sqrt{255}\, c_2}{16}$
  - Solve for c_1 and c_2

    $\left\{c_1 = 2, c_2 = \frac{2\sqrt{255}}{255}\right\}$
  - Substitute constant values into general solution and simplify

    $x = \frac{2\left(\sqrt{255} \sin\left(\frac{\sqrt{255}\, t}{16}\right) + 255 \cos\left(\frac{\sqrt{255}\, t}{16}\right)\right) \mathrm{e}^{-\frac{t}{16}}}{255}$

- Solution to the IVP

  $x = \frac{2\left(\sqrt{255} \sin\left(\frac{\sqrt{255}\, t}{16}\right) + 255 \cos\left(\frac{\sqrt{255}\, t}{16}\right)\right) \mathrm{e}^{-\frac{t}{16}}}{255}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 31

```
dsolve([diff(x(t),t$2)+125/1000*diff(x(t),t)+x(t)=0,x(0) = 2, D(x)(0) = 0],x(t), singsol=all
```

$$x(t) = \frac{2\,\mathrm{e}^{-\frac{t}{16}}\left(\sqrt{255}\,\sin\left(\frac{\sqrt{255}\,t}{16}\right)+255\cos\left(\frac{\sqrt{255}\,t}{16}\right)\right)}{255}$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 47

```
DSolve[{x''[t]+125/1000*x'[t]+x[t]==0,{x[0]==2,x'[0]==0}},x[t],t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{2}{255}e^{-t/16}\left(\sqrt{255}\sin\left(\frac{\sqrt{255}t}{16}\right)+255\cos\left(\frac{\sqrt{255}t}{16}\right)\right)$$

9 Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110

9.1 problem 1(a) 1125
9.2 problem 1(b) 1137
9.3 problem 1(c) 1149
9.4 problem 1(d) 1161
9.5 problem 1(e) 1173
9.6 problem 1(f) 1185
9.7 problem 1(g) 1197
9.8 problem 1(h) 1210
9.9 problem 1(i) 1223
9.10 problem 1(j) 1235
9.11 problem 1(k) 1247
9.12 problem 1(L) 1259
9.13 problem 2(a) 1271
9.14 problem 2(b) 1282
9.15 problem 2(c) 1302
9.16 problem 2(d) 1313
9.17 problem 2(e) 1324
9.18 problem 2(g) 1335
9.19 problem 2(h) 1346
9.20 problem 3 1358
9.21 problem 4 1372
9.22 problem 5 1385
9.23 problem 6 1409

9.1 problem 1(a)

9.1.1 Solving as second order linear constant coeff ode 1125
9.1.2 Solving using Kovacic algorithm 1129
9.1.3 Maple step by step solution 1134

Internal problem ID [11452]
Internal file name [`OUTPUT/10434_Thursday_May_18_2023_04_19_23_AM_45534682/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 1(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + x' + x = 3t^3 - 1}$$

9.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = 1, f(t) = 3t^3 - 1$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A=1, B=1, C=1$. Let the solution be $x=e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t}+\lambda\, \mathrm{e}^{\lambda t}+\mathrm{e}^{\lambda t}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2+\lambda+1=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=1, C=1$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-1}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{1^2-(4)\,(1)\,(1)}\\&=-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-\frac{1}{2}+\frac{i\sqrt{3}}{2}\\\lambda_2&=-\frac{1}{2}-\frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$$

$$\lambda_2=-\frac{1}{2}-\frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2}=\alpha\pm i\beta$$

Where $\alpha=-\frac{1}{2}$ and $\beta=\frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x=e^{\alpha t}(c_1\cos(\beta t)+c_2\sin(\beta t))$$

Which becomes

$$x=e^{-\frac{t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,t}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^3 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2, t^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right), \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_4 t^3 + A_3 t^2 + A_2 t + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_4 t^3 + A_3 t^2 + 3t^2 A_4 + A_2 t + 2t A_3 + 6t A_4 + A_1 + A_2 + 2A_3 = 3t^3 - 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 17, A_2 = 0, A_3 = -9, A_4 = 3]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 3t^3 - 9t^2 + 17$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(\mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)\right) + (3t^3 - 9t^2 + 17) \end{aligned}$$

Summary

The solution(s) found are the following

$$
x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right) \right) + 3t^3 - 9t^2 + 17 \tag{1}
$$

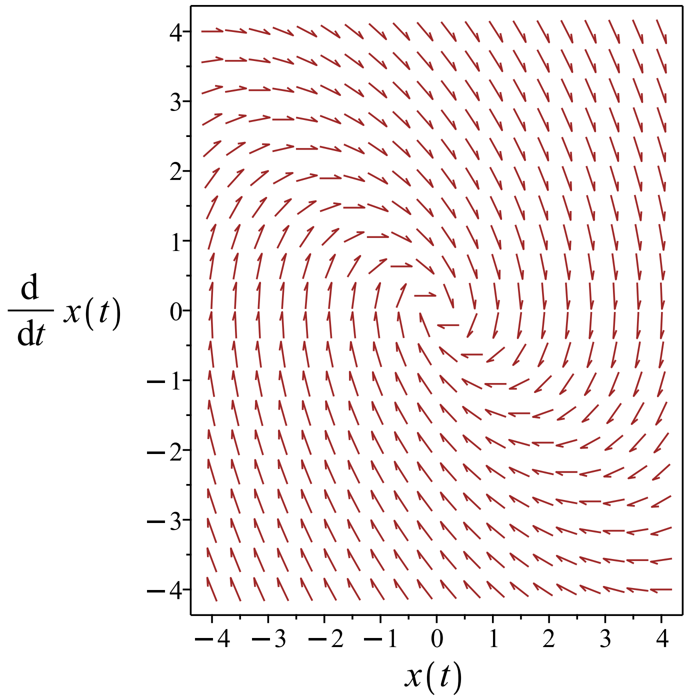


Figure 251: Slope field plot

Verification of solutions

$$
x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right) \right) + 3t^3 - 9t^2 + 17
$$

Verified OK.

9.1.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' + x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{3z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 216: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\frac{\sqrt{3}\, t}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''+x'+x=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^3+1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2, t^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right), \frac{2\,\mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_4 t^3 + A_3 t^2 + A_2 t + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_4 t^3 + A_3 t^2 + 3t^2 A_4 + A_2 t + 2t A_3 + 6t A_4 + A_1 + A_2 + 2A_3 = 3t^3 - 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 17, A_2 = 0, A_3 = -9, A_4 = 3]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 3t^3 - 9t^2 + 17$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right) + \left(3t^3 - 9t^2 + 17\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} + 3t^3 - 9t^2 + 17 \tag{1}$$

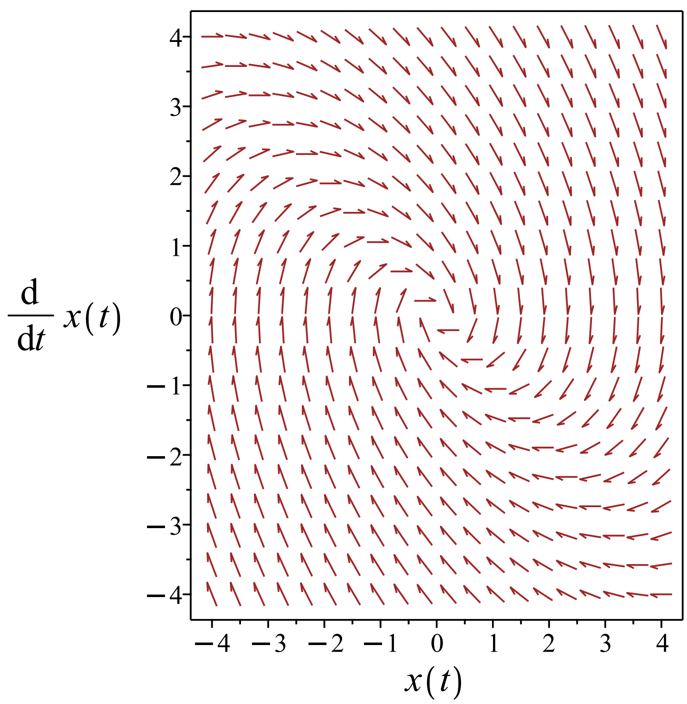


Figure 252: Slope field plot

Verification of solutions

$$x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2c_2 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} + 3t^3 - 9t^2 + 17$$

Verified OK.

9.1.3 Maple step by step solution

Let's solve

$x'' + x' + x = 3t^3 - 1$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t),x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t),x_2(t))} dt\right), f(t) = 3t^3 - 1\right]$

- ○ Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right) & e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right) \\ -\frac{e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} & -\frac{e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} + \frac{e^{-\frac{t}{2}}\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $$W(x_1(t), x_2(t)) = \frac{\sqrt{3}\,e^{-t}}{2}$$

- Substitute functions into equation for $x_p(t)$

  $$x_p(t) = -\frac{2\,e^{-\frac{t}{2}}\sqrt{3}\left(\cos\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int e^{\frac{t}{2}}(3t^3-1)\sin\left(\frac{\sqrt{3}\,t}{2}\right)dt\right) - \sin\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int e^{\frac{t}{2}}(3t^3-1)\cos\left(\frac{\sqrt{3}\,t}{2}\right)dt\right)\right)}{3}$$

- Compute integrals

  $$x_p(t) = 3t^3 - 9t^2 + 17$$

- Substitute particular solution into general solution to ODE

  $$x = c_1 e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right) + e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2 + 3t^3 - 9t^2 + 17$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+x(t)=3*t^3-1,x(t), singsol=all)
```

$$x(t) = e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2 + e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)c_1 + 3t^3 - 9t^2 + 17$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 59

```
DSolve[x''[t]+x'[t]+x[t]==3*t^3-1,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 3t^3 - 9t^2 + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}t}{2}\right) + c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}t}{2}\right) + 17$$

9.2 problem 1(b)

9.2.1 Solving as second order linear constant coeff ode 1137
9.2.2 Solving using Kovacic algorithm 1141
9.2.3 Maple step by step solution 1146

Internal problem ID [11453]
Internal file name [`OUTPUT/10435_Thursday_May_18_2023_04_19_24_AM_43720782/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 1(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + x' + x = 3\cos(t) - 2\sin(t)}$$

9.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = 1, f(t) = 3\cos(t) - 2\sin(t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A=1, B=1, C=1$. Let the solution be $x=e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t}+\lambda \mathrm{e}^{\lambda t}+\mathrm{e}^{\lambda t}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2+\lambda+1=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=1, C=1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2-(4)(1)(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2}+\frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2}-\frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2}+\frac{i\sqrt{3}}{2}$$

$$\lambda_2 = -\frac{1}{2}-\frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2}=\alpha \pm i\beta$$

Where $\alpha=-\frac{1}{2}$ and $\beta=\frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x=e^{\alpha t}(c_1\cos(\beta t)+c_2\sin(\beta t))$$

Which becomes

$$x=e^{-\frac{t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,t}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\cos(t) - 2\sin(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right), \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(t) + A_2 \cos(t) = 3\cos(t) - 2\sin(t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 3]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 2\cos(t) + 3\sin(t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(\mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)\right) + (2\cos(t) + 3\sin(t)) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right) \right) + 2\cos(t) + 3\sin(t) \tag{1}$$

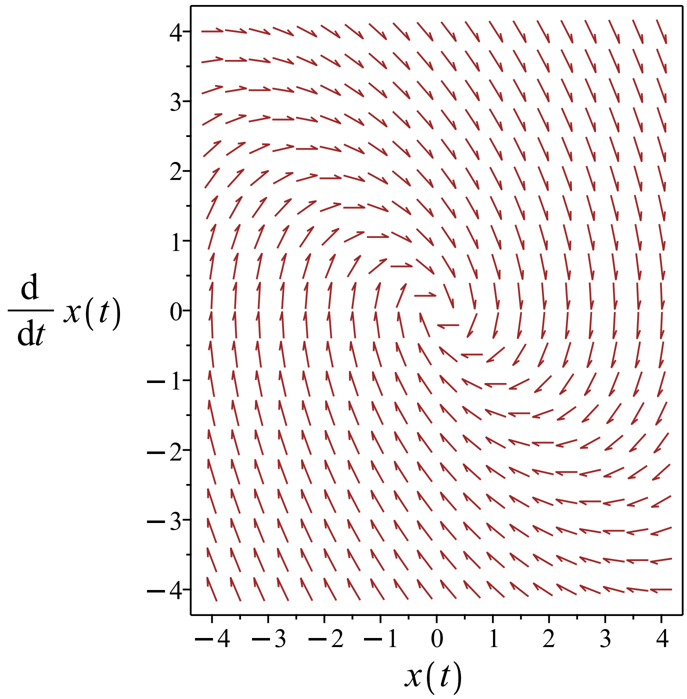


Figure 253: Slope field plot

Verification of solutions

$$x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right) \right) + 2\cos(t) + 3\sin(t)$$

Verified OK.

9.2.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
x'' + x' + x &= 0 \tag{1} \\
Ax'' + Bx' + Cx &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= 1 \tag{3} \\
C &= 1
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = xe^{\int \frac{B}{2A}\, dt}
$$

Then (2) becomes

$$
z''(t) = rz(t) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-3}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= -3 \\
t &= 4
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(t) = -\frac{3z(t)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 218: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos\left(\frac{\sqrt{3}\,t}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''+x'+x=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)+\frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\cos(t)-2\sin(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right), \frac{2\,\mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1\cos(t)+A_2\sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\sin(t)+A_2\cos(t)=3\cos(t)-2\sin(t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1=2, A_2=3]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 2\cos(t)+3\sin(t)$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3}\right) + (2\cos(t) + 3\sin(t))
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} + 2\cos(t) + 3\sin(t) \tag{1}
$$

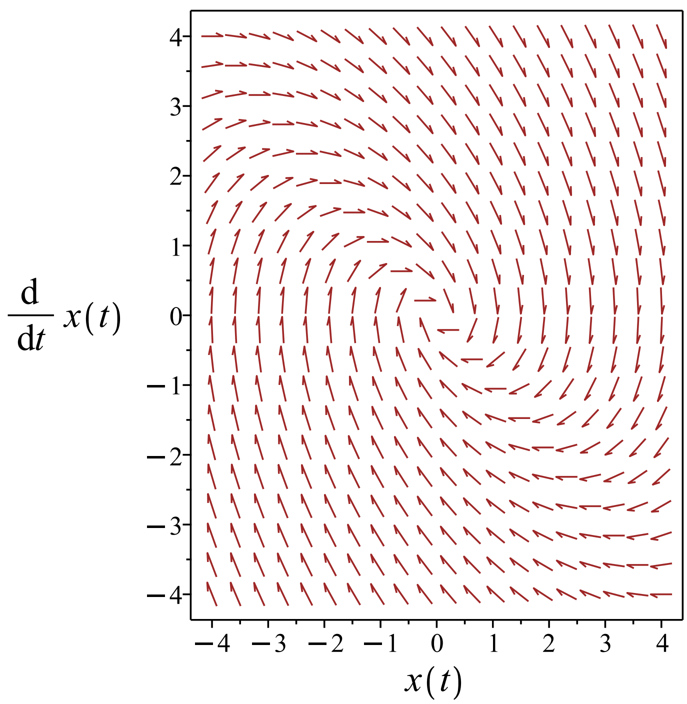


Figure 254: Slope field plot

Verification of solutions

$$
x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} + 2\cos(t) + 3\sin(t)
$$

Verified OK.

9.2.3 Maple step by step solution

Let's solve

$x'' + x' + x = 3\cos(t) - 2\sin(t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)$

- General solution of the ODE

  $x = c_1x_1(t) + c_2x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right) + e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = 3\cos(t) - 2\sin(t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right) & e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right) \\ -\frac{e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}}\sqrt{3}\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} & -\frac{e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} + \frac{e^{-\frac{t}{2}}\sqrt{3}\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \end{bmatrix}$

- ○ Compute Wronskian

$$W(x_1(t), x_2(t)) = \frac{\sqrt{3}\,e^{-t}}{2}$$

- Substitute functions into equation for $x_p(t)$

$$x_p(t) = \frac{2\sqrt{3}\,e^{-\frac{t}{2}}\left(\cos\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int e^{\frac{t}{2}}(-3\cos(t)+2\sin(t))\sin\left(\frac{\sqrt{3}\,t}{2}\right)dt\right)-\sin\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int e^{\frac{t}{2}}(-3\cos(t)+2\sin(t))\cos\left(\frac{\sqrt{3}\,t}{2}\right)dt\right)\right)}{3}$$

- Compute integrals

$$x_p(t) = 2\cos(t) + 3\sin(t)$$

- Substitute particular solution into general solution to ODE

$$x = c_1 e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right) + e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2 + 3\sin(t) + 2\cos(t)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+x(t)=3*cos(t)-2*sin(t),x(t), singsol=all)
```

$$x(t) = e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2 + e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)c_1 + 3\sin(t) + 2\cos(t)$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 56

```
DSolve[x''[t]+x'[t]+x[t]==3*Cos[t]-2*Sin[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 3\sin(t) + 2\cos(t) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}t}{2}\right) + c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}t}{2}\right)$$

9.3 problem 1(c)

9.3.1 Solving as second order linear constant coeff ode 1149
9.3.2 Solving using Kovacic algorithm 1153
9.3.3 Maple step by step solution 1158

Internal problem ID [11454]
Internal file name [`OUTPUT/10436_Thursday_May_18_2023_04_19_26_AM_4201971/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 1(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' + x' + x = 12}$$

9.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = 1, f(t) = 12$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \lambda \,\mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right), e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 12$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 12]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 12$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)\right) + (12) \end{aligned}$$

Summary
The solution(s) found are the following

$$x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right) + 12 \tag{1}$$

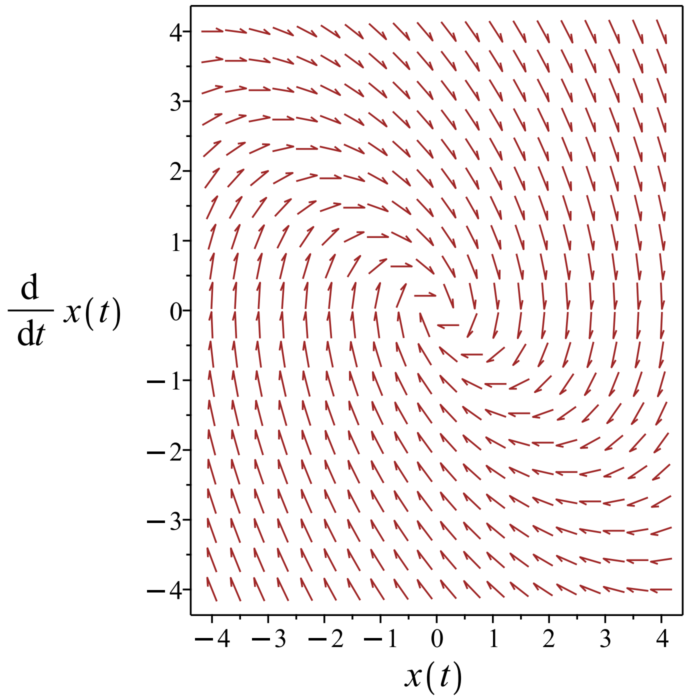


Figure 255: Slope field plot

Verification of solutions

$$x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right) + 12$$

Verified OK.

9.3.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{3z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 220: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos\left(\frac{\sqrt{3}\, t}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''+x'+x=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)+\frac{2c_2 \mathrm{e}^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right), \frac{2\,\mathrm{e}^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 12$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 12]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 12$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2 c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3}\right) + (12)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2 c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} + 12 \tag{1}
$$

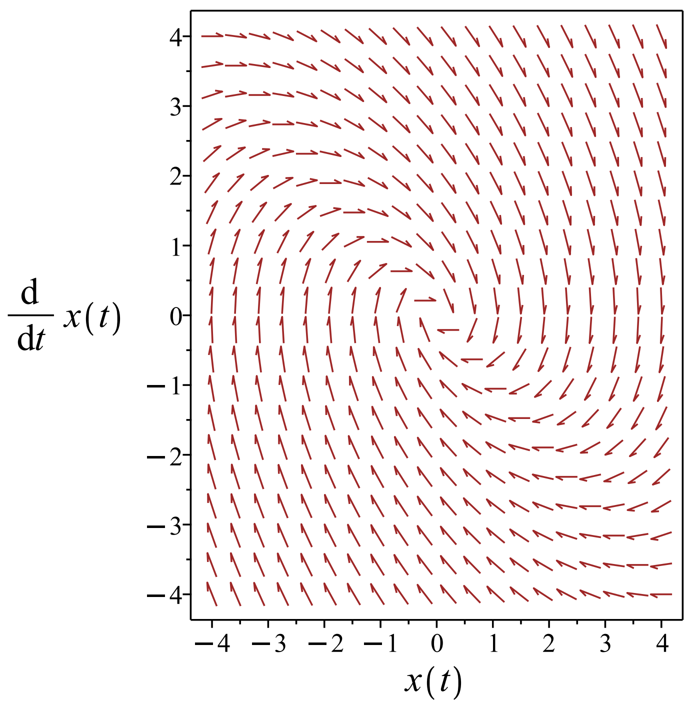


Figure 256: Slope field plot

Verification of solutions

$$
x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2 c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} + 12
$$

Verified OK.

9.3.3 Maple step by step solution

Let's solve

$x'' + x' + x = 12$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = 12\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) & e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) \\ -\frac{e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}} \sqrt{3} \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} & -\frac{e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} + \frac{e^{-\frac{t}{2}} \sqrt{3} \cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \end{bmatrix}$

- ○ Compute Wronskian

$$W(x_1(t)\,,x_2(t)) = \frac{\sqrt{3}\,\mathrm{e}^{-t}}{2}$$

- Substitute functions into equation for $x_p(t)$

$$x_p(t) = -8\,\mathrm{e}^{-\frac{t}{2}}\sqrt{3}\left(\cos\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int \mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)dt\right) - \sin\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int \mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)dt\right)\right)$$

- Compute integrals

$$x_p(t) = 12$$

- Substitute particular solution into general solution to ODE

$$x = c_1\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right) + \mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2 + 12$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+x(t)=12,x(t), singsol=all)
```

$$x(t) = \mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2 + \mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)c_1 + 12$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 49

```
DSolve[x''[t]+x'[t]+x[t]==12,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}t}{2}\right) + c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}t}{2}\right) + 12$$

9.4 problem 1(d)

9.4.1 Solving as second order linear constant coeff ode 1161
9.4.2 Solving using Kovacic algorithm 1165
9.4.3 Maple step by step solution 1170

Internal problem ID [11455]
Internal file name [`OUTPUT/10437_Thursday_May_18_2023_04_19_27_AM_39202564/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 1(d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + x' + x = t^2 e^{3t}}$$

9.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = 1, f(t) = t^2 e^{3t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A=1, B=1, C=1$. Let the solution be $x=e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t}+\lambda \,\mathrm{e}^{\lambda t}+\mathrm{e}^{\lambda t}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2+\lambda+1=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=1, C=1$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-1}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{1^2-(4)\,(1)\,(1)}\\&=-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-\frac{1}{2}+\frac{i\sqrt{3}}{2}\\\lambda_2&=-\frac{1}{2}-\frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=-\frac{1}{2}+\frac{i\sqrt{3}}{2}\\\lambda_2&=-\frac{1}{2}-\frac{i\sqrt{3}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2}=\alpha\pm i\beta$$

Where $\alpha=-\frac{1}{2}$ and $\beta=\frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x=e^{\alpha t}(c_1\cos(\beta t)+c_2\sin(\beta t))$$

Which becomes

$$x=e^{-\frac{t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,t}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right) \right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2 e^{3t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{t\, e^{3t}, t^2 e^{3t}, e^{3t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right), e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 t\, e^{3t} + A_2 t^2 e^{3t} + A_3 e^{3t}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$7A_1 e^{3t} + 13A_1 t\, e^{3t} + 2A_2 e^{3t} + 14A_2 t\, e^{3t} + 13A_2 t^2 e^{3t} + 13A_3 e^{3t} = t^2 e^{3t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{14}{169}, A_2 = \frac{1}{13}, A_3 = \frac{72}{2197}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{14t\, e^{3t}}{169} + \frac{t^2 e^{3t}}{13} + \frac{72\, e^{3t}}{2197}$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)\right) + \left(-\frac{14t\, e^{3t}}{169} + \frac{t^2 e^{3t}}{13} + \frac{72\, e^{3t}}{2197}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right) - \frac{14t\, e^{3t}}{169} + \frac{t^2 e^{3t}}{13} + \frac{72\, e^{3t}}{2197} \tag{1}
$$

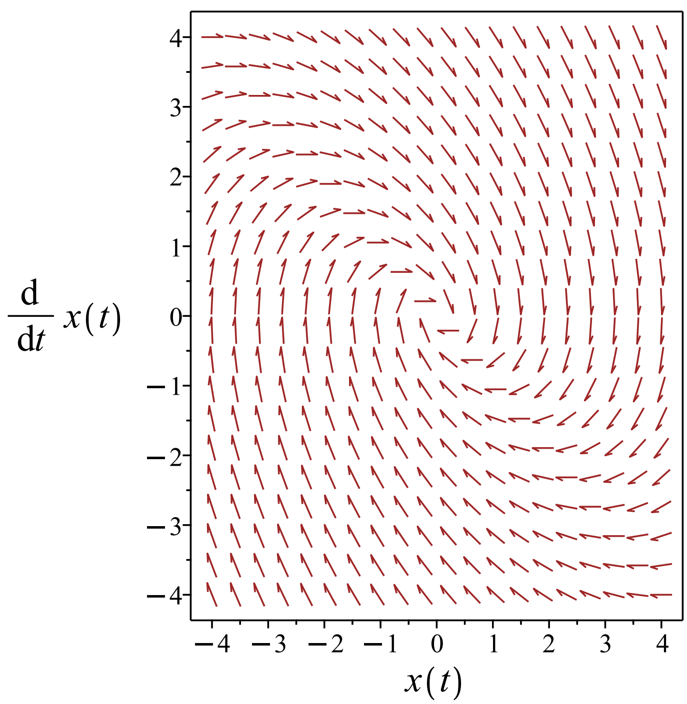


Figure 257: Slope field plot

Verification of solutions

$$
x = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right) - \frac{14t\, e^{3t}}{169} + \frac{t^2 e^{3t}}{13} + \frac{72\, e^{3t}}{2197}
$$

Verified OK.

9.4.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' + x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{3z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t) \, e^{-\int \frac{B}{2A} \, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 222: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\frac{\sqrt{3}\, t}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''+x'+x=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)+\frac{2c_2 e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2 e^{3t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{t\,e^{3t}, t^2 e^{3t}, e^{3t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right), \frac{2\,e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 t\,e^{3t}+A_2 t^2 e^{3t}+A_3 e^{3t}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$7A_1 e^{3t}+13A_1 t\,e^{3t}+2A_2 e^{3t}+14A_2 t\,e^{3t}+13A_2 t^2 e^{3t}+13A_3 e^{3t}=t^2 e^{3t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=-\frac{14}{169}, A_2=\frac{1}{13}, A_3=\frac{72}{2197}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{14t\,e^{3t}}{169} + \frac{t^2 e^{3t}}{13} + \frac{72\,e^{3t}}{2197}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 e^{-\frac{t}{2}} \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right) + \left(-\frac{14t\,e^{3t}}{169} + \frac{t^2 e^{3t}}{13} + \frac{72\,e^{3t}}{2197}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 e^{-\frac{t}{2}} \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} - \frac{14t\,e^{3t}}{169} + \frac{t^2 e^{3t}}{13} + \frac{72\,e^{3t}}{2197} \tag{1}$$

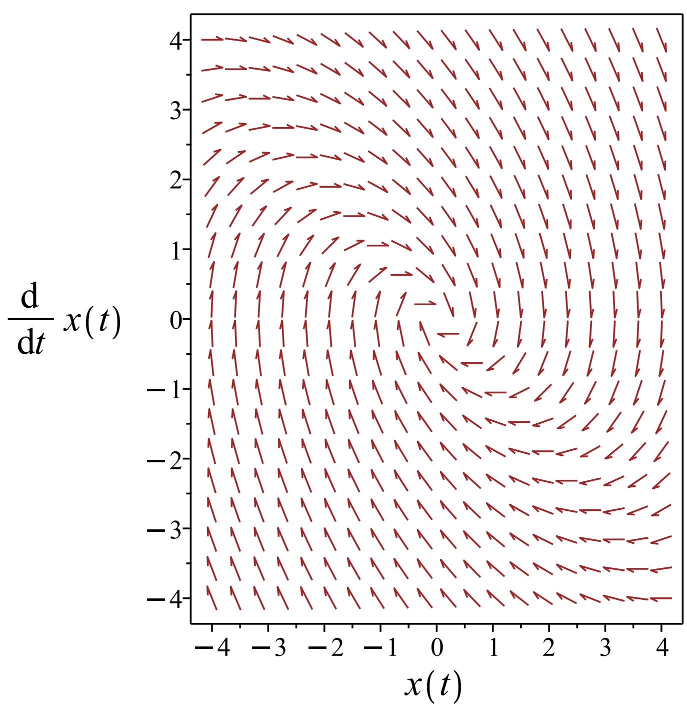


Figure 258: Slope field plot

Verification of solutions

$$x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2c_2 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} - \frac{14t\, e^{3t}}{169} + \frac{t^2 e^{3t}}{13} + \frac{72\, e^{3t}}{2197}$$

Verified OK.

9.4.3 Maple step by step solution

Let's solve

$x'' + x' + x = t^2 e^{3t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = t^2 e^{3t}\right]$

  ○ Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) & e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) \\ -\frac{e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} & -\frac{e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} + \frac{e^{-\frac{t}{2}} \sqrt{3}\, \cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = \frac{\sqrt{3}\, e^{-t}}{2}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{2\sqrt{3}\, e^{-\frac{t}{2}} \left(\cos\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int t^2 e^{\frac{7t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) dt\right) - \sin\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int t^2 e^{\frac{7t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) dt\right)\right)}{3}$

- Compute integrals

  $x_p(t) = \frac{e^{3t}(169t^2 - 182t + 72)}{2197}$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) c_2 + \frac{e^{3t}(169t^2 - 182t + 72)}{2197}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+x(t)=t^2*exp(3*t),x(t), singsol=all)
```

$$x(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) c_2 + e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) c_1 + \frac{(169t^2 - 182t + 72)\, e^{3t}}{2197}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 62

```
DSolve[x''[t]+x'[t]+x[t]==t^2*exp(3*t),x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-t/2}\left(3 \exp e^{t/2}\left(t^3-3t^2+6\right)+c_2 \cos\left(\frac{\sqrt{3}t}{2}\right)+c_1 \sin\left(\frac{\sqrt{3}t}{2}\right)\right)$$

9.5 problem 1(e)

9.5.1 Solving as second order linear constant coeff ode 1173
9.5.2 Solving using Kovacic algorithm 1177
9.5.3 Maple step by step solution 1182

Internal problem ID [11456]
Internal file name [`OUTPUT/10438_Thursday_May_18_2023_04_19_29_AM_9931015/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 1(e).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + x' + x = 5\sin(7t)}$$

9.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = 1, f(t) = 5\sin(7t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \lambda \, \mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)}\\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5 \sin(7t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(7t), \sin(7t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right), e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(7t) + A_2 \sin(7t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-48A_1 \cos(7t) - 48A_2 \sin(7t) - 7A_1 \sin(7t) + 7A_2 \cos(7t) = 5 \sin(7t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{35}{2353}, A_2 = -\frac{240}{2353}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{35 \cos(7t)}{2353} - \frac{240 \sin(7t)}{2353}$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(\mathrm{e}^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)\right) + \left(-\frac{35\cos(7t)}{2353} - \frac{240\sin(7t)}{2353}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = \mathrm{e}^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right) - \frac{35\cos(7t)}{2353} - \frac{240\sin(7t)}{2353} \tag{1}
$$

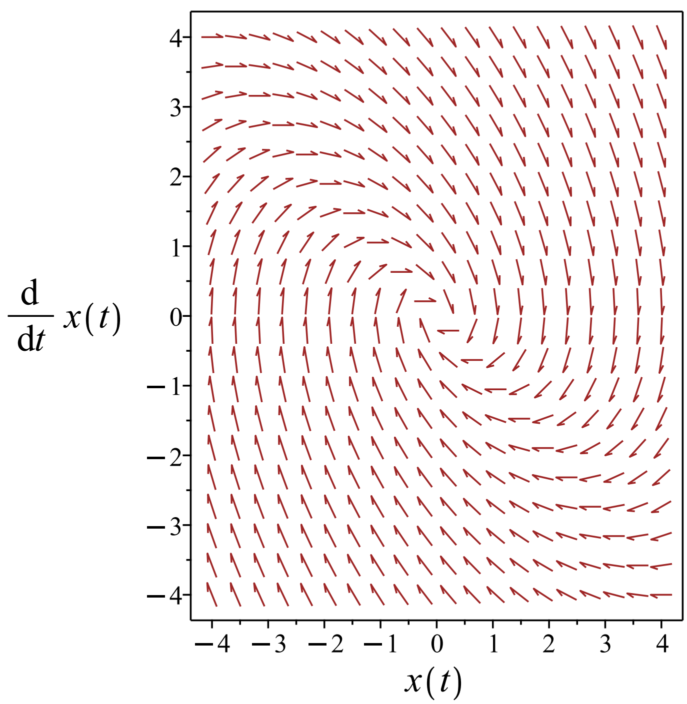


Figure 259: Slope field plot

Verification of solutions

$$
x = \mathrm{e}^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right) - \frac{35\cos(7t)}{2353} - \frac{240\sin(7t)}{2353}
$$

Verified OK.

9.5.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
x'' + x' + x &= 0 \tag{1} \\
Ax'' + Bx' + Cx &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= 1 \tag{3} \\
C &= 1
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = xe^{\int \frac{B}{2A}\,dt}
$$

Then (2) becomes

$$
z''(t) = rz(t) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-3}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= -3 \\
t &= 4
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(t) = -\frac{3z(t)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\,dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 224: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos\left(\frac{\sqrt{3}\, t}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\,dt \\
&= x_1\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5 \sin(7t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(7t), \sin(7t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right), \frac{2\, e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(7t) + A_2 \sin(7t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-48A_1 \cos(7t) - 48A_2 \sin(7t) - 7A_1 \sin(7t) + 7A_2 \cos(7t) = 5 \sin(7t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{35}{2353}, A_2 = -\frac{240}{2353}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{35\cos(7t)}{2353} - \frac{240\sin(7t)}{2353}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right) + \left(-\frac{35\cos(7t)}{2353} - \frac{240\sin(7t)}{2353}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} - \frac{35\cos(7t)}{2353} - \frac{240\sin(7t)}{2353} \tag{1}$$

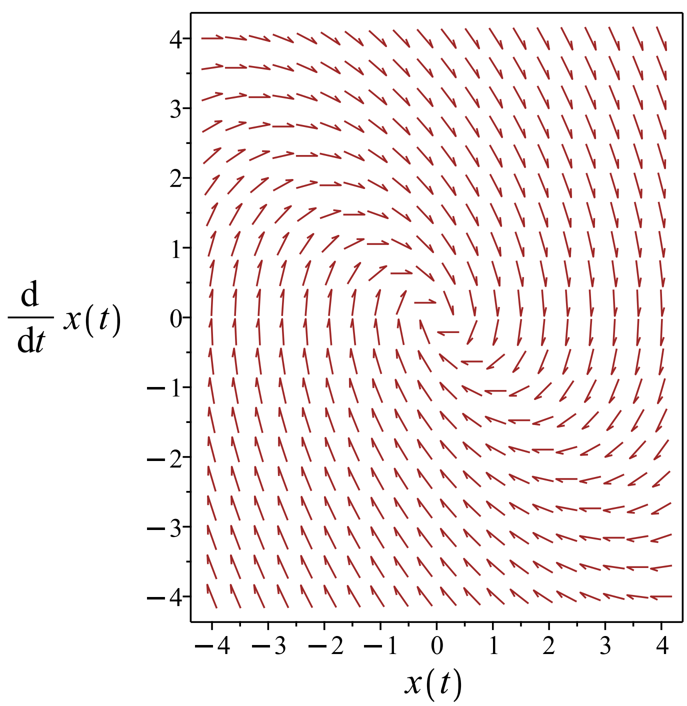


Figure 260: Slope field plot

Verification of solutions

$$x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2c_2 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} - \frac{35 \cos(7t)}{2353} - \frac{240 \sin(7t)}{2353}$$

Verified OK.

9.5.3 Maple step by step solution

Let's solve

$x'' + x' + x = 5 \sin(7t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t) \left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t) \left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = 5 \sin(7t)\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) & e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) \\ -\frac{e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} & -\frac{e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} + \frac{e^{-\frac{t}{2}} \sqrt{3}\, \cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = \frac{\sqrt{3}\, e^{-t}}{2}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{10\, e^{-\frac{t}{2}} \sqrt{3}\left(\cos\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int e^{\frac{t}{2}} \sin(7t) \sin\left(\frac{\sqrt{3}\,t}{2}\right)dt\right) - \sin\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int e^{\frac{t}{2}} \sin(7t) \cos\left(\frac{\sqrt{3}\,t}{2}\right)dt\right)\right)}{3}$

- Compute integrals

  $x_p(t) = -\frac{35 \cos(7t)}{2353} - \frac{240 \sin(7t)}{2353}$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) c_2 - \frac{240 \sin(7t)}{2353} - \frac{35 \cos(7t)}{2353}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 43

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+x(t)=5*sin(7*t),x(t), singsol=all)
```

$$x(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) c_2 + e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) c_1 - \frac{240 \sin(7t)}{2353} - \frac{35 \cos(7t)}{2353}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 65

```
DSolve[x''[t]+x'[t]+x[t]==5*Sin[7*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{5(48\sin(7t) + 7\cos(7t))}{2353} + c_2e^{-t/2}\cos\left(\frac{\sqrt{3}t}{2}\right) + c_1e^{-t/2}\sin\left(\frac{\sqrt{3}t}{2}\right)$$

9.6 problem 1(f)

9.6.1 Solving as second order linear constant coeff ode 1185
9.6.2 Solving using Kovacic algorithm 1189
9.6.3 Maple step by step solution 1194

Internal problem ID [11457]
Internal file name [`OUTPUT/10439_Thursday_May_18_2023_04_19_30_AM_56889426/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 1(f).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + x' + x = e^{2t} \cos(t) + t^2}$$

9.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = 1, f(t) = e^{2t} \cos(t) + t^2$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A=1, B=1, C=1$. Let the solution be $x=e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t}+\lambda \mathrm{e}^{\lambda t}+\mathrm{e}^{\lambda t}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2+\lambda+1=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=1, C=1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2-(4)(1)(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2}+\frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2}-\frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2}+\frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2}-\frac{i\sqrt{3}}{2}
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2}=\alpha \pm i\beta$$

Where $\alpha=-\frac{1}{2}$ and $\beta=\frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x=e^{\alpha t}(c_1\cos(\beta t)+c_2\sin(\beta t))$$

Which becomes

$$x=e^{-\frac{t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,t}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right) \right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2t} \cos(t) + t^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2t} \cos(t), e^{2t} \sin(t)\}, \{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right), e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 e^{2t} \cos(t) + A_2 e^{2t} \sin(t) + A_3 + A_4 t + A_5 t^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &6A_1 e^{2t} \cos(t) - 5A_1 e^{2t} \sin(t) + 6A_2 e^{2t} \sin(t) + 5A_2 e^{2t} \cos(t) \\ &+ 2A_5 + A_4 + 2A_5 t + A_3 + A_4 t + A_5 t^2 = e^{2t} \cos(t) + t^2 \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{6}{61}, A_2 = \frac{5}{61}, A_3 = 0, A_4 = -2, A_5 = 1 \right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{6\, e^{2t} \cos(t)}{61} + \frac{5\, e^{2t} \sin(t)}{61} - 2t + t^2$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(\mathrm{e}^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)\right) + \left(\frac{6\,\mathrm{e}^{2t}\cos(t)}{61} + \frac{5\,\mathrm{e}^{2t}\sin(t)}{61} - 2t + t^2\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
x = \mathrm{e}^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right) + \frac{6\,\mathrm{e}^{2t}\cos(t)}{61} + \frac{5\,\mathrm{e}^{2t}\sin(t)}{61} - 2t + t^2 \quad (1)
$$

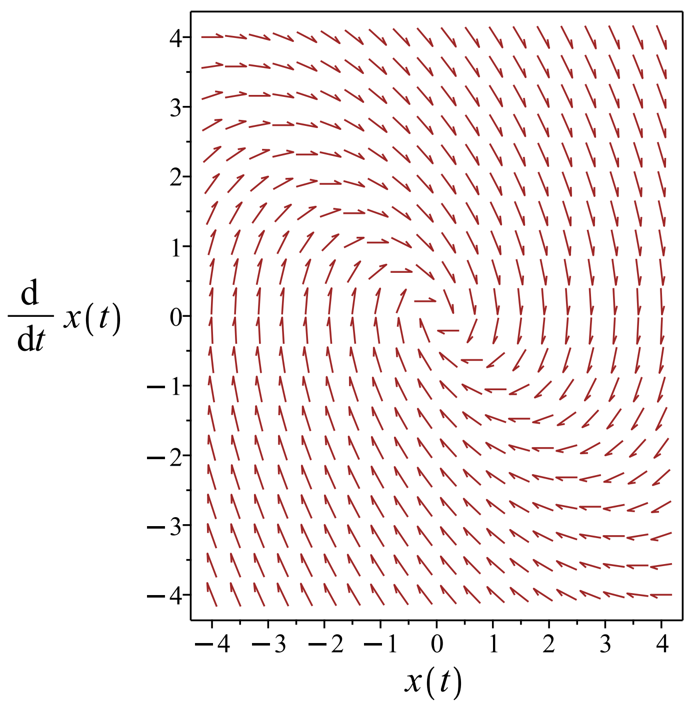


Figure 261: Slope field plot

Verification of solutions

$$
x = \mathrm{e}^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right) + \frac{6\,\mathrm{e}^{2t}\cos(t)}{61} + \frac{5\,\mathrm{e}^{2t}\sin(t)}{61} - 2t + t^2
$$

Verified OK.

9.6.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{3z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\,e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 226: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\frac{\sqrt{3}\, t}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\,dt \\
&= x_1\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''+x'+x=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)+\frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{2t}\cos(t)+t^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{2t}\cos(t), \mathrm{e}^{2t}\sin(t)\}, \{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right), \frac{2\,\mathrm{e}^{-\frac{t}{2}} \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1\mathrm{e}^{2t}\cos(t)+A_2\mathrm{e}^{2t}\sin(t)+A_3+A_4t+A_5t^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&6A_1\mathrm{e}^{2t}\cos(t)-5A_1\mathrm{e}^{2t}\sin(t)+6A_2\mathrm{e}^{2t}\sin(t)+5A_2\mathrm{e}^{2t}\cos(t)\\&+2A_5+A_4+2A_5t+A_3+A_4t+A_5t^2=\mathrm{e}^{2t}\cos(t)+t^2\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=\frac{6}{61}, A_2=\frac{5}{61}, A_3=0, A_4=-2, A_5=1\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{6\,\mathrm{e}^{2t}\cos(t)}{61} + \frac{5\,\mathrm{e}^{2t}\sin(t)}{61} - 2t + t^2$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right) + \left(\frac{6\,\mathrm{e}^{2t}\cos(t)}{61} + \frac{5\,\mathrm{e}^{2t}\sin(t)}{61} - 2t + t^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} + \frac{6\,\mathrm{e}^{2t}\cos(t)}{61} + \frac{5\,\mathrm{e}^{2t}\sin(t)}{61} - 2t + t^2 \tag{1}$$

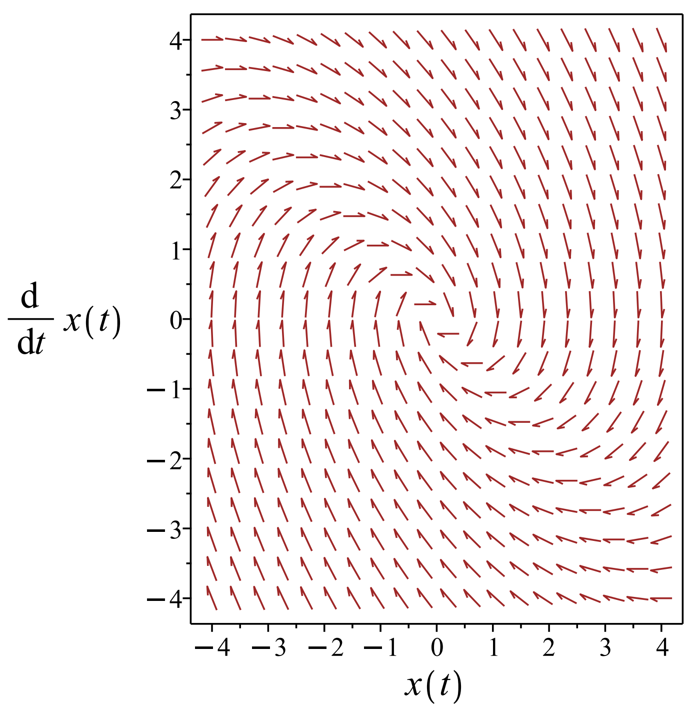


Figure 262: Slope field plot

Verification of solutions

$$x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2 c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} + \frac{6\, \mathrm{e}^{2t} \cos(t)}{61} + \frac{5\, \mathrm{e}^{2t} \sin(t)}{61} - 2t + t^2$$

Verified OK.

9.6.3 Maple step by step solution

Let's solve

$x'' + x' + x = \mathrm{e}^{2t} \cos(t) + t^2$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t) \left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t) \left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = \mathrm{e}^{2t} \cos(t) + t^2\right]$

- ○ Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) & e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) \\ -\frac{e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{2} & -\frac{e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{2} + \frac{e^{-\frac{t}{2}} \sqrt{3}\, \cos\left(\frac{\sqrt{3}\, t}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = \frac{\sqrt{3}\, e^{-t}}{2}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{2\sqrt{3}\, e^{-\frac{t}{2}} \left(\cos\left(\frac{\sqrt{3}\, t}{2}\right) \left(\int e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) (e^{2t} \cos(t) + t^2)\, dt\right) - \sin\left(\frac{\sqrt{3}\, t}{2}\right) \left(\int e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) (e^{2t} \cos(t) + t^2)\, dt\right)\right)}{3}$

- Compute integrals

  $x_p(t) = \frac{(6\cos(t) + 5\sin(t))e^{2t}}{61} + t^2 - 2t$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) c_2 + \frac{(6\cos(t) + 5\sin(t))e^{2t}}{61} + t^2 - 2t$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 52

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+x(t)=exp(2*t)*cos(t)+t^2,x(t), singsol=all)
```

$$x(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) c_2 + e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) c_1 + t^2 - 2t + \frac{(5\sin(t) + 6\cos(t))\, e^{2t}}{61}$$

✓ Solution by Mathematica

Time used: 2.836 (sec). Leaf size: 76

```
DSolve[x''[t]+x'[t]+x[t]==Exp[2*t]*Cos[t]+t^2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to t^2 - 2t + \frac{5}{61} e^{2t} \sin(t) + \frac{6}{61} e^{2t} \cos(t) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}t}{2}\right) + c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}t}{2}\right)$$

9.7 problem 1(g)

9.7.1 Solving as second order linear constant coeff ode 1197
9.7.2 Solving using Kovacic algorithm 1201
9.7.3 Maple step by step solution 1207

Internal problem ID [11458]
Internal file name [`OUTPUT/10440_Thursday_May_18_2023_04_19_32_AM_91792732/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 1(g).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + x' + x = t\,\mathrm{e}^{-t} \sin(\pi t)}$$

9.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = 1, f(t) = t\,\mathrm{e}^{-t} \sin(\pi t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)}\\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right) \right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t\,e^{-t} \sin(\pi t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\cos(\pi t), e^{-t}\sin(\pi t), t\,e^{-t}\cos(\pi t), t\,e^{-t}\sin(\pi t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right), e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 e^{-t}\cos(\pi t) + A_2 e^{-t}\sin(\pi t) + A_3 t\,e^{-t}\cos(\pi t) + A_4 t\,e^{-t}\sin(\pi t)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&A_3 t\,e^{-t}\cos(\pi t) - 2A_3 e^{-t}\pi\sin(\pi t) + 2A_4 e^{-t}\pi\cos(\pi t) - A_3 t\,e^{-t}\pi^2\cos(\pi t) \\
&+ A_4 t\,e^{-t}\sin(\pi t) - A_4 t\,e^{-t}\pi^2\sin(\pi t) + A_3 t\,e^{-t}\pi\sin(\pi t) - A_4 t\,e^{-t}\pi\cos(\pi t) \\
&- A_1 e^{-t}\pi^2\cos(\pi t) + A_1 e^{-t}\pi\sin(\pi t) - A_2 e^{-t}\pi\cos(\pi t) - A_3 e^{-t}\cos(\pi t) \\
&- A_4 e^{-t}\sin(\pi t) - A_2 e^{-t}\pi^2\sin(\pi t) + A_1 e^{-t}\cos(\pi t) + A_2 e^{-t}\sin(\pi t) = t\,e^{-t}\sin(\pi t)
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2\pi^3(\pi^2-2)}{(\pi^4-\pi^2+1)^2}, A_2 = \frac{-3\pi^4+\pi^2+1}{(\pi^4-\pi^2+1)^2}, A_3 = \frac{\pi}{\pi^4-\pi^2+1}, A_4 = \frac{-\pi^2+1}{\pi^4-\pi^2+1}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$
\begin{aligned}
x_p = &-\frac{2\pi^3(\pi^2-2)\,\mathrm{e}^{-t}\cos(\pi t)}{(\pi^4-\pi^2+1)^2}+\frac{(-3\pi^4+\pi^2+1)\,\mathrm{e}^{-t}\sin(\pi t)}{(\pi^4-\pi^2+1)^2} \\
&+\frac{\pi t\,\mathrm{e}^{-t}\cos(\pi t)}{\pi^4-\pi^2+1}+\frac{(-\pi^2+1)\,t\,\mathrm{e}^{-t}\sin(\pi t)}{\pi^4-\pi^2+1}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(\mathrm{e}^{-\frac{t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,t}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)\right)+\left(-\frac{2\pi^3(\pi^2-2)\,\mathrm{e}^{-t}\cos(\pi t)}{(\pi^4-\pi^2+1)^2}\right. \\
&\quad \left.+\frac{(-3\pi^4+\pi^2+1)\,\mathrm{e}^{-t}\sin(\pi t)}{(\pi^4-\pi^2+1)^2}+\frac{\pi t\,\mathrm{e}^{-t}\cos(\pi t)}{\pi^4-\pi^2+1}+\frac{(-\pi^2+1)\,t\,\mathrm{e}^{-t}\sin(\pi t)}{\pi^4-\pi^2+1}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
x = {}&\mathrm{e}^{-\frac{t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,t}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)-\frac{2\pi^3(\pi^2-2)\,\mathrm{e}^{-t}\cos(\pi t)}{(\pi^4-\pi^2+1)^2} \\
&+\frac{(-3\pi^4+\pi^2+1)\,\mathrm{e}^{-t}\sin(\pi t)}{(\pi^4-\pi^2+1)^2}+\frac{\pi t\,\mathrm{e}^{-t}\cos(\pi t)}{\pi^4-\pi^2+1}+\frac{(-\pi^2+1)\,t\,\mathrm{e}^{-t}\sin(\pi t)}{\pi^4-\pi^2+1}
\end{aligned} \tag{1}
$$

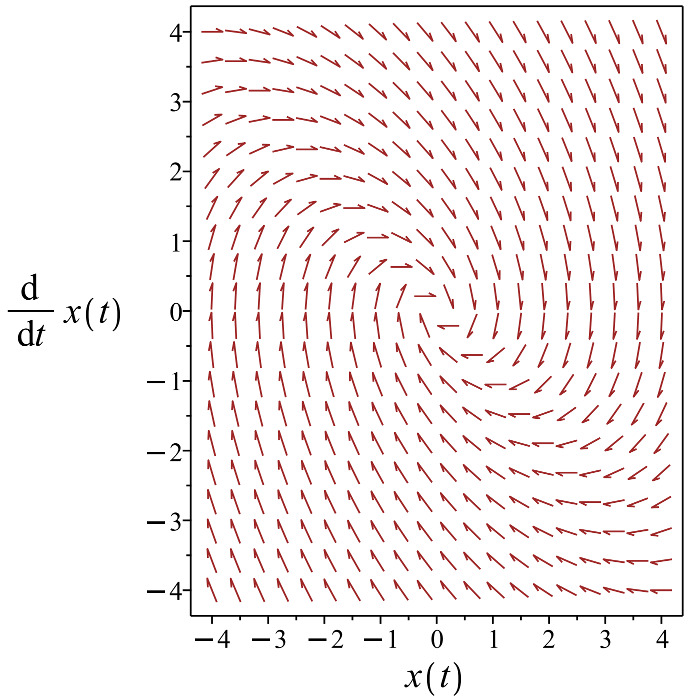


Figure 263: Slope field plot

Verification of solutions

$$x = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right) - \frac{2\pi^3(\pi^2-2)\, e^{-t}\cos(\pi t)}{(\pi^4-\pi^2+1)^2} + \frac{(-3\pi^4+\pi^2+1)\, e^{-t}\sin(\pi t)}{(\pi^4-\pi^2+1)^2} + \frac{\pi t\, e^{-t}\cos(\pi t)}{\pi^4-\pi^2+1} + \frac{(-\pi^2+1)\, t\, e^{-t}\sin(\pi t)}{\pi^4-\pi^2+1}$$

Verified OK.

9.7.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' + x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{3z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 228: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\frac{\sqrt{3}\,t}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2 c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t\,\mathrm{e}^{-t} \sin(\pi t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-t} \cos(\pi t), \mathrm{e}^{-t} \sin(\pi t), t\,\mathrm{e}^{-t} \cos(\pi t), t\,\mathrm{e}^{-t} \sin(\pi t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right), \frac{2\,\mathrm{e}^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1\mathrm{e}^{-t} \cos(\pi t) + A_2\mathrm{e}^{-t} \sin(\pi t) + A_3 t\,\mathrm{e}^{-t} \cos(\pi t) + A_4 t\,\mathrm{e}^{-t} \sin(\pi t)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&-A_3 t\,\mathrm{e}^{-t}\pi^2 \cos(\pi t) + A_3 t\,\mathrm{e}^{-t}\pi \sin(\pi t) - A_4 t\,\mathrm{e}^{-t}\pi \cos(\pi t) \\
&- A_1\mathrm{e}^{-t}\pi^2 \cos(\pi t) + A_3 t\,\mathrm{e}^{-t} \cos(\pi t) + A_1\mathrm{e}^{-t}\pi \sin(\pi t) \\
&- A_2\mathrm{e}^{-t}\pi \cos(\pi t) - A_3\mathrm{e}^{-t} \cos(\pi t) - A_4\mathrm{e}^{-t} \sin(\pi t) + A_4 t\,\mathrm{e}^{-t} \sin(\pi t) \\
&- A_2\mathrm{e}^{-t}\pi^2 \sin(\pi t) - 2A_3\mathrm{e}^{-t}\pi \sin(\pi t) + 2A_4\mathrm{e}^{-t}\pi \cos(\pi t) \\
&+ A_1\mathrm{e}^{-t} \cos(\pi t) + A_2\mathrm{e}^{-t} \sin(\pi t) - A_4 t\,\mathrm{e}^{-t}\pi^2 \sin(\pi t) = t\,\mathrm{e}^{-t} \sin(\pi t)
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2\pi^3(\pi^2-2)}{(\pi^4-\pi^2+1)^2}, A_2 = \frac{-3\pi^4+\pi^2+1}{(\pi^4-\pi^2+1)^2}, A_3 = \frac{\pi}{\pi^4-\pi^2+1}, A_4 = \frac{-\pi^2+1}{\pi^4-\pi^2+1}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$\begin{aligned}
x_p = &-\frac{2\pi^3(\pi^2-2)\,\mathrm{e}^{-t} \cos(\pi t)}{(\pi^4-\pi^2+1)^2} + \frac{(-3\pi^4+\pi^2+1)\,\mathrm{e}^{-t} \sin(\pi t)}{(\pi^4-\pi^2+1)^2} \\
&+ \frac{\pi t\,\mathrm{e}^{-t} \cos(\pi t)}{\pi^4-\pi^2+1} + \frac{(-\pi^2+1)\,t\,\mathrm{e}^{-t} \sin(\pi t)}{\pi^4-\pi^2+1}
\end{aligned}$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2 c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3}\right) + \left(-\frac{2\pi^3 (\pi^2 - 2)\, \mathrm{e}^{-t} \cos(\pi t)}{(\pi^4 - \pi^2 + 1)^2}\right. \\
&\quad \left. + \frac{(-3\pi^4 + \pi^2 + 1)\, \mathrm{e}^{-t} \sin(\pi t)}{(\pi^4 - \pi^2 + 1)^2} + \frac{\pi t\, \mathrm{e}^{-t} \cos(\pi t)}{\pi^4 - \pi^2 + 1} + \frac{(-\pi^2 + 1)\, t\, \mathrm{e}^{-t} \sin(\pi t)}{\pi^4 - \pi^2 + 1}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2 c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} - \frac{2\pi^3 (\pi^2 - 2)\, \mathrm{e}^{-t} \cos(\pi t)}{(\pi^4 - \pi^2 + 1)^2} \\
+ \frac{(-3\pi^4 + \pi^2 + 1)\, \mathrm{e}^{-t} \sin(\pi t)}{(\pi^4 - \pi^2 + 1)^2} + \frac{\pi t\, \mathrm{e}^{-t} \cos(\pi t)}{\pi^4 - \pi^2 + 1} + \frac{(-\pi^2 + 1)\, t\, \mathrm{e}^{-t} \sin(\pi t)}{\pi^4 - \pi^2 + 1}
\end{aligned} \tag{1}
$$

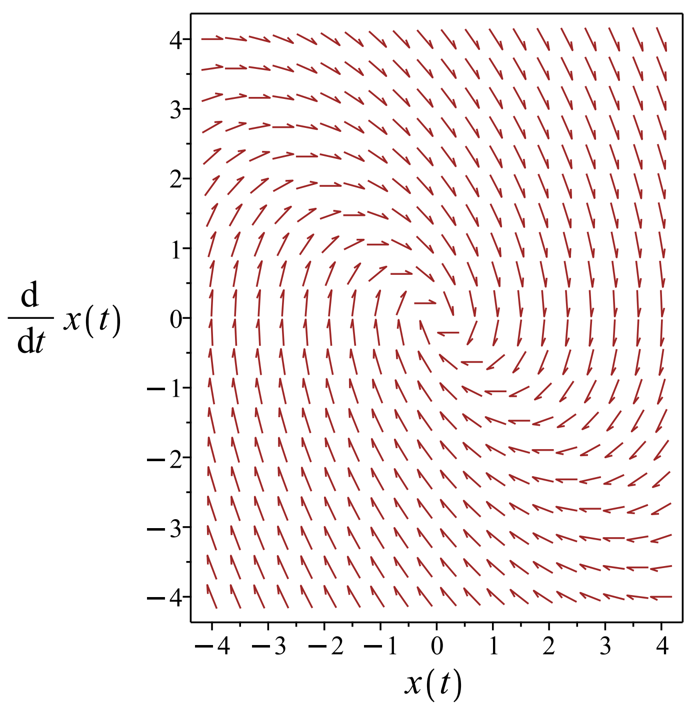


Figure 264: Slope field plot

Verification of solutions

$$x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2 c_2 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} - \frac{2\pi^3 (\pi^2 - 2)\, e^{-t} \cos(\pi t)}{(\pi^4 - \pi^2 + 1)^2} + \frac{(-3\pi^4 + \pi^2 + 1)\, e^{-t} \sin(\pi t)}{(\pi^4 - \pi^2 + 1)^2} + \frac{\pi t\, e^{-t} \cos(\pi t)}{\pi^4 - \pi^2 + 1} + \frac{(-\pi^2 + 1)\, t\, e^{-t} \sin(\pi t)}{\pi^4 - \pi^2 + 1}$$

Verified OK.

9.7.3 Maple step by step solution

Let's solve

$x'' + x' + x = t\, e^{-t} \sin(\pi t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

$$\left[x_p(t)=-x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right)+x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t)=t\,\mathrm{e}^{-t}\sin(\pi t)\right]$$

- Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t))=\left[\begin{array}{cc}\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right) & \mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right) \\ -\frac{\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2}-\frac{\mathrm{e}^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} & -\frac{\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2}+\frac{\mathrm{e}^{-\frac{t}{2}}\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2}\end{array}\right]$$

- Compute Wronskian

$$W(x_1(t), x_2(t))=\frac{\sqrt{3}\,\mathrm{e}^{-t}}{2}$$

- Substitute functions into equation for $x_p(t)$

$$x_p(t)=-\frac{2\,\mathrm{e}^{-\frac{t}{2}}\sqrt{3}\left(\cos\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int \mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)t\sin(\pi t)dt\right)-\sin\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int \mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)t\sin(\pi t)dt\right)\right)}{3}$$

- Compute integrals

$$x_p(t)=-\frac{\left(\left(t\,\pi^6+(-2t+3)\pi^4+(2t-1)\pi^2-t-1\right)\sin(\pi t)-\left((-2+t)\pi^4+(-t+4)\pi^2+t\right)\pi\cos(\pi t)\right)\mathrm{e}^{-t}}{\left(1-\pi\sqrt{3}+\pi^2\right)^2\left(1+\pi\sqrt{3}+\pi^2\right)^2}$$

- Substitute particular solution into general solution to ODE

$$x=c_1\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)+\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2-\frac{\left(\left(t\,\pi^6+(-2t+3)\pi^4+(2t-1)\pi^2-t-1\right)\sin(\pi t)-\left((-2+t)\pi^4+(-t+4)\pi^2+t\right)\pi}{\left(1-\pi\sqrt{3}+\pi^2\right)^2\left(1+\pi\sqrt{3}+\pi^2\right)^2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 131

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+x(t)=t*exp(-t)*sin(Pi*t),x(t), singsol=all)
```

$$x(t) = -\frac{e^{-\frac{t}{2}} \left(-c_1 (\pi^4 - \pi^2 + 1)^2 \cos\left(\frac{\sqrt{3}\, t}{2}\right) - c_2 (\pi^4 - \pi^2 + 1)^2 \sin\left(\frac{\sqrt{3}\, t}{2}\right) + ((\pi^6 t + (-2t + 3)\, \pi^4 + (2t - 1)\, \pi^2}{(\pi^4 - \pi^2 + 1)^2}$$

✓ Solution by Mathematica

Time used: 0.155 (sec). Leaf size: 123

```
DSolve[x''[t]+x'[t]+x[t]==t*Exp[-t]*Sin[Pi*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-t} \left(\frac{(\pi^2 (1 - 2t) - \pi^6 t + t + \pi^4 (2t - 3) + 1) \sin(\pi t) + \pi (-\pi^2 (t - 4) + \pi^4 (t - 2) + t) \cos(\pi t)}{(1 - \pi^2 + \pi^4)^2} + c_2 e^{t/2} \cos\left(\frac{\sqrt{3} t}{2}\right) + c_1 e^{t/2} \sin\left(\frac{\sqrt{3} t}{2}\right) \right)$$

9.8 problem 1(h)

9.8.1 Solving as second order linear constant coeff ode 1210
9.8.2 Solving using Kovacic algorithm 1214
9.8.3 Maple step by step solution 1220

Internal problem ID [11459]
Internal file name [`OUTPUT/10441_Thursday_May_18_2023_04_19_34_AM_37135740/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 1(h).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + x' + x = (t + 2)\sin(\pi t)}$$

9.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = 1, f(t) = (t + 2)\sin(\pi t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \lambda \,\mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right) \right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(t+2)\sin(\pi t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{t\cos(\pi t), \sin(\pi t)\, t, \cos(\pi t), \sin(\pi t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right), \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 t\cos(\pi t) + A_2 \sin(\pi t)\, t + A_3 \cos(\pi t) + A_4 \sin(\pi t)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&-2A_1\pi\sin(\pi t) - A_1 t\,\pi^2\cos(\pi t) - A_2\pi^2\sin(\pi t)\, t + 2A_2\pi\cos(\pi t) \\
&- A_3\pi^2\cos(\pi t) - A_4\pi^2\sin(\pi t) + A_1\cos(\pi t) - A_1 t\pi\sin(\pi t) \\
&+ A_2\pi\cos(\pi t)\, t + A_2\sin(\pi t) - A_3\pi\sin(\pi t) + A_4\pi\cos(\pi t) \\
&+ A_1 t\cos(\pi t) + A_2\sin(\pi t)\, t + A_3\cos(\pi t) + A_4\sin(\pi t) = (t+2)\sin(\pi t)
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{\pi}{\pi^4 - \pi^2 + 1}, A_2 = \frac{-\pi^2 + 1}{\pi^4 - \pi^2 + 1}, A_3 = \frac{-4\pi^5 + 6\pi^3 - 2\pi}{(\pi^4 - \pi^2 + 1)^2}, A_4 = \frac{-2\pi^6 + 7\pi^4 - 5\pi^2 + 1}{(\pi^4 - \pi^2 + 1)^2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{\pi t \cos(\pi t)}{\pi^4 - \pi^2 + 1} + \frac{(-\pi^2 + 1)\sin(\pi t)\, t}{\pi^4 - \pi^2 + 1} + \frac{(-4\pi^5 + 6\pi^3 - 2\pi)\cos(\pi t)}{(\pi^4 - \pi^2 + 1)^2} + \frac{(-2\pi^6 + 7\pi^4 - 5\pi^2 + 1)\sin(\pi t)}{(\pi^4 - \pi^2 + 1)^2}$$

Therefore the general solution is

$$\begin{aligned}
x &= x_h + x_p \\
&= \left(e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)\right) + \left(-\frac{\pi t \cos(\pi t)}{\pi^4 - \pi^2 + 1} + \frac{(-\pi^2 + 1)\sin(\pi t)\, t}{\pi^4 - \pi^2 + 1} \right. \\
&\quad \left. + \frac{(-4\pi^5 + 6\pi^3 - 2\pi)\cos(\pi t)}{(\pi^4 - \pi^2 + 1)^2} + \frac{(-2\pi^6 + 7\pi^4 - 5\pi^2 + 1)\sin(\pi t)}{(\pi^4 - \pi^2 + 1)^2}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
x = {}& e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right) - \frac{\pi t \cos(\pi t)}{\pi^4 - \pi^2 + 1} + \frac{(-\pi^2 + 1)\sin(\pi t)\, t}{\pi^4 - \pi^2 + 1} \\
&+ \frac{(-4\pi^5 + 6\pi^3 - 2\pi)\cos(\pi t)}{(\pi^4 - \pi^2 + 1)^2} + \frac{(-2\pi^6 + 7\pi^4 - 5\pi^2 + 1)\sin(\pi t)}{(\pi^4 - \pi^2 + 1)^2}
\end{aligned} \tag{1}$$

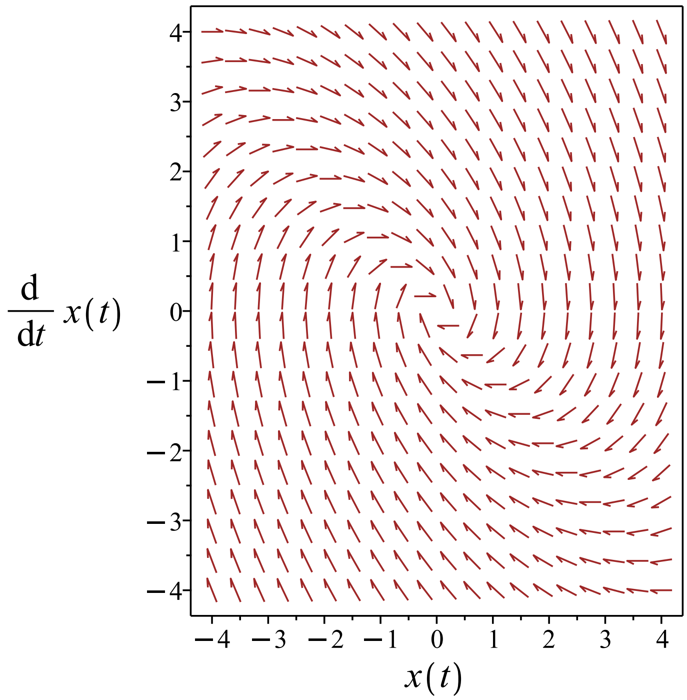


Figure 265: Slope field plot

Verification of solutions

$$x = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right) - \frac{\pi t \cos(\pi t)}{\pi^4 - \pi^2 + 1} + \frac{(-\pi^2 + 1)\sin(\pi t)\,t}{\pi^4 - \pi^2 + 1} + \frac{(-4\pi^5 + 6\pi^3 - 2\pi)\cos(\pi t)}{(\pi^4 - \pi^2 + 1)^2} + \frac{(-2\pi^6 + 7\pi^4 - 5\pi^2 + 1)\sin(\pi t)}{(\pi^4 - \pi^2 + 1)^2}$$

Verified OK.

9.8.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{3z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 230: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\frac{\sqrt{3}\,t}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\ &= z_1 e^{-\frac{t}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\, dt}}{(x_1)^2}\, dt \\ &= x_1 \int \frac{e^{-t}}{(x_1)^2}\, dt \\ &= x_1 \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, t}{2}\right)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, t}{2}\right)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2 c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
(t+2)\sin(\pi t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{t\cos(\pi t), \sin(\pi t)\, t, \cos(\pi t), \sin(\pi t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{ e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right), \frac{2\, e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} \right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_1 t\cos(\pi t) + A_2 \sin(\pi t)\, t + A_3 \cos(\pi t) + A_4 \sin(\pi t)
$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&-2A_1\pi\sin(\pi t) - A_1 t\,\pi^2\cos(\pi t) - A_2\pi^2\sin(\pi t)\, t + 2A_2\pi\cos(\pi t) \\
&- A_3\pi^2\cos(\pi t) - A_4\pi^2\sin(\pi t) + A_1\cos(\pi t) - A_1 t\pi\sin(\pi t) \\
&+ A_2\pi\cos(\pi t)\, t + A_2\sin(\pi t) - A_3\pi\sin(\pi t) + A_4\pi\cos(\pi t) \\
&+ A_1 t\cos(\pi t) + A_2\sin(\pi t)\, t + A_3\cos(\pi t) + A_4\sin(\pi t) = (t+2)\sin(\pi t)
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = -\frac{\pi}{\pi^4-\pi^2+1}, A_2 = \frac{-\pi^2+1}{\pi^4-\pi^2+1}, A_3 = \frac{-4\pi^5+6\pi^3-2\pi}{(\pi^4-\pi^2+1)^2}, A_4 = \frac{-2\pi^6+7\pi^4-5\pi^2+1}{(\pi^4-\pi^2+1)^2}\right]
$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$
\begin{aligned}
x_p = &-\frac{\pi t\cos(\pi t)}{\pi^4-\pi^2+1} + \frac{(-\pi^2+1)\sin(\pi t)\, t}{\pi^4-\pi^2+1} \\
&+ \frac{(-4\pi^5+6\pi^3-2\pi)\cos(\pi t)}{(\pi^4-\pi^2+1)^2} + \frac{(-2\pi^6+7\pi^4-5\pi^2+1)\sin(\pi t)}{(\pi^4-\pi^2+1)^2}
\end{aligned}
$$

Therefore the general solution is

$$x = x_h + x_p$$

$$= \left(c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right) + \left(-\frac{\pi t \cos(\pi t)}{\pi^4 - \pi^2 + 1} + \frac{(-\pi^2 + 1)\sin(\pi t)\, t}{\pi^4 - \pi^2 + 1} + \frac{(-4\pi^5 + 6\pi^3 - 2\pi)\cos(\pi t)}{(\pi^4 - \pi^2 + 1)^2} + \frac{(-2\pi^6 + 7\pi^4 - 5\pi^2 + 1)\sin(\pi t)}{(\pi^4 - \pi^2 + 1)^2}\right)$$

Summary

The solution(s) found are the following

$$x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} - \frac{\pi t \cos(\pi t)}{\pi^4 - \pi^2 + 1} + \frac{(-\pi^2 + 1)\sin(\pi t)\, t}{\pi^4 - \pi^2 + 1} + \frac{(-4\pi^5 + 6\pi^3 - 2\pi)\cos(\pi t)}{(\pi^4 - \pi^2 + 1)^2} + \frac{(-2\pi^6 + 7\pi^4 - 5\pi^2 + 1)\sin(\pi t)}{(\pi^4 - \pi^2 + 1)^2} \tag{1}$$

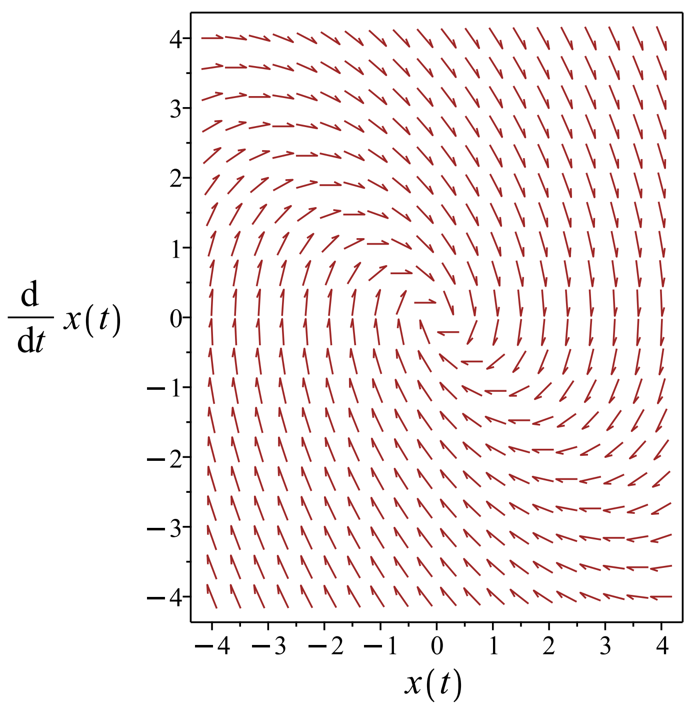


Figure 266: Slope field plot

Verification of solutions

$$x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2 c_2 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} - \frac{\pi t \cos(\pi t)}{\pi^4 - \pi^2 + 1} + \frac{(-\pi^2 + 1) \sin(\pi t)\, t}{\pi^4 - \pi^2 + 1} + \frac{(-4\pi^5 + 6\pi^3 - 2\pi) \cos(\pi t)}{(\pi^4 - \pi^2 + 1)^2} + \frac{(-2\pi^6 + 7\pi^4 - 5\pi^2 + 1) \sin(\pi t)}{(\pi^4 - \pi^2 + 1)^2}$$

Verified OK.

9.8.3 Maple step by step solution

Let's solve

$x'' + x' + x = (t + 2) \sin(\pi t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

$$\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = (t+2)\sin(\pi t)\right]$$

- Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right) & e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right) \\ -\frac{e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} & -\frac{e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} + \frac{e^{-\frac{t}{2}}\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

$$W(x_1(t), x_2(t)) = \frac{\sqrt{3}\,e^{-t}}{2}$$

- Substitute functions into equation for $x_p(t)$

$$x_p(t) = -\frac{2\sqrt{3}\,e^{-\frac{t}{2}}\left(\cos\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int (t+2)e^{\frac{t}{2}}\sin(\pi t)\sin\left(\frac{\sqrt{3}\,t}{2}\right)dt\right) - \sin\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int (t+2)e^{\frac{t}{2}}\sin(\pi t)\cos\left(\frac{\sqrt{3}\,t}{2}\right)dt\right)\right)}{3}$$

- Compute integrals

$$x_p(t) = \frac{((-t-2)\pi^6 + (2t+7)\pi^4 + (-2t-5)\pi^2 + t + 1)\sin(\pi t) - \pi\cos(\pi t)((t+4)\pi^4 + (-t-6)\pi^2 + t + 2)}{\left(1-\pi\sqrt{3}+\pi^2\right)^2\left(1+\pi\sqrt{3}+\pi^2\right)^2}$$

- Substitute particular solution into general solution to ODE

$$x = c_1 e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right) + e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2 + \frac{((-t-2)\pi^6 + (2t+7)\pi^4 + (-2t-5)\pi^2 + t + 1)\sin(\pi t) - \pi\cos(\pi t)((t+4)\pi^4 + (-}{\left(1-\pi\sqrt{3}+\pi^2\right)^2\left(1+\pi\sqrt{3}+\pi^2\right)^2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 129

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+x(t)=(t+2)*sin(Pi*t),x(t), singsol=all)
```

$$x(t) = \frac{c_1 e^{-\frac{t}{2}} (\pi^4 - \pi^2 + 1)^2 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 e^{-\frac{t}{2}} (\pi^4 - \pi^2 + 1)^2 \sin\left(\frac{\sqrt{3}\, t}{2}\right) + ((-t-2)\,\pi^6 + (2t+7)\,\pi^4 + (-2t -}{(\pi^4 - \pi^2 + 1)^2}$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 122

```
DSolve[x''[t]+x'[t]+x[t]==(t+2)*Sin[Pi*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{(t - \pi^6(t+2) - \pi^2(2t+5) + \pi^4(2t+7) + 1)\sin(\pi t) - \pi(t + \pi^4(t+4) - \pi^2(t+6) + 2)\cos(\pi t)}{(1 - \pi^2 + \pi^4)^2} + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}t}{2}\right) + c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}t}{2}\right)$$

9.9 problem 1(i)

9.9.1 Solving as second order linear constant coeff ode 1223
9.9.2 Solving using Kovacic algorithm 1227
9.9.3 Maple step by step solution 1232

Internal problem ID [11460]
Internal file name [`OUTPUT/10442_Thursday_May_18_2023_04_19_37_AM_83877305/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 1(i).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x'' + x' + x = 4t + 5\,\mathrm{e}^{-t}}$$

9.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = 1, f(t) = 4t + 5\,\mathrm{e}^{-t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)}\\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right) \right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4t + 5\,e^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}, \{1, t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right), e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 e^{-t} + A_2 + A_3 t$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{-t} + A_3 + A_2 + A_3 t = 4t + 5\,e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 5, A_2 = -4, A_3 = 4]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 5\,e^{-t} - 4 + 4t$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right) \right) \right) + (5\,e^{-t} - 4 + 4t) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right) \right) + 5\,e^{-t} - 4 + 4t \tag{1}$$

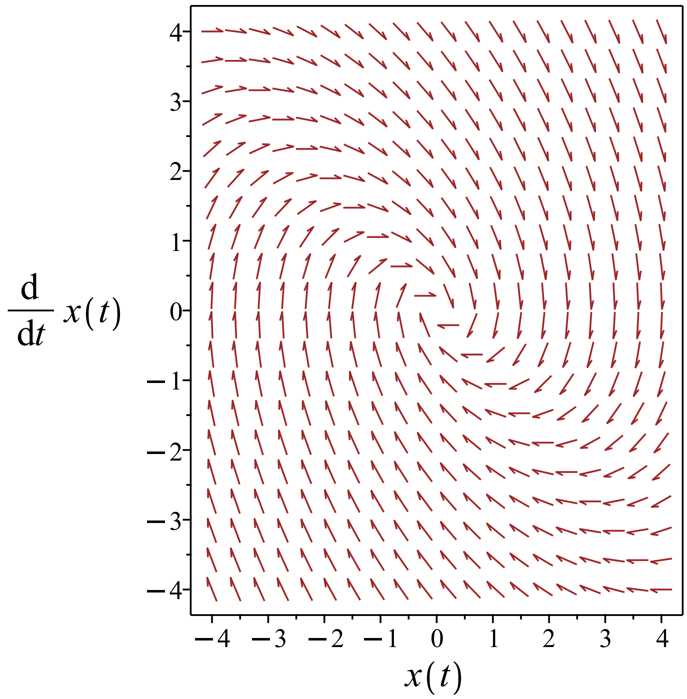


Figure 267: Slope field plot

Verification of solutions

$$x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right) \right) + 5\,e^{-t} - 4 + 4t$$

Verified OK.

9.9.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' + x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{3z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 232: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos\left(\frac{\sqrt{3}\, t}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4t + 5\,\mathrm{e}^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-t}\}, \{1, t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right), \frac{2\,\mathrm{e}^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1\mathrm{e}^{-t} + A_2 + A_3t$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1\mathrm{e}^{-t} + A_3 + A_2 + A_3t = 4t + 5\,\mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 5, A_2 = -4, A_3 = 4]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 5\,\mathrm{e}^{-t} - 4 + 4t$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right) + \left(5\,\mathrm{e}^{-t} - 4 + 4t\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} + 5\,\mathrm{e}^{-t} - 4 + 4t \tag{1}$$

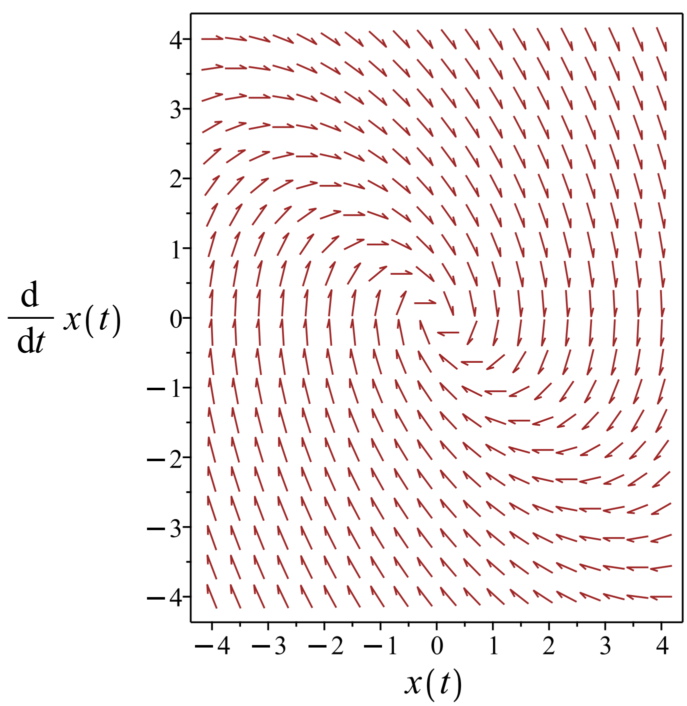


Figure 268: Slope field plot

Verification of solutions

$$x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2 c_2 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} + 5\, e^{-t} - 4 + 4t$$

Verified OK.

9.9.3 Maple step by step solution

Let's solve

$x'' + x' + x = 4t + 5\, e^{-t}$

- Highest derivative means the order of the ODE is 2

  x''
- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$
- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-3})}{2}$
- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$
- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)$
- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)$
- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$
- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t) \left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t) \left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = 4t + 5\, e^{-t}\right]$
- ○ Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) & e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) \\ -\frac{e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} & -\frac{e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} + \frac{e^{-\frac{t}{2}} \sqrt{3}\, \cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = \frac{\sqrt{3}\, e^{-t}}{2}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{2\sqrt{3}\, e^{-\frac{t}{2}} \left(\cos\left(\frac{\sqrt{3}\,t}{2}\right) \left(\int \sin\left(\frac{\sqrt{3}\,t}{2}\right) \left(4t\, e^{\frac{t}{2}} + 5\, e^{-\frac{t}{2}}\right) dt\right) - \sin\left(\frac{\sqrt{3}\,t}{2}\right) \left(\int \cos\left(\frac{\sqrt{3}\,t}{2}\right) \left(4t\, e^{\frac{t}{2}} + 5\, e^{-\frac{t}{2}}\right) dt\right)\right)}{3}$

- Compute integrals

  $x_p(t) = 5\, e^{-t} - 4 + 4t$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) c_2 + 5\, e^{-t} + 4t - 4$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+x(t)=4*t+5*exp(-t),x(t), singsol=all)
```

$$x(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) c_2 + e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) c_1 + 4t - 4 + 5\, e^{-t}$$

✓ Solution by Mathematica

Time used: 4.02 (sec). Leaf size: 59

```
DSolve[x''[t]+x'[t]+x[t]==4*t+5*Exp[-t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 4t + 5e^{-t} + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}t}{2}\right) + c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}t}{2}\right) - 4$$

9.10 problem 1(j)

9.10.1 Solving as second order linear constant coeff ode 1235
9.10.2 Solving using Kovacic algorithm 1239
9.10.3 Maple step by step solution 1244

Internal problem ID [11461]
Internal file name [`OUTPUT/10443_Thursday_May_18_2023_04_19_39_AM_73892930/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 1(j).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + x' + x = 5 \sin(2t) + t\,\mathrm{e}^{t}}$$

9.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = 1, f(t) = 5\sin(2t) + t\,\mathrm{e}^{t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)}\\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5 \sin(2t) + t\, e^{t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{t\, e^{t}, e^{t}\}, \{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right), e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 t\, e^{t} + A_2 e^{t} + A_3 \cos(2t) + A_4 \sin(2t)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&3A_1 e^{t} + 3A_1 t\, e^{t} + 3A_2 e^{t} - 3A_3 \cos(2t) - 3A_4 \sin(2t) - 2A_3 \sin(2t) + 2A_4 \cos(2t)\\ &= 5 \sin(2t) + t\, e^{t}\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = -\frac{1}{3}, A_3 = -\frac{10}{13}, A_4 = -\frac{15}{13}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{t\, e^{t}}{3} - \frac{e^{t}}{3} - \frac{10 \cos(2t)}{13} - \frac{15 \sin(2t)}{13}$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(\mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)\right) + \left(\frac{t\,\mathrm{e}^{t}}{3} - \frac{\mathrm{e}^{t}}{3} - \frac{10\cos(2t)}{13} - \frac{15\sin(2t)}{13}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = \mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right) + \frac{t\,\mathrm{e}^{t}}{3} - \frac{\mathrm{e}^{t}}{3} - \frac{10\cos(2t)}{13} - \frac{15\sin(2t)}{13} \tag{1}
$$

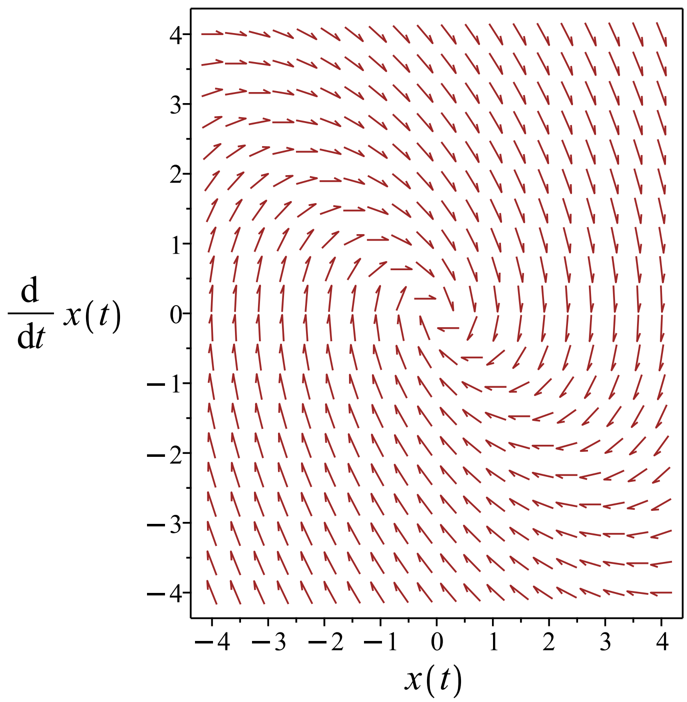


Figure 269: Slope field plot

Verification of solutions

$$
x = \mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right) + \frac{t\,\mathrm{e}^{t}}{3} - \frac{\mathrm{e}^{t}}{3} - \frac{10\cos(2t)}{13} - \frac{15\sin(2t)}{13}
$$

Verified OK.

9.10.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{3z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 234: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos\left(\frac{\sqrt{3}\, t}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''+x'+x=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5\sin(2t) + t\,\mathrm{e}^{t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{t\,\mathrm{e}^{t}, \mathrm{e}^{t}\}, \{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right), \frac{2\,\mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 t\,\mathrm{e}^{t} + A_2 \mathrm{e}^{t} + A_3 \cos(2t) + A_4 \sin(2t)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&3A_1\mathrm{e}^{t} + 3A_1 t\,\mathrm{e}^{t} + 3A_2\mathrm{e}^{t} - 3A_3\cos(2t) - 3A_4\sin(2t) - 2A_3\sin(2t) + 2A_4\cos(2t)\\ &\quad = 5\sin(2t) + t\,\mathrm{e}^{t}\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = -\frac{1}{3}, A_3 = -\frac{10}{13}, A_4 = -\frac{15}{13}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{t\,\mathrm{e}^t}{3} - \frac{\mathrm{e}^t}{3} - \frac{10\cos(2t)}{13} - \frac{15\sin(2t)}{13}$$

Therefore the general solution is

$$\begin{aligned}x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right) + \left(\frac{t\,\mathrm{e}^t}{3} - \frac{\mathrm{e}^t}{3} - \frac{10\cos(2t)}{13} - \frac{15\sin(2t)}{13}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} + \frac{t\,\mathrm{e}^t}{3} - \frac{\mathrm{e}^t}{3} - \frac{10\cos(2t)}{13} - \frac{15\sin(2t)}{13} \tag{1}$$

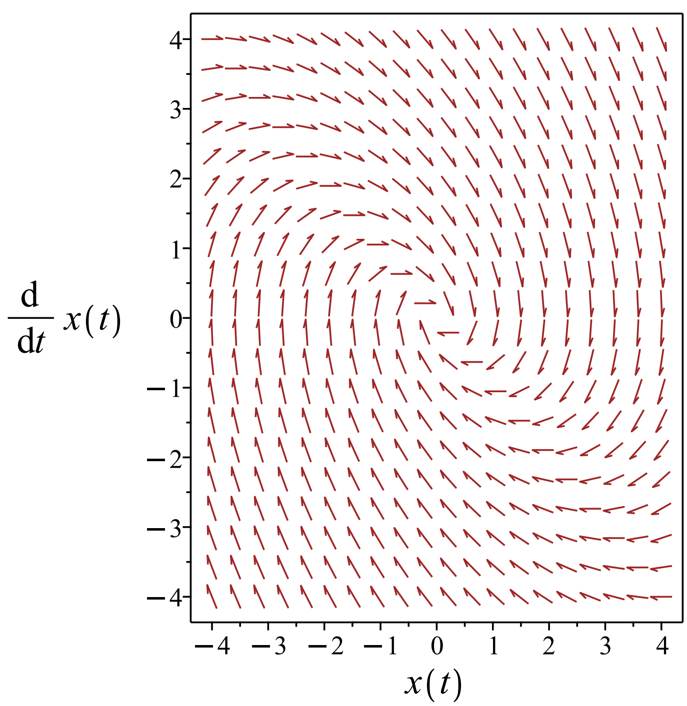


Figure 270: Slope field plot

Verification of solutions

$$x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2 c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} + \frac{t\, \mathrm{e}^{t}}{3} - \frac{\mathrm{e}^{t}}{3} - \frac{10 \cos(2t)}{13} - \frac{15 \sin(2t)}{13}$$

Verified OK.

9.10.3 Maple step by step solution

Let's solve

$x'' + x' + x = 5 \sin(2t) + t\, \mathrm{e}^{t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t) \left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t) \left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = 5 \sin(2t) + t\, \mathrm{e}^{t}\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) & e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) \\ -\frac{e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} & -\frac{e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} + \frac{e^{-\frac{t}{2}} \sqrt{3}\, \cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $$W(x_1(t), x_2(t)) = \frac{\sqrt{3}\, e^{-t}}{2}$$

- Substitute functions into equation for $x_p(t)$

  $$x_p(t) = -\frac{2\sqrt{3}\, e^{-\frac{t}{2}} \left(\cos\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)(5\sin(2t)+t\,e^{t})\,dt\right) - \sin\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)(5\sin(2t)+t\,e^{t})\,dt\right)\right)}{3}$$

- Compute integrals

  $$x_p(t) = -\frac{10\cos(2t)}{13} - \frac{15\sin(2t)}{13} + \frac{(t-1)e^{t}}{3}$$

- Substitute particular solution into general solution to ODE

  $$x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) c_2 - \frac{10\cos(2t)}{13} - \frac{15\sin(2t)}{13} + \frac{(t-1)e^{t}}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 50

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+x(t)=5*sin(2*t)+t*exp(t),x(t), singsol=all)
```

$$x(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) c_2 + e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) c_1 - \frac{10\cos(2t)}{13} - \frac{15\sin(2t)}{13} + \frac{e^{t}(t-1)}{3}$$

✓ Solution by Mathematica

Time used: 5.753 (sec). Leaf size: 83

```
DSolve[x''[t]+x'[t]+x[t]==5*Sin[2*t]+t*Exp[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{39}\left(-13e^t + 13e^t t + 30\sin^2(t) - 30\cos^2(t) - 90\sin(t)\cos(t) + 39c_2 e^{-t/2}\cos\left(\frac{\sqrt{3}t}{2}\right) + 39c_1 e^{-t/2}\sin\left(\frac{\sqrt{3}t}{2}\right)\right)$$

9.11 problem 1(k)

9.11.1 Solving as second order linear constant coeff ode 1247
9.11.2 Solving using Kovacic algorithm 1251
9.11.3 Maple step by step solution 1256

Internal problem ID [11462]
Internal file name [`OUTPUT/10444_Thursday_May_18_2023_04_19_41_AM_49005702/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 1(k).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + x' + x = t^3 + 1 - 4\cos(t)\, t}$$

9.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = 1, f(t) = t^3 + 1 - 4\cos(t)\, t$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \lambda \, \mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^3 + 1 - 4\cos(t)\, t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2, t^3\}, \{\cos(t)\, t, \sin(t)\, t, \cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right), \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_4 t^3 + A_3 t^2 + A_2 t + A_1 + A_5 \cos(t)\, t + A_6 \sin(t)\, t + A_7 \cos(t) + A_8 \sin(t)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &A_1 + A_2 + 2A_3 + A_5 \cos(t) - A_5 \sin(t)\, t + A_2 t + A_3 t^2 + A_4 t^3 \\ &+ 3A_4 t^2 + 2A_3 t - A_7 \sin(t) + A_8 \cos(t) + 6A_4 t + A_6 \cos(t)\, t \\ &+ A_6 \sin(t) - 2A_5 \sin(t) + 2A_6 \cos(t) = t^3 + 1 - 4\cos(t)\, t \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 7, A_2 = 0, A_3 = -3, A_4 = 1, A_5 = 0, A_6 = -4, A_7 = -4, A_8 = 8]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = t^3 - 3t^2 + 7 - 4\sin(t)\, t - 4\cos(t) + 8\sin(t)$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)\right) \\
&\quad + \left(t^3 - 3t^2 + 7 - 4\sin(t)\,t - 4\cos(t) + 8\sin(t)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
x = {} & e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right) + t^3 \\
& - 3t^2 + 7 - 4\sin(t)\,t - 4\cos(t) + 8\sin(t)
\end{aligned}
\tag{1}
$$

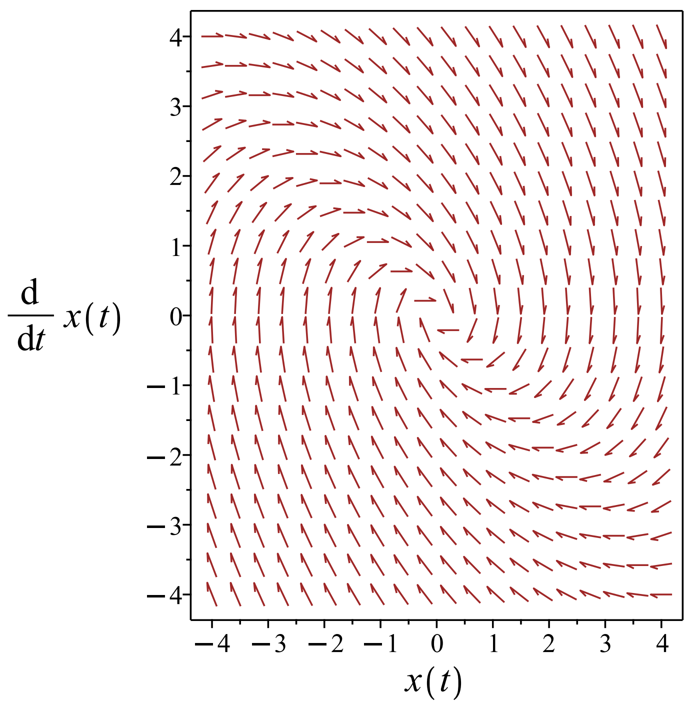


Figure 271: Slope field plot

Verification of solutions

$$
x = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right) + t^3 - 3t^2 + 7 - 4\sin(t)\,t - 4\cos(t) + 8\sin(t)
$$

Verified OK.

9.11.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' + x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{3z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 236: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos\left(\frac{\sqrt{3}\, t}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''+x'+x=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^3+1-4\cos(t)\,t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1,t,t^2,t^3\},\{\cos(t)\,t,\sin(t)\,t,\cos(t)\,,\sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right), \frac{2\,\mathrm{e}^{-\frac{t}{2}} \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_4t^3+A_3t^2+A_2t+A_1+A_5\cos(t)\,t+A_6\sin(t)\,t+A_7\cos(t)+A_8\sin(t)$$

The unknowns $\{A_1,A_2,A_3,A_4,A_5,A_6,A_7,A_8\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&A_2t+A_3t^2+A_4t^3+3A_4t^2+2A_3t-A_7\sin(t)+A_8\cos(t)\\&+6A_4t+A_1+A_2+2A_3+A_5\cos(t)-A_5\sin(t)\,t+A_6\cos(t)\,t\\&+A_6\sin(t)-2A_5\sin(t)+2A_6\cos(t)=t^3+1-4\cos(t)\,t\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 7, A_2 = 0, A_3 = -3, A_4 = 1, A_5 = 0, A_6 = -4, A_7 = -4, A_8 = 8]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = t^3 - 3t^2 + 7 - 4\sin(t)\, t - 4\cos(t) + 8\sin(t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3}\right) \\ &\quad + \left(t^3 - 3t^2 + 7 - 4\sin(t)\, t - 4\cos(t) + 8\sin(t)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} + t^3 \\ - 3t^2 + 7 - 4\sin(t)\, t - 4\cos(t) + 8\sin(t) \end{aligned} \tag{1}$$

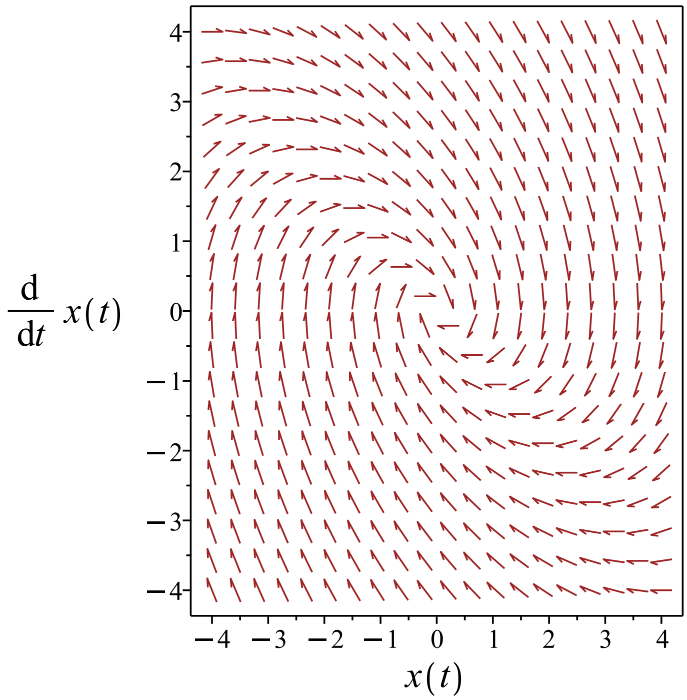


Figure 272: Slope field plot

Verification of solutions

$$x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2c_2 e^{-\frac{t}{2}}\sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} + t^3 - 3t^2 + 7 - 4\sin(t)\, t - 4\cos(t) + 8\sin(t)$$

Verified OK.

9.11.3 Maple step by step solution

Let's solve

$x'' + x' + x = t^3 + 1 - 4\cos(t)\, t$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

$r = \frac{(-1) \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) c_2 + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

- Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = t^3 + 1 - 4\cos(t)\, t\right]$

- Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) & e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) \\ -\frac{e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}} \sqrt{3} \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{2} & -\frac{e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{2} + \frac{e^{-\frac{t}{2}} \sqrt{3} \cos\left(\frac{\sqrt{3}\, t}{2}\right)}{2} \end{bmatrix}$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = \frac{\sqrt{3}\, e^{-t}}{2}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = \frac{2\sqrt{3}\, e^{-\frac{t}{2}} \left(\cos\left(\frac{\sqrt{3}\, t}{2}\right)\left(\int -e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)(t^3+1-4\cos(t) t) dt\right) - \sin\left(\frac{\sqrt{3}\, t}{2}\right)\left(\int -e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)(t^3+1-4\cos(t) t) dt\right)\right)}{3}$

- Compute integrals

  $x_p(t) = (-4t + 8)\sin(t) + t^3 - 3t^2 - 4\cos(t) + 7$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) c_2 + (-4t + 8)\sin(t) + t^3 - 3t^2 - 4\cos(t) + 7$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 52

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+x(t)=t^3+1-4*t*cos(t),x(t), singsol=all)
```

$$x(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) c_2 + e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) c_1 + (-4t+8)\sin(t) + t^3 - 3t^2 - 4\cos(t) + 7$$

✓ Solution by Mathematica

Time used: 4.528 (sec). Leaf size: 70

```
DSolve[x''[t]+x'[t]+x[t]==t^3+1-4*t*Cos[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to t^3 - 3t^2 - 4t\sin(t) + 8\sin(t) - 4\cos(t) + c_2 e^{-t/2}\cos\left(\frac{\sqrt{3}t}{2}\right) + c_1 e^{-t/2}\sin\left(\frac{\sqrt{3}t}{2}\right) + 7$$

9.12 problem 1(L)

9.12.1 Solving as second order linear constant coeff ode 1259
9.12.2 Solving using Kovacic algorithm 1263
9.12.3 Maple step by step solution 1268

Internal problem ID [11463]
Internal file name [`OUTPUT/10445_Thursday_May_18_2023_04_19_43_AM_67712321/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 1(L).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + x' + x = -6 + 2\,\mathrm{e}^{2t} \sin(t)}$$

9.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = 1, f(t) = -6 + 2\,\mathrm{e}^{2t} \sin(t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \lambda \, \mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-6 + 2\,e^{2t} \sin(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^{2t} \cos(t), e^{2t} \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right), e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 + A_2 e^{2t} \cos(t) + A_3 e^{2t} \sin(t)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_2 e^{2t} \cos(t) - 5A_2 e^{2t} \sin(t) + 6A_3 e^{2t} \sin(t) + 5A_3 e^{2t} \cos(t) + A_1 = -6 + 2\,e^{2t} \sin(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -6, A_2 = -\frac{10}{61}, A_3 = \frac{12}{61}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -6 - \frac{10\,e^{2t} \cos(t)}{61} + \frac{12\,e^{2t} \sin(t)}{61}$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(\mathrm{e}^{-\frac{t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,t}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)\right)+\left(-6-\frac{10\,\mathrm{e}^{2t}\cos\left(t\right)}{61}+\frac{12\,\mathrm{e}^{2t}\sin\left(t\right)}{61}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = \mathrm{e}^{-\frac{t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,t}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)-6-\frac{10\,\mathrm{e}^{2t}\cos\left(t\right)}{61}+\frac{12\,\mathrm{e}^{2t}\sin\left(t\right)}{61} \tag{1}
$$

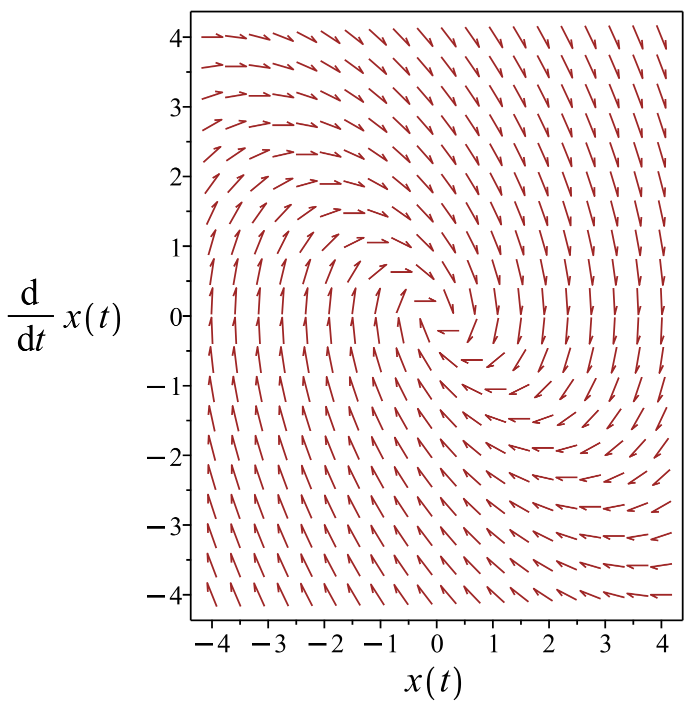


Figure 273: Slope field plot

Verification of solutions

$$
x = \mathrm{e}^{-\frac{t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,t}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)-6-\frac{10\,\mathrm{e}^{2t}\cos\left(t\right)}{61}+\frac{12\,\mathrm{e}^{2t}\sin\left(t\right)}{61}
$$

Verified OK.

9.12.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{3z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 238: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\frac{\sqrt{3}\,t}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''+x'+x=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h=c_1\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)+\frac{2c_2\mathrm{e}^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-6+2\,\mathrm{e}^{2t}\sin(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\},\{\mathrm{e}^{2t}\cos(t)\,,\mathrm{e}^{2t}\sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right),\frac{2\,\mathrm{e}^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p=A_1+A_2\mathrm{e}^{2t}\cos(t)+A_3\mathrm{e}^{2t}\sin(t)$$

The unknowns $\{A_1,A_2,A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_2\mathrm{e}^{2t}\cos(t)-5A_2\mathrm{e}^{2t}\sin(t)+6A_3\mathrm{e}^{2t}\sin(t)+5A_3\mathrm{e}^{2t}\cos(t)+A_1=-6+2\,\mathrm{e}^{2t}\sin(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=-6,A_2=-\frac{10}{61},A_3=\frac{12}{61}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -6 - \frac{10\,\mathrm{e}^{2t}\cos(t)}{61} + \frac{12\,\mathrm{e}^{2t}\sin(t)}{61}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right) + \left(-6 - \frac{10\,\mathrm{e}^{2t}\cos(t)}{61} + \frac{12\,\mathrm{e}^{2t}\sin(t)}{61}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 \mathrm{e}^{-\frac{t}{2}} \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} - 6 - \frac{10\,\mathrm{e}^{2t}\cos(t)}{61} + \frac{12\,\mathrm{e}^{2t}\sin(t)}{61} \tag{1}$$

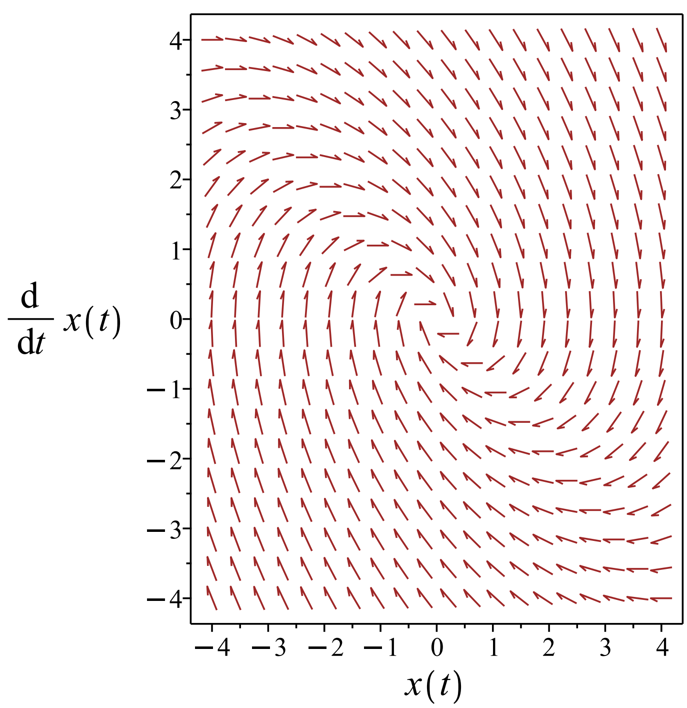


Figure 274: Slope field plot

Verification of solutions

$$x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2 c_2 e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} - 6 - \frac{10\, e^{2t} \cos(t)}{61} + \frac{12\, e^{2t} \sin(t)}{61}$$

Verified OK.

9.12.3 Maple step by step solution

Let's solve

$x'' + x' + x = -6 + 2\, e^{2t} \sin(t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t) \left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t) \left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = -6 + 2\, e^{2t} \sin(t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) & e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) \\ -\frac{e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} & -\frac{e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} + \frac{e^{-\frac{t}{2}} \sqrt{3}\, \cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = \frac{\sqrt{3}\, e^{-t}}{2}$
- Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{4\, e^{-\frac{t}{2}} \sqrt{3}\left(\cos\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)(-3+e^{2t}\sin(t))\,dt\right) - \sin\left(\frac{\sqrt{3}\,t}{2}\right)\left(\int e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)(-3+e^{2t}\sin(t))\,dt\right)\right)}{3}$
- Compute integrals

  $x_p(t) = -6 + \frac{2(-5\cos(t)+6\sin(t))\, e^{2t}}{61}$

- Substitute particular solution into general solution to ODE

  $x = -6 + c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) c_2 + \frac{2(-5\cos(t)+6\sin(t))\, e^{2t}}{61}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+x(t)=-6+2*exp(2*t)*sin(t),x(t), singsol=all)
```

$$x(t) = e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) c_2 + e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) c_1 - 6 + \frac{2(6\sin(t) - 5\cos(t))\, e^{2t}}{61}$$

✓ Solution by Mathematica

Time used: 1.794 (sec). Leaf size: 71

```
DSolve[x''[t]+x'[t]+x[t]==-6+2*Exp[2*t]*Sin[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{12}{61} e^{2t} \sin(t) - \frac{10}{61} e^{2t} \cos(t) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}t}{2}\right) + c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}t}{2}\right) - 6$$

9.13 problem 2(a)

9.13.1 Solving as second order linear constant coeff ode 1271
9.13.2 Solving using Kovacic algorithm 1274
9.13.3 Maple step by step solution 1279

Internal problem ID [11464]
Internal file name [`OUTPUT/10446_Thursday_May_18_2023_04_19_45_AM_93723987/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 2(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + 7x = t\,\mathrm{e}^{3t}}$$

9.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = 7, f(t) = t\,\mathrm{e}^{3t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 7x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 7$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 7\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 7 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 7$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(7)} \\ &= \pm i\sqrt{7}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{7} \\ \lambda_2 &= -i\sqrt{7}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{7} \\ \lambda_2 &= -i\sqrt{7}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{7}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^0\left(c_1 \cos\left(\sqrt{7}\,t\right) + c_2 \sin\left(\sqrt{7}\,t\right)\right)$$

Or

$$x = c_1 \cos\left(\sqrt{7}\,t\right) + c_2 \sin\left(\sqrt{7}\,t\right)$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \cos\left(\sqrt{7}\, t\right) + c_2 \sin\left(\sqrt{7}\, t\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t\, e^{3t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{t\, e^{3t}, e^{3t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(\sqrt{7}\, t\right), \sin\left(\sqrt{7}\, t\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 t\, e^{3t} + A_2 e^{3t}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 e^{3t} + 16A_1 t\, e^{3t} + 16A_2 e^{3t} = t\, e^{3t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{16}, A_2 = -\frac{3}{128}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{t\, e^{3t}}{16} - \frac{3\, e^{3t}}{128}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \cos\left(\sqrt{7}\, t\right) + c_2 \sin\left(\sqrt{7}\, t\right)\right) + \left(\frac{t\, e^{3t}}{16} - \frac{3\, e^{3t}}{128}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$x = c_1 \cos\left(\sqrt{7}\, t\right) + c_2 \sin\left(\sqrt{7}\, t\right) + \frac{t\, e^{3t}}{16} - \frac{3\, e^{3t}}{128} \tag{1}$$

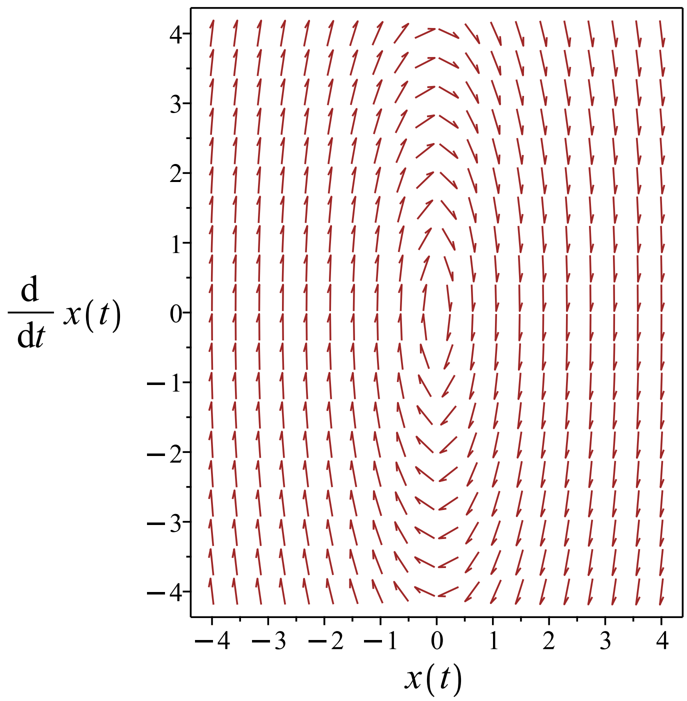


Figure 275: Slope field plot

Verification of solutions

$$x = c_1 \cos\left(\sqrt{7}\, t\right) + c_2 \sin\left(\sqrt{7}\, t\right) + \frac{t\, e^{3t}}{16} - \frac{3\, e^{3t}}{128}$$

Verified OK.

9.13.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 7x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -7 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -7z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 240: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -7$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\sqrt{7}\, t\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}x_1 &= z_1\\ &= \cos\left(\sqrt{7}\, t\right)\end{aligned}$$

Which simplifies to

$$x_1 = \cos\left(\sqrt{7}\, t\right)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\
&= \cos\left(\sqrt{7}\, t\right) \int \frac{1}{\cos\left(\sqrt{7}\, t\right)^2}\, dt \\
&= \cos\left(\sqrt{7}\, t\right) \left(\frac{\sqrt{7}\, \tan\left(\sqrt{7}\, t\right)}{7}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\cos\left(\sqrt{7}\, t\right)\right) + c_2 \left(\cos\left(\sqrt{7}\, t\right) \left(\frac{\sqrt{7}\, \tan\left(\sqrt{7}\, t\right)}{7}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 7x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \cos\left(\sqrt{7}\, t\right) + \frac{c_2 \sqrt{7}\, \sin\left(\sqrt{7}\, t\right)}{7}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t\,e^{3t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{t\,e^{3t}, e^{3t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{7}\,\sin\left(\sqrt{7}\,t\right)}{7}, \cos\left(\sqrt{7}\,t\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 t\,e^{3t} + A_2 e^{3t}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 e^{3t} + 16A_1 t\,e^{3t} + 16A_2 e^{3t} = t\,e^{3t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{16}, A_2 = -\frac{3}{128}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{t\,e^{3t}}{16} - \frac{3\,e^{3t}}{128}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \cos\left(\sqrt{7}\,t\right) + \frac{c_2\sqrt{7}\,\sin\left(\sqrt{7}\,t\right)}{7}\right) + \left(\frac{t\,e^{3t}}{16} - \frac{3\,e^{3t}}{128}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 \cos\left(\sqrt{7}\, t\right) + \frac{c_2 \sqrt{7}\, \sin\left(\sqrt{7}\, t\right)}{7} + \frac{t\, e^{3t}}{16} - \frac{3\, e^{3t}}{128} \tag{1}$$

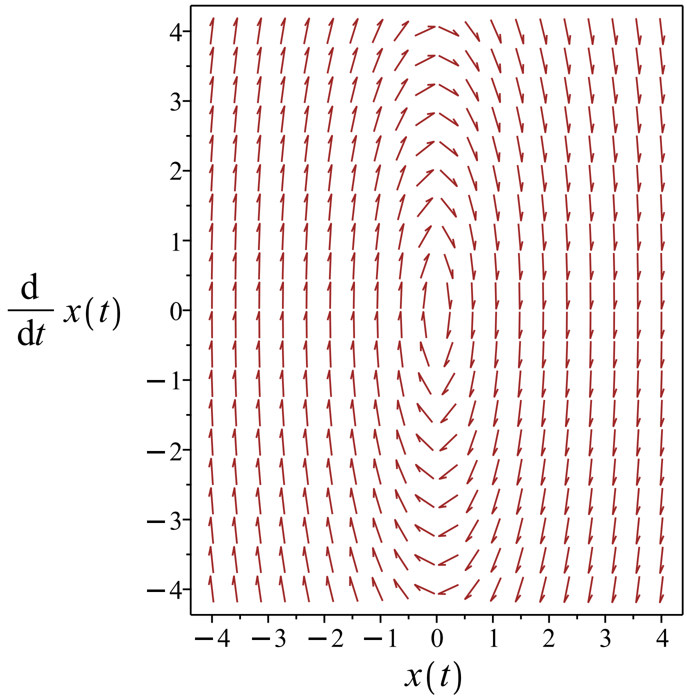


Figure 276: Slope field plot

Verification of solutions

$$x = c_1 \cos\left(\sqrt{7}\, t\right) + \frac{c_2 \sqrt{7}\, \sin\left(\sqrt{7}\, t\right)}{7} + \frac{t\, e^{3t}}{16} - \frac{3\, e^{3t}}{128}$$

Verified OK.

9.13.3 Maple step by step solution

Let's solve

$x'' + 7x = t\, e^{3t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

$r^2 + 7 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-28})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\mathrm{I}\sqrt{7}, \mathrm{I}\sqrt{7}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \cos\left(\sqrt{7}\, t\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin\left(\sqrt{7}\, t\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos\left(\sqrt{7}\, t\right) + c_2 \sin\left(\sqrt{7}\, t\right) + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t) \left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t) \left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = t\, \mathrm{e}^{3t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos\left(\sqrt{7}\, t\right) & \sin\left(\sqrt{7}\, t\right) \\ -\sqrt{7} \sin\left(\sqrt{7}\, t\right) & \sqrt{7} \cos\left(\sqrt{7}\, t\right) \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = \sqrt{7}$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -\frac{\sqrt{7}\left(\cos\left(\sqrt{7}\, t\right)\left(\int \sin\left(\sqrt{7}\, t\right) t\, \mathrm{e}^{3t} dt\right) - \sin\left(\sqrt{7}\, t\right)\left(\int \cos\left(\sqrt{7}\, t\right) t\, \mathrm{e}^{3t} dt\right)\right)}{7}$

  - Compute integrals

    $x_p(t) = \frac{\mathrm{e}^{3t}(8t-3)}{128}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \cos\left(\sqrt{7}\, t\right) + c_2 \sin\left(\sqrt{7}\, t\right) + \frac{\mathrm{e}^{3t}(8t-3)}{128}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(x(t),t$2)+7*x(t)=t*exp(3*t),x(t), singsol=all)
```

$$x(t) = \sin\left(\sqrt{7}\,t\right) c_2 + \cos\left(\sqrt{7}\,t\right) c_1 + \frac{(8t-3)\,e^{3t}}{128}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 42

```
DSolve[x''[t]+7*x[t]==t*Exp[3*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{128}e^{3t}(8t-3) + c_1\cos\left(\sqrt{7}t\right) + c_2\sin\left(\sqrt{7}t\right)$$

9.14 problem 2(b)

9.14.1 Solving as second order linear constant coeff ode 1282
9.14.2 Solving as second order integrable as is ode 1286
9.14.3 Solving as second order ode missing y ode 1288
9.14.4 Solving as type second_order_integrable_as_is (not using ABC version) 1290
9.14.5 Solving using Kovacic algorithm 1292
9.14.6 Solving as exact linear second order ode ode 1297
9.14.7 Maple step by step solution 1299

Internal problem ID [11465]
Internal file name [`OUTPUT/10447_Thursday_May_18_2023_04_19_46_AM_71081623/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 2(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x'' - x' = 6 + e^{2t}}$$

9.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = -1, C = 0, f(t) = 6 + e^{2t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''-x'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t)+Bx'(t)+Cx(t)=0$$

Where in the above $A=1, B=-1, C=0$. Let the solution be $x=e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t}-\lambda \,\mathrm{e}^{\lambda t}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2-\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=-1, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{1}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{-1^2-(4)\,(1)\,(0)}\\&=\frac{1}{2}\pm\frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1=\frac{1}{2}+\frac{1}{2}$$
$$\lambda_2=\frac{1}{2}-\frac{1}{2}$$

Which simplifies to

$$\lambda_1=1$$
$$\lambda_2=0$$

Since roots are real and distinct, then the solution is

$$x=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}$$
$$x=c_1e^{(1)t}+c_2e^{(0)t}$$

Or

$$
x = c_1 e^t + c_2
$$

Therefore the homogeneous solution x_h is

$$
x_h = c_1 e^t + c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
6 + e^{2t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}, \{e^{2t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, e^t\}
$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$
[\{t\}, \{e^{2t}\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
x_p = A_1 t + A_2 e^{2t}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_2 e^{2t} - A_1 = 6 + e^{2t}
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = -6, A_2 = \frac{1}{2}\right]
$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$
x_p = -6t + \frac{e^{2t}}{2}
$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(c_1 \mathrm{e}^t + c_2\right) + \left(-6t + \frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1 \mathrm{e}^t + c_2 - 6t + \frac{\mathrm{e}^{2t}}{2} \tag{1}
$$

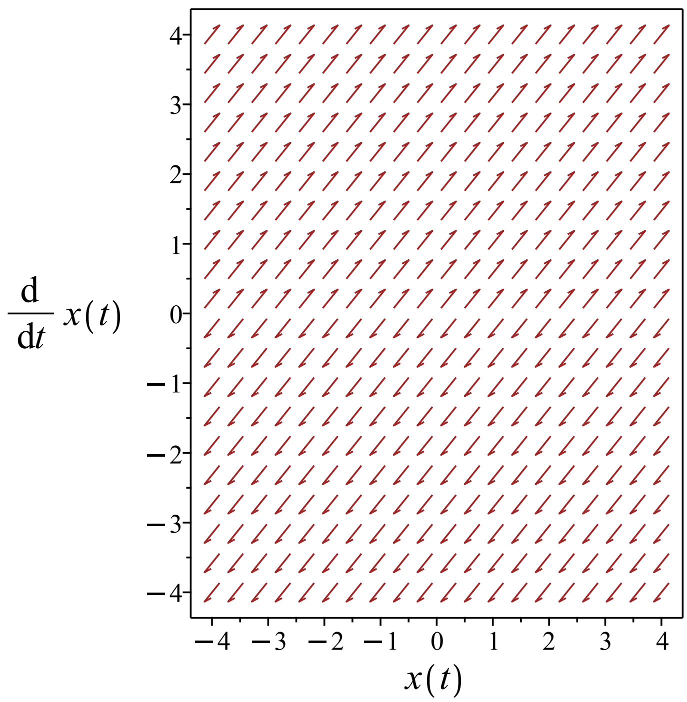


Figure 277: Slope field plot

Verification of solutions

$$
x = c_1 \mathrm{e}^t + c_2 - 6t + \frac{\mathrm{e}^{2t}}{2}
$$

Verified OK.

9.14.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int (x'' - x')\,dt = \int (6 + e^{2t})\,dt$$
$$-x + x' = 6t + \frac{e^{2t}}{2} + c_1$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = -1$$
$$q(t) = 6t + \frac{e^{2t}}{2} + c_1$$

Hence the ode is

$$-x + x' = 6t + \frac{e^{2t}}{2} + c_1$$

The integrating factor μ is

$$\mu = e^{\int (-1)dt}$$
$$= e^{-t}$$

The ode becomes

$$\frac{d}{dt}(\mu x) = (\mu)\left(6t + \frac{e^{2t}}{2} + c_1\right)$$
$$\frac{d}{dt}(e^{-t}x) = (e^{-t})\left(6t + \frac{e^{2t}}{2} + c_1\right)$$
$$d(e^{-t}x) = \left((c_1 + 6t)\,e^{-t} + \frac{e^t}{2}\right)\,dt$$

Integrating gives

$$e^{-t}x = \int (c_1 + 6t)\,e^{-t} + \frac{e^t}{2}\,dt$$
$$e^{-t}x = -c_1 e^{-t} - 6t\,e^{-t} - 6\,e^{-t} + \frac{e^t}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-t}$ results in

$$x = e^t \left(-c_1 e^{-t} - 6t\, e^{-t} - 6\, e^{-t} + \frac{e^t}{2} \right) + c_2 e^t$$

which simplifies to

$$x = -c_1 - 6t - 6 + \frac{e^{2t}}{2} + c_2 e^t$$

Summary

The solution(s) found are the following

$$x = -c_1 - 6t - 6 + \frac{e^{2t}}{2} + c_2 e^t \tag{1}$$

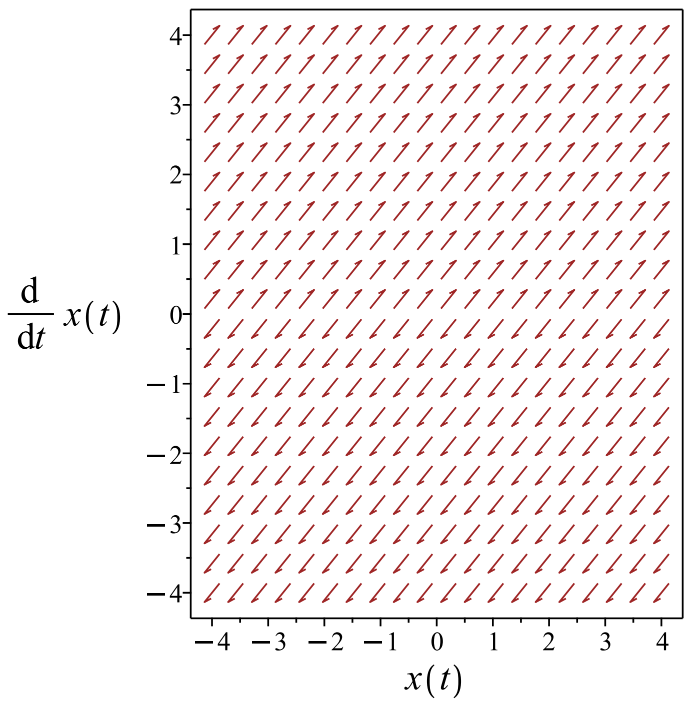


Figure 278: Slope field plot

Verification of solutions

$$x = -c_1 - 6t - 6 + \frac{e^{2t}}{2} + c_2 e^t$$

Verified OK.

9.14.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable x. Let

$$p(t) = x'$$

Then

$$p'(t) = x''$$

Hence the ode becomes

$$p'(t) - p(t) - 6 - e^{2t} = 0$$

Which is now solve for $p(t)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(t) + p(t)p(t) = q(t)$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= 6 + e^{2t} \end{aligned}$$

Hence the ode is

$$p'(t) - p(t) = 6 + e^{2t}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-1)dt} \\ &= e^{-t} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dt}(\mu p) &= (\mu)\left(6 + e^{2t}\right) \\ \frac{d}{dt}\left(e^{-t} p\right) &= \left(e^{-t}\right)\left(6 + e^{2t}\right) \\ d\left(e^{-t} p\right) &= \left(6\,e^{-t} + e^{t}\right)\, dt \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-t} p &= \int 6\,e^{-t} + e^{t}\, dt \\ e^{-t} p &= -6\,e^{-t} + e^{t} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-t}$ results in

$$p(t) = \mathrm{e}^{t}\left(-6\,\mathrm{e}^{-t} + \mathrm{e}^{t}\right) + c_1 \mathrm{e}^{t}$$

which simplifies to

$$p(t) = c_1 \mathrm{e}^{t} - 6 + \mathrm{e}^{2t}$$

Since $p = x'$ then the new first order ode to solve is

$$x' = c_1 \mathrm{e}^{t} - 6 + \mathrm{e}^{2t}$$

Integrating both sides gives

$$\begin{aligned} x &= \int c_1 \mathrm{e}^{t} - 6 + \mathrm{e}^{2t}\,\mathrm{d}t \\ &= c_1 \mathrm{e}^{t} + c_2 - 6t + \frac{\mathrm{e}^{2t}}{2} \end{aligned}$$

Summary
The solution(s) found are the following

$$x = c_1 \mathrm{e}^{t} + c_2 - 6t + \frac{\mathrm{e}^{2t}}{2} \tag{1}$$

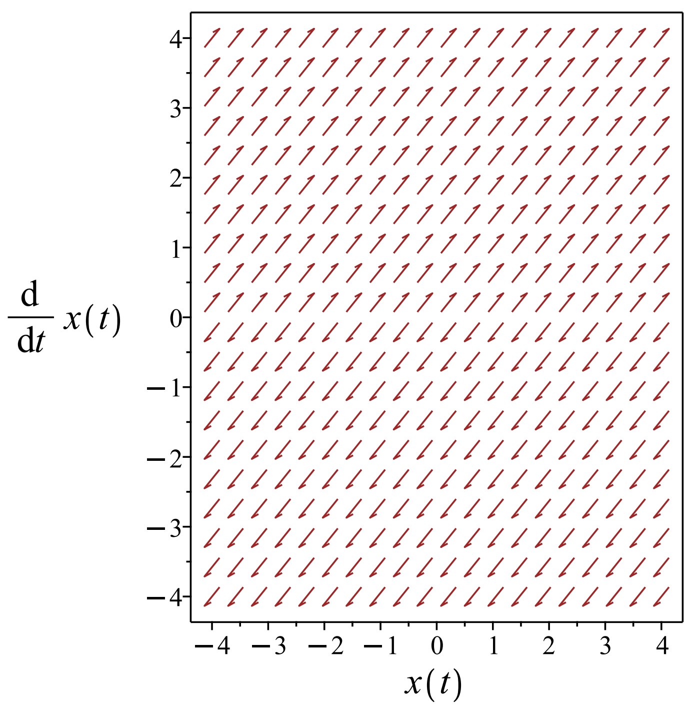


Figure 279: Slope field plot

Verification of solutions

$$
x = c_1 \mathrm{e}^{t} + c_2 - 6t + \frac{\mathrm{e}^{2t}}{2}
$$

Verified OK.

9.14.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
x'' - x' = 6 + \mathrm{e}^{2t}
$$

Integrating both sides of the ODE w.r.t t gives

$$
\begin{aligned}
\int (x'' - x')\, dt &= \int (6 + \mathrm{e}^{2t})\, dt \\
-x + x' &= 6t + \frac{\mathrm{e}^{2t}}{2} + c_1
\end{aligned}
$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
x' + p(t)x = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= -1 \\
q(t) &= 6t + \frac{\mathrm{e}^{2t}}{2} + c_1
\end{aligned}
$$

Hence the ode is

$$
-x + x' = 6t + \frac{\mathrm{e}^{2t}}{2} + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-1)dt} \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(6t + \frac{\mathrm{e}^{2t}}{2} + c_1\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-t} x\right) &= \left(\mathrm{e}^{-t}\right)\left(6t + \frac{\mathrm{e}^{2t}}{2} + c_1\right) \\
\mathrm{d}\left(\mathrm{e}^{-t} x\right) &= \left((c_1 + 6t)\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{t}}{2}\right) \mathrm{d}t
\end{aligned}
$$

Integrating gives

$$\mathrm{e}^{-t}x = \int (c_1 + 6t)\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{t}}{2}\,\mathrm{d}t$$
$$\mathrm{e}^{-t}x = -c_1\mathrm{e}^{-t} - 6t\,\mathrm{e}^{-t} - 6\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{t}}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-t}$ results in

$$x = \mathrm{e}^{t}\left(-c_1\mathrm{e}^{-t} - 6t\,\mathrm{e}^{-t} - 6\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{t}}{2}\right) + c_2\mathrm{e}^{t}$$

which simplifies to

$$x = -c_1 - 6t - 6 + \frac{\mathrm{e}^{2t}}{2} + c_2\mathrm{e}^{t}$$

Summary
The solution(s) found are the following

$$x = -c_1 - 6t - 6 + \frac{\mathrm{e}^{2t}}{2} + c_2\mathrm{e}^{t} \tag{1}$$

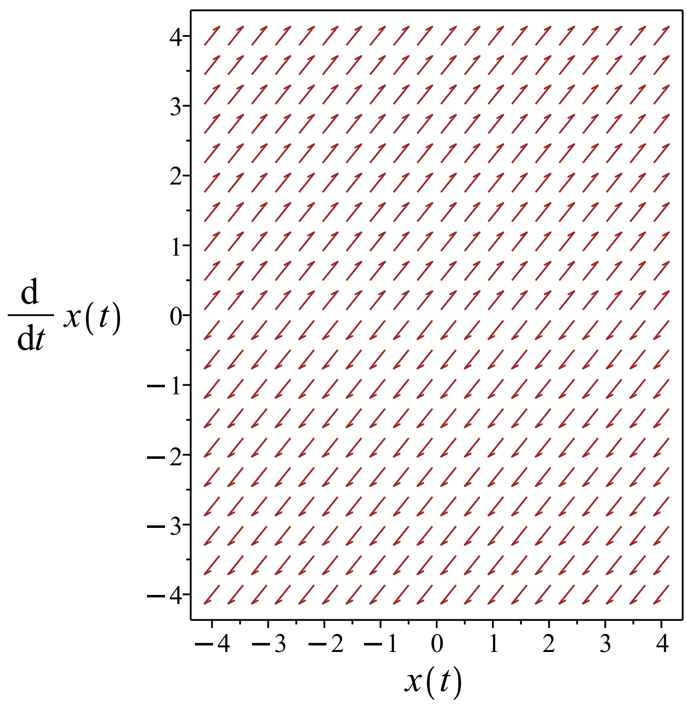


Figure 280: Slope field plot

Verification of solutions

$$x = -c_1 - 6t - 6 + \frac{e^{2t}}{2} + c_2 e^t$$

Verified OK.

9.14.5 Solving using Kovacic algorithm

Writing the ode as

$$x'' - x' = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 242: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dt} \\
&= z_1 e^{\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = 1$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-1}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^t}{(x_1)^2}\,dt \\
&= x_1 \left(\mathrm{e}^t\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(1) + c_2\left(1\left(\mathrm{e}^t\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''-x'=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 + c_2\mathrm{e}^t$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$6+\mathrm{e}^{2t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\},\{\mathrm{e}^{2t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1,\mathrm{e}^t\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\},\{\mathrm{e}^{2t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1t + A_2\mathrm{e}^{2t}$$

The unknowns $\{A_1,A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2\mathrm{e}^{2t}-A_1 = 6+\mathrm{e}^{2t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=-6, A_2=\frac{1}{2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -6t + \frac{e^{2t}}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1 + c_2 e^t) + \left(-6t + \frac{e^{2t}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 + c_2 e^t - 6t + \frac{e^{2t}}{2} \tag{1}$$

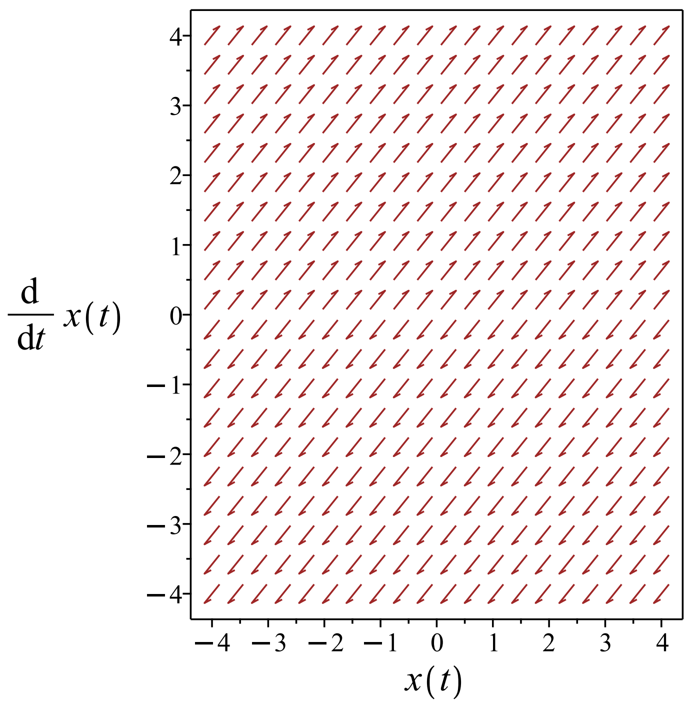


Figure 281: Slope field plot

Verification of solutions

$$x = c_1 + c_2 e^t - 6t + \frac{e^{2t}}{2}$$

Verified OK.

9.14.6 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\,x''+q(t)\,x'+r(t)\,x=s(t)$$

is exact if

$$p''(t)-q'(t)+r(t)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x)&=1\\
q(x)&=-1\\
r(x)&=0\\
s(x)&=6+\mathrm{e}^{2t}
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x)&=0\\
q'(x)&=0
\end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(t)\,x'+\left(q(t)-p'(t)\right)x\right)'=s(x)$$

Integrating gives

$$p(t)\,x'+\left(q(t)-p'(t)\right)x=\int s(t)\;dt$$

Substituting the above values for p,q,r,s gives

$$-x+x'=\int 6+\mathrm{e}^{2t}\,dt$$

We now have a first order ode to solve which is

$$-x+x'=6t+\frac{\mathrm{e}^{2t}}{2}+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned}
p(t) &= -1 \\
q(t) &= 6t + \frac{e^{2t}}{2} + c_1
\end{aligned}$$

Hence the ode is

$$-x + x' = 6t + \frac{e^{2t}}{2} + c_1$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int (-1)dt} \\
&= e^{-t}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dt}(\mu x) &= (\mu)\left(6t + \frac{e^{2t}}{2} + c_1\right) \\
\frac{d}{dt}\left(e^{-t}x\right) &= \left(e^{-t}\right)\left(6t + \frac{e^{2t}}{2} + c_1\right) \\
d\left(e^{-t}x\right) &= \left((c_1 + 6t)\,e^{-t} + \frac{e^{t}}{2}\right) dt
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{-t}x &= \int (c_1 + 6t)\,e^{-t} + \frac{e^{t}}{2}\, dt \\
e^{-t}x &= -c_1 e^{-t} - 6t\,e^{-t} - 6\,e^{-t} + \frac{e^{t}}{2} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-t}$ results in

$$x = e^{t}\left(-c_1 e^{-t} - 6t\,e^{-t} - 6\,e^{-t} + \frac{e^{t}}{2}\right) + c_2 e^{t}$$

which simplifies to

$$x = -c_1 - 6t - 6 + \frac{e^{2t}}{2} + c_2 e^{t}$$

Summary

The solution(s) found are the following

$$x = -c_1 - 6t - 6 + \frac{e^{2t}}{2} + c_2 e^t \tag{1}$$

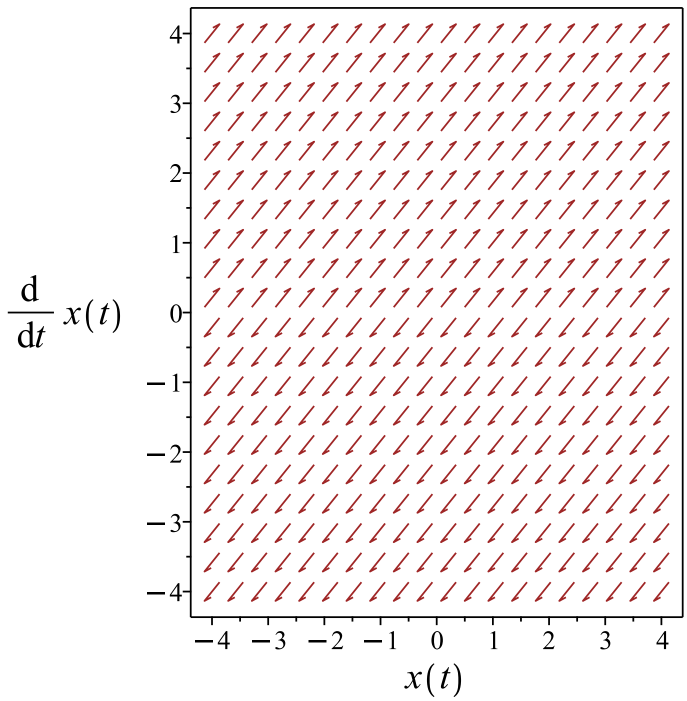


Figure 282: Slope field plot

Verification of solutions

$$x = -c_1 - 6t - 6 + \frac{e^{2t}}{2} + c_2 e^t$$

Verified OK.

9.14.7 Maple step by step solution

Let's solve

$x'' - x' = 6 + e^{2t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

$r^2 - r = 0$

- Factor the characteristic polynomial

  $r(r-1) = 0$

- Roots of the characteristic polynomial

  $r = (0, 1)$

- 1st solution of the homogeneous ODE

  $x_1(t) = 1$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^t$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 + c_2 e^t + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = 6 + e^{2t}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} 1 & e^t \\ 0 & e^t \end{bmatrix}$

- ○ Compute Wronskian

  $W(x_1(t), x_2(t)) = e^t$

- ○ Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\left(\int (6 + e^{2t})\,dt\right) + e^t\left(\int (6\,e^{-t} + e^t)\,dt\right)$

- ○ Compute integrals

  $x_p(t) = -6t + \frac{e^{2t}}{2} - 6$

- Substitute particular solution into general solution to ODE

  $x = c_1 + c_2 e^t - 6t + \frac{e^{2t}}{2} - 6$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _b(_a)+6+exp(2*_a), _b(_a)`   *** Suble
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(x(t),t$2)-diff(x(t),t)=6+exp(2*t),x(t), singsol=all)
```

$$x(t) = c_1 e^t + \frac{e^{2t}}{2} - 6t + c_2$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 26

```
DSolve[x''[t]-x'[t]==6+Exp[2*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -6t + \frac{e^{2t}}{2} + c_1 e^t + c_2$$

9.15 problem 2(c)

9.15.1 Solving as second order linear constant coeff ode 1302
9.15.2 Solving using Kovacic algorithm 1305
9.15.3 Maple step by step solution 1310

Internal problem ID [11466]
Internal file name [`OUTPUT/10448_Thursday_May_18_2023_04_19_48_AM_61795472/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 2(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x'' + x = t^2}$$

9.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = 1, f(t) = t^2$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\
&= \pm i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +i \\
\lambda_2 &= -i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= i \\
\lambda_2 &= -i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^0(c_1 \cos(t) + c_2 \sin(t))$$

Or

$$x = c_1 \cos(t) + c_2 \sin(t)$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \cos(t) + c_2 \sin(t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(t), \sin(t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_3 t^2 + A_2 t + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_3 t^2 + A_2 t + A_1 + 2A_3 = t^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0, A_3 = 1]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = t^2 - 2$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1 \cos(t) + c_2 \sin(t)) + (t^2 - 2) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 \cos(t) + c_2 \sin(t) + t^2 - 2 \tag{1}$$

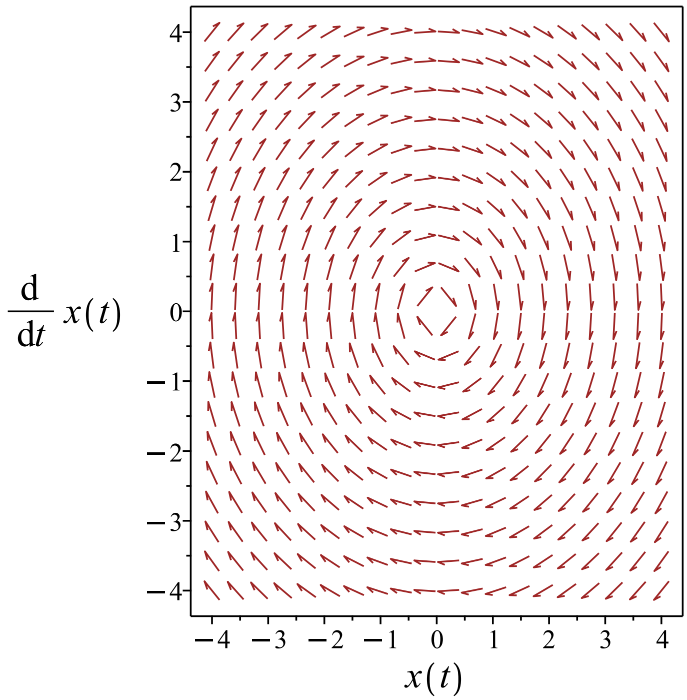


Figure 283: Slope field plot

Verification of solutions

$$x = c_1 \cos(t) + c_2 \sin(t) + t^2 - 2$$

Verified OK.

9.15.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\,e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 244: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \cos(t) \end{aligned}$$

Which simplifies to

$$x_1 = \cos(t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\,dt \\ &= \cos(t) \int \frac{1}{\cos(t)^2}\,dt \\ &= \cos(t)\,(\tan(t)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1(\cos(t)) + c_2(\cos(t)\,(\tan(t))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \cos(t) + c_2 \sin(t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(t), \sin(t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_3t^2 + A_2t + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_3t^2 + A_2t + A_1 + 2A_3 = t^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0, A_3 = 1]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = t^2 - 2$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1 \cos(t) + c_2 \sin(t)) + (t^2 - 2) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 \cos(t) + c_2 \sin(t) + t^2 - 2 \tag{1}$$

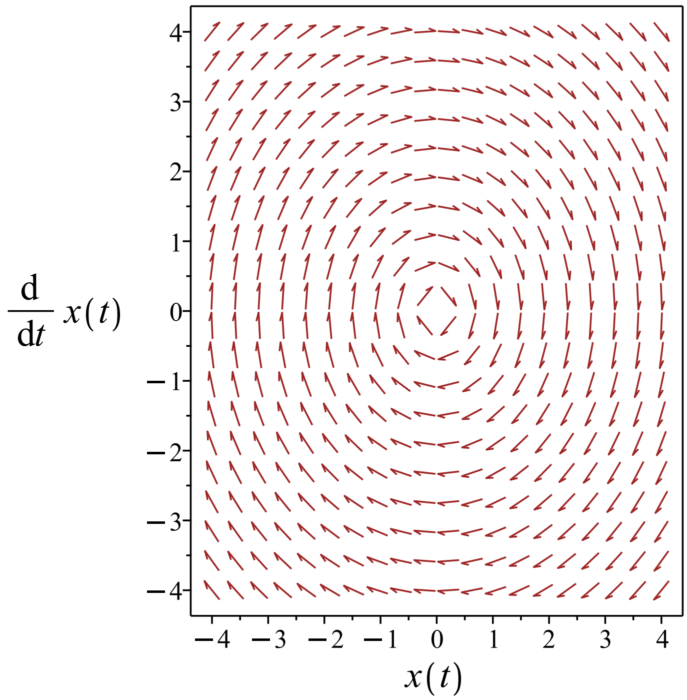


Figure 284: Slope field plot

Verification of solutions

$$x = c_1 \cos(t) + c_2 \sin(t) + t^2 - 2$$

Verified OK.

9.15.3 Maple step by step solution

Let's solve

$x'' + x = t^2$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(t) + c_2 \sin(t) + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = t^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 1$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -\cos(t)\left(\int \sin(t)\, t^2 dt\right) + \sin(t)\left(\int \cos(t)\, t^2 dt\right)$

  - Compute integrals

    $x_p(t) = t^2 - 2$

- Substitute particular solution into general solution to ODE

  $x = c_1 \cos(t) + c_2 \sin(t) + t^2 - 2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(x(t),t$2)+x(t)=t^2,x(t), singsol=all)
```

$$x(t) = \sin(t)\, c_2 + \cos(t)\, c_1 + t^2 - 2$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 20

```
DSolve[x''[t]+x[t]==t^2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to t^2 + c_1 \cos(t) + c_2 \sin(t) - 2$$

9.16 problem 2(d)

9.16.1 Solving as second order linear constant coeff ode 1313
9.16.2 Solving using Kovacic algorithm 1316
9.16.3 Maple step by step solution 1321

Internal problem ID [11467]
Internal file name [`OUTPUT/10449_Thursday_May_18_2023_04_19_49_AM_50905919/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 2(d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x'' - 3x' - 4x = 2t^2}$$

9.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = -3, C = -4, f(t) = 2t^2$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - 3x' - 4x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -3, C = -4$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - 3\lambda\,\mathrm{e}^{\lambda t} - 4\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 3\lambda - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = -4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(-4)} \\
&= \frac{3}{2} \pm \frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{3}{2} + \frac{5}{2} \\
\lambda_2 &= \frac{3}{2} - \frac{5}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 4 \\
\lambda_2 &= -1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
x &= c_1 e^{(4)t} + c_2 e^{(-1)t}
\end{aligned}$$

Or

$$x = c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{-t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-t}, e^{4t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_3 t^2 + A_2 t + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_3 t^2 - 4A_2 t - 6tA_3 - 4A_1 - 3A_2 + 2A_3 = 2t^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{13}{16}, A_2 = \frac{3}{4}, A_3 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{1}{2}t^2 + \frac{3}{4}t - \frac{13}{16}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{4t} + c_2 e^{-t}\right) + \left(-\frac{1}{2}t^2 + \frac{3}{4}t - \frac{13}{16}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{4t} + c_2 e^{-t} - \frac{t^2}{2} + \frac{3t}{4} - \frac{13}{16} \tag{1}$$

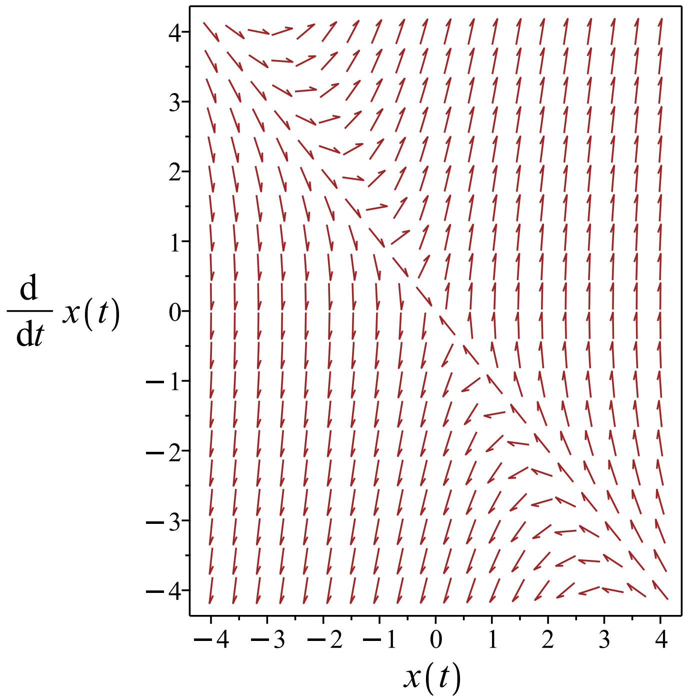


Figure 285: Slope field plot

Verification of solutions

$$x = c_1 e^{4t} + c_2 e^{-t} - \frac{t^2}{2} + \frac{3t}{4} - \frac{13}{16}$$

Verified OK.

9.16.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 3x' - 4x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 25$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(t) = \frac{25z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 246: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{5t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dt} \\
&= z_1 e^{\frac{3t}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{3t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-3}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{3t}}{(x_1)^2}\,dt \\
&= x_1\left(\frac{\mathrm{e}^{5t}}{5}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-t}\right) + c_2 \left(\mathrm{e}^{-t} \left(\frac{\mathrm{e}^{5t}}{5}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' - 3x' - 4x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{-t} + \frac{c_2 \mathrm{e}^{4t}}{5}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
t^2
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, t, t^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{4t}}{5}, \mathrm{e}^{-t}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_3 t^2 + A_2 t + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_3t^2 - 4A_2t - 6tA_3 - 4A_1 - 3A_2 + 2A_3 = 2t^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{13}{16}, A_2 = \frac{3}{4}, A_3 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{1}{2}t^2 + \frac{3}{4}t - \frac{13}{16}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{-t} + \frac{c_2 \mathrm{e}^{4t}}{5}\right) + \left(-\frac{1}{2}t^2 + \frac{3}{4}t - \frac{13}{16}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 \mathrm{e}^{-t} + \frac{c_2 \mathrm{e}^{4t}}{5} - \frac{t^2}{2} + \frac{3t}{4} - \frac{13}{16} \tag{1}$$

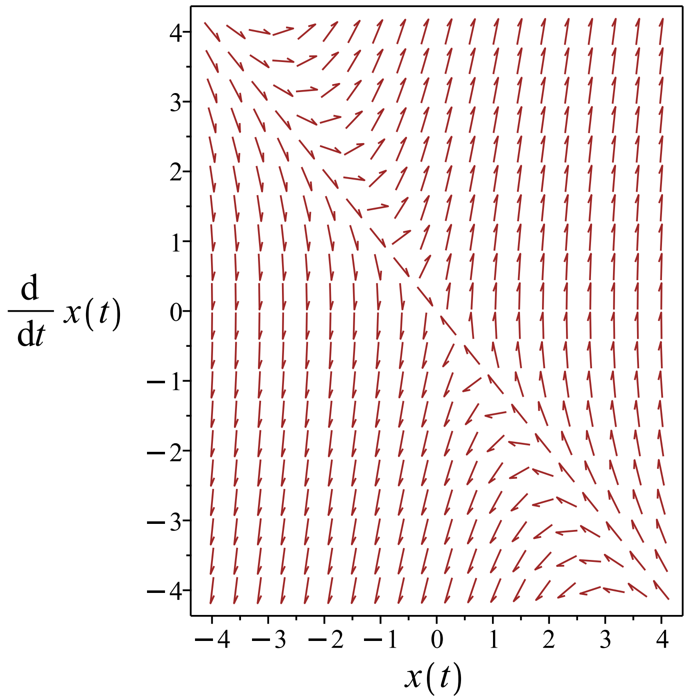


Figure 286: Slope field plot

Verification of solutions

$$x = c_1 e^{-t} + \frac{c_2 e^{4t}}{5} - \frac{t^2}{2} + \frac{3t}{4} - \frac{13}{16}$$

Verified OK.

9.16.3 Maple step by step solution

Let's solve

$x'' - 3x' - 4x = 2t^2$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r - 4 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-4) = 0$

- Roots of the characteristic polynomial

$r = (-1, 4)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{4t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{4t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = 2t^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} & \mathrm{e}^{4t} \\ -\mathrm{e}^{-t} & 4\,\mathrm{e}^{4t} \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 5\,\mathrm{e}^{3t}$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -\frac{2\,\mathrm{e}^{-t}\left(\int t^2 \mathrm{e}^{t} dt\right)}{5} + \frac{2\,\mathrm{e}^{4t}\left(\int t^2 \mathrm{e}^{-4t} dt\right)}{5}$

  - Compute integrals

    $x_p(t) = -\frac{1}{2}t^2 + \frac{3}{4}t - \frac{13}{16}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{4t} - \frac{t^2}{2} + \frac{3t}{4} - \frac{13}{16}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(x(t),t$2)-3*diff(x(t),t)-4*x(t)=2*t^2,x(t), singsol=all)
```

$$x(t) = \mathrm{e}^{4t}c_2 + \mathrm{e}^{-t}c_1 - \frac{t^2}{2} + \frac{3t}{4} - \frac{13}{16}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 37

```
DSolve[x''[t]-3*x'[t]-4*x[t]==2*t^2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{t^2}{2} + \frac{3t}{4} + c_1e^{-t} + c_2e^{4t} - \frac{13}{16}$$

9.17 problem 2(e)

9.17.1 Solving as second order linear constant coeff ode 1324
9.17.2 Solving using Kovacic algorithm 1327
9.17.3 Maple step by step solution 1332

Internal problem ID [11468]
Internal file name [`OUTPUT/10450_Thursday_May_18_2023_04_19_51_AM_18775969/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 2(e).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x'' + x = 9\,\mathrm{e}^{-t}}$$

9.17.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = 1, f(t) = 9\,\mathrm{e}^{-t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^0(c_1 \cos(t) + c_2 \sin(t))$$

Or

$$x = c_1 \cos(t) + c_2 \sin(t)$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \cos(t) + c_2 \sin(t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$9\,\mathrm{e}^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(t), \sin(t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \mathrm{e}^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^{-t} = 9\,\mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{9}{2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{9\,\mathrm{e}^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1 \cos(t) + c_2 \sin(t)) + \left(\frac{9\,\mathrm{e}^{-t}}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$x = c_1 \cos(t) + c_2 \sin(t) + \frac{9\,\mathrm{e}^{-t}}{2} \tag{1}$$

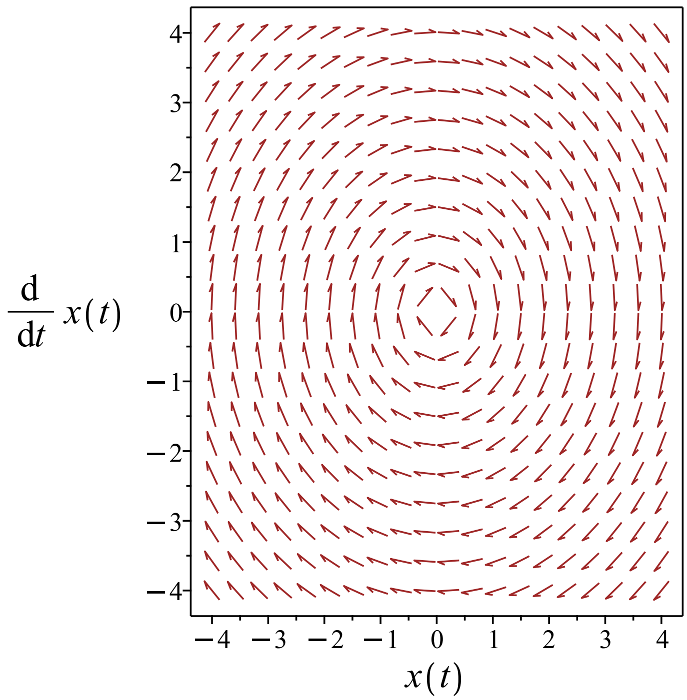


Figure 287: Slope field plot

Verification of solutions

$$x = c_1 \cos(t) + c_2 \sin(t) + \frac{9\,\mathrm{e}^{-t}}{2}$$

Verified OK.

9.17.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 248: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
x_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
x_1 &= z_1 \\
&= \cos(t)
\end{aligned}
$$

Which simplifies to

$$x_1 = \cos(t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\
&= \cos(t) \int \frac{1}{\cos(t)^2}\, dt \\
&= \cos(t)\,(\tan(t))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(\cos(t)) + c_2(\cos(t)\,(\tan(t)))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \cos(t) + c_2 \sin(t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$9\,\mathrm{e}^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(t), \sin(t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 e^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{-t} = 9\, e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{9}{2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{9\, e^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1 \cos(t) + c_2 \sin(t)) + \left(\frac{9\, e^{-t}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 \cos(t) + c_2 \sin(t) + \frac{9\, e^{-t}}{2} \tag{1}$$

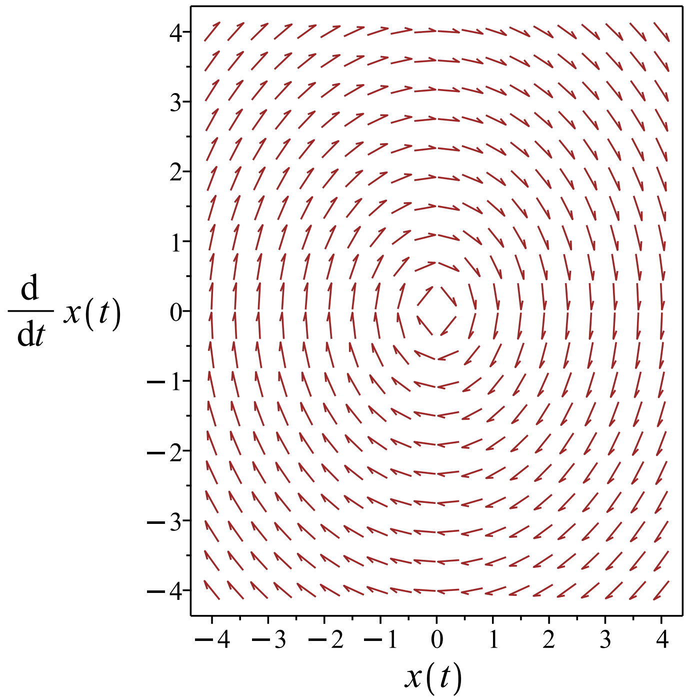


Figure 288: Slope field plot

<u>Verification of solutions</u>

$$x = c_1 \cos(t) + c_2 \sin(t) + \frac{9\,\mathrm{e}^{-t}}{2}$$

Verified OK.

9.17.3 Maple step by step solution

Let's solve

$x'' + x = 9\,\mathrm{e}^{-t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(t) + c_2 \sin(t) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = 9\,\mathrm{e}^{-t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \left[\begin{array}{cc} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{array}\right]$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 1$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -9\cos(t)\left(\int \mathrm{e}^{-t}\sin(t)\,dt\right) + 9\sin(t)\left(\int \mathrm{e}^{-t}\cos(t)\,dt\right)$

  - Compute integrals

    $x_p(t) = \frac{9\,\mathrm{e}^{-t}}{2}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \cos(t) + c_2 \sin(t) + \frac{9\,\mathrm{e}^{-t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(x(t),t$2)+x(t)=9*exp(-t),x(t), singsol=all)
```

$$x(t) = \sin(t)\, c_2 + \cos(t)\, c_1 + \frac{9\, e^{-t}}{2}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 25

```
DSolve[x''[t]+x[t]==9*Exp[-t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{9e^{-t}}{2} + c_1 \cos(t) + c_2 \sin(t)$$

9.18 problem 2(g)

9.18.1 Solving as second order linear constant coeff ode 1335
9.18.2 Solving using Kovacic algorithm 1338
9.18.3 Maple step by step solution 1343

Internal problem ID [11469]
Internal file name [`OUTPUT/10451_Thursday_May_18_2023_04_19_52_AM_15009311/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 2(g).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' - 4x = \cos(2t)}$$

9.18.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = -4, f(t) = \cos(2t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - 4x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - 4\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2\\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ x &= c_1 e^{(2)t} + c_2 e^{(-2)t}\end{aligned}$$

Or

$$x = c_1 e^{2t} + c_2 e^{-2t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 e^{2t} + c_2 e^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t}, e^{2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(2t) + A_2 \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_1 \cos(2t) - 8A_2 \sin(2t) = \cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{8}, A_2 = 0\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{\cos(2t)}{8}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{2t} + c_2 e^{-2t}\right) + \left(-\frac{\cos(2t)}{8}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$x = c_1 e^{2t} + c_2 e^{-2t} - \frac{\cos(2t)}{8} \tag{1}$$

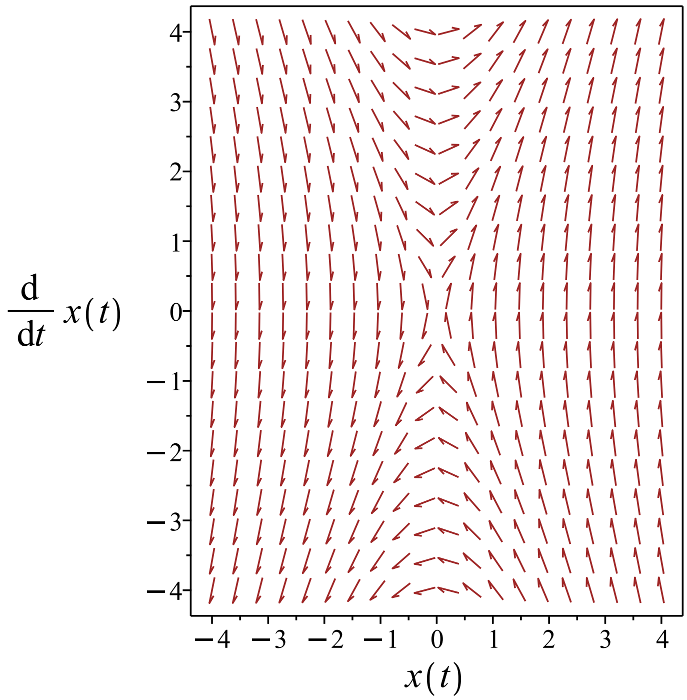


Figure 289: Slope field plot

Verification of solutions

$$x = c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{-2t} - \frac{\cos(2t)}{8}$$

Verified OK.

9.18.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 4x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 250: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-2t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
x_1 &= z_1 \\
&= \mathrm{e}^{-2t}
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-2t}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2}\,dt \\
&= \mathrm{e}^{-2t} \int \frac{1}{\mathrm{e}^{-4t}}\,dt \\
&= \mathrm{e}^{-2t} \left(\frac{\mathrm{e}^{4t}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-2t}\right) + c_2 \left(\mathrm{e}^{-2t} \left(\frac{\mathrm{e}^{4t}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' - 4x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{-2t} + \frac{c_2 \mathrm{e}^{2t}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\cos(2t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(2t), \sin(2t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{2t}}{4}, \mathrm{e}^{-2t}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_1 \cos(2t) + A_2 \sin(2t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_1 \cos(2t) - 8A_2 \sin(2t) = \cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{8}, A_2 = 0\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{\cos(2t)}{8}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-2t} + \frac{c_2 e^{2t}}{4}\right) + \left(-\frac{\cos(2t)}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{-2t} + \frac{c_2 e^{2t}}{4} - \frac{\cos(2t)}{8} \tag{1}$$

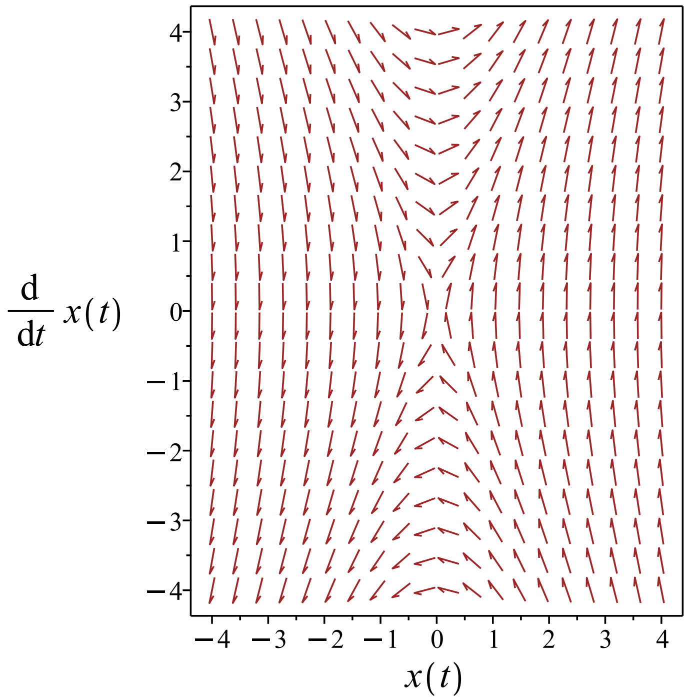


Figure 290: Slope field plot

Verification of solutions

$$x = c_1 e^{-2t} + \frac{c_2 e^{2t}}{4} - \frac{\cos(2t)}{8}$$

Verified OK.

9.18.3 Maple step by step solution

Let's solve

$x'' - 4x = \cos(2t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r-2)(r+2) = 0$

- Roots of the characteristic polynomial

$r = (-2, 2)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{2t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{2t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = \cos(2t)\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & \mathrm{e}^{2t} \\ -2\,\mathrm{e}^{-2t} & 2\,\mathrm{e}^{2t} \end{bmatrix}$

  ○ Compute Wronskian

  $W(x_1(t), x_2(t)) = 4$

  ○ Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{\mathrm{e}^{-2t}\left(\int \mathrm{e}^{2t} \cos(2t) dt\right)}{4} + \frac{\mathrm{e}^{2t}\left(\int \mathrm{e}^{-2t} \cos(2t) dt\right)}{4}$

  ○ Compute integrals

  $x_p(t) = -\frac{\cos(2t)}{8}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{2t} - \frac{\cos(2t)}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(x(t),t$2)-4*x(t)=cos(2*t),x(t), singsol=all)
```

$$x(t) = c_2 \mathrm{e}^{2t} + \mathrm{e}^{-2t} c_1 - \frac{\cos(2t)}{8}$$

✓ Solution by Mathematica

Time used: 0.115 (sec). Leaf size: 30

```
DSolve[x''[t]-4*x[t]==Cos[2*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{1}{8}\cos(2t) + c_1 e^{2t} + c_2 e^{-2t}$$

9.19 problem 2(h)

9.19.1 Solving as second order linear constant coeff ode 1346
9.19.2 Solving using Kovacic algorithm 1350
9.19.3 Maple step by step solution 1355

Internal problem ID [11470]
Internal file name [`OUTPUT/10452_Thursday_May_18_2023_04_19_54_AM_16605948/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 2(h).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + x' + 2x = \sin(2t)\, t}$$

9.19.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = 2, f(t) = \sin(2t)\, t$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + 2x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A=1, B=1, C=2$. Let the solution be $x=e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t}+\lambda\,\mathrm{e}^{\lambda t}+2\,\mathrm{e}^{\lambda t}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2+\lambda+2=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=1, C=2$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-1}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{1^2-(4)\,(1)\,(2)}\\&=-\frac{1}{2}\pm\frac{i\sqrt{7}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-\frac{1}{2}+\frac{i\sqrt{7}}{2}\\\lambda_2&=-\frac{1}{2}-\frac{i\sqrt{7}}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1=-\frac{1}{2}+\frac{i\sqrt{7}}{2}$$
$$\lambda_2=-\frac{1}{2}-\frac{i\sqrt{7}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2}=\alpha\pm i\beta$$

Where $\alpha=-\frac{1}{2}$ and $\beta=\frac{\sqrt{7}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x=e^{\alpha t}(c_1\cos(\beta t)+c_2\sin(\beta t))$$

Which becomes

$$x=e^{-\frac{t}{2}}\left(c_1\cos\left(\frac{\sqrt{7}\,t}{2}\right)+c_2\sin\left(\frac{\sqrt{7}\,t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{7}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{7}\, t}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2t)\, t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t)\, t, \sin(2t)\, t, \cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{7}\, t}{2}\right), e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(2t)\, t + A_2 \sin(2t)\, t + A_3 \cos(2t) + A_4 \sin(2t)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&-2A_1 \cos(2t)\, t - 4A_1 \sin(2t) - 2A_2 \sin(2t)\, t + 4A_2 \cos(2t) \\
&- 2A_3 \cos(2t) - 2A_4 \sin(2t) - 2A_1 \sin(2t)\, t + A_1 \cos(2t) \\
&+ 2A_2 \cos(2t)\, t + A_2 \sin(2t) - 2A_3 \sin(2t) + 2A_4 \cos(2t) = \sin(2t)\, t
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = -\frac{1}{4}, A_3 = -\frac{1}{8}, A_4 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{\cos(2t)\, t}{4} - \frac{\sin(2t)\, t}{4} - \frac{\cos(2t)}{8} + \frac{\sin(2t)}{2}$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(\mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{7}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{7}\, t}{2}\right)\right)\right) \\
&\quad + \left(-\frac{\cos(2t)\, t}{4} - \frac{\sin(2t)\, t}{4} - \frac{\cos(2t)}{8} + \frac{\sin(2t)}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
x &= \mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{7}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{7}\, t}{2}\right)\right) \\
&\quad - \frac{\cos(2t)\, t}{4} - \frac{\sin(2t)\, t}{4} - \frac{\cos(2t)}{8} + \frac{\sin(2t)}{2}
\end{aligned}
\tag{1}
$$

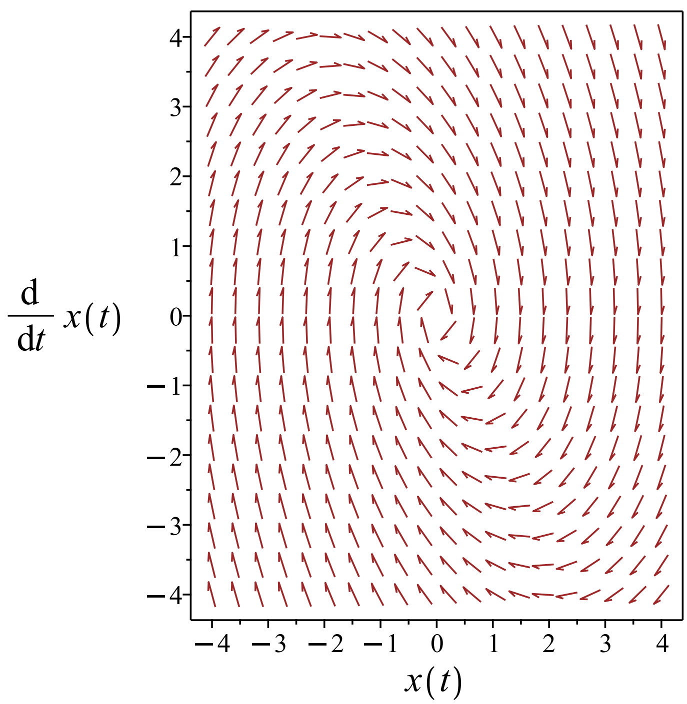


Figure 291: Slope field plot

Verification of solutions

$$x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{7}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{7}\, t}{2}\right)\right) - \frac{\cos(2t)\, t}{4} - \frac{\sin(2t)\, t}{4} - \frac{\cos(2t)}{8} + \frac{\sin(2t)}{2}$$

Verified OK.

9.19.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' + 2x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -7 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{7z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 252: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{7}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\frac{\sqrt{7}\, t}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\, dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{7}\, t}{2}\right)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\, dt \\
&= x_1 \left(\frac{2\sqrt{7}\, \tan\left(\frac{\sqrt{7}\, t}{2}\right)}{7}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{7}\, t}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{7}\, t}{2}\right) \left(\frac{2\sqrt{7}\, \tan\left(\frac{\sqrt{7}\, t}{2}\right)}{7}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' + 2x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = \cos\left(\frac{\sqrt{7}\,t}{2}\right) \mathrm{e}^{-\frac{t}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{7}\,t}{2}\right) \mathrm{e}^{-\frac{t}{2}} \sqrt{7}}{7}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2t)\,t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t)\,t, \sin(2t)\,t, \cos(2t)\,, \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{7}\,t}{2}\right), \frac{2\sin\left(\frac{\sqrt{7}\,t}{2}\right) \mathrm{e}^{-\frac{t}{2}} \sqrt{7}}{7}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(2t)\,t + A_2 \sin(2t)\,t + A_3 \cos(2t) + A_4 \sin(2t)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&-2A_1 \cos(2t)\,t - 4A_1 \sin(2t) - 2A_2 \sin(2t)\,t + 4A_2 \cos(2t) \\
&\quad - 2A_3 \cos(2t) - 2A_4 \sin(2t) - 2A_1 \sin(2t)\,t + A_1 \cos(2t) \\
&\quad + 2A_2 \cos(2t)\,t + A_2 \sin(2t) - 2A_3 \sin(2t) + 2A_4 \cos(2t) = \sin(2t)\,t
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = -\frac{1}{4}, A_2 = -\frac{1}{4}, A_3 = -\frac{1}{8}, A_4 = \frac{1}{2}\right]
$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$
x_p = -\frac{\cos(2t)\,t}{4} - \frac{\sin(2t)\,t}{4} - \frac{\cos(2t)}{8} + \frac{\sin(2t)}{2}
$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(\cos\left(\frac{\sqrt{7}\,t}{2}\right) \mathrm{e}^{-\frac{t}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{7}\,t}{2}\right) \mathrm{e}^{-\frac{t}{2}} \sqrt{7}}{7}\right) \\
&\quad + \left(-\frac{\cos(2t)\,t}{4} - \frac{\sin(2t)\,t}{4} - \frac{\cos(2t)}{8} + \frac{\sin(2t)}{2}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
x = \cos\left(\frac{\sqrt{7}\,t}{2}\right) \mathrm{e}^{-\frac{t}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{7}\,t}{2}\right) \mathrm{e}^{-\frac{t}{2}} \sqrt{7}}{7} \\
- \frac{\cos(2t)\,t}{4} - \frac{\sin(2t)\,t}{4} - \frac{\cos(2t)}{8} + \frac{\sin(2t)}{2}
\end{aligned} \tag{1}
$$

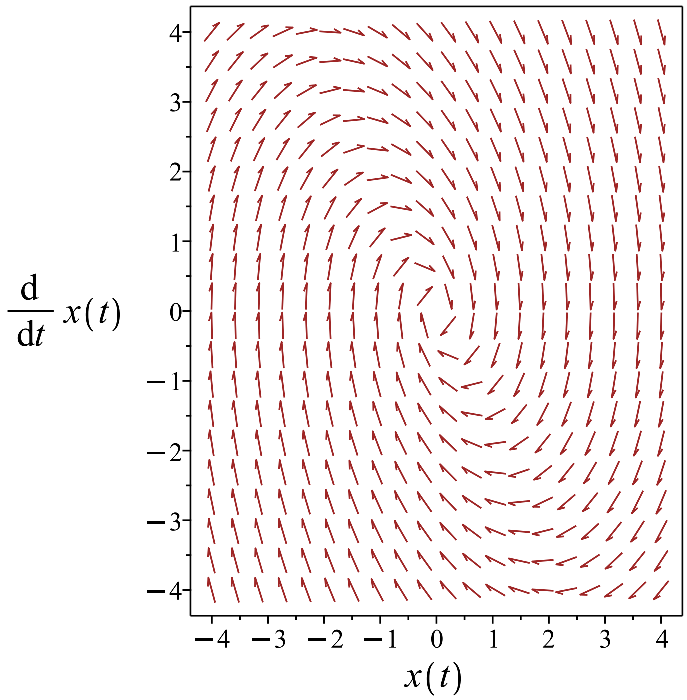


Figure 292: Slope field plot

Verification of solutions

$$x = \cos\left(\frac{\sqrt{7}\,t}{2}\right) e^{-\frac{t}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{7}\,t}{2}\right) e^{-\frac{t}{2}} \sqrt{7}}{7} - \frac{\cos(2t)\,t}{4} - \frac{\sin(2t)\,t}{4} - \frac{\cos(2t)}{8} + \frac{\sin(2t)}{2}$$

Verified OK.

9.19.3 Maple step by step solution

Let's solve

$x'' + x' + 2x = \sin(2t)\,t$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-7})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}, -\frac{1}{2}+\frac{\mathrm{I}\sqrt{7}}{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{7}\,t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{\sqrt{7}\,t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = \cos\left(\frac{\sqrt{7}\,t}{2}\right) \mathrm{e}^{-\frac{t}{2}} c_1 + \sin\left(\frac{\sqrt{7}\,t}{2}\right) \mathrm{e}^{-\frac{t}{2}} c_2 + x_p(t)$

- □ Find a particular solution $x_p(t)$ of the ODE
  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t),x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t),x_2(t))} dt\right), f(t) = \sin(2t)\, t\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(x_1(t), x_2(t)) = \left[\begin{array}{cc} \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{7}\,t}{2}\right) & \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{\sqrt{7}\,t}{2}\right) \\ -\frac{\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{7}\,t}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{7}\,t}{2}\right) \mathrm{e}^{-\frac{t}{2}} \sqrt{7}}{2} & -\frac{\mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{\sqrt{7}\,t}{2}\right)}{2} + \frac{\mathrm{e}^{-\frac{t}{2}} \sqrt{7} \cos\left(\frac{\sqrt{7}\,t}{2}\right)}{2} \end{array}\right]$$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = \frac{\sqrt{7}\,\mathrm{e}^{-t}}{2}$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -\frac{2\,\mathrm{e}^{-\frac{t}{2}} \sqrt{7}\left(\cos\left(\frac{\sqrt{7}\,t}{2}\right)\left(\int t\,\mathrm{e}^{\frac{t}{2}} \sin(2t) \sin\left(\frac{\sqrt{7}\,t}{2}\right) dt\right) - \sin\left(\frac{\sqrt{7}\,t}{2}\right)\left(\int t\,\mathrm{e}^{\frac{t}{2}} \sin(2t) \cos\left(\frac{\sqrt{7}\,t}{2}\right) dt\right)\right)}{7}$

  - Compute integrals

    $x_p(t) = \frac{(-2t-1)\cos(2t)}{8} - \frac{\sin(2t)(-2+t)}{4}$

- Substitute particular solution into general solution to ODE

  $x = \cos\left(\frac{\sqrt{7}\,t}{2}\right) \mathrm{e}^{-\frac{t}{2}} c_1 + \sin\left(\frac{\sqrt{7}\,t}{2}\right) \mathrm{e}^{-\frac{t}{2}} c_2 + \frac{(-2t-1)\cos(2t)}{8} - \frac{\sin(2t)(-2+t)}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 51

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+2*x(t)=t*sin(2*t),x(t), singsol=all)
```

$$x(t) = \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right) c_2 + \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{7}\, t}{2}\right) c_1 + \frac{(-2t-1)\cos(2t)}{8} - \frac{\sin(2t)(t-2)}{4}$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 72

```
DSolve[x''[t]+x'[t]+2*x[t]==t*Sin[2*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{1}{4}(t-2)\sin(2t) - \frac{1}{8}(2t+1)\cos(2t) + c_2 e^{-t/2}\cos\left(\frac{\sqrt{7}t}{2}\right) + c_1 e^{-t/2}\sin\left(\frac{\sqrt{7}t}{2}\right)$$

9.20 problem 3

9.20.1 Existence and uniqueness analysis 1358
9.20.2 Solving as second order linear constant coeff ode 1359
9.20.3 Solving using Kovacic algorithm 1363
9.20.4 Maple step by step solution 1369

Internal problem ID [11471]
Internal file name [`OUTPUT/10453_Thursday_May_18_2023_04_19_56_AM_61312231/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' - bx' + x = \sin(2t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

9.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -b \\ q(t) &= 1 \\ F &= \sin(2t) \end{aligned}$$

Hence the ode is

$$x'' - bx' + x = \sin(2t)$$

The domain of $p(t) = -b$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \sin(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.20.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = -b, C = 1, f(t) = \sin(2t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - bx' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -b, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - b\lambda\, \mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$-b\lambda + \lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = -b, C = 1$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{b}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-b^2 - (4)\,(1)\,(1)} \\
&= \frac{b}{2} \pm \frac{\sqrt{b^2 - 4}}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{b}{2} + \frac{\sqrt{b^2 - 4}}{2} \\
\lambda_2 &= \frac{b}{2} - \frac{\sqrt{b^2 - 4}}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= \frac{b}{2} + \frac{\sqrt{b^2 - 4}}{2} \\
\lambda_2 &= \frac{b}{2} - \frac{\sqrt{b^2 - 4}}{2}
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
x &= c_1 e^{\left(\frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right)t} + c_2 e^{\left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}\right)t}
\end{aligned}
$$

Or

$$
x = c_1 \mathrm{e}^{\left(\frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right)t} + c_2 \mathrm{e}^{\left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}\right)t}
$$

Therefore the homogeneous solution x_h is

$$
x_h = c_1 \mathrm{e}^{\left(\frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right)t} + c_2 \mathrm{e}^{\left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}\right)t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}\right)t}, e^{\left(\frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right)t} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(2t) + A_2 \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(2t) - 3A_2 \sin(2t) - b(-2A_1 \sin(2t) + 2A_2 \cos(2t)) = \sin(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2b}{4b^2+9}, A_2 = -\frac{3}{4b^2+9}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{2\cos(2t)\, b}{4b^2+9} - \frac{3\sin(2t)}{4b^2+9}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{\left(\frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right)t} + c_2 e^{\left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}\right)t}\right) + \left(\frac{2\cos(2t)\, b}{4b^2+9} - \frac{3\sin(2t)}{4b^2+9}\right) \end{aligned}$$

Which simplifies to

$$x = c_1 \mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}} + c_2 \mathrm{e}^{-\frac{\left(-b+\sqrt{b^2-4}\right)t}{2}} + \frac{2\cos(2t)\, b}{4b^2+9} - \frac{3\sin(2t)}{4b^2+9}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}} + c_2 \mathrm{e}^{-\frac{\left(-b+\sqrt{b^2-4}\right)t}{2}} + \frac{2\cos(2t)\, b}{4b^2+9} - \frac{3\sin(2t)}{4b^2+9} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = \frac{(4c_1 + 4c_2)\, b^2 + 2b + 9c_1 + 9c_2}{4b^2+9} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1 \left(\frac{b}{2} + \frac{\sqrt{b^2-4}}{2}\right) \mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}} + c_2 \left(\frac{b}{2} - \frac{\sqrt{b^2-4}}{2}\right) \mathrm{e}^{-\frac{\left(-b+\sqrt{b^2-4}\right)t}{2}} - \frac{4\sin(2t)\, b}{4b^2+9} - \frac{6\cos(2t)}{4b^2+9}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = \frac{4\left(b^2 + \frac{9}{4}\right)(c_1 - c_2)\sqrt{b^2-4} - 12 + (4c_1 + 4c_2)\, b^3 + (9c_1 + 9c_2)\, b}{8b^2+18} \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{\sqrt{b^2-4}\, b^2 - b^3 + 6\sqrt{b^2-4} + 4b}{(4b^2+9)(b^2-4)}$$
$$c_2 = -\frac{\sqrt{b^2-4}\, b^2 + b^3 + 6\sqrt{b^2-4} - 4b}{(4b^2+9)(b^2-4)}$$

Substituting these values back in above solution results in

$$x = \frac{2\cos(2t)\, b\sqrt{b^2-4} - \mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}} b\sqrt{b^2-4} + \mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}} b^2 - \mathrm{e}^{\frac{\left(b-\sqrt{b^2-4}\right)t}{2}} \sqrt{b^2-4}\, b - \mathrm{e}^{\frac{\left(b-\sqrt{b^2-4}\right)t}{2}} b^2 - 3\sin}{4\sqrt{b^2-4}\, b^2 + 9\sqrt{b^2-4}}$$

Which simplifies to

$$x = -\frac{\left(b^2 + b\sqrt{b^2-4} + 6\right)\mathrm{e}^{-\frac{\left(-b+\sqrt{b^2-4}\right)t}{2}} + \left(b\sqrt{b^2-4} - b^2 - 6\right)\mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}} - 2\left(\cos(2t)\, b - \frac{3\sin(2t)}{2}\right)\sqrt{b^2-4}}{\sqrt{b^2-4}\,(4b^2+9)}$$

Summary
The solution(s) found are the following

$$x = \tag{1}$$
$$-\frac{\left(b^{2}+b\sqrt{b^{2}-4}+6\right) \mathrm{e}^{-\frac{\left(-b+\sqrt{b^{2}-4}\right) t}{2}}+\left(b\sqrt{b^{2}-4}-b^{2}-6\right) \mathrm{e}^{\frac{\left(b+\sqrt{b^{2}-4}\right) t}{2}}-2\left(\cos \left(2t\right) b-\frac{3 \sin \left(2t\right)}{2}\right) \sqrt{b^{2}-4}}{\sqrt{b^{2}-4}\, \left(4b^{2}+9\right)}$$

Verification of solutions

$$x =$$
$$-\frac{\left(b^{2}+b\sqrt{b^{2}-4}+6\right) \mathrm{e}^{-\frac{\left(-b+\sqrt{b^{2}-4}\right) t}{2}}+\left(b\sqrt{b^{2}-4}-b^{2}-6\right) \mathrm{e}^{\frac{\left(b+\sqrt{b^{2}-4}\right) t}{2}}-2\left(\cos \left(2t\right) b-\frac{3 \sin \left(2t\right)}{2}\right) \sqrt{b^{2}-4}}{\sqrt{b^{2}-4}\, \left(4b^{2}+9\right)}$$

Verified OK.

9.20.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' - bx' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -b \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{b^2 - 4}{4} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= b^2 - 4 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{b^2}{4} - 1\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 254: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{b^2}{4} - 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{\frac{t\sqrt{b^2-4}}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-b}{1}\,dt} \\
&= z_1 e^{\frac{bt}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{bt}{2}}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-b}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{bt}}{(x_1)^2}\,dt \\
&= x_1\left(-\frac{\mathrm{e}^{-t\sqrt{b^2-4}}}{\sqrt{b^2-4}}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}}\right) + c_2 \left(\mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}} \left(-\frac{\mathrm{e}^{-t\sqrt{b^2-4}}}{\sqrt{b^2-4}}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' - bx' + x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}} - \frac{c_2 \mathrm{e}^{-\frac{\left(-b+\sqrt{b^2-4}\right)t}{2}}}{\sqrt{b^2-4}}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin(2t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(2t), \sin(2t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{ -\frac{\mathrm{e}^{-\frac{\left(-b+\sqrt{b^2-4}\right)t}{2}}}{\sqrt{b^2-4}}, \mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}} \right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_1 \cos(2t) + A_2 \sin(2t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1\cos(2t) - 3A_2\sin(2t) - b(-2A_1\sin(2t) + 2A_2\cos(2t)) = \sin(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2b}{4b^2+9}, A_2 = -\frac{3}{4b^2+9}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{2\cos(2t)\,b}{4b^2+9} - \frac{3\sin(2t)}{4b^2+9}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}} - \frac{c_2 \mathrm{e}^{-\frac{\left(-b+\sqrt{b^2-4}\right)t}{2}}}{\sqrt{b^2-4}}\right) + \left(\frac{2\cos(2t)\,b}{4b^2+9} - \frac{3\sin(2t)}{4b^2+9}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}} - \frac{c_2 \mathrm{e}^{-\frac{\left(-b+\sqrt{b^2-4}\right)t}{2}}}{\sqrt{b^2-4}} + \frac{2\cos(2t)\,b}{4b^2+9} - \frac{3\sin(2t)}{4b^2+9} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = \frac{4\sqrt{b^2-4}\,c_1 b^2 - 4b^2 c_2 + 9\sqrt{b^2-4}\,c_1 + 2b\sqrt{b^2-4} - 9c_2}{(4b^2+9)\sqrt{b^2-4}} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1\left(\frac{b}{2} + \frac{\sqrt{b^2-4}}{2}\right)\mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}} - \frac{c_2\left(\frac{b}{2} - \frac{\sqrt{b^2-4}}{2}\right)\mathrm{e}^{-\frac{\left(-b+\sqrt{b^2-4}\right)t}{2}}}{\sqrt{b^2-4}} - \frac{4\sin(2t)\,b}{4b^2+9} - \frac{6\cos(2t)}{4b^2+9}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = \frac{2\left(c_1 b^3 + b^2 c_2 + \frac{9}{4} c_1 b + \frac{9}{4} c_2 - 3\right) \sqrt{b^2 - 4} + 2(b^2 c_1 - b c_2 - 4 c_1)\left(b^2 + \frac{9}{4}\right)}{(4b^2 + 9)\sqrt{b^2 - 4}} \tag{2A}$$

Equations $\{1\mathrm{A}, 2\mathrm{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{\sqrt{b^2 - 4}\, b^2 - b^3 + 6\sqrt{b^2 - 4} + 4b}{4b^4 - 7b^2 - 36}$$
$$c_2 = \frac{b^2 + b\sqrt{b^2 - 4} + 6}{4b^2 + 9}$$

Substituting these values back in above solution results in

$$x = \frac{2\cos(2t)\, b(b^2 - 4)^{\frac{3}{2}} - \mathrm{e}^{\frac{\left(b - \sqrt{b^2 - 4}\right)t}{2}} b(b^2 - 4)^{\frac{3}{2}} - \mathrm{e}^{\frac{\left(b - \sqrt{b^2 - 4}\right)t}{2}} b^4 - \mathrm{e}^{\frac{\left(b + \sqrt{b^2 - 4}\right)t}{2}} (b^2 - 4)^{\frac{3}{2}}\, b + \mathrm{e}^{\frac{\left(b + \sqrt{b^2 - 4}\right)t}{2}} b^4 - 3}{4\,(b^2 - 4)^{\frac{3}{2}}\, b^2 + 9\,(b^2 -}$$

Which simplifies to

$$x = -\frac{\left(b^2 + b\sqrt{b^2 - 4} + 6\right)\mathrm{e}^{-\frac{\left(-b + \sqrt{b^2 - 4}\right)t}{2}} + \left(b\sqrt{b^2 - 4} - b^2 - 6\right)\mathrm{e}^{\frac{\left(b + \sqrt{b^2 - 4}\right)t}{2}} - 2\left(\cos(2t)\, b - \frac{3\sin(2t)}{2}\right)\sqrt{b^2 - 4}}{\sqrt{b^2 - 4}\,(4b^2 + 9)}$$

Summary

The solution(s) found are the following

$$x = -\frac{\left(b^2 + b\sqrt{b^2 - 4} + 6\right)\mathrm{e}^{-\frac{\left(-b + \sqrt{b^2 - 4}\right)t}{2}} + \left(b\sqrt{b^2 - 4} - b^2 - 6\right)\mathrm{e}^{\frac{\left(b + \sqrt{b^2 - 4}\right)t}{2}} - 2\left(\cos(2t)\, b - \frac{3\sin(2t)}{2}\right)\sqrt{b^2 - 4}}{\sqrt{b^2 - 4}\,(4b^2 + 9)} \tag{1}$$

Verification of solutions

$$x = -\frac{\left(b^2 + b\sqrt{b^2 - 4} + 6\right)\mathrm{e}^{-\frac{\left(-b + \sqrt{b^2 - 4}\right)t}{2}} + \left(b\sqrt{b^2 - 4} - b^2 - 6\right)\mathrm{e}^{\frac{\left(b + \sqrt{b^2 - 4}\right)t}{2}} - 2\left(\cos(2t)\, b - \frac{3\sin(2t)}{2}\right)\sqrt{b^2 - 4}}{\sqrt{b^2 - 4}\,(4b^2 + 9)}$$

Verified OK.

9.20.4 Maple step by step solution

Let's solve

$$\left[x''-bx'+x=\sin(2t), x(0)=0, x'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  x''
- Characteristic polynomial of homogeneous ODE

  $-br+r^2+1=0$
- Use quadratic formula to solve for r

  $r=\frac{b\pm\left(\sqrt{b^2-4}\right)}{2}$
- Roots of the characteristic polynomial

  $r=\left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}, \frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right)$
- 1st solution of the homogeneous ODE

  $x_1(t)=\mathrm{e}^{\left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}\right)t}$
- 2nd solution of the homogeneous ODE

  $x_2(t)=\mathrm{e}^{\left(\frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right)t}$
- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)+x_p(t)$
- Substitute in solutions of the homogeneous ODE

  $x=c_1\mathrm{e}^{\left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}\right)t}+c_2\mathrm{e}^{\left(\frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right)t}+x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t)=-x_1(t)\left(\int\frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right)+x_2(t)\left(\int\frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t)=\sin(2t)\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t))=\begin{bmatrix}\mathrm{e}^{\left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}\right)t} & \mathrm{e}^{\left(\frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right)t}\\ \left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}\right)\mathrm{e}^{\left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}\right)t} & \left(\frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right)\mathrm{e}^{\left(\frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right)t}\end{bmatrix}$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = \sqrt{b^2-4}\,\mathrm{e}^{bt}$
- Substitute functions into equation for $x_p(t)$

  $$x_p(t) = \frac{-\mathrm{e}^{-\frac{\left(-b+\sqrt{b^2-4}\right)t}{2}}\left(\int \sin(2t)\mathrm{e}^{\frac{\left(-b+\sqrt{b^2-4}\right)t}{2}}dt\right)+\mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}}\left(\int \sin(2t)\mathrm{e}^{-\frac{\left(b+\sqrt{b^2-4}\right)t}{2}}dt\right)}{\sqrt{b^2-4}}$$
- Compute integrals

  $$x_p(t) = \frac{-8\cos(2t)b+12\sin(2t)}{\left(b\sqrt{b^2-4}-b^2-6\right)\left(b^2+b\sqrt{b^2-4}+6\right)}$$

- Substitute particular solution into general solution to ODE

  $$x = c_1\mathrm{e}^{\left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}\right)t} + c_2\mathrm{e}^{\left(\frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right)t} + \frac{-8\cos(2t)b+12\sin(2t)}{\left(b\sqrt{b^2-4}-b^2-6\right)\left(b^2+b\sqrt{b^2-4}+6\right)}$$

☐ Check validity of solution $x = c_1\mathrm{e}^{\left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}\right)t} + c_2\mathrm{e}^{\left(\frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right)t} + \frac{-8\cos(2t)b+12\sin(2t)}{\left(b\sqrt{b^2-4}-b^2-6\right)\left(b^2+b\sqrt{b^2-4}+6\right)}$

- Use initial condition $x(0) = 0$

  $$0 = c_1 + c_2 - \frac{8b}{\left(b\sqrt{b^2-4}-b^2-6\right)\left(b^2+b\sqrt{b^2-4}+6\right)}$$
- Compute derivative of the solution

  $$x' = c_1\left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}\right)\mathrm{e}^{\left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}\right)t} + c_2\left(\frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right)\mathrm{e}^{\left(\frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right)t} + \frac{16\sin(2t)b+24\cos(2t)}{\left(b\sqrt{b^2-4}-b^2-6\right)\left(b^2+b\sqrt{b^2-4}+6\right)}$$
- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $$0 = c_1\left(\frac{b}{2}-\frac{\sqrt{b^2-4}}{2}\right) + c_2\left(\frac{b}{2}+\frac{\sqrt{b^2-4}}{2}\right) + \frac{24}{\left(b\sqrt{b^2-4}-b^2-6\right)\left(b^2+b\sqrt{b^2-4}+6\right)}$$
- Solve for c_1 and c_2

  $$\left\{c_1 = -\frac{\sqrt{b^2-4}\,b^2+b^3+6\sqrt{b^2-4}-4b}{(4b^2+9)(b^2-4)}, c_2 = -\frac{-\sqrt{b^2-4}\,b^2+b^3-6\sqrt{b^2-4}-4b}{(4b^2+9)(b^2-4)}\right\}$$
- Substitute constant values into general solution and simplify

  $$x = \frac{\left(-4b^3-4\sqrt{b^2-4}\,b^2+16b-24\sqrt{b^2-4}\right)\mathrm{e}^{-\frac{\left(-b+\sqrt{b^2-4}\right)t}{2}}+\left(-4b^3+4\sqrt{b^2-4}\,b^2+16b+24\sqrt{b^2-4}\right)\mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}}+8(-2+b)(2+b)\left(\mathrm{co}\right.}{(-b^4+4b^2)(b^2-4)+b^6+8b^4-12b^2-144}$$

- Solution to the IVP

  $$x = \frac{\left(-4b^3-4\sqrt{b^2-4}\,b^2+16b-24\sqrt{b^2-4}\right)\mathrm{e}^{-\frac{\left(-b+\sqrt{b^2-4}\right)t}{2}}+\left(-4b^3+4\sqrt{b^2-4}\,b^2+16b+24\sqrt{b^2-4}\right)\mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}}+8(-2+b)(2+b)\left(\mathrm{co}\right.}{(-b^4+4b^2)(b^2-4)+b^6+8b^4-12b^2-144}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.157 (sec). Leaf size: 135

```
dsolve([diff(x(t),t$2)-b*diff(x(t),t)+x(t)=sin(2*t),x(0) = 0, D(x)(0) = 0],x(t), singsol=all
```

$$x(t) = \frac{\left(-\sqrt{b^2-4}\, b^2 - b^3 - 6\sqrt{b^2-4} + 4b\right) \mathrm{e}^{-\frac{\left(-b+\sqrt{b^2-4}\right)t}{2}} + \left(\sqrt{b^2-4}\, b^2 - b^3 + 6\sqrt{b^2-4} + 4b\right) \mathrm{e}^{\frac{\left(b+\sqrt{b^2-4}\right)t}{2}} + 2(}{4b^4 - 7b^2 - 36}$$

✓ Solution by Mathematica

Time used: 0.873 (sec). Leaf size: 122

```
DSolve[{x''[t]-b*x'[t]+x[t]==Sin[2*t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{\frac{e^{\frac{1}{2}\left(b-\sqrt{b^2-4}\right)t}\left(b^2\left(e^{\sqrt{b^2-4}t}-1\right)-\sqrt{b^2-4}b\left(e^{\sqrt{b^2-4}t}+1\right)+6e^{\sqrt{b^2-4}t}-6\right)}{\sqrt{b^2-4}} + 2b\cos(2t) - 3\sin(2t)}{4b^2+9}$$

9.21 problem 4

9.21.1 Existence and uniqueness analysis 1372
9.21.2 Solving as second order linear constant coeff ode 1373
9.21.3 Solving using Kovacic algorithm 1377
9.21.4 Maple step by step solution 1382

Internal problem ID [11472]
Internal file name [`OUTPUT/10454_Thursday_May_18_2023_04_19_58_AM_5814105/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x'' - 3x' - 40x = 2\,\mathrm{e}^{-t}}$$

With initial conditions

$$[x(0) = 0, x'(0) = 1]$$

9.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -3 \\ q(t) &= -40 \\ F &= 2\,\mathrm{e}^{-t} \end{aligned}$$

Hence the ode is

$$x'' - 3x' - 40x = 2\,\mathrm{e}^{-t}$$

The domain of $p(t) = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -40$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 2\,\mathrm{e}^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.21.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = -3, C = -40, f(t) = 2\,\mathrm{e}^{-t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - 3x' - 40x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -3, C = -40$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - 3\lambda\,\mathrm{e}^{\lambda t} - 40\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 3\lambda - 40 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = -3, C = -40$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-3^2 - (4)(1)(-40)} \\
&= \frac{3}{2} \pm \frac{13}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{3}{2} + \frac{13}{2} \\
\lambda_2 &= \frac{3}{2} - \frac{13}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 8 \\
\lambda_2 &= -5
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
x &= c_1 e^{(8)t} + c_2 e^{(-5)t}
\end{aligned}
$$

Or

$$
x = c_1 \mathrm{e}^{8t} + c_2 \mathrm{e}^{-5t}
$$

Therefore the homogeneous solution x_h is

$$
x_h = c_1 \mathrm{e}^{8t} + c_2 \mathrm{e}^{-5t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2\,\mathrm{e}^{-t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-5t}, e^{8t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 e^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-36A_1 e^{-t} = 2\,e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{18}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{e^{-t}}{18}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{8t} + c_2 e^{-5t}\right) + \left(-\frac{e^{-t}}{18}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{8t} + c_2 e^{-5t} - \frac{e^{-t}}{18} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 - \frac{1}{18} \tag{1A}$$

Taking derivative of the solution gives

$$x' = 8c_1 e^{8t} - 5c_2 e^{-5t} + \frac{e^{-t}}{18}$$

substituting $x' = 1$ and $t = 0$ in the above gives

$$1 = 8c_1 - 5c_2 + \frac{1}{18} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{11}{117}$$
$$c_2 = -\frac{1}{26}$$

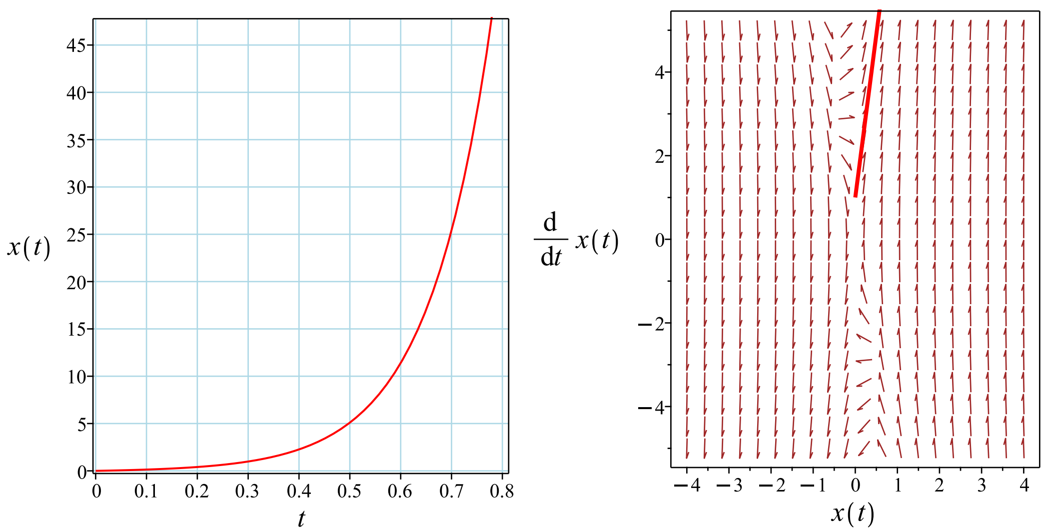
Substituting these values back in above solution results in

$$x = \frac{11\,e^{8t}}{117} - \frac{e^{-5t}}{26} - \frac{e^{-t}}{18}$$

Summary

The solution(s) found are the following

$$x = \frac{11\,e^{8t}}{117} - \frac{e^{-5t}}{26} - \frac{e^{-t}}{18} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \frac{11\,e^{8t}}{117} - \frac{e^{-5t}}{26} - \frac{e^{-t}}{18}$$

Verified OK.

9.21.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 3x' - 40x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= -40 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{169}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 169 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{169z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 256: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{169}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-\frac{13t}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\frac{3t}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-5t}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-3}{1}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{3t}}{(x_1)^2}\, dt \\
&= x_1 \left(\frac{\mathrm{e}^{13t}}{13}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-5t}\right) + c_2 \left(\mathrm{e}^{-5t} \left(\frac{\mathrm{e}^{13t}}{13}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' - 3x' - 40x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{-5t} + \frac{c_2 \mathrm{e}^{8t}}{13}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{8t}}{13}, \mathrm{e}^{-5t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \mathrm{e}^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-36A_1 \mathrm{e}^{-t} = 2\,\mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{18}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{\mathrm{e}^{-t}}{18}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{-5t} + \frac{c_2 \mathrm{e}^{8t}}{13}\right) + \left(-\frac{\mathrm{e}^{-t}}{18}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{-5t} + \frac{c_2 e^{8t}}{13} - \frac{e^{-t}}{18} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{13} - \frac{1}{18} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -5c_1 e^{-5t} + \frac{8c_2 e^{8t}}{13} + \frac{e^{-t}}{18}$$

substituting $x' = 1$ and $t = 0$ in the above gives

$$1 = -5c_1 + \frac{8c_2}{13} + \frac{1}{18} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{1}{26} \\ c_2 &= \frac{11}{9} \end{aligned}$$

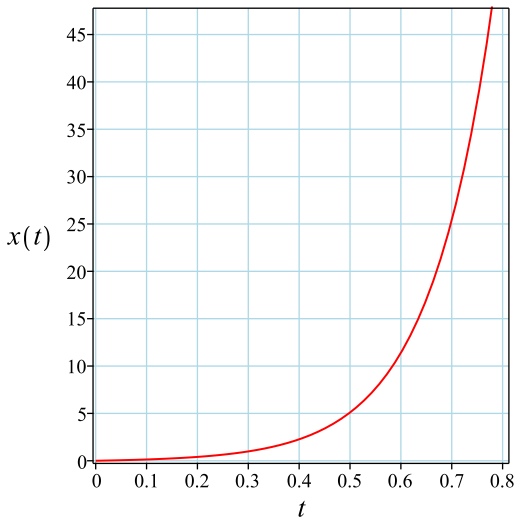
Substituting these values back in above solution results in

$$x = \frac{11\,e^{8t}}{117} - \frac{e^{-5t}}{26} - \frac{e^{-t}}{18}$$

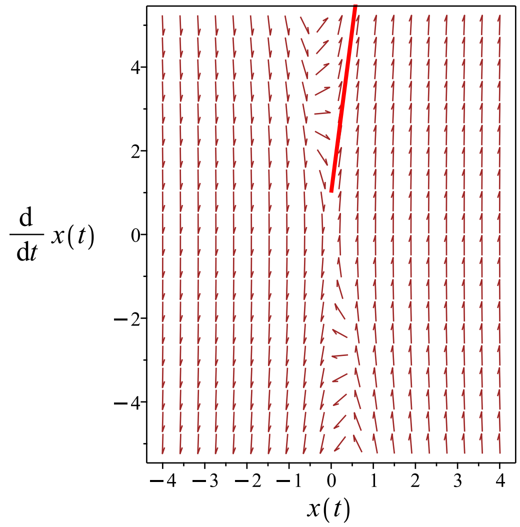
Summary

The solution(s) found are the following

$$x = \frac{11\,e^{8t}}{117} - \frac{e^{-5t}}{26} - \frac{e^{-t}}{18} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{11\,\mathrm{e}^{8t}}{117} - \frac{\mathrm{e}^{-5t}}{26} - \frac{\mathrm{e}^{-t}}{18}$$

Verified OK.

9.21.4 Maple step by step solution

Let's solve

$$\left[x'' - 3x' - 40x = 2\,\mathrm{e}^{-t}, x(0) = 0, x'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  x''
- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r - 40 = 0$
- Factor the characteristic polynomial

  $(r+5)(r-8) = 0$
- Roots of the characteristic polynomial

  $r = (-5, 8)$
- 1st solution of the homogeneous ODE

$x_1(t) = \mathrm{e}^{-5t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{8t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-5t} + c_2 \mathrm{e}^{8t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t) \left(\int \frac{x_2(t) f(t)}{W(x_1(t),x_2(t))} dt\right) + x_2(t) \left(\int \frac{x_1(t) f(t)}{W(x_1(t),x_2(t))} dt\right), f(t) = 2\,\mathrm{e}^{-t}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-5t} & \mathrm{e}^{8t} \\ -5\,\mathrm{e}^{-5t} & 8\,\mathrm{e}^{8t} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(x_1(t), x_2(t)) = 13\,\mathrm{e}^{3t}$

  ◦ Substitute functions into equation for $x_p(t)$

  $x_p(t) = \frac{2\left(\mathrm{e}^{13t}\left(\int \mathrm{e}^{-9t} dt\right) - \left(\int \mathrm{e}^{4t} dt\right)\right)\mathrm{e}^{-5t}}{13}$

  ◦ Compute integrals

  $x_p(t) = -\frac{\mathrm{e}^{-t}}{18}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \mathrm{e}^{-5t} + c_2 \mathrm{e}^{8t} - \frac{\mathrm{e}^{-t}}{18}$

☐ Check validity of solution $x = c_1 \mathrm{e}^{-5t} + c_2 \mathrm{e}^{8t} - \frac{\mathrm{e}^{-t}}{18}$

  ◦ Use initial condition $x(0) = 0$

  $0 = c_1 + c_2 - \frac{1}{18}$

  ◦ Compute derivative of the solution

  $x' = -5c_1 \mathrm{e}^{-5t} + 8c_2 \mathrm{e}^{8t} + \frac{\mathrm{e}^{-t}}{18}$

  ◦ Use the initial condition $x'\Big|_{\{t=0\}} = 1$

  $1 = -5c_1 + 8c_2 + \frac{1}{18}$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{26}, c_2 = \frac{11}{117}\right\}$

- Substitute constant values into general solution and simplify

  $x = -\frac{(-22\,\mathrm{e}^{13t}+13\,\mathrm{e}^{4t}+9)\,\mathrm{e}^{-5t}}{234}$

- Solution to the IVP

  $x = -\frac{(-22\,\mathrm{e}^{13t}+13\,\mathrm{e}^{4t}+9)\,\mathrm{e}^{-5t}}{234}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve([diff(x(t),t$2)-3*diff(x(t),t)-40*x(t)=2*exp(-t),x(0) = 0, D(x)(0) = 1],x(t), singsol
```

$$x(t) = -\frac{(-22\,\mathrm{e}^{13t}+13\,\mathrm{e}^{4t}+9)\,\mathrm{e}^{-5t}}{234}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 30

```
DSolve[{x''[t]-3*x'[t]-40*x[t]==2*Exp[-t],{x[0]==0,x'[0]==1}},x[t],t,IncludeSingularSolution
```

$$x(t) \to \frac{1}{234}e^{-5t}\left(-13e^{4t}+22e^{13t}-9\right)$$

9.22 problem 5

9.22.1 Existence and uniqueness analysis 1386
9.22.2 Solving as second order linear constant coeff ode 1386
9.22.3 Solving as second order integrable as is ode 1390
9.22.4 Solving as second order ode missing y ode 1392
9.22.5 Solving as type second_order_integrable_as_is (not using ABC version) 1394
9.22.6 Solving using Kovacic algorithm 1397
9.22.7 Solving as exact linear second order ode ode 1402
9.22.8 Maple step by step solution 1405

Internal problem ID [11473]
Internal file name [`OUTPUT/10455_Thursday_May_18_2023_04_19_59_AM_20984005/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' - 2x' = 4}$$

With initial conditions

$$[x(0) = 1, x'(0) = 0]$$

9.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 0 \\ F &= 4 \end{aligned}$$

Hence the ode is

$$x'' - 2x' = 4$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $F = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.22.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = -2, C = 0, f(t) = 4$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - 2x' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -2, C = 0$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - 2\lambda\, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 2\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(0)}\\ &= 1 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 1\\ \lambda_2 &= 1 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= 0\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ x &= c_1 e^{(2)t} + c_2 e^{(0)t}\end{aligned}$$

Or

$$x = c_1 \mathrm{e}^{2t} + c_2$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^{2t} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{2t}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 = 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -2t$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{2t} + c_2\right) + (-2t) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{2t} + c_2 - 2t \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = 2c_1 e^{2t} - 2$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = 2c_1 - 2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 0$$

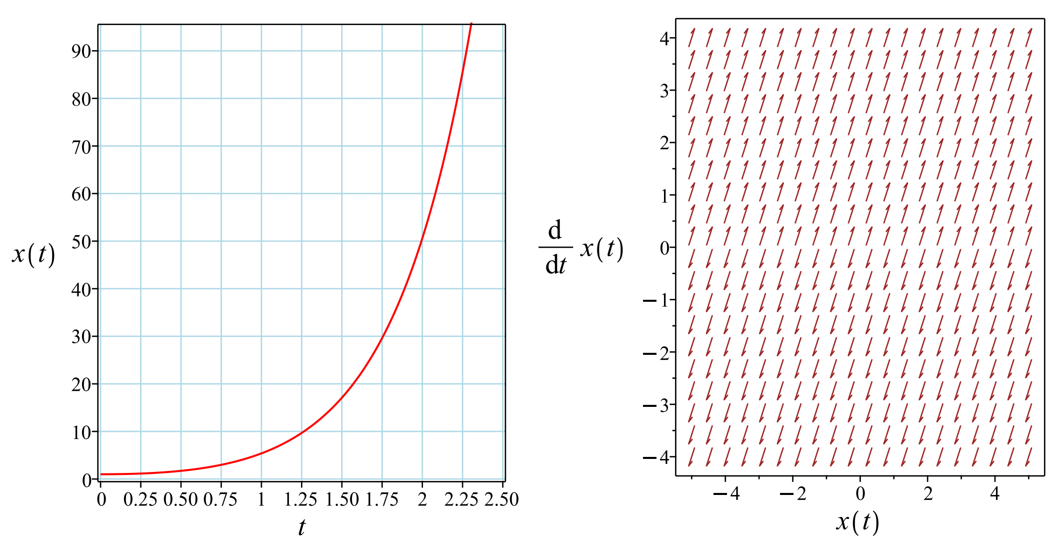
Substituting these values back in above solution results in

$$x = e^{2t} - 2t$$

Summary

The solution(s) found are the following

$$x = e^{2t} - 2t \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
x = e^{2t} - 2t
$$

Verified OK.

9.22.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$
\int (x'' - 2x')\,dt = \int 4dt
$$
$$
-2x + x' = 4t + c_1
$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
x' + p(t)x = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= -2 \\
q(t) &= 4t + c_1
\end{aligned}
$$

Hence the ode is

$$
-2x + x' = 4t + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int (-2)dt} \\
&= e^{-2t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu x) &= (\mu)(4t + c_1) \\
\frac{d}{dt}\left(e^{-2t}x\right) &= \left(e^{-2t}\right)(4t + c_1) \\
d\left(e^{-2t}x\right) &= \left((4t + c_1)\,e^{-2t}\right)\,dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-2t}x &= \int (4t + c_1)\,e^{-2t}\,dt \\
e^{-2t}x &= -\frac{(4t + c_1 + 2)\,e^{-2t}}{2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-2t}$ results in

$$x = -\frac{\mathrm{e}^{2t}(4t + c_1 + 2)\,\mathrm{e}^{-2t}}{2} + c_2\mathrm{e}^{2t}$$

which simplifies to

$$x = -2t - \frac{c_1}{2} - 1 + c_2\mathrm{e}^{2t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = -2t - \frac{c_1}{2} - 1 + c_2\mathrm{e}^{2t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = -1 - \frac{c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -2 + 2c_2\mathrm{e}^{2t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -2 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -2 \\ c_2 &= 1 \end{aligned}$$

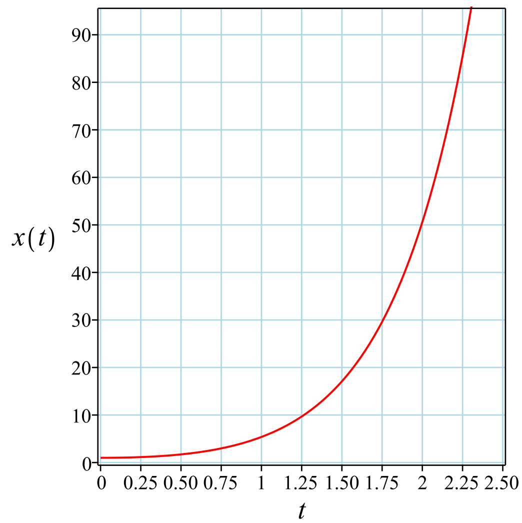
Substituting these values back in above solution results in

$$x = \mathrm{e}^{2t} - 2t$$

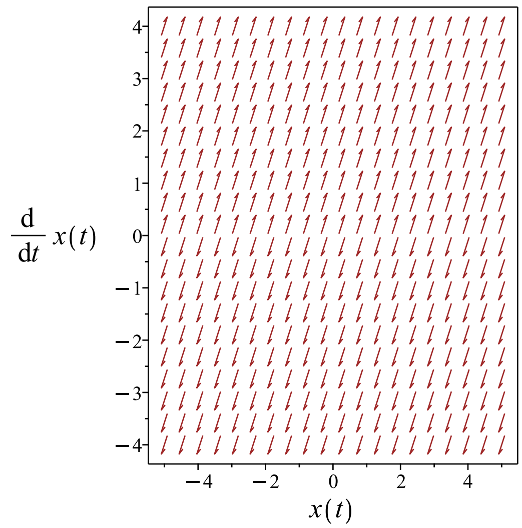
Summary

The solution(s) found are the following

$$x = \mathrm{e}^{2t} - 2t \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{2t} - 2t$$

Verified OK.

9.22.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable x. Let

$$p(t) = x'$$

Then

$$p'(t) = x''$$

Hence the ode becomes

$$p'(t) - 2p(t) - 4 = 0$$

Which is now solve for $p(t)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{2p+4} dp = \int dt$$
$$\frac{\ln(p+2)}{2} = t + c_1$$

Raising both side to exponential gives

$$\sqrt{p+2} = e^{t+c_1}$$

Which simplifies to

$$\sqrt{p+2} = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $p = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_2^2 - 2$$

$$c_2 = -\sqrt{2}$$

Substituting c_2 found above in the general solution gives

$$p(t) = 2\,e^{2t} - 2$$

Since $p = x'$ then the new first order ode to solve is

$$x' = 2\,e^{2t} - 2$$

Integrating both sides gives

$$\begin{aligned} x &= \int 2\,e^{2t} - 2\,dt \\ &= e^{2t} - 2t + c_3 \end{aligned}$$

Initial conditions are used to solve for c_3. Substituting $t = 0$ and $x = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 1 + c_3$$

$$c_3 = 0$$

Substituting c_3 found above in the general solution gives

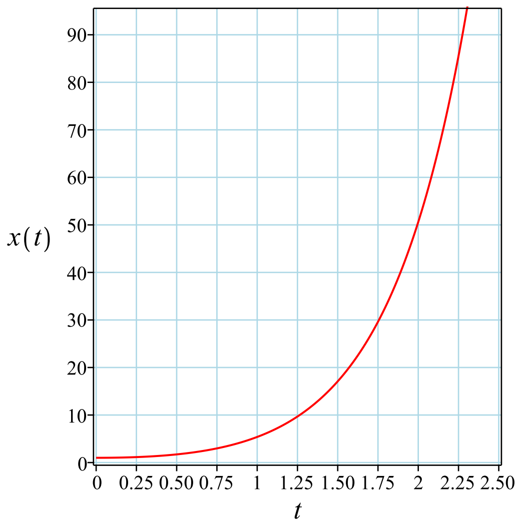
$$x = e^{2t} - 2t$$

Initial conditions are used to solve for the constants of integration.

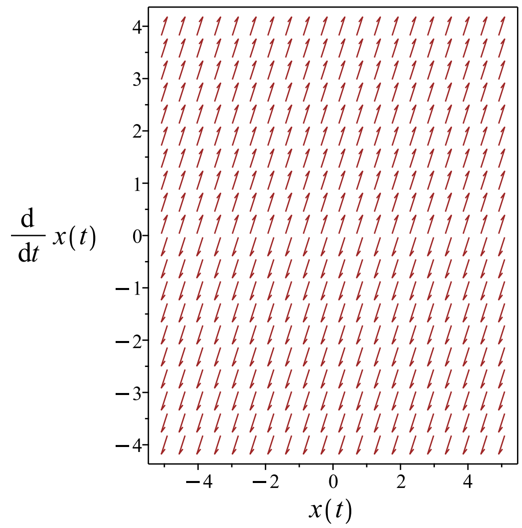
Summary

The solution(s) found are the following

$$x = e^{2t} - 2t \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$x = e^{2t} - 2t$$

Verified OK.

9.22.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x'' - 2x' = 4$$

Integrating both sides of the ODE w.r.t t gives

$$\int (x'' - 2x')\, dt = \int 4 dt$$

$$-2x + x' = 4t + c_1$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = -2$$
$$q(t) = 4t + c_1$$

Hence the ode is

$$-2x + x' = 4t + c_1$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int (-2)dt} \\
&= e^{-2t}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dt}(\mu x) &= (\mu)(4t + c_1) \\
\frac{d}{dt}\left(e^{-2t} x\right) &= \left(e^{-2t}\right)(4t + c_1) \\
d\left(e^{-2t} x\right) &= \left((4t + c_1) e^{-2t}\right) dt
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{-2t} x &= \int (4t + c_1) e^{-2t} dt \\
e^{-2t} x &= -\frac{(4t + c_1 + 2) e^{-2t}}{2} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-2t}$ results in

$$x = -\frac{e^{2t}(4t + c_1 + 2) e^{-2t}}{2} + c_2 e^{2t}$$

which simplifies to

$$x = -2t - \frac{c_1}{2} - 1 + c_2 e^{2t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = -2t - \frac{c_1}{2} - 1 + c_2 e^{2t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = -1 - \frac{c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -2 + 2c_2 e^{2t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -2 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 1$$

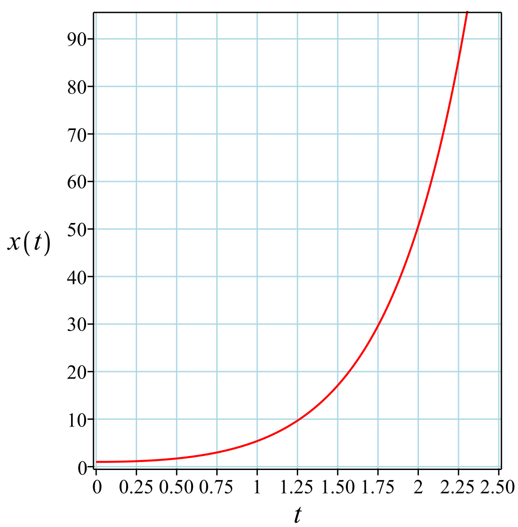
Substituting these values back in above solution results in

$$x = e^{2t} - 2t$$

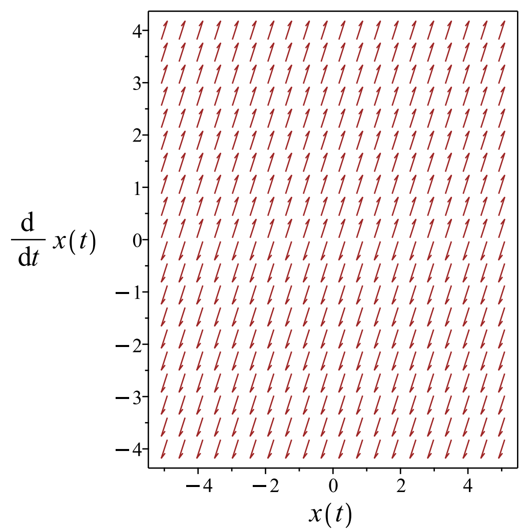
Summary

The solution(s) found are the following

$$x = e^{2t} - 2t \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{2t} - 2t$$

Verified OK.

9.22.6 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 2x' = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 258: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2}{1} \, dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^t \\
&= z_1 \left(\mathrm{e}^t\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = 1
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{x_1^2} \, dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-2}{1} \, dt}}{(x_1)^2} \, dt \\
&= x_1 \int \frac{e^{2t}}{(x_1)^2} \, dt \\
&= x_1 \left(\frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(1) + c_2 \left(1 \left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' - 2x' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 + \frac{c_2 \mathrm{e}^{2t}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, \frac{e^{2t}}{2}\right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 = 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -2t$$

Therefore the general solution is

$$\begin{aligned}
x &= x_h + x_p \\
&= \left(c_1 + \frac{c_2 e^{2t}}{2}\right) + (-2t)
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 + \frac{c_2 \mathrm{e}^{2t}}{2} - 2t \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 + \frac{c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_2 \mathrm{e}^{2t} - 2$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = c_2 - 2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 2 \end{aligned}$$

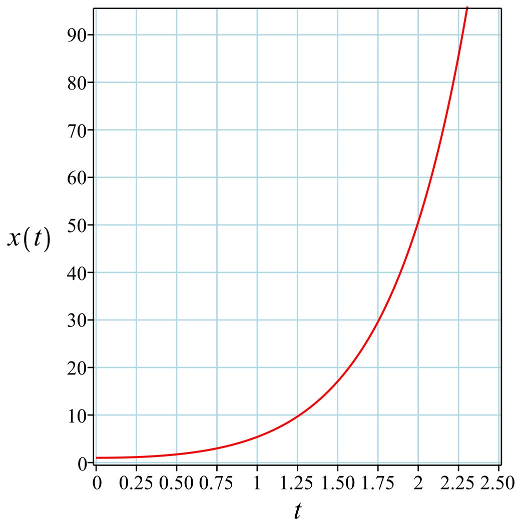
Substituting these values back in above solution results in

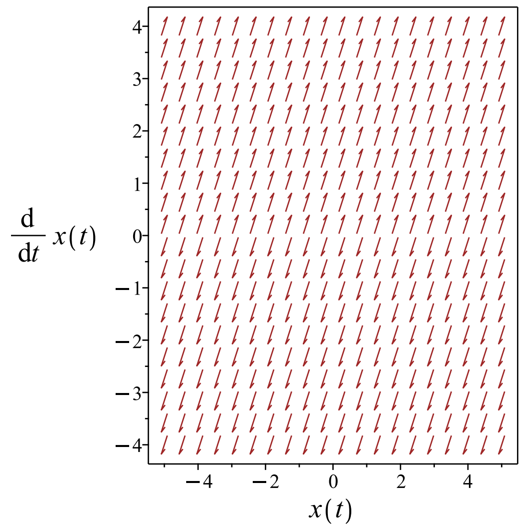
$$x = \mathrm{e}^{2t} - 2t$$

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{2t} - 2t \tag{1}$$





(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = e^{2t} - 2t$$

Verified OK.

9.22.7 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\,x'' + q(t)\,x' + r(t)\,x = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x) &= 1\\ q(x) &= -2\\ r(x) &= 0\\ s(x) &= 4\end{aligned}$$

Hence

$$\begin{aligned}p''(x) &= 0\\ q'(x) &= 0\end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,x'+(q(t)-p'(t))\,x)'=s(x)$$

Integrating gives

$$p(t)\,x'+(q(t)-p'(t))\,x=\int s(t)\;dt$$

Substituting the above values for p,q,r,s gives

$$-2x+x'=\int 4\,dt$$

We now have a first order ode to solve which is

$$-2x+x'=4t+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x'+p(t)x=q(t)$$

Where here

$$\begin{aligned}p(t)&=-2\\ q(t)&=4t+c_1\end{aligned}$$

Hence the ode is

$$-2x+x'=4t+c_1$$

The integrating factor μ is

$$\begin{aligned}\mu&=\mathrm{e}^{\int(-2)dt}\\ &=\mathrm{e}^{-2t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu x)&=(\mu)\,(4t+c_1)\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-2t}x\right)&=\left(\mathrm{e}^{-2t}\right)(4t+c_1)\\ \mathrm{d}\left(\mathrm{e}^{-2t}x\right)&=\left((4t+c_1)\,\mathrm{e}^{-2t}\right)\;\mathrm{d}t\end{aligned}$$

Integrating gives

$$
\begin{aligned}
e^{-2t}x &= \int (4t + c_1)\, e^{-2t}\, dt \\
e^{-2t}x &= -\frac{(4t + c_1 + 2)\, e^{-2t}}{2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-2t}$ results in

$$
x = -\frac{e^{2t}(4t + c_1 + 2)\, e^{-2t}}{2} + c_2 e^{2t}
$$

which simplifies to

$$
x = -2t - \frac{c_1}{2} - 1 + c_2 e^{2t}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = -2t - \frac{c_1}{2} - 1 + c_2 e^{2t} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$
1 = -1 - \frac{c_1}{2} + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = -2 + 2c_2 e^{2t}
$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$
0 = -2 + 2c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -2 \\
c_2 &= 1
\end{aligned}
$$

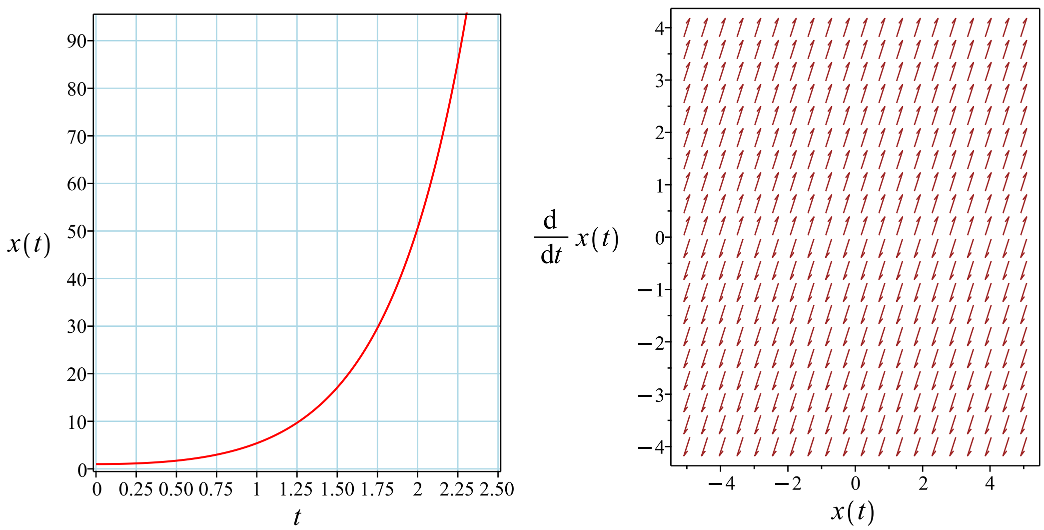
Substituting these values back in above solution results in

$$
x = e^{2t} - 2t
$$

Summary

The solution(s) found are the following

$$x = e^{2t} - 2t \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = e^{2t} - 2t$$

Verified OK.

9.22.8 Maple step by step solution

Let's solve

$\left[x'' - 2x' = 4, x(0) = 1, x'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r = 0$

- Factor the characteristic polynomial

  $r(r - 2) = 0$

- Roots of the characteristic polynomial

  $r = (0, 2)$
- 1st solution of the homogeneous ODE

  $x_1(t) = 1$
- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{2t}$
- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$
- Substitute in solutions of the homogeneous ODE

  $x = c_1 + c_2 \mathrm{e}^{2t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t) \left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t) \left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = 4\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} 1 & \mathrm{e}^{2t} \\ 0 & 2\,\mathrm{e}^{2t} \end{bmatrix}$
  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 2\,\mathrm{e}^{2t}$
  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -2\left(\int 1 dt\right) + 2\,\mathrm{e}^{2t}\left(\int \mathrm{e}^{-2t} dt\right)$
  - Compute integrals

    $x_p(t) = -2t - 1$

- Substitute particular solution into general solution to ODE

  $x = c_1 + c_2 \mathrm{e}^{2t} - 2t - 1$

☐ Check validity of solution $x = c_1 + c_2 \mathrm{e}^{2t} - 2t - 1$

  - Use initial condition $x(0) = 1$

    $1 = -1 + c_1 + c_2$
  - Compute derivative of the solution

    $x' = -2 + 2c_2 \mathrm{e}^{2t}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = -2 + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $x = e^{2t} - 2t$

- Solution to the IVP

  $x = e^{2t} - 2t$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 2*_b(_a)+4, _b(_a)`   *** Sublevel 2 **
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve([diff(x(t),t$2)-2*diff(x(t),t)=4,x(0) = 1, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = e^{2t} - 2t$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 14

```
DSolve[{x''[t]-2*x'[t]==4,{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{2t} - 2t$$

9.23 problem 6

9.23.1 Existence and uniqueness analysis 1409
9.23.2 Solving as second order linear constant coeff ode 1410
9.23.3 Solving using Kovacic algorithm 1414
9.23.4 Maple step by step solution 1420

Internal problem ID [11474]
Internal file name [`OUTPUT/10456_Thursday_May_18_2023_04_20_01_AM_70888592/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.1 Nonhomogeneous Equations: Undetermined Coefficients. Exercises page 110
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + 2x = \cos\left(\sqrt{2}\, t\right)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 1]$$

9.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 2 \\ F &= \cos\left(\sqrt{2}\, t\right) \end{aligned}$$

Hence the ode is

$$x'' + 2x = \cos\left(\sqrt{2}\,t\right)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \cos\left(\sqrt{2}\,t\right)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.23.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = 2, f(t) = \cos\left(\sqrt{2}\,t\right)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 2x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\,e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(2)} \\
&= \pm i\sqrt{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +i\sqrt{2} \\
\lambda_2 &= -i\sqrt{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= i\sqrt{2} \\
\lambda_2 &= -i\sqrt{2}
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^0\left(c_1 \cos\left(\sqrt{2}\, t\right) + c_2 \sin\left(\sqrt{2}\, t\right)\right)$$

Or

$$x = c_1 \cos\left(\sqrt{2}\, t\right) + c_2 \sin\left(\sqrt{2}\, t\right)$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \cos\left(\sqrt{2}\, t\right) + c_2 \sin\left(\sqrt{2}\, t\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos\left(\sqrt{2}\,t\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\cos\left(\sqrt{2}\,t\right), \sin\left(\sqrt{2}\,t\right)\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(\sqrt{2}\,t\right), \sin\left(\sqrt{2}\,t\right)\right\}$$

Since $\cos\left(\sqrt{2}\,t\right)$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$\left[\left\{t\cos\left(\sqrt{2}\,t\right), t\sin\left(\sqrt{2}\,t\right)\right\}\right]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t\cos\left(\sqrt{2}\,t\right) + A_2 t\sin\left(\sqrt{2}\,t\right)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sqrt{2}\,\sin\left(\sqrt{2}\,t\right) + 2A_2\sqrt{2}\,\cos\left(\sqrt{2}\,t\right) = \cos\left(\sqrt{2}\,t\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{\sqrt{2}}{4}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{\sqrt{2}\,t\sin\left(\sqrt{2}\,t\right)}{4}$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(c_1 \cos\left(\sqrt{2}\, t\right) + c_2 \sin\left(\sqrt{2}\, t\right)\right) + \left(\frac{\sqrt{2}\, t \sin\left(\sqrt{2}\, t\right)}{4}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 \cos\left(\sqrt{2}\, t\right) + c_2 \sin\left(\sqrt{2}\, t\right) + \frac{\sqrt{2}\, t \sin\left(\sqrt{2}\, t\right)}{4} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = -c_1 \sqrt{2}\, \sin\left(\sqrt{2}\, t\right) + c_2 \sqrt{2}\, \cos\left(\sqrt{2}\, t\right) + \frac{\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{4} + \frac{t \cos\left(\sqrt{2}\, t\right)}{2}
$$

substituting $x' = 1$ and $t = 0$ in the above gives

$$
1 = \sqrt{2}\, c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= \frac{\sqrt{2}}{2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
x = \frac{\sqrt{2}\, t \sin\left(\sqrt{2}\, t\right)}{4} + \frac{\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{2}
$$

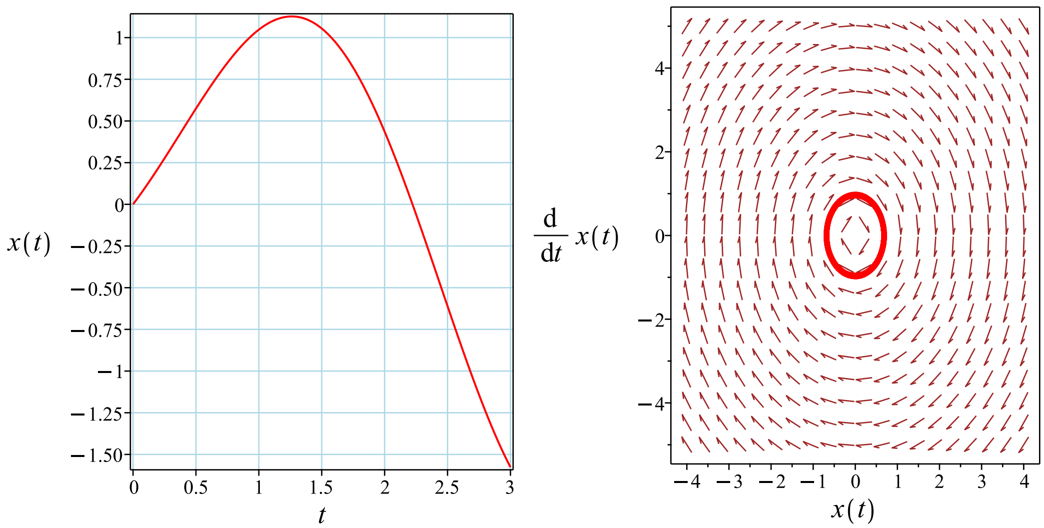
Which simplifies to

$$
x = \frac{(t + 2)\, \sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{4}
$$

Summary

The solution(s) found are the following

$$x = \frac{(t+2)\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \frac{(t+2)\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{4}$$

Verified OK.

9.23.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 2x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -2z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 260: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos\left(\sqrt{2}\, t\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
x_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
x_1 &= z_1 \\
&= \cos\left(\sqrt{2}\, t\right)
\end{aligned}
$$

Which simplifies to

$$x_1 = \cos\left(\sqrt{2}\, t\right)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\ &= \cos\left(\sqrt{2}\, t\right) \int \frac{1}{\cos\left(\sqrt{2}\, t\right)^2}\, dt \\ &= \cos\left(\sqrt{2}\, t\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, t\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1 \left(\cos\left(\sqrt{2}\, t\right)\right) + c_2 \left(\cos\left(\sqrt{2}\, t\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, t\right)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 2x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \cos\left(\sqrt{2}\, t\right) + \frac{c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos\left(\sqrt{2}\, t\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\cos\left(\sqrt{2}\, t\right), \sin\left(\sqrt{2}\, t\right)\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{2}, \cos\left(\sqrt{2}\, t\right)\right\}$$

Since $\cos\left(\sqrt{2}\, t\right)$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$\left[\left\{t\cos\left(\sqrt{2}\, t\right), t\sin\left(\sqrt{2}\, t\right)\right\}\right]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t\cos\left(\sqrt{2}\, t\right) + A_2 t\sin\left(\sqrt{2}\, t\right)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) + 2A_2\sqrt{2}\, \cos\left(\sqrt{2}\, t\right) = \cos\left(\sqrt{2}\, t\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{\sqrt{2}}{4}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{\sqrt{2}\, t\sin\left(\sqrt{2}\, t\right)}{4}$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(c_1 \cos\left(\sqrt{2}\, t\right) + \frac{c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{2}\right) + \left(\frac{\sqrt{2}\, t \sin\left(\sqrt{2}\, t\right)}{4}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 \cos\left(\sqrt{2}\, t\right) + \frac{c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{2} + \frac{\sqrt{2}\, t \sin\left(\sqrt{2}\, t\right)}{4} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = -c_1 \sqrt{2}\, \sin\left(\sqrt{2}\, t\right) + c_2 \cos\left(\sqrt{2}\, t\right) + \frac{\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{4} + \frac{t \cos\left(\sqrt{2}\, t\right)}{2}
$$

substituting $x' = 1$ and $t = 0$ in the above gives

$$
1 = c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 1
\end{aligned}
$$

Substituting these values back in above solution results in

$$
x = \frac{\sqrt{2}\, t \sin\left(\sqrt{2}\, t\right)}{4} + \frac{\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{2}
$$

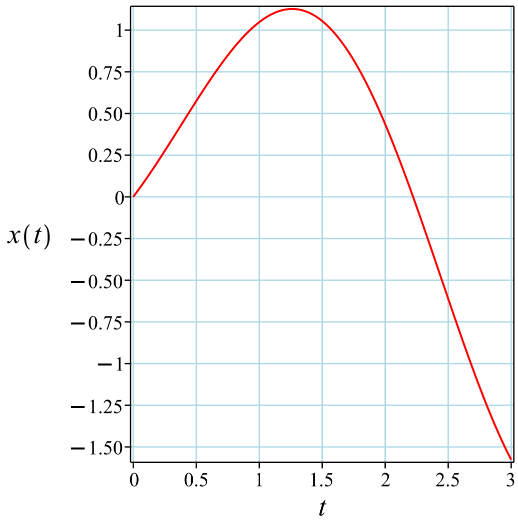
Which simplifies to

$$
x = \frac{(t+2)\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{4}
$$

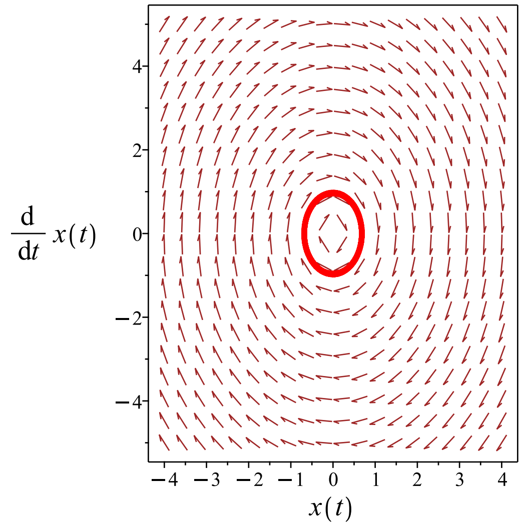
Summary

The solution(s) found are the following

$$
x = \frac{(t+2)\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{4} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{(t+2)\sqrt{2}\sin\left(\sqrt{2}\,t\right)}{4}$$

Verified OK.

9.23.4 Maple step by step solution

Let's solve

$$\left[x'' + 2x = \cos\left(\sqrt{2}\,t\right), x(0) = 0, x'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\mathrm{I}\sqrt{2}, \mathrm{I}\sqrt{2}\right)$

- 1st solution of the homogeneous ODE

$x_1(t) = \cos\left(\sqrt{2}\,t\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin\left(\sqrt{2}\,t\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos\left(\sqrt{2}\,t\right) + c_2 \sin\left(\sqrt{2}\,t\right) + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

- Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = \cos\left(\sqrt{2}\,t\right)\right]$

- Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos\left(\sqrt{2}\,t\right) & \sin\left(\sqrt{2}\,t\right) \\ -\sqrt{2}\sin\left(\sqrt{2}\,t\right) & \sqrt{2}\cos\left(\sqrt{2}\,t\right) \end{bmatrix}$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = \sqrt{2}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{\sqrt{2}\left(\cos\left(\sqrt{2}\,t\right)\left(\int \sin\left(2\sqrt{2}\,t\right) dt\right) - 2\sin\left(\sqrt{2}\,t\right)\left(\int \cos\left(\sqrt{2}\,t\right)^2 dt\right)\right)}{4}$

- Compute integrals

  $x_p(t) = \frac{\sqrt{2}\,t \sin\left(\sqrt{2}\,t\right)}{4} + \frac{\cos\left(\sqrt{2}\,t\right)}{8}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \cos\left(\sqrt{2}\,t\right) + c_2 \sin\left(\sqrt{2}\,t\right) + \frac{\sqrt{2}\,t \sin\left(\sqrt{2}\,t\right)}{4} + \frac{\cos\left(\sqrt{2}\,t\right)}{8}$

☐ Check validity of solution $x = c_1 \cos\left(\sqrt{2}\,t\right) + c_2 \sin\left(\sqrt{2}\,t\right) + \frac{\sqrt{2}\,t \sin\left(\sqrt{2}\,t\right)}{4} + \frac{\cos\left(\sqrt{2}\,t\right)}{8}$

- Use initial condition $x(0) = 0$

  $0 = c_1 + \frac{1}{8}$

- Compute derivative of the solution

  $x' = -c_1\sqrt{2}\sin\left(\sqrt{2}\,t\right) + c_2\sqrt{2}\cos\left(\sqrt{2}\,t\right) + \frac{\sqrt{2}\sin\left(\sqrt{2}\,t\right)}{8} + \frac{t\cos\left(\sqrt{2}\,t\right)}{2}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 1$

  $1 = \sqrt{2}\, c_2$
- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{8}, c_2 = \frac{\sqrt{2}}{2}\right\}$
- Substitute constant values into general solution and simplify

  $x = \frac{(t+2)\sqrt{2}\,\sin\left(\sqrt{2}\, t\right)}{4}$

- Solution to the IVP

  $x = \frac{(t+2)\sqrt{2}\,\sin\left(\sqrt{2}\, t\right)}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve([diff(x(t),t$2)+2*x(t)=cos(sqrt(2)*t),x(0) = 0, D(x)(0) = 1],x(t), singsol=all)
```

$$x(t) = \frac{\sqrt{2}\,\sin\left(\sqrt{2}\, t\right)(t+2)}{4}$$

✓ Solution by Mathematica

Time used: 0.264 (sec). Leaf size: 25

```
DSolve[{x''[t]+2*x[t]==Cos[Sqrt[2]*t],{x[0]==0,x'[0]==1}},x[t],t,IncludeSingularSolutions ->
```

$$x(t) \rightarrow \frac{(t+2)\sin\left(\sqrt{2}t\right)}{2\sqrt{2}}$$

10 Chapter 2, Second order linear equations. Section 2.3.2 Resonance Exercises page 114

10.1 problem 6 1425
10.2 problem 7(a) 1440
10.3 problem 7(c) 1453

10.1 problem 6

10.1.1 Existence and uniqueness analysis 1425
10.1.2 Solving as second order linear constant coeff ode 1426
10.1.3 Solving using Kovacic algorithm 1430
10.1.4 Maple step by step solution 1436

Internal problem ID [11475]
Internal file name [`OUTPUT/10457_Thursday_May_18_2023_04_20_03_AM_53632313/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.2 Resonance Exercises page 114
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + \frac{x'}{100} + 4x = \cos(2t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

10.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned}
p(t) &= \frac{1}{100} \\
q(t) &= 4 \\
F &= \cos(2t)
\end{aligned}$$

Hence the ode is

$$x'' + \frac{x'}{100} + 4x = \cos(2t)$$

The domain of $p(t) = \frac{1}{100}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \cos(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.1.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = \frac{1}{100}, C = 4, f(t) = \cos(2t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + \frac{x'}{100} + 4x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = \frac{1}{100}, C = 4$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \frac{\lambda e^{\lambda t}}{100} + 4 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \frac{1}{100}\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = \frac{1}{100}, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-\frac{1}{100}}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{\frac{1}{100}^2 - (4)\,(1)\,(4)} \\
&= -\frac{1}{200} \pm \frac{i\sqrt{159999}}{200}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{200} + \frac{i\sqrt{159999}}{200} \\
\lambda_2 &= -\frac{1}{200} - \frac{i\sqrt{159999}}{200}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -\frac{1}{200} + \frac{i\sqrt{159999}}{200} \\
\lambda_2 &= -\frac{1}{200} - \frac{i\sqrt{159999}}{200}
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{200}$ and $\beta = \frac{\sqrt{159999}}{200}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{t}{200}}\left(c_1 \cos\left(\frac{\sqrt{159999}\,t}{200}\right) + c_2 \sin\left(\frac{\sqrt{159999}\,t}{200}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = e^{-\frac{t}{200}} \left(c_1 \cos\left(\frac{\sqrt{159999}\, t}{200}\right) + c_2 \sin\left(\frac{\sqrt{159999}\, t}{200}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right), e^{-\frac{t}{200}} \sin\left(\frac{\sqrt{159999}\, t}{200}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(2t) + A_2 \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-\frac{A_1 \sin(2t)}{50} + \frac{A_2 \cos(2t)}{50} = \cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 50]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 50 \sin(2t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{-\frac{t}{200}} \left(c_1 \cos\left(\frac{\sqrt{159999}\, t}{200}\right) + c_2 \sin\left(\frac{\sqrt{159999}\, t}{200}\right)\right)\right) + (50 \sin(2t)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = e^{-\frac{t}{200}} \left(c_1 \cos\left(\frac{\sqrt{159999}\, t}{200}\right) + c_2 \sin\left(\frac{\sqrt{159999}\, t}{200}\right)\right) + 50 \sin\left(2t\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{e^{-\frac{t}{200}} \left(c_1 \cos\left(\frac{\sqrt{159999}\, t}{200}\right) + c_2 \sin\left(\frac{\sqrt{159999}\, t}{200}\right)\right)}{200} + e^{-\frac{t}{200}} \left(-\frac{c_1 \sqrt{159999}\, \sin\left(\frac{\sqrt{159999}\, t}{200}\right)}{200} + \frac{c_2 \sqrt{159999}\, c}{2}\right.$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{c_1}{200} + \frac{\sqrt{159999}\, c_2}{200} + 100 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

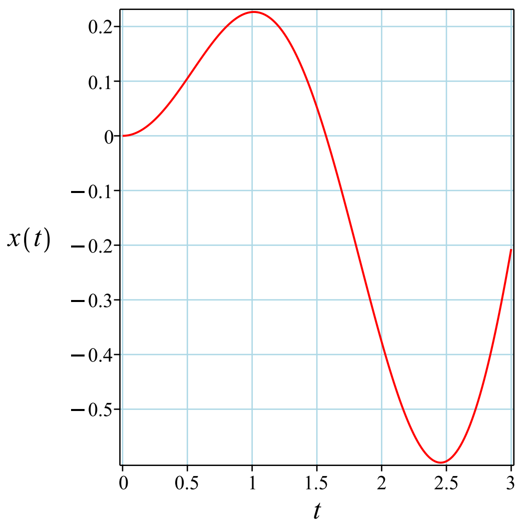
$$\begin{aligned} c_1 &= 0 \\ c_2 &= -\frac{20000\sqrt{159999}}{159999} \end{aligned}$$

Substituting these values back in above solution results in

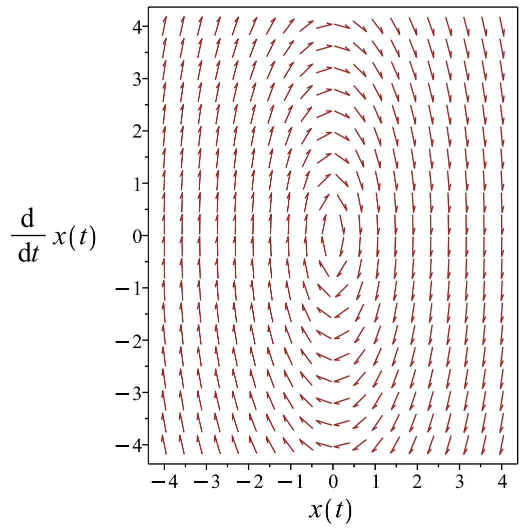
$$x = -\frac{20000 \sin\left(\frac{\sqrt{159999}\, t}{200}\right) e^{-\frac{t}{200}} \sqrt{159999}}{159999} + 50 \sin\left(2t\right)$$

<u>Summary</u>
The solution(s) found are the following

$$x = -\frac{20000 \sin\left(\frac{\sqrt{159999}\, t}{200}\right) e^{-\frac{t}{200}} \sqrt{159999}}{159999} + 50 \sin\left(2t\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -\frac{20000 \sin\left(\frac{\sqrt{159999}\, t}{200}\right) \mathrm{e}^{-\frac{t}{200}} \sqrt{159999}}{159999} + 50 \sin(2t)$$

Verified OK.

10.1.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + \frac{x'}{100} + 4x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{1}{100} \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-159999}{40000} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -159999 \\ t &= 40000 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{159999z(t)}{40000} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 262: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{159999}{40000}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\frac{\sqrt{159999}\, t}{200}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{1}{100}}{1} \, dt} \\
&= z_1 e^{-\frac{t}{200}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{200}}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{x_1^2} \, dt$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{\frac{1}{100}}{1} dt}}{(x_1)^2} dt \\
&= x_1 \int \frac{e^{-\frac{t}{100}}}{(x_1)^2} dt \\
&= x_1 \left(\frac{200\sqrt{159999}\, \tan\left(\frac{\sqrt{159999}\, t}{200}\right)}{159999} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right) \right) \\
&\quad + c_2 \left(\mathrm{e}^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right) \left(\frac{200\sqrt{159999}\, \tan\left(\frac{\sqrt{159999}\, t}{200}\right)}{159999} \right) \right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + \frac{x'}{100} + 4x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right) + \frac{200 c_2 \sin\left(\frac{\sqrt{159999}\, t}{200}\right) \mathrm{e}^{-\frac{t}{200}} \sqrt{159999}}{159999}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\cos(2t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right), \frac{200 \sin\left(\frac{\sqrt{159999}\, t}{200}\right) e^{-\frac{t}{200}} \sqrt{159999}}{159999} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(2t) + A_2 \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-\frac{A_1 \sin(2t)}{50} + \frac{A_2 \cos(2t)}{50} = \cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 50]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 50 \sin(2t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right) + \frac{200 c_2 \sin\left(\frac{\sqrt{159999}\, t}{200}\right) e^{-\frac{t}{200}} \sqrt{159999}}{159999}\right) + (50 \sin(2t)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right) + \frac{200 c_2 \sin\left(\frac{\sqrt{159999}\, t}{200}\right) e^{-\frac{t}{200}} \sqrt{159999}}{159999} + 50 \sin(2t) \quad (1)$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{c_1 e^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\,t}{200}\right)}{200} - \frac{c_1 e^{-\frac{t}{200}} \sqrt{159999}\, \sin\left(\frac{\sqrt{159999}\,t}{200}\right)}{200} + c_2 \cos\left(\frac{\sqrt{159999}\,t}{200}\right) e^{-\frac{t}{200}} - \frac{c_2 \sin\left(\sqrt{1\cdots\right.}{\cdots}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = 100 + c_2 - \frac{c_1}{200} \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 0 \\ c_2 &= -100 \end{aligned}$$

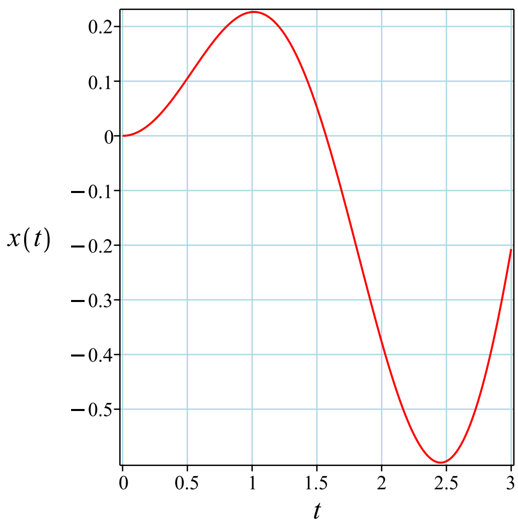
Substituting these values back in above solution results in

$$x = -\frac{20000 \sin\left(\frac{\sqrt{159999}\,t}{200}\right) e^{-\frac{t}{200}} \sqrt{159999}}{159999} + 50 \sin(2t)$$

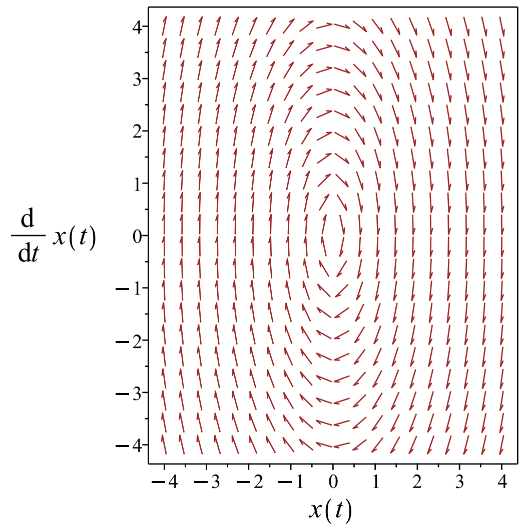
Summary

The solution(s) found are the following

$$x = -\frac{20000 \sin\left(\frac{\sqrt{159999}\,t}{200}\right) e^{-\frac{t}{200}} \sqrt{159999}}{159999} + 50 \sin(2t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$x = -\frac{20000 \sin\left(\frac{\sqrt{159999}\, t}{200}\right) e^{-\frac{t}{200}} \sqrt{159999}}{159999} + 50 \sin(2t)$$

Verified OK.

10.1.4 Maple step by step solution

Let's solve

$\left[x'' + \frac{x'}{100} + 4x = \cos(2t)\, , x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + \frac{1}{100} r + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{\left(-\frac{1}{100}\right) \pm \left(\sqrt{-\frac{159999}{10000}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{200} - \frac{\mathrm{I}\sqrt{159999}}{200}, -\frac{1}{200} + \frac{\mathrm{I}\sqrt{159999}}{200}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-\frac{t}{200}} \sin\left(\frac{\sqrt{159999}\, t}{200}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right) + \sin\left(\frac{\sqrt{159999}\, t}{200}\right) \mathrm{e}^{-\frac{t}{200}} c_2 + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t) \left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t) \left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = \cos(2t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \left[\begin{array}{cc} \mathrm{e}^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right) & \mathrm{e}^{-\frac{t}{200}} \sin\left(\frac{\sqrt{1599}}{200}\right. \\ -\frac{\mathrm{e}^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right)}{200} - \frac{\sin\left(\frac{\sqrt{159999}\, t}{200}\right) \mathrm{e}^{-\frac{t}{200}} \sqrt{159999}}{200} & -\frac{\mathrm{e}^{-\frac{t}{200}} \sin\left(\frac{\sqrt{159999}\, t}{200}\right)}{200} + \frac{\mathrm{e}^{-\frac{t}{200}} \sqrt{15}}{} \end{array}\right.$

  ◦ Compute Wronskian

  $W(x_1(t), x_2(t)) = \frac{\sqrt{159999}\, \mathrm{e}^{-\frac{t}{100}}}{200}$

  ◦ Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{200\, \mathrm{e}^{-\frac{t}{200}} \sqrt{159999} \left(\cos\left(\frac{\sqrt{159999}\, t}{200}\right) \left(\int \cos(2t) \mathrm{e}^{\frac{t}{200}} \sin\left(\frac{\sqrt{159999}\, t}{200}\right) dt\right) - \sin\left(\frac{\sqrt{159999}\, t}{200}\right) \left(\int \cos(2t) \mathrm{e}^{\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right)\right.\right.}{159999}$

  ◦ Compute integrals

  $x_p(t) = 50 \sin(2t)$

- Substitute particular solution into general solution to ODE

  $x = c_1 \mathrm{e}^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right) + \sin\left(\frac{\sqrt{159999}\, t}{200}\right) \mathrm{e}^{-\frac{t}{200}} c_2 + 50 \sin(2t)$

☐ Check validity of solution $x = c_1 \mathrm{e}^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right) + \sin\left(\frac{\sqrt{159999}\, t}{200}\right) \mathrm{e}^{-\frac{t}{200}} c_2 + 50 \sin(2t)$

  ◦ Use initial condition $x(0) = 0$

  $0 = c_1$

  ◦ Compute derivative of the solution

$$x' = -\frac{c_1 \mathrm{e}^{-\frac{t}{200}} \cos\left(\frac{\sqrt{159999}\, t}{200}\right)}{200} - \frac{c_1 \mathrm{e}^{-\frac{t}{200}} \sqrt{159999}\, \sin\left(\frac{\sqrt{159999}\, t}{200}\right)}{200} + \frac{\sqrt{159999}\, \cos\left(\frac{\sqrt{159999}\, t}{200}\right) \mathrm{e}^{-\frac{t}{200}} c_2}{200} - \frac{\sin\left(\frac{\sqrt{159999}\, t}{200}\right) \mathrm{e}^{-}}{200}$$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $$0 = -\frac{c_1}{200} + \frac{\sqrt{159999}\, c_2}{200} + 100$$

- Solve for c_1 and c_2

  $$\left\{c_1 = 0, c_2 = -\frac{20000\sqrt{159999}}{159999}\right\}$$

- Substitute constant values into general solution and simplify

  $$x = -\frac{20000 \sin\left(\frac{\sqrt{159999}\, t}{200}\right) \mathrm{e}^{-\frac{t}{200}} \sqrt{159999}}{159999} + 50 \sin(2t)$$

- Solution to the IVP

  $$x = -\frac{20000 \sin\left(\frac{\sqrt{159999}\, t}{200}\right) \mathrm{e}^{-\frac{t}{200}} \sqrt{159999}}{159999} + 50 \sin(2t)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 27

```
dsolve([diff(x(t),t$2)+1/100*diff(x(t),t)+4*x(t)=cos(2*t),x(0) = 0, D(x)(0) = 0],x(t), sings
```

$$x(t) = -\frac{20000 \,\mathrm{e}^{-\frac{t}{200}} \sqrt{159999}\, \sin\left(\frac{\sqrt{159999}\, t}{200}\right)}{159999} + 50 \sin(2t)$$

✓ Solution by Mathematica

Time used: 1.827 (sec). Leaf size: 37

```
DSolve[{x''[t]+1/100*x'[t]+4*x[t]==Cos[2*t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSoluti
```

$$x(t) \to 50\sin(2t) - \frac{20000e^{-t/200}\sin\left(\frac{\sqrt{159999}t}{200}\right)}{\sqrt{159999}}$$

10.2 problem 7(a)

10.2.1 Existence and uniqueness analysis 1440
10.2.2 Solving as second order linear constant coeff ode 1441
10.2.3 Solving using Kovacic algorithm 1444
10.2.4 Maple step by step solution 1450

Internal problem ID [11476]
Internal file name [`OUTPUT/10458_Thursday_May_18_2023_04_20_04_AM_33949814/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.2 Resonance Exercises page 114
Problem number: 7(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + w^2 x = \cos(\beta t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

10.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= w^2 \\ F &= \cos(\beta t) \end{aligned}$$

Hence the ode is

$$x'' + w^2x = \cos(\beta t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = w^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \cos(\beta t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.2.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = w^2, f(t) = \cos(\beta t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + w^2x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = w^2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + w^2 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + w^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = w^2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(w^2)} \\
&= \pm\sqrt{-w^2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +\sqrt{-w^2} \\
\lambda_2 &= -\sqrt{-w^2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= \sqrt{-w^2} \\
\lambda_2 &= -\sqrt{-w^2}
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
x &= c_1 e^{\left(\sqrt{-w^2}\right)t} + c_2 e^{\left(-\sqrt{-w^2}\right)t}
\end{aligned}$$

Or

$$x = c_1 \mathrm{e}^{\sqrt{-w^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-w^2}\, t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^{\sqrt{-w^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-w^2}\, t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(\beta t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(\beta t), \sin(\beta t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{\sqrt{-w^2}\,t}, e^{-\sqrt{-w^2}\,t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(\beta t) + A_2 \sin(\beta t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\beta^2 \cos(\beta t) - A_2\beta^2 \sin(\beta t) + w^2(A_1 \cos(\beta t) + A_2 \sin(\beta t)) = \cos(\beta t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{-\beta^2 + w^2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{\cos(\beta t)}{-\beta^2 + w^2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{\sqrt{-w^2}\,t} + c_2 e^{-\sqrt{-w^2}\,t}\right) + \left(\frac{\cos(\beta t)}{-\beta^2 + w^2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{\sqrt{-w^2}\,t} + c_2 e^{-\sqrt{-w^2}\,t} + \frac{\cos(\beta t)}{-\beta^2 + w^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = \frac{(-c_1 - c_2)\beta^2 + 1 + (c_1 + c_2) w^2}{-\beta^2 + w^2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1\sqrt{-w^2}\,\mathrm{e}^{\sqrt{-w^2}\,t} - c_2\sqrt{-w^2}\,\mathrm{e}^{-\sqrt{-w^2}\,t} - \frac{\beta\sin(\beta t)}{-\beta^2+w^2}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = (c_1 - c_2)\sqrt{-w^2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{1}{2\beta^2 - 2w^2} \\ c_2 &= \frac{1}{2\beta^2 - 2w^2} \end{aligned}$$

Substituting these values back in above solution results in

$$x = \frac{-2\cos(\beta t) + \mathrm{e}^{\sqrt{-w^2}\,t} + \mathrm{e}^{-\sqrt{-w^2}\,t}}{2\beta^2 - 2w^2}$$

Summary

The solution(s) found are the following

$$x = \frac{-2\cos(\beta t) + \mathrm{e}^{\sqrt{-w^2}\,t} + \mathrm{e}^{-\sqrt{-w^2}\,t}}{2\beta^2 - 2w^2} \tag{1}$$

Verification of solutions

$$x = \frac{-2\cos(\beta t) + \mathrm{e}^{\sqrt{-w^2}\,t} + \mathrm{e}^{-\sqrt{-w^2}\,t}}{2\beta^2 - 2w^2}$$

Verified OK.

10.2.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + w^2 x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= w^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-w^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -w^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-w^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 264: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -w^2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{\sqrt{-w^2}\, t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
x_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
x_1 &= z_1 \\
&= \mathrm{e}^{\sqrt{-w^2}\, t}
\end{aligned}
$$

Which simplifies to

$$x_1 = \mathrm{e}^{\sqrt{-w^2}\, t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\
&= \mathrm{e}^{\sqrt{-w^2}\, t} \int \frac{1}{\mathrm{e}^{2\sqrt{-w^2}\, t}}\, dt \\
&= \mathrm{e}^{\sqrt{-w^2}\, t} \left(\frac{\sqrt{-w^2}\, \mathrm{e}^{-2\sqrt{-w^2}\, t}}{2w^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{\sqrt{-w^2}\, t}\right) + c_2 \left(\mathrm{e}^{\sqrt{-w^2}\, t} \left(\frac{\sqrt{-w^2}\, \mathrm{e}^{-2\sqrt{-w^2}\, t}}{2w^2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + w^2 x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{\sqrt{-w^2}\, t} + \frac{c_2 \sqrt{-w^2}\, \mathrm{e}^{-\sqrt{-w^2}\, t}}{2w^2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(\beta t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(\beta t), \sin(\beta t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{-w^2}\, e^{-\sqrt{-w^2}\, t}}{2w^2}, e^{\sqrt{-w^2}\, t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(\beta t) + A_2 \sin(\beta t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\beta^2 \cos(\beta t) - A_2\beta^2 \sin(\beta t) + w^2(A_1 \cos(\beta t) + A_2 \sin(\beta t)) = \cos(\beta t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{-\beta^2 + w^2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{\cos(\beta t)}{-\beta^2 + w^2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{\sqrt{-w^2}\, t} + \frac{c_2\sqrt{-w^2}\, e^{-\sqrt{-w^2}\, t}}{2w^2}\right) + \left(\frac{\cos(\beta t)}{-\beta^2 + w^2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{\sqrt{-w^2}\, t} + \frac{c_2 \sqrt{-w^2}\, e^{-\sqrt{-w^2}\, t}}{2w^2} + \frac{\cos(\beta t)}{-\beta^2 + w^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = \frac{(-\beta^2 c_2 + c_2 w^2)\sqrt{-w^2} + 2w^2(-\beta^2 c_1 + c_1 w^2 + 1)}{-2\beta^2 w^2 + 2w^4} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1 \sqrt{-w^2}\, e^{\sqrt{-w^2}\, t} + \frac{c_2 e^{-\sqrt{-w^2}\, t}}{2} - \frac{\beta \sin(\beta t)}{-\beta^2 + w^2}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = c_1 \sqrt{-w^2} + \frac{c_2}{2} \tag{2A}$$

Equations $\{1\text{A}, 2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{2\beta^2 - 2w^2}$$
$$c_2 = -\frac{\sqrt{-w^2}}{\beta^2 - w^2}$$

Substituting these values back in above solution results in

$$x = \frac{-2\cos(\beta t) + e^{\sqrt{-w^2}\, t} + e^{-\sqrt{-w^2}\, t}}{2\beta^2 - 2w^2}$$

Summary

The solution(s) found are the following

$$x = \frac{-2\cos(\beta t) + e^{\sqrt{-w^2}\, t} + e^{-\sqrt{-w^2}\, t}}{2\beta^2 - 2w^2} \tag{1}$$

Verification of solutions

$$x = \frac{-2\cos(\beta t) + e^{\sqrt{-w^2}\, t} + e^{-\sqrt{-w^2}\, t}}{2\beta^2 - 2w^2}$$

Verified OK.

10.2.4 Maple step by step solution

Let's solve

$$\left[x''+w^2x=\cos(\beta t),x(0)=0,x'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2+w^2=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm\left(\sqrt{-4w^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(\sqrt{-w^2},-\sqrt{-w^2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t)=\mathrm{e}^{\sqrt{-w^2}\,t}$

- 2nd solution of the homogeneous ODE

  $x_2(t)=\mathrm{e}^{-\sqrt{-w^2}\,t}$

- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)+x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x=c_1\mathrm{e}^{\sqrt{-w^2}\,t}+c_2\mathrm{e}^{-\sqrt{-w^2}\,t}+x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t)=-x_1(t)\left(\int\frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right)+x_2(t)\left(\int\frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right),f(t)=\cos(\beta t)\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(x_1(t),x_2(t))=\left[\begin{array}{cc}\mathrm{e}^{\sqrt{-w^2}\,t} & \mathrm{e}^{-\sqrt{-w^2}\,t}\\ \sqrt{-w^2}\,\mathrm{e}^{\sqrt{-w^2}\,t} & -\sqrt{-w^2}\,\mathrm{e}^{-\sqrt{-w^2}\,t}\end{array}\right]$

- ◦ Compute Wronskian

  $W(x_1(t),x_2(t))=-2\sqrt{-w^2}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = \frac{e^{\sqrt{-w^2}\, t}\left(\int e^{-\sqrt{-w^2}\, t} \cos(\beta t) dt\right) - e^{-\sqrt{-w^2}\, t}\left(\int \cos(\beta t) e^{\sqrt{-w^2}\, t} dt\right)}{2\sqrt{-w^2}}$

- Compute integrals

  $x_p(t) = \frac{\cos(\beta t)}{-\beta^2 + w^2}$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{\sqrt{-w^2}\, t} + c_2 e^{-\sqrt{-w^2}\, t} + \frac{\cos(\beta t)}{-\beta^2 + w^2}$

☐ Check validity of solution $x = c_1 e^{\sqrt{-w^2}\, t} + c_2 e^{-\sqrt{-w^2}\, t} + \frac{\cos(\beta t)}{-\beta^2 + w^2}$

- Use initial condition $x(0) = 0$

  $0 = c_1 + c_2 + \frac{1}{-\beta^2 + w^2}$

- Compute derivative of the solution

  $x' = c_1 \sqrt{-w^2}\, e^{\sqrt{-w^2}\, t} - c_2 \sqrt{-w^2}\, e^{-\sqrt{-w^2}\, t} - \frac{\beta \sin(\beta t)}{-\beta^2 + w^2}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = c_1 \sqrt{-w^2} - c_2 \sqrt{-w^2}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{2(\beta^2 - w^2)}, c_2 = \frac{1}{2(\beta^2 - w^2)}\right\}$

- Substitute constant values into general solution and simplify

  $x = \frac{-2\cos(\beta t) + e^{\sqrt{-w^2}\, t} + e^{-\sqrt{-w^2}\, t}}{2\beta^2 - 2w^2}$

- Solution to the IVP

  $x = \frac{-2\cos(\beta t) + e^{\sqrt{-w^2}\, t} + e^{-\sqrt{-w^2}\, t}}{2\beta^2 - 2w^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve([diff(x(t),t$2)+w^2*x(t)=cos(beta*t),x(0) = 0, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \frac{\cos(tw) - \cos(\beta t)}{\beta^2 - w^2}$$

✓ Solution by Mathematica

Time used: 0.367 (sec). Leaf size: 28

```
DSolve[{x''[t]+w^2*x[t]==Cos[\[Beta]*t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions
```

$$x(t) \to \frac{\cos(\beta t) - \cos(tw)}{w^2 - \beta^2}$$

10.3 problem 7(c)

10.3.1 Existence and uniqueness analysis 1453
10.3.2 Solving as second order linear constant coeff ode 1454
10.3.3 Solving using Kovacic algorithm 1458
10.3.4 Maple step by step solution 1463

Internal problem ID [11477]
Internal file name [`OUTPUT/10459_Thursday_May_18_2023_04_20_06_AM_92603263/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.3.2 Resonance Exercises page 114
Problem number: 7(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + 3025x = \cos(45t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

10.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 3025 \\ F &= \cos(45t) \end{aligned}$$

Hence the ode is

$$x'' + 3025x = \cos(45t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3025$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \cos(45t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.3.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = 3025, f(t) = \cos(45t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 3025x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 3025$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 3025\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 3025 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 3025$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(3025)} \\
&= \pm 55i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +55i \\
\lambda_2 &= -55i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 55i \\
\lambda_2 &= -55i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 55$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^0(c_1 \cos(55t) + c_2 \sin(55t))$$

Or

$$x = c_1 \cos(55t) + c_2 \sin(55t)$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \cos(55t) + c_2 \sin(55t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(45t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(45t), \sin(45t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(55t), \sin(55t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(45t) + A_2 \sin(45t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$1000A_1 \cos(45t) + 1000A_2 \sin(45t) = \cos(45t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{1000}, A_2 = 0\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{\cos(45t)}{1000}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1 \cos(55t) + c_2 \sin(55t)) + \left(\frac{\cos(45t)}{1000}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \cos(55t) + c_2 \sin(55t) + \frac{\cos(45t)}{1000} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{1}{1000} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -55c_1 \sin(55t) + 55c_2 \cos(55t) - \frac{9\sin(45t)}{200}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = 55c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

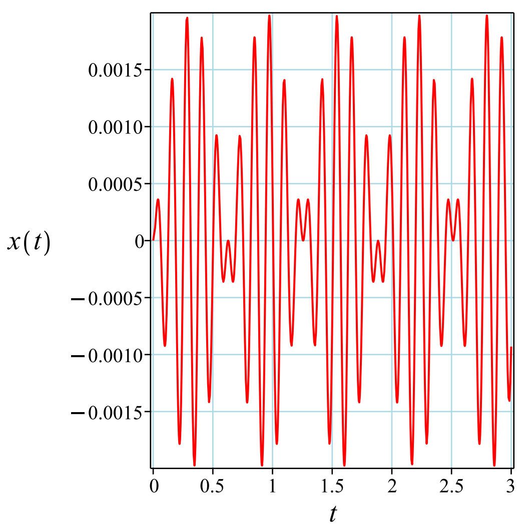
$$c_1 = -\frac{1}{1000}$$
$$c_2 = 0$$

Substituting these values back in above solution results in

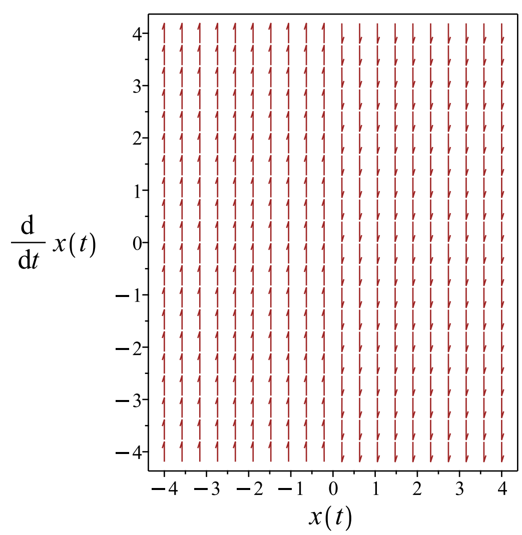
$$x = -\frac{\cos(55t)}{1000} + \frac{\cos(45t)}{1000}$$

Summary
The solution(s) found are the following

$$x = -\frac{\cos(55t)}{1000} + \frac{\cos(45t)}{1000} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -\frac{\cos(55t)}{1000} + \frac{\cos(45t)}{1000}$$

Verified OK.

10.3.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 3025x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 3025 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3025}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3025 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -3025z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 266: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -3025$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(55t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \cos(55t) \end{aligned}$$

Which simplifies to

$$x_1 = \cos(55t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\ &= \cos(55t) \int \frac{1}{\cos(55t)^2}\, dt \\ &= \cos(55t) \left(\frac{\tan(55t)}{55}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1(\cos(55t)) + c_2\left(\cos(55t)\left(\frac{\tan(55t)}{55}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x'' + 3025x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \cos(55t) + \frac{c_2 \sin(55t)}{55}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(45t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(45t), \sin(45t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(55t)}{55}, \cos(55t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(45t) + A_2 \sin(45t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$1000A_1 \cos(45t) + 1000A_2 \sin(45t) = \cos(45t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{1000}, A_2 = 0\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{\cos(45t)}{1000}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \cos(55t) + \frac{c_2 \sin(55t)}{55}\right) + \left(\frac{\cos(45t)}{1000}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \cos(55t) + \frac{c_2 \sin(55t)}{55} + \frac{\cos(45t)}{1000} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{1}{1000} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -55c_1 \sin(55t) + c_2 \cos(55t) - \frac{9 \sin(45t)}{200}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{1}{1000} \\ c_2 &= 0 \end{aligned}$$

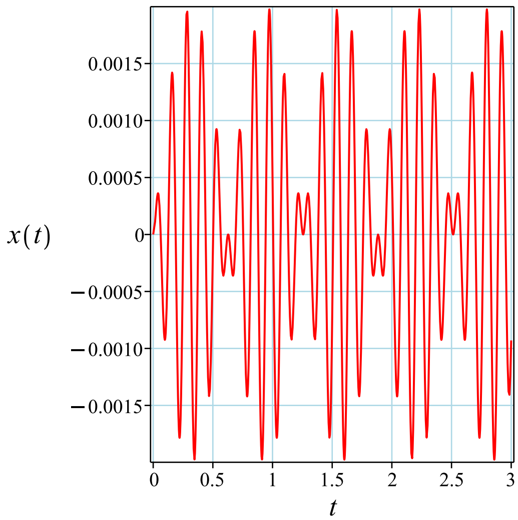
Substituting these values back in above solution results in

$$x = -\frac{\cos(55t)}{1000} + \frac{\cos(45t)}{1000}$$

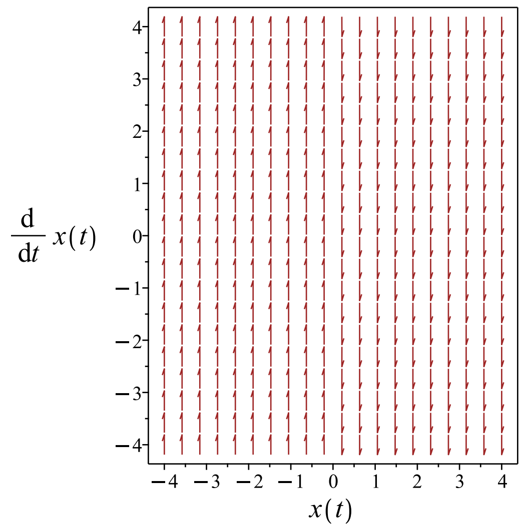
Summary

The solution(s) found are the following

$$x = -\frac{\cos(55t)}{1000} + \frac{\cos(45t)}{1000} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -\frac{\cos(55t)}{1000} + \frac{\cos(45t)}{1000}$$

Verified OK.

10.3.4 Maple step by step solution

Let's solve

$$\left[x'' + 3025x = \cos(45t), x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3025 = 0$

- Use quadratic formula to solve for r

$r = \frac{0 \pm (\sqrt{-12100})}{2}$

- Roots of the characteristic polynomial

  $r = (-55\,\mathrm{I}, 55\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \cos(55t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(55t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(55t) + c_2 \sin(55t) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t) \left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t) \left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = \cos(45t)\right]$

- Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(55t) & \sin(55t) \\ -55\sin(55t) & 55\cos(55t) \end{bmatrix}$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = 55$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{\cos(55t)\left(\int \sin(55t)\cos(45t) dt\right)}{55} + \frac{\sin(55t)\left(\int \cos(55t)\cos(45t) dt\right)}{55}$

- Compute integrals

  $x_p(t) = \frac{\cos(45t)}{1000}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \cos(55t) + c_2 \sin(55t) + \frac{\cos(45t)}{1000}$

□ Check validity of solution $x = c_1 \cos(55t) + c_2 \sin(55t) + \frac{\cos(45t)}{1000}$

- Use initial condition $x(0) = 0$

  $0 = c_1 + \frac{1}{1000}$

- Compute derivative of the solution

$$x' = -55c_1 \sin(55t) + 55c_2 \cos(55t) - \frac{9\sin(45t)}{200}$$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $$0 = 55c_2$$

- Solve for c_1 and c_2

  $$\left\{c_1 = -\frac{1}{1000}, c_2 = 0\right\}$$

- Substitute constant values into general solution and simplify

  $$x = -\frac{\cos(55t)}{1000} + \frac{\cos(45t)}{1000}$$

- Solution to the IVP

  $$x = -\frac{\cos(55t)}{1000} + \frac{\cos(45t)}{1000}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 17

```
dsolve([diff(x(t),t$2)+(55)^2*x(t)=cos(45*t),x(0) = 0, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = -\frac{\cos(55t)}{1000} + \frac{\cos(45t)}{1000}$$

✓ Solution by Mathematica

Time used: 0.338 (sec). Leaf size: 36

```
DSolve[{x''[t]+55^2*x[t]==Cos[45*t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions -> T
```

$$x(t) \to \frac{1}{250}\sin^2(5t)(\cos(5t)+\cos(15t)+\cos(25t)+\cos(35t)+\cos(45t))$$

11 Chapter 2, Second order linear equations. Section 2.4.1 Cauchy-Euler equations. Exercises page 120

11.1 problem 1(a) 1468
11.2 problem 1(b) 1478
11.3 problem 1(c) 1488
11.4 problem 1(d) 1510
11.5 problem 1(e) 1532
11.6 problem 1(f) 1548
11.7 problem 1(g) 1568
11.8 problem 1(h) 1583
11.9 problem 2 1607

11.1 problem 1(a)

11.1.1 Solving as second order euler ode ode 1468
11.1.2 Solving using Kovacic algorithm 1470
11.1.3 Maple step by step solution 1475

Internal problem ID [11478]
Internal file name [`OUTPUT/10460_Thursday_May_18_2023_04_20_08_AM_35420635/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.1 Cauchy-Euler equations. Exercises page 120
Problem number: 1(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x'' + \frac{x}{t^2} = 0}$$

The ode can be written as

$$t^2x'' + x = 0$$

Which shows it is a Euler ODE.

11.1.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 0rt^{r-1} + t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 0\,t^r + t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1)+0+1=0$$

Or

$$r^2-r+1=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= \frac{1}{2}-\frac{i\sqrt{3}}{2} \\ r_2 &= \frac{1}{2}+\frac{i\sqrt{3}}{2} \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{\sqrt{3}}{2}$. Hence the solution becomes

$$\begin{aligned} x &= c_1 t^{r_1} + c_2 t^{r_2} \\ &= c_1 t^{\alpha+i\beta} + c_2 t^{\alpha-i\beta} \\ &= t^{\alpha}\left(c_1 t^{i\beta} + c_2 t^{-i\beta}\right) \\ &= t^{\alpha}\left(c_1 e^{\ln\left(t^{i\beta}\right)} + c_2 e^{\ln\left(t^{-i\beta}\right)}\right) \\ &= t^{\alpha}\left(c_1 e^{i(\beta \ln t)} + c_2 e^{-i(\beta \ln t)}\right) \end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{\sqrt{3}}{2}$, the above becomes

$$x = t^{\frac{1}{2}}\left(c_1 e^{-\frac{i\sqrt{3}\,\ln(t)}{2}} + c_2 e^{\frac{i\sqrt{3}\,\ln(t)}{2}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$x = \sqrt{t}\left(c_1 \cos\left(\frac{\sqrt{3}\,\ln(t)}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,\ln(t)}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$x = \sqrt{t}\left(c_1 \cos\left(\frac{\sqrt{3}\,\ln(t)}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,\ln(t)}{2}\right)\right) \tag{1}$$

Verification of solutions

$$x = \sqrt{t}\left(c_1 \cos\left(\frac{\sqrt{3}\,\ln(t)}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,\ln(t)}{2}\right)\right)$$

Verified OK.

11.1.2 Solving using Kovacic algorithm

Writing the ode as

$$t^2x'' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{1}{t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 268: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -1$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{t^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -1$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \frac{i\sqrt{3}}{2} - \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{t} + (-)\,(0) \\ &= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{t} \\ &= \frac{1 - i\sqrt{3}}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$
(0)+2\left(\frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{t}\right)(0)+\left(\left(-\frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{t^2}\right)+\left(\frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{t}\right)^2-\left(-\frac{1}{t^2}\right)\right)=0
$$
$$
0=0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{t} dt} \\
&= t^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}
\end{aligned}
$$

The first solution to the original ode in x is found from

$$
x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt}
$$

Since $B=0$ then the above reduces to

$$
\begin{aligned}
x_1 &= z_1 \\
&= t^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}
\end{aligned}
$$

Which simplifies to

$$
x_1 = t^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{x_1^2} \, dt
$$

Since $B=0$ then the above becomes

$$
\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2} \, dt \\
&= t^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \int \frac{1}{t^{1-i\sqrt{3}}} \, dt \\
&= t^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{i t^{i\sqrt{3}} \sqrt{3}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1x_1 + c_2x_2 \\
&= c_1\left(t^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right) + c_2\left(t^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\left(-\frac{it^{i\sqrt{3}}\sqrt{3}}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1t^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} - \frac{ic_2\sqrt{3}\,t^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{3} \tag{1}
$$

Verification of solutions

$$
x = c_1t^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} - \frac{ic_2\sqrt{3}\,t^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{3}
$$

Verified OK.

11.1.3 Maple step by step solution

Let's solve

$t^2x'' + x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = -\frac{x}{t^2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + \frac{x}{t^2} = 0$

- Multiply by denominators of the ODE

  $t^2x'' + x = 0$

- Make a change of variables

  $s = \ln(t)$

- ☐ Substitute the change of variables back into the ODE

  - ◦ Calculate the 1st derivative of x with respect to t , using the chain rule

    $x' = \left(\frac{d}{ds}x(s)\right) s'(t)$

- Compute derivative

  $x' = \frac{\frac{d}{ds}x(s)}{t}$

- Calculate the 2nd derivative of x with respect to t , using the chain rule

  $x'' = \left(\frac{d^2}{ds^2}x(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}x(s)\right)$

- Compute derivative

  $x'' = \frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $t^2\left(\frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}\right) + x(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}x(s) - \frac{d}{ds}x(s) + x(s) = 0$

- Characteristic polynomial of ODE

  $r^2 - r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{1 \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - \frac{I\sqrt{3}}{2}, \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the ODE

  $x_1(s) = e^{\frac{s}{2}} \cos\left(\frac{\sqrt{3}\, s}{2}\right)$

- 2nd solution of the ODE

  $x_2(s) = e^{\frac{s}{2}} \sin\left(\frac{\sqrt{3}\, s}{2}\right)$

- General solution of the ODE

  $x(s) = c_1 x_1(s) + c_2 x_2(s)$

- Substitute in solutions

  $x(s) = c_1 e^{\frac{s}{2}} \cos\left(\frac{\sqrt{3}\, s}{2}\right) + c_2 e^{\frac{s}{2}} \sin\left(\frac{\sqrt{3}\, s}{2}\right)$

- Change variables back using $s = \ln(t)$

  $x = c_1\sqrt{t}\, \cos\left(\frac{\sqrt{3}\, \ln(t)}{2}\right) + c_2\sqrt{t}\, \sin\left(\frac{\sqrt{3}\, \ln(t)}{2}\right)$

- Simplify

  $$x = \sqrt{t}\left(c_1 \cos\left(\frac{\sqrt{3}\,\ln(t)}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,\ln(t)}{2}\right)\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(x(t),t$2)=-1/t^2*x(t),x(t), singsol=all)
```

$$x(t) = \sqrt{t}\left(c_1 \sin\left(\frac{\ln(t)\sqrt{3}}{2}\right) + c_2 \cos\left(\frac{\ln(t)\sqrt{3}}{2}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 42

```
DSolve[x''[t]==-1/t^2*x[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \sqrt{t}\left(c_1 \cos\left(\frac{1}{2}\sqrt{3}\log(t)\right) + c_2 \sin\left(\frac{1}{2}\sqrt{3}\log(t)\right)\right)$$

11.2 problem 1(b)

11.2.1 Solving as second order euler ode ode 1478
11.2.2 Solving using Kovacic algorithm 1479
11.2.3 Maple step by step solution 1485

Internal problem ID [11479]
Internal file name [`OUTPUT/10461_Thursday_May_18_2023_04_20_10_AM_17741855/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.1 Cauchy-Euler equations. Exercises page 120
Problem number: 1(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{x'' - \frac{4x}{t^2} = 0}$$

The ode can be written as

$$t^2x'' - 4x = 0$$

Which shows it is a Euler ODE.

11.2.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 0rt^{r-1} - 4t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 0\,t^r - 4t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1)+0-4=0$$

Or

$$r^2 - r - 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= \frac{1}{2} - \frac{\sqrt{17}}{2} \\ r_2 &= \frac{1}{2} + \frac{\sqrt{17}}{2} \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$x = c_1 x_1 + c_2 x_2$$

Where $x_1 = t^{r_1}$ and $x_2 = t^{r_2}$. Hence

$$x = c_1 t^{\frac{1}{2}-\frac{\sqrt{17}}{2}} + c_2 t^{\frac{1}{2}+\frac{\sqrt{17}}{2}}$$

Summary
The solution(s) found are the following

$$x = c_1 t^{\frac{1}{2}-\frac{\sqrt{17}}{2}} + c_2 t^{\frac{1}{2}+\frac{\sqrt{17}}{2}} \tag{1}$$

Verification of solutions

$$x = c_1 t^{\frac{1}{2}-\frac{\sqrt{17}}{2}} + c_2 t^{\frac{1}{2}+\frac{\sqrt{17}}{2}}$$

Verified OK.

11.2.2 Solving using Kovacic algorithm

Writing the ode as

$$t^2 x'' - 4x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{4}{t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t) \, e^{-\int \frac{B}{2A} \, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 270: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{4}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition

of r given above. Therefore $b = 4$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{17}}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{17}}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{4}{t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 4$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{17}}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{17}}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4}{t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + \frac{\sqrt{17}}{2}$	$\frac{1}{2} - \frac{\sqrt{17}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + \frac{\sqrt{17}}{2}$	$\frac{1}{2} - \frac{\sqrt{17}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2} - \frac{\sqrt{17}}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \frac{\sqrt{17}}{2} - \left(\frac{1}{2} - \frac{\sqrt{17}}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{1}{2} - \frac{\sqrt{17}}{2}}{t} + (-)\,(0) \\
&= \frac{\frac{1}{2} - \frac{\sqrt{17}}{2}}{t} \\
&= -\frac{-1 + \sqrt{17}}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{17}}{2}}{t}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{17}}{2}}{t^2} \right) + \left(\frac{\frac{1}{2} - \frac{\sqrt{17}}{2}}{t} \right)^2 - \left(\frac{4}{t^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2} - \frac{\sqrt{17}}{2}}{t} dt} \\
&= t^{\frac{1}{2} - \frac{\sqrt{17}}{2}}
\end{aligned}
$$

The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}x_1 &= z_1\\ &= t^{\frac{1}{2}-\frac{\sqrt{17}}{2}}\end{aligned}$$

Which simplifies to

$$x_1 = t^{\frac{1}{2}-\frac{\sqrt{17}}{2}}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}x_2 &= x_1 \int \frac{1}{x_1^2}\,dt\\ &= t^{\frac{1}{2}-\frac{\sqrt{17}}{2}} \int \frac{1}{t^{1-\sqrt{17}}}\,dt\\ &= t^{\frac{1}{2}-\frac{\sqrt{17}}{2}} \left(\frac{t^{\sqrt{17}}\sqrt{17}}{17}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}x &= c_1 x_1 + c_2 x_2\\ &= c_1\left(t^{\frac{1}{2}-\frac{\sqrt{17}}{2}}\right) + c_2\left(t^{\frac{1}{2}-\frac{\sqrt{17}}{2}}\left(\frac{t^{\sqrt{17}}\sqrt{17}}{17}\right)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 t^{\frac{1}{2}-\frac{\sqrt{17}}{2}} + \frac{c_2\sqrt{17}\,t^{\frac{1}{2}+\frac{\sqrt{17}}{2}}}{17} \tag{1}$$

Verification of solutions

$$x = c_1 t^{\frac{1}{2}-\frac{\sqrt{17}}{2}} + \frac{c_2\sqrt{17}\, t^{\frac{1}{2}+\frac{\sqrt{17}}{2}}}{17}$$

Verified OK.

11.2.3 Maple step by step solution

Let's solve

$t^2x'' - 4x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = \frac{4x}{t^2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' - \frac{4x}{t^2} = 0$

- Multiply by denominators of the ODE

  $t^2x'' - 4x = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of x with respect to t , using the chain rule

    $x' = \left(\frac{d}{ds}x(s)\right) s'(t)$

  - Compute derivative

    $x' = \frac{\frac{d}{ds}x(s)}{t}$

  - Calculate the 2nd derivative of x with respect to t , using the chain rule

    $x'' = \left(\frac{d^2}{ds^2}x(s)\right) s'(t)^2 + s''(t)\left(\frac{d}{ds}x(s)\right)$

  - Compute derivative

    $x'' = \frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}$

  Substitute the change of variables back into the ODE

$$t^2\left(\frac{\frac{d^2}{ds^2}x(s)}{t^2}-\frac{\frac{d}{ds}x(s)}{t^2}\right)-4x(s)=0$$

- Simplify

$$\frac{d^2}{ds^2}x(s)-\frac{d}{ds}x(s)-4x(s)=0$$

- Characteristic polynomial of ODE

$$r^2-r-4=0$$

- Use quadratic formula to solve for r

$$r=\frac{1\pm\left(\sqrt{17}\right)}{2}$$

- Roots of the characteristic polynomial

$$r=\left(\frac{1}{2}-\frac{\sqrt{17}}{2},\frac{1}{2}+\frac{\sqrt{17}}{2}\right)$$

- 1st solution of the ODE

$$x_1(s)=\mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{17}}{2}\right)s}$$

- 2nd solution of the ODE

$$x_2(s)=\mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{17}}{2}\right)s}$$

- General solution of the ODE

$$x(s)=c_1x_1(s)+c_2x_2(s)$$

- Substitute in solutions

$$x(s)=c_1\mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{17}}{2}\right)s}+c_2\mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{17}}{2}\right)s}$$

- Change variables back using $s=\ln(t)$

$$x=c_1\mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{17}}{2}\right)\ln(t)}+c_2\mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{17}}{2}\right)\ln(t)}$$

- Simplify

$$x=\sqrt{t}\left(t^{-\frac{\sqrt{17}}{2}}c_1+t^{\frac{\sqrt{17}}{2}}c_2\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(x(t),t$2)=4/t^2*x(t),x(t), singsol=all)
```

$$x(t) = \sqrt{t}\left(t^{\frac{\sqrt{17}}{2}} c_1 + t^{-\frac{\sqrt{17}}{2}} c_2\right)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 34

```
DSolve[x''[t]==4/t^2*x[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to t^{\frac{1}{2}-\frac{\sqrt{17}}{2}}\left(c_2 t^{\sqrt{17}} + c_1\right)$$

11.3 problem 1(c)

11.3.1 Solving as second order euler ode ode 1489
11.3.2 Solving as second order change of variable on x method 2 ode . 1490
11.3.3 Solving as second order change of variable on x method 1 ode . 1492
11.3.4 Solving as second order change of variable on y method 2 ode . 1494
11.3.5 Solving as second order integrable as is ode 1497
11.3.6 Solving as type second_order_integrable_as_is (not using ABC version) 1498
11.3.7 Solving using Kovacic algorithm 1500
11.3.8 Solving as exact linear second order ode ode 1505
11.3.9 Maple step by step solution 1507

Internal problem ID [11480]
Internal file name [`OUTPUT/10462_Thursday_May_18_2023_04_20_11_AM_85414870/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.1 Cauchy-Euler equations. Exercises page 120
Problem number: 1(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{t^2x'' + 3tx' + x = 0}$$

11.3.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 3trt^{r-1} + t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 3r\,t^r + t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 3r + 1 = 0$$

Or

$$r^2 + 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = -1$$

Since the roots are equal, then the general solution is

$$x = c_1 x_1 + c_2 x_2$$

Where $x_1 = t^r$ and $x_2 = t^r \ln(t)$. Hence

$$x = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t}$$

Summary

The solution(s) found are the following

$$x = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t} \tag{1}$$

Verification of solutions

$$x = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t}$$

Verified OK.

11.3.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2x'' + 3tx' + x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{3}{t} \\ q(t) &= \frac{1}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1 x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)}dt \\ &= \int \mathrm{e}^{-\left(\int \frac{3}{t}dt\right)}dt \\ &= \int e^{-3\ln(t)}\, dt \\ &= \int \frac{1}{t^3}dt \\ &= -\frac{1}{2t^2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{\frac{1}{t^2}}{\frac{1}{t^6}} \\
&= t^4
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}x(\tau) + q_1 x(\tau) = 0 \\
\frac{d^2}{d\tau^2}x(\tau) + t^4 x(\tau) = 0
\end{aligned}
$$

But in terms of τ

$$
t^4 = \frac{1}{4\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}x(\tau) + \frac{x(\tau)}{4\tau^2} = 0
$$

The above ode is now solved for $x(\tau)$. The ode can be written as

$$
4\left(\frac{d^2}{d\tau^2}x(\tau)\right)\tau^2 + x(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $x(\tau) = \tau^r$, then $x' = r\tau^{r-1}$ and $x'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0
$$

Simplifying gives

$$
4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
4r(r-1) + 0 + 1 = 0
$$

Or

$$
4r^2 - 4r + 1 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{2} \\
r_2 &= \frac{1}{2}
\end{aligned}
$$

Since the roots are equal, then the general solution is

$$
x(\tau) = c_1 x_1 + c_2 x_2
$$

Where $x_1 = \tau^r$ and $x_2 = \tau^r \ln(\tau)$. Hence

$$
x(\tau) = c_1 \sqrt{\tau} + c_2 \sqrt{\tau}\, \ln(\tau)
$$

The above solution is now transformed back to x using (6) which results in

$$
x = \frac{\sqrt{2}\,\sqrt{-\frac{1}{t^2}}\,\left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{t^2}\right)\right)}{2}
$$

Summary

The solution(s) found are the following

$$
x = \frac{\sqrt{2}\,\sqrt{-\frac{1}{t^2}}\,\left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{t^2}\right)\right)}{2} \tag{1}
$$

Verification of solutions

$$
x = \frac{\sqrt{2}\,\sqrt{-\frac{1}{t^2}}\,\left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{t^2}\right)\right)}{2}
$$

Verified OK.

11.3.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$
t^2 x'' + 3 t x' + x = 0 \tag{1}
$$

Becomes

$$
x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(t) &= \frac{3}{t} \\
q(t) &= \frac{1}{t^2}
\end{aligned}
$$

Applying change of variables $\tau = g(t)$ to (2) results

$$
\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1 x(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{\frac{1}{t^2}}}{c} \\
\tau'' &= -\frac{1}{c\sqrt{\frac{1}{t^2}}\,t^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\
&= \frac{-\frac{1}{c\sqrt{\frac{1}{t^2}}\,t^3} + \frac{3}{t}\frac{\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\
&= 2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
x(\tau)'' + p_1 x(\tau)' + q_1 x(\tau) &= 0 \\
\frac{d^2}{d\tau^2}x(\tau) + 2c\left(\frac{d}{d\tau}x(\tau)\right) + c^2 x(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $x(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$x(\tau) = \mathrm{e}^{-c\tau} c_1$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dt \\
&= \frac{\int \sqrt{\frac{1}{t^2}}dt}{c} \\
&= \frac{\sqrt{\frac{1}{t^2}}\, t \ln(t)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$x = \frac{c_1}{t}$$

Summary

The solution(s) found are the following

$$x = \frac{c_1}{t} \tag{1}$$

Verification of solutions

$$x = \frac{c_1}{t}$$

Verified OK.

11.3.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2 x'' + 3t x' + x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}$$

Where

$$\begin{aligned}
p(t) &= \frac{3}{t} \\
q(t) &= \frac{1}{t^2}
\end{aligned}$$

Applying change of variables on the depndent variable $x = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not x.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{3n}{t^2} + \frac{1}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \frac{v'(t)}{t} &= 0 \\ v''(t) + \frac{v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{u}{t} \end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{t}\,dt \\
\int \frac{1}{u}\,du &= \int -\frac{1}{t}\,dt \\
\ln(u) &= -\ln(t) + c_1 \\
u &= \mathrm{e}^{-\ln(t)+c_1} \\
&= \frac{c_1}{t}
\end{aligned}
$$

Now that $u(t)$ is known, then

$$
\begin{aligned}
v'(t) &= u(t) \\
v(t) &= \int u(t)\,dt + c_2 \\
&= c_1 \ln(t) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
x &= v(t)\,t^n \\
&= \frac{c_1 \ln(t) + c_2}{t} \\
&= \frac{c_1 \ln(t) + c_2}{t}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = \frac{c_1 \ln(t) + c_2}{t} \tag{1}
$$

Verification of solutions

$$
x = \frac{c_1 \ln(t) + c_2}{t}
$$

Verified OK.

11.3.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int \left(t^2x'' + 3tx' + x\right) dt = 0$$
$$t^2x' + xt = c_1$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = \frac{1}{t}$$
$$q(t) = \frac{c_1}{t^2}$$

Hence the ode is

$$x' + \frac{x}{t} = \frac{c_1}{t^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{t}dt}$$
$$= t$$

The ode becomes

$$\frac{d}{dt}(\mu x) = (\mu)\left(\frac{c_1}{t^2}\right)$$
$$\frac{d}{dt}(tx) = (t)\left(\frac{c_1}{t^2}\right)$$
$$d(tx) = \left(\frac{c_1}{t}\right) dt$$

Integrating gives

$$tx = \int \frac{c_1}{t}\, dt$$
$$tx = c_1 \ln(t) + c_2$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$x = \frac{c_1 \ln(t)}{t} + \frac{c_2}{t}$$

which simplifies to

$$x = \frac{c_1 \ln(t) + c_2}{t}$$

Summary
The solution(s) found are the following

$$x = \frac{c_1 \ln(t) + c_2}{t} \tag{1}$$

Verification of solutions

$$x = \frac{c_1 \ln(t) + c_2}{t}$$

Verified OK.

11.3.6 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$t^2 x'' + 3tx' + x = 0$$

Integrating both sides of the ODE w.r.t t gives

$$\int (t^2 x'' + 3tx' + x)\, dt = 0$$
$$t^2 x' + xt = c_1$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = \frac{1}{t}$$
$$q(t) = \frac{c_1}{t^2}$$

Hence the ode is

$$x' + \frac{x}{t} = \frac{c_1}{t^2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{t}dt} \\
&= t
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(\frac{c_1}{t^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}(tx) &= (t)\left(\frac{c_1}{t^2}\right) \\
\mathrm{d}(tx) &= \left(\frac{c_1}{t}\right)\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
tx &= \int \frac{c_1}{t}\,\mathrm{d}t \\
tx &= c_1 \ln(t) + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$
x = \frac{c_1 \ln(t)}{t} + \frac{c_2}{t}
$$

which simplifies to

$$
x = \frac{c_1 \ln(t) + c_2}{t}
$$

Summary

The solution(s) found are the following

$$
x = \frac{c_1 \ln(t) + c_2}{t} \tag{1}
$$

Verification of solutions

$$
x = \frac{c_1 \ln(t) + c_2}{t}
$$

Verified OK.

11.3.7 Solving using Kovacic algorithm

Writing the ode as

$$t^2x'' + 3tx' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= 3t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 272: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4t^2}
$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2t} + (-)\,(0) \\
&= \frac{1}{2t} \\
&= \frac{1}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2t}\right)^2 - \left(-\frac{1}{4t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \frac{1}{2t} dt} \\
&= \sqrt{t}
\end{aligned}
$$

The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3t}{t^2}\,dt} \\
&= z_1 e^{-\frac{3\ln(t)}{2}} \\
&= z_1\left(\frac{1}{t^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \frac{1}{t}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{3t}{t^2}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-3\ln(t)}}{(x_1)^2}\,dt \\
&= x_1(\ln(t))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\frac{1}{t}\right) + c_2\left(\frac{1}{t}(\ln(t))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = \frac{c_1}{t} + \frac{c_2\ln(t)}{t} \tag{1}
$$

Verification of solutions

$$
x = \frac{c_1}{t} + \frac{c_2\ln(t)}{t}
$$

Verified OK.

11.3.8 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\,x''+q(t)\,x'+r(t)\,x=s(t)$$

is exact if

$$p''(t)-q'(t)+r(t)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x)&=t^2\\
q(x)&=3t\\
r(x)&=1\\
s(x)&=0
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x)&=2\\
q'(x)&=3
\end{aligned}$$

Therefore (1) becomes

$$2-(3)+(1)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,x'+(q(t)-p'(t))\,x)'=s(x)$$

Integrating gives

$$p(t)\,x'+(q(t)-p'(t))\,x=\int s(t)\,dt$$

Substituting the above values for p,q,r,s gives

$$t^2x'+xt=c_1$$

We now have a first order ode to solve which is

$$t^2x'+xt=c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x'+p(t)x=q(t)$$

Where here

$$
\begin{aligned}
p(t) &= \frac{1}{t} \\
q(t) &= \frac{c_1}{t^2}
\end{aligned}
$$

Hence the ode is

$$
x' + \frac{x}{t} = \frac{c_1}{t^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{t} dt} \\
&= t
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu x) &= (\mu)\left(\frac{c_1}{t^2}\right) \\
\frac{d}{dt}(tx) &= (t)\left(\frac{c_1}{t^2}\right) \\
d(tx) &= \left(\frac{c_1}{t}\right) dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
tx &= \int \frac{c_1}{t}\, dt \\
tx &= c_1 \ln(t) + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$
x = \frac{c_1 \ln(t)}{t} + \frac{c_2}{t}
$$

which simplifies to

$$
x = \frac{c_1 \ln(t) + c_2}{t}
$$

Summary

The solution(s) found are the following

$$
x = \frac{c_1 \ln(t) + c_2}{t} \tag{1}
$$

Verification of solutions

$$
x = \frac{c_1 \ln(t) + c_2}{t}
$$

Verified OK.

11.3.9 Maple step by step solution

Let's solve

$t^2x'' + 3tx' + x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = -\frac{3x'}{t} - \frac{x}{t^2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + \frac{3x'}{t} + \frac{x}{t^2} = 0$

- Multiply by denominators of the ODE

  $t^2x'' + 3tx' + x = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of x with respect to t , using the chain rule

    $x' = \left(\frac{d}{ds}x(s)\right) s'(t)$

  - Compute derivative

    $x' = \frac{\frac{d}{ds}x(s)}{t}$

  - Calculate the 2nd derivative of x with respect to t , using the chain rule

    $x'' = \left(\frac{d^2}{ds^2}x(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}x(s)\right)$

  - Compute derivative

    $x'' = \frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $t^2\left(\frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}\right) + 3\frac{d}{ds}x(s) + x(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}x(s) + 2\frac{d}{ds}x(s) + x(s) = 0$

- Characteristic polynomial of ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$
- Root of the characteristic polynomial

  $r = -1$
- 1st solution of the ODE

  $x_1(s) = e^{-s}$
- Repeated root, multiply $x_1(s)$ by s to ensure linear independence

  $x_2(s) = s\,e^{-s}$
- General solution of the ODE

  $x(s) = c_1 x_1(s) + c_2 x_2(s)$
- Substitute in solutions

  $x(s) = c_1 e^{-s} + c_2 s\,e^{-s}$
- Change variables back using $s = \ln(t)$

  $x = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t}$
- Simplify

  $x = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(t^2*diff(x(t),t$2)+3*t*diff(x(t),t)+x(t)=0,x(t), singsol=all)
```

$$x(t) = \frac{c_2 \ln(t) + c_1}{t}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 17

```
DSolve[t^2*x''[t]+3*t*x'[t]+x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{c_2 \log(t) + c_1}{t}$$

11.4 problem 1(d)

11.4.1 Solving as second order euler ode ode 1511
11.4.2 Solving as linear second order ode solved by an integrating factor ode 1512
11.4.3 Solving as second order change of variable on x method 2 ode . 1513
11.4.4 Solving as second order change of variable on x method 1 ode . 1515
11.4.5 Solving as second order change of variable on y method 1 ode . 1517
11.4.6 Solving as second order change of variable on y method 2 ode . 1519
11.4.7 Solving as second order integrable as is ode 1521
11.4.8 Solving as type second_order_integrable_as_is (not using ABC version) 1523
11.4.9 Solving using Kovacic algorithm 1524
11.4.10 Solving as exact linear second order ode ode 1527
11.4.11 Maple step by step solution 1529

Internal problem ID [11481]
Internal file name [`OUTPUT/10463_Thursday_May_18_2023_04_20_13_AM_79251219/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.1 Cauchy-Euler equations. Exercises page 120
Problem number: 1(d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{tx'' + 4x' + \frac{2x}{t} = 0}$$

The ode can be written as

$$t^2x'' + 4tx' + 2x = 0$$

Which shows it is a Euler ODE.

11.4.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 4trt^{r-1} + 2t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 4r\,t^r + 2t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 4r + 2 = 0$$

Or

$$r^2 + 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -2$$
$$r_2 = -1$$

Since the roots are real and distinct, then the general solution is

$$x = c_1x_1 + c_2x_2$$

Where $x_1 = t^{r_1}$ and $x_2 = t^{r_2}$. Hence

$$x = \frac{c_1}{t^2} + \frac{c_2}{t}$$

Summary

The solution(s) found are the following

$$x = \frac{c_1}{t^2} + \frac{c_2}{t} \tag{1}$$

Verification of solutions

$$x = \frac{c_1}{t^2} + \frac{c_2}{t}$$

Verified OK.

11.4.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$x'' + p(t)\,x' + \frac{\left(p(t)^2 + p'(t)\right)x}{2} = f(t)$$

Where $p(t) = \frac{4}{t}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int \frac{4}{t}\,dx} \\ &= t^2 \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)x)'' &= 0 \\ \left(t^2x\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(t^2x\right)' = c_1$$

Integrating again gives

$$\left(t^2x\right) = c_1t + c_2$$

Hence the solution is

$$x = \frac{c_1t + c_2}{t^2}$$

Or

$$x = \frac{c_1}{t} + \frac{c_2}{t^2}$$

Summary

The solution(s) found are the following

$$x = \frac{c_1}{t} + \frac{c_2}{t^2} \tag{1}$$

Verification of solutions

$$x = \frac{c_1}{t} + \frac{c_2}{t^2}$$

Verified OK.

11.4.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2x'' + 4tx' + 2x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{4}{t} \\ q(t) &= \frac{2}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)}\,dt \\ &= \int \mathrm{e}^{-\left(\int \frac{4}{t}dt\right)}\,dt \\ &= \int e^{-4\ln(t)}\,dt \\ &= \int \frac{1}{t^4}dt \\ &= -\frac{1}{3t^3} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{\frac{2}{t^2}}{\frac{1}{t^8}} \\
&= 2t^6
\end{aligned}
\tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}x(\tau) + q_1 x(\tau) = 0 \\
\frac{d^2}{d\tau^2}x(\tau) + 2t^6 x(\tau) = 0
\end{aligned}
$$

But in terms of τ

$$
2t^6 = \frac{2}{9\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}x(\tau) + \frac{2x(\tau)}{9\tau^2} = 0
$$

The above ode is now solved for $x(\tau)$. The ode can be written as

$$
9\left(\frac{d^2}{d\tau^2}x(\tau)\right)\tau^2 + 2x(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $x(\tau) = \tau^r$, then $x' = r\tau^{r-1}$ and $x'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0
$$

Simplifying gives

$$
9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
9r(r-1) + 0 + 2 = 0
$$

Or

$$
9r^2 - 9r + 2 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{3} \\
r_2 &= \frac{2}{3}
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$
x(\tau) = c_1 x_1 + c_2 x_2
$$

Where $x_1 = \tau^{r_1}$ and $x_2 = \tau^{r_2}$. Hence

$$
x(\tau) = c_1 \tau^{\frac{1}{3}} + c_2 \tau^{\frac{2}{3}}
$$

The above solution is now transformed back to x using (6) which results in

$$
x = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{t^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{t^3}\right)^{\frac{2}{3}}}{3}
$$

<u>Summary</u>
The solution(s) found are the following

$$
x = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{t^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{t^3}\right)^{\frac{2}{3}}}{3} \tag{1}
$$

<u>Verification of solutions</u>

$$
x = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{t^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{t^3}\right)^{\frac{2}{3}}}{3}
$$

Verified OK.

11.4.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$
t^2 x'' + 4 t x' + 2 x = 0 \tag{1}
$$

Becomes

$$
x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(t) &= \frac{4}{t} \\
q(t) &= \frac{2}{t^2}
\end{aligned}
$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1 x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{2}\,\sqrt{\frac{1}{t^2}}}{c} \\ \tau'' &= -\frac{\sqrt{2}}{c\sqrt{\frac{1}{t^2}}\,t^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\ &= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{t^2}}\,t^3} + \frac{4}{t}\frac{\sqrt{2}\,\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\ &= \frac{3c\sqrt{2}}{2}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}x(\tau)'' + p_1 x(\tau)' + q_1 x(\tau) &= 0 \\ \frac{d^2}{d\tau^2}x(\tau) + \frac{3c\sqrt{2}\left(\frac{d}{d\tau}x(\tau)\right)}{2} + c^2 x(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $x(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$x(\tau) = \mathrm{e}^{-\frac{3\sqrt{2}\,c\tau}{4}}\left(c_1\cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2\sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\
&= \frac{\int \sqrt{2}\,\sqrt{\frac{1}{t^2}}dt}{c} \\
&= \frac{\sqrt{2}\,\sqrt{\frac{1}{t^2}}\,t\ln\left(t\right)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
x = \frac{c_1\cosh\left(\frac{\ln(t)}{2}\right) + ic_2\sinh\left(\frac{\ln(t)}{2}\right)}{t^{\frac{3}{2}}}
$$

Summary

The solution(s) found are the following

$$
x = \frac{c_1\cosh\left(\frac{\ln(t)}{2}\right) + ic_2\sinh\left(\frac{\ln(t)}{2}\right)}{t^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$
x = \frac{c_1\cosh\left(\frac{\ln(t)}{2}\right) + ic_2\sinh\left(\frac{\ln(t)}{2}\right)}{t^{\frac{3}{2}}}
$$

Verified OK.

11.4.5 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$
x'' + p(t)\,x' + q(t)\,x = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(t) &= \frac{4}{t} \\
q(t) &= \frac{2}{t^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{2}{t^2} - \frac{\left(\frac{4}{t}\right)'}{2} - \frac{\left(\frac{4}{t}\right)^2}{4} \\
&= \frac{2}{t^2} - \frac{\left(-\frac{4}{t^2}\right)}{2} - \frac{\left(\frac{16}{t^2}\right)}{4} \\
&= \frac{2}{t^2} - \left(-\frac{2}{t^2}\right) - \frac{4}{t^2} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable t then the transformation

$$
x = v(t)\, z(t) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(t)$ is given by

$$
\begin{aligned}
z(t) &= \mathrm{e}^{-\left(\int \frac{p(t)}{2} dt\right)} \\
&= e^{-\int \frac{\frac{4}{t}}{2}} \\
&= \frac{1}{t^2}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
x = \frac{v(t)}{t^2} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(t) = 0
$$

Which is now solved for $v(t)$ Integrating twice gives the solution

$$
v(t) = c_1 t + c_2
$$

Now that $v(t)$ is known, then

$$
\begin{aligned}
x &= v(t)\, z(t) \\
&= (c_1 t + c_2)\, (z(t))
\end{aligned} \tag{7}
$$

But from (5)

$$z(t) = \frac{1}{t^2}$$

Hence (7) becomes

$$x = \frac{c_1 t + c_2}{t^2}$$

Summary

The solution(s) found are the following

$$x = \frac{c_1 t + c_2}{t^2} \tag{1}$$

Verification of solutions

$$x = \frac{c_1 t + c_2}{t^2}$$

Verified OK.

11.4.6 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2 x'' + 4tx' + 2x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{4}{t} \\ q(t) &= \frac{2}{t^2} \end{aligned}$$

Applying change of variables on the depndent variable $x = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not x.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{4n}{t^2} + \frac{2}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \frac{2v'(t)}{t} &= 0 \\ v''(t) + \frac{2v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{2u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{2u}{t} \end{aligned}$$

Where $f(t) = -\frac{2}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{2}{t}\, dt \\ \int \frac{1}{u}\, du &= \int -\frac{2}{t}\, dt \\ \ln(u) &= -2\ln(t) + c_1 \\ u &= e^{-2\ln(t) + c_1} \\ &= \frac{c_1}{t^2} \end{aligned}$$

Now that $u(t)$ is known, then

$$\begin{aligned}
v'(t) &= u(t) \\
v(t) &= \int u(t)\, dt + c_2 \\
&= -\frac{c_1}{t} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
x &= v(t)\, t^n \\
&= \frac{-\frac{c_1}{t} + c_2}{t} \\
&= \frac{c_2 t - c_1}{t^2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$x = \frac{-\frac{c_1}{t} + c_2}{t} \tag{1}$$

Verification of solutions

$$x = \frac{-\frac{c_1}{t} + c_2}{t}$$

Verified OK.

11.4.7 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\begin{aligned}
\int \left(t^2 x'' + 4tx' + 2x\right) dt &= 0 \\
t^2 x' + 2xt &= c_1
\end{aligned}$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned}
p(t) &= \frac{2}{t} \\
q(t) &= \frac{c_1}{t^2}
\end{aligned}$$

Hence the ode is

$$x' + \frac{2x}{t} = \frac{c_1}{t^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{t}dt} \\ &= t^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(\frac{c_1}{t^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}(t^2 x) &= (t^2)\left(\frac{c_1}{t^2}\right) \\ \mathrm{d}(t^2 x) &= c_1\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}t^2 x &= \int c_1\,\mathrm{d}t \\ t^2 x &= c_1 t + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$x = \frac{c_1}{t} + \frac{c_2}{t^2}$$

which simplifies to

$$x = \frac{c_1 t + c_2}{t^2}$$

Summary
The solution(s) found are the following

$$x = \frac{c_1 t + c_2}{t^2} \tag{1}$$

Verification of solutions

$$x = \frac{c_1 t + c_2}{t^2}$$

Verified OK.

11.4.8 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
t^2x'' + 4tx' + 2x = 0
$$

Integrating both sides of the ODE w.r.t t gives

$$
\begin{aligned}
\int (t^2x'' + 4tx' + 2x)\, dt &= 0 \\
t^2x' + 2xt &= c_1
\end{aligned}
$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
x' + p(t)x = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= \frac{2}{t} \\
q(t) &= \frac{c_1}{t^2}
\end{aligned}
$$

Hence the ode is

$$
x' + \frac{2x}{t} = \frac{c_1}{t^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{2}{t}dt} \\
&= t^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu x) &= (\mu)\left(\frac{c_1}{t^2}\right) \\
\frac{d}{dt}(t^2x) &= (t^2)\left(\frac{c_1}{t^2}\right) \\
d(t^2x) &= c_1\, dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
t^2x &= \int c_1\, dt \\
t^2x &= c_1t + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$x = \frac{c_1}{t} + \frac{c_2}{t^2}$$

which simplifies to

$$x = \frac{c_1 t + c_2}{t^2}$$

Summary

The solution(s) found are the following

$$x = \frac{c_1 t + c_2}{t^2} \tag{1}$$

Verification of solutions

$$x = \frac{c_1 t + c_2}{t^2}$$

Verified OK.

11.4.9 Solving using Kovacic algorithm

Writing the ode as

$$t^2 x'' + 4tx' + 2x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= 4t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 274: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4t}{t^2}\,dt} \\
&= z_1 e^{-2\ln(t)} \\
&= z_1 \left(\frac{1}{t^2}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = \frac{1}{t^2}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{4t}{t^2}\,dt}}{\left(x_1\right)^2}\,dt \\
&= x_1 \int \frac{e^{-4\ln(t)}}{\left(x_1\right)^2}\,dt \\
&= x_1(t)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1x_1 + c_2x_2 \\
&= c_1\left(\frac{1}{t^2}\right) + c_2\left(\frac{1}{t^2}(t)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = \frac{c_1}{t^2} + \frac{c_2}{t} \tag{1}
$$

Verification of solutions

$$
x = \frac{c_1}{t^2} + \frac{c_2}{t}
$$

Verified OK.

11.4.10 Solving as exact linear second order ode ode

An ode of the form

$$
p(t)\,x'' + q(t)\,x' + r(t)\,x = s(t)
$$

is exact if

$$
p''(t) - q'(t) + r(t) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= t^2 \\
q(x) &= 4t \\
r(x) &= 2 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 4
\end{aligned}
$$

Therefore (1) becomes

$$
2 - (4) + (2) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(t)\,x' + (q(t) - p'(t))\,x)' = s(x)
$$

Integrating gives

$$p(t)\,x' + (q(t) - p'(t))\,x = \int s(t)\,dt$$

Substituting the above values for p, q, r, s gives

$$t^2x' + 2xt = c_1$$

We now have a first order ode to solve which is

$$t^2x' + 2xt = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= \frac{2}{t} \\ q(t) &= \frac{c_1}{t^2} \end{aligned}$$

Hence the ode is

$$x' + \frac{2x}{t} = \frac{c_1}{t^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{2}{t}dt} \\ &= t^2 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(\frac{c_1}{t^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(t^2x\right) &= \left(t^2\right)\left(\frac{c_1}{t^2}\right) \\ \mathrm{d}\left(t^2x\right) &= c_1\,\mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} t^2x &= \int c_1\,\mathrm{d}t \\ t^2x &= c_1t + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$x = \frac{c_1}{t} + \frac{c_2}{t^2}$$

which simplifies to

$$x = \frac{c_1 t + c_2}{t^2}$$

<u>Summary</u>
The solution(s) found are the following

$$x = \frac{c_1 t + c_2}{t^2} \tag{1}$$

<u>Verification of solutions</u>

$$x = \frac{c_1 t + c_2}{t^2}$$

Verified OK.

11.4.11 Maple step by step solution

Let's solve

$t^2 x'' + 4tx' + 2x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = -\frac{4x'}{t} - \frac{2x}{t^2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + \frac{4x'}{t} + \frac{2x}{t^2} = 0$

- Multiply by denominators of the ODE

  $t^2 x'' + 4tx' + 2x = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of x with respect to t , using the chain rule

  $x' = \left(\frac{d}{ds}x(s)\right) s'(t)$

- Compute derivative

  $x' = \frac{\frac{d}{ds}x(s)}{t}$

- Calculate the 2nd derivative of x with respect to t , using the chain rule

  $x'' = \left(\frac{d^2}{ds^2}x(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}x(s)\right)$

- Compute derivative

  $x'' = \frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $t^2\left(\frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}\right) + 4\frac{d}{ds}x(s) + 2x(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}x(s) + 3\frac{d}{ds}x(s) + 2x(s) = 0$
- Characteristic polynomial of ODE

  $r^2 + 3r + 2 = 0$
- Factor the characteristic polynomial

  $(r + 2)(r + 1) = 0$
- Roots of the characteristic polynomial

  $r = (-2, -1)$
- 1st solution of the ODE

  $x_1(s) = \mathrm{e}^{-2s}$
- 2nd solution of the ODE

  $x_2(s) = \mathrm{e}^{-s}$
- General solution of the ODE

  $x(s) = c_1x_1(s) + c_2x_2(s)$
- Substitute in solutions

  $x(s) = c_1\mathrm{e}^{-2s} + c_2\mathrm{e}^{-s}$
- Change variables back using $s = \ln(t)$

  $x = \frac{c_1}{t^2} + \frac{c_2}{t}$
- Simplify

  $x = \frac{c_1}{t^2} + \frac{c_2}{t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 13

```
dsolve(t*diff(x(t),t$2)+4*diff(x(t),t)+2/t*x(t)=0,x(t), singsol=all)
```

$$x(t) = \frac{c_1 t + c_2}{t^2}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 16

```
DSolve[t*x''[t]+4*x'[t]+2/t*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{c_2 t + c_1}{t^2}$$

11.5 problem 1(e)

11.5.1 Solving as second order euler ode ode 1532
11.5.2 Solving as second order change of variable on x method 2 ode . 1533
11.5.3 Solving as second order change of variable on x method 1 ode . 1536
11.5.4 Solving as second order change of variable on y method 2 ode . 1538
11.5.5 Solving using Kovacic algorithm 1540
11.5.6 Maple step by step solution 1545

Internal problem ID [11482]
Internal file name [`OUTPUT/10464_Thursday_May_18_2023_04_20_14_AM_52449874/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.1 Cauchy-Euler equations. Exercises page 120
Problem number: 1(e).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{t^2x'' - 7tx' + 16x = 0}$$

11.5.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} - 7trt^{r-1} + 16t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r - 7r\,t^r + 16t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) - 7r + 16 = 0$$

Or

$$r^2 - 8r + 16 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 4 \\ r_2 &= 4 \end{aligned}$$

Since the roots are equal, then the general solution is

$$x = c_1 x_1 + c_2 x_2$$

Where $x_1 = t^r$ and $x_2 = t^r \ln(t)$. Hence

$$x = t^4 c_1 + c_2 t^4 \ln(t)$$

Summary

The solution(s) found are the following

$$x = t^4 c_1 + c_2 t^4 \ln(t) \tag{1}$$

Verification of solutions

$$x = t^4 c_1 + c_2 t^4 \ln(t)$$

Verified OK.

11.5.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2 x'' - 7t x' + 16x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{7}{t} \\ q(t) &= \frac{16}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1 x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)} dt \\
&= \int \mathrm{e}^{-\left(\int -\frac{7}{t}dt\right)} dt \\
&= \int e^{7\ln(t)}\, dt \\
&= \int t^7 dt \\
&= \frac{t^8}{8}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{\frac{16}{t^2}}{t^{14}} \\
&= \frac{16}{t^{16}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}x(\tau) + q_1 x(\tau) &= 0 \\
\frac{d^2}{d\tau^2}x(\tau) + \frac{16x(\tau)}{t^{16}} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{16}{t^{16}} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}x(\tau) + \frac{x(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $x(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}x(\tau)\right)\tau^2 + x(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $x(\tau) = \tau^r$, then $x' = r\tau^{r-1}$ and $x'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$x(\tau) = c_1x_1 + c_2x_2$$

Where $x_1 = \tau^r$ and $x_2 = \tau^r \ln(\tau)$. Hence

$$x(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to x using (6) which results in

$$x = \frac{\sqrt{2}\,\sqrt{t^8}\,(c_1 + c_2\ln(t^8) - 3c_2\ln(2))}{4}$$

Summary
The solution(s) found are the following

$$x = \frac{\sqrt{2}\,\sqrt{t^8}\,(c_1 + c_2 \ln(t^8) - 3c_2 \ln(2))}{4} \tag{1}$$

Verification of solutions

$$x = \frac{\sqrt{2}\,\sqrt{t^8}\,(c_1 + c_2 \ln(t^8) - 3c_2 \ln(2))}{4}$$

Verified OK.

11.5.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$t^2x'' - 7tx' + 16x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{7}{t} \\ q(t) &= \frac{16}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{4\sqrt{\frac{1}{t^2}}}{c} \qquad (6) \\ \tau'' &= -\frac{4}{c\sqrt{\frac{1}{t^2}}\,t^3} \end{aligned}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\
&= \frac{-\frac{4}{c\sqrt{\frac{1}{t^2}}\,t^3} - \frac{7}{t}\frac{4\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{4\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\
&= -2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
x(\tau)'' + p_1 x(\tau)' + q_1 x(\tau) &= 0 \\
\frac{d^2}{d\tau^2}x(\tau) - 2c\left(\frac{d}{d\tau}x(\tau)\right) + c^2 x(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $x(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
x(\tau) = \mathrm{e}^{c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\
&= \frac{\int 4\sqrt{\frac{1}{t^2}}dt}{c} \\
&= \frac{4\sqrt{\frac{1}{t^2}}\,t\ln(t)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
x = t^4 c_1
$$

Summary

The solution(s) found are the following

$$
x = t^4 c_1 \tag{1}
$$

Verification of solutions

$$
x = t^4 c_1
$$

Verified OK.

11.5.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2x'' - 7tx' + 16x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = 0 \tag{2}$$

Where

$$\begin{aligned}p(t) &= -\frac{7}{t}\\ q(t) &= \frac{16}{t^2}\end{aligned}$$

Applying change of variables on the depndent variable $x = v(t)\,t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not x.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} - \frac{7n}{t^2} + \frac{16}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 4 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}v''(t) + \frac{v'(t)}{t} &= 0\\ v''(t) + \frac{v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t)+\frac{u(t)}{t}=0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(t,u)\\ &= f(t)g(u)\\ &= -\frac{u}{t}\end{aligned}$$

Where $f(t)=-\frac{1}{t}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= -\frac{1}{t}\,dt\\ \int\frac{1}{u}\,du &= \int-\frac{1}{t}\,dt\\ \ln(u) &= -\ln(t)+c_1\\ u &= \mathrm{e}^{-\ln(t)+c_1}\\ &= \frac{c_1}{t}\end{aligned}$$

Now that $u(t)$ is known, then

$$\begin{aligned}v'(t) &= u(t)\\ v(t) &= \int u(t)\,dt+c_2\\ &= c_1\ln(t)+c_2\end{aligned}$$

Hence

$$\begin{aligned}x &= v(t)\,t^n\\ &= (c_1\ln(t)+c_2)\,t^4\\ &= (c_1\ln(t)+c_2)\,t^4\end{aligned}$$

Summary

The solution(s) found are the following

$$x=(c_1\ln(t)+c_2)\,t^4 \tag{1}$$

Verification of solutions

$$x=(c_1\ln(t)+c_2)\,t^4$$

Verified OK.

11.5.5 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
t^2x'' - 7tx' + 16x &= 0 \tag{1}\\
Ax'' + Bx' + Cx &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= t^2\\
B &= -7t \tag{3}\\
C &= 16
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = xe^{\int \frac{B}{2A}\,dt}
$$

Then (2) becomes

$$
z''(t) = rz(t) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{4t^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= -1\\
t &= 4t^2
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(-\frac{1}{4t^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\,dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 276: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4t^2}
$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2t} + (-)\,(0) \\
&= \frac{1}{2t} \\
&= \frac{1}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2t}\right)^2 - \left(-\frac{1}{4t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \frac{1}{2t} dt} \\
&= \sqrt{t}
\end{aligned}
$$

The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-7t}{t^2} \, dt} \\
&= z_1 e^{\frac{7 \ln(t)}{2}} \\
&= z_1 \left(t^{\frac{7}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = t^4
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{x_1^2} \, dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-7t}{t^2} \, dt}}{(x_1)^2} \, dt \\
&= x_1 \int \frac{e^{7 \ln(t)}}{(x_1)^2} \, dt \\
&= x_1(\ln(t))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(t^4\right) + c_2 \left(t^4(\ln(t))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = t^4 c_1 + c_2 t^4 \ln(t) \tag{1}
$$

Verification of solutions

$$
x = t^4 c_1 + c_2 t^4 \ln(t)
$$

Verified OK.

11.5.6 Maple step by step solution

Let's solve

$t^2x'' - 7tx' + 16x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = \frac{7x'}{t} - \frac{16x}{t^2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' - \frac{7x'}{t} + \frac{16x}{t^2} = 0$

- Multiply by denominators of the ODE

  $t^2x'' - 7tx' + 16x = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of x with respect to t , using the chain rule

    $x' = \left(\frac{d}{ds}x(s)\right) s'(t)$

  ◦ Compute derivative

    $x' = \frac{\frac{d}{ds}x(s)}{t}$

  ◦ Calculate the 2nd derivative of x with respect to t , using the chain rule

    $x'' = \left(\frac{d^2}{ds^2}x(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}x(s)\right)$

  ◦ Compute derivative

    $x'' = \frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $t^2\left(\frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}\right) - 7\frac{d}{ds}x(s) + 16x(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}x(s) - 8\frac{d}{ds}x(s) + 16x(s) = 0$

- Characteristic polynomial of ODE

  $r^2 - 8r + 16 = 0$

- Factor the characteristic polynomial

  $(r-4)^2 = 0$
- Root of the characteristic polynomial

  $r = 4$
- 1st solution of the ODE

  $x_1(s) = e^{4s}$
- Repeated root, multiply $x_1(s)$ by s to ensure linear independence

  $x_2(s) = s\,e^{4s}$
- General solution of the ODE

  $x(s) = c_1 x_1(s) + c_2 x_2(s)$
- Substitute in solutions

  $x(s) = c_1 e^{4s} + c_2 s\,e^{4s}$
- Change variables back using $s = \ln(t)$

  $x = t^4 c_1 + c_2 t^4 \ln(t)$
- Simplify

  $x = t^4(c_2 \ln(t) + c_1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(t^2*diff(x(t),t$2)-7*t*diff(x(t),t)+16*x(t)=0,x(t), singsol=all)
```

$$x(t) = t^4(c_2 \ln(t) + c_1)$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 18

```
DSolve[t^2*x''[t]-7*t*x'[t]+16*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to t^4(4c_2 \log(t) + c_1)$$

11.6 problem 1(f)

11.6.1 Existence and uniqueness analysis 1549
11.6.2 Solving as second order euler ode ode 1549
11.6.3 Solving as second order change of variable on x method 2 ode . 1551
11.6.4 Solving as second order change of variable on y method 2 ode . 1555
11.6.5 Solving using Kovacic algorithm 1558
11.6.6 Maple step by step solution 1565

Internal problem ID [11483]
Internal file name [`OUTPUT/10465_Thursday_May_18_2023_04_20_16_AM_91015671/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.1 Cauchy-Euler equations. Exercises page 120
Problem number: 1(f).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{t^2x'' + 3tx' - 8x = 0}$$

With initial conditions

$$[x(1) = 0, x'(1) = 2]$$

11.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= \frac{3}{t} \\ q(t) &= -\frac{8}{t^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + \frac{3x'}{t} - \frac{8x}{t^2} = 0$$

The domain of $p(t) = \frac{3}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = -\frac{8}{t^2}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

11.6.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 3trt^{r-1} - 8t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 3r\,t^r - 8t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 3r - 8 = 0$$

Or

$$r^2 + 2r - 8 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= -4 \\
r_2 &= 2
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$x = c_1 x_1 + c_2 x_2$$

Where $x_1 = t^{r_1}$ and $x_2 = t^{r_2}$. Hence

$$x = \frac{c_1}{t^4} + c_2 t^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \frac{c_1}{t^4} + c_2 t^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{4c_1}{t^5} + 2c_2 t$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = -4c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{1}{3} \\
c_2 &= \frac{1}{3}
\end{aligned}
$$

Substituting these values back in above solution results in

$$x = \frac{t^6 - 1}{3t^4}$$

Summary
The solution(s) found are the following

$$x = \frac{t^6 - 1}{3t^4} \tag{1}$$

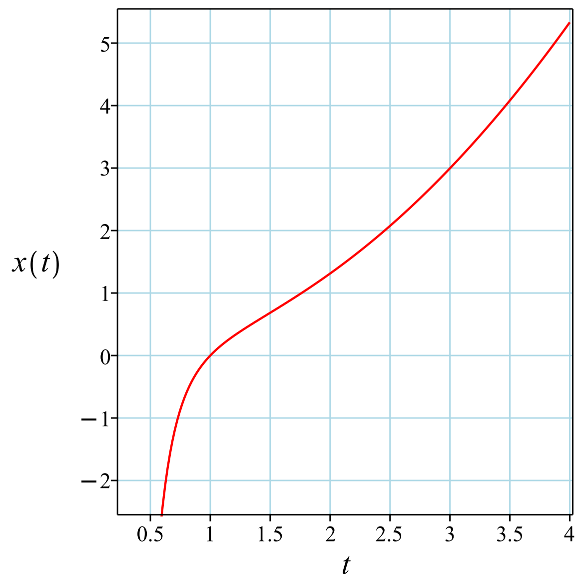


Figure 307: Solution plot

Verification of solutions

$$x = \frac{t^6 - 1}{3t^4}$$

Verified OK.

11.6.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2 x'' + 3tx' - 8x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}$$

Where

$$p(t) = \frac{3}{t}$$
$$q(t) = -\frac{8}{t^2}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1 x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(t)dt\right)}\,dt \\
&= \int e^{-\left(\int \frac{3}{t}dt\right)}\,dt \\
&= \int e^{-3\ln(t)}\,dt \\
&= \int \frac{1}{t^3}dt \\
&= -\frac{1}{2t^2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{-\frac{8}{t^2}}{\frac{1}{t^6}} \\
&= -8t^4
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}x(\tau) + q_1 x(\tau) = 0$$
$$\frac{d^2}{d\tau^2}x(\tau) - 8t^4 x(\tau) = 0$$

But in terms of τ

$$-8t^4 = -\frac{2}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}x(\tau) - \frac{2x(\tau)}{\tau^2} = 0$$

The above ode is now solved for $x(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}x(\tau)\right)\tau^2 - 2x(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $x(\tau) = \tau^r$, then $x' = r\tau^{r-1}$ and $x'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 2\tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r - 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= 2 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$x(\tau) = c_1x_1 + c_2x_2$$

Where $x_1 = \tau^{r_1}$ and $x_2 = \tau^{r_2}$. Hence

$$x(\tau) = \frac{c_1}{\tau} + c_2\tau^2$$

The above solution is now transformed back to x using (6) which results in

$$x = -2t^2c_1 + \frac{c_2}{4t^4}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = -2t^2c_1 + \frac{c_2}{4t^4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$0 = -2c_1 + \frac{c_2}{4} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -4c_1t - \frac{c_2}{t^5}$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = -4c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{1}{6} \\ c_2 &= -\frac{4}{3} \end{aligned}$$

Substituting these values back in above solution results in

$$x = \frac{t^6 - 1}{3t^4}$$

Summary

The solution(s) found are the following

$$x = \frac{t^6 - 1}{3t^4} \tag{1}$$

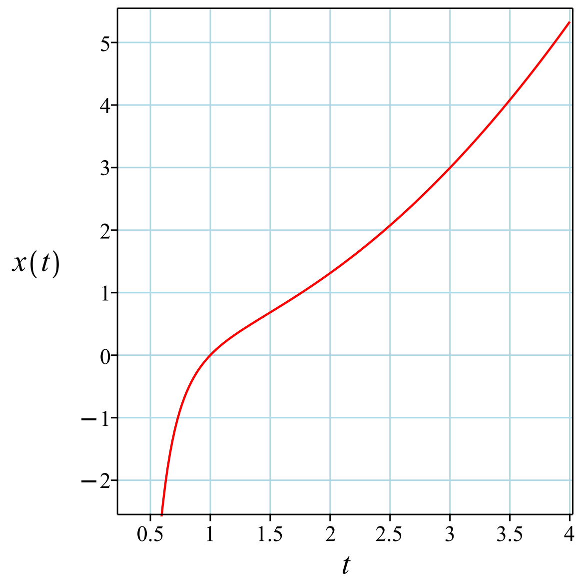


Figure 308: Solution plot

Verification of solutions

$$x=\frac{t^6-1}{3t^4}$$

Verified OK.

11.6.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2x''+3tx'-8x=0 \tag{1}$$

Becomes

$$x''+p(t)\,x'+q(t)\,x=0 \tag{2}$$

Where

$$p(t)=\frac{3}{t}$$
$$q(t)=-\frac{8}{t^2}$$

Applying change of variables on the depndent variable $x=v(t)\,t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not x.

$$v''(t)+\left(\frac{2n}{t}+p\right)v'(t)+\left(\frac{n(n-1)}{t^2}+\frac{np}{t}+q\right)v(t)=0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{3n}{t^2} - \frac{8}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \frac{7v'(t)}{t} &= 0 \\ v''(t) + \frac{7v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{7u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{7u}{t} \end{aligned}$$

Where $f(t) = -\frac{7}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{7}{t}\, dt \\ \int \frac{1}{u}\, du &= \int -\frac{7}{t}\, dt \\ \ln(u) &= -7\ln(t) + c_1 \\ u &= e^{-7\ln(t)+c_1} \\ &= \frac{c_1}{t^7} \end{aligned}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\, dt + c_2 \\ &= -\frac{c_1}{6t^6} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} x &= v(t)\, t^n \\ &= \left(-\frac{c_1}{6t^6} + c_2\right) t^2 \\ &= \frac{6c_2t^6 - c_1}{6t^4} \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \left(-\frac{c_1}{6t^6} + c_2\right) t^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$0 = -\frac{c_1}{6} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{c_1}{t^5} + 2\left(-\frac{c_1}{6t^6} + c_2\right) t$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = \frac{2c_1}{3} + 2c_2 \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 2 \\ c_2 &= \frac{1}{3} \end{aligned}$$

Substituting these values back in above solution results in

$$x = \frac{t^6 - 1}{3t^4}$$

Summary

The solution(s) found are the following

$$x = \frac{t^6 - 1}{3t^4} \tag{1}$$

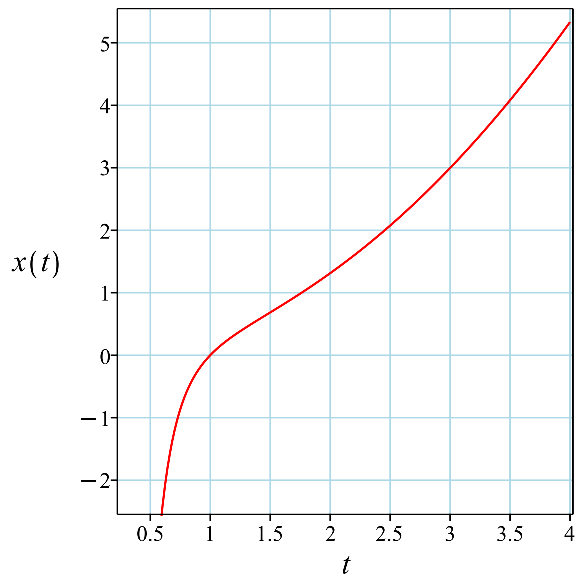


Figure 309: Solution plot

Verification of solutions

$$x = \frac{t^6 - 1}{3t^4}$$

Verified OK.

11.6.5 Solving using Kovacic algorithm

Writing the ode as

$$t^2x'' + 3tx' - 8x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= t^2 \\
B &= 3t \\
C &= -8
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = x e^{\int \frac{B}{2A}\,dt}
$$

Then (2) becomes

$$
z''(t) = r z(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{35}{4t^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 35 \\
t &= 4t^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{35}{4t^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\,dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 278: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{35}{4t^2}
$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition

of r given above. Therefore $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{35}{4t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{35}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{5}{2} - \left(-\frac{5}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2t} + (-)\,(0) \\
&= -\frac{5}{2t} \\
&= -\frac{5}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{5}{2t}\right)(0) + \left(\left(\frac{5}{2t^2}\right) + \left(-\frac{5}{2t}\right)^2 - \left(\frac{35}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int -\frac{5}{2t} dt} \\
&= \frac{1}{t^{\frac{5}{2}}}
\end{aligned}
$$

The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3t}{t^2}\,dt} \\
&= z_1 e^{-\frac{3\ln(t)}{2}} \\
&= z_1 \left(\frac{1}{t^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \frac{1}{t^4}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{3t}{t^2}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-3\ln(t)}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{t^6}{6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\frac{1}{t^4}\right) + c_2 \left(\frac{1}{t^4}\left(\frac{t^6}{6}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = \frac{c_1}{t^4} + \frac{c_2 t^2}{6} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$0 = c_1 + \frac{c_2}{6} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{4c_1}{t^5} + \frac{c_2 t}{3}$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = -4c_1 + \frac{c_2}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{3}$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$x = \frac{t^6 - 1}{3t^4}$$

Summary

The solution(s) found are the following

$$x = \frac{t^6 - 1}{3t^4} \tag{1}$$

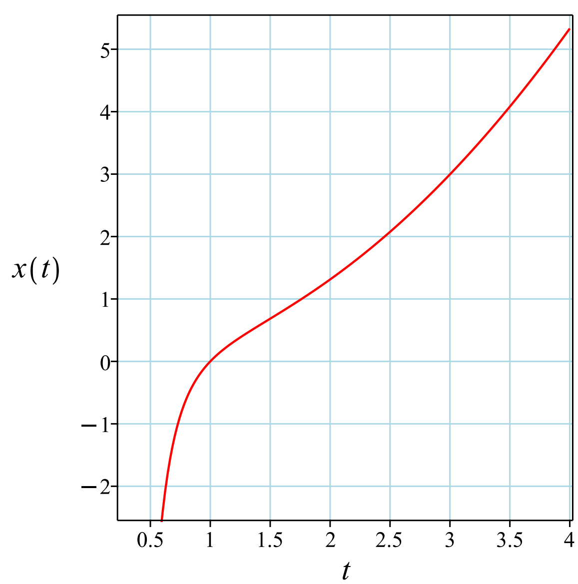


Figure 310: Solution plot

Verification of solutions

$$x = \frac{t^6 - 1}{3t^4}$$

Verified OK.

11.6.6 Maple step by step solution

Let's solve

$$\left[t^2x'' + 3tx' - 8x = 0, x(1) = 0, x'\Big|_{\{t=1\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = -\frac{3x'}{t} + \frac{8x}{t^2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + \frac{3x'}{t} - \frac{8x}{t^2} = 0$

- Multiply by denominators of the ODE

  $t^2x'' + 3tx' - 8x = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of x with respect to t , using the chain rule

    $x' = \left(\frac{d}{ds}x(s)\right) s'(t)$

  - Compute derivative

    $x' = \frac{\frac{d}{ds}x(s)}{t}$

  - Calculate the 2nd derivative of x with respect to t , using the chain rule

    $x'' = \left(\frac{d^2}{ds^2}x(s)\right) s'(t)^2 + s''(t)\left(\frac{d}{ds}x(s)\right)$

  - Compute derivative

    $x'' = \frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}$

  Substitute the change of variables back into the ODE

$$t^2\left(\frac{\frac{d^2}{ds^2}x(s)}{t^2}-\frac{\frac{d}{ds}x(s)}{t^2}\right)+3\frac{d}{ds}x(s)-8x(s)=0$$

- Simplify

  $\frac{d^2}{ds^2}x(s)+2\frac{d}{ds}x(s)-8x(s)=0$
- Characteristic polynomial of ODE

  $r^2+2r-8=0$
- Factor the characteristic polynomial

  $(r+4)(r-2)=0$
- Roots of the characteristic polynomial

  $r=(-4,2)$
- 1st solution of the ODE

  $x_1(s)=\mathrm{e}^{-4s}$
- 2nd solution of the ODE

  $x_2(s)=\mathrm{e}^{2s}$
- General solution of the ODE

  $x(s)=c_1x_1(s)+c_2x_2(s)$
- Substitute in solutions

  $x(s)=c_1\mathrm{e}^{-4s}+c_2\mathrm{e}^{2s}$
- Change variables back using $s=\ln(t)$

  $x=\frac{c_1}{t^4}+c_2t^2$
- Simplify

  $x=\frac{c_1}{t^4}+c_2t^2$

□ Check validity of solution $x=\frac{c_1}{t^4}+c_2t^2$

- Use initial condition $x(1)=0$

  $0=c_1+c_2$
- Compute derivative of the solution

  $x'=-\frac{4c_1}{t^5}+2c_2t$
- Use the initial condition $x'\Big|_{\{t=1\}}=2$

  $2=-4c_1+2c_2$

- Solve for c_1 and c_2

  $\{c_1 = -\frac{1}{3}, c_2 = \frac{1}{3}\}$

- Substitute constant values into general solution and simplify

  $x = \frac{t^6-1}{3t^4}$

- Solution to the IVP

  $x = \frac{t^6-1}{3t^4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([t^2*diff(x(t),t$2)+3*t*diff(x(t),t)-8*x(t)=0,x(1) = 0, D(x)(1) = 2],x(t), singsol=al
```

$$x(t) = \frac{t^6 - 1}{3t^4}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 17

```
DSolve[{t^2*x''[t]+3*t*x'[t]-8*x[t]==0,{x[1]==0,x'[1]==2}},x[t],t,IncludeSingularSolutions -
```

$$x(t) \to \frac{t^6 - 1}{3t^4}$$

11.7 problem 1(g)

11.7.1 Existence and uniqueness analysis 1569
11.7.2 Solving as second order euler ode ode 1569
11.7.3 Solving as second order ode missing y ode 1571
11.7.4 Solving using Kovacic algorithm 1573
11.7.5 Maple step by step solution 1580

Internal problem ID [11484]
Internal file name [`OUTPUT/10466_Thursday_May_18_2023_04_20_17_AM_50766969/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.1 Cauchy-Euler equations. Exercises page 120
Problem number: 1(g).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{t^2x'' + tx' = 0}$$

With initial conditions

$$[x(1) = 0, x'(1) = 2]$$

The ODE is

$$t^2x'' + tx' = 0$$

Or

$$t(x' + tx'') = 0$$

For $t \neq 0$ the above simplifies to

$$x' + tx'' = 0$$

11.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{t} \\ q(t) &= 0 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + \frac{x'}{t} = 0$$

The domain of $p(t) = \frac{1}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. Hence solution exists and is unique.

11.7.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + trt^{r-1} + 0 = 0$$

Simplifying gives

$$r(r-1)\,t^r + r\,t^r + 0 = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + r + 0 = 0$$

Or

$$r^2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 0 \\ r_2 &= 0 \end{aligned}$$

Since the roots are equal, then the general solution is

$$x = c_1 x_1 + c_2 x_2$$

Where $x_1 = t^r$ and $x_2 = t^r \ln(t)$. Hence

$$x = c_2 \ln(t) + c_1$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_2 \ln(t) + c_1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{c_2}{t}$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$x = 2 \ln(t)$$

Summary

The solution(s) found are the following

$$x = 2 \ln(t) \tag{1}$$

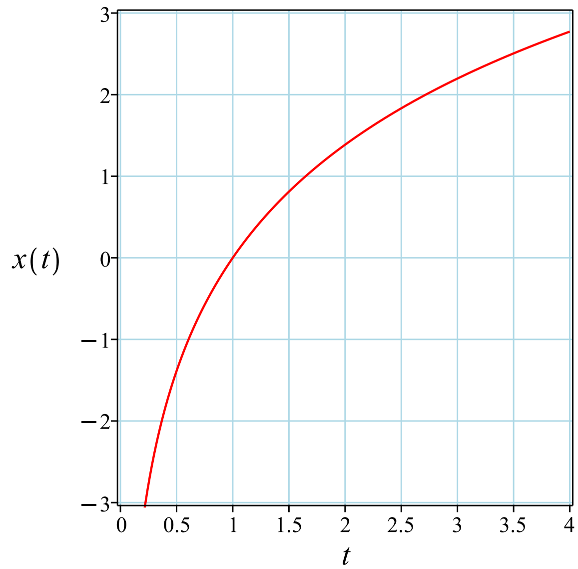


Figure 311: Solution plot

Verification of solutions

$$x = 2\ln(t)$$

Verified OK. `{t <> 0}`

11.7.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable x. Let

$$p(t) = x'$$

Then

$$p'(t) = x''$$

Hence the ode becomes

$$t^2 p'(t) + p(t)\, t = 0$$

Which is now solve for $p(t)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(t, p) \\ &= f(t)g(p) \\ &= -\frac{p}{t} \end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\,dp &= -\frac{1}{t}\,dt \\
\int \frac{1}{p}\,dp &= \int -\frac{1}{t}\,dt \\
\ln(p) &= -\ln(t) + c_1 \\
p &= \mathrm{e}^{-\ln(t)+c_1} \\
&= \frac{c_1}{t}
\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $p = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$p(t) = \frac{2}{t}$$

Since $p = x'$ then the new first order ode to solve is

$$x' = \frac{2}{t}$$

Integrating both sides gives

$$\begin{aligned}
x &= \int \frac{2}{t}\,\mathrm{d}t \\
&= 2\ln(t) + c_2
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $t = 1$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_2$$

$$c_2 = 0$$

Substituting c_2 found above in the general solution gives

$$x = 2\ln(t)$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$x = 2 \ln (t) \tag{1}$$

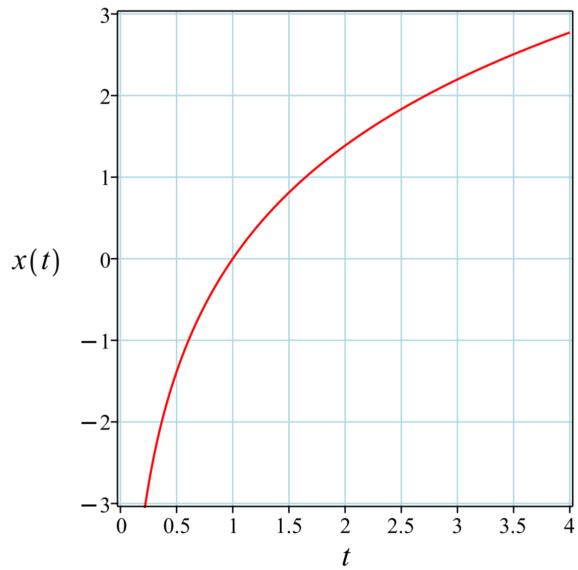


Figure 312: Solution plot

Verification of solutions

$$x = 2 \ln (t)$$

Verified OK. `{t <> 0}`

11.7.4 Solving using Kovacic algorithm

Writing the ode as

$$t^2 x'' + t x' = 0 \tag{1}$$

$$A x'' + B x' + C x = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 280: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition

of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2t} + (-)\,(0) \\ &= \frac{1}{2t} \\ &= \frac{1}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2t}\right)^2 - \left(-\frac{1}{4t^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \frac{1}{2t} dt} \\ &= \sqrt{t} \end{aligned}$$

The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t}{t^2}\,dt} \\
&= z_1 e^{-\frac{\ln(t)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{t}}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = 1$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{t}{t^2}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-\ln(t)}}{(x_1)^2}\,dt \\
&= x_1(\ln(t))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(1) + c_2(1(\ln(t)))
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_2 \ln(t) + c_1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{c_2}{t}$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$x = 2\ln(t)$$

Summary

The solution(s) found are the following

$$x = 2\ln(t) \tag{1}$$

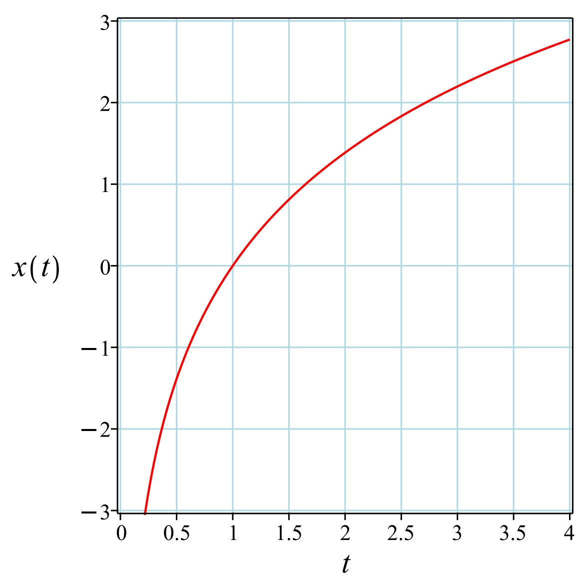


Figure 313: Solution plot

Verification of solutions

$$x = 2\ln(t)$$

Verified OK. `{t <> 0}`

11.7.5 Maple step by step solution

Let's solve

$$\left[t^2 x'' + t x' = 0, x(1) = 0, x'\Big|_{\{t=1\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = -\frac{x'}{t}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + \frac{x'}{t} = 0$

- Multiply by denominators of the ODE

  $x' + t x'' = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of x with respect to t , using the chain rule

    $x' = \left(\frac{d}{ds}x(s)\right) s'(t)$

  - Compute derivative

    $x' = \frac{\frac{d}{ds}x(s)}{t}$

  - Calculate the 2nd derivative of x with respect to t , using the chain rule

    $x'' = \left(\frac{d^2}{ds^2}x(s)\right) s'(t)^2 + s''(t)\left(\frac{d}{ds}x(s)\right)$

  - Compute derivative

    $x'' = \frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $\frac{\frac{d}{ds}x(s)}{t} + t\left(\frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}\right) = 0$

- Simplify

  $\frac{\frac{d^2}{ds^2}x(s)}{t} = 0$

- Isolate 2nd derivative

$\frac{d^2}{ds^2}x(s) = 0$

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the ODE

  $x_1(s) = 1$

- Repeated root, multiply $x_1(s)$ by s to ensure linear independence

  $x_2(s) = s$

- General solution of the ODE

  $x(s) = c_1 x_1(s) + c_2 x_2(s)$

- Substitute in solutions

  $x(s) = c_2 s + c_1$

- Change variables back using $s = \ln(t)$

  $x = c_2 \ln(t) + c_1$

- ☐ Check validity of solution $x = c_2 \ln(t) + c_1$
  - Use initial condition $x(1) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $x' = \frac{c_2}{t}$

  - Use the initial condition $x'\Big|_{\{t=1\}} = 2$

    $2 = c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 2\}$

  - Substitute constant values into general solution and simplify

    $x = 2\ln(t)$

- Solution to the IVP

$$x = 2\ln(t)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve([t^2*diff(x(t),t$2)+t*diff(x(t),t)=0,x(1) = 0, D(x)(1) = 2],x(t), singsol=all)
```

$$x(t) = 2\ln(t)$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 9

```
DSolve[{t^2*x''[t]+t*x'[t]==0,{x[1]==0,x'[1]==2}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 2\log(t)$$

11.8 problem 1(h)

11.8.1 Existence and uniqueness analysis 1584
11.8.2 Solving as second order euler ode ode 1584
11.8.3 Solving as second order change of variable on x method 2 ode . 1587
11.8.4 Solving as second order change of variable on x method 1 ode . 1591
11.8.5 Solving as second order change of variable on y method 2 ode . 1594
11.8.6 Solving using Kovacic algorithm 1597
11.8.7 Maple step by step solution 1603

Internal problem ID [11485]
Internal file name [`OUTPUT/10467_Thursday_May_18_2023_04_20_19_AM_25054400/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.1 Cauchy-Euler equations. Exercises page 120
Problem number: 1(h).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{t^2x'' - tx' + 2x = 0}$$

With initial conditions

$$[x(1) = 0, x'(1) = 1]$$

11.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -\frac{1}{t} \\ q(t) &= \frac{2}{t^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' - \frac{x'}{t} + \frac{2x}{t^2} = 0$$

The domain of $p(t) = -\frac{1}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = \frac{2}{t^2}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

11.8.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} - trt^{r-1} + 2t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r - r\,t^r + 2t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) - r + 2 = 0$$

Or

$$r^2 - 2r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= 1 - i \\
r_2 &= 1 + i
\end{aligned}
$$

The roots are complex conjugate of each others. Let the roots be

$$
\begin{aligned}
r_1 &= \alpha + i\beta \\
r_2 &= \alpha - i\beta
\end{aligned}
$$

Where in this case $\alpha = 1$ and $\beta = -1$. Hence the solution becomes

$$
\begin{aligned}
x &= c_1 t^{r_1} + c_2 t^{r_2} \\
&= c_1 t^{\alpha + i\beta} + c_2 t^{\alpha - i\beta} \\
&= t^{\alpha}\left(c_1 t^{i\beta} + c_2 t^{-i\beta}\right) \\
&= t^{\alpha}\left(c_1 e^{\ln\left(t^{i\beta}\right)} + c_2 e^{\ln\left(t^{-i\beta}\right)}\right) \\
&= t^{\alpha}\left(c_1 e^{i(\beta \ln t)} + c_2 e^{-i(\beta \ln t)}\right)
\end{aligned}
$$

Using the values for $\alpha = 1, \beta = -1$, the above becomes

$$
x = t^1\left(c_1 e^{-i\ln(t)} + c_2 e^{i\ln(t)}\right)
$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$
x = t(c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)))
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = t(c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$x' = c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)) + t\left(-\frac{c_1 \sin(\ln(t))}{t} + \frac{c_2 \cos(\ln(t))}{t}\right)$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$x = t \sin(\ln(t))$$

Summary
The solution(s) found are the following

$$x = t \sin(\ln(t)) \tag{1}$$

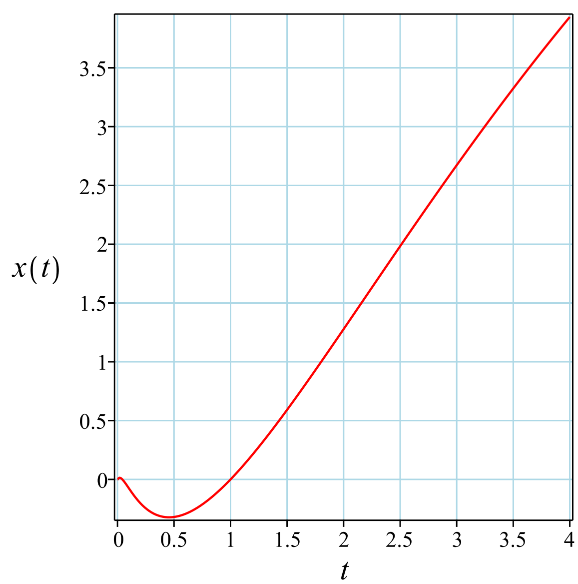


Figure 314: Solution plot

Verification of solutions

$$x = t \sin(\ln(t))$$

Verified OK.

11.8.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2x'' - tx' + 2x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{1}{t} \\ q(t) &= \frac{2}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)}dt \\ &= \int \mathrm{e}^{-\left(\int -\frac{1}{t}dt\right)}dt \\ &= \int e^{\ln(t)}\,dt \\ &= \int t dt \\ &= \frac{t^2}{2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{\frac{2}{t^2}}{t^2} \\ &= \frac{2}{t^4} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}x(\tau) + q_1 x(\tau) &= 0 \\ \frac{d^2}{d\tau^2}x(\tau) + \frac{2x(\tau)}{t^4} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{2}{t^4} = \frac{1}{2\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}x(\tau) + \frac{x(\tau)}{2\tau^2} = 0$$

The above ode is now solved for $x(\tau)$. The ode can be written as

$$2\left(\frac{d^2}{d\tau^2}x(\tau)\right)\tau^2 + x(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $x(\tau) = \tau^r$, then $x' = r\tau^{r-1}$ and $x'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$2\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$2r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$2r(r-1) + 0 + 1 = 0$$

Or

$$2r^2 - 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - \frac{i}{2}$$
$$r_2 = \frac{1}{2} + \frac{i}{2}$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}$. Hence the solution becomes

$$\begin{aligned}
x(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\
&= c_1 \tau^{\alpha+i\beta} + c_2 \tau^{\alpha-i\beta} \\
&= \tau^{\alpha}\left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta}\right) \\
&= \tau^{\alpha}\left(c_1 e^{\ln\left(\tau^{i\beta}\right)} + c_2 e^{\ln\left(\tau^{-i\beta}\right)}\right) \\
&= \tau^{\alpha}\left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)}\right)
\end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$, the above becomes

$$x(\tau) = \tau^{\frac{1}{2}}\left(c_1 e^{-\frac{i\ln(\tau)}{2}} + c_2 e^{\frac{i\ln(\tau)}{2}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$x(\tau) = \sqrt{\tau}\left(c_1 \cos\left(\frac{\ln(\tau)}{2}\right) + c_2 \sin\left(\frac{\ln(\tau)}{2}\right)\right)$$

The above solution is now transformed back to x using (6) which results in

$$x = \frac{\sqrt{2}\, t\left(c_1 \cos\left(-\frac{\ln(2)}{2} + \ln(t)\right) + c_2 \sin\left(-\frac{\ln(2)}{2} + \ln(t)\right)\right)}{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \frac{\sqrt{2}\, t\left(c_1 \cos\left(-\frac{\ln(2)}{2} + \ln(t)\right) + c_2 \sin\left(-\frac{\ln(2)}{2} + \ln(t)\right)\right)}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$0 = \frac{\left(c_1 \cos\left(\frac{\ln(2)}{2}\right) - c_2 \sin\left(\frac{\ln(2)}{2}\right)\right)\sqrt{2}}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{\sqrt{2}\left(c_1 \cos\left(-\frac{\ln(2)}{2} + \ln(t)\right) + c_2 \sin\left(-\frac{\ln(2)}{2} + \ln(t)\right)\right)}{2} + \frac{\sqrt{2}\, t \left(-\frac{c_1 \sin\left(-\frac{\ln(2)}{2} + \ln(t)\right)}{t} + \frac{c_2 \cos\left(-\frac{\ln(2)}{2} + \ln(t)\right)}{t}\right)}{2}$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = \frac{\left((c_1 + c_2)\cos\left(\frac{\ln(2)}{2}\right) + \sin\left(\frac{\ln(2)}{2}\right)(c_1 - c_2)\right)\sqrt{2}}{2} \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \sqrt{2}\, \sin\left(\frac{\ln(2)}{2}\right)$$
$$c_2 = \sqrt{2}\, \cos\left(\frac{\ln(2)}{2}\right)$$

Substituting these values back in above solution results in

$$x = \sin\left(-\frac{\ln(2)}{2} + \ln(t)\right)\cos\left(\frac{\ln(2)}{2}\right) t + \frac{t \sin(\ln(t))}{2} - \frac{\sin(-\ln(2) + \ln(t))\, t}{2}$$

Which simplifies to

$$x = \frac{\left(2 \sin\left(-\frac{\ln(2)}{2} + \ln(t)\right)\cos\left(\frac{\ln(2)}{2}\right) + \sin(\ln(t)) - \sin(-\ln(2) + \ln(t))\right) t}{2}$$

Summary

The solution(s) found are the following

$$x = \frac{\left(2 \sin\left(-\frac{\ln(2)}{2} + \ln(t)\right)\cos\left(\frac{\ln(2)}{2}\right) + \sin(\ln(t)) - \sin(-\ln(2) + \ln(t))\right) t}{2} \tag{1}$$

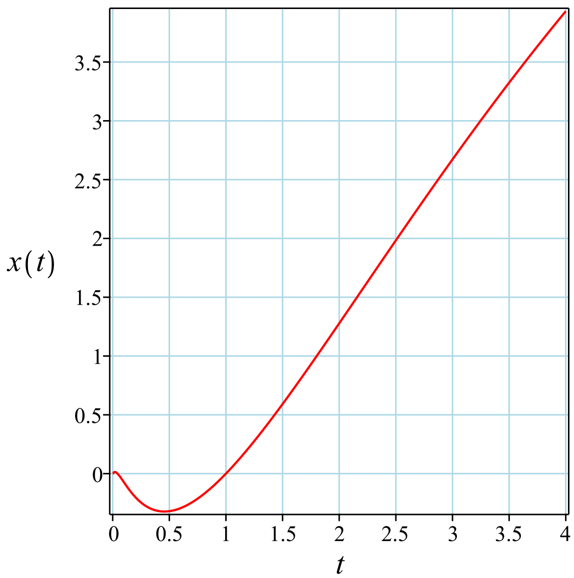


Figure 315: Solution plot

Verification of solutions

$$x = \frac{\left(2\sin\left(-\frac{\ln(2)}{2}+\ln(t)\right)\cos\left(\frac{\ln(2)}{2}\right)+\sin(\ln(t))-\sin(-\ln(2)+\ln(t))\right)t}{2}$$

Verified OK.

11.8.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$t^2x''-tx'+2x=0 \tag{1}$$

Becomes

$$x''+p(t)\,x'+q(t)\,x=0 \tag{2}$$

Where

$$p(t)=-\frac{1}{t}$$
$$q(t)=\frac{2}{t^2}$$

Applying change of variables $\tau=g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}x(\tau)+p_1\left(\frac{d}{d\tau}x(\tau)\right)+q_1x(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{2}\,\sqrt{\frac{1}{t^2}}}{c} \\
\tau'' &= -\frac{\sqrt{2}}{c\sqrt{\frac{1}{t^2}}\,t^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\
&= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{t^2}}\,t^3} - \frac{1}{t}\frac{\sqrt{2}\,\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\
&= -c\sqrt{2}
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
x(\tau)'' + p_1 x(\tau)' + q_1 x(\tau) &= 0 \\
\frac{d^2}{d\tau^2}x(\tau) - c\sqrt{2}\left(\frac{d}{d\tau}x(\tau)\right) + c^2 x(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $x(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$x(\tau) = e^{\frac{\sqrt{2}\,c\tau}{2}}\left(c_1 \cos\left(\frac{\sqrt{2}\,c\tau}{2}\right) + c_2 \sin\left(\frac{\sqrt{2}\,c\tau}{2}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\
&= \frac{\int \sqrt{2}\sqrt{\frac{1}{t^2}}dt}{c} \\
&= \frac{\sqrt{2}\sqrt{\frac{1}{t^2}}\,t\ln(t)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$x = t(c_1\cos(\ln(t)) + c_2\sin(\ln(t)))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = t(c_1\cos(\ln(t)) + c_2\sin(\ln(t))) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1\cos(\ln(t)) + c_2\sin(\ln(t)) + t\left(-\frac{c_1\sin(\ln(t))}{t} + \frac{c_2\cos(\ln(t))}{t}\right)$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}
c_1 &= 0 \\
c_2 &= 1
\end{aligned}$$

Substituting these values back in above solution results in

$$x = t\sin(\ln(t))$$

Summary
The solution(s) found are the following

$$x = t \sin (\ln (t)) \tag{1}$$

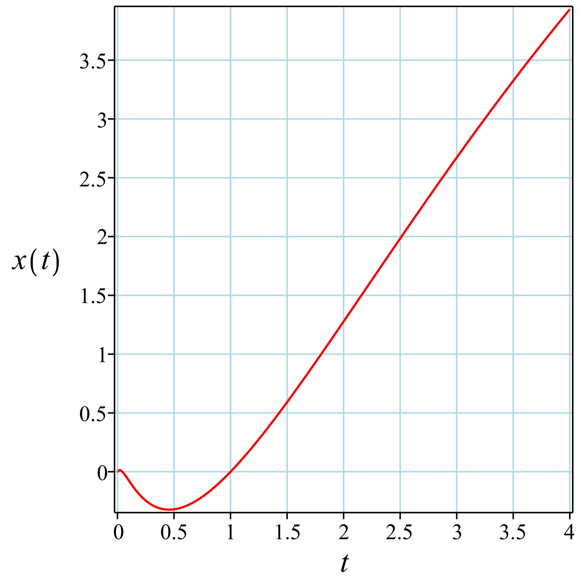


Figure 316: Solution plot

Verification of solutions

$$x = t \sin (\ln (t))$$

Verified OK.

11.8.5 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2 x'' - t x' + 2x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}$$

Where

$$p(t) = -\frac{1}{t}$$
$$q(t) = \frac{2}{t^2}$$

Applying change of variables on the depndent variable $x = v(t)\,t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not x.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} - \frac{n}{t^2} + \frac{2}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 + i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \left(\frac{2+2i}{t} - \frac{1}{t}\right) v'(t) &= 0 \\ v''(t) + \frac{(1+2i)\, v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{(1+2i)\, u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= \frac{(-1-2i)\, u}{t} \end{aligned}$$

Where $f(t) = \frac{-1-2i}{t}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{-1-2i}{t}\,dt \\
\int \frac{1}{u}\,du &= \int \frac{-1-2i}{t}\,dt \\
\ln(u) &= (-1-2i)\ln(t) + c_1 \\
u &= \mathrm{e}^{(-1-2i)\ln(t)+c_1} \\
&= c_1 \mathrm{e}^{(-1-2i)\ln(t)}
\end{aligned}
$$

Which simplifies to

$$
u(t) = \frac{c_1 t^{-2i}}{t}
$$

Now that $u(t)$ is known, then

$$
\begin{aligned}
v'(t) &= u(t) \\
v(t) &= \int u(t)\,dt + c_2 \\
&= \frac{i c_1 t^{-2i}}{2} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
x &= v(t)\,t^n \\
&= \left(\frac{i c_1 t^{-2i}}{2} + c_2\right) t^{1+i} \\
&= t^{1+i} c_2 + \frac{i t^{1-i} c_1}{2}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = \left(\frac{i c_1 t^{-2i}}{2} + c_2\right) t^{1+i} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$
0 = \frac{i c_1}{2} + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$x' = \frac{c_1 t^{-2i} t^{1+i}}{t} + \frac{(1+i)\left(\frac{i c_1 t^{-2i}}{2} + c_2\right) t^{1+i}}{t}$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = (1+i)\left(\frac{c_1}{2} + c_2\right) \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= -\frac{i}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$x = \frac{i t^{1+i} t^{-2i}}{2} - \frac{i t^{1+i}}{2}$$

Summary
The solution(s) found are the following

$$x = -\frac{i(-t^{1-i} + t^{1+i})}{2} \tag{1}$$

Verification of solutions

$$x = -\frac{i(-t^{1-i} + t^{1+i})}{2}$$

Verified OK.

11.8.6 Solving using Kovacic algorithm

Writing the ode as

$$t^2 x'' - t x' + 2x = 0 \tag{1}$$
$$A x'' + B x' + C x = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{5}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 282: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{5}{4t^2}
$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition

of r given above. Therefore $b = -\frac{5}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{5}{4t^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{5}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2} - i$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - i - \left(\frac{1}{2} - i\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - i}{t} + (-)\,(0) \\
&= \frac{\frac{1}{2} - i}{t} \\
&= \frac{\frac{1}{2} - i}{t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - i}{t}\right)(0) + \left(\left(\frac{-\frac{1}{2} + i}{t^2} \right) + \left(\frac{\frac{1}{2} - i}{t} \right)^2 - \left(-\frac{5}{4t^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \frac{\frac{1}{2} - i}{t} dt} \\
&= t^{\frac{1}{2} - i}
\end{aligned}
$$

The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t}{t^2}\,dt} \\
&= z_1 e^{\frac{\ln(t)}{2}} \\
&= z_1\left(\sqrt{t}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = t^{1-i}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-t}{t^2}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{\ln(t)}}{(x_1)^2}\,dt \\
&= x_1\left(-\frac{i t^{2i}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(t^{1-i}\right) + c_2\left(t^{1-i}\left(-\frac{i t^{2i}}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = t^{1-i} c_1 - \frac{i c_2 t^{1+i}}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 1$ in the above gives

$$0 = c_1 - \frac{ic_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{(1-i)\, t^{1-i} c_1}{t} + \frac{\left(\frac{1}{2} - \frac{i}{2}\right) c_2 t^{1+i}}{t}$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = (1-i)\left(c_1 + \frac{c_2}{2}\right) \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{i}{2} \\ c_2 &= 1 \end{aligned}$$

Substituting these values back in above solution results in

$$x = \frac{it^{1-i}}{2} - \frac{it^{1+i}}{2}$$

<u>Summary</u>

The solution(s) found are the following

$$x = -\frac{i(-t^{1-i} + t^{1+i})}{2} \tag{1}$$

<u>Verification of solutions</u>

$$x = -\frac{i(-t^{1-i} + t^{1+i})}{2}$$

Verified OK.

11.8.7 Maple step by step solution

Let's solve

$$\left[t^2 x'' - tx' + 2x = 0, x(1) = 0, x'\Big|_{\{t=1\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  x''
- Isolate 2nd derivative

  $x''=\frac{x'}{t}-\frac{2x}{t^2}$
- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x''-\frac{x'}{t}+\frac{2x}{t^2}=0$
- Multiply by denominators of the ODE

  $t^2x''-tx'+2x=0$
- Make a change of variables

  $s=\ln(t)$
- ☐ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of x with respect to t , using the chain rule

    $x'=\left(\frac{d}{ds}x(s)\right)s'(t)$
  - Compute derivative

    $x'=\frac{\frac{d}{ds}x(s)}{t}$
  - Calculate the 2nd derivative of x with respect to t , using the chain rule

    $x''=\left(\frac{d^2}{ds^2}x(s)\right)s'(t)^2+s''(t)\left(\frac{d}{ds}x(s)\right)$
  - Compute derivative

    $x''=\frac{\frac{d^2}{ds^2}x(s)}{t^2}-\frac{\frac{d}{ds}x(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $t^2\left(\frac{\frac{d^2}{ds^2}x(s)}{t^2}-\frac{\frac{d}{ds}x(s)}{t^2}\right)-\frac{d}{ds}x(s)+2x(s)=0$
- Simplify

  $\frac{d^2}{ds^2}x(s)-2\frac{d}{ds}x(s)+2x(s)=0$
- Characteristic polynomial of ODE

  $r^2-2r+2=0$
- Use quadratic formula to solve for r

  $r=\frac{2\pm(\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

$r = (1 - \mathrm{I}, 1 + \mathrm{I})$

- 1st solution of the ODE

  $x_1(s) = \mathrm{e}^s \cos(s)$

- 2nd solution of the ODE

  $x_2(s) = \mathrm{e}^s \sin(s)$

- General solution of the ODE

  $x(s) = c_1 x_1(s) + c_2 x_2(s)$

- Substitute in solutions

  $x(s) = c_1 \mathrm{e}^s \cos(s) + c_2 \mathrm{e}^s \sin(s)$

- Change variables back using $s = \ln(t)$

  $x = c_1 t \cos(\ln(t)) + c_2 \sin(\ln(t))\, t$

- Simplify

  $x = t(c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)))$

□ Check validity of solution $x = t(c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)))$

  - Use initial condition $x(1) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $x' = c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)) + t\left(-\frac{c_1 \sin(\ln(t))}{t} + \frac{c_2 \cos(\ln(t))}{t}\right)$

  - Use the initial condition $x'\Big|_{\{t=1\}} = 1$

    $1 = c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 1\}$

  - Substitute constant values into general solution and simplify

    $x = t \sin(\ln(t))$

- Solution to the IVP

  $x = t \sin(\ln(t))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve([t^2*diff(x(t),t$2)-t*diff(x(t),t)+2*x(t)=0,x(1) = 0, D(x)(1) = 1],x(t), singsol=all)
```

$$x(t) = t\sin(\ln(t))$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 10

```
DSolve[{t^2*x''[t]-t*x'[t]+2*x[t]==0,{x[1]==0,x'[1]==1}},x[t],t,IncludeSingularSolutions ->
```

$$x(t) \to t\sin(\log(t))$$

11.9 problem 2

11.9.1 Existence and uniqueness analysis 1607
11.9.2 Solving as second order ode missing y ode 1608
11.9.3 Solving using Kovacic algorithm 1610
11.9.4 Maple step by step solution 1617

Internal problem ID [11486]
Internal file name [`OUTPUT/10468_Thursday_May_18_2023_04_20_21_AM_74297487/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.1 Cauchy-Euler equations. Exercises page 120
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_y]]`

$$\boxed{x'' + t^2x' = 0}$$

With initial conditions

$$[x(0) = 0, x'(0) = 1]$$

11.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= t^2 \\ q(t) &= 0 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + t^2 x' = 0$$

The domain of $p(t) = t^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

11.9.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable x. Let

$$p(t) = x'$$

Then

$$p'(t) = x''$$

Hence the ode becomes

$$p'(t) + t^2 p(t) = 0$$

Which is now solve for $p(t)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(t, p) \\ &= f(t)g(p) \\ &= -t^2 p \end{aligned}$$

Where $f(t) = -t^2$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned} \frac{1}{p}\, dp &= -t^2\, dt \\ \int \frac{1}{p}\, dp &= \int -t^2\, dt \\ \ln(p) &= -\frac{t^3}{3} + c_1 \\ p &= \mathrm{e}^{-\frac{t^3}{3} + c_1} \\ &= c_1 \mathrm{e}^{-\frac{t^3}{3}} \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$p(t) = \mathrm{e}^{-\frac{t^3}{3}}$$

Since $p = x'$ then the new first order ode to solve is

$$x' = \mathrm{e}^{-\frac{t^3}{3}}$$

Integrating both sides gives

$$\begin{aligned}
x &= \int \mathrm{e}^{-\frac{t^3}{3}}\, \mathrm{d}t \\
&= \frac{3^{\frac{1}{3}} \left(\frac{3t 3^{\frac{5}{6}} \mathrm{e}^{-\frac{t^3}{6}} \operatorname{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right)}{4(t^3)^{\frac{1}{6}}} + \frac{3\, 3^{\frac{5}{6}} \mathrm{e}^{-\frac{t^3}{6}} \operatorname{WhittakerM}\left(\frac{7}{6}, \frac{2}{3}, \frac{t^3}{3}\right)}{t^2 (t^3)^{\frac{1}{6}}}\right)}{3} + c_2
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_2$$

$$c_2 = 0$$

Substituting c_2 found above in the general solution gives

$$x = \frac{3\, 3^{\frac{1}{6}} \mathrm{e}^{-\frac{t^3}{6}} \operatorname{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right) t^3 + 12\, 3^{\frac{1}{6}} \mathrm{e}^{-\frac{t^3}{6}} \operatorname{WhittakerM}\left(\frac{7}{6}, \frac{2}{3}, \frac{t^3}{3}\right)}{4 \left(t^3\right)^{\frac{1}{6}} t^2}$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$x = \frac{3\, 3^{\frac{1}{6}} \mathrm{e}^{-\frac{t^3}{6}} \operatorname{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right) t^3 + 12\, 3^{\frac{1}{6}} \mathrm{e}^{-\frac{t^3}{6}} \operatorname{WhittakerM}\left(\frac{7}{6}, \frac{2}{3}, \frac{t^3}{3}\right)}{4 \left(t^3\right)^{\frac{1}{6}} t^2} \tag{1}$$

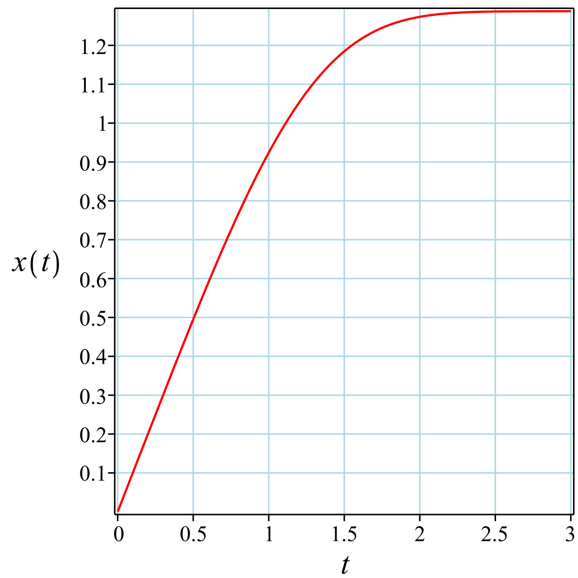


Figure 317: Solution plot

Verification of solutions

$$x = \frac{3\, 3^{\frac{1}{6}} \mathrm{e}^{-\frac{t^3}{6}} \operatorname{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right) t^3 + 12\, 3^{\frac{1}{6}} \mathrm{e}^{-\frac{t^3}{6}} \operatorname{WhittakerM}\left(\frac{7}{6}, \frac{2}{3}, \frac{t^3}{3}\right)}{4\, (t^3)^{\frac{1}{6}}\, t^2}$$

Verified OK.

11.9.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + t^2 x' = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= t^2 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t(t^3 + 4)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t(t^3 + 4) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t(t^3 + 4)}{4}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 284: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 4 \\
&= -4
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{2} a_i t^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $t^v = t^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{t^2}{2} + \frac{1}{t} - \frac{1}{t^4} + \frac{2}{t^7} - \frac{5}{t^{10}} + \frac{14}{t^{13}} - \frac{42}{t^{16}} + \frac{132}{t^{19}} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 2$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i t^i \\
&= \frac{t^2}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^1 = t$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{t^4}{4}$$

This shows that the coefficient of t in the above is 0. Now we need to find the coefficient of t in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of t in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t(t^3 + 4)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}t^4 + t\right) + (0) \\
&= \frac{1}{4}t^4 + t
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{t}$ in the quotient is 1. Now b can be found.

$$\begin{aligned}
b &= (1) - (0) \\
&= 1
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{t^2}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{1}{\frac{1}{2}} - 2\right) &= 0 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{1}{\frac{1}{2}} - 2\right) &= -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t(t^3 + 4)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{t^2}{2}$	0	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{t^2}{2}\right) \\ &= \frac{t^2}{2} \\ &= \frac{t^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{t^2}{2}\right)(0) + \left((t) + \left(\frac{t^2}{2}\right)^2 - \left(\frac{t(t^3+4)}{4}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= pe^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \frac{t^2}{2} dt} \\
&= \mathrm{e}^{\frac{t^3}{6}}
\end{aligned}$$

The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t^2}{1} \, dt} \\
&= z_1 e^{-\frac{t^3}{6}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t^3}{6}}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = 1$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{x_1^2} \, dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{t^2}{1} \, dt}}{(x_1)^2} \, dt \\
&= x_1 \int \frac{e^{-\frac{t^3}{3}}}{(x_1)^2} \, dt \\
&= x_1 \left(\int \mathrm{e}^{-\frac{t^3}{3}} dt\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(1) + c_2 \left(1 \left(\int \mathrm{e}^{-\frac{t^3}{3}} dt\right)\right)
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 + c_2 \left(\int e^{-\frac{t^3}{3}}dt\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = \lim_{t \to 0} \left(c_1 + c_2 \left(\int e^{-\frac{t^3}{3}}dt\right)\right) \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_2 e^{-\frac{t^3}{3}}$$

substituting $x' = 1$ and $t = 0$ in the above gives

$$1 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\left(\int^0 e^{-\frac{\_a^3}{3}}d\_a\right)$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$x = -\left(\int^0 e^{-\frac{\_a^3}{3}}d\_a\right) + \int e^{-\frac{t^3}{3}}dt$$

<u>Summary</u>
The solution(s) found are the following

$$x = -\left(\int^0 e^{-\frac{\_a^3}{3}}d\_a\right) + \int e^{-\frac{t^3}{3}}dt \tag{1}$$

<u>Verification of solutions</u>

$$x = -\left(\int^0 e^{-\frac{\_a^3}{3}}d\_a\right) + \int e^{-\frac{t^3}{3}}dt$$

Verified OK.

11.9.4 Maple step by step solution

Let's solve

$\left[x''+t^2x'=0, x(0)=0, x'\Big|_{\{t=0\}}=1\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Assume series solution for x

  $x=\sum_{k=0}^{\infty} a_k t^k$

□ Rewrite ODE with series expansions

  - Convert $t^2 \cdot x'$ to series expansion

    $t^2 \cdot x'=\sum_{k=0}^{\infty} a_k k\, t^{k+1}$

  - Shift index using $k->k-1$

    $t^2 \cdot x'=\sum_{k=1}^{\infty} a_{k-1}(k-1)\, t^k$

  - Convert x'' to series expansion

    $x''=\sum_{k=2}^{\infty} a_k k(k-1)\, t^{k-2}$

  - Shift index using $k->k+2$

    $x''=\sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, t^k$

  Rewrite ODE with series expansions

  $2a_2+\left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1)+a_{k-1}(k-1)\right) t^k\right)=0$

- Each term must be 0

  $2a_2=0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2}+a_{k-1}(k-1)=0$

- Shift index using $k->k+1$

  $\left((k+1)^2+3k+5\right) a_{k+3}+a_k k=0$

- Recursion relation that defines the series solution to the ODE

$$\left[x=\sum_{k=0}^{\infty} a_k t^k, a_{k+3}=-\frac{a_k k}{k^2+5k+6}, 2a_2=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
<- LODE missing y successful`
```

✓ Solution by Maple

Time used: 1.297 (sec). Leaf size: 63

```
dsolve([diff(x(t),t$2)+t^2*diff(x(t),t)=0,x(0) = 0, D(x)(0) = 1],x(t), singsol=all)
```

$$x(t)=\frac{\mathrm{e}^{-\frac{t^3}{3}}\sqrt{t}\left(4\,3^{\frac{5}{6}}(t^3)^{\frac{1}{6}}+9\,\mathrm{e}^{\frac{t^3}{6}}\,\mathrm{WhittakerM}\left(\frac{1}{6},\frac{2}{3},\frac{t^3}{3}\right)\right)3^{\frac{1}{6}}\left(\begin{cases}\frac{1}{1-i\sqrt{3}} & t<0\\ \frac{1}{2} & 0\leq t\end{cases}\right)}{6}$$

✓ Solution by Mathematica

Time used: 0.153 (sec). Leaf size: 43

```
DSolve[{x''[t]+t^2*x'[t]==0,{x[0]==0,x'[0]==1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t)\to \frac{t^2\,\mathrm{Gamma}\left(\frac{1}{3}\right)-(t^3)^{2/3}\,\Gamma\left(\frac{1}{3},\frac{t^3}{3}\right)}{3^{2/3}t^2}$$

12 Chapter 2, Second order linear equations. Section 2.4.2 Variation of parameters. Exercises page 124

12.1 problem 1(a) 1620
12.2 problem 1(b) 1633
12.3 problem 1(c) 1645
12.4 problem 1(d) 1658
12.5 problem 1(e) 1676
12.6 problem 1(f) 1689
12.7 problem 2 1704
12.8 problem 3 1720
12.9 problem 7 1752

12.1 problem 1(a)

12.1.1 Solving as second order linear constant coeff ode 1620
12.1.2 Solving using Kovacic algorithm 1625
12.1.3 Maple step by step solution 1630

Internal problem ID [11487]
Internal file name [`OUTPUT/10469_Thursday_May_18_2023_04_20_23_AM_69700110/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.2 Variation of parameters. Exercises page 124
Problem number: 1(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + x = \tan(t)}$$

12.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = 1, f(t) = \tan(t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^0(c_1 \cos(t) + c_2 \sin(t))$$

Or

$$x = c_1 \cos(t) + c_2 \sin(t)$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \cos(t) + c_2 \sin(t)$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} x_1 &= \cos(t) \\ x_2 &= \sin(t) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ \frac{d}{dt}(\cos(t)) & \frac{d}{dt}(\sin(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}$$

Therefore

$$W = (\cos(t))(\cos(t)) - (\sin(t))(-\sin(t))$$

Which simplifies to

$$W = \sin(t)^2 + \cos(t)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(t)\tan(t)}{1}\,dt$$

Which simplifies to

$$u_1 = -\int \sin(t)\tan(t)\,dt$$

Hence

$$u_1 = \sin(t) - \ln(\sec(t) + \tan(t))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(t)\tan(t)}{1}\,dt$$

Which simplifies to

$$u_2 = \int \sin(t)\,dt$$

Hence

$$u_2 = -\cos(t)$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = (\sin(t) - \ln(\sec(t) + \tan(t)))\cos(t) - \sin(t)\cos(t)$$

Which simplifies to

$$x_p(t) = -\cos(t)\ln(\sec(t) + \tan(t))$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= (c_1 \cos(t) + c_2 \sin(t)) + (-\cos(t) \ln(\sec(t) + \tan(t)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1 \cos(t) + c_2 \sin(t) - \cos(t) \ln(\sec(t) + \tan(t)) \tag{1}
$$

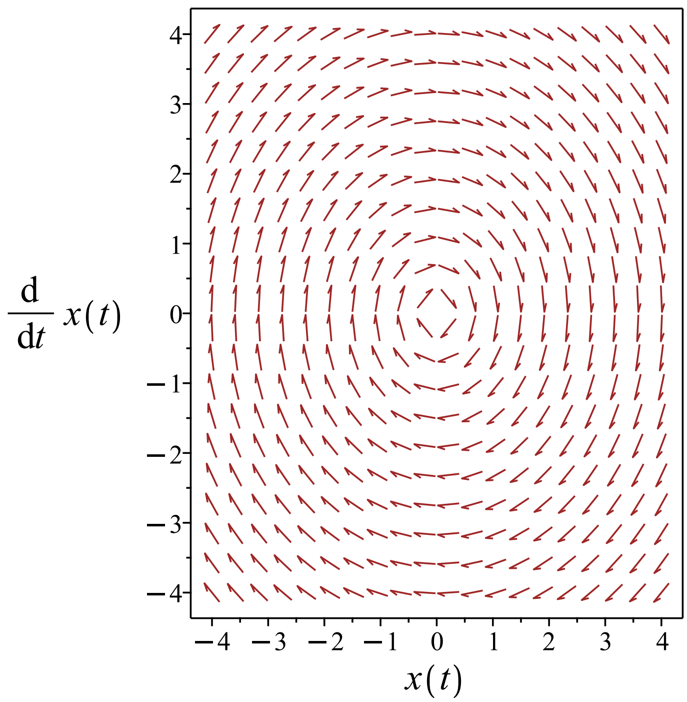


Figure 318: Slope field plot

Verification of solutions

$$
x = c_1 \cos(t) + c_2 \sin(t) - \cos(t) \ln(\sec(t) + \tan(t))
$$

Verified OK.

12.1.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align}
x'' + x &= 0 \tag{1}\\
Ax'' + Bx' + Cx &= 0 \tag{2}
\end{align}
$$

Comparing (1) and (2) shows that

$$
\begin{align}
A &= 1 \nonumber\\
B &= 0 \tag{3}\\
C &= 1 \nonumber
\end{align}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = xe^{\int \frac{B}{2A}\, dt}
$$

Then (2) becomes

$$
z''(t) = rz(t) \tag{4}
$$

Where r is given by

$$
\begin{align}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \nonumber
\end{align}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= -1\\
t &= 1
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(t) = -z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 286: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \cos(t) \end{aligned}$$

Which simplifies to

$$x_1 = \cos(t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\ &= \cos(t) \int \frac{1}{\cos(t)^2}\, dt \\ &= \cos(t)(\tan(t)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1(\cos(t)) + c_2(\cos(t)(\tan(t))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \cos(t) + c_2 \sin(t)$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = \cos(t)$$
$$x_2 = \sin(t)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ \frac{d}{dt}(\cos(t)) & \frac{d}{dt}(\sin(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}$$

Therefore

$$W = (\cos(t))(\cos(t)) - (\sin(t))(-\sin(t))$$

Which simplifies to

$$W = \sin(t)^2 + \cos(t)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(t)\tan(t)}{1}\,dt$$

Which simplifies to

$$u_1 = -\int \sin(t)\tan(t)\,dt$$

Hence

$$u_1 = \sin(t) - \ln(\sec(t) + \tan(t))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(t)\tan(t)}{1}\,dt$$

Which simplifies to

$$u_2 = \int \sin(t)\,dt$$

Hence

$$u_2 = -\cos(t)$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = (\sin(t) - \ln(\sec(t) + \tan(t)))\cos(t) - \sin(t)\cos(t)$$

Which simplifies to

$$x_p(t) = -\cos(t)\ln(\sec(t) + \tan(t))$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1\cos(t) + c_2\sin(t)) + (-\cos(t)\ln(\sec(t) + \tan(t))) \end{aligned}$$

Summary
The solution(s) found are the following

$$x = c_1 \cos(t) + c_2 \sin(t) - \cos(t) \ln(\sec(t) + \tan(t)) \tag{1}$$

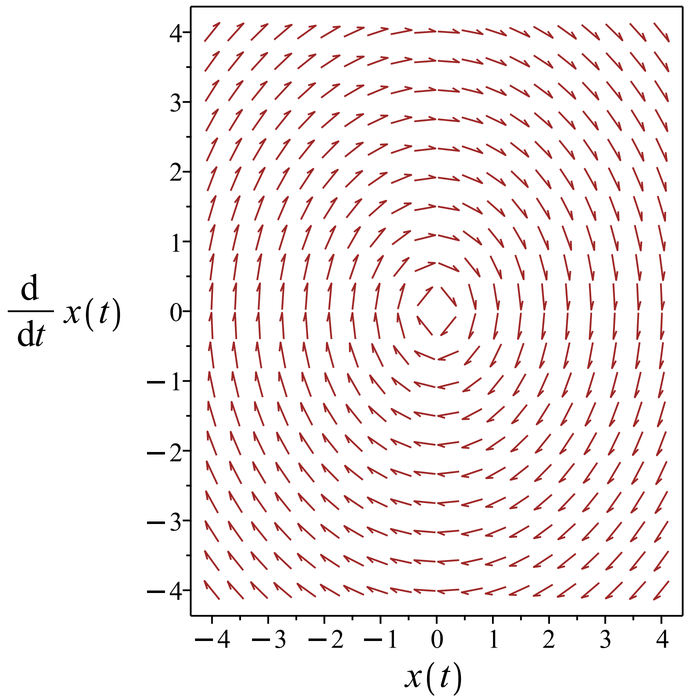


Figure 319: Slope field plot

Verification of solutions

$$x = c_1 \cos(t) + c_2 \sin(t) - \cos(t) \ln(\sec(t) + \tan(t))$$

Verified OK.

12.1.3 Maple step by step solution

Let's solve

$x'' + x = \tan(t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0\pm(\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $x_1(t) = \cos(t)$
- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(t)$
- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$
- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(t) + c_2 \sin(t) + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \tan(t)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$
  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 1$
  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -\cos(t)\left(\int \sin(t)\tan(t)\,dt\right) + \sin(t)\left(\int \sin(t)\,dt\right)$
  - Compute integrals

    $x_p(t) = -\cos(t)\ln(\sec(t) + \tan(t))$

- Substitute particular solution into general solution to ODE

  $x = c_1 \cos(t) + c_2 \sin(t) - \cos(t)\ln(\sec(t) + \tan(t))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(x(t),t$2)+x(t)=tan(t),x(t), singsol=all)
```

$$x(t) = \sin(t)\, c_2 + \cos(t)\, c_1 - \cos(t) \ln(\sec(t) + \tan(t))$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 23

```
DSolve[x''[t]+x[t]==Tan[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \cos(t)(-\text{arctanh}(\sin(t))) + c_1 \cos(t) + c_2 \sin(t)$$

12.2 problem 1(b)

12.2.1 Solving as second order linear constant coeff ode 1633
12.2.2 Solving using Kovacic algorithm 1636
12.2.3 Maple step by step solution 1642

Internal problem ID [11488]
Internal file name [`OUTPUT/10470_Thursday_May_18_2023_04_20_25_AM_47398927/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.2 Variation of parameters. Exercises page 124
Problem number: 1(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' - x = t\,e^{t}}$$

12.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = -1, f(t) = t\,e^{t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ x &= c_1 e^{(1)t} + c_2 e^{(-1)t}\end{aligned}$$

Or

$$x = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{-t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t\,e^{t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{t\,e^{t}, e^{t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{t}, e^{-t}\}$$

Since e^{t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,e^{t}, t^{2}e^{t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t\,e^{t} + A_2 t^{2} e^{t}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{t} + 2A_2 e^{t} + 4A_2 t\,e^{t} = t\,e^{t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{t\,e^{t}}{4} + \frac{t^{2}e^{t}}{4}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{t} + c_2 e^{-t}\right) + \left(-\frac{t\,e^{t}}{4} + \frac{t^{2}e^{t}}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
x = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-t} - \frac{t\,\mathrm{e}^{t}}{4} + \frac{t^2 \mathrm{e}^{t}}{4} \tag{1}
$$

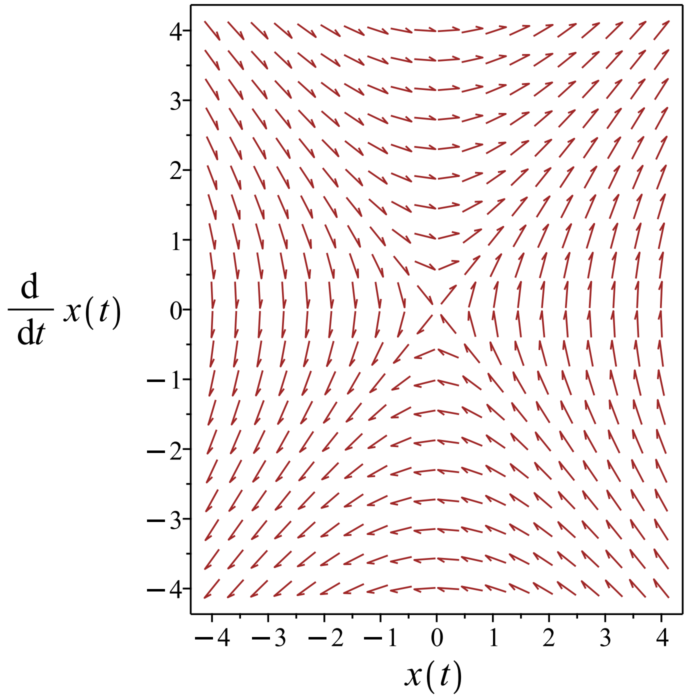


Figure 320: Slope field plot

Verification of solutions

$$
x = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-t} - \frac{t\,\mathrm{e}^{t}}{4} + \frac{t^2 \mathrm{e}^{t}}{4}
$$

Verified OK.

12.2.2 Solving using Kovacic algorithm

Writing the ode as

$$
x'' - x = 0 \tag{1}
$$

$$
Ax'' + Bx' + Cx = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= -1
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = x e^{\int \frac{B}{2A}\, dt}
$$

Then (2) becomes

$$
z''(t) = r z(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 288: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
x_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
x_1 &= z_1 \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\
&= \mathrm{e}^{-t} \int \frac{1}{\mathrm{e}^{-2t}}\, dt \\
&= \mathrm{e}^{-t} \left(\frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-t}\right) + c_2 \left(\mathrm{e}^{-t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-t} + \frac{c_2 \mathrm{e}^{t}}{2}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = \mathrm{e}^{-t}$$
$$x_2 = \frac{\mathrm{e}^{t}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{a W(t)} \tag{2}$$
$$u_2 = \int \frac{x_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-t} & \frac{\mathrm{e}^{t}}{2} \\ \frac{d}{dt}(\mathrm{e}^{-t}) & \frac{d}{dt}\left(\frac{\mathrm{e}^{t}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-t} & \frac{\mathrm{e}^{t}}{2} \\ -\mathrm{e}^{-t} & \frac{\mathrm{e}^{t}}{2} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{-t})\left(\frac{\mathrm{e}^{t}}{2}\right) - \left(\frac{\mathrm{e}^{t}}{2}\right)(-\mathrm{e}^{-t})$$

Which simplifies to

$$W = \mathrm{e}^{-t}\mathrm{e}^{t}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{2t}t}{2}}{1} dt$$

Which simplifies to

$$u_1 = -\int \frac{e^{2t}t}{2} dt$$

Hence

$$u_1 = -\frac{(2t-1)\, e^{2t}}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-t}t\, e^{t}}{1} dt$$

Which simplifies to

$$u_2 = \int t dt$$

Hence

$$u_2 = \frac{t^2}{2}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = -\frac{(2t-1)\, e^{2t}e^{-t}}{8} + \frac{t^2e^{t}}{4}$$

Which simplifies to

$$x_p(t) = \frac{e^{t}(2t^2-2t+1)}{8}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-t} + \frac{c_2 e^{t}}{2}\right) + \left(\frac{e^{t}(2t^2-2t+1)}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{-t} + \frac{c_2 e^t}{2} + \frac{e^t (2t^2 - 2t + 1)}{8} \tag{1}$$

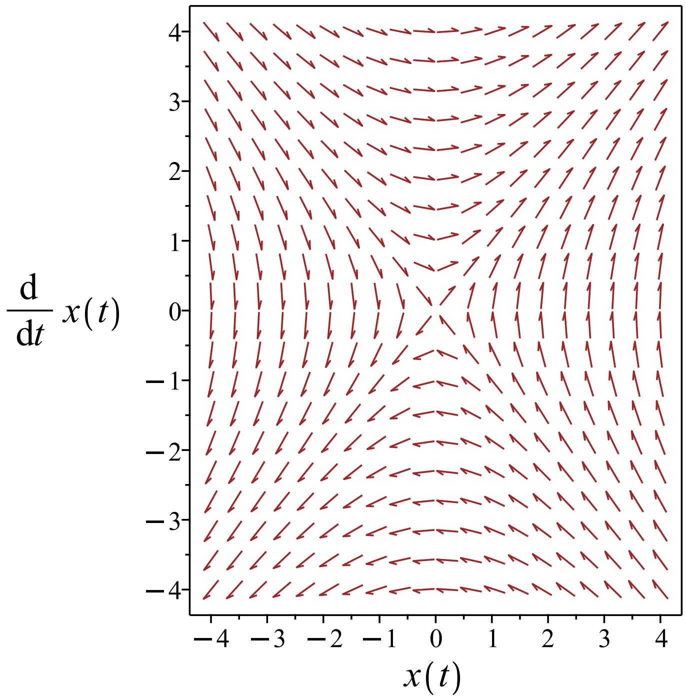


Figure 321: Slope field plot

Verification of solutions

$$x = c_1 e^{-t} + \frac{c_2 e^t}{2} + \frac{e^t (2t^2 - 2t + 1)}{8}$$

Verified OK.

12.2.3 Maple step by step solution

Let's solve

$x'' - x = t\,e^t$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

$r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = t\,\mathrm{e}^{t}\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \left[\begin{array}{cc} \mathrm{e}^{-t} & \mathrm{e}^{t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{t} \end{array}\right]$

  ○ Compute Wronskian

  $W(x_1(t), x_2(t)) = 2$

  ○ Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{\mathrm{e}^{-t}\left(\int \mathrm{e}^{2t} t dt\right)}{2} + \frac{\mathrm{e}^{t}\left(\int t dt\right)}{2}$

  ○ Compute integrals

  $x_p(t) = \frac{\mathrm{e}^{t}(2t^2 - 2t + 1)}{8}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} + \frac{\mathrm{e}^{t}(2t^2 - 2t + 1)}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(x(t),t$2)-x(t)=t*exp(t),x(t), singsol=all)
```

$$x(t) = c_2 \mathrm{e}^{-t} + \frac{\mathrm{e}^{t}(t^2 + 4c_1 - t)}{4}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 35

```
DSolve[x''[t]-x[t]==t*Exp[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{8} e^t \left(2t^2 - 2t + 1 + 8c_1\right) + c_2 e^{-t}$$

12.3 problem 1(c)

12.3.1 Solving as second order linear constant coeff ode 1645
12.3.2 Solving using Kovacic algorithm 1649
12.3.3 Maple step by step solution 1655

Internal problem ID [11489]
Internal file name [`OUTPUT/10471_Thursday_May_18_2023_04_20_27_AM_11978487/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.2 Variation of parameters. Exercises page 124
Problem number: 1(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' - x = \frac{1}{t}}$$

12.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = -1, f(t) = \frac{1}{t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ x &= c_1 e^{(1)t} + c_2 e^{(-1)t}\end{aligned}$$

Or

$$x = c_1 e^t + c_2 e^{-t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 e^t + c_2 e^{-t}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = \mathrm{e}^t$$
$$x_2 = \mathrm{e}^{-t}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^t & \mathrm{e}^{-t} \\ \frac{d}{dt}(\mathrm{e}^t) & \frac{d}{dt}(\mathrm{e}^{-t}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^t & \mathrm{e}^{-t} \\ \mathrm{e}^t & -\mathrm{e}^{-t} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^t)(-\mathrm{e}^{-t}) - (\mathrm{e}^{-t})(\mathrm{e}^t)$$

Which simplifies to

$$W = -2\,\mathrm{e}^{-t}\mathrm{e}^t$$

Which simplifies to

$$W = -2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-t}}{t}}{-2}\,dt$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-t}}{2t}dt$$

Hence

$$u_1 = -\frac{\text{expIntegral}_1(t)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{t}}{t}}{-2}\,dt$$

Which simplifies to

$$u_2 = \int -\frac{e^{t}}{2t}dt$$

Hence

$$u_2 = \frac{\text{expIntegral}_1(-t)}{2}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = -\frac{\text{expIntegral}_1(t)\,e^{t}}{2} + \frac{\text{expIntegral}_1(-t)\,e^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{t} + c_2 e^{-t}\right) + \left(-\frac{\text{expIntegral}_1(t)\,e^{t}}{2} + \frac{\text{expIntegral}_1(-t)\,e^{-t}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-t} - \frac{\mathrm{expIntegral}_1(t)\,\mathrm{e}^{t}}{2} + \frac{\mathrm{expIntegral}_1(-t)\,\mathrm{e}^{-t}}{2} \tag{1}$$

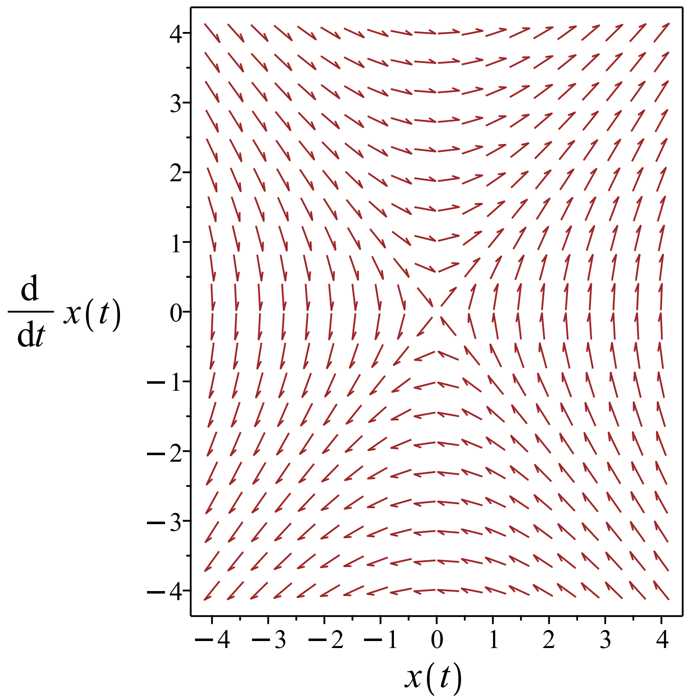


Figure 322: Slope field plot

Verification of solutions

$$x = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-t} - \frac{\mathrm{expIntegral}_1(t)\,\mathrm{e}^{t}}{2} + \frac{\mathrm{expIntegral}_1(-t)\,\mathrm{e}^{-t}}{2}$$

Verified OK.

12.3.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' - x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= -1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = xe^{\int \frac{B}{2A}\,dt}
$$

Then (2) becomes

$$
z''(t) = rz(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\,dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 290: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
x_1 &= z_1 \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\
&= \mathrm{e}^{-t} \int \frac{1}{\mathrm{e}^{-2t}}\, dt \\
&= \mathrm{e}^{-t} \left(\frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-t}\right) + c_2 \left(\mathrm{e}^{-t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-t} + \frac{c_2 \mathrm{e}^{t}}{2}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = \mathrm{e}^{-t}$$
$$x_2 = \frac{\mathrm{e}^{t}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$
$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-t} & \frac{\mathrm{e}^{t}}{2} \\ \frac{d}{dt}\left(\mathrm{e}^{-t}\right) & \frac{d}{dt}\left(\frac{\mathrm{e}^{t}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-t} & \frac{\mathrm{e}^{t}}{2} \\ -\mathrm{e}^{-t} & \frac{\mathrm{e}^{t}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-t}\right)\left(\frac{\mathrm{e}^{t}}{2}\right) - \left(\frac{\mathrm{e}^{t}}{2}\right)\left(-\mathrm{e}^{-t}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-t}\mathrm{e}^{t}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^t}{2t}}{1} dt$$

Which simplifies to

$$u_1 = -\int \frac{e^t}{2t} dt$$

Hence

$$u_1 = \frac{\text{expIntegral}_1(-t)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-t}}{t}}{1} dt$$

Which simplifies to

$$u_2 = \int \frac{e^{-t}}{t} dt$$

Hence

$$u_2 = -\text{expIntegral}_1(t)$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = -\frac{\text{expIntegral}_1(t)\, e^t}{2} + \frac{\text{expIntegral}_1(-t)\, e^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-t} + \frac{c_2 e^t}{2}\right) + \left(-\frac{\text{expIntegral}_1(t)\, e^t}{2} + \frac{\text{expIntegral}_1(-t)\, e^{-t}}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$x = c_1 e^{-t} + \frac{c_2 e^t}{2} - \frac{\text{expIntegral}_1(t)\, e^t}{2} + \frac{\text{expIntegral}_1(-t)\, e^{-t}}{2} \tag{1}$$

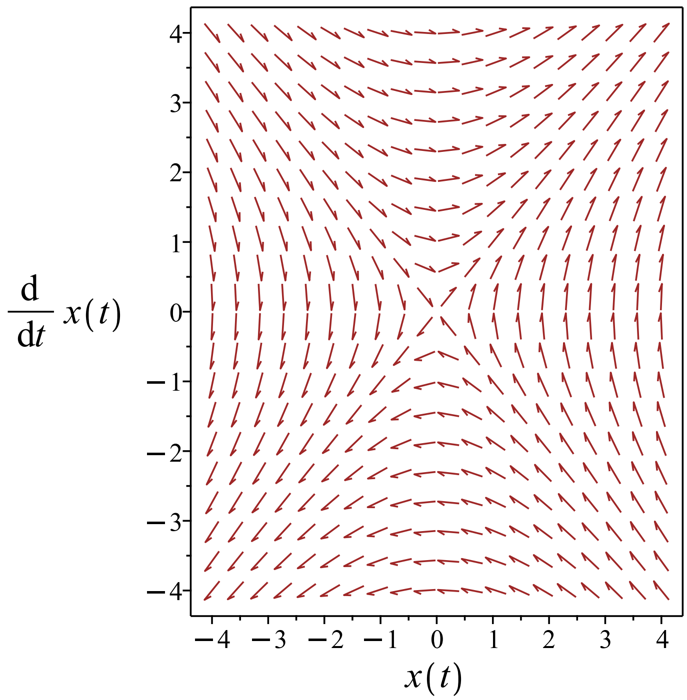


Figure 323: Slope field plot

Verification of solutions

$$x = c_1 e^{-t} + \frac{c_2 e^t}{2} - \frac{\text{expIntegral}_1(t)\, e^t}{2} + \frac{\text{expIntegral}_1(-t)\, e^{-t}}{2}$$

Verified OK.

12.3.3 Maple step by step solution

Let's solve

$x'' - x = \frac{1}{t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

$r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \frac{1}{t}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} & \mathrm{e}^{t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{t} \end{bmatrix}$

  ◦ Compute Wronskian

    $W(x_1(t), x_2(t)) = 2$

  ◦ Substitute functions into equation for $x_p(t)$

    $x_p(t) = -\frac{\mathrm{e}^{-t}\left(\int \frac{\mathrm{e}^{t}}{t}dt\right)}{2} + \frac{\mathrm{e}^{t}\left(\int \frac{\mathrm{e}^{-t}}{t}dt\right)}{2}$

  ◦ Compute integrals

    $x_p(t) = -\frac{\mathrm{Ei}_1(t)\mathrm{e}^{t}}{2} + \frac{\mathrm{Ei}_1(-t)\mathrm{e}^{-t}}{2}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} - \frac{\mathrm{Ei}_1(t)\mathrm{e}^{t}}{2} + \frac{\mathrm{Ei}_1(-t)\mathrm{e}^{-t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(x(t),t$2)-x(t)=1/t,x(t), singsol=all)
```

$$x(t) = \frac{\text{expIntegral}_1\left(-t\right) e^{-t}}{2} + c_2 e^{-t} + e^{t}\left(c_1 - \frac{\text{expIntegral}_1\left(t\right)}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 42

```
DSolve[x''[t]-x[t]==1/t,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2} e^{-t}\left(e^{2t}\,\text{ExpIntegralEi}(-t) - \text{ExpIntegralEi}(t) + 2\left(c_1 e^{2t} + c_2\right)\right)$$

12.4 problem 1(d)

12.4.1 Solving as second order euler ode ode 1658
12.4.2 Solving as second order integrable as is ode 1662
12.4.3 Solving as type second_order_integrable_as_is (not using ABC version) 1663
12.4.4 Solving using Kovacic algorithm 1665
12.4.5 Solving as exact linear second order ode ode 1672

Internal problem ID [11490]
Internal file name [`OUTPUT/10472_Thursday_May_18_2023_04_20_28_AM_30712311/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.2 Variation of parameters. Exercises page 124
Problem number: 1(d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{t^2x'' - 2x = t^3}$$

12.4.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = t^2, B = 0, C = -2, f(t) = t^3$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. Solving for x_h from

$$t^2x'' - 2x = 0$$

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 0rt^{r-1} - 2t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 0\,t^r - 2t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= 2 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$x = c_1x_1 + c_2x_2$$

Where $x_1 = t^{r_1}$ and $x_2 = t^{r_2}$. Hence

$$x = \frac{c_1}{t} + c_2t^2$$

Next, we find the particular solution to the ODE

$$t^2x'' - 2x = t^3$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1x_1 + u_2x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} x_1 &= \frac{1}{t} \\ x_2 &= t^2 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{t} & t^2 \\ \frac{d}{dt}\left(\frac{1}{t}\right) & \frac{d}{dt}(t^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{t} & t^2 \\ -\frac{1}{t^2} & 2t \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{t}\right)(2t) - (t^2)\left(-\frac{1}{t^2}\right)$$

Which simplifies to

$$W = 3$$

Which simplifies to

$$W = 3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{t^5}{3t^2}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{t^3}{3}\,dt$$

Hence

$$u_1 = -\frac{t^4}{12}$$

And Eq. (3) becomes

$$u_2 = \int \frac{t^2}{3t^2}\, dt$$

Which simplifies to

$$u_2 = \int \frac{1}{3} dt$$

Hence

$$u_2 = \frac{t}{3}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \frac{t^3}{4}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \frac{t^3}{4} + \frac{c_1}{t} + c_2 t^2 \end{aligned}$$

Summary

The solution(s) found are the following

$$x = \frac{t^3}{4} + \frac{c_1}{t} + c_2 t^2 \tag{1}$$

Verification of solutions

$$x = \frac{t^3}{4} + \frac{c_1}{t} + c_2 t^2$$

Verified OK.

12.4.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$
\begin{aligned}
\int \left(t^2x'' - 2x\right) dt &= \int t^3 dt \\
t^2x' - 2xt &= \frac{t^4}{4} + c_1
\end{aligned}
$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
x' + p(t)x = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= -\frac{2}{t} \\
q(t) &= \frac{t^4 + 4c_1}{4t^2}
\end{aligned}
$$

Hence the ode is

$$
x' - \frac{2x}{t} = \frac{t^4 + 4c_1}{4t^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{2}{t}dt} \\
&= \frac{1}{t^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(\frac{t^4 + 4c_1}{4t^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{t^2}\right) &= \left(\frac{1}{t^2}\right)\left(\frac{t^4 + 4c_1}{4t^2}\right) \\
\mathrm{d}\left(\frac{x}{t^2}\right) &= \left(\frac{t^4 + 4c_1}{4t^4}\right) \mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{x}{t^2} &= \int \frac{t^4 + 4c_1}{4t^4} \mathrm{d}t \\
\frac{x}{t^2} &= \frac{t}{4} - \frac{c_1}{3t^3} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$x = t^2\left(\frac{t}{4} - \frac{c_1}{3t^3}\right) + c_2 t^2$$

Summary
The solution(s) found are the following

$$x = t^2\left(\frac{t}{4} - \frac{c_1}{3t^3}\right) + c_2 t^2 \tag{1}$$

Verification of solutions

$$x = t^2\left(\frac{t}{4} - \frac{c_1}{3t^3}\right) + c_2 t^2$$

Verified OK.

12.4.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$t^2 x'' - 2x = t^3$$

Integrating both sides of the ODE w.r.t t gives

$$\int \left(t^2 x'' - 2x\right) dt = \int t^3 dt$$
$$t^2 x' - 2xt = \frac{t^4}{4} + c_1$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = -\frac{2}{t}$$
$$q(t) = \frac{t^4 + 4c_1}{4t^2}$$

Hence the ode is

$$x' - \frac{2x}{t} = \frac{t^4 + 4c_1}{4t^2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{2}{t}dt} \\
&= \frac{1}{t^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu x) &= (\mu)\left(\frac{t^4+4c_1}{4t^2}\right) \\
\frac{d}{dt}\left(\frac{x}{t^2}\right) &= \left(\frac{1}{t^2}\right)\left(\frac{t^4+4c_1}{4t^2}\right) \\
d\left(\frac{x}{t^2}\right) &= \left(\frac{t^4+4c_1}{4t^4}\right) dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{x}{t^2} &= \int \frac{t^4+4c_1}{4t^4}\, dt \\
\frac{x}{t^2} &= \frac{t}{4} - \frac{c_1}{3t^3} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$
x = t^2\left(\frac{t}{4} - \frac{c_1}{3t^3}\right) + c_2 t^2
$$

<u>Summary</u>

The solution(s) found are the following

$$
x = t^2\left(\frac{t}{4} - \frac{c_1}{3t^3}\right) + c_2 t^2 \tag{1}
$$

<u>Verification of solutions</u>

$$
x = t^2\left(\frac{t}{4} - \frac{c_1}{3t^3}\right) + c_2 t^2
$$

Verified OK.

12.4.4 Solving using Kovacic algorithm

Writing the ode as

$$t^2x'' - 2x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{2}{t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 292: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{2}{t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2}{t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -1$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{t} + (-)\,(0) \\
&= -\frac{1}{t} \\
&= -\frac{1}{t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{t}\right)(0) + \left(\left(\frac{1}{t^2}\right) + \left(-\frac{1}{t}\right)^2 - \left(\frac{2}{t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int -\frac{1}{t} dt} \\
&= \frac{1}{t}
\end{aligned}
$$

The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \frac{1}{t} \end{aligned}$$

Which simplifies to

$$x_1 = \frac{1}{t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\,dt \\ &= \frac{1}{t} \int \frac{1}{\frac{1}{t^2}}\,dt \\ &= \frac{1}{t}\left(\frac{t^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1\left(\frac{1}{t}\right) + c_2\left(\frac{1}{t}\left(\frac{t^3}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$t^2x''-2x=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h=\frac{c_1}{t}+\frac{c_2t^2}{3}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t)=u_1x_1+u_2x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1=\frac{1}{t}$$

$$x_2=\frac{t^2}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1=-\int\frac{x_2f(t)}{aW(t)} \tag{2}$$

$$u_2=\int\frac{x_1f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W=\begin{vmatrix} \frac{1}{t} & \frac{t^2}{3} \\ \frac{d}{dt}\left(\frac{1}{t}\right) & \frac{d}{dt}\left(\frac{t^2}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{t} & \frac{t^2}{3} \\ -\frac{1}{t^2} & \frac{2t}{3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{t}\right)\left(\frac{2t}{3}\right) - \left(\frac{t^2}{3}\right)\left(-\frac{1}{t^2}\right)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{t^5}{3}}{t^2}\, dt$$

Which simplifies to

$$u_1 = -\int \frac{t^3}{3} dt$$

Hence

$$u_1 = -\frac{t^4}{12}$$

And Eq. (3) becomes

$$u_2 = \int \frac{t^2}{t^2}\, dt$$

Which simplifies to

$$u_2 = \int 1 dt$$

Hence

$$u_2 = t$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \frac{t^3}{4}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(\frac{c_1}{t} + \frac{c_2 t^2}{3}\right) + \left(\frac{t^3}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$x = \frac{c_1}{t} + \frac{c_2 t^2}{3} + \frac{t^3}{4} \tag{1}$$

Verification of solutions

$$x = \frac{c_1}{t} + \frac{c_2 t^2}{3} + \frac{t^3}{4}$$

Verified OK.

12.4.5 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\, x'' + q(t)\, x' + r(t)\, x = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= t^2 \\ q(x) &= 0 \\ r(x) &= -2 \\ s(x) &= t^3 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$2-(0)+(-2)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,x'+(q(t)-p'(t))\,x)'=s(x)$$

Integrating gives

$$p(t)\,x'+(q(t)-p'(t))\,x=\int s(t)\;dt$$

Substituting the above values for p,q,r,s gives

$$t^2x'-2xt=\int t^3\,dt$$

We now have a first order ode to solve which is

$$t^2x'-2xt=\frac{t^4}{4}+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x'+p(t)x=q(t)$$

Where here

$$\begin{aligned}p(t)&=-\frac{2}{t}\\ q(t)&=\frac{t^4+4c_1}{4t^2}\end{aligned}$$

Hence the ode is

$$x'-\frac{2x}{t}=\frac{t^4+4c_1}{4t^2}$$

The integrating factor μ is

$$\begin{aligned}\mu&=\mathrm{e}^{\int -\frac{2}{t}dt}\\ &=\frac{1}{t^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu x)&=(\mu)\left(\frac{t^4+4c_1}{4t^2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{t^2}\right)&=\left(\frac{1}{t^2}\right)\left(\frac{t^4+4c_1}{4t^2}\right)\\ \mathrm{d}\left(\frac{x}{t^2}\right)&=\left(\frac{t^4+4c_1}{4t^4}\right)\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$
\begin{aligned}
\frac{x}{t^2} &= \int \frac{t^4+4c_1}{4t^4}\,\mathrm{d}t \\
\frac{x}{t^2} &= \frac{t}{4}-\frac{c_1}{3t^3}+c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$
x = t^2\left(\frac{t}{4}-\frac{c_1}{3t^3}\right)+c_2t^2
$$

Summary

The solution(s) found are the following

$$
x = t^2\left(\frac{t}{4}-\frac{c_1}{3t^3}\right)+c_2t^2 \tag{1}
$$

Verification of solutions

$$
x = t^2\left(\frac{t}{4}-\frac{c_1}{3t^3}\right)+c_2t^2
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(t^2*diff(x(t),t$2)-2*x(t)=t^3,x(t), singsol=all)
```

$$
x(t) = t^2c_2+\frac{t^3}{4}+\frac{c_1}{t}
$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 25

```
DSolve[t^2*x''[t]-2*x[t]==t^3,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{t^3}{4} + c_2 t^2 + \frac{c_1}{t}$$

12.5 problem 1(e)

12.5.1 Solving as second order linear constant coeff ode 1676
12.5.2 Solving using Kovacic algorithm 1681
12.5.3 Maple step by step solution 1686

Internal problem ID [11491]
Internal file name [`OUTPUT/10473_Thursday_May_18_2023_04_20_30_AM_24728072/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.2 Variation of parameters. Exercises page 124
Problem number: 1(e).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + x = \frac{1}{1+t}}$$

12.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = 1, f(t) = \frac{1}{1+t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^0(c_1 \cos(t) + c_2 \sin(t))$$

Or

$$x = c_1 \cos(t) + c_2 \sin(t)$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \cos(t) + c_2 \sin(t)$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} x_1 &= \cos(t) \\ x_2 &= \sin(t) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ \frac{d}{dt}(\cos(t)) & \frac{d}{dt}(\sin(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}$$

Therefore

$$W = (\cos(t))(\cos(t)) - (\sin(t))(-\sin(t))$$

Which simplifies to

$$W = \sin(t)^2 + \cos(t)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(t)}{1+t}}{1} dt$$

Which simplifies to

$$u_1 = -\int \frac{\sin(t)}{1+t} dt$$

Hence

$$u_1 = -\operatorname{Si}(1+t)\cos(1) + \operatorname{Ci}(1+t)\sin(1)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos(t)}{1+t}}{1} dt$$

Which simplifies to

$$u_2 = \int \frac{\cos(t)}{1+t} dt$$

Hence

$$u_2 = \operatorname{Si}(1+t)\sin(1) + \operatorname{Ci}(1+t)\cos(1)$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} x_p(t) = {} & (-\operatorname{Si}(1+t)\cos(1) + \operatorname{Ci}(1+t)\sin(1))\cos(t) \\ & + (\operatorname{Si}(1+t)\sin(1) + \operatorname{Ci}(1+t)\cos(1))\sin(t) \end{aligned}$$

Which simplifies to

$$x_p(t) = (-\cos(1)\cos(t) + \sin(t)\sin(1))\,\mathrm{Si}(1+t) \\ + (\sin(t)\cos(1) + \sin(1)\cos(t))\,\mathrm{Ci}(1+t)$$

Therefore the general solution is

$$x = x_h + x_p$$
$$= (c_1\cos(t) + c_2\sin(t)) + ((-\cos(1)\cos(t) + \sin(t)\sin(1))\,\mathrm{Si}(1+t) \\ + (\sin(t)\cos(1) + \sin(1)\cos(t))\,\mathrm{Ci}(1+t))$$

Summary

The solution(s) found are the following

$$x = c_1\cos(t) + c_2\sin(t) + (-\cos(1)\cos(t) + \sin(t)\sin(1))\,\mathrm{Si}(1+t) \\ + (\sin(t)\cos(1) + \sin(1)\cos(t))\,\mathrm{Ci}(1+t) \tag{1}$$

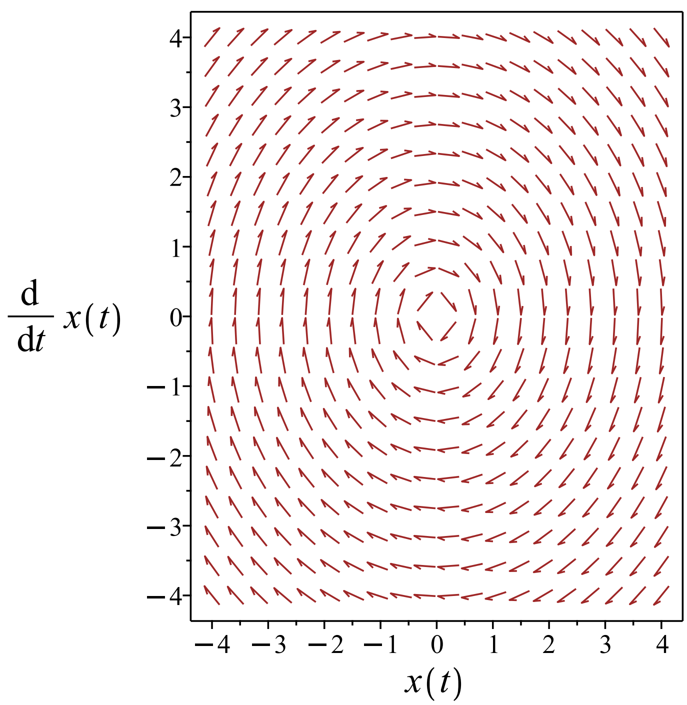


Figure 324: Slope field plot

Verification of solutions

$$x = c_1\cos(t) + c_2\sin(t) + (-\cos(1)\cos(t) + \sin(t)\sin(1))\,\mathrm{Si}(1+t) \\ + (\sin(t)\cos(1) + \sin(1)\cos(t))\,\mathrm{Ci}(1+t)$$

Verified OK.

12.5.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
x'' + x &= 0 \tag{1} \\
Ax'' + Bx' + Cx &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= 0 \tag{3} \\
C &= 1
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = x e^{\int \frac{B}{2A}\, dt}
$$

Then (2) becomes

$$
z''(t) = r z(t) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= -1 \\
t &= 1
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(t) = -z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 293: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
x_1 &= z_1 \\
&= \cos(t)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \cos(t)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A} dt}}{x_1^2} dt
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2} dt \\
&= \cos(t) \int \frac{1}{\cos(t)^2} dt \\
&= \cos(t) (\tan(t))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(\cos(t)) + c_2(\cos(t) (\tan(t)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \cos(t) + c_2 \sin(t)
$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = \cos(t)$$
$$x_2 = \sin(t)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ \frac{d}{dt}(\cos(t)) & \frac{d}{dt}(\sin(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}$$

Therefore

$$W = (\cos(t))(\cos(t)) - (\sin(t))(-\sin(t))$$

Which simplifies to

$$W = \sin(t)^2 + \cos(t)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(t)}{1+t}}{1} dt$$

Which simplifies to

$$u_1 = -\int \frac{\sin(t)}{1+t} dt$$

Hence

$$u_1 = -\operatorname{Si}(1+t)\cos(1) + \operatorname{Ci}(1+t)\sin(1)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos(t)}{1+t}}{1} dt$$

Which simplifies to

$$u_2 = \int \frac{\cos(t)}{1+t} dt$$

Hence

$$u_2 = \operatorname{Si}(1+t)\sin(1) + \operatorname{Ci}(1+t)\cos(1)$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} x_p(t) = {} & (-\operatorname{Si}(1+t)\cos(1) + \operatorname{Ci}(1+t)\sin(1))\cos(t) \\ & + (\operatorname{Si}(1+t)\sin(1) + \operatorname{Ci}(1+t)\cos(1))\sin(t) \end{aligned}$$

Which simplifies to

$$\begin{aligned} x_p(t) = {} & (-\cos(1)\cos(t) + \sin(t)\sin(1))\operatorname{Si}(1+t) \\ & + (\sin(t)\cos(1) + \sin(1)\cos(t))\operatorname{Ci}(1+t) \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1\cos(t) + c_2\sin(t)) + ((-\cos(1)\cos(t) + \sin(t)\sin(1))\operatorname{Si}(1+t) \\ &\quad + (\sin(t)\cos(1) + \sin(1)\cos(t))\operatorname{Ci}(1+t)) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
x = c_1 \cos(t) + c_2 \sin(t) + (-\cos(1)\cos(t) + \sin(t)\sin(1))\,\mathrm{Si}(1+t) \\
+ (\sin(t)\cos(1) + \sin(1)\cos(t))\,\mathrm{Ci}(1+t)
\end{aligned} \tag{1}
$$

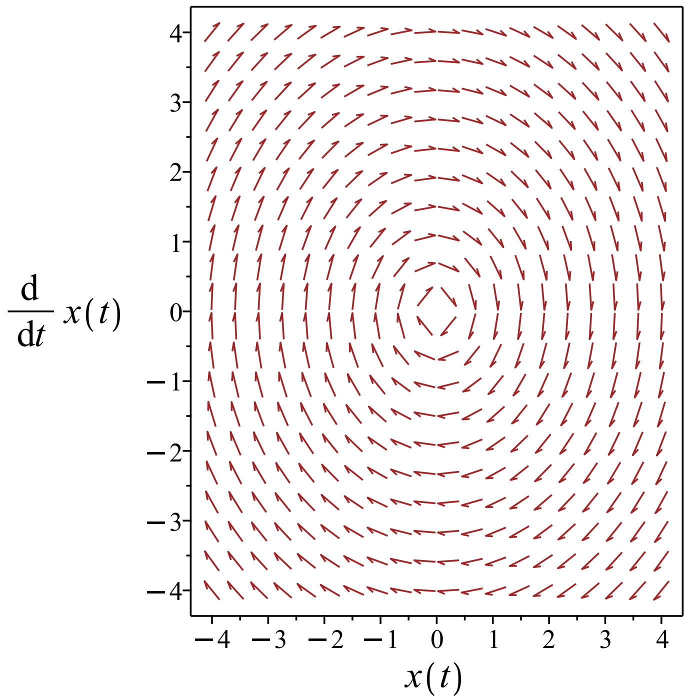


Figure 325: Slope field plot

Verification of solutions

$$
\begin{aligned}
x = c_1 \cos(t) + c_2 \sin(t) + (-\cos(1)\cos(t) + \sin(t)\sin(1))\,\mathrm{Si}(1+t) \\
+ (\sin(t)\cos(1) + \sin(1)\cos(t))\,\mathrm{Ci}(1+t)
\end{aligned}
$$

Verified OK.

12.5.3 Maple step by step solution

Let's solve

$x'' + x = \frac{1}{1+t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

$r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(t) + c_2 \sin(t) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t) \left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t) \left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = \frac{1}{1+t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 1$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -\cos(t) \left(\int \frac{\sin(t)}{1+t} dt\right) + \sin(t) \left(\int \frac{\cos(t)}{1+t} dt\right)$

  - Compute integrals

    $x_p(t) = (-\cos(1)\cos(t) + \sin(t)\sin(1))\,\mathrm{Si}(1+t) + (\sin(t)\cos(1) + \sin(1)\cos(t))\,\mathrm{Ci}(1+t)$

- Substitute particular solution into general solution to ODE

  $x = c_1 \cos(t) + c_2 \sin(t) + (-\cos(1)\cos(t) + \sin(t)\sin(1))\,\mathrm{Si}(1+t) + (\sin(t)\cos(1) + \sin(1)\,\mathrm{c}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(x(t),t$2)+x(t)=1/(1+t),x(t), singsol=all)
```

$$x(t) = \sin(t)\, c_2 + \cos(t)\, c_1 - \mathrm{Si}(t+1)\cos(t+1) + \mathrm{Ci}(t+1)\sin(t+1)$$

✓ Solution by Mathematica

Time used: 0.125 (sec). Leaf size: 35

```
DSolve[x''[t]+x[t]==1/(1+t),x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \mathrm{CosIntegral}(t+1)\sin(t+1) - \mathrm{Si}(t+1)\cos(t+1) + c_1\cos(t) + c_2\sin(t)$$

12.6 problem 1(f)

12.6.1 Solving as second order linear constant coeff ode 1689
12.6.2 Solving as linear second order ode solved by an integrating factor ode 1693
12.6.3 Solving using Kovacic algorithm 1695
12.6.4 Maple step by step solution 1701

Internal problem ID [11492]
Internal file name [`OUTPUT/10474_Thursday_May_18_2023_04_20_32_AM_38498164/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.2 Variation of parameters. Exercises page 124
Problem number: 1(f).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' - 2x' + x = \frac{e^{t}}{2t}}$$

12.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = -2, C = 1, f(t) = \frac{e^{t}}{2t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - 2x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - 2\lambda\, e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-2)^2 - (4)\,(1)\,(1)}\\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$x = c_1 e^{t} + c_2 t\, e^{t} \tag{1}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 e^{t} + c_2 t\, e^{t}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}x_1 &= e^{t}\\ x_2 &= t\, e^{t}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{t} & t\,e^{t} \\ \frac{d}{dt}\left(e^{t}\right) & \frac{d}{dt}\left(t\,e^{t}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{t} & t\,e^{t} \\ e^{t} & t\,e^{t}+e^{t} \end{vmatrix}$$

Therefore

$$W = \left(e^{t}\right)\left(t\,e^{t}+e^{t}\right) - \left(t\,e^{t}\right)\left(e^{t}\right)$$

Which simplifies to

$$W = e^{2t}$$

Which simplifies to

$$W = e^{2t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{2t}}{2}}{e^{2t}}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{1}{2} dt$$

Hence

$$u_1 = -\frac{t}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{2t}}{2t}}{e^{2t}} dt$$

Which simplifies to

$$u_2 = \int \frac{1}{2t} dt$$

Hence

$$u_2 = \frac{\ln(t)}{2}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = -\frac{t\,e^t}{2} + \frac{\ln(t)\,t\,e^t}{2}$$

Which simplifies to

$$x_p(t) = \frac{t\,e^t(-1+\ln(t))}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^t + c_2 t\,e^t\right) + \left(\frac{t\,e^t(-1+\ln(t))}{2}\right) \end{aligned}$$

Which simplifies to

$$x = e^t(c_2 t + c_1) + \frac{t\,e^t(-1+\ln(t))}{2}$$

Summary

The solution(s) found are the following

$$x = e^t(c_2 t + c_1) + \frac{t\,e^t(-1+\ln(t))}{2} \tag{1}$$

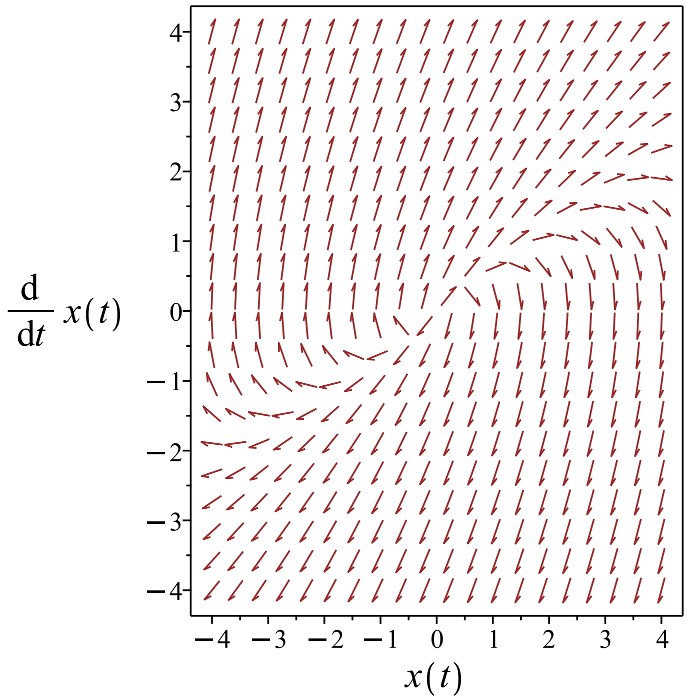


Figure 326: Slope field plot

Verification of solutions

$$x = e^t(c_2t + c_1) + \frac{t\,e^t(-1 + \ln(t))}{2}$$

Verified OK.

12.6.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$x'' + p(t)\,x' + \frac{\left(p(t)^2 + p'(t)\right)x}{2} = f(t)$$

Where $p(t) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= e^{-t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)x)'' &= \frac{\mathrm{e}^{-t}\mathrm{e}^{t}}{2t} \\
\left(\mathrm{e}^{-t}x\right)'' &= \frac{\mathrm{e}^{-t}\mathrm{e}^{t}}{2t}
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{-t}x\right)' = \frac{\ln(t)}{2} + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{-t}x\right) = \frac{t(2c_1 + \ln(t) - 1)}{2} + c_2
$$

Hence the solution is

$$
x = \frac{\frac{t(2c_1+\ln(t)-1)}{2} + c_2}{\mathrm{e}^{-t}}
$$

Or

$$
x = c_1 t\,\mathrm{e}^{t} + \frac{t\,\mathrm{e}^{t}\ln(t)}{2} + c_2\mathrm{e}^{t} - \frac{t\,\mathrm{e}^{t}}{2}
$$

Summary

The solution(s) found are the following

$$
x = c_1 t\,\mathrm{e}^{t} + \frac{t\,\mathrm{e}^{t}\ln(t)}{2} + c_2\mathrm{e}^{t} - \frac{t\,\mathrm{e}^{t}}{2} \tag{1}
$$

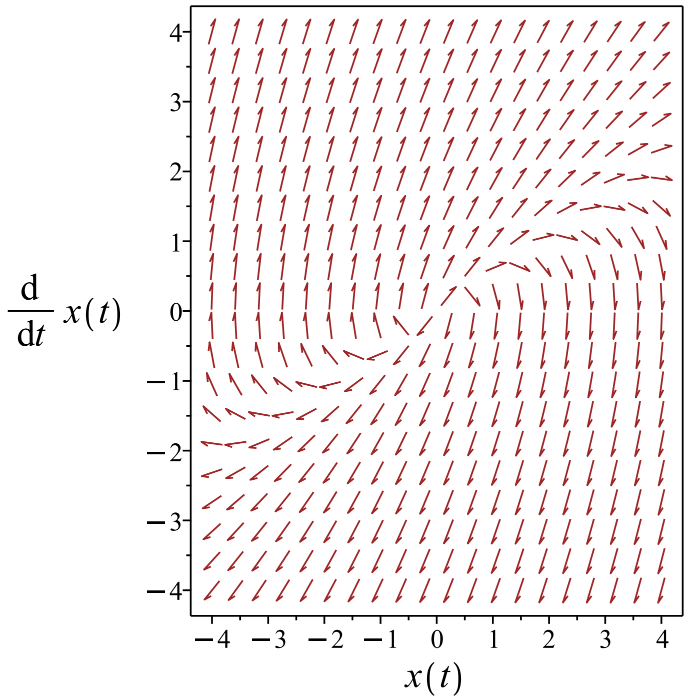


Figure 327: Slope field plot

Verification of solutions

$$x = c_1 t\, e^t + \frac{t\, e^t \ln(t)}{2} + c_2 e^t - \frac{t\, e^t}{2}$$

Verified OK.

12.6.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 2x' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 295: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dt} \\
&= z_1 e^{t} \\
&= z_1 \left(\mathrm{e}^{t}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{t}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-2}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{2t}}{(x_1)^2}\,dt \\
&= x_1(t)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{t}\right) + c_2 \left(\mathrm{e}^{t} (t)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' - 2x' + x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{t} + c_2 t\,\mathrm{e}^{t}
$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}
$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
x_1 &= \mathrm{e}^{t} \\
x_2 &= t\,\mathrm{e}^{t}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}
$$

$$
u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}
$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^t & t\,e^t \\ \frac{d}{dt}(e^t) & \frac{d}{dt}(t\,e^t) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^t & t\,e^t \\ e^t & t\,e^t + e^t \end{vmatrix}$$

Therefore

$$W = (e^t)\,(t\,e^t + e^t) - (t\,e^t)\,(e^t)$$

Which simplifies to

$$W = e^{2t}$$

Which simplifies to

$$W = e^{2t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{2t}}{2}}{e^{2t}}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{1}{2}dt$$

Hence

$$u_1 = -\frac{t}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{2t}}{2t}}{e^{2t}}\,dt$$

Which simplifies to

$$u_2 = \int \frac{1}{2t} dt$$

Hence

$$u_2 = \frac{\ln(t)}{2}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = -\frac{t\,e^t}{2} + \frac{t\,e^t \ln(t)}{2}$$

Which simplifies to

$$x_p(t) = \frac{t\,e^t(-1 + \ln(t))}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^t + c_2 t\,e^t\right) + \left(\frac{t\,e^t(-1 + \ln(t))}{2}\right) \end{aligned}$$

Which simplifies to

$$x = e^t(c_2 t + c_1) + \frac{t\,e^t(-1 + \ln(t))}{2}$$

Summary

The solution(s) found are the following

$$x = e^t(c_2 t + c_1) + \frac{t\,e^t(-1 + \ln(t))}{2} \tag{1}$$

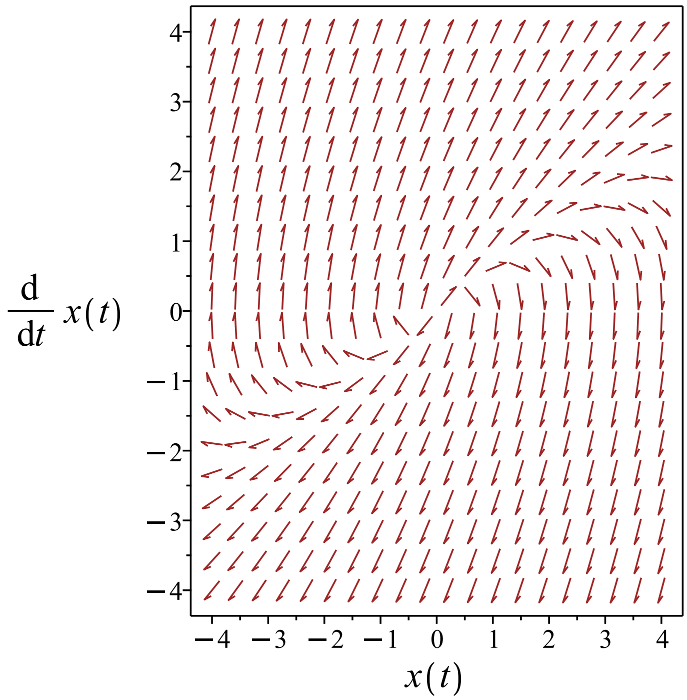


Figure 328: Slope field plot

Verification of solutions

$$x = e^t(c_2t + c_1) + \frac{t\,e^t(-1 + \ln(t))}{2}$$

Verified OK.

12.6.4 Maple step by step solution

Let's solve

$x'' - 2x' + x = \frac{e^t}{2t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

$r=1$

- 1st solution of the homogeneous ODE

  $x_1(t)=\mathrm{e}^t$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t)=t\,\mathrm{e}^t$

- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)+x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x=c_1\mathrm{e}^t+c_2t\,\mathrm{e}^t+x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t)=-x_1(t)\left(\int\frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right)+x_2(t)\left(\int\frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right),f(t)=\frac{\mathrm{e}^t}{2t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t),x_2(t))=\begin{bmatrix}\mathrm{e}^t & t\,\mathrm{e}^t\\ \mathrm{e}^t & t\,\mathrm{e}^t+\mathrm{e}^t\end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t),x_2(t))=\mathrm{e}^{2t}$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t)=-\frac{\mathrm{e}^t\left(\int 1dt-\left(\int\frac{1}{t}dt\right)t\right)}{2}$

  - Compute integrals

    $x_p(t)=\frac{t\,\mathrm{e}^t(-1+\ln(t))}{2}$

- Substitute particular solution into general solution to ODE

  $x=c_1\mathrm{e}^t+c_2t\,\mathrm{e}^t+\frac{t\,\mathrm{e}^t(-1+\ln(t))}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 23

```
dsolve(diff(x(t),t$2)-2*diff(x(t),t)+x(t)=1/(2*t)*exp(t),x(t), singsol=all)
```

$$x(t) = \frac{(t\ln(t) + t(2c_1 - 1) + 2c_2)\,e^t}{2}$$

✓ Solution by Mathematica
Time used: 0.036 (sec). Leaf size: 29

```
DSolve[x''[t]-2*x'[t]+x[t]==1/(2*t)*Exp[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2}e^t(t\log(t) + (-1 + 2c_2)t + 2c_1)$$

12.7 problem 2

12.7.1 Solving as second order ode missing y ode 1704
12.7.2 Solving as second order ode non constant coeff transformation on B ode 1706
12.7.3 Solving using Kovacic algorithm 1710
12.7.4 Maple step by step solution 1718

Internal problem ID [11493]
Internal file name [`OUTPUT/10475_Thursday_May_18_2023_04_20_34_AM_3578107/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.2 Variation of parameters. Exercises page 124
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x'' + \frac{x'}{t} = a}$$

12.7.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable x. Let

$$p(t) = x'$$

Then

$$p'(t) = x''$$

Hence the ode becomes

$$p'(t) + \frac{p(t)}{t} - a = 0$$

Which is now solve for $p(t)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(t) + p(t)p(t) = q(t)$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{t} \\ q(t) &= a \end{aligned}$$

Hence the ode is

$$p'(t) + \frac{p(t)}{t} = a$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{t} dt} \\ &= t \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dt}(\mu p) &= (\mu)(a) \\ \frac{d}{dt}(tp) &= (t)(a) \\ d(tp) &= (ta)\, dt \end{aligned}$$

Integrating gives

$$\begin{aligned} tp &= \int ta \, dt \\ tp &= \frac{t^2 a}{2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$p(t) = \frac{ta}{2} + \frac{c_1}{t}$$

Since $p = x'$ then the new first order ode to solve is

$$x' = \frac{ta}{2} + \frac{c_1}{t}$$

Integrating both sides gives

$$
\begin{aligned}
x &= \int \frac{t^2 a + 2c_1}{2t}\, \mathrm{d}t \\
&= \frac{t^2 a}{4} + c_1 \ln(t) + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = \frac{t^2 a}{4} + c_1 \ln(t) + c_2 \tag{1}
$$

Verification of solutions

$$
x = \frac{t^2 a}{4} + c_1 \ln(t) + c_2
$$

Verified OK.

12.7.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ax'' + Bx' + Cx = F(t)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
x = Bv
$$

This results in

$$
\begin{aligned}
x' &= B'v + v'B \\
x'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0
\end{aligned} \tag{1}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)\, u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $x = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The ODE is now normalized to

$$x' + tx'' = a$$

Where now

$$\begin{aligned} A &= t \\ B &= 1 \\ C &= 0 \\ F &= a \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (t)\,(0) + (1)\,(0) + (0)\,(1) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$tv'' + (1)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$tu'(t) + u(t) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{u}{t} \end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{t}\, dt \\ \int \frac{1}{u}\, du &= \int -\frac{1}{t}\, dt \\ \ln(u) &= -\ln(t) + c_1 \\ u &= e^{-\ln(t) + c_1} \\ &= \frac{c_1}{t} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{t} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(t) &= \int \frac{c_1}{t}\, \mathrm{d}t \\ &= c_1 \ln(t) + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} x_h(t) &= Bv \\ &= (1)\,(c_1 \ln(t) + c_2) \\ &= c_1 \ln(t) + c_2 \end{aligned}$$

And now the particular solution $x_p(t)$ will be found. The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} x_1 &= 1 \\ x_2 &= \ln(t) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(t) \\ \frac{d}{dt}(1) & \frac{d}{dt}(\ln(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(t) \\ 0 & \frac{1}{t} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{t}\right) - (\ln(t))(0)$$

Which simplifies to

$$W = \frac{1}{t}$$

Which simplifies to

$$W = \frac{1}{t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(t)\, a}{1}\, dt$$

Which simplifies to

$$u_1 = -\int \ln(t)\, a dt$$

Hence

$$u_1 = -a(t\ln(t) - t)$$

And Eq. (3) becomes

$$u_2 = \int \frac{a}{1}\, dt$$

Which simplifies to

$$u_2 = \int a dt$$

Hence

$$u_2 = ta$$

Which simplifies to

$$
\begin{aligned}
u_1 &= -at(-1+\ln(t)) \\
u_2 &= ta
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
x_p(t) = -at(-1+\ln(t)) + ta\ln(t)
$$

Which simplifies to

$$
x_p(t) = ta
$$

Hence the complete solution is

$$
\begin{aligned}
x(t) &= x_h + x_p \\
&= (c_1\ln(t) + c_2) + (ta) \\
&= c_1\ln(t) + c_2 + ta
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1\ln(t) + c_2 + ta \tag{1}
$$

Verification of solutions

$$
x = c_1\ln(t) + c_2 + ta
$$

Verified OK.

12.7.3 Solving using Kovacic algorithm

Writing the ode as

$$
x' + tx'' = 0 \tag{1}
$$
$$
Ax'' + Bx' + Cx = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= t \\
B &= 1 \\
C &= 0
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 297: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition

of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2t} + (-)\,(0) \\
&= \frac{1}{2t} \\
&= \frac{1}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2t}\right)^2 - \left(-\frac{1}{4t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \frac{1}{2t} dt} \\
&= \sqrt{t}
\end{aligned}
$$

The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{t}\,dt} \\
&= z_1 e^{-\frac{\ln(t)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = 1
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{t}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-\ln(t)}}{(x_1)^2}\,dt \\
&= x_1(\ln(t))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(1) + c_2(1(\ln(t)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$
x' + tx'' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 + c_2 \ln(t)$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} x_1 &= 1 \\ x_2 &= \ln(t) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(t) \\ \frac{d}{dt}(1) & \frac{d}{dt}(\ln(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(t) \\ 0 & \frac{1}{t} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{t}\right) - (\ln(t))(0)$$

Which simplifies to

$$W = \frac{1}{t}$$

Which simplifies to

$$W = \frac{1}{t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(t)\, a}{1}\, dt$$

Which simplifies to

$$u_1 = -\int \ln(t)\, a dt$$

Hence

$$u_1 = -a(t\ln(t) - t)$$

And Eq. (3) becomes

$$u_2 = \int \frac{a}{1}\, dt$$

Which simplifies to

$$u_2 = \int a dt$$

Hence

$$u_2 = ta$$

Which simplifies to

$$\begin{aligned} u_1 &= -at(-1 + \ln(t)) \\ u_2 &= ta \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = -at(-1 + \ln(t)) + ta\ln(t)$$

Which simplifies to

$$x_p(t) = ta$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1 + c_2 \ln(t)) + (ta) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 + c_2 \ln(t) + ta \tag{1}$$

Verification of solutions

$$x = c_1 + c_2 \ln(t) + ta$$

Verified OK.

12.7.4 Maple step by step solution

Let's solve

$x' + tx'' = a$

- Highest derivative means the order of the ODE is 2

  x''

- Make substitution $u = x'$ to reduce order of ODE

  $u(t) + tu'(t) = a$

- Integrate both sides with respect to t

  $\int (u(t) + tu'(t))\, dt = \int a dt + c_1$

- Evaluate integral

  $tu(t) = ta + c_1$

- Solve for $u(t)$

  $u(t) = \frac{ta+c_1}{t}$

- Solve 1st ODE for $u(t)$

  $u(t) = \frac{ta+c_1}{t}$

- Make substitution $u = x'$

$x' = \frac{ta+c_1}{t}$

- Integrate both sides to solve for x

  $\int x' dt = \int \frac{ta+c_1}{t} dt + c_2$

- Compute integrals

  $x = c_1 \ln(t) + c_2 + ta$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(-a*_a+_b(_a))/_a, _b(_a)`   *** Suble
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(x(t),t$2)+1/t*diff(x(t),t)=a,x(t), singsol=all)
```

$$x(t) = \frac{a\,t^2}{4} + c_1 \ln(t) + c_2$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 21

```
DSolve[x''[t]+1/t*x'[t]==a,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{at^2}{4} + c_1 \log(t) + c_2$$

12.8 problem 3

12.8.1 Solving as second order euler ode ode 1721
12.8.2 Solving as second order change of variable on x method 2 ode . 1724
12.8.3 Solving as second order change of variable on x method 1 ode . 1729
12.8.4 Solving as second order change of variable on y method 2 ode . 1734
12.8.5 Solving as second order ode non constant coeff transformation on B ode 1738
12.8.6 Solving using Kovacic algorithm 1743

Internal problem ID [11494]
Internal file name [`OUTPUT/10476_Thursday_May_18_2023_04_20_35_AM_42897556/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.2 Variation of parameters. Exercises page 124
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{t^2x'' - 3tx' + 3x = 4t^7}$$

12.8.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = t^2, B = -3t, C = 3, f(t) = 4t^7$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. Solving for x_h from

$$t^2x'' - 3tx' + 3x = 0$$

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} - 3trt^{r-1} + 3t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r - 3r\,t^r + 3t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) - 3r + 3 = 0$$

Or

$$r^2 - 4r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$x = c_1x_1 + c_2x_2$$

Where $x_1 = t^{r_1}$ and $x_2 = t^{r_2}$. Hence

$$x = c_2t^3 + c_1t$$

Next, we find the particular solution to the ODE

$$t^2x'' - 3tx' + 3x = 4t^7$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = t$$
$$x_2 = t^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} t & t^3 \\ \frac{d}{dt}(t) & \frac{d}{dt}(t^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix}$$

Therefore

$$W = (t)\left(3t^2\right) - \left(t^3\right)(1)$$

Which simplifies to

$$W = 2t^3$$

Which simplifies to

$$W = 2t^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4t^{10}}{2t^5}\,dt$$

Which simplifies to

$$u_1 = -\int 2t^5 dt$$

Hence

$$u_1 = -\frac{t^6}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4t^8}{2t^5}\,dt$$

Which simplifies to

$$u_2 = \int 2t^3 dt$$

Hence

$$u_2 = \frac{t^4}{2}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \frac{t^7}{6}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \frac{1}{6}t^7 + c_2 t^3 + c_1 t \end{aligned}$$

Summary
The solution(s) found are the following

$$x = \frac{1}{6}t^7 + c_2 t^3 + c_1 t \tag{1}$$

Verification of solutions

$$x = \frac{1}{6}t^7 + c_2 t^3 + c_1 t$$

Verified OK.

12.8.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$t^2x'' - 3tx' + 3x = 0$$

In normal form the ode

$$t^2x'' - 3tx' + 3x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = 0 \tag{2}$$

Where

$$p(t) = -\frac{3}{t}$$
$$q(t) = \frac{3}{t^2}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1 x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(t)dt\right)} dt \\
&= \int e^{-\left(\int -\frac{3}{t}dt\right)} dt \\
&= \int e^{3\ln(t)}\, dt \\
&= \int t^3 dt \\
&= \frac{t^4}{4}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{\frac{3}{t^2}}{t^6} \\
&= \frac{3}{t^8}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}x(\tau) + q_1 x(\tau) &= 0 \\
\frac{d^2}{d\tau^2}x(\tau) + \frac{3x(\tau)}{t^8} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{3}{t^8} = \frac{3}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}x(\tau) + \frac{3x(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $x(\tau)$. The ode can be written as

$$16\left(\frac{d^2}{d\tau^2}x(\tau)\right)\tau^2 + 3x(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $x(\tau) = \tau^r$, then $x' = r\tau^{r-1}$ and $x'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 3\tau^r = 0$$

Simplifying gives

$$16r(r-1)\,\tau^r + 0\,\tau^r + 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$16r(r-1) + 0 + 3 = 0$$

Or

$$16r^2 - 16r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{4}$$
$$r_2 = \frac{3}{4}$$

Since the roots are real and distinct, then the general solution is

$$x(\tau) = c_1x_1 + c_2x_2$$

Where $x_1 = \tau^{r_1}$ and $x_2 = \tau^{r_2}$. Hence

$$x(\tau) = c_1\tau^{\frac{1}{4}} + c_2\tau^{\frac{3}{4}}$$

The above solution is now transformed back to x using (6) which results in

$$x = \frac{\sqrt{2}\,(t^4)^{\frac{1}{4}}\left(c_2\sqrt{t^4} + 2c_1\right)}{4}$$

Therefore the homogeneous solution x_h is

$$x_h = \frac{\sqrt{2}\,(t^4)^{\frac{1}{4}}\left(c_2\sqrt{t^4} + 2c_1\right)}{4}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = \left(t^4\right)^{\frac{1}{4}}$$
$$x_2 = \left(t^4\right)^{\frac{3}{4}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$
$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(t^4\right)^{\frac{1}{4}} & \left(t^4\right)^{\frac{3}{4}} \\ \frac{d}{dt}\left(\left(t^4\right)^{\frac{1}{4}}\right) & \frac{d}{dt}\left(\left(t^4\right)^{\frac{3}{4}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(t^4\right)^{\frac{1}{4}} & \left(t^4\right)^{\frac{3}{4}} \\ \frac{t^3}{\left(t^4\right)^{\frac{3}{4}}} & \frac{3t^3}{\left(t^4\right)^{\frac{1}{4}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(t^4\right)^{\frac{1}{4}}\right)\left(\frac{3t^3}{\left(t^4\right)^{\frac{1}{4}}}\right) - \left(\left(t^4\right)^{\frac{3}{4}}\right)\left(\frac{t^3}{\left(t^4\right)^{\frac{3}{4}}}\right)$$

Which simplifies to

$$W = 2t^3$$

Which simplifies to

$$W = 2t^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4(t^4)^{\frac{3}{4}} t^7}{2t^5} \, dt$$

Which simplifies to

$$u_1 = -\int 2\left(t^4\right)^{\frac{3}{4}} t^2 dt$$

Hence

$$u_1 = -\frac{t^3 (t^4)^{\frac{3}{4}}}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4(t^4)^{\frac{1}{4}} t^7}{2t^5} \, dt$$

Which simplifies to

$$u_2 = \int 2\left(t^4\right)^{\frac{1}{4}} t^2 dt$$

Hence

$$u_2 = \frac{t^3 (t^4)^{\frac{1}{4}}}{2}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \frac{t^7}{6}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(\frac{\sqrt{2}\,(t^4)^{\frac{1}{4}} \left(c_2 \sqrt{t^4} + 2c_1\right)}{4}\right) + \left(\frac{t^7}{6}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$x = \frac{\sqrt{2}\,(t^4)^{\frac{1}{4}}\left(c_2\sqrt{t^4}+2c_1\right)}{4}+\frac{t^7}{6} \tag{1}$$

Verification of solutions

$$x = \frac{\sqrt{2}\,(t^4)^{\frac{1}{4}}\left(c_2\sqrt{t^4}+2c_1\right)}{4}+\frac{t^7}{6}$$

Verified OK.

12.8.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t)+Bx'(t)+Cx(t)=f(t)$$

Where $A=t^2, B=-3t, C=3, f(t)=4t^7$. Let the solution be

$$x=x_h+x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. Solving for x_h from

$$t^2x''-3tx'+3x=0$$

In normal form the ode

$$t^2x''-3tx'+3x=0 \tag{1}$$

Becomes

$$x''+p(t)\,x'+q(t)\,x=0 \tag{2}$$

Where

$$p(t)=-\frac{3}{t}$$
$$q(t)=\frac{3}{t^2}$$

Applying change of variables $\tau=g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}x(\tau)+p_1\left(\frac{d}{d\tau}x(\tau)\right)+q_1x(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{3}\,\sqrt{\frac{1}{t^2}}}{c} \\ \tau'' &= -\frac{\sqrt{3}}{c\sqrt{\frac{1}{t^2}}\,t^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\ &= \frac{-\frac{\sqrt{3}}{c\sqrt{\frac{1}{t^2}}\,t^3} - \frac{3}{t}\frac{\sqrt{3}\,\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{3}\,\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\ &= -\frac{4c\sqrt{3}}{3} \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} x(\tau)'' + p_1 x(\tau)' + q_1 x(\tau) &= 0 \\ \frac{d^2}{d\tau^2}x(\tau) - \frac{4c\sqrt{3}\,\left(\frac{d}{d\tau}x(\tau)\right)}{3} + c^2 x(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $x(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$x(\tau) = \mathrm{e}^{\frac{2\sqrt{3}\,c\tau}{3}}\left(c_1 \cosh\left(\frac{\sqrt{3}\,c\tau}{3}\right) + ic_2 \sinh\left(\frac{\sqrt{3}\,c\tau}{3}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dt \\
&= \frac{\int \sqrt{3}\,\sqrt{\frac{1}{t^2}}dt}{c} \\
&= \frac{\sqrt{3}\,\sqrt{\frac{1}{t^2}}\,t\ln(t)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$x = \frac{t((ic_2 + c_1)\,t^2 - ic_2 + c_1)}{2}$$

Now the particular solution to this ODE is found

$$t^2x'' - 3tx' + 3x = 4t^7$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1x_1 + u_2x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
x_1 &= \left(t^4\right)^{\frac{1}{4}} \\
x_2 &= \left(t^4\right)^{\frac{3}{4}}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (t^4)^{\frac{1}{4}} & (t^4)^{\frac{3}{4}} \\ \frac{d}{dt}\left((t^4)^{\frac{1}{4}}\right) & \frac{d}{dt}\left((t^4)^{\frac{3}{4}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (t^4)^{\frac{1}{4}} & (t^4)^{\frac{3}{4}} \\ \frac{t^3}{(t^4)^{\frac{3}{4}}} & \frac{3t^3}{(t^4)^{\frac{1}{4}}} \end{vmatrix}$$

Therefore

$$W = \left((t^4)^{\frac{1}{4}}\right)\left(\frac{3t^3}{(t^4)^{\frac{1}{4}}}\right) - \left((t^4)^{\frac{3}{4}}\right)\left(\frac{t^3}{(t^4)^{\frac{3}{4}}}\right)$$

Which simplifies to

$$W = 2t^3$$

Which simplifies to

$$W = 2t^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4(t^4)^{\frac{3}{4}}\, t^7}{2t^5}\, dt$$

Which simplifies to

$$u_1 = -\int 2(t^4)^{\frac{3}{4}}\, t^2 dt$$

Hence

$$u_1 = -\frac{t^3 (t^4)^{\frac{3}{4}}}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4(t^4)^{\frac{1}{4}}\, t^7}{2t^5}\, dt$$

Which simplifies to

$$u_2 = \int 2\left(t^4\right)^{\frac{1}{4}} t^2 dt$$

Hence

$$u_2 = \frac{t^3 \left(t^4\right)^{\frac{1}{4}}}{2}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \frac{t^7}{6}$$

Therefore the general solution is

$$\begin{aligned}
x &= x_h + x_p \\
&= \left(\frac{t\left(\left(ic_2 + c_1\right) t^2 - ic_2 + c_1\right)}{2}\right) + \left(\frac{t^7}{6}\right) \\
&= \frac{t^7}{6} + \frac{t\left(\left(ic_2 + c_1\right) t^2 - ic_2 + c_1\right)}{2}
\end{aligned}$$

Which simplifies to

$$x = \frac{t^7}{6} + \frac{t\left(\left(ic_2 + c_1\right) t^2 - ic_2 + c_1\right)}{2}$$

Summary

The solution(s) found are the following

$$x = \frac{t^7}{6} + \frac{t\left(\left(ic_2 + c_1\right) t^2 - ic_2 + c_1\right)}{2} \tag{1}$$

Verification of solutions

$$x = \frac{t^7}{6} + \frac{t\left(\left(ic_2 + c_1\right) t^2 - ic_2 + c_1\right)}{2}$$

Verified OK.

12.8.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = t^2, B = -3t, C = 3, f(t) = 4t^7$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. Solving for x_h from

$$t^2x'' - 3tx' + 3x = 0$$

In normal form the ode

$$t^2x'' - 3tx' + 3x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{3}{t} \\ q(t) &= \frac{3}{t^2} \end{aligned}$$

Applying change of variables on the depndent variable $x = v(t)\,t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not x.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} - \frac{3n}{t^2} + \frac{3}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \frac{3v'(t)}{t} &= 0 \\ v''(t) + \frac{3v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{3u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{3u}{t} \end{aligned}$$

Where $f(t) = -\frac{3}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{3}{t}\,dt \\ \int \frac{1}{u}\,du &= \int -\frac{3}{t}\,dt \\ \ln(u) &= -3\ln(t) + c_1 \\ u &= \mathrm{e}^{-3\ln(t)+c_1} \\ &= \frac{c_1}{t^3} \end{aligned}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\,dt + c_2 \\ &= -\frac{c_1}{2t^2} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
x &= v(t)\, t^n \\
&= \left(-\frac{c_1}{2t^2} + c_2\right) t^3 \\
&= c_2 t^3 - \frac{1}{2} c_1 t
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
t^2 x'' - 3tx' + 3x = 4t^7
$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}
$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
x_1 &= t \\
x_2 &= t^3
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{x_2 f(t)}{a W(t)} \tag{2}
$$

$$
u_2 = \int \frac{x_1 f(t)}{a W(t)} \tag{3}
$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} t & t^3 \\ \frac{d}{dt}(t) & \frac{d}{dt}(t^3) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix}$$

Therefore

$$W = (t)\left(3t^2\right) - \left(t^3\right)(1)$$

Which simplifies to

$$W = 2t^3$$

Which simplifies to

$$W = 2t^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4t^{10}}{2t^5}\, dt$$

Which simplifies to

$$u_1 = -\int 2t^5 dt$$

Hence

$$u_1 = -\frac{t^6}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4t^8}{2t^5}\, dt$$

Which simplifies to

$$u_2 = \int 2t^3 dt$$

Hence

$$u_2 = \frac{t^4}{2}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \frac{t^7}{6}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(\left(-\frac{c_1}{2t^2} + c_2\right) t^3\right) + \left(\frac{t^7}{6}\right) \\ &= \frac{t^7}{6} + \left(-\frac{c_1}{2t^2} + c_2\right) t^3 \end{aligned}$$

Which simplifies to

$$x = -\frac{t(-t^6 - 6c_2t^2 + 3c_1)}{6}$$

Summary

The solution(s) found are the following

$$x = -\frac{t(-t^6 - 6c_2t^2 + 3c_1)}{6} \tag{1}$$

Verification of solutions

$$x = -\frac{t(-t^6 - 6c_2t^2 + 3c_1)}{6}$$

Verified OK.

12.8.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ax'' + Bx' + Cx = F(t)$$

This method reduces the order ode the ODE by one by applying the transformation

$$x = Bv$$

This results in

$$
\begin{aligned}
x' &= B'v + v'B \\
x'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $x = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= t^2 \\
B &= -3t \\
C &= 3 \\
F &= 4t^7
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(t^2\right)(0) + (-3t)(-3) + (3)(-3t) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-3t^3 v'' + \left(3t^2\right) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
-3t^2 (u'(t)\, t - u(t)) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(t,u) \\
&= f(t)g(u) \\
&= \frac{u}{t}
\end{aligned}
$$

Where $f(t) = \frac{1}{t}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{1}{t}\,dt \\
\int \frac{1}{u}\,du &= \int \frac{1}{t}\,dt \\
\ln(u) &= \ln(t) + c_1 \\
u &= \mathrm{e}^{\ln(t)+c_1} \\
&= c_1 t
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1 t
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(t) &= \int c_1 t \,\mathrm{d}t \\
&= \frac{t^2 c_1}{2} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
x_h(t) &= Bv \\
&= (-3t)\left(\frac{t^2 c_1}{2} + c_2\right) \\
&= -\frac{3t(t^2 c_1 + 2c_2)}{2}
\end{aligned}
$$

And now the particular solution $x_p(t)$ will be found. The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}
$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = t$$
$$x_2 = t^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} t & t^3 \\ \frac{d}{dt}(t) & \frac{d}{dt}(t^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix}$$

Therefore

$$W = (t)\left(3t^2\right) - \left(t^3\right)(1)$$

Which simplifies to

$$W = 2t^3$$

Which simplifies to

$$W = 2t^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4t^{10}}{2t^5}\, dt$$

Which simplifies to

$$u_1 = -\int 2t^5 dt$$

Hence

$$u_1 = -\frac{t^6}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4t^8}{2t^5}\, dt$$

Which simplifies to

$$u_2 = \int 2t^3 dt$$

Hence

$$u_2 = \frac{t^4}{2}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \frac{t^7}{6}$$

Hence the complete solution is

$$\begin{aligned} x(t) &= x_h + x_p \\ &= \left(-\frac{3t(t^2c_1 + 2c_2)}{2}\right) + \left(\frac{t^7}{6}\right) \\ &= \frac{t(t^6 - 9t^2c_1 - 18c_2)}{6} \end{aligned}$$

Summary

The solution(s) found are the following

$$x = \frac{t(t^6 - 9t^2c_1 - 18c_2)}{6} \tag{1}$$

Verification of solutions

$$x = \frac{t(t^6 - 9t^2c_1 - 18c_2)}{6}$$

Verified OK.

12.8.6 Solving using Kovacic algorithm

Writing the ode as

$$t^2x'' - 3tx' + 3x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -3t \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 299: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3}{4t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + (-)\,(0) \\ &= -\frac{1}{2t} \\ &= -\frac{1}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(-\frac{1}{2t}\right)^2 - \left(\frac{3}{4t^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int -\frac{1}{2t} dt} \\ &= \frac{1}{\sqrt{t}} \end{aligned}$$

The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3t}{t^2}\,dt} \\
&= z_1 e^{\frac{3\ln(t)}{2}} \\
&= z_1 \left(t^{\frac{3}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = t
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-3t}{t^2}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{3\ln(t)}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{t^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(t) + c_2\left(t\left(\frac{t^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$
t^2x'' - 3tx' + 3x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 t + \frac{1}{2} c_2 t^3$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = t$$
$$x_2 = \frac{t^3}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{a W(t)} \tag{2}$$
$$u_2 = \int \frac{x_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} t & \frac{t^3}{2} \\ \frac{d}{dt}(t) & \frac{d}{dt}\left(\frac{t^3}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} t & \frac{t^3}{2} \\ 1 & \frac{3t^2}{2} \end{vmatrix}$$

Therefore

$$W = (t)\left(\frac{3t^2}{2}\right) - \left(\frac{t^3}{2}\right)(1)$$

Which simplifies to

$$W = t^3$$

Which simplifies to

$$W = t^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2t^{10}}{t^5}\,dt$$

Which simplifies to

$$u_1 = -\int 2t^5 dt$$

Hence

$$u_1 = -\frac{t^6}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4t^8}{t^5}\,dt$$

Which simplifies to

$$u_2 = \int 4t^3 dt$$

Hence

$$u_2 = t^4$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \frac{t^7}{6}$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(c_1 t + \frac{1}{2} c_2 t^3\right) + \left(\frac{t^7}{6}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$x = c_1 t + \frac{1}{2} c_2 t^3 + \frac{1}{6} t^7 \tag{1}$$

Verification of solutions

$$x = c_1 t + \frac{1}{2} c_2 t^3 + \frac{1}{6} t^7$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(t^2*diff(x(t),t$2)-3*t*diff(x(t),t)+3*x(t)=4*t^7,x(t), singsol=all)
```

$$x(t) = \frac{(t^6 + 3c_1 t^2 + 6c_2)\, t}{6}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 23

```
DSolve[t^2*x''[t]-3*t*x'[t]+3*x[t]==4*t^7,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{t^7}{6} + c_2 t^3 + c_1 t$$

12.9 problem 7

12.9.1 Solving as second order linear constant coeff ode 1752
12.9.2 Solving using Kovacic algorithm 1757
12.9.3 Maple step by step solution 1763

Internal problem ID [11495]
Internal file name [`OUTPUT/10477_Thursday_May_18_2023_04_20_37_AM_21015621/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.2 Variation of parameters. Exercises page 124
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' - x = \frac{e^t}{1 + e^t}}$$

12.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = -1, f(t) = \frac{e^t}{1+e^t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$x = c_1 e^{(1)t} + c_2 e^{(-1)t}$$

Or

$$x = c_1 e^t + c_2 e^{-t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 e^t + c_2 e^{-t}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} x_1 &= \mathrm{e}^t \\ x_2 &= \mathrm{e}^{-t} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^t & \mathrm{e}^{-t} \\ \frac{d}{dt}(\mathrm{e}^t) & \frac{d}{dt}(\mathrm{e}^{-t}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^t & \mathrm{e}^{-t} \\ \mathrm{e}^t & -\mathrm{e}^{-t} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^t)(-\mathrm{e}^{-t}) - (\mathrm{e}^{-t})(\mathrm{e}^t)$$

Which simplifies to

$$W = -2\,\mathrm{e}^{-t}\mathrm{e}^t$$

Which simplifies to

$$W = -2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-t} e^t}{1+e^t}}{-2} dt$$

Which simplifies to

$$u_1 = -\int -\frac{1}{2 + 2\,e^t} dt$$

Hence

$$u_1 = -\frac{\ln(1+e^t)}{2} + \frac{\ln(e^t)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{2t}}{1+e^t}}{-2} dt$$

Which simplifies to

$$u_2 = \int -\frac{e^{2t}}{2 + 2\,e^t} dt$$

Hence

$$u_2 = -\frac{e^t}{2} + \frac{\ln(1+e^t)}{2}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \left(-\frac{\ln(1+e^t)}{2} + \frac{\ln(e^t)}{2}\right) e^t + \left(-\frac{e^t}{2} + \frac{\ln(1+e^t)}{2}\right) e^{-t}$$

Which simplifies to

$$x_p(t) = \frac{(-e^t + e^{-t}) \ln(1+e^t)}{2} + \frac{e^t \ln(e^t)}{2} - \frac{1}{2}$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(c_1 \mathrm{e}^t + c_2 \mathrm{e}^{-t}\right) + \left(\frac{\left(-\mathrm{e}^t + \mathrm{e}^{-t}\right) \ln\left(1+\mathrm{e}^t\right)}{2} + \frac{\mathrm{e}^t \ln\left(\mathrm{e}^t\right)}{2} - \frac{1}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{-t} + \frac{\left(-\mathrm{e}^t + \mathrm{e}^{-t}\right) \ln\left(1+\mathrm{e}^t\right)}{2} + \frac{\mathrm{e}^t \ln\left(\mathrm{e}^t\right)}{2} - \frac{1}{2} \tag{1}
$$

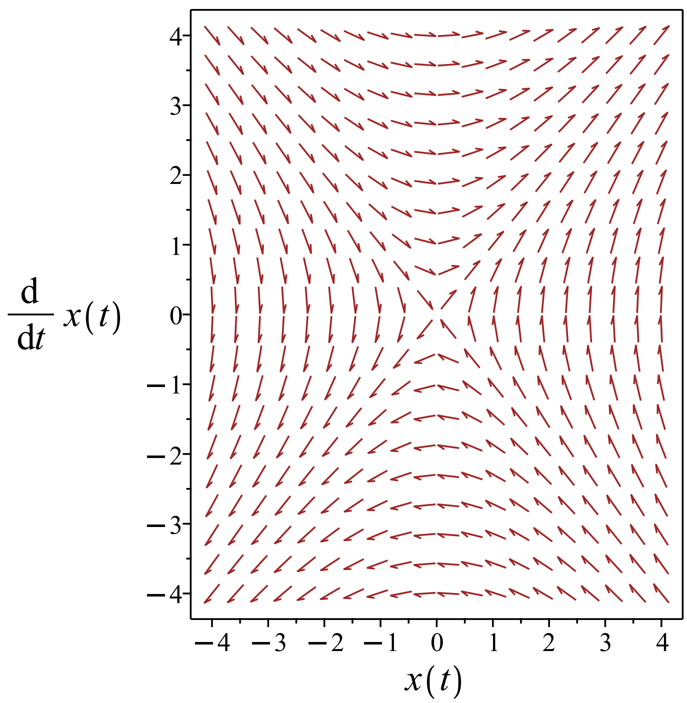


Figure 329: Slope field plot

Verification of solutions

$$
x = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{-t} + \frac{\left(-\mathrm{e}^t + \mathrm{e}^{-t}\right) \ln\left(1+\mathrm{e}^t\right)}{2} + \frac{\mathrm{e}^t \ln\left(\mathrm{e}^t\right)}{2} - \frac{1}{2}
$$

Verified OK.

12.9.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' - x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 300: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
x_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
x_1 &= z_1 \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-t}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A} dt}}{x_1^2} dt
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2} dt \\
&= \mathrm{e}^{-t} \int \frac{1}{\mathrm{e}^{-2t}} dt \\
&= \mathrm{e}^{-t} \left(\frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-t}\right) + c_2 \left(\mathrm{e}^{-t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' - x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-t} + \frac{c_2 \mathrm{e}^{t}}{2}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} x_1 &= \mathrm{e}^{-t} \\ x_2 &= \frac{\mathrm{e}^{t}}{2} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-t} & \frac{\mathrm{e}^{t}}{2} \\ \frac{d}{dt}\left(\mathrm{e}^{-t}\right) & \frac{d}{dt}\left(\frac{\mathrm{e}^{t}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-t} & \frac{\mathrm{e}^{t}}{2} \\ -\mathrm{e}^{-t} & \frac{\mathrm{e}^{t}}{2} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{-t})\left(\frac{\mathrm{e}^{t}}{2}\right) - \left(\frac{\mathrm{e}^{t}}{2}\right)(-\mathrm{e}^{-t})$$

Which simplifies to

$$W = \mathrm{e}^{-t}\mathrm{e}^{t}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{2t}}{2+2\,\mathrm{e}^{t}}}{1}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{\mathrm{e}^{2t}}{2+2\,\mathrm{e}^{t}}dt$$

Hence

$$u_1 = -\frac{\mathrm{e}^{t}}{2} + \frac{\ln\left(1+\mathrm{e}^{t}\right)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{-t}\mathrm{e}^{t}}{1+\mathrm{e}^{t}}}{1}\,dt$$

Which simplifies to

$$u_2 = \int \frac{1}{1+\mathrm{e}^{t}}dt$$

Hence

$$u_2 = -\ln\left(1+\mathrm{e}^{t}\right) + \ln\left(\mathrm{e}^{t}\right)$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \left(-\frac{\mathrm{e}^{t}}{2} + \frac{\ln\left(1+\mathrm{e}^{t}\right)}{2}\right)\mathrm{e}^{-t} + \frac{\left(-\ln\left(1+\mathrm{e}^{t}\right) + \ln\left(\mathrm{e}^{t}\right)\right)\mathrm{e}^{t}}{2}$$

Which simplifies to

$$x_p(t) = \frac{(-e^t + e^{-t}) \ln(1 + e^t)}{2} + \frac{e^t \ln(e^t)}{2} - \frac{1}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-t} + \frac{c_2 e^t}{2}\right) + \left(\frac{(-e^t + e^{-t}) \ln(1 + e^t)}{2} + \frac{e^t \ln(e^t)}{2} - \frac{1}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{-t} + \frac{c_2 e^t}{2} + \frac{(-e^t + e^{-t}) \ln(1 + e^t)}{2} + \frac{e^t \ln(e^t)}{2} - \frac{1}{2} \tag{1}$$

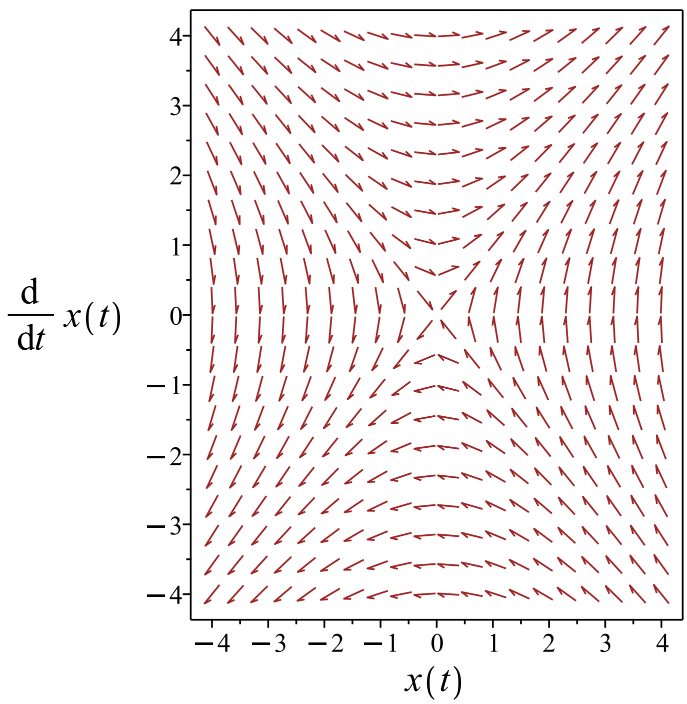


Figure 330: Slope field plot

Verification of solutions

$$x = c_1 e^{-t} + \frac{c_2 e^t}{2} + \frac{(-e^t + e^{-t}) \ln(1 + e^t)}{2} + \frac{e^t \ln(e^t)}{2} - \frac{1}{2}$$

Verified OK.

12.9.3 Maple step by step solution

Let's solve

$x'' - x = \frac{e^t}{1+e^t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^t$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{-t} + c_2 e^t + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \frac{e^t}{1+e^t}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{bmatrix}$

  ◦ Compute Wronskian

    $W(x_1(t), x_2(t)) = 2$

  ◦ Substitute functions into equation for $x_p(t)$

$$x_p(t) = -\frac{e^{-t}\left(\int \frac{e^{2t}}{1+e^t}dt\right)}{2} + \frac{e^t\left(\int \frac{1}{1+e^t}dt\right)}{2}$$

- Compute integrals

$$x_p(t) = \frac{(-e^t+e^{-t})\ln(1+e^t)}{2} + \frac{e^t\ln(e^t)}{2} - \frac{1}{2}$$

- Substitute particular solution into general solution to ODE

$$x = c_1e^{-t} + c_2e^t + \frac{(-e^t+e^{-t})\ln(1+e^t)}{2} + \frac{e^t\ln(e^t)}{2} - \frac{1}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(x(t),t$2)-x(t)=exp(t)/(1+exp(t)),x(t), singsol=all)
```

$$x(t) = c_2e^{-t} + c_1e^t + \frac{(-e^t + e^{-t})\ln(1+e^t)}{2} + \frac{e^t\ln(e^t)}{2} - \frac{1}{2}$$

✓ Solution by Mathematica

Time used: 0.123 (sec). Leaf size: 51

```
DSolve[x''[t]-x[t]==Exp[t]/(1+Exp[t]),x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -e^t\operatorname{arctanh}(2e^t+1) + \frac{1}{2}e^{-t}\log\left(e^t+1\right) + c_1e^t + c_2e^{-t} - \frac{1}{2}$$

13 Chapter 2, Second order linear equations. Section 2.4.3 Reduction of order. Exercises page 125

13.1 problem 1 1766
13.2 problem 2 1770
13.3 problem 4 1774
13.4 problem 5 1777
13.5 problem 6 1782

13.1 problem 1

13.1.1 Maple step by step solution 1767

Internal problem ID [11496]
Internal file name [`OUTPUT/10478_Thursday_May_18_2023_04_20_39_AM_11046977/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.3 Reduction of order. Exercises page 125
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x'' + tx' + x = 0}$$

Given that one solution of the ode is

$$x_1 = \mathrm{e}^{-\frac{t^2}{2}}$$

Given one basis solution $x_1(t)$, then the second basis solution is given by

$$x_2(t) = x_1 \left(\int \frac{\mathrm{e}^{-(\int p dt)}}{x_1^2} dt \right)$$

Where $p(x)$ is the coefficient of x' when the ode is written in the normal form

$$x'' + p(t)\, x' + q(t)\, x = f(t)$$

Looking at the ode to solve shows that

$$p(t) = t$$

Therefore

$$x_2(t) = \mathrm{e}^{-\frac{t^2}{2}} \left(\int \mathrm{e}^{-\left(\int t dt\right)} \mathrm{e}^{t^2} dt\right)$$

$$x_2(t) = \mathrm{e}^{-\frac{t^2}{2}} \int \frac{\mathrm{e}^{-\frac{t^2}{2}}}{\mathrm{e}^{-t^2}}, dt$$

$$x_2(t) = \mathrm{e}^{-\frac{t^2}{2}} \left(\int \mathrm{e}^{\frac{t^2}{2}} dt\right)$$

$$x_2(t) = -\frac{i \mathrm{e}^{-\frac{t^2}{2}} \sqrt{\pi}\, \sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\, t}{2}\right)}{2}$$

Hence the solution is

$$\begin{aligned} x &= c_1 x_1(t) + c_2 x_2(t) \\ &= \mathrm{e}^{-\frac{t^2}{2}} c_1 - \frac{i c_2 \mathrm{e}^{-\frac{t^2}{2}} \sqrt{\pi}\, \sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\, t}{2}\right)}{2} \end{aligned}$$

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{-\frac{t^2}{2}} c_1 - \frac{i c_2 \mathrm{e}^{-\frac{t^2}{2}} \sqrt{\pi}\, \sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\, t}{2}\right)}{2} \tag{1}$$

Verification of solutions

$$x = \mathrm{e}^{-\frac{t^2}{2}} c_1 - \frac{i c_2 \mathrm{e}^{-\frac{t^2}{2}} \sqrt{\pi}\, \sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\, t}{2}\right)}{2}$$

Verified OK.

13.1.1 Maple step by step solution

Let's solve

$x'' + tx' + x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Assume series solution for x

$$x = \sum_{k=0}^{\infty} a_k t^k$$

- □ Rewrite DE with series expansions
  - Convert $t \cdot x'$ to series expansion

    $$t \cdot x' = \sum_{k=0}^{\infty} a_k k \, t^k$$

  - Convert x'' to series expansion

    $$x'' = \sum_{k=2}^{\infty} a_k k(k-1) \, t^{k-2}$$

  - Shift index using $k- >k+2$

    $$x'' = \sum_{k=0}^{\infty} a_{k+2}(k+2) \, (k+1) \, t^k$$

  Rewrite DE with series expansions

  $$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2) \, (k+1) + a_k(k+1)\right) t^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+1) \left(a_{k+2}(k+2) + a_k\right) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[x = \sum_{k=0}^{\infty} a_k t^k, a_{k+2} = -\frac{a_k}{k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve([diff(x(t),t$2)+t*diff(x(t),t)+x(t)=0,exp(-t^2/2)],singsol=all)
```

$$x(t) = \left(\operatorname{erf}\left(\frac{i\sqrt{2}\,t}{2}\right) c_1 + c_2\right) \mathrm{e}^{-\frac{t^2}{2}}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 41

```
DSolve[x''[t]+t*x'[t]+x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2} e^{-\frac{t^2}{2}} \left(\sqrt{2\pi} c_1 \operatorname{erfi}\left(\frac{t}{\sqrt{2}}\right) + 2c_2\right)$$

13.2 problem 2

13.2.1 Maple step by step solution 1772

Internal problem ID [11497]
Internal file name [`OUTPUT/10479_Thursday_May_18_2023_04_20_40_AM_49692987/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.3 Reduction of order. Exercises page 125
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[_Hermite]
```

$$\boxed{x'' - tx' + x = 0}$$

Given that one solution of the ode is

$$x_1 = t$$

Given one basis solution $x_1(t)$, then the second basis solution is given by

$$x_2(t) = x_1 \left(\int \frac{e^{-(\int p dt)}}{x_1^2} dt \right)$$

Where $p(x)$ is the coefficient of x' when the ode is written in the normal form

$$x'' + p(t)\, x' + q(t)\, x = f(t)$$

Looking at the ode to solve shows that

$$p(t) = -t$$

Therefore

$$x_2(t) = t\left(\int \frac{\mathrm{e}^{-\left(\int -t dt\right)}}{t^2} dt\right)$$

$$x_2(t) = t \int \frac{\mathrm{e}^{\frac{t^2}{2}}}{t^2}, dt$$

$$x_2(t) = t\left(\int \frac{\mathrm{e}^{\frac{t^2}{2}}}{t^2} dt\right)$$

$$x_2(t) = t\left(-\frac{\mathrm{e}^{\frac{t^2}{2}}}{t} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,t}{2}\right)}{2}\right)$$

Hence the solution is

$$\begin{aligned} x &= c_1 x_1(t) + c_2 x_2(t) \\ &= c_1 t + c_2 t\left(-\frac{\mathrm{e}^{\frac{t^2}{2}}}{t} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,t}{2}\right)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 t + c_2 t\left(-\frac{\mathrm{e}^{\frac{t^2}{2}}}{t} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,t}{2}\right)}{2}\right) \tag{1}$$

Verification of solutions

$$x = c_1 t + c_2 t\left(-\frac{\mathrm{e}^{\frac{t^2}{2}}}{t} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,t}{2}\right)}{2}\right)$$

Verified OK.

13.2.1 Maple step by step solution

Let's solve

$x'' - tx' + x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Assume series solution for x

  $x = \sum_{k=0}^{\infty} a_k t^k$

□ Rewrite DE with series expansions

- ∘ Convert $t \cdot x'$ to series expansion

  $t \cdot x' = \sum_{k=0}^{\infty} a_k k\, t^k$

- ∘ Convert x'' to series expansion

  $x'' = \sum_{k=2}^{\infty} a_k k(k-1)\, t^{k-2}$

- ∘ Shift index using $k- >k+2$

  $x'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, t^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(k-1)\right) t^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} - a_k(k-1) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[x = \sum_{k=0}^{\infty} a_k t^k, a_{k+2} = \frac{a_k(k-1)}{k^2+3k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 38

```
dsolve([diff(x(t),t$2)-t*diff(x(t),t)+x(t)=0,t],singsol=all)
```

$$x(t) = c_2 \mathrm{e}^{\frac{t^2}{2}} + \frac{\left(ic_2\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,t}{2}\right) + 2c_1\right)t}{2}$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 61

```
DSolve[x''[t]-t*x'[t]+x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\sqrt{\frac{\pi}{2}} c_2 \sqrt{t^2} \operatorname{erfi}\left(\frac{\sqrt{t^2}}{\sqrt{2}}\right) + c_2 e^{\frac{t^2}{2}} + \sqrt{2} c_1 t$$

13.3 problem 4

13.3.1 Maple step by step solution 1775

Internal problem ID [11498]
Internal file name [`OUTPUT/10480_Thursday_May_18_2023_04_20_42_AM_19877591/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.3 Reduction of order. Exercises page 125
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{x'' - 2ax' + a^2x = 0}$$

Given that one solution of the ode is

$$x_1 = \mathrm{e}^{ta}$$

Given one basis solution $x_1(t)$, then the second basis solution is given by

$$x_2(t) = x_1 \left(\int \frac{\mathrm{e}^{-(\int p dt)}}{x_1^2} dt \right)$$

Where $p(x)$ is the coefficient of x' when the ode is written in the normal form

$$x'' + p(t)\, x' + q(t)\, x = f(t)$$

Looking at the ode to solve shows that

$$p(t) = -2a$$

Therefore

$$x_2(t) = e^{ta} \left(\int e^{-(\int -2adt)} e^{-2ta} dt \right)$$

$$x_2(t) = e^{ta} \int \frac{e^{2ta}}{e^{2ta}}, dt$$

$$x_2(t) = e^{ta} \left(\int 1dt \right)$$

$$x_2(t) = e^{ta} t$$

Hence the solution is

$$\begin{aligned} x &= c_1 x_1(t) + c_2 x_2(t) \\ &= e^{ta} c_1 + c_2 e^{ta} t \end{aligned}$$

Summary

The solution(s) found are the following

$$x = e^{ta} c_1 + c_2 e^{ta} t \tag{1}$$

Verification of solutions

$$x = e^{ta} c_1 + c_2 e^{ta} t$$

Verified OK.

13.3.1 Maple step by step solution

Let's solve

$x'' - 2ax' + a^2 x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $a^2 - 2ar + r^2 = 0$

- Factor the characteristic polynomial

  $(a - r)^2 = 0$

- Root of the characteristic polynomial

  $r = a$

- 1st solution of the ODE

  $x_1(t) = e^{ta}$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t) = e^{ta}t$

- General solution of the ODE

  $x = c_1x_1(t) + c_2x_2(t)$

- Substitute in solutions

  $x = e^{ta}c_1 + c_2e^{ta}t$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([diff(x(t),t$2)-2*a*diff(x(t),t)+a^2*x(t)=0,exp(a*t)],singsol=all)
```

$$x(t) = e^{at}(c_2t + c_1)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 18

```
DSolve[x''[t]-2*a*x'[t]+a^2*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{at}(c_2t + c_1)$$

13.4 problem 5

13.4.1 Maple step by step solution 1778

Internal problem ID [11499]
Internal file name [`OUTPUT/10481_Thursday_May_18_2023_04_20_43_AM_83109919/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.3 Reduction of order. Exercises page 125
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x'' - \frac{(t+2)\,x'}{t} + \frac{(t+2)\,x}{t^2} = 0}$$

Given that one solution of the ode is

$$x_1 = t$$

Given one basis solution $x_1(t)$, then the second basis solution is given by

$$x_2(t) = x_1 \left(\int \frac{\mathrm{e}^{-(\int p dt)}}{x_1^2} dt \right)$$

Where $p(x)$ is the coefficient of x' when the ode is written in the normal form

$$x'' + p(t)\,x' + q(t)\,x = f(t)$$

Looking at the ode to solve shows that

$$p(t) = -1 - \frac{2}{t}$$

Therefore

$$x_2(t) = t\left(\int \frac{e^{-\left(\int \left(-1-\frac{2}{t}\right)dt\right)}}{t^2}dt\right)$$

$$x_2(t) = t\int \frac{e^{t+2\ln(t)}}{t^2}, dt$$

$$x_2(t) = t\left(\int e^t dt\right)$$

$$x_2(t) = t\,e^t$$

Hence the solution is

$$\begin{aligned} x &= c_1x_1(t) + c_2x_2(t) \\ &= c_1t + c_2t\,e^t \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1t + c_2t\,e^t \tag{1}$$

Verification of solutions

$$x = c_1t + c_2t\,e^t$$

Verified OK.

13.4.1 Maple step by step solution

Let's solve

$x'' + \frac{(-t-2)x'}{t} + \frac{(t+2)x}{t^2} = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = \frac{(t+2)x'}{t} - \frac{(t+2)x}{t^2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' - \frac{(t+2)x'}{t} + \frac{(t+2)x}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

- ○ Define functions

$\left[P_2(t) = -\frac{t+2}{t}, P_3(t) = \frac{t+2}{t^2}\right]$

- $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -2$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 2$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $t^2x'' - (t+2)\,tx' + x(t+2) = 0$

- Assume series solution for x

  $x = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

- Convert $t^m \cdot x$ to series expansion for $m = 0..1$

  $t^m \cdot x = \sum_{k=0}^{\infty} a_k t^{k+r+m}$

- Shift index using $k- >k - m$

  $t^m \cdot x = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$

- Convert $t^m \cdot x'$ to series expansion for $m = 1..2$

  $t^m \cdot x' = \sum_{k=0}^{\infty} a_k(k+r)\,t^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $t^m \cdot x' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,t^{k+r}$

- Convert $t^2 \cdot x''$ to series expansion

  $t^2 \cdot x'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,t^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+r)(-2+r)t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)(k+r-2) - a_{k-1}(k+r-2)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-2)\left(a_k(k+r-1) - a_{k-1}\right) = 0$$

- Shift index using $k- > k+1$

$$(k+r-1)\left(a_{k+1}(k+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+r}$$

- Recursion relation for $r = 1$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 1$

$$\left[x = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+2}$$

- Solution for $r = 2$

$$\left[x = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+2}\right]$$

- Combine solutions and rename parameters

$$\left[x = \left(\sum_{k=0}^{\infty} a_k t^{1+k}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{1+k} = \frac{a_k}{1+k}, b_{1+k} = \frac{b_k}{k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve([diff(x(t),t$2)-(t+2)/t*diff(x(t),t)+(t+2)/t^2*x(t)=0,t],singsol=all)
```

$$x(t) = t(c_1 + c_2 e^t)$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 16

```
DSolve[x''[t]-(t+2)/t*x'[t]+(t+2)/t^2*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to t(c_2 e^t + c_1)$$

13.5 problem 6

13.5.1 Maple step by step solution 1783

Internal problem ID [11500]
Internal file name [`OUTPUT/10482_Thursday_May_18_2023_04_20_45_AM_67513854/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.4.3 Reduction of order. Exercises page 125
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{t^2x'' + tx' + \left(t^2 - \frac{1}{4}\right)x = 0}$$

Given that one solution of the ode is

$$x_1 = \frac{\cos(t)}{\sqrt{t}}$$

Given one basis solution $x_1(t)$, then the second basis solution is given by

$$x_2(t) = x_1\left(\int \frac{\mathrm{e}^{-(\int p dt)}}{x_1^2} dt\right)$$

Where $p(x)$ is the coefficient of x' when the ode is written in the normal form

$$x'' + p(t)\,x' + q(t)\,x = f(t)$$

Looking at the ode to solve shows that

$$p(t) = \frac{1}{t}$$

Therefore

$$x_2(t) = \frac{\cos(t)\left(\int \frac{e^{-\left(\int \frac{1}{t}dt\right)}t}{\cos(t)^2}dt\right)}{\sqrt{t}}$$

$$x_2(t) = \frac{\cos(t)}{\sqrt{t}} \int \frac{\frac{1}{t}}{\frac{\cos(t)^2}{t}}, dt$$

$$x_2(t) = \frac{\cos(t)\left(\int \sec(t)^2 \, dt\right)}{\sqrt{t}}$$

$$x_2(t) = \frac{\cos(t)\tan(t)}{\sqrt{t}}$$

Hence the solution is

$$\begin{aligned} x &= c_1x_1(t) + c_2x_2(t) \\ &= \frac{\cos(t)\,c_1}{\sqrt{t}} + \frac{c_2\cos(t)\tan(t)}{\sqrt{t}} \end{aligned}$$

Summary

The solution(s) found are the following

$$x = \frac{\cos(t)\,c_1}{\sqrt{t}} + \frac{c_2\cos(t)\tan(t)}{\sqrt{t}} \tag{1}$$

Verification of solutions

$$x = \frac{\cos(t)\,c_1}{\sqrt{t}} + \frac{c_2\cos(t)\tan(t)}{\sqrt{t}}$$

Verified OK.

13.5.1 Maple step by step solution

Let's solve

$t^2x'' + tx' + \left(t^2 - \frac{1}{4}\right)x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = -\frac{(4t^2-1)x}{4t^2} - \frac{x'}{t}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + \frac{x'}{t} + \frac{(4t^2-1)x}{4t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(t) = \frac{1}{t}, P_3(t) = \frac{4t^2-1}{4t^2}\right]$

  - $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left.(t \cdot P_2(t))\right|_{t=0} = 1$

  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $\left.(t^2 \cdot P_3(t))\right|_{t=0} = -\frac{1}{4}$

  - $t = 0$is a regular singular point

    Check to see if $t_0 = 0$ is a regular singular point

    $t_0 = 0$

- Multiply by denominators

  $4t^2x'' + 4tx' + (4t^2 - 1)\, x = 0$

- Assume series solution for x

  $x = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $t^m \cdot x$ to series expansion for $m = 0..2$

    $t^m \cdot x = \sum_{k=0}^{\infty} a_k t^{k+r+m}$

  - Shift index using $k- >k - m$

    $t^m \cdot x = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$

  - Convert $t \cdot x'$ to series expansion

    $t \cdot x' = \sum_{k=0}^{\infty} a_k (k + r)\, t^{k+r}$

  - Convert $t^2 \cdot x''$ to series expansion

    $t^2 \cdot x'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, t^{k+r}$

Rewrite ODE with series expansions

$$a_0(1+2r)(-1+2r)t^r + a_1(3+2r)(1+2r)t^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)(2k+2r-1)+4a_{k-2}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)(1+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2+8kr+4r^2-1)+4a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2+8(k+2)r+4r^2-1\right)+4a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+2}=-\frac{4a_k}{4k^2+12k+8}$

- Solution for $r=-\frac{1}{2}$

  $\left[x=\sum_{k=0}^{\infty} a_k t^{k-\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=-\frac{4a_k}{4k^2+20k+24}$

- Solution for $r=\frac{1}{2}$

  $\left[x=\sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[x=\left(\sum_{k=0}^{\infty} a_k t^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 17

```
dsolve([t^2*diff(x(t),t$2)+t*diff(x(t),t)+(t^2-1/4)*x(t)=0,cos(t)/sqrt(t)],singsol=all)
```

$$x(t) = \frac{c_1 \sin(t) + c_2 \cos(t)}{\sqrt{t}}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 39

```
DSolve[t^2*x''[t]+t*x'[t]+(t^2-1/4)*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{e^{-it}(2c_1 - ic_2 e^{2it})}{2\sqrt{t}}$$

14 Chapter 2, Second order linear equations. Section 2.5 Higher order equations. Exercises page 130

14.1 problem 1(a) 1788
14.2 problem 1(b) 1793
14.3 problem 1(c) 1801
14.4 problem 1(d) 1806
14.5 problem 1(e) 1815
14.6 problem 1(f) 1819
14.7 problem 2 1824

14.1 problem 1(a)

14.1.1 Maple step by step solution 1789

Internal problem ID [11501]
Internal file name [`OUTPUT/10483_Thursday_May_18_2023_04_20_47_AM_77934714/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.5 Higher order equations. Exercises page 130
Problem number: 1(a).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{x''' + x' = 0}$$

The characteristic equation is

$$\lambda^3 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$x_h(t) = c_1 + e^{it}c_2 + e^{-it}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$x_1 = 1$$
$$x_2 = e^{it}$$
$$x_3 = e^{-it}$$

Summary

The solution(s) found are the following

$$x = c_1 + e^{it} c_2 + e^{-it} c_3 \tag{1}$$

Verification of solutions

$$x = c_1 + e^{it} c_2 + e^{-it} c_3$$

Verified OK.

14.1.1 Maple step by step solution

Let's solve

$x''' + x' = 0$

- Highest derivative means the order of the ODE is 3

  x'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $x_1(t)$

    $x_1(t) = x$

  ◦ Define new variable $x_2(t)$

    $x_2(t) = x'$

  ◦ Define new variable $x_3(t)$

    $x_3(t) = x''$

  ◦ Isolate for $x_3'(t)$ using original ODE

    $x_3'(t) = -x_2(t)$

  Convert linear ODE into a system of first order ODEs

  $[x_2(t) = x_1'(t), x_3(t) = x_2'(t), x_3'(t) = -x_2(t)]$

- Define vector

  $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}t} \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(t) + \mathrm{I}\sin(t) \\ \mathrm{I}(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_2(t) = \begin{bmatrix} -\cos(t) \\ \sin(t) \\ \cos(t) \end{bmatrix}, \vec{x}_3(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$

- Substitute solutions into the general solution

$$\vec{x} = \begin{bmatrix} c_3\sin(t) - c_2\cos(t) + c_1 \\ c_3\cos(t) + c_2\sin(t) \\ -c_3\sin(t) + c_2\cos(t) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$x = c_3\sin(t) - c_2\cos(t) + c_1$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(x(t),t$3)+diff(x(t),t)=0,x(t), singsol=all)
```

$$x(t) = c_1 + \sin(t) c_2 + c_3 \cos(t)$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 19

```
DSolve[x'''[t]+x'[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -c_2 \cos(t) + c_1 \sin(t) + c_3$$

14.2 problem 1(b)

14.2.1 Maple step by step solution 1795

Internal problem ID [11502]
Internal file name [`OUTPUT/10484_Thursday_May_18_2023_04_20_48_AM_44934501/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.5 Higher order equations. Exercises page 130
Problem number: 1(b).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{x''' + x' = 1}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE And x_p is a particular solution to the nonhomogeneous ODE. x_h is the solution to

$$x''' + x' = 0$$

The characteristic equation is

$$\lambda^3 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$x_h(t) = c_1 + e^{it}c_2 + e^{-it}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$x_1 = 1$$
$$x_2 = e^{it}$$
$$x_3 = e^{-it}$$

Now the particular solution to the given ODE is found

$$x''' + x' = 1$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{it}, e^{-it}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = t$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 + e^{it} c_2 + e^{-it} c_3\right) + (t) \end{aligned}$$

Summary
The solution(s) found are the following

$$x = c_1 + e^{it} c_2 + e^{-it} c_3 + t \tag{1}$$

Verification of solutions

$$x = c_1 + e^{it} c_2 + e^{-it} c_3 + t$$

Verified OK.

14.2.1 Maple step by step solution

Let's solve

$x''' + x' = 1$

- Highest derivative means the order of the ODE is 3

  x'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $x_1(t)$

    $x_1(t) = x$

  ◦ Define new variable $x_2(t)$

    $x_2(t) = x'$

  ◦ Define new variable $x_3(t)$

    $x_3(t) = x''$

  ◦ Isolate for $x_3'(t)$ using original ODE

$x_3'(t) = 1 - x_2(t)$

Convert linear ODE into a system of first order ODEs

$[x_2(t) = x_1'(t)\,, x_3(t) = x_2'(t)\,, x_3'(t) = 1 - x_2(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(t) + \mathrm{I}\sin(t) \\ \mathrm{I}(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_2(t) = \begin{bmatrix} -\cos(t) \\ \sin(t) \\ \cos(t) \end{bmatrix}, \vec{x}_3(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + \vec{x}_p(t)$

- ☐ Fundamental matrix
  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $$\phi(t) = \begin{bmatrix} 1 & -\cos(t) & \sin(t) \\ 0 & \sin(t) & \cos(t) \\ 0 & \cos(t) & -\sin(t) \end{bmatrix}$$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

    $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

    $$\Phi(t) = \begin{bmatrix} 1 & -\cos(t) & \sin(t) \\ 0 & \sin(t) & \cos(t) \\ 0 & \cos(t) & -\sin(t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

    $$\Phi(t) = \begin{bmatrix} 1 & \sin(t) & -\cos(t) + 1 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

    $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

  - Take the derivative of the particular solution

    $\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

  - Cancel like terms

    $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(t) = \frac{1}{\Phi(t)} \cdot \overrightarrow{f}(t)$

- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{x}_p(t) = \begin{bmatrix} t - \sin(t) \\ -\cos(t) + 1 \\ \sin(t) \end{bmatrix}$$

- Plug particular solution back into general solution

  $$\overrightarrow{x}(t) = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2(t) + c_3 \overrightarrow{x}_3(t) + \begin{bmatrix} t - \sin(t) \\ -\cos(t) + 1 \\ \sin(t) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $x = (c_3 - 1)\sin(t) - c_2 \cos(t) + t + c_1$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -_b(_a)+1, _b(_a)`   *** Subl
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(x(t),t$3)+diff(x(t),t)=1,x(t), singsol=all)
```

$$x(t) = c_1 \sin(t) - c_2 \cos(t) + t + c_3$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 20

```
DSolve[x'''[t]+x'[t]==1,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to t - c_2 \cos(t) + c_1 \sin(t) + c_3$$

14.3 problem 1(c)

14.3.1 Maple step by step solution 1802

Internal problem ID [11503]
Internal file name [`OUTPUT/10485_Thursday_May_18_2023_04_20_49_AM_799892/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.5 Higher order equations. Exercises page 130
Problem number: 1(c).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{x''' + x'' = 0}$$

The characteristic equation is

$$\lambda^3 + \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$x_h(t) = c_1 \mathrm{e}^{-t} + c_2 + c_3 t$$

The fundamental set of solutions for the homogeneous solution are the following

$$x_1 = \mathrm{e}^{-t}$$
$$x_2 = 1$$
$$x_3 = t$$

Summary
The solution(s) found are the following

$$x = c_1 e^{-t} + c_2 + c_3 t \tag{1}$$

Verification of solutions

$$x = c_1 e^{-t} + c_2 + c_3 t$$

Verified OK.

14.3.1 Maple step by step solution

Let's solve

$x''' + x'' = 0$

- Highest derivative means the order of the ODE is 3

  x'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $x_1(t)$

    $x_1(t) = x$
  - Define new variable $x_2(t)$

    $x_2(t) = x'$
  - Define new variable $x_3(t)$

    $x_3(t) = x''$
  - Isolate for $x_3'(t)$ using original ODE

    $x_3'(t) = -x_3(t)$

  Convert linear ODE into a system of first order ODEs

  $[x_2(t) = x_1'(t), x_3(t) = x_2'(t), x_3'(t) = -x_3(t)]$

- Define vector

  $$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{array}\right] \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{array}\right]$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$x = c_1 \mathrm{e}^{-t} + c_2$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(x(t),t$3)+diff(x(t),t$2)=0,x(t), singsol=all)
```

$$x(t) = c_1 + c_2 t + c_3 e^{-t}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 20

```
DSolve[x'''[t]+x''[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^{-t} + c_3 t + c_2$$

14.4 problem 1(d)

14.4.1 Maple step by step solution 1808

Internal problem ID [11504]
Internal file name [`OUTPUT/10486_Thursday_May_18_2023_04_20_51_AM_23119982/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.5 Higher order equations. Exercises page 130
Problem number: 1(d).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{x''' - x' - 8x = 0}$$

The characteristic equation is

$$\lambda^3 - \lambda - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = \frac{\left(108 + 3\sqrt{1293}\right)^{\frac{1}{3}}}{3} + \frac{1}{\left(108 + 3\sqrt{1293}\right)^{\frac{1}{3}}}$$

$$\lambda_2 = -\frac{\left(108 + 3\sqrt{1293}\right)^{\frac{1}{3}}}{6} - \frac{1}{2\left(108 + 3\sqrt{1293}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3} - \frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)}{2}$$

$$\lambda_3 = -\frac{\left(108 + 3\sqrt{1293}\right)^{\frac{1}{3}}}{6} - \frac{1}{2\left(108 + 3\sqrt{1293}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3} - \frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)}{2}$$

Therefore the homogeneous solution is

$$
x_h(t) = \mathrm{e}^{\left(-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}-\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)}{2}\right)t} c_1 + \mathrm{e}^{\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}+\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)t} c_2
$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
x_1 &= \mathrm{e}^{\left(-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}-\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)}{2}\right)t} \\
x_2 &= \mathrm{e}^{\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}+\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)t} \\
x_3 &= \mathrm{e}^{\left(-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}-\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)}{2}\right)t}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
x = {} & \mathrm{e}^{\left(-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}-\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)}{2}\right)t} c_1 \\
& + \mathrm{e}^{\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}+\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)t} c_2 \\
& + \mathrm{e}^{\left(-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}-\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)}{2}\right)t} c_3
\end{aligned} \tag{1}
$$

Verification of solutions

$$x = \mathrm{e}^{\left(-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}-\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)}{2}\right)t} c_1 + \mathrm{e}^{\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}+\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)t} c_2 + \mathrm{e}^{\left(-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}-\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)}{2}\right)t} c_3$$

Verified OK.

14.4.1 Maple step by step solution

Let's solve

$x''' - x' - 8x = 0$

- Highest derivative means the order of the ODE is 3

  x'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $x_1(t)$

    $x_1(t) = x$
  - Define new variable $x_2(t)$

    $x_2(t) = x'$
  - Define new variable $x_3(t)$

    $x_3(t) = x''$
  - Isolate for $x_3'(t)$ using original ODE

    $x_3'(t) = x_2(t) + 8x_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[x_2(t) = x_1'(t), x_3(t) = x_2'(t), x_3'(t) = x_2(t) + 8x_1(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 1 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3} + \frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}, \begin{bmatrix} \frac{1}{\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3} + \frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)^2} \\ \frac{1}{\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3} + \frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}} \\ 1 \end{bmatrix}\right], \left[-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6} - \frac{\quad}{2\left(108+\right.}\right.\right.$$

- Consider eigenpair

$$\left[\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}+\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}, \left[\begin{array}{c} \frac{1}{\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}+\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)^{2}} \\ \frac{1}{\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}+\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}+\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)t} \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}+\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)^{2}} \\ \frac{1}{\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}+\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}-\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)}{2}, \left[\begin{array}{c} \frac{\cdots}{\left(-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{\cdots}\right.}\cdots\right.} \\ \frac{\cdots}{-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{\cdots}\right.}\cdots} \end{array}\right.\right.$$

- Solution from eigenpair

$$
e^{\left(-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}-\frac{I\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}-\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)}{2}\right)t} \cdot \left[\frac{\left(-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}-\right.}{-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}-}\right.
$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$
e^{\left(-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)t} \cdot \left(\cos\left(\frac{\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}-\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)t}{2}\right)-I\sin\left(\frac{\sqrt{3}\left(\frac{\left(108+3\sqrt{129\right.}}{3}\right.}{}\right.\right.
$$

- Simplify expression

$$
\mathrm{e}^{\left(-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)t} \cdot \left[\begin{array}{c}
\frac{\cos\left(\frac{\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}-\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}\cdots\right.}{2\cdots}\right.}{\left(-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}-\frac{\cdots}{\left(108+\cdots\right.}\right.}{2}\cdots\right.} \\
\frac{\cos\left(\frac{\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}-\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}\cdots\right.}{2\cdots}\right.}{-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}-\frac{\cdots}{\left(108+\cdots\right.}\right.}{2}\cdots} \\
\cos\left(\frac{\sqrt{3}\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}-\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(\frac{\left(108+3\sqrt{12\cdots}\right.}{3}\cdots\right.}{\cdots}\right.
\end{array}\right.
$$

- Both real and imaginary parts are solutions to the homogeneous system

$$
\vec{x}_2(t) = \mathrm{e}^{\left(-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{1}{2\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)t} \cdot \left[\begin{array}{c}
-\frac{9\left(108+3\sqrt{1293}\right)^{\frac{2}{3}}\left(\sin\left(\frac{\sqrt{3}\left(\left(108+3\sqrt{1293}\right)^{\frac{2}{3}}-3\right)t}{6\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)\left(108+3\sqrt{129\cdots}\right.\right.}{\cdots} \\
-\frac{3\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}\left(-\sin\left(\frac{\sqrt{3}\left(\left(108+3\sqrt{\cdots}\right.\right.}{6\left(108+3\cdots\right.}\right.\right.}{\cdots} \\
\vdots
\end{array}\right.
$$

- General solution to the system of ODEs
  $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$
- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}+\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)t} \cdot \begin{bmatrix} \frac{1}{\left(\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}+\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}\right)^2} \\ \frac{1}{\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{3}+\frac{1}{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{\left(-\frac{\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}{6}-\frac{}{2(10}\right.}$$

- First component of the vector is the solution to the ODE

$$x = \frac{2\left(108+3\sqrt{3}\,\sqrt{431}\right)^{\frac{2}{3}}\left(\mathrm{e}^{-\frac{\left(\left(108+3\sqrt{1293}\right)^{\frac{2}{3}}+3\right)t}{6\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}}\left(\left(\left(-4c_2\sqrt{3}-12c_3\right)\sqrt{431}-\frac{287\sqrt{3}\,c_3}{2}-\frac{287c_2}{2}\right)\left(108+3\sqrt{3}\,\sqrt{431}\right)^{\frac{2}{3}}+c_2\left(\sqrt{3}\,\sqrt{4}\right.\right.\right.}{}$$

<u>Maple trace</u>

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 150

```
dsolve(diff(x(t),t$3)-diff(x(t),t)-8*x(t)=0,x(t), singsol=all)
```

$$
\begin{aligned}
x(t) = {} & c_1 \mathrm{e}^{\frac{\left(\left(108+3\sqrt{1293}\right)^{\frac{2}{3}}+3\right)t}{3\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}} \\
& - c_2 \mathrm{e}^{-\frac{\left(\left(108+3\sqrt{1293}\right)^{\frac{2}{3}}+3\right)t}{6\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}} \sin\left(\frac{\sqrt{3}\left(\left(108+3\sqrt{3}\,\sqrt{431}\right)^{\frac{2}{3}}-3\right)t}{6\left(108+3\sqrt{3}\,\sqrt{431}\right)^{\frac{1}{3}}}\right) \\
& + c_3 \mathrm{e}^{-\frac{\left(\left(108+3\sqrt{1293}\right)^{\frac{2}{3}}+3\right)t}{6\left(108+3\sqrt{1293}\right)^{\frac{1}{3}}}} \cos\left(\frac{\sqrt{3}\left(\left(108+3\sqrt{3}\,\sqrt{431}\right)^{\frac{2}{3}}-3\right)t}{6\left(108+3\sqrt{3}\,\sqrt{431}\right)^{\frac{1}{3}}}\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 69

```
DSolve[x'''[t]-x'[t]-8*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
x(t) \to {} & c_2 \exp\left(t\text{Root}\left[\#1^3-\#1-8\&,2\right]\right)+c_3 \exp\left(t\text{Root}\left[\#1^3-\#1-8\&,3\right]\right) \\
& + c_1 \exp\left(t\text{Root}\left[\#1^3-\#1-8\&,1\right]\right)
\end{aligned}
$$

14.5 problem 1(e)

Internal problem ID [11505]
Internal file name [`OUTPUT/10487_Thursday_May_18_2023_04_20_52_AM_35911259/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.5 Higher order equations. Exercises page 130
Problem number: 1(e).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{x''' + x'' = 2\,\mathrm{e}^{t} + 3t^{2}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE And x_p is a particular solution to the nonhomogeneous ODE. x_h is the solution to

$$x''' + x'' = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$x_h(t) = c_1\mathrm{e}^{-t} + c_2 + c_3 t$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
x_1 &= \mathrm{e}^{-t} \\
x_2 &= 1 \\
x_3 &= t
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
x''' + x'' = 2\,\mathrm{e}^{t} + 3t^2
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2\,\mathrm{e}^{t} + 3t^2
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{t}\}, \{1, t, t^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, t, \mathrm{e}^{-t}\}
$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$
[\{\mathrm{e}^{t}\}, \{t, t^2, t^3\}]
$$

Since t is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$
[\{\mathrm{e}^{t}\}, \{t^2, t^3, t^4\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
x_p = A_1\mathrm{e}^{t} + A_2t^2 + A_3t^3 + A_4t^4
$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_1\mathrm{e}^{t} + 6A_3 + 24A_4t + 2A_2 + 6A_3t + 12A_4t^2 = 2\,\mathrm{e}^{t} + 3t^2
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 3, A_3 = -1, A_4 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = e^t + 3t^2 - t^3 + \frac{t^4}{4}$$

Therefore the general solution is

$$\begin{aligned}
x &= x_h + x_p \\
&= (c_1 e^{-t} + c_2 + c_3 t) + \left(e^t + 3t^2 - t^3 + \frac{t^4}{4}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{-t} + c_2 + c_3 t + e^t + 3t^2 - t^3 + \frac{t^4}{4} \tag{1}$$

Verification of solutions

$$x = c_1 e^{-t} + c_2 + c_3 t + e^t + 3t^2 - t^3 + \frac{t^4}{4}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 3*_a^2+2*exp(_a)-_b(_a), _b(_a)`   ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve(diff(x(t),t$3)+diff(x(t),t$2)=2*exp(t)+3*t^2,x(t), singsol=all)
```

$$x(t) = \frac{t^4}{4} + 3t^2 - t^3 + \mathrm{e}^{-t} c_1 + \mathrm{e}^t + c_2 t + c_3$$

✓ Solution by Mathematica

Time used: 0.308 (sec). Leaf size: 40

```
DSolve[x'''[t]+x''[t]==2*Exp[t]+3*t^2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{t^4}{4} - t^3 + 3t^2 + e^t + c_3 t + c_1 e^{-t} + c_2$$

14.6 problem 1(f)

14.6.1 Maple step by step solution 1820

Internal problem ID [11506]
Internal file name [`OUTPUT/10488_Thursday_May_18_2023_04_20_54_AM_14995609/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.5 Higher order equations. Exercises page 130
Problem number: 1(f).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{x''' - 8x = 0}$$

The characteristic equation is

$$\lambda^3 - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = i\sqrt{3} - 1$$
$$\lambda_3 = -i\sqrt{3} - 1$$

Therefore the homogeneous solution is

$$x_h(t) = c_1 e^{2t} + e^{\left(i\sqrt{3}-1\right)t} c_2 + e^{\left(-i\sqrt{3}-1\right)t} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$x_1 = e^{2t}$$
$$x_2 = e^{\left(i\sqrt{3}-1\right)t}$$
$$x_3 = e^{\left(-i\sqrt{3}-1\right)t}$$

Summary
The solution(s) found are the following

$$x = c_1 e^{2t} + e^{\left(i\sqrt{3}-1\right)t} c_2 + e^{\left(-i\sqrt{3}-1\right)t} c_3 \tag{1}$$

Verification of solutions

$$x = c_1 e^{2t} + e^{\left(i\sqrt{3}-1\right)t} c_2 + e^{\left(-i\sqrt{3}-1\right)t} c_3$$

Verified OK.

14.6.1 Maple step by step solution

Let's solve

$x''' - 8x = 0$

- Highest derivative means the order of the ODE is 3

  x'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $x_1(t)$

    $x_1(t) = x$

  ◦ Define new variable $x_2(t)$

    $x_2(t) = x'$

  ◦ Define new variable $x_3(t)$

    $x_3(t) = x''$

  ◦ Isolate for $x_3'(t)$ using original ODE

    $x_3'(t) = 8x_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[x_2(t) = x_1'(t), x_3(t) = x_2'(t), x_3'(t) = 8x_1(t)]$

- Define vector

  $$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- System to solve

$$\overrightarrow{x}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 0 & 0 \end{bmatrix} \cdot \overrightarrow{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}\sqrt{3}-1, \begin{bmatrix} \frac{1}{\left(-\mathrm{I}\sqrt{3}-1\right)^2} \\ \frac{1}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{3}-1, \begin{bmatrix} \frac{1}{\left(\mathrm{I}\sqrt{3}-1\right)^2} \\ \frac{1}{\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}\sqrt{3}-1, \begin{bmatrix} \frac{1}{\left(-\mathrm{I}\sqrt{3}-1\right)^2} \\ \frac{1}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(-I\sqrt{3}-1\right)t}\cdot\left[\begin{array}{c}\frac{1}{\left(-I\sqrt{3}-1\right)^{2}}\\ \frac{1}{-I\sqrt{3}-1}\\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-t}\cdot\left(\cos\left(\sqrt{3}\,t\right)-I\sin\left(\sqrt{3}\,t\right)\right)\cdot\left[\begin{array}{c}\frac{1}{\left(-I\sqrt{3}-1\right)^{2}}\\ \frac{1}{-I\sqrt{3}-1}\\ 1\end{array}\right]$$

- Simplify expression

$$e^{-t}\cdot\left[\begin{array}{c}\frac{\cos\left(\sqrt{3}\,t\right)-I\sin\left(\sqrt{3}\,t\right)}{\left(-I\sqrt{3}-1\right)^{2}}\\ \frac{\cos\left(\sqrt{3}\,t\right)-I\sin\left(\sqrt{3}\,t\right)}{-I\sqrt{3}-1}\\ \cos\left(\sqrt{3}\,t\right)-I\sin\left(\sqrt{3}\,t\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_{2}(t)=e^{-t}\cdot\left[\begin{array}{c}-\frac{\cos\left(\sqrt{3}\,t\right)}{8}-\frac{\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{8}\\ -\frac{\cos\left(\sqrt{3}\,t\right)}{4}+\frac{\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{4}\\ \cos\left(\sqrt{3}\,t\right)\end{array}\right],\vec{x}_{3}(t)=e^{-t}\cdot\left[\begin{array}{c}-\frac{\cos\left(\sqrt{3}\,t\right)\sqrt{3}}{8}+\frac{\sin\left(\sqrt{3}\,t\right)}{8}\\ \frac{\cos\left(\sqrt{3}\,t\right)\sqrt{3}}{4}+\frac{\sin\left(\sqrt{3}\,t\right)}{4}\\ -\sin\left(\sqrt{3}\,t\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{x}=c_{1}\vec{x}_{1}+c_{2}\vec{x}_{2}(t)+c_{3}\vec{x}_{3}(t)$$

- Substitute solutions into the general solution

$$\vec{x}=c_{1}e^{2t}\cdot\left[\begin{array}{c}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{array}\right]+c_{2}e^{-t}\cdot\left[\begin{array}{c}-\frac{\cos\left(\sqrt{3}\,t\right)}{8}-\frac{\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{8}\\ -\frac{\cos\left(\sqrt{3}\,t\right)}{4}+\frac{\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{4}\\ \cos\left(\sqrt{3}\,t\right)\end{array}\right]+c_{3}e^{-t}\cdot\left[\begin{array}{c}-\frac{\cos\left(\sqrt{3}\,t\right)\sqrt{3}}{8}+\frac{\sin\left(\sqrt{3}\,t\right)}{8}\\ \frac{\cos\left(\sqrt{3}\,t\right)\sqrt{3}}{4}+\frac{\sin\left(\sqrt{3}\,t\right)}{4}\\ -\sin\left(\sqrt{3}\,t\right)\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$x=-\frac{e^{-t}\left(\sqrt{3}\,c_{3}+c_{2}\right)\cos\left(\sqrt{3}\,t\right)}{8}-\frac{e^{-t}\left(c_{2}\sqrt{3}-c_{3}\right)\sin\left(\sqrt{3}\,t\right)}{8}+\frac{c_{1}e^{2t}}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(x(t),t$3)-8*x(t)=0,x(t), singsol=all)
```

$$x(t) = c_1 e^{2t} + c_2 e^{-t} \sin\left(\sqrt{3}\, t\right) + c_3 e^{-t} \cos\left(\sqrt{3}\, t\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 52

```
DSolve[x'''[t]-x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-t/2}\left(c_1 e^{3t/2} + c_2 \cos\left(\frac{\sqrt{3}t}{2}\right) + c_3 \sin\left(\frac{\sqrt{3}t}{2}\right)\right)$$

14.7 problem 2

14.7.1 Maple step by step solution 1827

Internal problem ID [11507]
Internal file name [`OUTPUT/10489_Thursday_May_18_2023_04_20_55_AM_52111856/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 2, Second order linear equations. Section 2.5 Higher order equations. Exercises page 130
Problem number: 2.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _missing_x]]`

$$\boxed{x''' + x'' - x' - 4x = 0}$$

With initial conditions

$$[x(0) = 1, x'(0) = 0, x''(0) = -1]$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - \lambda - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = \frac{\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}}}{6} + \frac{8}{3\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}}} - \frac{1}{3}$$

$$\lambda_2 = -\frac{\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}}}{12} - \frac{4}{3\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}}} - \frac{1}{3} + \frac{i\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6} - \frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}\right)}{2}$$

$$\lambda_3 = -\frac{\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}}}{12} - \frac{4}{3\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}}} - \frac{1}{3} - \frac{i\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6} - \frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}\right)}{2}$$

Therefore the homogeneous solution is

$$x_h(t) = \mathrm{e}^{\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}+\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)t} c_1 + \mathrm{e}^{\left(-\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{12}-\frac{4}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}+\frac{i\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}-\frac{8}{3\left(388+36\sqrt{113}\right)}\right.}{2}\right.}$$

The fundamental set of solutions for the homogeneous solution are the following

$$x_1 = \mathrm{e}^{\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}+\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)t}$$

$$x_2 = \mathrm{e}^{\left(-\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{12}-\frac{4}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}+\frac{i\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}-\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}\right)}{2}\right)t}$$

$$x_3 = \mathrm{e}^{\left(-\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{12}-\frac{4}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}-\frac{i\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}-\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}\right)}{2}\right)t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \mathrm{e}^{\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}+\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)t} c_1 + \mathrm{e}^{\left(-\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{12}-\frac{4}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}+\frac{i\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}-\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}\right)}{2}\right.} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 + c_2 + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$x' = \left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}+\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)\mathrm{e}^{\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}+\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)t} c_1 + \left(-\frac{\left(388+3\right.}{1}\right.$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = \frac{\left(i(c_2 - c_3)\sqrt{3} + 2c_1 - c_2 - c_3\right)\left(388 + 36\sqrt{113}\right)^{\frac{2}{3}} + 4(-c_1 - c_2 - c_3)\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}} + 16i(-c_2 + c}{12\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}}} \tag{2A}$$

Taking two derivatives of the solution gives

$$x'' = \left(\frac{\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}}}{6} + \frac{8}{3\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}}} - \frac{1}{3}\right)^2 \mathrm{e}^{\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6} + \frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}} - \frac{1}{3}\right)t} c_1 + \left(-\frac{(388 +}{}\right.$$

substituting $x'' = -1$ and $t = 0$ in the above gives

$$-1 = \frac{\frac{9\left(-\frac{(-2c_1+c_2+c_3)\sqrt{113}}{9} - \frac{i(c_2-c_3)\sqrt{339}}{9} - i(c_2-c_3)\sqrt{3} + 2c_1 - c_2 - c_3\right)\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{2} - (-c_1 - c_2 - c_3)\left(388 + 36\sqrt{113}\right)^{\frac{2}{3}}}{\left(388 + 36\sqrt{113}\right)} \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$c_1 = \frac{153228 + 2\left(34239 + 3223\sqrt{113}\right)\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}} - 4\left(97\sqrt{113} + 1017\right)\left(388 + 36\sqrt{113}\right)^{\frac{2}{3}} + 14444}{\left(97\sqrt{113} + 1017\right)\left(9\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}}\sqrt{113} + 4\left(388 + 36\sqrt{113}\right)^{\frac{2}{3}} + 97\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}} + 64\right.}$$

$$c_2 = \frac{\sqrt{3}\left(-2034i\sqrt{3}\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}} + 4850\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}} + 97i\sqrt{3}\left(388 + 36\sqrt{113}\right)^{\frac{2}{3}}\sqrt{113} + 105(\right.}{}$$

$$c_3 = \frac{\sqrt{3}\left(-\sqrt{339}\left(388 + 36\sqrt{113}\right)^{\frac{2}{3}} + 32\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}}\sqrt{339} - 25\sqrt{3}\left(388 + 36\sqrt{113}\right)^{\frac{2}{3}} + 776\sqrt{339} +\right.}{}$$

Substituting these values back in above solution results in

$$x = \text{Expression too large to display}$$

Summary
The solution(s) found are the following

$$x \tag{1}$$

$$= \frac{\left(\left(-10582601696i\sqrt{3} - 995527392i\sqrt{339} - 4134180912\sqrt{113} - 43946945712\right)\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}} + (-1\right.}{}$$

Verification of solutions

$$x$$

$$= \frac{\left(\left(-10582601696i\sqrt{3} - 995527392i\sqrt{339} - 4134180912\sqrt{113} - 43946945712\right)\left(388 + 36\sqrt{113}\right)^{\frac{1}{3}} + (-1\right.}{}$$

Verified OK.

14.7.1 Maple step by step solution

Let's solve

$$\left[x''' + x'' - x' - 4x = 0, x(0) = 1, x'\Big|_{\{t=0\}} = 0, x''\Big|_{\{t=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 3

  x'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $x_1(t)$

    $x_1(t) = x$
  - Define new variable $x_2(t)$

    $x_2(t) = x'$
  - Define new variable $x_3(t)$

    $x_3(t) = x''$
  - Isolate for $x_3'(t)$ using original ODE

    $x_3'(t) = -x_3(t) + x_2(t) + 4x_1(t)$

  Convert linear ODE into a system of first order ODEs

$[x_2(t) = x_1'(t)\,, x_3(t) = x_2'(t)\,, x_3'(t) = -x_3(t) + x_2(t) + 4x_1(t)]$

- Define vector

$$\vec{x}(t) = \left[\begin{array}{c} x_1(t) \\ x_2(t) \\ x_3(t) \end{array}\right]$$

- System to solve

$$\vec{x}'(t) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 1 & -1 \end{array}\right] \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 1 & -1 \end{array}\right]$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6} + \frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}} - \frac{1}{3}, \left[\begin{array}{c} \frac{1}{\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6} + \frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}} - \frac{1}{3}\right)^2} \\ \frac{1}{\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6} + \frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}} - \frac{1}{3}} \\ 1 \end{array}\right]\right], \left[-\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{12}\right.\right.$$

- Consider eigenpair

$$\left[\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}+\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}, \left[\begin{array}{c} \frac{1}{\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}+\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)^{2}} \\ \frac{1}{\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}+\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_{1} = \mathrm{e}^{\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}+\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)t} \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}+\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)^{2}} \\ \frac{1}{\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}+\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{12}-\frac{4}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}-\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}\right)}{2}, \ldots\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{12}-\frac{4}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}-\frac{I\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}-\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}\right)}{2}\right)t} \cdot \left[\frac{\left(-\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{12}-\frac{4}{3\left(388+36\sqrt{113}\right.}\right.}{-\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{12}-\frac{4}{3\left(388+36\sqrt{113}\right.}}\right.$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{\left(-\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{12}-\frac{4}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)t} \cdot \left(\cos\left(\frac{\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}-\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}\right)t}{2}\right)-I\sin\left(\frac{\sqrt{3}\left(\frac{\left(388+3\right.}{}\right.}{}\right.\right.$$

- Simplify expression

$$
e^{\left(-\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{12}-\frac{4}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)t}\cdot\left[\begin{array}{c}
\frac{\cos\left(\frac{\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}-\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}\right)t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}\cdots\right.}{\cdots}\right.}{\left(-\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{12}-\frac{4}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}-\cdots\right.}{2}\right.} \\
\frac{\cos\left(\frac{\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}-\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}\right)t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}\cdots\right.}{\cdots}\right.}{-\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{12}-\frac{4}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}-\cdots\right.}{2}} \\
\cos\left(\frac{\sqrt{3}\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}-\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}\right)t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(\frac{\left(388+\cdots\right.}{\cdots}\right.}{\cdots}\right.
\end{array}\right.
$$

- Both real and imaginary parts are solutions to the homogeneous system

$$
\left[\vec{x}_2(t)=e^{\left(-\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{12}-\frac{4}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)t}\cdot\left[-\frac{18\left(388+36\sqrt{113}\right)^{\frac{2}{3}}\left(\sqrt{3}\left(388+36\sqrt{113}\right)^{\frac{4}{3}}\sin\left(\frac{\sqrt{3}\left(\left(388+36\sqrt{\cdots}\right.\right.}{12\left(388+36\cdots\right.}\right.\right.\right.}{\cdots}\right.\right.
$$

- General solution to the system of ODEs

$\vec{x}=c_1\vec{x}_1+c_2\vec{x}_2(t)+c_3\vec{x}_3(t)$

- Substitute solutions into the general solution

$$\vec{x} = \mathrm{e}^{\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}+\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)t} c_1 \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}+\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)^{2}} \\ \frac{1}{\frac{\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}{6}+\frac{8}{3\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}-\frac{1}{3}} \\ 1 \end{array}\right] + c_2 \mathrm{e}^{\left(-\frac{\left(388+36\sqrt{113}\right.}{12}\right.}$$

- First component of the vector is the solution to the ODE

$$x = -\frac{23571\left(\left(\left(-\frac{11\left(\sqrt{113}+\frac{115}{11}\right)\left(\sqrt{3}\,c_3+c_2\right)\left(388+36\sqrt{113}\right)^{\frac{2}{3}}}{108}+c_2\left(\sqrt{113}+\frac{97}{9}\right)\left(388+36\sqrt{113}\right)^{\frac{1}{3}}+\frac{53\left(\sqrt{113}+\frac{557}{53}\right)\left(-\sqrt{3}\,c_3+c_2\right)}{27}\right)\cos\right.\right.}{}$$

- Use the initial condition $x(0) = 1$

$$1 = -\frac{23571\left(-\frac{11\left(\sqrt{113}+\frac{115}{11}\right)\left(\sqrt{3}\,c_3+c_2\right)\left(388+36\sqrt{113}\right)^{\frac{2}{3}}}{108}+c_2\left(\sqrt{113}+\frac{97}{9}\right)\left(388+36\sqrt{113}\right)^{\frac{1}{3}}+\frac{53\left(\sqrt{113}+\frac{557}{53}\right)\left(-\sqrt{3}\,c_3+c_2\right)}{27}+\left(-\frac{111}{27}\right.\right.}{2097152}$$

- Calculate the 1st derivative of the solution

$$x' = -\frac{23571\left(\left(-\frac{\left(-\frac{11\left(\sqrt{113}+\frac{115}{11}\right)\left(\sqrt{3}\,c_3+c_2\right)\left(388+36\sqrt{113}\right)^{\frac{2}{3}}}{108}+c_2\left(\sqrt{113}+\frac{97}{9}\right)\left(388+36\sqrt{113}\right)^{\frac{1}{3}}+\frac{53\left(\sqrt{113}+\frac{557}{53}\right)\left(-\sqrt{3}\,c_3+c_2\right)}{27}\right)\sqrt{3}\left(\left(\right.\right.}{12\left(388+36\sqrt{113}\right)^{\frac{1}{3}}}\right.\right.}{}$$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

$$0 = -\frac{23571 \Big(-\frac{11 \big((c_2\sqrt{3}-c_3)(\sqrt{113}+\frac{115}{11})(388+36\sqrt{113})^{\frac{2}{3}} + \frac{108(\sqrt{113}+\frac{97}{9})c_3(388+36\sqrt{113})^{\frac{1}{3}}}{11} + \frac{212(\sqrt{113}+\frac{557}{53})(c_2\sqrt{3}+c_3)}{11} \big)\sqrt{3}\big(\big(388 \cdots}{1296(388+36\sqrt{113})^{\frac{1}{3}}} \cdots}{\cdots}$$

- Calculate the 2nd derivative of the solution

$$x'' = -\frac{23571 \Big(\Big(-\frac{\big(-\frac{11(\sqrt{113}+\frac{115}{11})(\sqrt{3}\,c_3+c_2)(388+36\sqrt{113})^{\frac{2}{3}}}{108} + c_2(\sqrt{113}+\frac{97}{9})(388+36\sqrt{113})^{\frac{1}{3}} + \frac{53(\sqrt{113}+\frac{557}{53})(-\sqrt{3}\,c_3+c_2)}{27} \big)\big(\big(388 \cdots}{48(388+36\sqrt{113})^{\frac{2}{3}}} \cdots}{\cdots}$$

- Use the initial condition $x''\Big|_{\{t=0\}} = -1$

$$-1 = -\frac{23571 \Big(-\frac{\big(-\frac{11(\sqrt{113}+\frac{115}{11})(\sqrt{3}\,c_3+c_2)(388+36\sqrt{113})^{\frac{2}{3}}}{108} + c_2(\sqrt{113}+\frac{97}{9})(388+36\sqrt{113})^{\frac{1}{3}} + \frac{53(\sqrt{113}+\frac{557}{53})(-\sqrt{3}\,c_3+c_2)}{27} \big)\big(\big(388 \cdots}{48(388+36\sqrt{113})^{\frac{2}{3}}} \cdots}{\cdots}$$

- Solve for the unknown coefficients
- Solution to the IVP

$$x = \frac{\Big(\Big(\big(47061043431710185670634348234223852 9\sqrt{113}+500265753778517505901362980996237679 3\big)(388+36\sqrt{113})^{\frac{4}{3}} + 2\big(1026 \cdots}{\cdots}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 296

```
dsolve([diff(x(t),t$3)+diff(x(t),t$2)-diff(x(t),t)-4*x(t)=0,x(0) = 1, D(x)(0) = 0, (D@@2)(x)
```

$x(t)$

$$= \frac{\left(\left((32\sqrt{113}+352)\left(388+36\sqrt{113}\right)^{\frac{1}{3}}+\left(-\sqrt{113}-25\right)\left(388+36\sqrt{113}\right)^{\frac{2}{3}}+776\sqrt{113}+8136\right)\cos\left(\frac{\sqrt{3}}{}\right.\right.}{}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 748

```
DSolve[{x'''[t]+x''[t]-x'[t]-4*x[t]==0,{x[0]==1,x'[0]==0,x''[0]==-1}},x[t],t,IncludeSingular
```

$x(t)$

$$\rightarrow \frac{\mathrm{Root}\left[\#1^3+\#1^2-\#1-4\&,1\right]\exp\left(t\mathrm{Root}\left[\#1^3+\#1^2-\#1-4\&,2\right]\right)-\mathrm{Root}\left[\#1^3+\#1^2-\#1-4\right.}{}$$

15 Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156

15.1 problem 6(a) 1836
15.2 problem 6(b) 1841
15.3 problem 6(c) 1846
15.4 problem 6(d) 1851
15.5 problem 6(e) 1856
15.6 problem 6(f) 1862
15.7 problem 6(g) 1867
15.8 problem 6(h) 1873
15.9 problem 6(i) 1879
15.10 problem 6(j) 1885
15.11 problem 11 1890
15.12 problem 12 1894
15.13 problem 14 1899
15.14 problem 15 1904

15.1 problem 6(a)

15.1.1 Existence and uniqueness analysis 1836
15.1.2 Solving as laplace ode 1837
15.1.3 Maple step by step solution 1838

Internal problem ID [11508]
Internal file name [`OUTPUT/10490_Thursday_May_18_2023_04_20_58_AM_59524997/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156
Problem number: 6(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{x' + 5x = \text{Heaviside}\,(-2+t)}$$

With initial conditions

$$[x(0) = 1]$$

15.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= 5 \\ q(t) &= \text{Heaviside}\,(-2+t) \end{aligned}$$

Hence the ode is

$$x' + 5x = \text{Heaviside}(-2 + t)$$

The domain of $p(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \text{Heaviside}(-2 + t)$ is

$$\{t < 2 \lor 2 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.1.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - x(0) + 5Y(s) = \frac{e^{-2s}}{s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 1 + 5Y(s) = \frac{e^{-2s}}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{e^{-2s} + s}{s(s + 5)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} x &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{e^{-2s} + s}{s(s + 5)}\right) \\ &= e^{-5t} + \frac{\text{Heaviside}(-2 + t)\left(1 - e^{-5t+10}\right)}{5} \end{aligned}$$

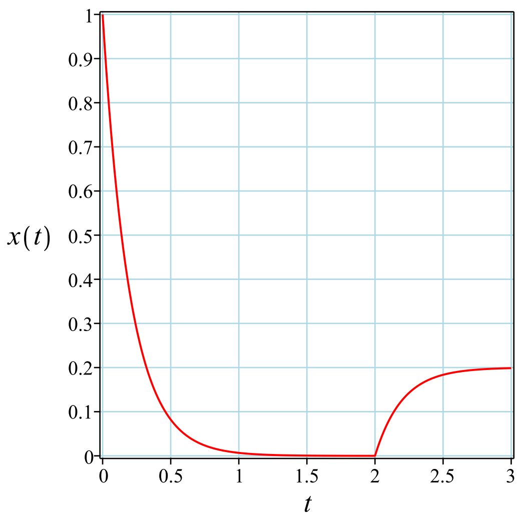
Hence the final solution is

$$x = e^{-5t} + \frac{\text{Heaviside}\,(-2+t)\,(1-e^{-5t+10})}{5}$$

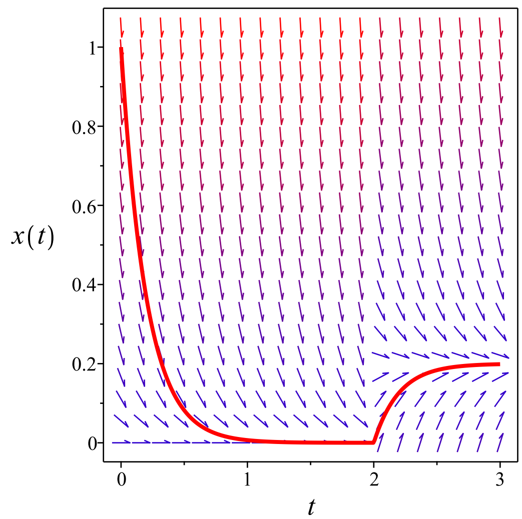
Summary

The solution(s) found are the following

$$x = e^{-5t} + \frac{\text{Heaviside}\,(-2+t)\,(1-e^{-5t+10})}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{-5t} + \frac{\text{Heaviside}\,(-2+t)\,(1-e^{-5t+10})}{5}$$

Verified OK.

15.1.3 Maple step by step solution

Let's solve

$[x' + 5x = \mathit{Heaviside}(-2+t)\,, x(0) = 1]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -5x + \mathit{Heaviside}(-2+t)$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + 5x = \mathit{Heaviside}(-2+t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(x' + 5x\right) = \mu(t)\,\mathit{Heaviside}(-2+t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

  $\mu(t)\left(x' + 5x\right) = \mu'(t)\,x + \mu(t)\,x'$

- Isolate $\mu'(t)$

  $\mu'(t) = 5\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{5t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,x)\right) dt = \int \mu(t)\,\mathit{Heaviside}(-2+t)\,dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,x = \int \mu(t)\,\mathit{Heaviside}(-2+t)\,dt + c_1$

- Solve for x

  $x = \frac{\int \mu(t)\mathit{Heaviside}(-2+t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{5t}$

  $x = \frac{\int \mathrm{e}^{5t}\mathit{Heaviside}(-2+t)dt + c_1}{\mathrm{e}^{5t}}$

- Evaluate the integrals on the rhs

  $x = \frac{\frac{\mathrm{e}^{5t}\mathit{Heaviside}(-2+t)}{5} - \frac{\mathit{Heaviside}(-2+t)\mathrm{e}^{10}}{5} + c_1}{\mathrm{e}^{5t}}$

- Simplify

  $x = -\frac{\mathit{Heaviside}(-2+t)\mathrm{e}^{-5t+10}}{5} + \frac{\mathit{Heaviside}(-2+t)}{5} + c_1\mathrm{e}^{-5t}$

- Use initial condition $x(0) = 1$

  $1 = c_1$

- Solve for c_1

  $c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

$$x = -\frac{Heaviside(-2+t)\,\mathrm{e}^{-5t+10}}{5} + \frac{Heaviside(-2+t)}{5} + \mathrm{e}^{-5t}$$

- Solution to the IVP

$$x = -\frac{Heaviside(-2+t)\,\mathrm{e}^{-5t+10}}{5} + \frac{Heaviside(-2+t)}{5} + \mathrm{e}^{-5t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 5.344 (sec). Leaf size: 25

```
dsolve([diff(x(t),t)+5*x(t)=Heaviside(t-2),x(0) = 1],x(t), singsol=all)
```

$$x(t) = -\frac{\mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{-5t+10}}{5} + \frac{\mathrm{Heaviside}\,(t-2)}{5} + \mathrm{e}^{-5t}$$

✓ Solution by Mathematica

Time used: 0.093 (sec). Leaf size: 37

```
DSolve[{x'[t]+5*x[t]==UnitStep[t-2],{x[0]==1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \begin{cases} e^{-5t} & t \leq 2 \\ \frac{1}{5}e^{-5t}(5 - e^{10} + e^{5t}) & \text{True} \end{cases}$$

15.2 problem 6(b)

15.2.1 Existence and uniqueness analysis 1841
15.2.2 Solving as laplace ode 1842
15.2.3 Maple step by step solution 1844

Internal problem ID [11509]
Internal file name [`OUTPUT/10491_Thursday_May_18_2023_04_21_00_AM_63822021/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156
Problem number: 6(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{x' + x = \sin(2t)}$$

With initial conditions

$$[x(0) = 0]$$

15.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= \sin(2t) \end{aligned}$$

Hence the ode is

$$x' + x = \sin(2t)$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \sin(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.2.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - x(0) + Y(s) = \frac{2}{s^2 + 4} \tag{1}$$

Replacing initial condition gives

$$sY(s) + Y(s) = \frac{2}{s^2 + 4}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{2}{(s^2 + 4)(s + 1)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{2}{5(s + 1)} + \frac{-\frac{1}{5} - \frac{i}{10}}{s - 2i} + \frac{-\frac{1}{5} + \frac{i}{10}}{s + 2i}$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{2}{5\,(s+1)}\right) &= \frac{2\,\mathrm{e}^{-t}}{5} \\
\mathcal{L}^{-1}\left(\frac{-\frac{1}{5}-\frac{i}{10}}{s-2i}\right) &= \left(-\frac{1}{5}-\frac{i}{10}\right)\mathrm{e}^{2it} \\
\mathcal{L}^{-1}\left(\frac{-\frac{1}{5}+\frac{i}{10}}{s+2i}\right) &= \left(-\frac{1}{5}+\frac{i}{10}\right)\mathrm{e}^{-2it}
\end{aligned}
$$

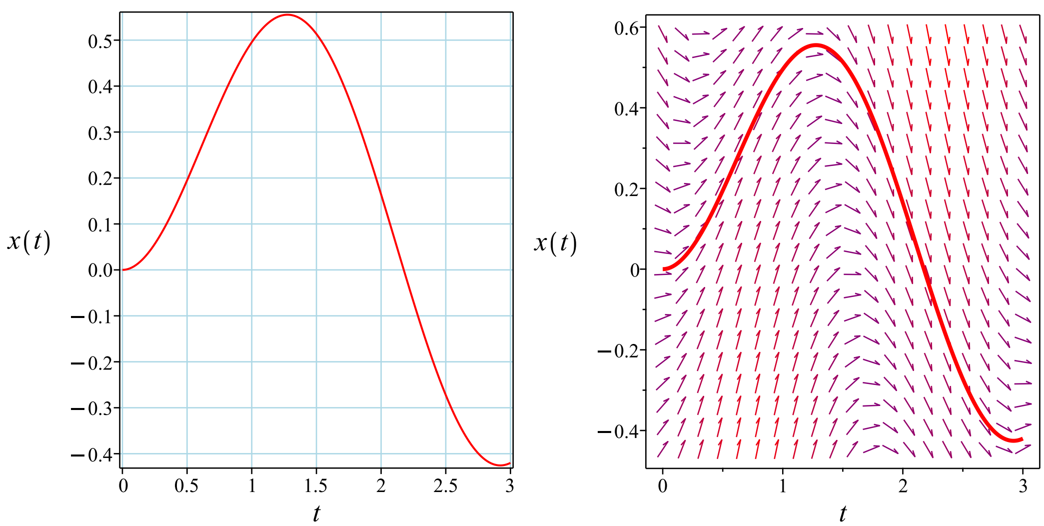
Adding the above results and simplifying gives

$$
x = \frac{2\,\mathrm{e}^{-t}}{5} - \frac{2\cos(2t)}{5} + \frac{\sin(2t)}{5}
$$

Summary

The solution(s) found are the following

$$
x = \frac{2\,\mathrm{e}^{-t}}{5} - \frac{2\cos(2t)}{5} + \frac{\sin(2t)}{5} \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
x = \frac{2\,\mathrm{e}^{-t}}{5} - \frac{2\cos(2t)}{5} + \frac{\sin(2t)}{5}
$$

Verified OK.

15.2.3 Maple step by step solution

Let's solve

$[x' + x = \sin(2t), x(0) = 0]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -x + \sin(2t)$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + x = \sin(2t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)(x' + x) = \mu(t)\sin(2t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, x)$

  $\mu(t)(x' + x) = \mu'(t)\, x + \mu(t)\, x'$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, x)\right) dt = \int \mu(t)\sin(2t)\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, x = \int \mu(t)\sin(2t)\, dt + c_1$

- Solve for x

  $x = \frac{\int \mu(t)\sin(2t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{t}$

  $x = \frac{\int \mathrm{e}^{t}\sin(2t)dt + c_1}{\mathrm{e}^{t}}$

- Evaluate the integrals on the rhs

  $x = \frac{-\frac{2\,\mathrm{e}^{t}\cos(2t)}{5} + \frac{\mathrm{e}^{t}\sin(2t)}{5} + c_1}{\mathrm{e}^{t}}$

- Simplify

  $x = \frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5} + c_1\mathrm{e}^{-t}$

- Use initial condition $x(0) = 0$

  $0 = -\frac{2}{5} + c_1$

- Solve for c_1

  $c_1 = \frac{2}{5}$

- Substitute $c_1 = \frac{2}{5}$ into general solution and simplify

  $x = \frac{2\,\mathrm{e}^{-t}}{5} - \frac{2\cos(2t)}{5} + \frac{\sin(2t)}{5}$

- Solution to the IVP

  $x = \frac{2\,\mathrm{e}^{-t}}{5} - \frac{2\cos(2t)}{5} + \frac{\sin(2t)}{5}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 4.485 (sec). Leaf size: 23

```
dsolve([diff(x(t),t)+x(t)=sin(2*t),x(0) = 0],x(t), singsol=all)
```

$$x(t) = \frac{2\,\mathrm{e}^{-t}}{5} - \frac{2\cos(2t)}{5} + \frac{\sin(2t)}{5}$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 27

```
DSolve[{x'[t]+x[t]==Sin[2*t],{x[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{5}\left(2e^{-t} + \sin(2t) - 2\cos(2t)\right)$$

15.3 problem 6(c)

15.3.1 Existence and uniqueness analysis 1846
15.3.2 Maple step by step solution 1849

Internal problem ID [11510]
Internal file name [`OUTPUT/10492_Thursday_May_18_2023_04_21_02_AM_42539011/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156
Problem number: 6(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' - x' - 6x = 0}$$

With initial conditions

$$[x(0) = 2, x'(0) = -1]$$

15.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= -6 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' - x' - 6x = 0$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(x') &= sY(s) - x(0)\\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) - sY(s) + x(0) - 6Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}x(0) &= 2\\ x'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 3 - 2s - sY(s) - 6Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2s - 3}{s^2 - s - 6}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{7}{5(s + 2)} + \frac{3}{5(s - 3)}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{7}{5\,(s+2)}\right) = \frac{7\,\mathrm{e}^{-2t}}{5}$$
$$\mathcal{L}^{-1}\left(\frac{3}{5\,(s-3)}\right) = \frac{3\,\mathrm{e}^{3t}}{5}$$

Adding the above results and simplifying gives

$$x = \frac{7\,\mathrm{e}^{-2t}}{5} + \frac{3\,\mathrm{e}^{3t}}{5}$$

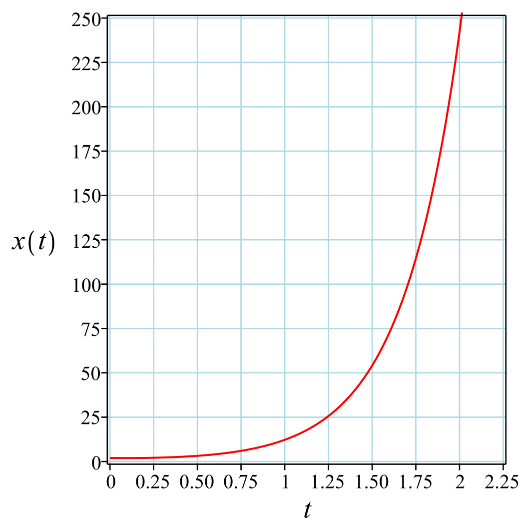
Simplifying the solution gives

$$x = \frac{(3\,\mathrm{e}^{5t}+7)\,\mathrm{e}^{-2t}}{5}$$

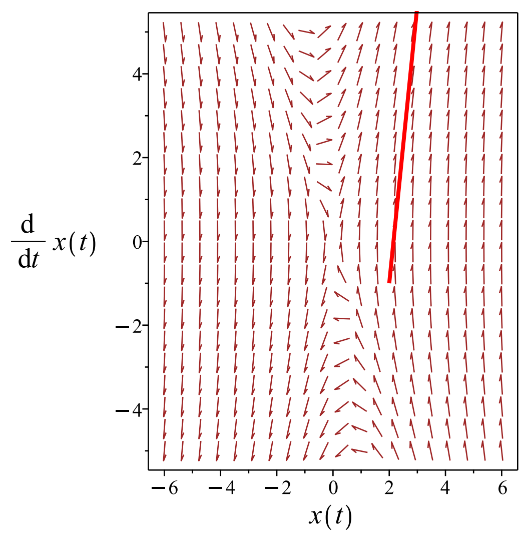
Summary

The solution(s) found are the following

$$x = \frac{(3\,\mathrm{e}^{5t}+7)\,\mathrm{e}^{-2t}}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{(3\,\mathrm{e}^{5t}+7)\,\mathrm{e}^{-2t}}{5}$$

Verified OK.

15.3.2 Maple step by step solution

Let's solve

$$\left[x'' - x' - 6x = 0, x(0) = 2, x'\Big|_{\{t=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 - r - 6 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 3)$

- 1st solution of the ODE

  $x_1(t) = e^{-2t}$

- 2nd solution of the ODE

  $x_2(t) = e^{3t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 e^{-2t} + c_2 e^{3t}$

□ Check validity of solution $x = c_1 e^{-2t} + c_2 e^{3t}$

  - Use initial condition $x(0) = 2$

    $2 = c_1 + c_2$

  - Compute derivative of the solution

    $x' = -2c_1 e^{-2t} + 3c_2 e^{3t}$

  - Use the initial condition $x'\Big|_{\{t=0\}} = -1$

    $-1 = -2c_1 + 3c_2$

  - Solve for c_1 and c_2

    $\left\{c_1 = \frac{7}{5}, c_2 = \frac{3}{5}\right\}$

- Substitute constant values into general solution and simplify

  $x = \frac{(3\,e^{5t}+7)\,e^{-2t}}{5}$

- Solution to the IVP

  $x = \frac{(3\,e^{5t}+7)\,e^{-2t}}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.625 (sec). Leaf size: 17

```
dsolve([diff(x(t),t$2)-diff(x(t),t)-6*x(t)=0,x(0) = 2, D(x)(0) = -1],x(t), singsol=all)
```

$$x(t) = \frac{(3\,e^{5t}+7)\,e^{-2t}}{5}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 23

```
DSolve[{x''[t]-x'[t]-6*x[t]==0,{x[0]==2,x'[0]==-1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{5}e^{-2t}\left(3e^{5t}+7\right)$$

15.4 problem 6(d)

15.4.1 Existence and uniqueness analysis 1851
15.4.2 Maple step by step solution 1854

Internal problem ID [11511]
Internal file name [`OUTPUT/10493_Thursday_May_18_2023_04_21_03_AM_81125574/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156
Problem number: 6(d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' - 2x' + 2x = 0}$$

With initial conditions

$$[x(0) = 0, x'(0) = 1]$$

15.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' - 2x' + 2x = 0$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$
$$\mathcal{L}(x'') = s^2Y(s) - x'(0) - sx(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) - 2sY(s) + 2x(0) + 2Y(s) = 0 \tag{1}$$

But the initial conditions are

$$x(0) = 0$$
$$x'(0) = 1$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - 2sY(s) + 2Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{1}{s^2 - 2s + 2}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{i}{2(s-1-i)} + \frac{i}{2s - 2 + 2i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{i}{2\left(s-1-i\right)}\right) = -\frac{i\,\mathrm{e}^{(1+i)t}}{2}$$
$$\mathcal{L}^{-1}\left(\frac{i}{2s-2+2i}\right) = \frac{i\,\mathrm{e}^{(1-i)t}}{2}$$

Adding the above results and simplifying gives

$$x = \mathrm{e}^{t}\sin\left(t\right)$$

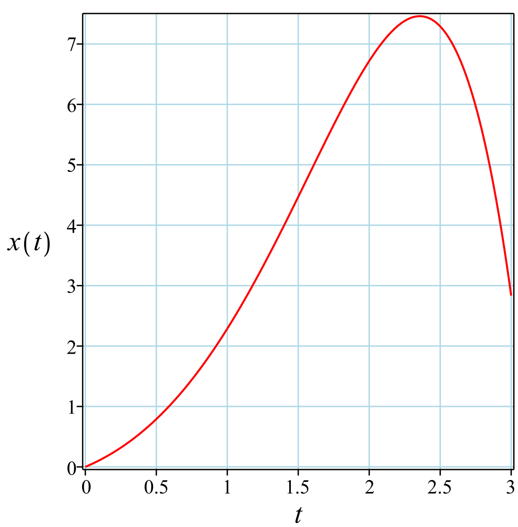
Simplifying the solution gives

$$x = \mathrm{e}^{t}\sin\left(t\right)$$

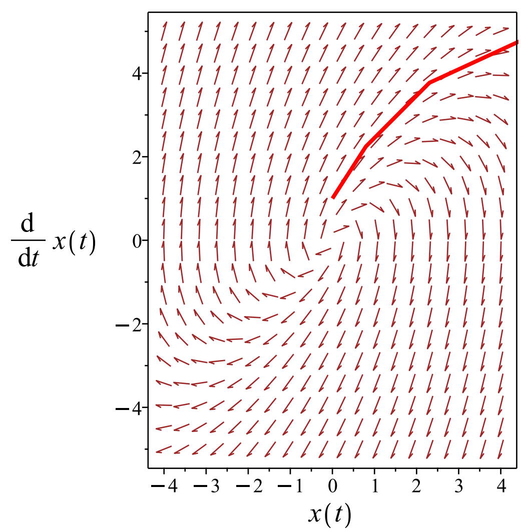
Summary

The solution(s) found are the following

$$x = \mathrm{e}^{t}\sin\left(t\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \mathrm{e}^{t}\sin\left(t\right)$$

Verified OK.

15.4.2 Maple step by step solution

Let's solve

$$\left[x'' - 2x' + 2x = 0, x(0) = 0, x'\bigg|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - \mathrm{I}, 1 + \mathrm{I})$

- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^t \cos(t)$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^t \sin(t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 \mathrm{e}^t \cos(t) + c_2 \mathrm{e}^t \sin(t)$

☐ Check validity of solution $x = c_1 \mathrm{e}^t \cos(t) + c_2 \mathrm{e}^t \sin(t)$

  - Use initial condition $x(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $x' = c_1 \mathrm{e}^t \cos(t) - c_1 \mathrm{e}^t \sin(t) + c_2 \mathrm{e}^t \sin(t) + c_2 \mathrm{e}^t \cos(t)$

  - Use the initial condition $x'\bigg|_{\{t=0\}} = 1$

    $1 = c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $$x = e^t \sin(t)$$

- Solution to the IVP

  $$x = e^t \sin(t)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.594 (sec). Leaf size: 9

```
dsolve([diff(x(t),t$2)-2*diff(x(t),t)+2*x(t)=0,x(0) = 0, D(x)(0) = 1],x(t), singsol=all)
```

$$x(t) = e^t \sin(t)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 11

```
DSolve[{x''[t]-2*x'[t]+2*x[t]==0,{x[0]==0,x'[0]==1}},x[t],t,IncludeSingularSolutions -> True
```

$$x(t) \to e^t \sin(t)$$

15.5 problem 6(e)

15.5.1 Existence and uniqueness analysis 1856
15.5.2 Maple step by step solution 1859

Internal problem ID [11512]
Internal file name [`OUTPUT/10494_Thursday_May_18_2023_04_21_05_AM_22319864/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156
Problem number: 6(e).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x'' - 2x' + 2x = \mathrm{e}^{-t}}$$

With initial conditions

$$[x(0) = 0, x'(0) = 1]$$

15.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 2 \\ F &= \mathrm{e}^{-t} \end{aligned}$$

Hence the ode is

$$x'' - 2x' + 2x = e^{-t}$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(x') &= sY(s) - x(0) \\
\mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) - 2sY(s) + 2x(0) + 2Y(s) = \frac{1}{s+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
x(0) &= 0 \\
x'(0) &= 1
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - 2sY(s) + 2Y(s) = \frac{1}{s+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2+s}{(s+1)(s^2 - 2s + 2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{-\frac{1}{10} - \frac{7i}{10}}{s - 1 - i} + \frac{-\frac{1}{10} + \frac{7i}{10}}{s - 1 + i} + \frac{1}{5s + 5}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{10} - \frac{7i}{10}}{s - 1 - i}\right) = \left(-\frac{1}{10} - \frac{7i}{10}\right) e^{(1+i)t}$$

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{10} + \frac{7i}{10}}{s - 1 + i}\right) = \left(-\frac{1}{10} + \frac{7i}{10}\right) e^{(1-i)t}$$

$$\mathcal{L}^{-1}\left(\frac{1}{5s + 5}\right) = \frac{e^{-t}}{5}$$

Adding the above results and simplifying gives

$$x = \frac{e^{-t}}{5} + \frac{(-\cos(t) + 7\sin(t))\, e^{t}}{5}$$

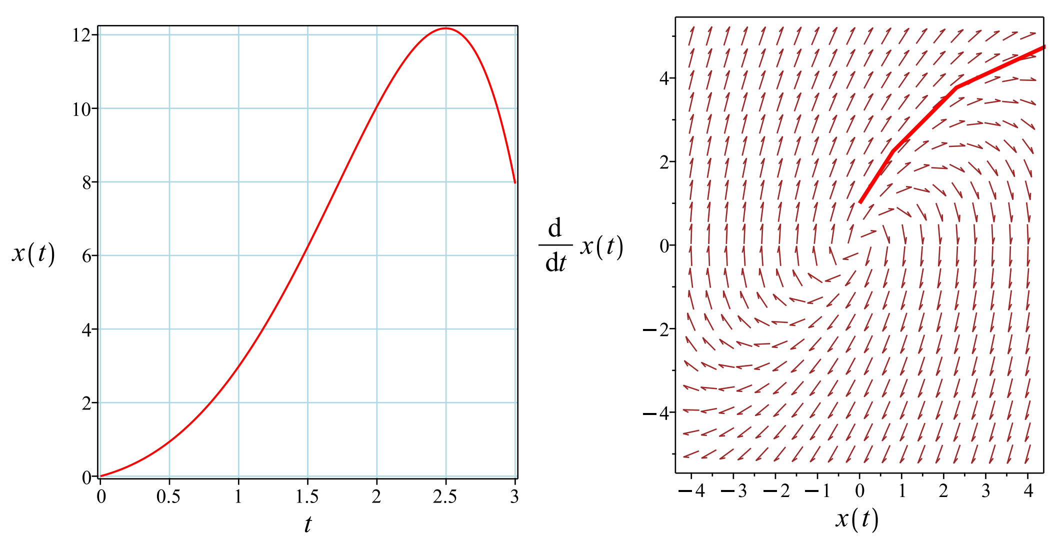
Simplifying the solution gives

$$x = \frac{e^{-t}}{5} + \frac{(-\cos(t) + 7\sin(t))\, e^{t}}{5}$$

Summary

The solution(s) found are the following

$$x = \frac{e^{-t}}{5} + \frac{(-\cos(t) + 7\sin(t))\, e^{t}}{5} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \frac{\mathrm{e}^{-t}}{5} + \frac{(-\cos(t) + 7\sin(t))\,\mathrm{e}^{t}}{5}$$

Verified OK.

15.5.2 Maple step by step solution

Let's solve

$\left[x'' - 2x' + 2x = \mathrm{e}^{-t}, x(0) = 0, x'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - \mathrm{I}, 1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{t}\cos(t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{t}\sin(t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{t}\cos(t) + c_2 \mathrm{e}^{t}\sin(t) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \mathrm{e}^{-t}\right]$

- ○ Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} e^t \cos(t) & e^t \sin(t) \\ e^t \cos(t) - e^t \sin(t) & e^t \sin(t) + e^t \cos(t) \end{bmatrix}$$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = e^{2t}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = -e^t\left(\cos(t)\left(\int \sin(t)\, e^{-2t}dt\right) - \sin(t)\left(\int \cos(t)\, e^{-2t}dt\right)\right)$

- Compute integrals

  $x_p(t) = \frac{e^{-t}}{5}$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^t \cos(t) + c_2 e^t \sin(t) + \frac{e^{-t}}{5}$

☐ Check validity of solution $x = c_1 e^t \cos(t) + c_2 e^t \sin(t) + \frac{e^{-t}}{5}$

- Use initial condition $x(0) = 0$

  $0 = c_1 + \frac{1}{5}$

- Compute derivative of the solution

  $x' = c_1 e^t \cos(t) - c_1 e^t \sin(t) + c_2 e^t \sin(t) + c_2 e^t \cos(t) - \frac{e^{-t}}{5}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 1$

  $1 = c_1 - \frac{1}{5} + c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{5}, c_2 = \frac{7}{5}\right\}$

- Substitute constant values into general solution and simplify

  $x = \frac{e^{-t}}{5} + \frac{(-\cos(t) + 7\sin(t))e^t}{5}$

- Solution to the IVP

  $x = \frac{e^{-t}}{5} + \frac{(-\cos(t) + 7\sin(t))e^t}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.125 (sec). Leaf size: 24

```
dsolve([diff(x(t),t$2)-2*diff(x(t),t)+2*x(t)=exp(-t),x(0) = 0, D(x)(0) = 1],x(t), singsol=al
```

$$x(t) = \frac{e^{-t}}{5} + \frac{(-\cos(t) + 7\sin(t))\, e^{t}}{5}$$

✓ Solution by Mathematica

Time used: 0.108 (sec). Leaf size: 29

```
DSolve[{x''[t]-2*x'[t]+2*x[t]==Exp[-t],{x[0]==0,x'[0]==1}},x[t],t,IncludeSingularSolutions -
```

$$x(t) \to \frac{1}{5}\left(e^{-t} + 7e^{t}\sin(t) - e^{t}\cos(t)\right)$$

15.6 problem 6(f)

15.6.1 Existence and uniqueness analysis 1862
15.6.2 Maple step by step solution 1864

Internal problem ID [11513]
Internal file name [`OUTPUT/10495_Thursday_May_18_2023_04_21_06_AM_36915675/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156
Problem number: 6(f).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' - x' = 0}$$

With initial conditions

$$[x(0) = 1, x'(0) = 0]$$

15.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= 0 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' - x' = 0$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(x') &= sY(s) - x(0)\\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) - sY(s) + x(0) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}x(0) &= 1\\ x'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 1 - s - sY(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{1}{s}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}x &= \mathcal{L}^{-1}(Y(s))\\ &= \mathcal{L}^{-1}\left(\frac{1}{s}\right)\\ &= 1\end{aligned}$$

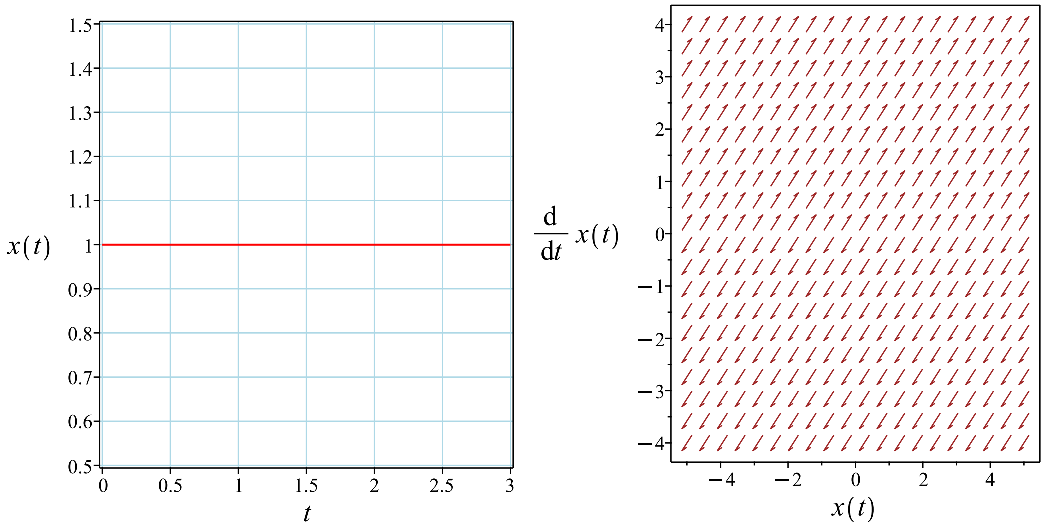
Simplifying the solution gives

$$x = 1$$

Summary

The solution(s) found are the following

$$x = 1 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = 1$$

Verified OK.

15.6.2 Maple step by step solution

Let's solve

$$\left[x'' - x' = 0, x(0) = 1, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 - r = 0$

- Factor the characteristic polynomial

  $r(r-1) = 0$

- Roots of the characteristic polynomial

  $r = (0, 1)$

- 1st solution of the ODE

  $x_1(t) = 1$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^t$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 + c_2 \mathrm{e}^t$

□ Check validity of solution $x = c_1 + c_2 \mathrm{e}^t$

  - Use initial condition $x(0) = 1$

    $1 = c_1 + c_2$

  - Compute derivative of the solution

    $x' = c_2 \mathrm{e}^t$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 0$

    $0 = c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = 0\}$

  - Substitute constant values into general solution and simplify

    $x = 1$

- Solution to the IVP

  $x = 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 3.766 (sec). Leaf size: 5

```
dsolve([diff(x(t),t$2)-diff(x(t),t)=0,x(0) = 1, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = 1$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 6

```
DSolve[{x''[t]-x'[t]==0,{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 1$$

15.7 problem 6(g)

15.7.1 Existence and uniqueness analysis 1867
15.7.2 Maple step by step solution 1870

Internal problem ID [11514]
Internal file name [`OUTPUT/10496_Thursday_May_18_2023_04_21_07_AM_2733894/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156
Problem number: 6(g).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + \frac{2x'}{5} + 2x = 1 - \text{Heaviside}\left(t - 5\right)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

15.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned}
p(t) &= \frac{2}{5} \\
q(t) &= 2 \\
F &= 1 - \text{Heaviside}\left(t - 5\right)
\end{aligned}$$

Hence the ode is

$$x'' + \frac{2x'}{5} + 2x = 1 - \text{Heaviside}(t-5)$$

The domain of $p(t) = \frac{2}{5}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 1 - \text{Heaviside}(t-5)$ is

$$\{t < 5 \lor 5 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(x') &= sY(s) - x(0)\\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) + \frac{2sY(s)}{5} - \frac{2x(0)}{5} + 2Y(s) = \frac{1-\mathrm{e}^{-5s}}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}x(0) &= 0\\ x'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + \frac{2sY(s)}{5} + 2Y(s) = \frac{1-\mathrm{e}^{-5s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{5(-1+\mathrm{e}^{-5s})}{s(5s^2+2s+10)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
x &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{5(-1+\mathrm{e}^{-5s})}{s\,(5s^2+2s+10)}\right) \\
&= \frac{1}{2} - \frac{\mathrm{e}^{-\frac{t}{5}}\left(7\cos\left(\frac{7t}{5}\right)+\sin\left(\frac{7t}{5}\right)\right)}{14} + \left(\frac{1}{100}+\frac{i}{700}\right)\left(-49+7i+25\,\mathrm{e}^{\left(-\frac{1}{5}-\frac{7i}{5}\right)(t-5)} + (24-7i)\,\mathrm{e}^{\left(-\frac{1}{5}+\frac{7i}{5}\right)(t-5}\right.
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
x = \frac{1}{2} - \frac{\mathrm{e}^{-\frac{t}{5}}\left(7\cos\left(\frac{7t}{5}\right)+\sin\left(\frac{7t}{5}\right)\right)}{14} + \left(\frac{1}{100}+\frac{i}{700}\right)&\left(-49+7i+25\,\mathrm{e}^{\left(-\frac{1}{5}-\frac{7i}{5}\right)(t-5)}\right. \\
&\left.+ (24-7i)\,\mathrm{e}^{\left(-\frac{1}{5}+\frac{7i}{5}\right)(t-5)}\right)\mathrm{Heaviside}\,(t-5)
\end{aligned}
$$

Simplifying the solution gives

$$
\begin{aligned}
x = \frac{1}{2} &+ \left(\frac{1}{4}+\frac{i}{28}\right)\mathrm{Heaviside}\,(t-5)\,\mathrm{e}^{\left(-\frac{1}{5}-\frac{7i}{5}\right)(t-5)} \\
&+ \left(\frac{1}{4}-\frac{i}{28}\right)\mathrm{Heaviside}\,(t-5)\,\mathrm{e}^{\left(-\frac{1}{5}+\frac{7i}{5}\right)(t-5)} \\
&+ \frac{\left(-7\cos\left(\frac{7t}{5}\right)-\sin\left(\frac{7t}{5}\right)\right)\mathrm{e}^{-\frac{t}{5}}}{14} - \frac{\mathrm{Heaviside}\,(t-5)}{2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
x = \frac{1}{2} &+ \left(\frac{1}{4}+\frac{i}{28}\right)\mathrm{Heaviside}\,(t-5)\,\mathrm{e}^{\left(-\frac{1}{5}-\frac{7i}{5}\right)(t-5)} \\
&+ \left(\frac{1}{4}-\frac{i}{28}\right)\mathrm{Heaviside}\,(t-5)\,\mathrm{e}^{\left(-\frac{1}{5}+\frac{7i}{5}\right)(t-5)} \\
&+ \frac{\left(-7\cos\left(\frac{7t}{5}\right)-\sin\left(\frac{7t}{5}\right)\right)\mathrm{e}^{-\frac{t}{5}}}{14} - \frac{\mathrm{Heaviside}\,(t-5)}{2}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
x = \frac{1}{2} &+ \left(\frac{1}{4}+\frac{i}{28}\right)\mathrm{Heaviside}\,(t-5)\,\mathrm{e}^{\left(-\frac{1}{5}-\frac{7i}{5}\right)(t-5)} \\
&+ \left(\frac{1}{4}-\frac{i}{28}\right)\mathrm{Heaviside}\,(t-5)\,\mathrm{e}^{\left(-\frac{1}{5}+\frac{7i}{5}\right)(t-5)} \\
&+ \frac{\left(-7\cos\left(\frac{7t}{5}\right)-\sin\left(\frac{7t}{5}\right)\right)\mathrm{e}^{-\frac{t}{5}}}{14} - \frac{\mathrm{Heaviside}\,(t-5)}{2}
\end{aligned}
$$

Verified OK.

15.7.2 Maple step by step solution

Let's solve

$$\left[x'' + \frac{2x'}{5} + 2x = 1 - \mathit{Heaviside}(t-5), x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + \frac{2}{5}r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{\left(-\frac{2}{5}\right) \pm \left(\sqrt{-\frac{196}{25}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{5} - \frac{7\mathrm{I}}{5}, -\frac{1}{5} + \frac{7\mathrm{I}}{5}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-\frac{t}{5}} \cos\left(\frac{7t}{5}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-\frac{t}{5}} \sin\left(\frac{7t}{5}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-\frac{t}{5}} \cos\left(\frac{7t}{5}\right) + c_2 \mathrm{e}^{-\frac{t}{5}} \sin\left(\frac{7t}{5}\right) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $$\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = 1 - \mathit{Heaviside}(t-5)\right]$$

- ◦ Wronskian of solutions of the homogeneous equation

  $$W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-\frac{t}{5}} \cos\left(\frac{7t}{5}\right) & \mathrm{e}^{-\frac{t}{5}} \sin\left(\frac{7t}{5}\right) \\ -\frac{\mathrm{e}^{-\frac{t}{5}} \cos\left(\frac{7t}{5}\right)}{5} - \frac{7\,\mathrm{e}^{-\frac{t}{5}} \sin\left(\frac{7t}{5}\right)}{5} & -\frac{\mathrm{e}^{-\frac{t}{5}} \sin\left(\frac{7t}{5}\right)}{5} + \frac{7\,\mathrm{e}^{-\frac{t}{5}} \cos\left(\frac{7t}{5}\right)}{5} \end{bmatrix}$$

- ◦ Compute Wronskian

  $W(x_1(t), x_2(t)) = \frac{7\,\mathrm{e}^{-\frac{2t}{5}}}{5}$

- Substitute functions into equation for $x_p(t)$

$$x_p(t) = \frac{5\,\mathrm{e}^{-\frac{t}{5}}\left(\cos\left(\frac{7t}{5}\right)\left(\int(-1+\mathit{Heaviside}(t-5))\sin\left(\frac{7t}{5}\right)\mathrm{e}^{\frac{t}{5}}dt\right)-\sin\left(\frac{7t}{5}\right)\left(\int(-1+\mathit{Heaviside}(t-5))\cos\left(\frac{7t}{5}\right)\mathrm{e}^{\frac{t}{5}}dt\right)\right)}{7}$$

- Compute integrals

$$x_p(t) = \frac{\left(\left(\cos(7)-\frac{\sin(7)}{7}\right)\cos\left(\frac{7t}{5}\right)+\frac{\sin\left(\frac{7t}{5}\right)(\cos(7)+7\sin(7))}{7}\right)\mathit{Heaviside}(t-5)\mathrm{e}^{1-\frac{t}{5}}}{2} - \frac{\mathit{Heaviside}(t-5)}{2} + \frac{1}{2}$$

- Substitute particular solution into general solution to ODE

$$x = c_1\mathrm{e}^{-\frac{t}{5}}\cos\left(\frac{7t}{5}\right) + c_2\mathrm{e}^{-\frac{t}{5}}\sin\left(\frac{7t}{5}\right) + \frac{\left(\left(\cos(7)-\frac{\sin(7)}{7}\right)\cos\left(\frac{7t}{5}\right)+\frac{\sin\left(\frac{7t}{5}\right)(\cos(7)+7\sin(7))}{7}\right)\mathit{Heaviside}(t-5)\mathrm{e}^{1-\frac{t}{5}}}{2} - \frac{\mathit{He}}{}$$

☐ Check validity of solution $x = c_1\mathrm{e}^{-\frac{t}{5}}\cos\left(\frac{7t}{5}\right) + c_2\mathrm{e}^{-\frac{t}{5}}\sin\left(\frac{7t}{5}\right) + \frac{\left(\left(\cos(7)-\frac{\sin(7)}{7}\right)\cos\left(\frac{7t}{5}\right)+\frac{\sin\left(\frac{7t}{5}\right)(\cos(7)+}{7}\right.}{2}$

- Use initial condition $x(0) = 0$

$$0 = \frac{1}{2} + c_1$$

- Compute derivative of the solution

$$x' = -\frac{c_1\mathrm{e}^{-\frac{t}{5}}\cos\left(\frac{7t}{5}\right)}{5} - \frac{7c_1\mathrm{e}^{-\frac{t}{5}}\sin\left(\frac{7t}{5}\right)}{5} - \frac{c_2\mathrm{e}^{-\frac{t}{5}}\sin\left(\frac{7t}{5}\right)}{5} + \frac{7c_2\mathrm{e}^{-\frac{t}{5}}\cos\left(\frac{7t}{5}\right)}{5} + \frac{\left(-\frac{7\left(\cos(7)-\frac{\sin(7)}{7}\right)\sin\left(\frac{7t}{5}\right)}{5}+\frac{\cos\left(\frac{7t}{5}\right)(\cos(7}{5}\right.}{2}$$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

$$0 = -\frac{c_1}{5} + \frac{7c_2}{5}$$

- Solve for c_1 and c_2

$$\left\{c_1 = -\frac{1}{2}, c_2 = -\frac{1}{14}\right\}$$

- Substitute constant values into general solution and simplify

$$x = -\frac{\mathrm{e}^{-\frac{t}{5}}\cos\left(\frac{7t}{5}\right)}{2} - \frac{\mathrm{e}^{-\frac{t}{5}}\sin\left(\frac{7t}{5}\right)}{14} + \frac{\left(\left(\cos(7)-\frac{\sin(7)}{7}\right)\cos\left(\frac{7t}{5}\right)+\frac{\sin\left(\frac{7t}{5}\right)(\cos(7)+7\sin(7))}{7}\right)\mathit{Heaviside}(t-5)\mathrm{e}^{1-\frac{t}{5}}}{2} - \frac{\mathit{Heaviside}(}{2}$$

- Solution to the IVP

$$x = -\frac{\mathrm{e}^{-\frac{t}{5}}\cos\left(\frac{7t}{5}\right)}{2} - \frac{\mathrm{e}^{-\frac{t}{5}}\sin\left(\frac{7t}{5}\right)}{14} + \frac{\left(\left(\cos(7)-\frac{\sin(7)}{7}\right)\cos\left(\frac{7t}{5}\right)+\frac{\sin\left(\frac{7t}{5}\right)(\cos(7)+7\sin(7))}{7}\right)\mathit{Heaviside}(t-5)\mathrm{e}^{1-\frac{t}{5}}}{2} - \frac{\mathit{Heaviside}(}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.672 (sec). Leaf size: 57

```
dsolve([diff(x(t),t$2)+4/10*diff(x(t),t)+2*x(t)=1-Heaviside(t-5),x(0) = 0, D(x)(0) = 0],x(t)
```

$$
\begin{aligned}
x(t) = {} & \frac{1}{2}+\left(\frac{1}{4}+\frac{i}{28}\right) \mathrm{Heaviside}\,(t-5)\,\mathrm{e}^{\left(-\frac{1}{5}-\frac{7i}{5}\right)(t-5)} \\
& +\left(\frac{1}{4}-\frac{i}{28}\right) \mathrm{Heaviside}\,(t-5)\,\mathrm{e}^{\left(-\frac{1}{5}+\frac{7i}{5}\right)(t-5)} \\
& +\frac{\left(-7\cos\left(\frac{7t}{5}\right)-\sin\left(\frac{7t}{5}\right)\right)\mathrm{e}^{-\frac{t}{5}}}{14}-\frac{\mathrm{Heaviside}\,(t-5)}{2}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 91

```
DSolve[{x''[t]+4/10*x'[t]+2*x[t]==1-UnitStep[t-5],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingular
```

$$
\begin{aligned}
& x(t) \rightarrow \\
& -\frac{1}{14}e^{-t/5}\left(-\theta(5-t)\left(7e^{t/5}+e\sin\left(7-\frac{7t}{5}\right)-7e\cos\left(7-\frac{7t}{5}\right)\right)+e\sin\left(7-\frac{7t}{5}\right)+\sin\left(\frac{7t}{5}\right)-7e\cos\left(7-\right.\right.
\end{aligned}
$$

15.8 problem 6(h)

15.8.1 Existence and uniqueness analysis 1873
15.8.2 Maple step by step solution 1876

Internal problem ID [11515]
Internal file name [`OUTPUT/10497_Thursday_May_18_2023_04_21_09_AM_5809768/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156
Problem number: 6(h).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + 9x = \sin(3t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

15.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 9 \\ F &= \sin(3t) \end{aligned}$$

Hence the ode is

$$x'' + 9x = \sin(3t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \sin(3t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(x') &= sY(s) - x(0)\\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) + 9Y(s) = \frac{3}{s^2+9} \tag{1}$$

But the initial conditions are

$$\begin{aligned}x(0) &= 0\\ x'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 9Y(s) = \frac{3}{s^2+9}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{3}{(s^2+9)^2}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{12\left(s-3i\right)^{2}}-\frac{1}{12\left(s+3i\right)^{2}}-\frac{i}{36\left(s-3i\right)}+\frac{i}{36s+108i}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(-\frac{1}{12\left(s-3i\right)^{2}}\right) &= -\frac{t\,\mathrm{e}^{3it}}{12} \\
\mathcal{L}^{-1}\left(-\frac{1}{12\left(s+3i\right)^{2}}\right) &= -\frac{t\,\mathrm{e}^{-3it}}{12} \\
\mathcal{L}^{-1}\left(-\frac{i}{36\left(s-3i\right)}\right) &= -\frac{i\mathrm{e}^{3it}}{36} \\
\mathcal{L}^{-1}\left(\frac{i}{36s+108i}\right) &= \frac{i\mathrm{e}^{-3it}}{36}
\end{aligned}$$

Adding the above results and simplifying gives

$$x = \frac{\sin\left(3t\right)}{18}-\frac{t\cos\left(3t\right)}{6}$$

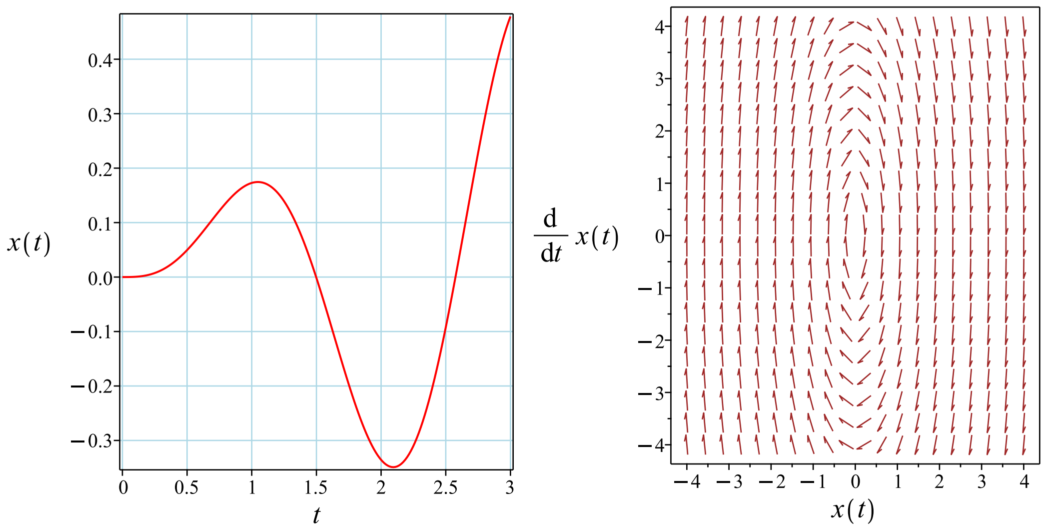
Simplifying the solution gives

$$x = \frac{\sin\left(3t\right)}{18}-\frac{t\cos\left(3t\right)}{6}$$

Summary

The solution(s) found are the following

$$x = \frac{\sin\left(3t\right)}{18}-\frac{t\cos\left(3t\right)}{6} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$x = \frac{\sin(3t)}{18} - \frac{t\cos(3t)}{6}$$

Verified OK.

15.8.2 Maple step by step solution

Let's solve

$$\left[x'' + 9x = \sin(3t), x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$x_1(t) = \cos(3t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(3t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(3t) + c_2 \sin(3t) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \sin(3t)\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$

  ○ Compute Wronskian

  $W(x_1(t), x_2(t)) = 3$

  ○ Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{\cos(3t)\left(\int \sin(3t)^2 dt\right)}{3} + \frac{\sin(3t)\left(\int \sin(6t)dt\right)}{6}$

  ○ Compute integrals

  $x_p(t) = \frac{\sin(3t)}{36} - \frac{t\cos(3t)}{6}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \cos(3t) + c_2 \sin(3t) + \frac{\sin(3t)}{36} - \frac{t\cos(3t)}{6}$

□ Check validity of solution $x = c_1 \cos(3t) + c_2 \sin(3t) + \frac{\sin(3t)}{36} - \frac{t\cos(3t)}{6}$

  ○ Use initial condition $x(0) = 0$

  $0 = c_1$

  ○ Compute derivative of the solution

  $x' = -3c_1 \sin(3t) + 3c_2 \cos(3t) - \frac{\cos(3t)}{12} + \frac{t\sin(3t)}{2}$

  ○ Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = -\frac{1}{12} + 3c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = \frac{1}{36}\}$

- Substitute constant values into general solution and simplify

  $x = \frac{\sin(3t)}{18} - \frac{t\cos(3t)}{6}$

- Solution to the IVP

  $x = \frac{\sin(3t)}{18} - \frac{t\cos(3t)}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.282 (sec). Leaf size: 18

```
dsolve([diff(x(t),t$2)+9*x(t)=sin(3*t),x(0) = 0, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \frac{\sin(3t)}{18} - \frac{\cos(3t)\,t}{6}$$

✓ Solution by Mathematica

Time used: 0.14 (sec). Leaf size: 21

```
DSolve[{x''[t]+9*x[t]==Sin[3*t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{18}(\sin(3t) - 3t\cos(3t))$$

15.9 problem 6(i)

15.9.1 Existence and uniqueness analysis 1879
15.9.2 Maple step by step solution 1882

Internal problem ID [11516]
Internal file name [`OUTPUT/10498_Thursday_May_18_2023_04_21_10_AM_77327639/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156
Problem number: 6(i).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' - 2x = 1}$$

With initial conditions

$$[x(0) = 1, x'(0) = 0]$$

15.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= -2 \\ F &= 1 \end{aligned}$$

Hence the ode is

$$x'' - 2x = 1$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(x') &= sY(s) - x(0)\\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) - 2Y(s) = \frac{1}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}x(0) &= 1\\ x'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - s - 2Y(s) = \frac{1}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2 + 1}{s\,(s^2 - 2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{3}{4\left(s-\sqrt{2}\right)} + \frac{3}{4\left(s+\sqrt{2}\right)} - \frac{1}{2s}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{3}{4\left(s-\sqrt{2}\right)}\right) = \frac{3\,\mathrm{e}^{\sqrt{2}\,t}}{4}$$

$$\mathcal{L}^{-1}\left(\frac{3}{4\left(s+\sqrt{2}\right)}\right) = \frac{3\,\mathrm{e}^{-\sqrt{2}\,t}}{4}$$

$$\mathcal{L}^{-1}\left(-\frac{1}{2s}\right) = -\frac{1}{2}$$

Adding the above results and simplifying gives

$$x = -\frac{1}{2} + \frac{3\cosh\left(\sqrt{2}\,t\right)}{2}$$

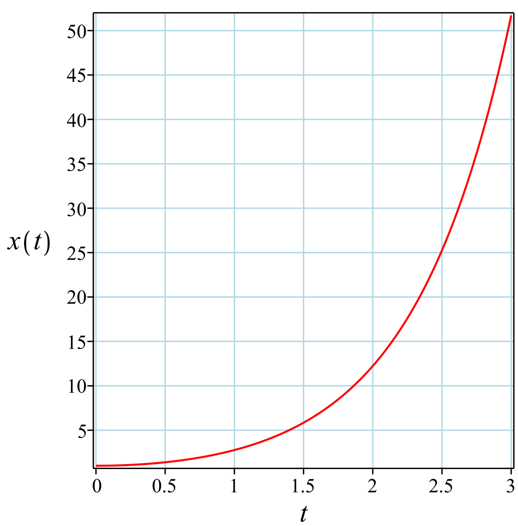
Simplifying the solution gives

$$x = -\frac{1}{2} + \frac{3\cosh\left(\sqrt{2}\,t\right)}{2}$$

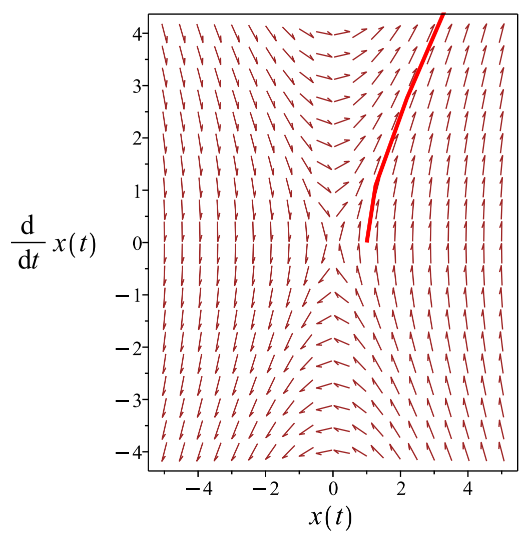
Summary

The solution(s) found are the following

$$x = -\frac{1}{2} + \frac{3\cosh\left(\sqrt{2}\,t\right)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -\frac{1}{2} + \frac{3\cosh\left(\sqrt{2}\,t\right)}{2}$$

Verified OK.

15.9.2 Maple step by step solution

Let's solve

$\left[x'' - 2x = 1, x(0) = 1, x'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0\pm\left(\sqrt{8}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\sqrt{2}, -\sqrt{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{\sqrt{2}\,t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-\sqrt{2}\,t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{\sqrt{2}\,t} + c_2 \mathrm{e}^{-\sqrt{2}\,t} + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = 1\right]$

- ◦ Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \left[\begin{array}{cc} e^{\sqrt{2}\,t} & e^{-\sqrt{2}\,t} \\ \sqrt{2}\,e^{\sqrt{2}\,t} & -\sqrt{2}\,e^{-\sqrt{2}\,t} \end{array}\right]$$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = -2\sqrt{2}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = \frac{\sqrt{2}\left(e^{\sqrt{2}\,t}\left(\int e^{-\sqrt{2}\,t}dt\right) - e^{-\sqrt{2}\,t}\left(\int e^{\sqrt{2}\,t}dt\right)\right)}{4}$

- Compute integrals

  $x_p(t) = -\frac{1}{2}$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{\sqrt{2}\,t} + c_2 e^{-\sqrt{2}\,t} - \frac{1}{2}$

☐ Check validity of solution $x = c_1 e^{\sqrt{2}\,t} + c_2 e^{-\sqrt{2}\,t} - \frac{1}{2}$

- Use initial condition $x(0) = 1$

  $1 = c_1 + c_2 - \frac{1}{2}$

- Compute derivative of the solution

  $x' = c_1\sqrt{2}\,e^{\sqrt{2}\,t} - c_2\sqrt{2}\,e^{-\sqrt{2}\,t}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = c_1\sqrt{2} - \sqrt{2}\,c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{3}{4}, c_2 = \frac{3}{4}\right\}$

- Substitute constant values into general solution and simplify

  $x = \frac{3\,e^{\sqrt{2}\,t}}{4} + \frac{3\,e^{-\sqrt{2}\,t}}{4} - \frac{1}{2}$

- Solution to the IVP

  $x = \frac{3\,e^{\sqrt{2}\,t}}{4} + \frac{3\,e^{-\sqrt{2}\,t}}{4} - \frac{1}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.172 (sec). Leaf size: 14

```
dsolve([diff(x(t),t$2)-2*x(t)=1,x(0) = 1, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = -\frac{1}{2} + \frac{3\cosh\left(\sqrt{2}\, t\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 34

```
DSolve[{x''[t]-2*x[t]==1,{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{4}\left(3e^{-\sqrt{2}t} + 3e^{\sqrt{2}t} - 2\right)$$

15.10 problem 6(j)

15.10.1 Existence and uniqueness analysis 1885
15.10.2 Solving as laplace ode 1886
15.10.3 Maple step by step solution 1887

Internal problem ID [11517]
Internal file name [`OUTPUT/10499_Thursday_May_18_2023_04_21_12_AM_8183189/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156
Problem number: 6(j).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{x' - 2x = \text{Heaviside}(t - 1)}$$

With initial conditions

$$[x(0) = 0]$$

15.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= \text{Heaviside}(t - 1) \end{aligned}$$

Hence the ode is

$$x' - 2x = \text{Heaviside}(t - 1)$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \text{Heaviside}(t - 1)$ is

$$\{t < 1 \lor 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.10.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - x(0) - 2Y(s) = \frac{e^{-s}}{s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 2Y(s) = \frac{e^{-s}}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{e^{-s}}{s(s-2)}$$

Taking the inverse Laplace transform gives

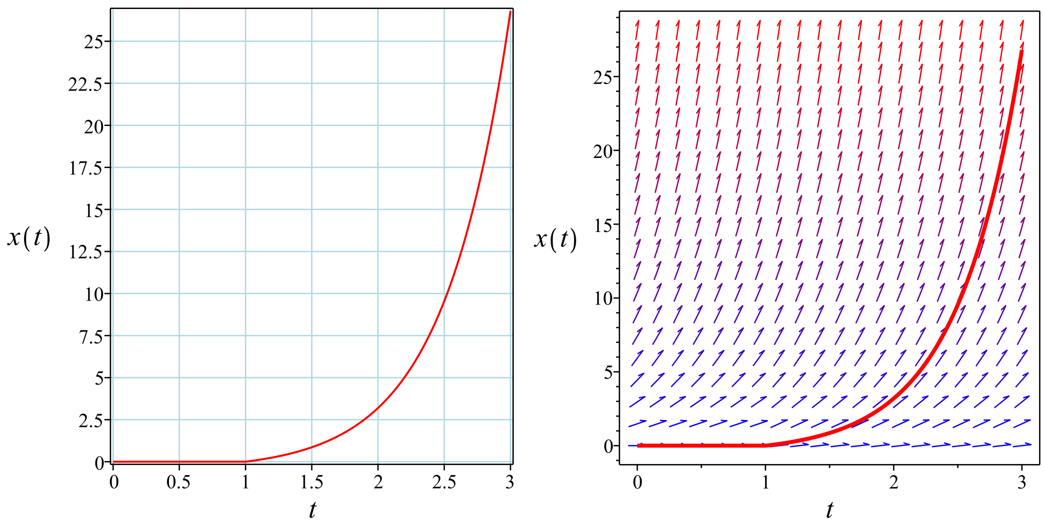
$$\begin{aligned} x &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{e^{-s}}{s(s-2)}\right) \\ &= \frac{\text{Heaviside}(t-1)\left(-1 + e^{2t-2}\right)}{2} \end{aligned}$$

Hence the final solution is

$$x = \frac{\text{Heaviside}\,(t-1)\,(-1+\mathrm{e}^{2t-2})}{2}$$

Summary
The solution(s) found are the following

$$x = \frac{\text{Heaviside}\,(t-1)\,(-1+\mathrm{e}^{2t-2})}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \frac{\text{Heaviside}\,(t-1)\,(-1+\mathrm{e}^{2t-2})}{2}$$

Verified OK.

15.10.3 Maple step by step solution

Let's solve

$[x' - 2x = \mathit{Heaviside}(t-1)\,, x(0) = 0]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = 2x + Heaviside(t-1)$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' - 2x = Heaviside(t-1)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(x' - 2x) = \mu(t)\, Heaviside(t-1)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

  $\mu(t)\,(x' - 2x) = \mu'(t)\,x + \mu(t)\,x'$

- Isolate $\mu'(t)$

  $\mu'(t) = -2\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-2t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,x)\right) dt = \int \mu(t)\, Heaviside(t-1)\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,x = \int \mu(t)\, Heaviside(t-1)\, dt + c_1$

- Solve for x

  $x = \frac{\int \mu(t) Heaviside(t-1) dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-2t}$

  $x = \frac{\int \mathrm{e}^{-2t} Heaviside(t-1) dt + c_1}{\mathrm{e}^{-2t}}$

- Evaluate the integrals on the rhs

  $x = \frac{-\frac{\mathrm{e}^{-2t} Heaviside(t-1)}{2} + \frac{Heaviside(t-1)\mathrm{e}^{-2}}{2} + c_1}{\mathrm{e}^{-2t}}$

- Simplify

  $x = -\frac{Heaviside(t-1)}{2} + \frac{Heaviside(t-1)\mathrm{e}^{2t-2}}{2} + c_1 \mathrm{e}^{2t}$

- Use initial condition $x(0) = 0$

  $0 = c_1$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

$$x = \frac{Heaviside(t-1)\left(-1+e^{2t-2}\right)}{2}$$

- Solution to the IVP

$$x = \frac{Heaviside(t-1)\left(-1+e^{2t-2}\right)}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 4.844 (sec). Leaf size: 18

```
dsolve([diff(x(t),t)=2*x(t)+Heaviside(t-1),x(0) = 0],x(t), singsol=all)
```

$$x(t) = \frac{\text{Heaviside}\,(t-1)\left(-1+\mathrm{e}^{2t-2}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.08 (sec). Leaf size: 25

```
DSolve[{x'[t]==2*x[t]+UnitStep[t-1],{x[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \left\{ \begin{array}{cc} \frac{1}{2}\left(-1+e^{2t-2}\right) & t>1 \\ 0 & \text{True} \end{array} \right.$$

15.11 problem 11

Internal problem ID [11518]
Internal file name [`OUTPUT/10500_Thursday_May_18_2023_04_21_14_AM_69301582/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_linear, `class A`]]`

Unable to solve or complete the solution.

$$\boxed{x' + 4x - \cos(2t)\,\mathrm{Heaviside}(2\pi - t) = 0}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$

The given ode becomes an algebraic equation in the Laplace domain

$$sY(s) - x(0) + 4Y(s) + \frac{s(\mathrm{e}^{-2\pi s} - 1)}{s^2 + 4} = 0 \tag{1}$$

But the initial conditions are

$$x(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$sY(s) + 4Y(s) + \frac{s(\mathrm{e}^{-2\pi s} - 1)}{s^2 + 4} = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{s(\mathrm{e}^{-2\pi s} - 1)}{(s^2 + 4)(s + 4)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
x &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{s(\mathrm{e}^{-2\pi s} - 1)}{(s^2 + 4)(s + 4)}\right) \\
&= -\frac{1}{5} + \frac{2\,\mathrm{Heaviside}(-2\pi + t)\,\mathrm{e}^{4\pi - 2t}\cosh(-4\pi + 2t)}{5} - \frac{\mathrm{e}^{-4t}}{5} + \frac{\mathrm{Heaviside}(2\pi - t)\cos(t)(2\cos(t) + \sin}{5}
\end{aligned}$$

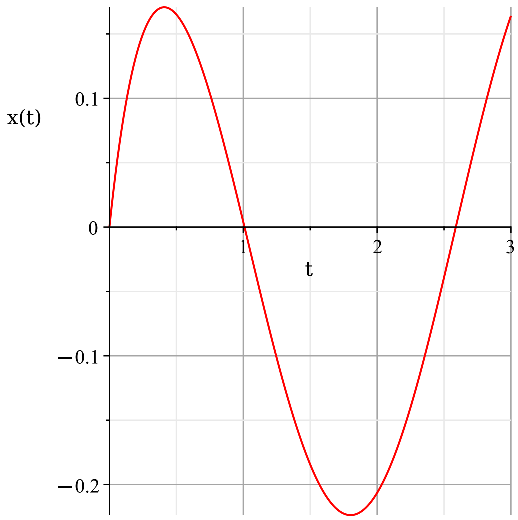
Hence the final solution is

$$\begin{aligned}
x = &-\frac{1}{5} + \frac{2\,\mathrm{Heaviside}(-2\pi + t)\,\mathrm{e}^{4\pi - 2t}\cosh(-4\pi + 2t)}{5} \\
&- \frac{\mathrm{e}^{-4t}}{5} + \frac{\mathrm{Heaviside}(2\pi - t)\cos(t)(2\cos(t) + \sin(t))}{5}
\end{aligned}$$

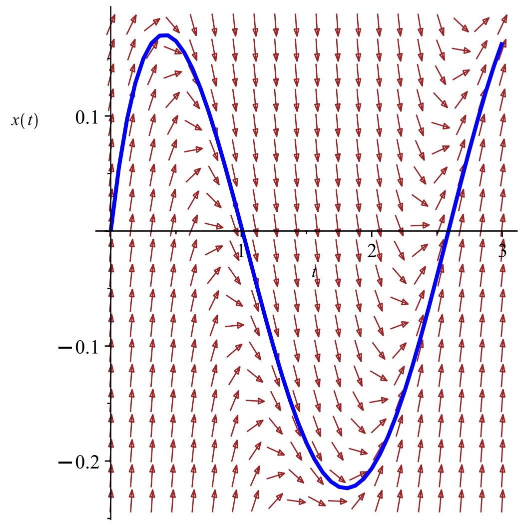
Summary

The solution(s) found are the following

$$\begin{aligned}
x = &-\frac{1}{5} + \frac{2\,\mathrm{Heaviside}(-2\pi + t)\,\mathrm{e}^{4\pi - 2t}\cosh(-4\pi + 2t)}{5} \\
&- \frac{\mathrm{e}^{-4t}}{5} + \frac{\mathrm{Heaviside}(2\pi - t)\cos(t)(2\cos(t) + \sin(t))}{5}
\end{aligned} \tag{1}$$



(a) Solution plot



(b) Phase plot

Verification of solutions

$$x = -\frac{1}{5} + \frac{2\,\text{Heaviside}\,(-2\pi + t)\,e^{4\pi - 2t} \cosh\,(-4\pi + 2t)}{5} - \frac{e^{-4t}}{5} + \frac{\text{Heaviside}\,(2\pi - t) \cos\,(t)\,(2\cos\,(t) + \sin\,(t))}{5}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✗ Solution by Maple

```
dsolve([diff(x(t),t)+4*x(t)=cos(2*t)*Heaviside(2*Pi-t),x(0) = 0, D(x)(0) = 0],x(t), singsol=
```

No solution found

✓ Solution by Mathematica

Time used: 0.168 (sec). Leaf size: 28

```
DSolve[{x''[t]+4*x[t]==Cos[2*t]*UnitStep[2*Pi-t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularS
```

$$x(t) \to \begin{cases} \pi \cos(t) \sin(t) & t > 2\pi \\ \frac{1}{2} t \cos(t) \sin(t) & \text{True} \end{cases}$$

15.12 problem 12

15.12.1 Existence and uniqueness analysis 1894
15.12.2 Solving as laplace ode 1895
15.12.3 Maple step by step solution 1896

Internal problem ID [11519]
Internal file name [`OUTPUT/10501_Thursday_May_18_2023_04_21_16_AM_57828492/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{-x + x' = -2\,\text{Heaviside}\,(t-1)}$$

With initial conditions

$$[x(0) = 1]$$

15.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= -2\,\text{Heaviside}\,(t-1) \end{aligned}$$

Hence the ode is

$$-x + x' = -2\,\mathrm{Heaviside}\,(t-1)$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -2\,\mathrm{Heaviside}\,(t-1)$ is

$$\{t < 1 \lor 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.12.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$-Y(s) + sY(s) - x(0) = -\frac{2\,\mathrm{e}^{-s}}{s} \tag{1}$$

Replacing initial condition gives

$$-Y(s) + sY(s) - 1 = -\frac{2\,\mathrm{e}^{-s}}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = -\frac{2\,\mathrm{e}^{-s} - s}{s\,(s-1)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
x &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{2\,\mathrm{e}^{-s} - s}{s\,(s-1)}\right) \\
&= 2\,\mathrm{Heaviside}\,(t-1) + \mathrm{e}^{t} + 2\,\mathrm{e}^{t-1}(-1 + \mathrm{Heaviside}\,(1-t))
\end{aligned}$$

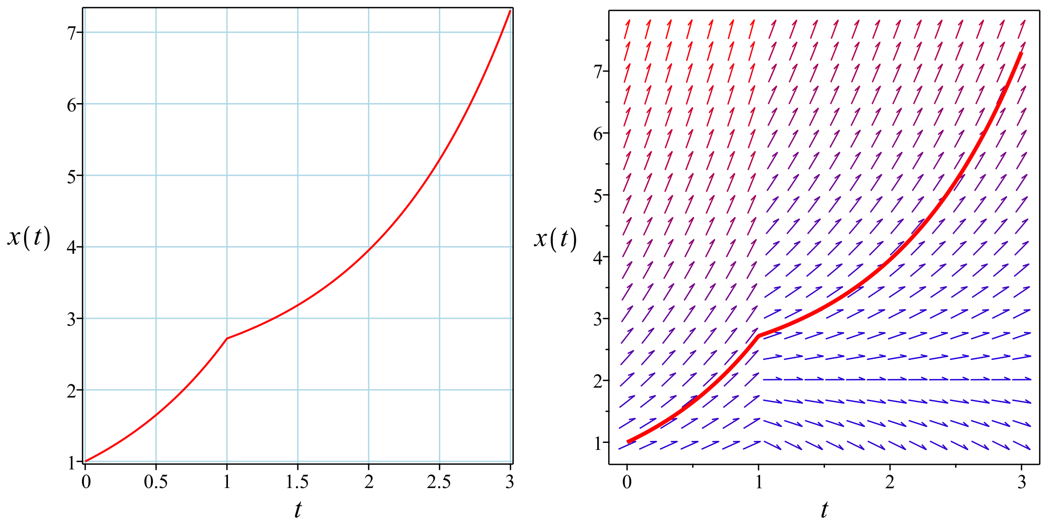
Hence the final solution is

$$x = 2\,\mathrm{Heaviside}\,(t-1) + \mathrm{e}^{t} + 2\,\mathrm{e}^{t-1}(-1 + \mathrm{Heaviside}\,(1-t))$$

Summary

The solution(s) found are the following

$$x = 2\,\mathrm{Heaviside}\,(t-1) + \mathrm{e}^{t} + 2\,\mathrm{e}^{t-1}(-1 + \mathrm{Heaviside}\,(1-t)) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = 2\,\mathrm{Heaviside}\,(t-1) + \mathrm{e}^{t} + 2\,\mathrm{e}^{t-1}(-1 + \mathrm{Heaviside}\,(1-t))$$

Verified OK.

15.12.3 Maple step by step solution

Let's solve

$[-x + x' = -2\mathit{Heaviside}(t-1)\,, x(0) = 1]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = x - 2\mathit{Heaviside}(t-1)$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $-x + x' = -2 Heaviside(t-1)$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)(-x + x') = -2\mu(t)\ Heaviside(t-1)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, x)$

  $\mu(t)(-x + x') = \mu'(t)\, x + \mu(t)\, x'$
- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, x)\right) dt = \int -2\mu(t)\ Heaviside(t-1)\, dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, x = \int -2\mu(t)\ Heaviside(t-1)\, dt + c_1$
- Solve for x

  $x = \frac{\int -2\mu(t) Heaviside(t-1) dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{-t}$

  $x = \frac{\int -2\,\mathrm{e}^{-t} Heaviside(t-1) dt + c_1}{\mathrm{e}^{-t}}$
- Evaluate the integrals on the rhs

  $x = \frac{2\,\mathrm{e}^{-t} Heaviside(t-1) - 2 Heaviside(t-1)\mathrm{e}^{-1} + c_1}{\mathrm{e}^{-t}}$
- Simplify

  $x = (2 - 2\,\mathrm{e}^{t-1})\ Heaviside(t-1) + c_1 \mathrm{e}^{t}$
- Use initial condition $x(0) = 1$

  $1 = c_1$
- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $x = (2 - 2\,\mathrm{e}^{t-1})\ Heaviside(t-1) + \mathrm{e}^{t}$
- Solution to the IVP

$$x = (2 - 2\,\mathrm{e}^{t-1})\,\mathit{Heaviside}(t-1) + \mathrm{e}^{t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 5.11 (sec). Leaf size: 27

```
dsolve([diff(x(t),t)=x(t)-2*Heaviside(t-1),x(0) = 1],x(t), singsol=all)
```

$$x(t) = \left(-2\,\mathrm{e}^{t-1} + 2\right)\mathrm{Heaviside}\,(t-1) + \mathrm{e}^{t}$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 26

```
DSolve[{x'[t]==x[t]-2*UnitStep[t-1],{x[0]==1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \begin{cases} e^{t} & t \leq 1 \\ 2 - 2e^{t-1} + e^{t} & \text{True} \end{cases}$$

15.13 problem 14

15.13.1 Existence and uniqueness analysis 1899
15.13.2 Solving as laplace ode 1900
15.13.3 Maple step by step solution 1901

Internal problem ID [11520]
Internal file name [`OUTPUT/10502_Thursday_May_18_2023_04_21_18_AM_23789009/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{x' + x = \text{Heaviside}\,(t-1) - \text{Heaviside}\,(-2+t)}$$

With initial conditions

$$[x(0) = 1]$$

15.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned}
p(t) &= 1 \\
q(t) &= \text{Heaviside}\,(t-1) - \text{Heaviside}\,(-2+t)
\end{aligned}$$

Hence the ode is

$$x' + x = \text{Heaviside}\,(t-1) - \text{Heaviside}\,(-2+t)$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \text{Heaviside}\,(t-1) - \text{Heaviside}\,(-2+t)$ is

$$\{1 \leq t \leq 2, 2 \leq t \leq \infty, -\infty \leq t \leq 1\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.13.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - x(0) + Y(s) = \frac{\mathrm{e}^{-s} - \mathrm{e}^{-2s}}{s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 1 + Y(s) = \frac{\mathrm{e}^{-s} - \mathrm{e}^{-2s}}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{\mathrm{e}^{-s} - \mathrm{e}^{-2s} + s}{s\,(s+1)}$$

Taking the inverse Laplace transform gives

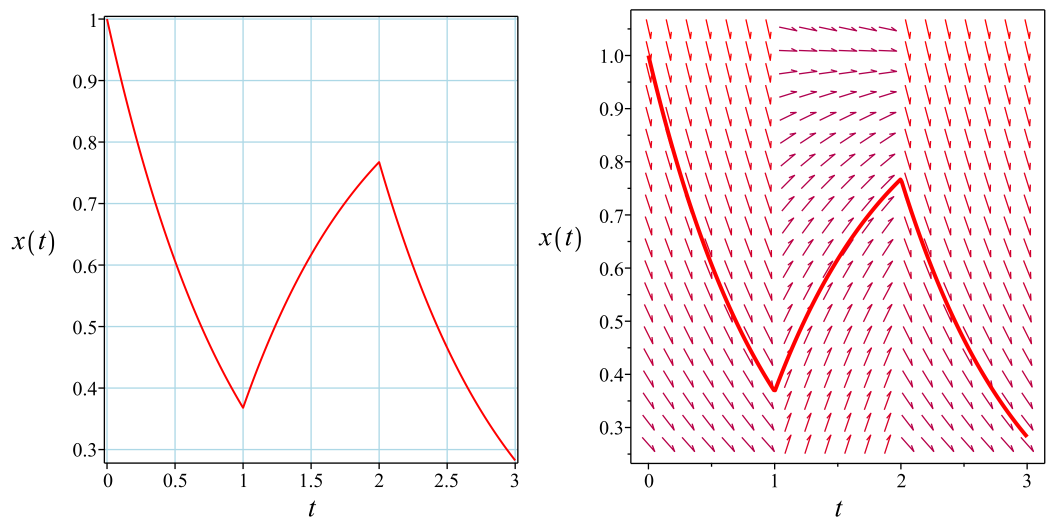
$$\begin{aligned}
x &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-s} - \mathrm{e}^{-2s} + s}{s\,(s+1)}\right) \\
&= \mathrm{e}^{-t} + \text{Heaviside}\,(t-1)\left(1 - \mathrm{e}^{1-t}\right) - \text{Heaviside}\,(-2+t)\left(1 - \mathrm{e}^{-t+2}\right)
\end{aligned}$$

Hence the final solution is

$$x = e^{-t} + \text{Heaviside}(t-1)\left(1-e^{1-t}\right) - \text{Heaviside}(-2+t)\left(1-e^{-t+2}\right)$$

Summary
The solution(s) found are the following

$$x = e^{-t} + \text{Heaviside}(t-1)\left(1-e^{1-t}\right) - \text{Heaviside}(-2+t)\left(1-e^{-t+2}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = e^{-t} + \text{Heaviside}(t-1)\left(1-e^{1-t}\right) - \text{Heaviside}(-2+t)\left(1-e^{-t+2}\right)$$

Verified OK.

15.13.3 Maple step by step solution

Let's solve

$[x' + x = \mathit{Heaviside}(t-1) - \mathit{Heaviside}(-2+t), x(0) = 1]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -x + \mathit{Heaviside}(t-1) - \mathit{Heaviside}(-2+t)$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + x = \mathit{Heaviside}(t-1) - \mathit{Heaviside}(-2+t)$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(x' + x\right) = \mu(t)\left(\mathit{Heaviside}(t-1) - \mathit{Heaviside}(-2+t)\right)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

  $\mu(t)\left(x' + x\right) = \mu'(t)\,x + \mu(t)\,x'$
- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,x)\right) dt = \int \mu(t)\left(\mathit{Heaviside}(t-1) - \mathit{Heaviside}(-2+t)\right) dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,x = \int \mu(t)\left(\mathit{Heaviside}(t-1) - \mathit{Heaviside}(-2+t)\right) dt + c_1$
- Solve for x

  $x = \frac{\int \mu(t)(\mathit{Heaviside}(t-1) - \mathit{Heaviside}(-2+t))dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{t}$

  $x = \frac{\int \mathrm{e}^{t}(\mathit{Heaviside}(t-1) - \mathit{Heaviside}(-2+t))dt + c_1}{\mathrm{e}^{t}}$
- Evaluate the integrals on the rhs

  $x = \frac{\mathrm{e}^{t}\,\mathit{Heaviside}(t-1) - \mathit{Heaviside}(t-1)\mathrm{e} - \mathrm{e}^{t}\,\mathit{Heaviside}(-2+t) + \mathit{Heaviside}(-2+t)\mathrm{e}^{2} + c_1}{\mathrm{e}^{t}}$
- Simplify

  $x = \mathit{Heaviside}(t-1) - \mathit{Heaviside}(-2+t) + \mathit{Heaviside}(-2+t)\,\mathrm{e}^{-t+2} - \mathit{Heaviside}(t-1)\,\mathrm{e}^{1-t} + c_1\mathrm{e}$
- Use initial condition $x(0) = 1$

  $1 = c_1$
- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $x = \mathit{Heaviside}(t-1) - \mathit{Heaviside}(-2+t) + \mathit{Heaviside}(-2+t)\,\mathrm{e}^{-t+2} - \mathit{Heaviside}(t-1)\,\mathrm{e}^{1-t} + \mathrm{e}^{-}$
- Solution to the IVP

$$x = Heaviside(t-1) - Heaviside(-2+t) + Heaviside(-2+t)\,e^{-t+2} - Heaviside(t-1)\,e^{1-t} + e^{-}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 5.11 (sec). Leaf size: 40

```
dsolve([diff(x(t),t)=-x(t)+Heaviside(t-1)-Heaviside(t-2),x(0) = 1],x(t), singsol=all)
```

$$x(t) = \text{Heaviside}\,(t-2)\,e^{2-t} - \text{Heaviside}\,(t-1)\,e^{-t+1} + e^{-t} - \text{Heaviside}\,(t-2) + \text{Heaviside}\,(t-1)$$

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 48

```
DSolve[{x'[t]==-x[t]+UnitStep[t-1]-UnitStep[t-2],{x[0]==1}},x[t],t,IncludeSingularSolutions
```

$$x(t) \to \begin{cases} e^{-t} & t \leq 1 \\ e^{-t}(1-e+e^2) & t > 2 \\ e^{-t}(1-e+e^t) & \text{True} \end{cases}$$

15.14 problem 15

15.14.1 Existence and uniqueness analysis 1904
15.14.2 Maple step by step solution 1907

Internal problem ID [11521]
Internal file name [`OUTPUT/10503_Thursday_May_18_2023_04_21_20_AM_58101082/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.2.1 Initial value problems. Exercises page 156
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + \pi^2 x = \pi^2 \operatorname{Heaviside}(1-t)}$$

With initial conditions

$$[x(0) = 1, x'(0) = 0]$$

15.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= \pi^2 \\ F &= \pi^2(1 - \operatorname{Heaviside}(t-1)) \end{aligned}$$

Hence the ode is

$$x'' + \pi^2 x = \pi^2 (1 - \text{Heaviside}\,(t-1))$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \pi^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \pi^2(1 - \text{Heaviside}\,(t-1))$ is

$$\{t < 1 \lor 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(x') &= sY(s) - x(0) \\
\mathcal{L}(x'') &= s^2 Y(s) - x'(0) - s x(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2 Y(s) - x'(0) - s x(0) + \pi^2 Y(s) = \frac{(-\mathrm{e}^{-s} + 1)\,\pi^2}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
x(0) &= 1 \\
x'(0) &= 0
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) - s + \pi^2 Y(s) = \frac{(-\mathrm{e}^{-s} + 1)\,\pi^2}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{\pi^2 \mathrm{e}^{-s} - \pi^2 - s^2}{s\,(\pi^2 + s^2)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
x &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{\pi^2 e^{-s} - \pi^2 - s^2}{s\,(\pi^2 + s^2)}\right) \\
&= -2\,\text{Heaviside}\,(t-1)\sin\left(\frac{\pi(t-1)}{2}\right)^2 + 1
\end{aligned}
$$

Hence the final solution is

$$
x = -2\,\text{Heaviside}\,(t-1)\sin\left(\frac{\pi(t-1)}{2}\right)^2 + 1
$$

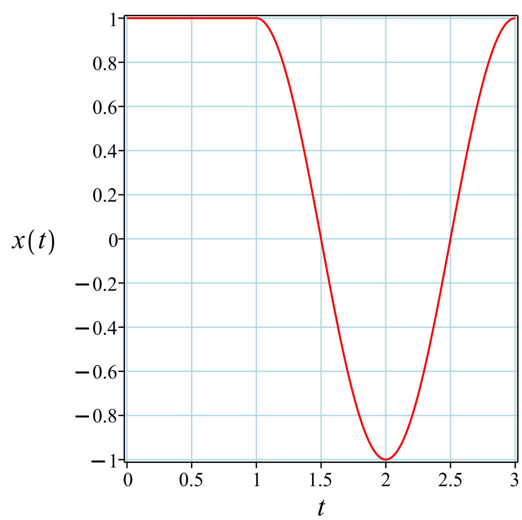
Simplifying the solution gives

$$
x = (-1 - \cos(\pi t))\,\text{Heaviside}\,(t-1) + 1
$$

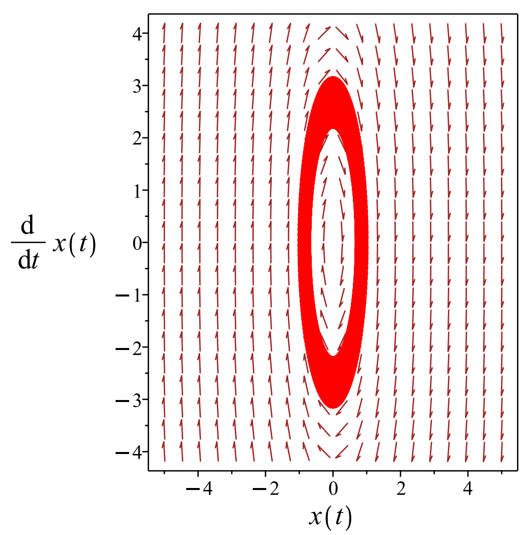
Summary

The solution(s) found are the following

$$
x = (-1 - \cos(\pi t))\,\text{Heaviside}\,(t-1) + 1 \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
x = (-1 - \cos(\pi t))\,\text{Heaviside}\,(t-1) + 1
$$

Verified OK.

15.14.2 Maple step by step solution

Let's solve

$$\left[x''+\pi^2 x=\pi^2(1-\mathit{Heaviside}(t-1)), x(0)=1, x'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2
  x''
- Isolate 2nd derivative
  $x''=-\pi^2 x-\pi^2(-1+\mathit{Heaviside}(t-1))$
- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $x''+\pi^2 x=-\pi^2(-1+\mathit{Heaviside}(t-1))$
- Characteristic polynomial of homogeneous ODE
  $\pi^2+r^2=0$
- Use quadratic formula to solve for r
  $r=\frac{0\pm\left(\sqrt{-4\pi^2}\right)}{2}$
- Roots of the characteristic polynomial
  $r=(-\mathrm{I}\pi, \mathrm{I}\pi)$
- 1st solution of the homogeneous ODE
  $x_1(t)=\cos(\pi t)$
- 2nd solution of the homogeneous ODE
  $x_2(t)=\sin(\pi t)$
- General solution of the ODE
  $x=c_1x_1(t)+c_2x_2(t)+x_p(t)$
- Substitute in solutions of the homogeneous ODE
  $x=c_1\cos(\pi t)+c_2\sin(\pi t)+x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function
  $\left[x_p(t)=-x_1(t)\left(\int\frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right)+x_2(t)\left(\int\frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t)=-\pi^2(-1+\mathit{Heaviside}(t-\right.$
- ○ Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(\pi t) & \sin(\pi t) \\ -\pi \sin(\pi t) & \pi \cos(\pi t) \end{bmatrix}$$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = \pi$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = \pi\left(\cos(\pi t)\left(\int \sin(\pi t)\left(-1 + \mathit{Heaviside}(t-1)\right) dt\right) - \sin(\pi t)\left(\int \cos(\pi t)\left(-1 + \mathit{Heaviside}(\right.\right.\right.$

- Compute integrals

  $x_p(t) = (-1 - \cos(\pi t))\, \mathit{Heaviside}(t-1) + 1$

- Substitute particular solution into general solution to ODE

  $x = c_1 \cos(\pi t) + c_2 \sin(\pi t) + (-1 - \cos(\pi t))\, \mathit{Heaviside}(t-1) + 1$

☐ Check validity of solution $x = c_1 \cos(\pi t) + c_2 \sin(\pi t) + (-1 - \cos(\pi t))\, \mathit{Heaviside}(t-1) + 1$

- Use initial condition $x(0) = 1$

  $1 = 1 + c_1$

- Compute derivative of the solution

  $x' = -c_1 \pi \sin(\pi t) + c_2 \pi \cos(\pi t) + \pi \sin(\pi t)\, \mathit{Heaviside}(t-1) + (-1 - \cos(\pi t))\, \mathit{Dirac}(t-1)$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = c_2 \pi$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $x = (-1 - \cos(\pi t))\, \mathit{Heaviside}(t-1) + 1$

- Solution to the IVP

  $x = (-1 - \cos(\pi t))\, \mathit{Heaviside}(t-1) + 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.062 (sec). Leaf size: 21

```
dsolve([diff(x(t),t$2)+Pi^2*x(t)=Pi^2*Heaviside(1-t),x(0) = 1, D(x)(0) = 0],x(t), singsol=al
```

$$x(t) = 1 + (-\cos(\pi t) - 1)\,\mathrm{Heaviside}(t - 1)$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 18

```
DSolve[{x''[t]+Pi^2*x[t]==Pi^2*UnitStep[1-t],{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolut
```

$$x(t) \to \begin{cases} 1 & t \leq 1 \\ -\cos(\pi t) & \text{True} \end{cases}$$

16 Chapter 3, Laplace transform. Section 3.3 The convolution property. Exercises page 162

16.1 problem 7 1911
16.2 problem 8 1917

16.1 problem 7

16.1.1 Existence and uniqueness analysis 1911
16.1.2 Maple step by step solution 1914

Internal problem ID [11522]
Internal file name [`OUTPUT/10504_Thursday_May_18_2023_04_21_21_AM_49106921/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.3 The convolution property. Exercises page 162
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' - 4x = 1 - \text{Heaviside}\,(t-1)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

16.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= -4 \\ F &= 1 - \text{Heaviside}\,(t-1) \end{aligned}$$

Hence the ode is

$$x'' - 4x = 1 - \text{Heaviside}\left(t - 1\right)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 1 - \text{Heaviside}\left(t - 1\right)$ is

$$\{t < 1 \vee 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(x') &= sY(s) - x(0) \\
\mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) - 4Y(s) = \frac{-\mathrm{e}^{-s} + 1}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
x(0) &= 0 \\
x'(0) &= 0
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 4Y(s) = \frac{-\mathrm{e}^{-s} + 1}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{\mathrm{e}^{-s} - 1}{s\left(s^2 - 4\right)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
x &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{e^{-s}-1}{s\left(s^{2}-4\right)}\right) \\
&= -\frac{\text{Heaviside}\left(t-1\right)\sinh\left(t-1\right)^{2}}{2}-\frac{1}{4}+\frac{\cosh\left(2t\right)}{4}
\end{aligned}
$$

Hence the final solution is

$$
x = -\frac{\text{Heaviside}\left(t-1\right)\sinh\left(t-1\right)^{2}}{2}-\frac{1}{4}+\frac{\cosh\left(2t\right)}{4}
$$

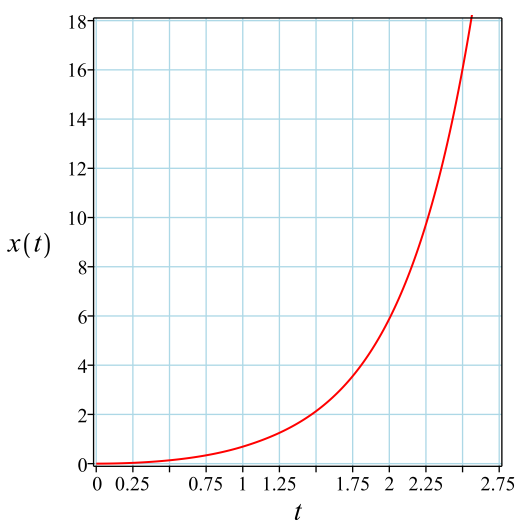
Simplifying the solution gives

$$
x = -\frac{\text{Heaviside}\left(t-1\right)\sinh\left(t-1\right)^{2}}{2}-\frac{1}{4}+\frac{\cosh\left(2t\right)}{4}
$$

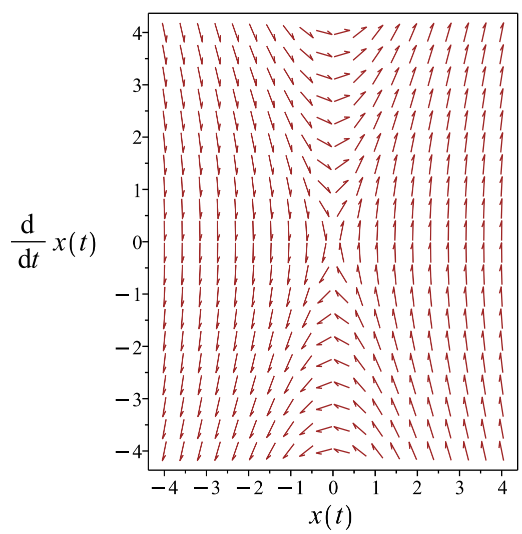
<u>Summary</u>

The solution(s) found are the following

$$
x = -\frac{\text{Heaviside}\left(t-1\right)\sinh\left(t-1\right)^{2}}{2}-\frac{1}{4}+\frac{\cosh\left(2t\right)}{4} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -\frac{\text{Heaviside}\,(t-1)\sinh\,(t-1)^2}{2} - \frac{1}{4} + \frac{\cosh\,(2t)}{4}$$

Verified OK.

16.1.2 Maple step by step solution

Let's solve

$\left[x'' - 4x = 1 - \mathit{Heaviside}(t-1)\,, x(0) = 0, x'\big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r-2)\,(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 2)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{2t}$

- General solution of the ODE

  $x = c_1x_1(t) + c_2x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{2t} + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = 1 - \mathit{Heaviside}(t-1)\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-2t} & e^{2t} \\ -2\,e^{-2t} & 2\,e^{2t} \end{bmatrix}$$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = 4$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = \frac{e^{-2t}\left(\int e^{2t}(-1+Heaviside(t-1))dt\right)}{4} - \frac{e^{2t}\left(\int e^{-2t}(-1+Heaviside(t-1))dt\right)}{4}$

- Compute integrals

  $x_p(t) = \frac{Heaviside(t-1)}{4} - \frac{Heaviside(t-1)e^{-2t+2}}{8} - \frac{1}{4} - \frac{Heaviside(t-1)e^{2t-2}}{8}$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{-2t} + c_2 e^{2t} + \frac{Heaviside(t-1)}{4} - \frac{Heaviside(t-1)e^{-2t+2}}{8} - \frac{1}{4} - \frac{Heaviside(t-1)e^{2t-2}}{8}$

□ Check validity of solution $x = c_1 e^{-2t} + c_2 e^{2t} + \frac{Heaviside(t-1)}{4} - \frac{Heaviside(t-1)e^{-2t+2}}{8} - \frac{1}{4} - \frac{Heaviside(t-}{8}$

- Use initial condition $x(0) = 0$

  $0 = c_1 + c_2 - \frac{1}{4}$

- Compute derivative of the solution

  $x' = -2c_1 e^{-2t} + 2c_2 e^{2t} + \frac{Dirac(t-1)}{4} - \frac{Dirac(t-1)e^{-2t+2}}{8} + \frac{Heaviside(t-1)e^{-2t+2}}{4} - \frac{Dirac(t-1)e^{2t-2}}{8} - \frac{Heavisid}{}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = -2c_1 + 2c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{8}, c_2 = \frac{1}{8}\right\}$

- Substitute constant values into general solution and simplify

  $x = \frac{e^{-2t}}{8} + \frac{e^{2t}}{8} + \frac{Heaviside(t-1)}{4} - \frac{Heaviside(t-1)e^{-2t+2}}{8} - \frac{1}{4} - \frac{Heaviside(t-1)e^{2t-2}}{8}$

- Solution to the IVP

  $x = \frac{e^{-2t}}{8} + \frac{e^{2t}}{8} + \frac{Heaviside(t-1)}{4} - \frac{Heaviside(t-1)e^{-2t+2}}{8} - \frac{1}{4} - \frac{Heaviside(t-1)e^{2t-2}}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.437 (sec). Leaf size: 24

```
dsolve([diff(x(t),t$2)-4*x(t)=1-Heaviside(t-1),x(0) = 0, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = -\frac{1}{4} + \frac{\cosh(2t)}{4} - \frac{\text{Heaviside}(t-1)\sinh(t-1)^2}{2}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 54

```
DSolve[{x''[t]-4*x[t]==1-UnitStep[t-1],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions -
```

$$x(t) \to \frac{1}{8}e^{-2(t+1)}\left(\left(e^2 - e^{2t}\right)^2 \theta(1-t) + \left(e^2 - 1\right)\left(e^{4t} - e^2\right)\right)$$

16.2 problem 8

16.2.1 Existence and uniqueness analysis 1917
16.2.2 Maple step by step solution 1920

Internal problem ID [11523]
Internal file name [`OUTPUT/10505_Thursday_May_18_2023_04_21_23_AM_33382183/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.3 The convolution property. Exercises page 162
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x'' + 3x' + 2x = \mathrm{e}^{-4t}}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

16.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= 2 \\ F &= \mathrm{e}^{-4t} \end{aligned}$$

Hence the ode is

$$x'' + 3x' + 2x = e^{-4t}$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-4t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(x') &= sY(s) - x(0)\\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) + 3sY(s) - 3x(0) + 2Y(s) = \frac{1}{s+4} \tag{1}$$

But the initial conditions are

$$\begin{aligned}x(0) &= 0\\ x'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s+4}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{1}{(s+4)(s^2+3s+2)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{2(s+2)} + \frac{1}{3s+3} + \frac{1}{6s+24}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{2(s+2)}\right) = -\frac{e^{-2t}}{2}$$
$$\mathcal{L}^{-1}\left(\frac{1}{3s+3}\right) = \frac{e^{-t}}{3}$$
$$\mathcal{L}^{-1}\left(\frac{1}{6s+24}\right) = \frac{e^{-4t}}{6}$$

Adding the above results and simplifying gives

$$x = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{3} + \frac{e^{-4t}}{6}$$

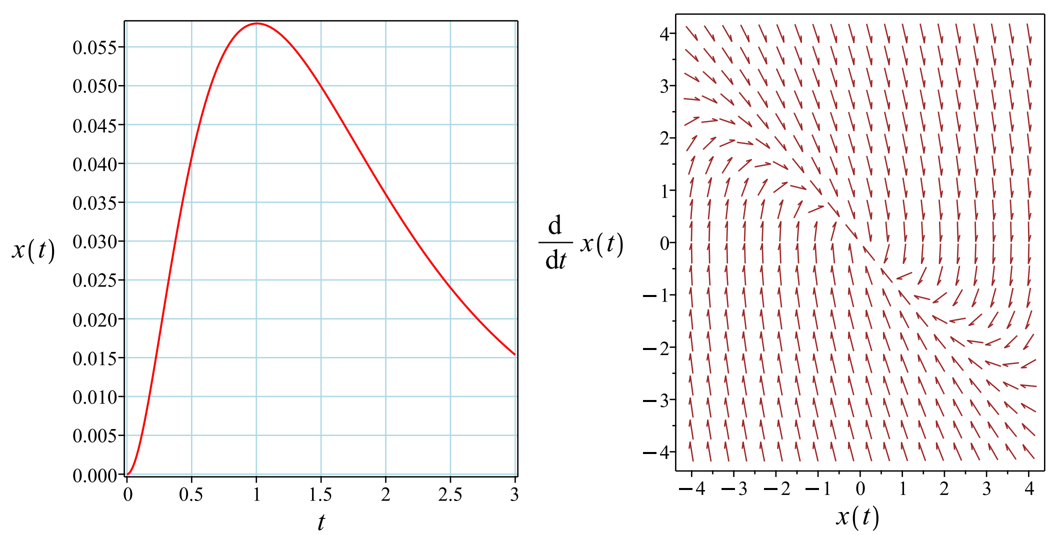
Simplifying the solution gives

$$x = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{3} + \frac{e^{-4t}}{6}$$

Summary

The solution(s) found are the following

$$x = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{3} + \frac{e^{-4t}}{6} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{3} + \frac{e^{-4t}}{6}$$

Verified OK.

16.2.2 Maple step by step solution

Let's solve

$$\left[x'' + 3x' + 2x = e^{-4t}, x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-2t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{-t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{-2t} + c_2 e^{-t} + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = e^{-4t}\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-2t} & e^{-t} \\ -2\,e^{-2t} & -e^{-t} \end{bmatrix}$$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = e^{-3t}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = -e^{-2t}\left(\int e^{-2t}dt\right) + e^{-t}\left(\int e^{-3t}dt\right)$

- Compute integrals

  $x_p(t) = \frac{e^{-4t}}{6}$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{-2t} + c_2 e^{-t} + \frac{e^{-4t}}{6}$

☐ Check validity of solution $x = c_1 e^{-2t} + c_2 e^{-t} + \frac{e^{-4t}}{6}$

- Use initial condition $x(0) = 0$

  $0 = c_1 + c_2 + \frac{1}{6}$

- Compute derivative of the solution

  $x' = -2c_1 e^{-2t} - c_2 e^{-t} - \frac{2\,e^{-4t}}{3}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = -2c_1 - c_2 - \frac{2}{3}$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{2}, c_2 = \frac{1}{3}\right\}$

- Substitute constant values into general solution and simplify

  $x = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{3} + \frac{e^{-4t}}{6}$

- Solution to the IVP

  $x = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{3} + \frac{e^{-4t}}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 4.719 (sec). Leaf size: 23

```
dsolve([diff(x(t),t$2)+3*diff(x(t),t)+2*x(t)=exp(-4*t),x(0) = 0, D(x)(0) = 0],x(t), singsol=
```

$$x(t) = \frac{e^{-4t}}{6} + \frac{e^{-t}}{3} - \frac{e^{-2t}}{2}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 28

```
DSolve[{x''[t]+3*x'[t]+2*x[t]==Exp[-4*t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions
```

$$x(t) \to \frac{1}{6} e^{-4t} \left(e^t - 1\right)^2 \left(2e^t + 1\right)$$

17 Chapter 3, Laplace transform. Section 3.4 Impulsive sources. Exercises page 173

17.1 problem 2 1924
17.2 problem 3 1929
17.3 problem 4 1935
17.4 problem 6 1941
17.5 problem 7 1947
17.6 problem 9 1953
17.7 problem 10 1959

17.1 problem 2

17.1.1 Existence and uniqueness analysis 1924
17.1.2 Solving as laplace ode 1925
17.1.3 Maple step by step solution 1926

Internal problem ID [11524]
Internal file name [`OUTPUT/10506_Thursday_May_18_2023_04_21_26_AM_62940236/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.4 Impulsive sources. Exercises page 173
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[[_linear, `class A`]]`

$$\boxed{x' + 3x = \delta(t-1) + \text{Heaviside}\,(t-4)}$$

With initial conditions

$$[x(0) = 1]$$

17.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= \delta(t-1) + \text{Heaviside}\,(t-4) \end{aligned}$$

Hence the ode is

$$x' + 3x = \delta(t-1) + \text{Heaviside}\,(t-4)$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \delta(t-1) + \text{Heaviside}\,(t-4)$ is

$$\{1 \leq t \leq 4, 4 \leq t \leq \infty, -\infty \leq t \leq 1\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

17.1.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - x(0) + 3Y(s) = \mathrm{e}^{-s} + \frac{\mathrm{e}^{-4s}}{s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 1 + 3Y(s) = \mathrm{e}^{-s} + \frac{\mathrm{e}^{-4s}}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{\mathrm{e}^{-s}s + \mathrm{e}^{-4s} + s}{s\,(s+3)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
x &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-s}s + \mathrm{e}^{-4s} + s}{s\,(s+3)}\right) \\
&= \text{Heaviside}\,(t-1)\,\mathrm{e}^{-3t+3} + \frac{\text{Heaviside}\,(t-4)\,(1-\mathrm{e}^{-3t+12})}{3} + \mathrm{e}^{-3t}
\end{aligned}$$

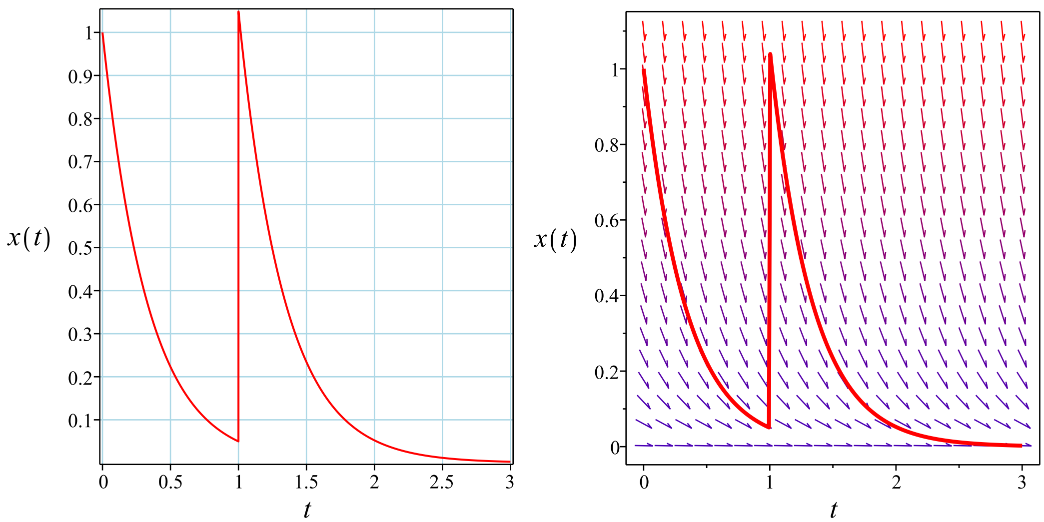
Hence the final solution is

$$x = \text{Heaviside}\,(t-1)\,\mathrm{e}^{-3t+3} + \frac{\text{Heaviside}\,(t-4)\,(1-\mathrm{e}^{-3t+12})}{3} + \mathrm{e}^{-3t}$$

Summary

The solution(s) found are the following

$$x = \text{Heaviside}(t-1)\,e^{-3t+3} + \frac{\text{Heaviside}(t-4)\left(1-e^{-3t+12}\right)}{3} + e^{-3t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \text{Heaviside}(t-1)\,e^{-3t+3} + \frac{\text{Heaviside}(t-4)\left(1-e^{-3t+12}\right)}{3} + e^{-3t}$$

Verified OK.

17.1.3 Maple step by step solution

Let's solve

$[x' + 3x = \mathit{Dirac}(t-1) + \mathit{Heaviside}(t-4), x(0) = 1]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -3x + \mathit{Dirac}(t-1) + \mathit{Heaviside}(t-4)$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + 3x = \mathit{Dirac}(t-1) + \mathit{Heaviside}(t-4)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(x'+3x\right)=\mu(t)\left(\mathit{Dirac}(t-1)+\mathit{Heaviside}(t-4)\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

  $\mu(t)\left(x'+3x\right)=\mu'(t)\,x+\mu(t)\,x'$

- Isolate $\mu'(t)$

  $\mu'(t)=3\mu(t)$

- Solve to find the integrating factor

  $\mu(t)=\mathrm{e}^{3t}$

- Integrate both sides with respect to t

  $\int\left(\frac{d}{dt}(\mu(t)\,x)\right)dt=\int\mu(t)\left(\mathit{Dirac}(t-1)+\mathit{Heaviside}(t-4)\right)dt+c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,x=\int\mu(t)\left(\mathit{Dirac}(t-1)+\mathit{Heaviside}(t-4)\right)dt+c_1$

- Solve for x

  $x=\frac{\int\mu(t)(\mathit{Dirac}(t-1)+\mathit{Heaviside}(t-4))dt+c_1}{\mu(t)}$

- Substitute $\mu(t)=\mathrm{e}^{3t}$

  $x=\frac{\int\mathrm{e}^{3t}(\mathit{Dirac}(t-1)+\mathit{Heaviside}(t-4))dt+c_1}{\mathrm{e}^{3t}}$

- Evaluate the integrals on the rhs

  $x=\frac{\mathrm{e}^3\mathit{Heaviside}(t-1)+\frac{\mathrm{e}^{3t}\mathit{Heaviside}(t-4)}{3}-\frac{\mathit{Heaviside}(t-4)\mathrm{e}^{12}}{3}+c_1}{\mathrm{e}^{3t}}$

- Simplify

  $x=-\frac{\mathit{Heaviside}(t-4)\mathrm{e}^{-3t+12}}{3}+\mathit{Heaviside}(t-1)\,\mathrm{e}^{-3t+3}+\frac{\mathit{Heaviside}(t-4)}{3}+\mathrm{e}^{-3t}c_1$

- Use initial condition $x(0)=1$

  $1=c_1$

- Solve for c_1

  $c_1=1$

- Substitute $c_1=1$ into general solution and simplify

  $x=-\frac{\mathit{Heaviside}(t-4)\mathrm{e}^{-3t+12}}{3}+\mathit{Heaviside}(t-1)\,\mathrm{e}^{-3t+3}+\frac{\mathit{Heaviside}(t-4)}{3}+\mathrm{e}^{-3t}$

- Solution to the IVP

  $x=-\frac{\mathit{Heaviside}(t-4)\mathrm{e}^{-3t+12}}{3}+\mathit{Heaviside}(t-1)\,\mathrm{e}^{-3t+3}+\frac{\mathit{Heaviside}(t-4)}{3}+\mathrm{e}^{-3t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 5.422 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)+3*x(t)=Dirac(t-1)+Heaviside(t-4),x(0) = 1],x(t), singsol=all)
```

$$x(t) = \text{Heaviside}(t-1)\,\mathrm{e}^{-3t+3} - \frac{\text{Heaviside}(t-4)\,\mathrm{e}^{-3t+12}}{3} + \frac{\text{Heaviside}(t-4)}{3} + \mathrm{e}^{-3t}$$

✓ Solution by Mathematica

Time used: 0.201 (sec). Leaf size: 53

```
DSolve[{x'[t]+3*x[t]==DiracDelta[t-1]+UnitStep[t-4],{x[0]==1}},x[t],t,IncludeSingularSolutio
```

$$x(t) \to \frac{1}{3}e^{-3t}\left(3e^{3}\theta(t-1) + \left(e^{12}-e^{3t}\right)\theta(4-t) + e^{3t} - e^{12} + 3\right)$$

17.2 problem 3

17.2.1 Existence and uniqueness analysis 1929
17.2.2 Maple step by step solution 1932

Internal problem ID [11525]
Internal file name [`OUTPUT/10507_Thursday_May_18_2023_04_21_29_AM_85428771/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.4 Impulsive sources. Exercises page 173
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' - x = \delta(t-5)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

17.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= -1 \\ F &= \delta(t-5) \end{aligned}$$

Hence the ode is

$$x'' - x = \delta(t - 5)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t - 5)$ is

$$\{t < 5 \lor 5 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(x') &= sY(s) - x(0)\\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) - Y(s) = e^{-5s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}x(0) &= 0\\ x'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - Y(s) = e^{-5s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-5s}}{s^2 - 1}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
x &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-5s}}{s^2-1}\right) \\
&= \text{Heaviside}\left(t-5\right)\sinh\left(t-5\right)
\end{aligned}
$$

Hence the final solution is

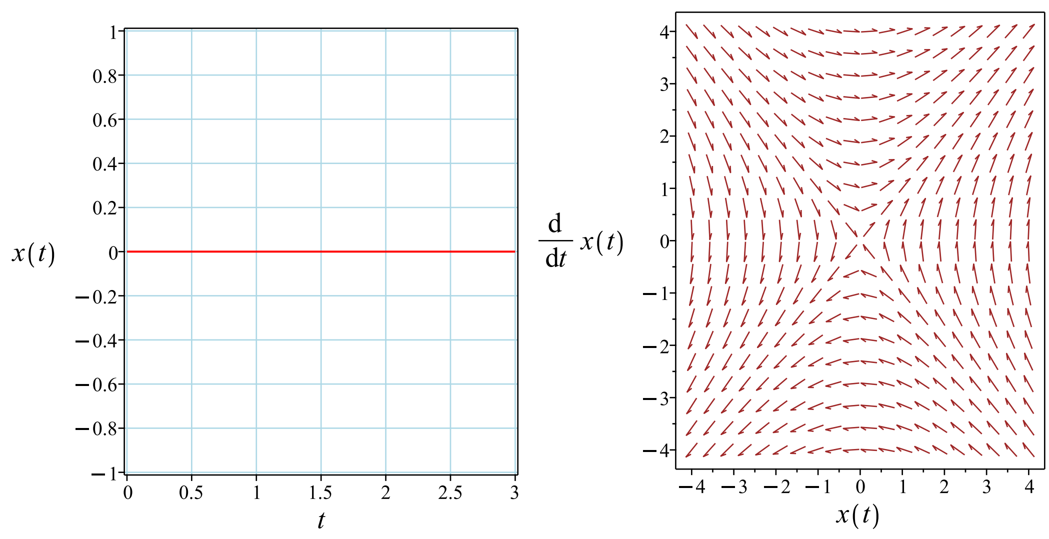
$$
x = \text{Heaviside}\left(t-5\right)\sinh\left(t-5\right)
$$

Simplifying the solution gives

$$
x = \text{Heaviside}\left(t-5\right)\sinh\left(t-5\right)
$$

Summary
The solution(s) found are the following

$$
x = \text{Heaviside}\left(t-5\right)\sinh\left(t-5\right) \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$
x = \text{Heaviside}\left(t-5\right)\sinh\left(t-5\right)
$$

Verified OK.

17.2.2 Maple step by step solution

Let's solve

$$\left[x'' - x = Dirac(t-5)\,, x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{-t} + c_2 e^{t} + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = Dirac(t-5)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-t} & e^{t} \\ -e^{-t} & e^{t} \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 2$

  - Substitute functions into equation for $x_p(t)$

$$x_p(t) = -\frac{\left(\int \mathit{Dirac}(t-5)dt\right)\left(\mathrm{e}^{-t+5}-\mathrm{e}^{t-5}\right)}{2}$$

- Compute integrals

$$x_p(t) = -\frac{\mathit{Heaviside}(t-5)\left(\mathrm{e}^{-t+5}-\mathrm{e}^{t-5}\right)}{2}$$

- Substitute particular solution into general solution to ODE

$$x = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{t} - \frac{\mathit{Heaviside}(t-5)\left(\mathrm{e}^{-t+5}-\mathrm{e}^{t-5}\right)}{2}$$

□ Check validity of solution $x = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{t} - \frac{\mathit{Heaviside}(t-5)\left(\mathrm{e}^{-t+5}-\mathrm{e}^{t-5}\right)}{2}$

- Use initial condition $x(0) = 0$

$$0 = c_1 + c_2$$

- Compute derivative of the solution

$$x' = -c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{t} - \frac{\mathit{Dirac}(t-5)\left(\mathrm{e}^{-t+5}-\mathrm{e}^{t-5}\right)}{2} - \frac{\mathit{Heaviside}(t-5)\left(-\mathrm{e}^{-t+5}-\mathrm{e}^{t-5}\right)}{2}$$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

$$0 = -c_1 + c_2$$

- Solve for c_1 and c_2

$$\{c_1 = 0, c_2 = 0\}$$

- Substitute constant values into general solution and simplify

$$x = -\frac{\mathit{Heaviside}(t-5)\left(\mathrm{e}^{-t+5}-\mathrm{e}^{t-5}\right)}{2}$$

- Solution to the IVP

$$x = -\frac{\mathit{Heaviside}(t-5)\left(\mathrm{e}^{-t+5}-\mathrm{e}^{t-5}\right)}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.828 (sec). Leaf size: 13

```
dsolve([diff(x(t),t$2)-x(t)=Dirac(t-5),x(0) = 0, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \text{Heaviside}\,(t-5)\sinh\,(t-5)$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 31

```
DSolve[{x''[t]-x[t]==DiracDelta[t-5],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{1}{2}e^{-t-5}\left(e^{2t}-e^{10}\right)\theta(t-5)$$

17.3 problem 4

17.3.1 Existence and uniqueness analysis 1935
17.3.2 Maple step by step solution 1938

Internal problem ID [11526]
Internal file name [`OUTPUT/10508_Thursday_May_18_2023_04_21_31_AM_57129417/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.4 Impulsive sources. Exercises page 173
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + x = \delta(-2+t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

17.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= \delta(-2+t) \end{aligned}$$

Hence the ode is

$$x'' + x = \delta(-2 + t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(-2 + t)$ is

$$\{t < 2 \lor 2 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$
$$\mathcal{L}(x'') = s^2Y(s) - x'(0) - sx(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) + Y(s) = e^{-2s} \tag{1}$$

But the initial conditions are

$$x(0) = 0$$
$$x'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + Y(s) = e^{-2s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-2s}}{s^2 + 1}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
x &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+1}\right) \\
&= \text{Heaviside}(-2+t)\sin(-2+t)
\end{aligned}
$$

Hence the final solution is

$$x = \text{Heaviside}(-2+t)\sin(-2+t)$$

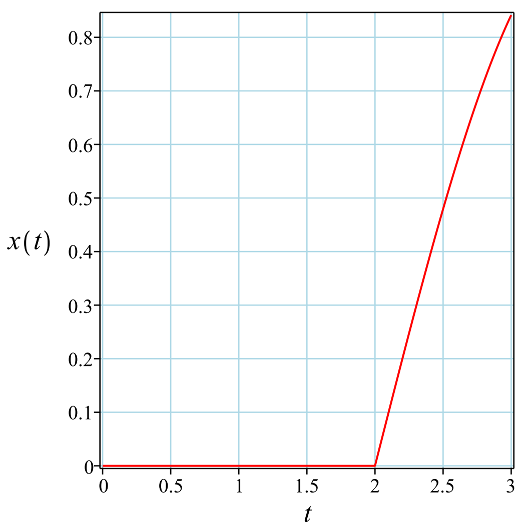
Simplifying the solution gives

$$x = \text{Heaviside}(-2+t)\sin(-2+t)$$

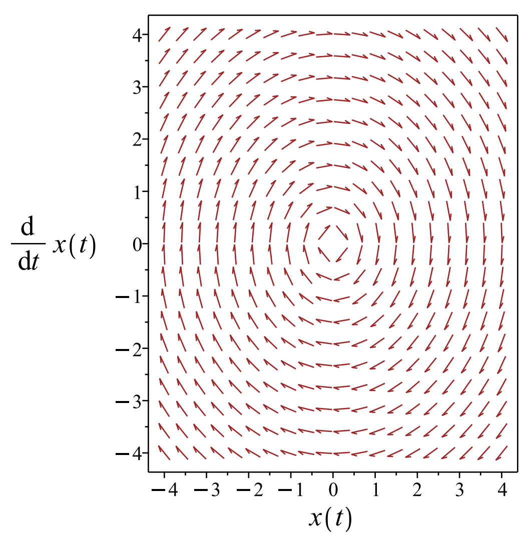
Summary

The solution(s) found are the following

$$x = \text{Heaviside}(-2+t)\sin(-2+t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \text{Heaviside}(-2+t)\sin(-2+t)$$

Verified OK.

17.3.2 Maple step by step solution

Let's solve

$$\left[x'' + x = Dirac(-2+t)\,, x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(t) + c_2 \sin(t) + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $$\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = Dirac(-2+t)\right]$$

  ○ Wronskian of solutions of the homogeneous equation

  $$W(x_1(t)\,, x_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

  ○ Compute Wronskian

  $W(x_1(t)\,, x_2(t)) = 1$

  ○ Substitute functions into equation for $x_p(t)$

$x_p(t) = \left(\int \mathit{Dirac}(-2+t)\,dt\right)(-\sin(2)\cos(t)+\cos(2)\sin(t))$

- Compute integrals

  $x_p(t) = \mathit{Heaviside}(-2+t)(-\sin(2)\cos(t)+\cos(2)\sin(t))$

- Substitute particular solution into general solution to ODE

  $x = c_1\cos(t)+c_2\sin(t)+\mathit{Heaviside}(-2+t)(-\sin(2)\cos(t)+\cos(2)\sin(t))$

☐ Check validity of solution $x = c_1\cos(t)+c_2\sin(t)+\mathit{Heaviside}(-2+t)(-\sin(2)\cos(t)+\cos($

- Use initial condition $x(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $x' = -c_1\sin(t)+c_2\cos(t)+\mathit{Dirac}(-2+t)(-\sin(2)\cos(t)+\cos(2)\sin(t))+\mathit{Heaviside}(-2+$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $x = \mathit{Heaviside}(-2+t)(-\sin(2)\cos(t)+\cos(2)\sin(t))$

- Solution to the IVP

  $x = \mathit{Heaviside}(-2+t)(-\sin(2)\cos(t)+\cos(2)\sin(t))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.922 (sec). Leaf size: 13

```
dsolve([diff(x(t),t$2)+x(t)=Dirac(t-2),x(0) = 0, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \text{Heaviside}\,(t-2)\sin\,(t-2)$$

✓ Solution by Mathematica

Time used: 0.206 (sec). Leaf size: 17

```
DSolve[{x''[t]+x[t]==DiracDelta[t-2],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions ->
```

$$x(t) \to -\theta(t-2)\sin(2-t)$$

17.4 problem 6

17.4.1 Existence and uniqueness analysis 1941
17.4.2 Maple step by step solution 1944

Internal problem ID [11527]
Internal file name [`OUTPUT/10509_Thursday_May_18_2023_04_21_33_AM_78067971/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.4 Impulsive sources. Exercises page 173
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + 4x = \delta(-2+t) - \delta(t-5)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

17.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= \delta(-2+t) - \delta(t-5) \end{aligned}$$

Hence the ode is

$$x'' + 4x = \delta(-2+t) - \delta(t-5)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(-2+t) - \delta(t-5)$ is

$$\{2 \leq t \leq 5, 5 \leq t \leq \infty, -\infty \leq t \leq 2\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(x') &= sY(s) - x(0) \\
\mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) + 4Y(s) = e^{-2s} - e^{-5s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
x(0) &= 0 \\
x'(0) &= 0
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 4Y(s) = e^{-2s} - e^{-5s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-2s} - e^{-5s}}{s^2 + 4}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
x &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-2s}-e^{-5s}}{s^2+4}\right) \\
&= \frac{\text{Heaviside}\,(-2+t)\sin\,(-4+2t)}{2} - \frac{\text{Heaviside}\,(t-5)\sin\,(2t-10)}{2}
\end{aligned}
$$

Hence the final solution is

$$
x = \frac{\text{Heaviside}\,(-2+t)\sin\,(-4+2t)}{2} - \frac{\text{Heaviside}\,(t-5)\sin\,(2t-10)}{2}
$$

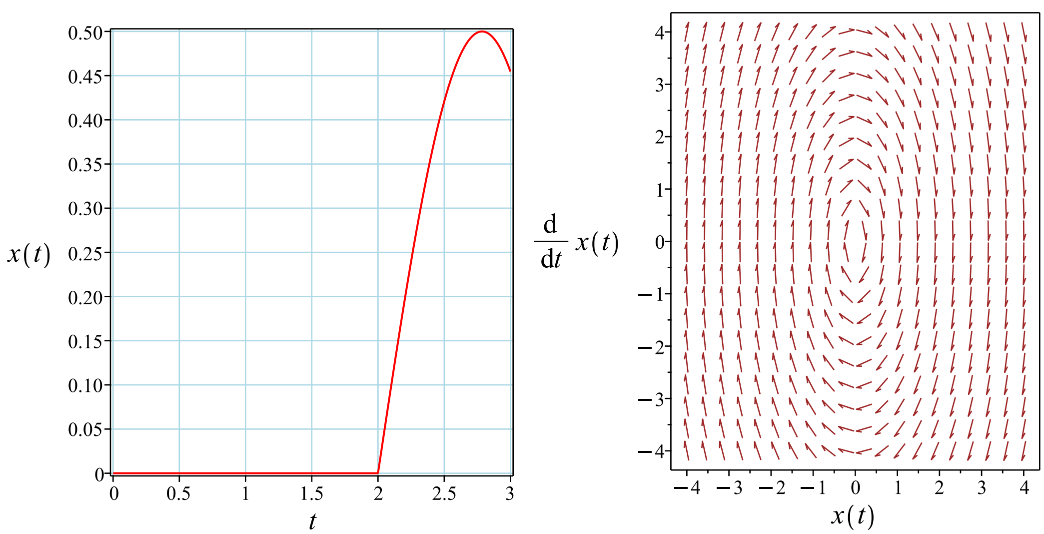
Simplifying the solution gives

$$
x = \frac{\text{Heaviside}\,(-2+t)\sin\,(-4+2t)}{2} - \frac{\text{Heaviside}\,(t-5)\sin\,(2t-10)}{2}
$$

Summary

The solution(s) found are the following

$$
x = \frac{\text{Heaviside}\,(-2+t)\sin\,(-4+2t)}{2} - \frac{\text{Heaviside}\,(t-5)\sin\,(2t-10)}{2} \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
x = \frac{\text{Heaviside}\,(-2+t)\sin\,(-4+2t)}{2} - \frac{\text{Heaviside}\,(t-5)\sin\,(2t-10)}{2}
$$

Verified OK.

17.4.2 Maple step by step solution

Let's solve

$$\left[x'' + 4x = Dirac(-2+t) - Dirac(t-5)\,, x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(2t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(2t) + c_2 \sin(2t) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = Dirac(-2+t) - Dirac(t\right.$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 2$

  - Substitute functions into equation for $x_p(t)$

$$x_p(t) = -\frac{\cos(2t)\left(\int(\sin(4)\,Dirac(-2+t)-\sin(10)\,Dirac(t-5))dt\right)}{2} + \frac{\sin(2t)\left(\int(Dirac(-2+t)\cos(4)-Dirac(t-5)\cos(10))dt\right)}{2}$$

- Compute integrals

  $$x_p(t) = \frac{(-\sin(4)\cos(2t)+\cos(4)\sin(2t))\,Heaviside(-2+t)}{2} - \frac{Heaviside(t-5)(\sin(2t)\cos(10)-\cos(2t)\sin(10))}{2}$$

- Substitute particular solution into general solution to ODE

  $$x = c_1\cos(2t) + c_2\sin(2t) + \frac{(-\sin(4)\cos(2t)+\cos(4)\sin(2t))\,Heaviside(-2+t)}{2} - \frac{Heaviside(t-5)(\sin(2t)\cos(10)-\cos(2}{2}$$

☐ Check validity of solution $x = c_1\cos(2t) + c_2\sin(2t) + \frac{(-\sin(4)\cos(2t)+\cos(4)\sin(2t))\,Heaviside(-2+t)}{2} -$

- Use initial condition $x(0) = 0$

  $$0 = c_1$$

- Compute derivative of the solution

  $$x' = -2c_1\sin(2t) + 2c_2\cos(2t) + \frac{(2\sin(4)\sin(2t)+2\cos(4)\cos(2t))\,Heaviside(-2+t)}{2} + \frac{(-\sin(4)\cos(2t)+\cos(4)\sin(2t}{2}$$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $$0 = 2c_2$$

- Solve for c_1 and c_2

  $$\{c_1 = 0, c_2 = 0\}$$

- Substitute constant values into general solution and simplify

  $$x = \frac{(-\sin(4)\cos(2t)+\cos(4)\sin(2t))\,Heaviside(-2+t)}{2} - \frac{Heaviside(t-5)(\sin(2t)\cos(10)-\cos(2t)\sin(10))}{2}$$

- Solution to the IVP

  $$x = \frac{(-\sin(4)\cos(2t)+\cos(4)\sin(2t))\,Heaviside(-2+t)}{2} - \frac{Heaviside(t-5)(\sin(2t)\cos(10)-\cos(2t)\sin(10))}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.703 (sec). Leaf size: 29

```
dsolve([diff(x(t),t$2)+4*x(t)=Dirac(t-2)-Dirac(t-5),x(0) = 0, D(x)(0) = 0],x(t), singsol=all
```

$$x(t) = -\frac{\text{Heaviside}\,(t-5)\sin\,(2t-10)}{2} + \frac{\text{Heaviside}\,(t-2)\sin\,(2t-4)}{2}$$

✓ Solution by Mathematica

Time used: 0.128 (sec). Leaf size: 33

```
DSolve[{x''[t]+4*x[t]==DiracDelta[t-2]-DiracDelta[t-5],{x[0]==0,x'[0]==0}},x[t],t,IncludeSin
```

$$x(t) \to \frac{1}{2}(\theta(t-5)\sin(10-2t) - \theta(t-2)\sin(4-2t))$$

17.5 problem 7

17.5.1 Existence and uniqueness analysis 1947
17.5.2 Maple step by step solution 1950

Internal problem ID [11528]
Internal file name [`OUTPUT/10510_Thursday_May_18_2023_04_21_36_AM_84742480/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.4 Impulsive sources. Exercises page 173
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + x = 3\delta(-2\pi + t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 1]$$

17.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= 3\delta(-2\pi + t) \end{aligned}$$

Hence the ode is

$$x'' + x = 3\delta(-2\pi + t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 3\delta(-2\pi + t)$ is

$$\{t < 2\pi \lor 2\pi < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(x') &= sY(s) - x(0)\\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) + Y(s) = 3\,\mathrm{e}^{-2s\pi} \tag{1}$$

But the initial conditions are

$$\begin{aligned}x(0) &= 0\\ x'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + Y(s) = 3\,\mathrm{e}^{-2s\pi}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{3\,\mathrm{e}^{-2s\pi} + 1}{s^2 + 1}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
x &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{3\,\mathrm{e}^{-2s\pi}+1}{s^2+1}\right) \\
&= \sin(t)\,(3\,\mathrm{Heaviside}\,(-2\pi+t)+1)
\end{aligned}
$$

Hence the final solution is

$$x = \sin(t)\,(3\,\mathrm{Heaviside}\,(-2\pi+t)+1)$$

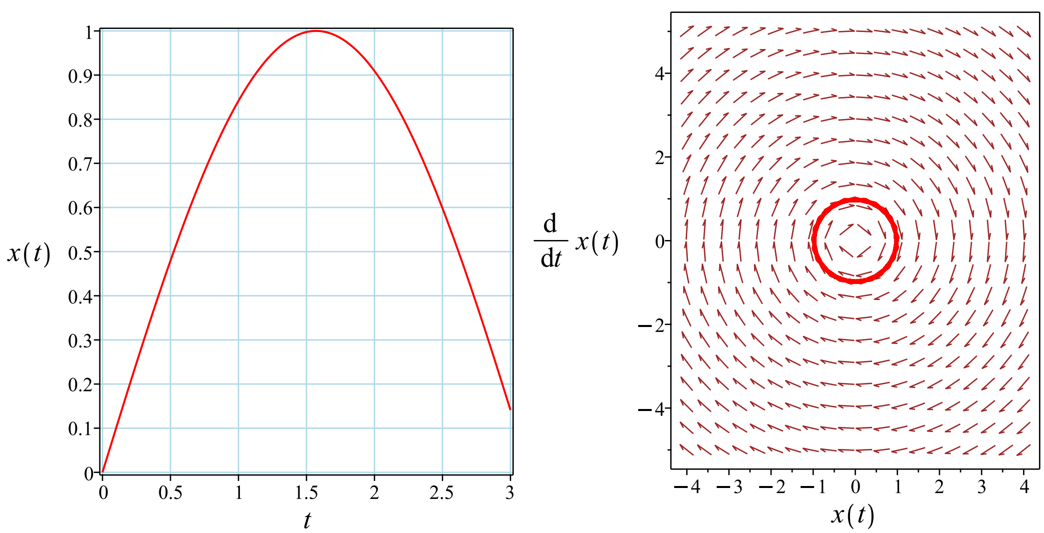
Simplifying the solution gives

$$x = \sin(t)\,(3\,\mathrm{Heaviside}\,(-2\pi+t)+1)$$

Summary

The solution(s) found are the following

$$x = \sin(t)\,(3\,\mathrm{Heaviside}\,(-2\pi+t)+1) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \sin(t)\,(3\,\mathrm{Heaviside}\,(-2\pi+t)+1)$$

Verified OK.

17.5.2 Maple step by step solution

Let's solve

$$\left[x'' + x = 3Dirac(-2\pi + t), x(0) = 0, x'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(t) + c_2 \sin(t) + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = 3Dirac(-2\pi + t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$

- ○ Compute Wronskian

  $W(x_1(t), x_2(t)) = 1$

- ○ Substitute functions into equation for $x_p(t)$

$x_p(t) = 3\sin(t)\left(\int Dirac(-2\pi + t)\,dt\right)$

- Compute integrals

  $x_p(t) = 3\sin(t)\,Heaviside(-2\pi + t)$

- Substitute particular solution into general solution to ODE

  $x = c_1 \cos(t) + c_2 \sin(t) + 3\sin(t)\,Heaviside(-2\pi + t)$

□ Check validity of solution $x = c_1 \cos(t) + c_2 \sin(t) + 3\sin(t)\,Heaviside(-2\pi + t)$

- Use initial condition $x(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $x' = -c_1 \sin(t) + c_2 \cos(t) + 3\cos(t)\,Heaviside(-2\pi + t) + 3\sin(t)\,Dirac(-2\pi + t)$

- Use the initial condition $x'\Big|_{\{t=0\}} = 1$

  $1 = c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $x = \sin(t)\,(3Heaviside(-2\pi + t) + 1)$

- Solution to the IVP

  $x = \sin(t)\,(3Heaviside(-2\pi + t) + 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.969 (sec). Leaf size: 17

```
dsolve([diff(x(t),t$2)+x(t)=3*Dirac(t-2*Pi),x(0) = 0, D(x)(0) = 1],x(t), singsol=all)
```

$$x(t) = \sin(t)\,(3\,\mathrm{Heaviside}(-2\pi + t) + 1)$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 18

```
DSolve[{x''[t]+x[t]==3*DiracDelta[t-2*Pi],{x[0]==0,x'[0]==1}},x[t],t,IncludeSingularSolution
```

$$x(t) \to (3\theta(t - 2\pi) + 1)\sin(t)$$

17.6 problem 9

17.6.1 Existence and uniqueness analysis 1953
17.6.2 Maple step by step solution 1956

Internal problem ID [11529]
Internal file name [`OUTPUT/10511_Thursday_May_18_2023_04_21_38_AM_35632445/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.4 Impulsive sources. Exercises page 173
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' + y = \delta(t - 1)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

17.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= 1 \\ F &= \delta(t - 1) \end{aligned}$$

Hence the ode is

$$y'' + y' + y = \delta(t - 1)$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t-1)$ is

$$\{t < 1 \lor 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + sY(s) - y(0) + Y(s) = \mathrm{e}^{-s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + sY(s) + Y(s) = \mathrm{e}^{-s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-s}}{s^2 + s + 1}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+s+1}\right) \\
&= \frac{2\,\mathrm{Heaviside}(t-1)\sqrt{3}\,e^{-\frac{t}{2}+\frac{1}{2}}\sin\left(\frac{\sqrt{3}\,(t-1)}{2}\right)}{3}
\end{aligned}
$$

Hence the final solution is

$$
y = \frac{2\,\mathrm{Heaviside}(t-1)\sqrt{3}\,e^{-\frac{t}{2}+\frac{1}{2}}\sin\left(\frac{\sqrt{3}\,(t-1)}{2}\right)}{3}
$$

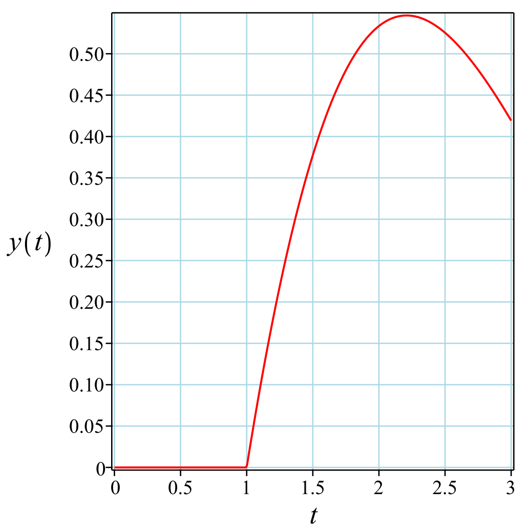
Simplifying the solution gives

$$
y = \frac{2\,\mathrm{Heaviside}(t-1)\sqrt{3}\,e^{-\frac{t}{2}+\frac{1}{2}}\sin\left(\frac{\sqrt{3}\,(t-1)}{2}\right)}{3}
$$

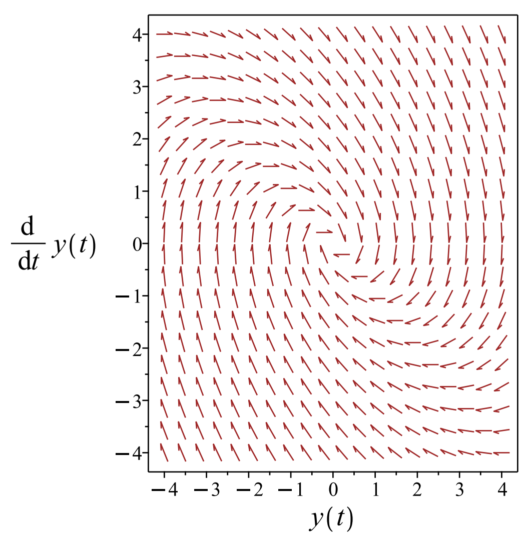
Summary

The solution(s) found are the following

$$
y = \frac{2\,\mathrm{Heaviside}(t-1)\sqrt{3}\,e^{-\frac{t}{2}+\frac{1}{2}}\sin\left(\frac{\sqrt{3}\,(t-1)}{2}\right)}{3} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2\,\mathrm{Heaviside}\left(t-1\right)\sqrt{3}\,\mathrm{e}^{-\frac{t}{2}+\frac{1}{2}}\sin\left(\frac{\sqrt{3}\,(t-1)}{2}\right)}{3}$$

Verified OK.

17.6.2 Maple step by step solution

Let's solve

$\left[y''+y'+y=\mathit{Dirac}(t-1)\,,y(0)=0,y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+r+1=0$

- Use quadratic formula to solve for r

  $r=\frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r=\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2},-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)+\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right),f(t)=\mathit{Dirac}(t-1)\right]$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right) & e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right) \\ -\frac{e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}}\sqrt{3}\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} & -\frac{e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} + \frac{e^{-\frac{t}{2}}\sqrt{3}\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(t), y_2(t)) = \frac{\sqrt{3}\,e^{-t}}{2}$$

- Substitute functions into equation for $y_p(t)$

$$y_p(t) = \frac{2\sqrt{3}\,e^{-\frac{t}{2}+\frac{1}{2}}\left(\int \mathit{Dirac}(t-1)dt\right)\left(\sin\left(\frac{\sqrt{3}\,t}{2}\right)\cos\left(\frac{\sqrt{3}}{2}\right)-\cos\left(\frac{\sqrt{3}\,t}{2}\right)\sin\left(\frac{\sqrt{3}}{2}\right)\right)}{3}$$

- Compute integrals

$$y_p(t) = \frac{2\sqrt{3}\,e^{-\frac{t}{2}+\frac{1}{2}}\,\mathit{Heaviside}(t-1)\left(\sin\left(\frac{\sqrt{3}\,t}{2}\right)\cos\left(\frac{\sqrt{3}}{2}\right)-\cos\left(\frac{\sqrt{3}\,t}{2}\right)\sin\left(\frac{\sqrt{3}}{2}\right)\right)}{3}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right) + e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2 + \frac{2\sqrt{3}\,e^{-\frac{t}{2}+\frac{1}{2}}\,\mathit{Heaviside}(t-1)\left(\sin\left(\frac{\sqrt{3}\,t}{2}\right)\cos\left(\frac{\sqrt{3}}{2}\right)-\cos\left(\frac{\sqrt{3}\,t}{2}\right)\sin\left(\frac{\sqrt{3}}{2}\right)\right)}{3}$$

☐ Check validity of solution $y = c_1 e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right) + e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2 + \frac{2\sqrt{3}\,e^{-\frac{t}{2}+\frac{1}{2}}\,\mathit{Heaviside}(t-1)\left(\sin\left(\frac{\sqrt{3}\right.\right.}{3}$

- Use initial condition $y(0) = 0$

$$0 = c_1$$

- Compute derivative of the solution

$$y' = -\frac{c_1 e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} - \frac{c_1 e^{-\frac{t}{2}}\sqrt{3}\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2}{2} + \frac{e^{-\frac{t}{2}}\sqrt{3}\cos\left(\frac{\sqrt{3}\,t}{2}\right)c_2}{2} - \frac{\sqrt{3}\,e^{-\frac{t}{2}+\frac{1}{2}}\,\mathit{Heaviside}(t-1)\left(\right.}{}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$$0 = -\frac{c_1}{2} + \frac{c_2\sqrt{3}}{2}$$

- Solve for c_1 and c_2

$$\{c_1 = 0, c_2 = 0\}$$

- Substitute constant values into general solution and simplify

$$y = \frac{2\sqrt{3}\,e^{-\frac{t}{2}+\frac{1}{2}}\,\mathit{Heaviside}(t-1)\left(\sin\left(\frac{\sqrt{3}\,t}{2}\right)\cos\left(\frac{\sqrt{3}}{2}\right)-\cos\left(\frac{\sqrt{3}\,t}{2}\right)\sin\left(\frac{\sqrt{3}}{2}\right)\right)}{3}$$

- Solution to the IVP

$$y = \frac{2\sqrt{3}\,e^{-\frac{t}{2}+\frac{1}{2}}\,\mathit{Heaviside}(t-1)\left(\sin\left(\frac{\sqrt{3}\,t}{2}\right)\cos\left(\frac{\sqrt{3}}{2}\right)-\cos\left(\frac{\sqrt{3}\,t}{2}\right)\sin\left(\frac{\sqrt{3}}{2}\right)\right)}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.844 (sec). Leaf size: 28

```
dsolve([diff(y(t),t$2)+diff(y(t),t)+y(t)=Dirac(t-1),y(0) = 0, D(y)(0) = 0],y(t), singsol=all
```

$$y(t) = \frac{2\sqrt{3}\,\mathrm{e}^{\frac{1}{2}-\frac{t}{2}}\,\mathrm{Heaviside}\,(t-1)\sin\left(\frac{\sqrt{3}\,(t-1)}{2}\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.124 (sec). Leaf size: 40

```
DSolve[{y''[t]+y'[t]+y[t]==DiracDelta[t-1],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutio
```

$$y(t) \to \frac{2e^{\frac{1}{2}-\frac{t}{2}}\theta(t-1)\sin\left(\frac{1}{2}\sqrt{3}(t-1)\right)}{\sqrt{3}}$$

17.7 problem 10

17.7.1 Existence and uniqueness analysis 1959
17.7.2 Maple step by step solution 1962

Internal problem ID [11530]
Internal file name [`OUTPUT/10512_Thursday_May_18_2023_04_21_40_AM_82357209/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 3, Laplace transform. Section 3.4 Impulsive sources. Exercises page 173
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + 4x = \frac{(t-5)\,\mathrm{Heaviside}\,(t-5)}{5} + \left(2 - \frac{t}{5}\right)\mathrm{Heaviside}\,(-10+t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

17.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned}
p(t) &= 0 \\
q(t) &= 4 \\
F &= \frac{(10-t)\,\mathrm{Heaviside}\,(-10+t)}{5} + \frac{(t-5)\,\mathrm{Heaviside}\,(t-5)}{5}
\end{aligned}$$

Hence the ode is

$$x'' + 4x = \frac{(10 - t)\,\mathrm{Heaviside}\,(-10 + t)}{5} + \frac{(t - 5)\,\mathrm{Heaviside}\,(t - 5)}{5}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \frac{(10-t)\,\mathrm{Heaviside}(-10+t)}{5} + \frac{(t-5)\,\mathrm{Heaviside}(t-5)}{5}$ is

$$\{5 \leq t \leq 10, 10 \leq t \leq \infty, -\infty \leq t \leq 5\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$
$$\mathcal{L}(x'') = s^2Y(s) - x'(0) - sx(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) + 4Y(s) = \frac{-\mathrm{e}^{-10s} + \mathrm{e}^{-5s}}{5s^2} \tag{1}$$

But the initial conditions are

$$x(0) = 0$$
$$x'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 4Y(s) = \frac{-\mathrm{e}^{-10s} + \mathrm{e}^{-5s}}{5s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{\mathrm{e}^{-10s} - \mathrm{e}^{-5s}}{5s^2\,(s^2 + 4)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
x &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{\mathrm{e}^{-10s}-\mathrm{e}^{-5s}}{5s^2\left(s^2+4\right)}\right) \\
&= -\frac{\text{Heaviside}\left(-10+t\right)\left(-20+2t-\sin\left(-20+2t\right)\right)}{40}+\frac{\text{Heaviside}\left(t-5\right)\left(2t-10-\sin\left(2t-10\right)\right)}{40}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
x = &-\frac{\text{Heaviside}\left(-10+t\right)\left(-20+2t-\sin\left(-20+2t\right)\right)}{40} \\
&+\frac{\text{Heaviside}\left(t-5\right)\left(2t-10-\sin\left(2t-10\right)\right)}{40}
\end{aligned}
$$

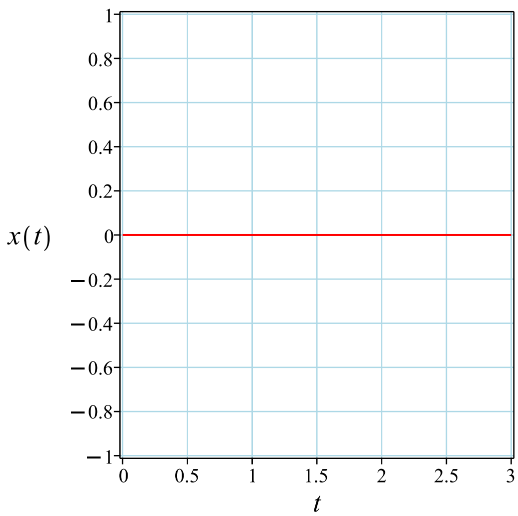
Simplifying the solution gives

$$
\begin{aligned}
x = &\frac{\text{Heaviside}\left(-10+t\right)\sin\left(-20+2t\right)}{40}-\frac{\text{Heaviside}\left(t-5\right)\sin\left(2t-10\right)}{40} \\
&+\frac{\left(20-2t\right)\text{Heaviside}\left(-10+t\right)}{40}+\frac{\left(t-5\right)\text{Heaviside}\left(t-5\right)}{20}
\end{aligned}
$$

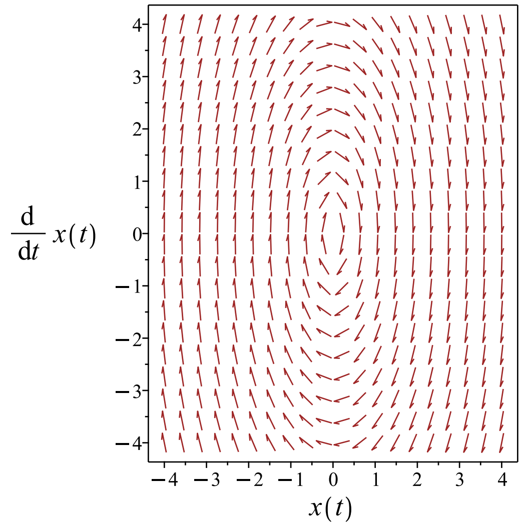
Summary

The solution(s) found are the following

$$
\begin{aligned}
x = &\frac{\text{Heaviside}\left(-10+t\right)\sin\left(-20+2t\right)}{40}-\frac{\text{Heaviside}\left(t-5\right)\sin\left(2t-10\right)}{40} \\
&+\frac{\left(20-2t\right)\text{Heaviside}\left(-10+t\right)}{40}+\frac{\left(t-5\right)\text{Heaviside}\left(t-5\right)}{20}
\end{aligned}
\tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{\text{Heaviside}\,(-10+t)\sin\,(-20+2t)}{40} - \frac{\text{Heaviside}\,(t-5)\sin\,(2t-10)}{40} + \frac{(20-2t)\,\text{Heaviside}\,(-10+t)}{40} + \frac{(t-5)\,\text{Heaviside}\,(t-5)}{20}$$

Verified OK.

17.7.2 Maple step by step solution

Let's solve

$$\left[x'' + 4x = \frac{(10-t)\mathit{Heaviside}(-10+t)}{5} + \frac{(t-5)\mathit{Heaviside}(t-5)}{5}, x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = \frac{\mathit{Heaviside}(t-5)t}{5} - \frac{\mathit{Heaviside}(-10+t)t}{5} - 4x - \mathit{Heaviside}(t-5) + 2\mathit{Heaviside}(-10+t)$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + 4x = \frac{\mathit{Heaviside}(t-5)t}{5} - \mathit{Heaviside}(t-5) + 2\mathit{Heaviside}(-10+t) - \frac{\mathit{Heaviside}(-10+t)t}{5}$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$
- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$
- 1st solution of the homogeneous ODE

  $x_1(t) = \cos(2t)$
- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(2t)$
- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$
- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(2t) + c_2 \sin(2t) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \frac{\mathit{Heaviside}(t-5)t}{5} - \mathit{Heaviside}(\right.$
- Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$
- Compute Wronskian

  $W(x_1(t), x_2(t)) = 2$
- Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{\cos(2t)\left(\int \sin(2t)((10-t)\mathit{Heaviside}(-10+t)+(t-5)\mathit{Heaviside}(t-5))dt\right)}{10} + \frac{\sin(2t)\left(\int \cos(2t)((10-t)\mathit{Heaviside}(-10+t)+\right.}{10}$
- Compute integrals

  $x_p(t) = \frac{(\cos(20)\sin(2t)-\sin(20)\cos(2t)-2t+20)\mathit{Heaviside}(-10+t)}{40} + \frac{\mathit{Heaviside}(t-5)\left(-\frac{\sin(2t)\cos(10)}{2}+\frac{\cos(2t)\sin(10)}{2}+t-5\right)}{20}$

- Substitute particular solution into general solution to ODE

  $x = c_1\cos(2t) + c_2\sin(2t) + \frac{(\cos(20)\sin(2t)-\sin(20)\cos(2t)-2t+20)\mathit{Heaviside}(-10+t)}{40} + \frac{\mathit{Heaviside}(t-5)\left(-\frac{\sin(2t)\cos}{2}\right.}{20}$

□ Check validity of solution $x = c_1\cos(2t) + c_2\sin(2t) + \frac{(\cos(20)\sin(2t)-\sin(20)\cos(2t)-2t+20)\mathit{Heaviside}(-10}{40}$

- Use initial condition $x(0) = 0$

$$0 = c_1$$

- Compute derivative of the solution

$$x' = -2c_1 \sin(2t) + 2c_2 \cos(2t) + \frac{(2\cos(20)\cos(2t) + 2\sin(20)\sin(2t) - 2)\, Heaviside(-10+t)}{40} + \frac{(\cos(20)\sin(2t) - \sin(20)}{}$$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

$$0 = 2c_2$$

- Solve for c_1 and c_2

$$\{c_1 = 0, c_2 = 0\}$$

- Substitute constant values into general solution and simplify

$$x = \frac{(\cos(20)\sin(2t) - \sin(20)\cos(2t) - 2t + 20)\, Heaviside(-10+t)}{40} + \frac{Heaviside(t-5)\left(-\frac{\sin(2t)\cos(10)}{2} + \frac{\cos(2t)\sin(10)}{2} + t - 5\right)}{20}$$

- Solution to the IVP

$$x = \frac{(\cos(20)\sin(2t) - \sin(20)\cos(2t) - 2t + 20)\, Heaviside(-10+t)}{40} + \frac{Heaviside(t-5)\left(-\frac{\sin(2t)\cos(10)}{2} + \frac{\cos(2t)\sin(10)}{2} + t - 5\right)}{20}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.594 (sec). Leaf size: 43

```
dsolve([diff(x(t),t$2)+4*x(t)=1/5*(t-5)*Heaviside(t-5)+(1-1/5*(t-5))*Heaviside(t-10),x(0) =
```

$$x(t) = \frac{\text{Heaviside}(t-10)\sin(2t-20)}{40} - \frac{\text{Heaviside}(t-5)\sin(2t-10)}{40} + \frac{(-2t+20)\,\text{Heaviside}(t-10)}{40} + \frac{(t-5)\,\text{Heaviside}(t-5)}{20}$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 55

```
DSolve[{x''[t]+4*x[t]==1/5*(t-5)*UnitStep[t-5]+(1-1/5*(t-5))*UnitStep[t-10],{x[0]==0,x'[0]==
```

$$x(t) \to \begin{cases} \frac{1}{40}(2(t-5)+\sin(10-2t)) & 5 < t \leq 10 \\ \frac{1}{40}(\sin(10-2t)-\sin(20-2t)+10) & t > 10 \end{cases}$$

18 Chapter 4, Linear Systems. Exercises page 190

18.1 problem 2(a) 1967
18.2 problem 2(b) 1976
18.3 problem 2(c) 1985
18.4 problem 2(d) 1994
18.5 problem 3(a) 2003
18.6 problem 3(b) 2012
18.7 problem 3(c) 2020
18.8 problem 3(d) 2029

18.1 problem 2(a)

18.1.1 Solution using Matrix exponential method 1967
18.1.2 Solution using explicit Eigenvalue and Eigenvector method . . . 1968
18.1.3 Maple step by step solution 1973

Internal problem ID [11531]
Internal file name [`OUTPUT/10513_Thursday_May_18_2023_04_21_41_AM_97803593/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 190
Problem number: 2(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= -3y(t) \\
y'(t) &= 2x
\end{aligned}
$$

18.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \cos\left(\sqrt{6}\,t\right) & -\frac{\sin\left(\sqrt{6}\,t\right)\sqrt{6}}{2} \\ \frac{\sin\left(\sqrt{6}\,t\right)\sqrt{6}}{3} & \cos\left(\sqrt{6}\,t\right) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos\left(\sqrt{6}\,t\right) & -\frac{\sin\left(\sqrt{6}\,t\right)\sqrt{6}}{2} \\ \frac{\sin\left(\sqrt{6}\,t\right)\sqrt{6}}{3} & \cos\left(\sqrt{6}\,t\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \cos\left(\sqrt{6}\,t\right) c_1 - \frac{\sin\left(\sqrt{6}\,t\right)\sqrt{6}\,c_2}{2} \\ \frac{\sin\left(\sqrt{6}\,t\right)\sqrt{6}\,c_1}{3} + \cos\left(\sqrt{6}\,t\right) c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -3 \\ 2 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i\sqrt{6}$$
$$\lambda_2 = -i\sqrt{6}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-i\sqrt{6}$	1	complex eigenvalue
$i\sqrt{6}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i\sqrt{6}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} - \left(-i\sqrt{6}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} i\sqrt{6} & -3 \\ 2 & i\sqrt{6} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} i\sqrt{6} & -3 & 0 \\ 2 & i\sqrt{6} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{i\sqrt{6}\,R_1}{3} \implies \left[\begin{array}{cc|c} i\sqrt{6} & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i\sqrt{6} & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{it\sqrt{6}}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{\mathrm{I}}{2}t\sqrt{6} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{it\sqrt{6}}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{\mathrm{I}}{2}t\sqrt{6} \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{i\sqrt{6}}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{\mathrm{I}}{2}t\sqrt{6} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{i\sqrt{6}}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{\mathrm{I}}{2}t\sqrt{6} \\ t \end{bmatrix} = \begin{bmatrix} -i\sqrt{6} \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i\sqrt{6}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} - \left(i\sqrt{6}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i\sqrt{6} & -3 \\ 2 & -i\sqrt{6} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -i\sqrt{6} & -3 & 0 \\ 2 & -i\sqrt{6} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{i\sqrt{6}\,R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} -i\sqrt{6} & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i\sqrt{6} & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{it\sqrt{6}}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{\mathrm{I}}{2}t\sqrt{6} \\ t \end{bmatrix} = \begin{bmatrix} \frac{it\sqrt{6}}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{\mathrm{I}}{2}t\sqrt{6} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{i\sqrt{6}}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{\mathrm{I}}{2}t\sqrt{6} \\ t \end{bmatrix} = \begin{bmatrix} \frac{i\sqrt{6}}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{\mathrm{I}}{2}t\sqrt{6} \\ t \end{bmatrix} = \begin{bmatrix} i\sqrt{6} \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$i\sqrt{6}$	1	1	No	$\begin{bmatrix} \frac{i\sqrt{6}}{2} \\ 1 \end{bmatrix}$
$-i\sqrt{6}$	1	1	No	$\begin{bmatrix} -\frac{i\sqrt{6}}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{i\mathrm{e}^{i\sqrt{6}\,t}\sqrt{6}}{2} \\ \mathrm{e}^{i\sqrt{6}\,t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{i\mathrm{e}^{-i\sqrt{6}\,t}\sqrt{6}}{2} \\ \mathrm{e}^{-i\sqrt{6}\,t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{i\sqrt{6}\left(c_2\mathrm{e}^{-i\sqrt{6}\,t} - c_1\mathrm{e}^{i\sqrt{6}\,t}\right)}{2} \\ c_1\mathrm{e}^{i\sqrt{6}\,t} + c_2\mathrm{e}^{-i\sqrt{6}\,t} \end{bmatrix}$$

The following is the phase plot of the system.

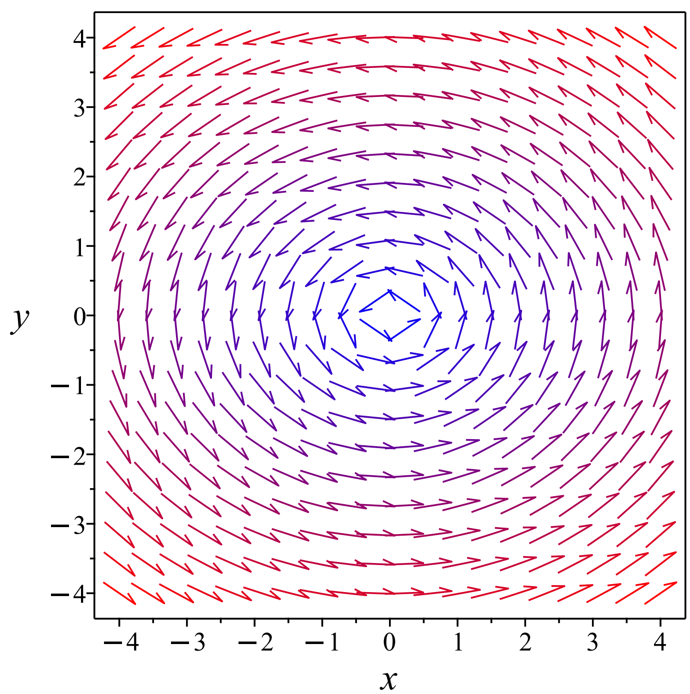


Figure 353: Phase plot

18.1.3 Maple step by step solution

Let's solve

$[x' = -3y(t), y'(t) = 2x]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}\sqrt{6}, \begin{bmatrix} -\frac{\mathrm{I}}{2}\sqrt{6} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{6}, \begin{bmatrix} \frac{\mathrm{I}}{2}\sqrt{6} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}\sqrt{6}, \begin{bmatrix} -\frac{\mathrm{I}}{2}\sqrt{6} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}\sqrt{6}\,t} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{2}\sqrt{6} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\left(\cos\left(\sqrt{6}\, t\right)-\mathrm{I} \sin\left(\sqrt{6}\, t\right)\right)\cdot\left[\begin{array}{c}-\frac{\mathrm{I}}{2}\sqrt{6} \\ 1\end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c}-\frac{\mathrm{I}}{2}\left(\cos\left(\sqrt{6}\, t\right)-\mathrm{I} \sin\left(\sqrt{6}\, t\right)\right)\sqrt{6} \\ \cos\left(\sqrt{6}\, t\right)-\mathrm{I} \sin\left(\sqrt{6}\, t\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t)=\left[\begin{array}{c}-\frac{\sin\left(\sqrt{6}\, t\right)\sqrt{6}}{2} \\ \cos\left(\sqrt{6}\, t\right)\end{array}\right], \vec{x}_2(t)=\left[\begin{array}{c}-\frac{\sqrt{6}\, \cos\left(\sqrt{6}\, t\right)}{2} \\ -\sin\left(\sqrt{6}\, t\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{x}=c_1 \vec{x}_1(t)+c_2 \vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x}=\left[\begin{array}{c}-\frac{c_2\sqrt{6}\, \cos\left(\sqrt{6}\, t\right)}{2}-\frac{c_1 \sin\left(\sqrt{6}\, t\right)\sqrt{6}}{2} \\ -c_2 \sin\left(\sqrt{6}\, t\right)+c_1 \cos\left(\sqrt{6}\, t\right)\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x \\ y(t)\end{array}\right]=\left[\begin{array}{c}-\frac{\sqrt{6}\left(c_1 \sin\left(\sqrt{6}\, t\right)+c_2 \cos\left(\sqrt{6}\, t\right)\right)}{2} \\ -c_2 \sin\left(\sqrt{6}\, t\right)+c_1 \cos\left(\sqrt{6}\, t\right)\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x=-\frac{\sqrt{6}\left(c_1 \sin\left(\sqrt{6}\, t\right)+c_2 \cos\left(\sqrt{6}\, t\right)\right)}{2}, y(t)=-c_2 \sin\left(\sqrt{6}\, t\right)+c_1 \cos\left(\sqrt{6}\, t\right)\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 48

```
dsolve([diff(x(t),t)=-3*y(t),diff(y(t),t)=2*x(t)],singsol=all)
```

$$x(t)=c_1 \sin\left(\sqrt{6}\, t\right)+c_2 \cos\left(\sqrt{6}\, t\right)$$
$$y(t)=-\frac{\sqrt{6}\left(\cos\left(\sqrt{6}\, t\right)c_1-\sin\left(\sqrt{6}\, t\right)c_2\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 69

```
DSolve[{x'[t]==-3*y[t],y'[t]==2*x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 \cos\left(\sqrt{6}t\right) - \sqrt{\frac{3}{2}}c_2 \sin\left(\sqrt{6}t\right)$$
$$y(t) \to c_2 \cos\left(\sqrt{6}t\right) + \sqrt{\frac{2}{3}}c_1 \sin\left(\sqrt{6}t\right)$$

18.2 problem 2(b)

18.2.1 Solution using Matrix exponential method 1976
18.2.2 Solution using explicit Eigenvalue and Eigenvector method . . . 1977
18.2.3 Maple step by step solution 1982

Internal problem ID [11532]
Internal file name [`OUTPUT/10514_Thursday_May_18_2023_04_21_44_AM_81332127/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 190
Problem number: 2(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= -2y(t) \\
y'(t) &= -4x
\end{aligned}
$$

18.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-2\sqrt{2}\,t}}{2} + \frac{e^{2\sqrt{2}\,t}}{2} & \frac{\left(-e^{2\sqrt{2}\,t}+e^{-2\sqrt{2}\,t}\right)\sqrt{2}}{4} \\ \frac{\left(-e^{2\sqrt{2}\,t}+e^{-2\sqrt{2}\,t}\right)\sqrt{2}}{2} & \frac{e^{-2\sqrt{2}\,t}}{2} + \frac{e^{2\sqrt{2}\,t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-2\sqrt{2}\,t}}{2}+\frac{e^{2\sqrt{2}\,t}}{2} & \frac{\left(-e^{2\sqrt{2}\,t}+e^{-2\sqrt{2}\,t}\right)\sqrt{2}}{4} \\ \frac{\left(-e^{2\sqrt{2}\,t}+e^{-2\sqrt{2}\,t}\right)\sqrt{2}}{2} & \frac{e^{-2\sqrt{2}\,t}}{2}+\frac{e^{2\sqrt{2}\,t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-2\sqrt{2}\,t}}{2}+\frac{e^{2\sqrt{2}\,t}}{2}\right)c_1 + \frac{\left(-e^{2\sqrt{2}\,t}+e^{-2\sqrt{2}\,t}\right)\sqrt{2}\,c_2}{4} \\ \frac{\left(-e^{2\sqrt{2}\,t}+e^{-2\sqrt{2}\,t}\right)\sqrt{2}\,c_1}{2} + \left(\frac{e^{-2\sqrt{2}\,t}}{2}+\frac{e^{2\sqrt{2}\,t}}{2}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\left(c_2\sqrt{2}+2c_1\right)e^{-2\sqrt{2}\,t}}{4} + \frac{e^{2\sqrt{2}\,t}\left(-\frac{c_2\sqrt{2}}{2}+c_1\right)}{2} \\ \frac{\left(c_1\sqrt{2}+c_2\right)e^{-2\sqrt{2}\,t}}{2} - \frac{e^{2\sqrt{2}\,t}\left(c_1\sqrt{2}-c_2\right)}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A-\lambda I\right)=0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 0 & -2 \\ -4 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -\lambda & -2 \\ -4 & -\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2\sqrt{2}$$
$$\lambda_2 = -2\sqrt{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-2\sqrt{2}$	1	real eigenvalue
$2\sqrt{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -2 \\ -4 & 0 \end{bmatrix} - \left(-2\sqrt{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2\sqrt{2} & -2 \\ -4 & 2\sqrt{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2\sqrt{2} & -2 & 0 \\ -4 & 2\sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \sqrt{2}\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 2\sqrt{2} & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2\sqrt{2} & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t\sqrt{2}}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t\sqrt{2}}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t\sqrt{2}}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t\sqrt{2}}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t\sqrt{2}}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t\sqrt{2}}{2} \\ t \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -2 \\ -4 & 0 \end{bmatrix} - \left(2\sqrt{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2\sqrt{2} & -2 \\ -4 & -2\sqrt{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2\sqrt{2} & -2 & 0 \\ -4 & -2\sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \sqrt{2}\,R_1 \implies \left[\begin{array}{cc|c} -2\sqrt{2} & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2\sqrt{2} & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t\sqrt{2}}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t\sqrt{2}}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t\sqrt{2}}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t\sqrt{2}}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t\sqrt{2}}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t\sqrt{2}}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
$2\sqrt{2}$	1	1	No	$\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$
$-2\sqrt{2}$	1	1	No	$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $2\sqrt{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{2\sqrt{2}\,t} \\ &= \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix} e^{2\sqrt{2}\,t}\end{aligned}$$

Since eigenvalue $-2\sqrt{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-2\sqrt{2}\,t} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix} e^{-2\sqrt{2}\,t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{\sqrt{2}\,\mathrm{e}^{2\sqrt{2}\,t}}{2} \\ \mathrm{e}^{2\sqrt{2}\,t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{\sqrt{2}\,\mathrm{e}^{-2\sqrt{2}\,t}}{2} \\ \mathrm{e}^{-2\sqrt{2}\,t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}\left(c_2\mathrm{e}^{-2\sqrt{2}\,t} - c_1\mathrm{e}^{2\sqrt{2}\,t}\right)}{2} \\ c_1\mathrm{e}^{2\sqrt{2}\,t} + c_2\mathrm{e}^{-2\sqrt{2}\,t} \end{bmatrix}$$

The following is the phase plot of the system.

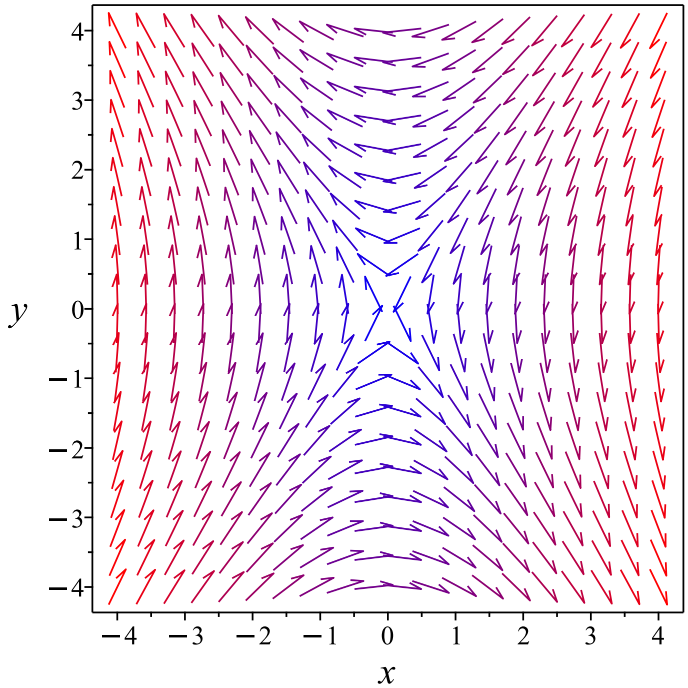


Figure 354: Phase plot

18.2.3 Maple step by step solution

Let's solve

$[x' = -2y(t), y'(t) = -4x]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & -2 \\ -4 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & -2 \\ -4 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 0 & -2 \\ -4 & 0 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2\sqrt{2}, \left[\begin{array}{c} \frac{\sqrt{2}}{2} \\ 1 \end{array}\right]\right], \left[2\sqrt{2}, \left[\begin{array}{c} -\frac{\sqrt{2}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2\sqrt{2}, \left[\begin{array}{c} \frac{\sqrt{2}}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{-2\sqrt{2}\,t} \cdot \left[\begin{array}{c} \frac{\sqrt{2}}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[2\sqrt{2}, \left[\begin{array}{c} -\frac{\sqrt{2}}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{2\sqrt{2}\,t} \cdot \left[\begin{array}{c} -\frac{\sqrt{2}}{2} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1\mathrm{e}^{-2\sqrt{2}\,t} \cdot \left[\begin{array}{c} \frac{\sqrt{2}}{2} \\ 1 \end{array}\right] + c_2\mathrm{e}^{2\sqrt{2}\,t} \cdot \left[\begin{array}{c} -\frac{\sqrt{2}}{2} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}\left(c_1 e^{-2\sqrt{2}\,t} - c_2 e^{2\sqrt{2}\,t}\right)}{2} \\ c_1 e^{-2\sqrt{2}\,t} + c_2 e^{2\sqrt{2}\,t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x = \frac{\sqrt{2}\left(c_1 e^{-2\sqrt{2}\,t} - c_2 e^{2\sqrt{2}\,t}\right)}{2}, y(t) = c_1 e^{-2\sqrt{2}\,t} + c_2 e^{2\sqrt{2}\,t}\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 52

```
dsolve([diff(x(t),t)=-2*y(t),diff(y(t),t)=-4*x(t)],singsol=all)
```

$$x(t) = c_1 e^{2\sqrt{2}\,t} + c_2 e^{-2\sqrt{2}\,t}$$
$$y(t) = -\sqrt{2}\left(c_1 e^{2\sqrt{2}\,t} - c_2 e^{-2\sqrt{2}\,t}\right)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 111

```
DSolve[{x'[t]==-2*y[t],y'[t]==-4*x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{4} e^{-2\sqrt{2}t}\left(2c_1\left(e^{4\sqrt{2}t}+1\right) - \sqrt{2}c_2\left(e^{4\sqrt{2}t}-1\right)\right)$$
$$y(t) \to \frac{1}{2} e^{-2\sqrt{2}t}\left(c_2\left(e^{4\sqrt{2}t}+1\right) - \sqrt{2}c_1\left(e^{4\sqrt{2}t}-1\right)\right)$$

18.3 problem 2(c)

18.3.1 Solution using Matrix exponential method 1985
18.3.2 Solution using explicit Eigenvalue and Eigenvector method . . . 1986
18.3.3 Maple step by step solution 1991

Internal problem ID [11533]
Internal file name [`OUTPUT/10515_Thursday_May_18_2023_04_21_45_AM_47094322/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 190
Problem number: 2(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x' = -3x$$
$$y'(t) = 2y(t)$$

18.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-3t} & 0 \\ 0 & \mathrm{e}^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-3t}c_1 \\ e^{2t}c_2 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-3-\lambda)(2-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 5 & 0 \end{array}\right]$$

Since the current pivot $A(1, 2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 0 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -5 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t} \end{aligned}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-3t} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-3t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-3t} \\ 0 \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2 \mathrm{e}^{-3t} \\ c_1 \mathrm{e}^{2t} \end{bmatrix}
$$

The following is the phase plot of the system.

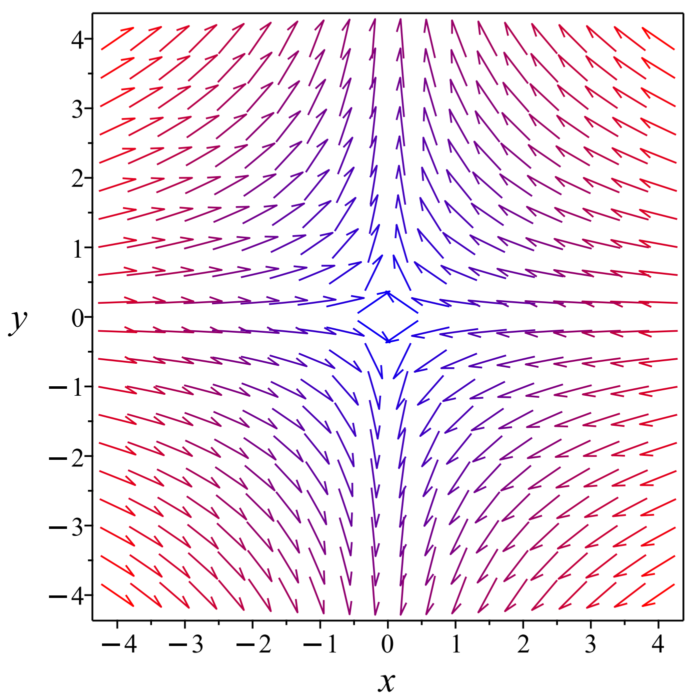


Figure 355: Phase plot

18.3.3 Maple step by step solution

Let's solve

$[x' = -3x, y'(t) = 2y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-3t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{2t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = e^{-3t} c_1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{-3t} c_1 \\ c_2 e^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x = e^{-3t} c_1, y(t) = c_2 e^{2t}\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve([diff(x(t),t)=-3*x(t),diff(y(t),t)=2*y(t)],singsol=all)
```

$$x(t) = c_2 e^{-3t}$$
$$y(t) = c_1 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 65

```
DSolve[{x'[t]==-3*x[t],y'[t]==3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
x(t) &\to c_1 e^{-3t} \\
y(t) &\to c_2 e^{3t} \\
x(t) &\to c_1 e^{-3t} \\
y(t) &\to 0 \\
x(t) &\to 0 \\
y(t) &\to c_2 e^{3t} \\
x(t) &\to 0 \\
y(t) &\to 0
\end{aligned}
$$

18.4 problem 2(d)

18.4.1 Solution using Matrix exponential method 1994
18.4.2 Solution using explicit Eigenvalue and Eigenvector method . . . 1995
18.4.3 Maple step by step solution 2000

Internal problem ID [11534]
Internal file name [`OUTPUT/10516_Thursday_May_18_2023_04_21_47_AM_60535053/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 190
Problem number: 2(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= 4y(t) \\
y'(t) &= 2y(t)
\end{aligned}
$$

18.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1 & 2\,\mathrm{e}^{2t} - 2 \\ 0 & \mathrm{e}^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 1 & 2\,\mathrm{e}^{2t}-2 \\ 0 & \mathrm{e}^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} c_1 + \left(2\,\mathrm{e}^{2t}-2\right) c_2 \\ \mathrm{e}^{2t} c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -\lambda & 4 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$
(-\lambda)(2-\lambda) = 0
$$

The roots of the above are the eigenvalues.

$$
\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= 2
\end{aligned}
$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$
\begin{aligned}
\left(\begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$
\left[\begin{array}{cc|c} 0 & 4 & 0 \\ 0 & 2 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{cc|c} 0 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	1	1	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{0} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{0} \end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2\,\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2\mathrm{e}^{2t} \\ c_2\mathrm{e}^{2t} \end{bmatrix}$$

The following is the phase plot of the system.

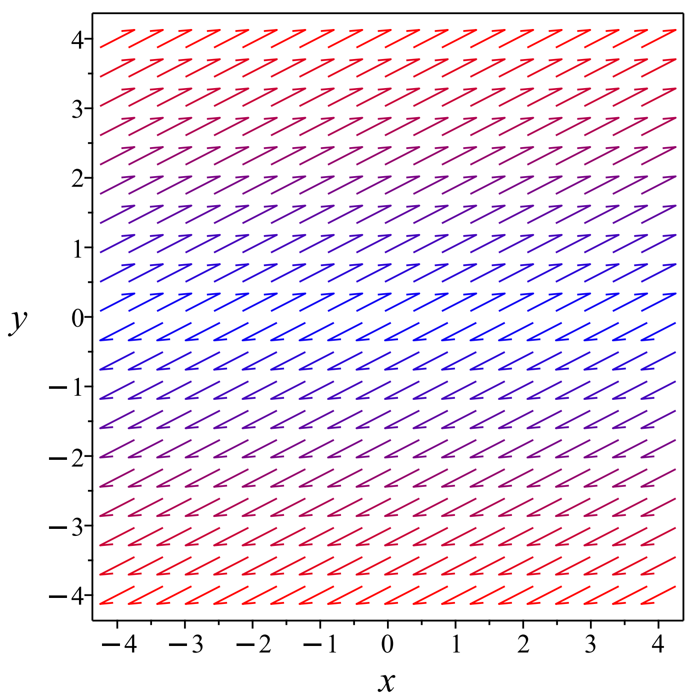


Figure 356: Phase plot

18.4.3 Maple step by step solution

Let's solve

$[x' = 4y(t), y'(t) = 2y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{2t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_2 e^{2t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 e^{2t} \\ c_2 e^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x = c_1 + 2c_2 e^{2t}, y(t) = c_2 e^{2t}\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([diff(x(t),t)=4*y(t),diff(y(t),t)=2*y(t)],singsol=all)
```

$$x(t) = 2c_2 e^{2t} + c_1$$
$$y(t) = c_2 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 65

```
DSolve[{x'[t]==4*x[t],y'[t]==2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
x(t) &\to c_1 e^{4t} \\
y(t) &\to c_2 e^{2t} \\
x(t) &\to c_1 e^{4t} \\
y(t) &\to 0 \\
x(t) &\to 0 \\
y(t) &\to c_2 e^{2t} \\
x(t) &\to 0 \\
y(t) &\to 0
\end{aligned}
$$

18.5 problem 3(a)

18.5.1 Solution using Matrix exponential method 2003
18.5.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2004
18.5.3 Maple step by step solution 2009

Internal problem ID [11535]
Internal file name [`OUTPUT/10517_Thursday_May_18_2023_04_21_48_AM_50486559/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 190
Problem number: 3(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x' = x$$
$$y'(t) = x + 2y(t)$$

18.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{t} & 0 \\ \mathrm{e}^{2t} - \mathrm{e}^{t} & \mathrm{e}^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{t} & 0 \\ \mathrm{e}^{2t}-\mathrm{e}^{t} & \mathrm{e}^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t} c_1 \\ (\mathrm{e}^{2t}-\mathrm{e}^{t})\, c_1 + \mathrm{e}^{2t} c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t} c_1 \\ (c_1+c_2)\, \mathrm{e}^{2t} - \mathrm{e}^{t} c_1 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix}\right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(1-\lambda)(2-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \implies \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t\end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^t \\ \mathrm{e}^t \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^t \\ c_1 \mathrm{e}^t + c_2 \mathrm{e}^{2t} \end{bmatrix}$$

The following is the phase plot of the system.

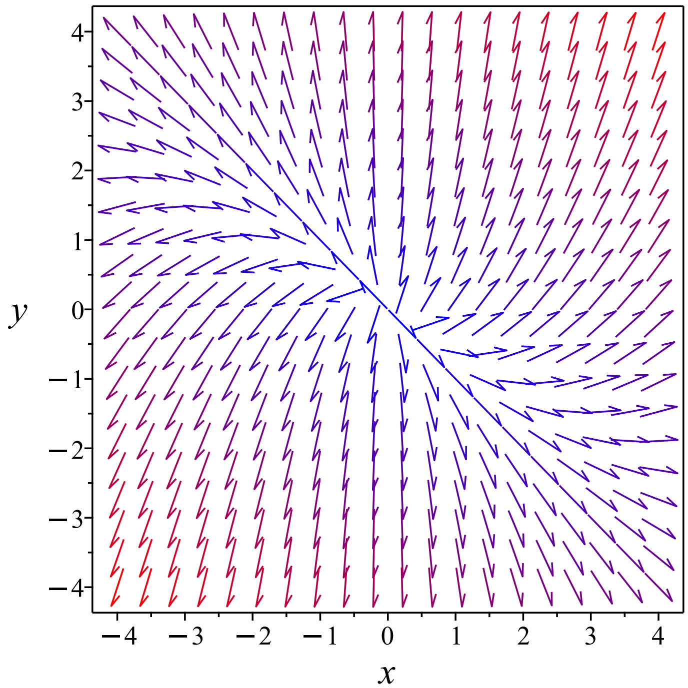


Figure 357: Phase plot

18.5.3 Maple step by step solution

Let's solve

$[x' = x, y'(t) = x + 2y(t)]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$

- Convert system into a vector equation

$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- System to solve

$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \cdot \vec{x}(t)$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

  $$\left[\left[1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = \mathrm{e}^{t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[2, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = \mathrm{e}^{2t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = c_1 \mathrm{e}^{t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{2t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^t \\ c_1 \mathrm{e}^t + c_2 \mathrm{e}^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x = -c_1 \mathrm{e}^t, y(t) = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{2t}\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve([diff(x(t),t)=x(t),diff(y(t),t)=x(t)+2*y(t)],singsol=all)
```

$$x(t) = c_2 \mathrm{e}^t$$
$$y(t) = -c_2 \mathrm{e}^t + c_1 \mathrm{e}^{2t}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 33

```
DSolve[{x'[t]==x[t],y'[t]==x[t]+2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^t$$
$$y(t) \to e^t (c_1 (e^t - 1) + c_2 e^t)$$

18.6 problem 3(b)

18.6.1 Solution using Matrix exponential method 2012
18.6.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2013
18.6.3 Maple step by step solution 2017

Internal problem ID [11536]
Internal file name [`OUTPUT/10518_Thursday_May_18_2023_04_21_50_AM_75493388/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 190
Problem number: 3(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x' &= x - y(t) \\ y'(t) &= x + y(t) \end{aligned}$$

18.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{t} \cos(t) & -\mathrm{e}^{t} \sin(t) \\ \mathrm{e}^{t} \sin(t) & \mathrm{e}^{t} \cos(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t \cos(t) & -e^t \sin(t) \\ e^t \sin(t) & e^t \cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t \cos(t) c_1 - e^t \sin(t) c_2 \\ e^t \sin(t) c_1 + e^t \cos(t) c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t(\cos(t) c_1 - \sin(t) c_2) \\ e^t(\sin(t) c_1 + \cos(t) c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + i$$
$$\lambda_2 = 1 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1+i$	1	complex eigenvalue
$1-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - (1-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - (1+i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$1 + i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$
$1 - i$	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} i\mathrm{e}^{(1+i)t} \\ \mathrm{e}^{(1+i)t} \end{bmatrix} + c_2 \begin{bmatrix} -i\mathrm{e}^{(1-i)t} \\ \mathrm{e}^{(1-i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -i\left(c_2\mathrm{e}^{(1-i)t} - c_1\mathrm{e}^{(1+i)t}\right) \\ c_1\mathrm{e}^{(1+i)t} + c_2\mathrm{e}^{(1-i)t} \end{bmatrix}$$

The following is the phase plot of the system.

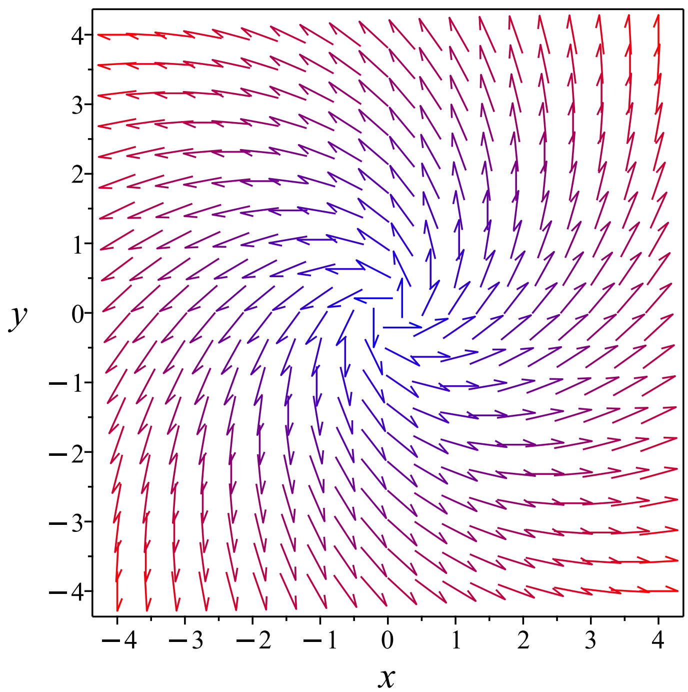


Figure 358: Phase plot

18.6.3 Maple step by step solution

Let's solve

$[x' = x - y(t), y'(t) = x + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1 - \mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[1 + \mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1 - \mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(1-\mathrm{I})t} \cdot \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{t} \cdot (\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^t \cdot \begin{bmatrix} -I(\cos(t) - I\sin(t)) \\ \cos(t) - I\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = e^t \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}, \vec{x}_2(t) = e^t \cdot \begin{bmatrix} -\cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^t \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + c_2 e^t \cdot \begin{bmatrix} -\cos(t) \\ -\sin(t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -e^t(c_1\sin(t) + c_2\cos(t)) \\ e^t(c_1\cos(t) - c_2\sin(t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x = -e^t(c_1\sin(t) + c_2\cos(t)), y(t) = e^t(c_1\cos(t) - c_2\sin(t))\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve([diff(x(t),t)=x(t)-y(t),diff(y(t),t)=x(t)+y(t)],singsol=all)
```

$$x(t) = e^t(c_1\sin(t) + c_2\cos(t))$$
$$y(t) = -e^t(c_1\cos(t) - c_2\sin(t))$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 39

```
DSolve[{x'[t]==x[t]-y[t],y'[t]==x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^t(c_1\cos(t) - c_2\sin(t))$$
$$y(t) \to e^t(c_2\cos(t) + c_1\sin(t))$$

18.7 problem 3(c)

18.7.1 Solution using Matrix exponential method 2020
18.7.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2021
18.7.3 Maple step by step solution 2026

Internal problem ID [11537]
Internal file name [`OUTPUT/10519_Thursday_May_18_2023_04_21_52_AM_74856101/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 190
Problem number: 3(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= x + 2y(t) \\
y'(t) &= x
\end{aligned}
$$

18.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{e^{-t}}{3} + \frac{2\,e^{2t}}{3} & \frac{2\,e^{2t}}{3} - \frac{2\,e^{-t}}{3} \\ \frac{e^{2t}}{3} - \frac{e^{-t}}{3} & \frac{2\,e^{-t}}{3} + \frac{e^{2t}}{3} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-t}}{3}+\frac{2\,e^{2t}}{3} & \frac{2\,e^{2t}}{3}-\frac{2\,e^{-t}}{3} \\ \frac{e^{2t}}{3}-\frac{e^{-t}}{3} & \frac{2\,e^{-t}}{3}+\frac{e^{2t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-t}}{3}+\frac{2\,e^{2t}}{3}\right)c_1+\left(\frac{2\,e^{2t}}{3}-\frac{2\,e^{-t}}{3}\right)c_2 \\ \left(\frac{e^{2t}}{3}-\frac{e^{-t}}{3}\right)c_1+\left(\frac{2\,e^{-t}}{3}+\frac{e^{2t}}{3}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-2c_2)e^{-t}}{3}+\frac{2(c_1+c_2)e^{2t}}{3} \\ \frac{(-c_1+2c_2)e^{-t}}{3}+\frac{(c_1+c_2)e^{2t}}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A-\lambda I\right)=0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - \lambda - 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{2t} \\ = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-t} \\ = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} 2c_1\mathrm{e}^{2t} - c_2\mathrm{e}^{-t} \\ c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

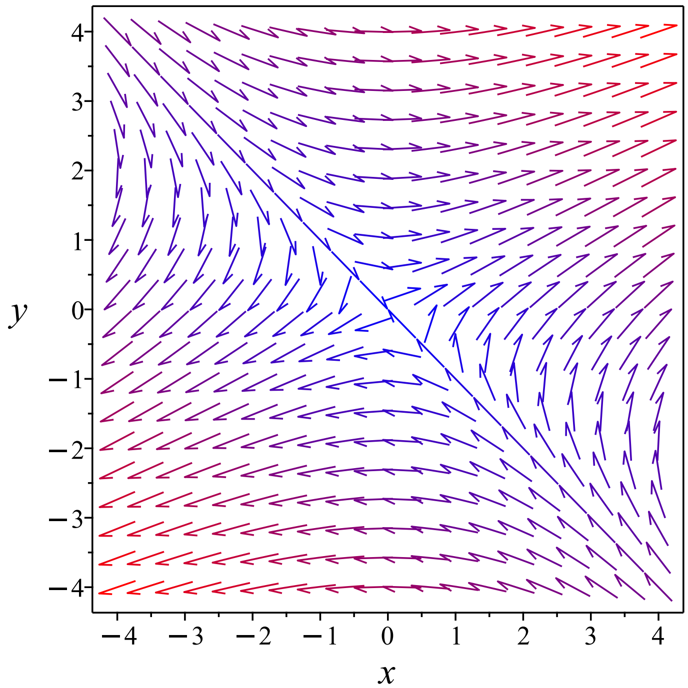


Figure 359: Phase plot

18.7.3 Maple step by step solution

Let's solve

$[x' = x + 2y(t)\,, y'(t) = x]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[2, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_2 = \mathrm{e}^{2t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

- Substitute solutions into the general solution

  $$\vec{x} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{2t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{-t} + 2c_2 \mathrm{e}^{2t} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x = -c_1 \mathrm{e}^{-t} + 2c_2 \mathrm{e}^{2t}, y(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{2t}\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=x(t)+2*y(t),diff(y(t),t)=x(t)],singsol=all)
```

$$x(t) = -\mathrm{e}^{-t} c_1 + 2c_2 \mathrm{e}^{2t}$$
$$y(t) = \mathrm{e}^{-t} c_1 + c_2 \mathrm{e}^{2t}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 71

```
DSolve[{x'[t]==x[t]+2*y[t],y'[t]==x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{3} e^{-t} \left(c_1 \left(2e^{3t} + 1\right) + 2c_2 \left(e^{3t} - 1\right)\right)$$
$$y(t) \to \frac{1}{3} e^{-t} \left(c_1 \left(e^{3t} - 1\right) + c_2 \left(e^{3t} + 2\right)\right)$$

18.8 problem 3(d)

18.8.1 Solution using Matrix exponential method 2029
18.8.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2030
18.8.3 Maple step by step solution 2034

Internal problem ID [11538]
Internal file name [`OUTPUT/10520_Thursday_May_18_2023_04_21_54_AM_13052924/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 190
Problem number: 3(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= -x - 2y(t) \\
y'(t) &= 2x - y(t)
\end{aligned}
$$

18.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{-t}\cos(2t) & -\mathrm{e}^{-t}\sin(2t) \\ \mathrm{e}^{-t}\sin(2t) & \mathrm{e}^{-t}\cos(2t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}\cos(2t) & -e^{-t}\sin(2t) \\ e^{-t}\sin(2t) & e^{-t}\cos(2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}\cos(2t)\, c_1 - e^{-t}\sin(2t)\, c_2 \\ e^{-t}\sin(2t)\, c_1 + e^{-t}\cos(2t)\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(\cos(2t)\, c_1 - \sin(2t)\, c_2) \\ e^{-t}(\sin(2t)\, c_1 + \cos(2t)\, c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -1-\lambda & -2 \\ 2 & -1-\lambda \end{bmatrix} \right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1 + 2i$$
$$\lambda_2 = -1 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1 - 2i$	1	complex eigenvalue
$-1 + 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix} - (-1 - 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2i & -2 & 0 \\ 2 & 2i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} 2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix} - (-1 + 2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2i & -2 & 0 \\ 2 & -2i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-1+2i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$
$-1-2i$	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} i\mathrm{e}^{(-1+2i)t} \\ \mathrm{e}^{(-1+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} -i\mathrm{e}^{(-1-2i)t} \\ \mathrm{e}^{(-1-2i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -i\left(c_2\mathrm{e}^{(-1-2i)t} - c_1\mathrm{e}^{(-1+2i)t}\right) \\ c_1\mathrm{e}^{(-1+2i)t} + c_2\mathrm{e}^{(-1-2i)t} \end{bmatrix}$$

The following is the phase plot of the system.

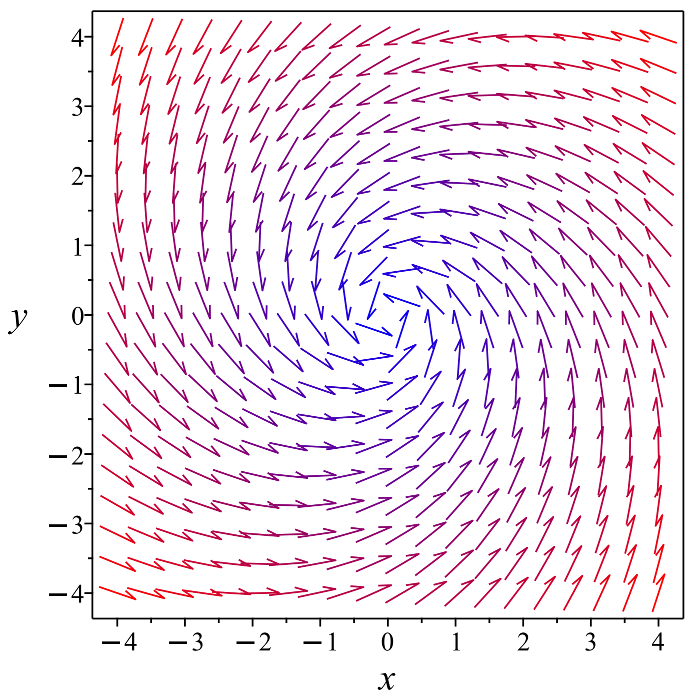


Figure 360: Phase plot

18.8.3 Maple step by step solution

Let's solve

$[x' = -x - 2y(t)\,, y'(t) = 2x - y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1-2\,\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[-1+2\,\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-2\,\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-2\,\mathrm{I})t} \cdot \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-t} \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-t} \cdot \begin{bmatrix} -\mathrm{I}(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = e^{-t} \cdot \begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix}, \vec{x}_2(t) = e^{-t} \cdot \begin{bmatrix} -\cos(2t) \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-t} \cdot \begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 e^{-t} \cdot \begin{bmatrix} -\cos(2t) \\ -\sin(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -e^{-t}(c_2\cos(2t) + c_1\sin(2t)) \\ e^{-t}(c_1\cos(2t) - c_2\sin(2t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x = -e^{-t}(c_2\cos(2t) + c_1\sin(2t)), y(t) = e^{-t}(c_1\cos(2t) - c_2\sin(2t))\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 46

```
dsolve([diff(x(t),t)=-x(t)-2*y(t),diff(y(t),t)=2*x(t)-y(t)],singsol=all)
```

$$x(t) = e^{-t}(c_1\sin(2t) + c_2\cos(2t))$$
$$y(t) = -e^{-t}(c_1\cos(2t) - c_2\sin(2t))$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 51

```
DSolve[{x'[t]==-x[t]-2*y[t],y'[t]==2*x[t]-y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tr
```

$$x(t) \to e^{-t}(c_1\cos(2t) - c_2\sin(2t))$$
$$y(t) \to e^{-t}(c_2\cos(2t) + c_1\sin(2t))$$

19 Chapter 4, Linear Systems. Exercises page 202

19.1 problem 1(a) 2038
19.2 problem 1(b) 2047
19.3 problem 1(c) 2056
19.4 problem 1(d) 2065
19.5 problem 1(e) 2074
19.6 problem 1(f) 2083
19.7 problem 3(a) 2091
19.8 problem 3(b) 2104

19.1 problem 1(a)

19.1.1 Solution using Matrix exponential method 2038
19.1.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2039
19.1.3 Maple step by step solution 2044

Internal problem ID [11539]
Internal file name [`OUTPUT/10521_Thursday_May_18_2023_04_21_55_AM_38449395/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 202
Problem number: 1(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= -2x - 3y(t) \\
y'(t) &= -x + 4y(t)
\end{aligned}
$$

19.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(2+\sqrt{3})\,\mathrm{e}^{-2\sqrt{3}\,t+t}}{4} - \frac{\mathrm{e}^{(1+2\sqrt{3})t}(\sqrt{3}-2)}{4} & -\frac{\left(\mathrm{e}^{(1+2\sqrt{3})t}-\mathrm{e}^{-2\sqrt{3}\,t+t}\right)\sqrt{3}}{4} \\ -\frac{\left(\mathrm{e}^{(1+2\sqrt{3})t}-\mathrm{e}^{-2\sqrt{3}\,t+t}\right)\sqrt{3}}{12} & \frac{(2-\sqrt{3})\,\mathrm{e}^{-2\sqrt{3}\,t+t}}{4} + \frac{(2+\sqrt{3})\,\mathrm{e}^{(1+2\sqrt{3})t}}{4} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{\left(2+\sqrt{3}\right)\mathrm{e}^{-2\sqrt{3}\,t+t}}{4} - \frac{\mathrm{e}^{\left(1+2\sqrt{3}\right)t}\left(\sqrt{3}-2\right)}{4} & -\frac{\left(\mathrm{e}^{\left(1+2\sqrt{3}\right)t}-\mathrm{e}^{-2\sqrt{3}\,t+t}\right)\sqrt{3}}{4} \\ -\frac{\left(\mathrm{e}^{\left(1+2\sqrt{3}\right)t}-\mathrm{e}^{-2\sqrt{3}\,t+t}\right)\sqrt{3}}{12} & \frac{\left(2-\sqrt{3}\right)\mathrm{e}^{-2\sqrt{3}\,t+t}}{4} + \frac{\left(2+\sqrt{3}\right)\mathrm{e}^{\left(1+2\sqrt{3}\right)t}}{4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\left(2+\sqrt{3}\right)\mathrm{e}^{-2\sqrt{3}\,t+t}}{4} - \frac{\mathrm{e}^{\left(1+2\sqrt{3}\right)t}\left(\sqrt{3}-2\right)}{4}\right)c_1 - \frac{\left(\mathrm{e}^{\left(1+2\sqrt{3}\right)t}-\mathrm{e}^{-2\sqrt{3}\,t+t}\right)\sqrt{3}\,c_2}{4} \\ -\frac{\left(\mathrm{e}^{\left(1+2\sqrt{3}\right)t}-\mathrm{e}^{-2\sqrt{3}\,t+t}\right)\sqrt{3}\,c_1}{12} + \left(\frac{\left(2-\sqrt{3}\right)\mathrm{e}^{-2\sqrt{3}\,t+t}}{4} + \frac{\left(2+\sqrt{3}\right)\mathrm{e}^{\left(1+2\sqrt{3}\right)t}}{4}\right)c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left((c_1+c_2)\sqrt{3}+2c_1\right)\mathrm{e}^{-2\sqrt{3}\,t+t}}{4} - \frac{\mathrm{e}^{\left(1+2\sqrt{3}\right)t}\left((c_1+c_2)\sqrt{3}-2c_1\right)}{4} \\ \frac{\left((c_1-3c_2)\sqrt{3}+6c_2\right)\mathrm{e}^{-2\sqrt{3}\,t+t}}{12} - \frac{\mathrm{e}^{\left(1+2\sqrt{3}\right)t}\left((c_1-3c_2)\sqrt{3}-6c_2\right)}{12} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & -3 \\ -1 & 4 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$
\det\left(\begin{bmatrix} -2-\lambda & -3 \\ -1 & 4-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 2\lambda - 11 = 0
$$

The roots of the above are the eigenvalues.

$$
\lambda_1 = 1 + 2\sqrt{3}
$$
$$
\lambda_2 = 1 - 2\sqrt{3}
$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1 + 2\sqrt{3}$	1	real eigenvalue
$1 - 2\sqrt{3}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - 2\sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$
\left(\begin{bmatrix} -2 & -3 \\ -1 & 4 \end{bmatrix} - \left(1 - 2\sqrt{3}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$
$$
\begin{bmatrix} -3 + 2\sqrt{3} & -3 \\ -1 & 3 + 2\sqrt{3} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$
\left[\begin{array}{cc|c} -3 + 2\sqrt{3} & -3 & 0 \\ -1 & 3 + 2\sqrt{3} & 0 \end{array}\right]
$$

$$
R_2 = R_2 + \frac{R_1}{-3 + 2\sqrt{3}} \Longrightarrow \left[\begin{array}{cc|c} -3 + 2\sqrt{3} & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3+2\sqrt{3} & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{-3+2\sqrt{3}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{-3+2\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{-3+2\sqrt{3}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{-3+2\sqrt{3}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{-3+2\sqrt{3}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{-3+2\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{-3+2\sqrt{3}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + 2\sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -3 \\ -1 & 4 \end{bmatrix} - \left(1+2\sqrt{3}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3-2\sqrt{3} & -3 \\ -1 & 3-2\sqrt{3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3-2\sqrt{3} & -3 & 0 \\ -1 & 3-2\sqrt{3} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{-3-2\sqrt{3}} \Longrightarrow \left[\begin{array}{cc|c} -3-2\sqrt{3} & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -3-2\sqrt{3} & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{3t}{3+2\sqrt{3}}\right\}$

Hence the solution is

$$
\begin{bmatrix} -\frac{3t}{3+2\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3t}{3+2\sqrt{3}} \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} -\frac{3t}{3+2\sqrt{3}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{3+2\sqrt{3}} \\ 1 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} -\frac{3t}{3+2\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{3+2\sqrt{3}} \\ 1 \end{bmatrix}
$$

Which is normalized to

$$
\begin{bmatrix} -\frac{3t}{3+2\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{3+2\sqrt{3}} \\ 1 \end{bmatrix}
$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$1+2\sqrt{3}$	1	1	No	$\begin{bmatrix} -\frac{3}{3+2\sqrt{3}} \\ 1 \end{bmatrix}$
$1-2\sqrt{3}$	1	1	No	$\begin{bmatrix} -\frac{3}{3-2\sqrt{3}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $1+2\sqrt{3}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(1+2\sqrt{3}\right)t} \\
&= \begin{bmatrix} -\frac{3}{3+2\sqrt{3}} \\ 1 \end{bmatrix} e^{\left(1+2\sqrt{3}\right)t}
\end{aligned}
$$

Since eigenvalue $1-2\sqrt{3}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(1-2\sqrt{3}\right)t} \\
&= \begin{bmatrix} -\frac{3}{3-2\sqrt{3}} \\ 1 \end{bmatrix} e^{\left(1-2\sqrt{3}\right)t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{3\,\mathrm{e}^{\left(1+2\sqrt{3}\right)t}}{3+2\sqrt{3}} \\ \mathrm{e}^{\left(1+2\sqrt{3}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{3\,\mathrm{e}^{\left(1-2\sqrt{3}\right)t}}{3-2\sqrt{3}} \\ \mathrm{e}^{\left(1-2\sqrt{3}\right)t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2\left(3+2\sqrt{3}\right)\mathrm{e}^{-2\sqrt{3}\,t+t} - 2\,\mathrm{e}^{\left(1+2\sqrt{3}\right)t}\left(\sqrt{3}-\frac{3}{2}\right)c_1 \\ c_1\mathrm{e}^{\left(1+2\sqrt{3}\right)t} + c_2\mathrm{e}^{-2\sqrt{3}\,t+t} \end{bmatrix}
$$

The following is the phase plot of the system.

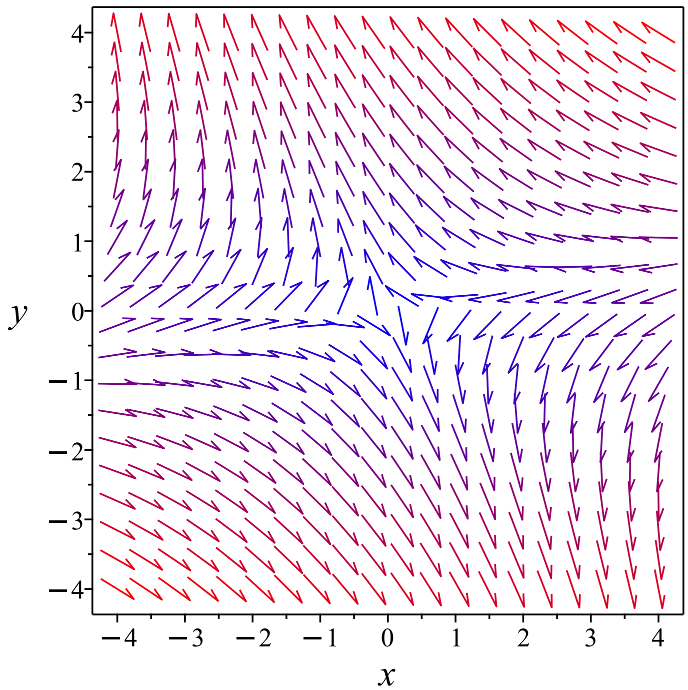


Figure 361: Phase plot

19.1.3 Maple step by step solution

Let's solve

$[x' = -2x - 3y(t), y'(t) = -x + 4y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -2 & -3 \\ -1 & 4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -2 & -3 \\ -1 & 4 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} -2 & -3 \\ -1 & 4 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1-2\sqrt{3}, \left[\begin{array}{c} -\frac{3}{3-2\sqrt{3}} \\ 1 \end{array}\right]\right], \left[1+2\sqrt{3}, \left[\begin{array}{c} -\frac{3}{3+2\sqrt{3}} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1-2\sqrt{3}, \left[\begin{array}{c} -\frac{3}{3-2\sqrt{3}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{1} = \mathrm{e}^{\left(1-2\sqrt{3}\right)t} \cdot \left[\begin{array}{c} -\frac{3}{3-2\sqrt{3}} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1+2\sqrt{3}, \left[\begin{array}{c} -\frac{3}{3+2\sqrt{3}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{2} = \mathrm{e}^{\left(1+2\sqrt{3}\right)t} \cdot \left[\begin{array}{c} -\frac{3}{3+2\sqrt{3}} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_{1}\overrightarrow{x}_{1} + c_{2}\overrightarrow{x}_{2}$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_{1}\mathrm{e}^{\left(1-2\sqrt{3}\right)t} \cdot \left[\begin{array}{c} -\frac{3}{3-2\sqrt{3}} \\ 1 \end{array}\right] + c_{2}\mathrm{e}^{\left(1+2\sqrt{3}\right)t} \cdot \left[\begin{array}{c} -\frac{3}{3+2\sqrt{3}} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 \left(3 + 2\sqrt{3}\right) \mathrm{e}^{-2\sqrt{3}\, t+t} - 2c_2 \mathrm{e}^{\left(1+2\sqrt{3}\right) t} \left(\sqrt{3} - \frac{3}{2}\right) \\ c_1 \mathrm{e}^{-2\sqrt{3}\, t+t} + c_2 \mathrm{e}^{\left(1+2\sqrt{3}\right) t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x = c_1 \left(3 + 2\sqrt{3}\right) \mathrm{e}^{-2\sqrt{3}\, t+t} - 2c_2 \mathrm{e}^{\left(1+2\sqrt{3}\right) t} \left(\sqrt{3} - \frac{3}{2}\right), y(t) = c_1 \mathrm{e}^{-2\sqrt{3}\, t+t} + c_2 \mathrm{e}^{\left(1+2\sqrt{3}\right) t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 95

```
dsolve([diff(x(t),t)=-2*x(t)-3*y(t),diff(y(t),t)=-x(t)+4*y(t)],singsol=all)
```

$$x(t) = c_1 \mathrm{e}^{\left(1+2\sqrt{3}\right) t} + c_2 \mathrm{e}^{-\left(-1+2\sqrt{3}\right) t}$$

$$y(t) = -\frac{2c_1 \mathrm{e}^{\left(1+2\sqrt{3}\right) t} \sqrt{3}}{3} + \frac{2c_2 \mathrm{e}^{-\left(-1+2\sqrt{3}\right) t} \sqrt{3}}{3} - c_1 \mathrm{e}^{\left(1+2\sqrt{3}\right) t} - c_2 \mathrm{e}^{-\left(-1+2\sqrt{3}\right) t}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 144

```
DSolve[{x'[t]==-2*x[t]-3*y[t],y'[t]==-x[t]+4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to -\frac{1}{4} e^{t-2\sqrt{3}t} \left(c_1 \left(\left(\sqrt{3} - 2\right) e^{4\sqrt{3}t} - 2 - \sqrt{3}\right) + \sqrt{3} c_2 \left(e^{4\sqrt{3}t} - 1\right)\right)$$

$$y(t) \to \frac{1}{12} e^{t-2\sqrt{3}t} \left(3c_2 \left(\left(2 + \sqrt{3}\right) e^{4\sqrt{3}t} + 2 - \sqrt{3}\right) - \sqrt{3} c_1 \left(e^{4\sqrt{3}t} - 1\right)\right)$$

19.2 problem 1(b)

19.2.1 Solution using Matrix exponential method 2047
19.2.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2048
19.2.3 Maple step by step solution 2053

Internal problem ID [11540]
Internal file name [`OUTPUT/10522_Thursday_May_18_2023_04_21_57_AM_90026720/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 202
Problem number: 1(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= -3y(t) \\
y'(t) &= -2x + y(t)
\end{aligned}
$$

19.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{(2\,\mathrm{e}^{5t}+3)\,\mathrm{e}^{-2t}}{5} & -\frac{3(\mathrm{e}^{5t}-1)\,\mathrm{e}^{-2t}}{5} \\ -\frac{2(\mathrm{e}^{5t}-1)\,\mathrm{e}^{-2t}}{5} & \frac{(3\,\mathrm{e}^{5t}+2)\,\mathrm{e}^{-2t}}{5} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(2\,e^{5t}+3)\,e^{-2t}}{5} & -\frac{3(e^{5t}-1)\,e^{-2t}}{5} \\ -\frac{2(e^{5t}-1)\,e^{-2t}}{5} & \frac{(3\,e^{5t}+2)\,e^{-2t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2\,e^{5t}+3)\,e^{-2t}c_1}{5} - \frac{3(e^{5t}-1)\,e^{-2t}c_2}{5} \\ -\frac{2(e^{5t}-1)\,e^{-2t}c_1}{5} + \frac{(3\,e^{5t}+2)\,e^{-2t}c_2}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-2t}((2c_1-3c_2)e^{5t}+3c_1+3c_2)}{5} \\ -\frac{2\,e^{-2t}\left(\left(c_1-\frac{3c_2}{2}\right)e^{5t}-c_1-c_2\right)}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -3 \\ -2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - \lambda - 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -3 & 0 \\ -2 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ -2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} -3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \left[\begin{array}{c}\frac{3}{2} \\ 1\end{array}\right] e^{-2t}\end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{3t} \\ &= \left[\begin{array}{c}-1 \\ 1\end{array}\right] e^{3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}x \\ y(t)\end{array}\right] = c_1 \left[\begin{array}{c}\frac{3\,\mathrm{e}^{-2t}}{2} \\ \mathrm{e}^{-2t}\end{array}\right] + c_2 \left[\begin{array}{c}-\mathrm{e}^{3t} \\ \mathrm{e}^{3t}\end{array}\right]$$

Which becomes

$$\left[\begin{array}{c}x \\ y(t)\end{array}\right] = \left[\begin{array}{c}-\frac{(2c_2\mathrm{e}^{5t} - 3c_1)\mathrm{e}^{-2t}}{2} \\ (c_2\mathrm{e}^{5t} + c_1)\,\mathrm{e}^{-2t}\end{array}\right]$$

The following is the phase plot of the system.

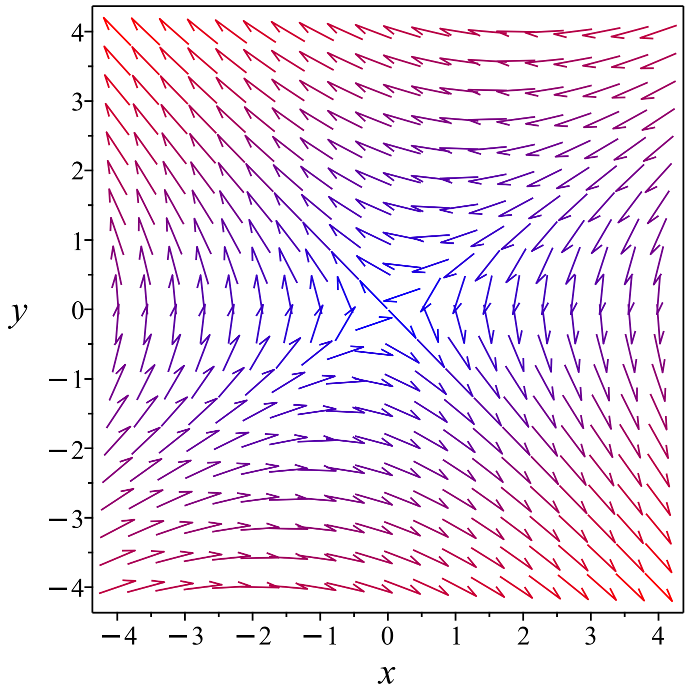


Figure 362: Phase plot

19.2.3 Maple step by step solution

Let's solve

$[x' = -3y(t), y'(t) = -2x + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{-2t} \cdot \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{(2c_2 \mathrm{e}^{5t} - 3c_1)\mathrm{e}^{-2t}}{2} \\ (c_2 \mathrm{e}^{5t} + c_1)\,\mathrm{e}^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs
  $\left\{x = -\frac{(2c_2 \mathrm{e}^{5t} - 3c_1)\mathrm{e}^{-2t}}{2}, y(t) = (c_2 \mathrm{e}^{5t} + c_1)\,\mathrm{e}^{-2t}\right\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=-3*y(t),diff(y(t),t)=-2*x(t)+y(t)],singsol=all)
```

$$x(t) = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-2t}$$
$$y(t) = -c_1 \mathrm{e}^{3t} + \frac{2c_2 \mathrm{e}^{-2t}}{3}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 74

```
DSolve[{x'[t]==-3*y[t],y'[t]==-2*x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{5} e^{-2t} \left(c_1 \left(2e^{5t} + 3\right) - 3c_2 \left(e^{5t} - 1\right)\right)$$
$$y(t) \to \frac{1}{5} e^{-2t} \left(c_2 \left(3e^{5t} + 2\right) - 2c_1 \left(e^{5t} - 1\right)\right)$$

19.3 problem 1(c)

19.3.1 Solution using Matrix exponential method 2056
19.3.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2057
19.3.3 Maple step by step solution 2062

Internal problem ID [11541]
Internal file name [`OUTPUT/10523_Thursday_May_18_2023_04_21_59_AM_87830062/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 202
Problem number: 1(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x' = -2x$$
$$y'(t) = x$$

19.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-2t} & 0 \\ \frac{1}{2} - \frac{\mathrm{e}^{-2t}}{2} & 1 \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{-2t} & 0 \\ \frac{1}{2} - \frac{\mathrm{e}^{-2t}}{2} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-2t} c_1 \\ \left(\frac{1}{2} - \frac{\mathrm{e}^{-2t}}{2}\right) c_1 + c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -2-\lambda & 0 \\ 1 & -\lambda \end{bmatrix}\right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$
(-2-\lambda)(-\lambda) = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 2 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-2t} \end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^0 \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^0\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -2\,\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -2c_1\mathrm{e}^{-2t} \\ c_1\mathrm{e}^{-2t} + c_2 \end{bmatrix}$$

The following is the phase plot of the system.

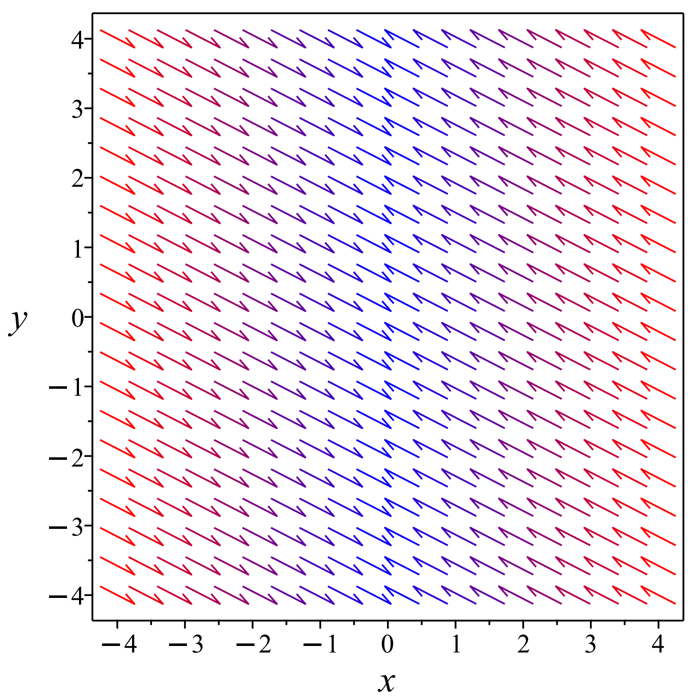


Figure 363: Phase plot

19.3.3 Maple step by step solution

Let's solve

$[x' = -2x, y'(t) = x]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{-2t} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -2c_1 \mathrm{e}^{-2t} \\ c_1 \mathrm{e}^{-2t} + c_2 \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x = -2c_1 \mathrm{e}^{-2t}, y(t) = c_1 \mathrm{e}^{-2t} + c_2\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve([diff(x(t),t)=-2*x(t),diff(y(t),t)=x(t)],singsol=all)
```

$$x(t) = c_2 \mathrm{e}^{-2t}$$
$$y(t) = -\frac{c_2 \mathrm{e}^{-2t}}{2} + c_1$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 35

```
DSolve[{x'[t]==-2*x[t],y'[t]==x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^{-2t}$$
$$y(t) \to c_1\left(\frac{1}{2} - \frac{e^{-2t}}{2}\right) + c_2$$

19.4 problem 1(d)

19.4.1 Solution using Matrix exponential method 2065
19.4.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2066
19.4.3 Maple step by step solution 2071

Internal problem ID [11542]
Internal file name [`OUTPUT/10524_Thursday_May_18_2023_04_22_01_AM_4512424/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 202
Problem number: 1(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= -2x - y(t) \\
y'(t) &= -4y(t)
\end{aligned}
$$

19.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-2t} & -\frac{e^{-2t}}{2} + \frac{e^{-4t}}{2} \\ 0 & e^{-4t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-2t} & -\frac{e^{-2t}}{2}+\frac{e^{-4t}}{2} \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t}c_1 + \left(-\frac{e^{-2t}}{2}+\frac{e^{-4t}}{2}\right) c_2 \\ e^{-4t}c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2c_1-c_2)e^{-2t}}{2} + \frac{e^{-4t}c_2}{2} \\ e^{-4t}c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -2 & -1 \\ 0 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -2-\lambda & -1 \\ 0 & -4-\lambda \end{bmatrix}\right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-2-\lambda)(-4-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-2$$
$$\lambda_2=-4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
-4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-4$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-2 & -1\\ 0 & -4\end{bmatrix}-(-4)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}2 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}2 & -1 & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}2 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2=t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1=\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -1 \\ 0 & -4 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
-4	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the

corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t}
\end{aligned}
$$

Since eigenvalue -4 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-4t} \\
&= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{-4t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-2t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \frac{\mathrm{e}^{-4t}}{2} \\ \mathrm{e}^{-4t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{-2t} + \frac{c_2 \mathrm{e}^{-4t}}{2} \\ c_2 \mathrm{e}^{-4t} \end{bmatrix}
$$

The following is the phase plot of the system.

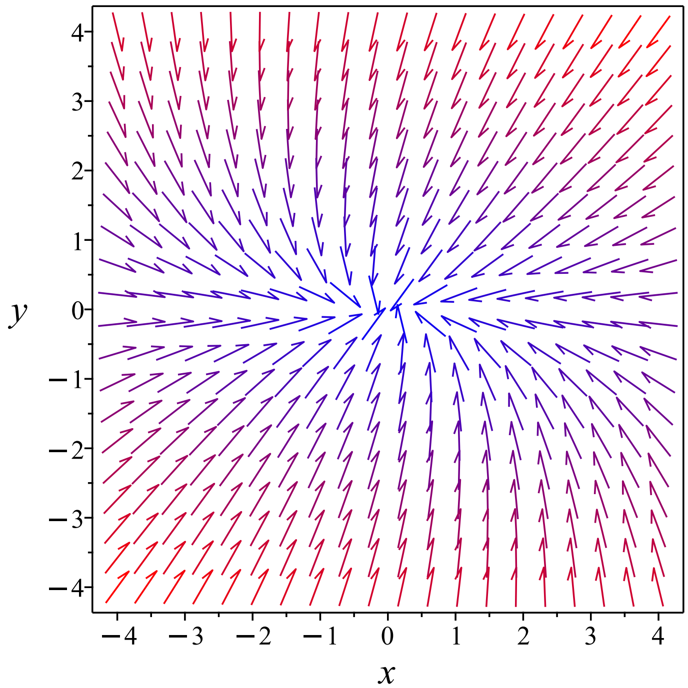


Figure 364: Phase plot

19.4.3 Maple step by step solution

Let's solve

$[x' = -2x - y(t), y'(t) = -4y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -2 & -1 \\ 0 & -4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -2 & -1 \\ 0 & -4 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & -1 \\ 0 & -4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-4, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-4t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{-4t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-2t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{-4t}}{2} + c_2 e^{-2t} \\ c_1 e^{-4t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x = \frac{c_1 e^{-4t}}{2} + c_2 e^{-2t}, y(t) = c_1 e^{-4t}\right\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve([diff(x(t),t)=-2*x(t)-y(t),diff(y(t),t)=-4*y(t)],singsol=all)
```

$$x(t) = \frac{c_2 e^{-4t}}{2} + c_1 e^{-2t}$$
$$y(t) = c_2 e^{-4t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 43

```
DSolve[{x'[t]==-2*x[t]-y[t],y'[t]==-4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2} e^{-4t} \left((2c_1 - c_2) e^{2t} + c_2\right)$$
$$y(t) \to c_2 e^{-4t}$$

19.5 problem 1(e)

19.5.1 Solution using Matrix exponential method 2074
19.5.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2075
19.5.3 Maple step by step solution 2080

Internal problem ID [11543]
Internal file name [`OUTPUT/10525_Thursday_May_18_2023_04_22_03_AM_34931340/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 202
Problem number: 1(e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= x - 2y(t) \\
y'(t) &= -2x + 4y(t)
\end{aligned}
$$

19.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{4}{5} + \frac{e^{5t}}{5} & -\frac{2\,e^{5t}}{5} + \frac{2}{5} \\ -\frac{2\,e^{5t}}{5} + \frac{2}{5} & \frac{1}{5} + \frac{4\,e^{5t}}{5} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{4}{5}+\frac{e^{5t}}{5} & -\frac{2\,e^{5t}}{5}+\frac{2}{5} \\ -\frac{2\,e^{5t}}{5}+\frac{2}{5} & \frac{1}{5}+\frac{4\,e^{5t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{4}{5}+\frac{e^{5t}}{5}\right)c_1+\left(-\frac{2\,e^{5t}}{5}+\frac{2}{5}\right)c_2 \\ \left(-\frac{2\,e^{5t}}{5}+\frac{2}{5}\right)c_1+\left(\frac{1}{5}+\frac{4\,e^{5t}}{5}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-2c_2)e^{5t}}{5}+\frac{4c_1}{5}+\frac{2c_2}{5} \\ \frac{(-2c_1+4c_2)e^{5t}}{5}+\frac{2c_1}{5}+\frac{c_2}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 5\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & -2 & 0 \\ -2 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{5t} \\ &= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{5t}\end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{0} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{0}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{e^{5t}}{2} \\ e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{5t}}{2} + 2c_2 \\ c_1 e^{5t} + c_2 \end{bmatrix}$$

The following is the phase plot of the system.

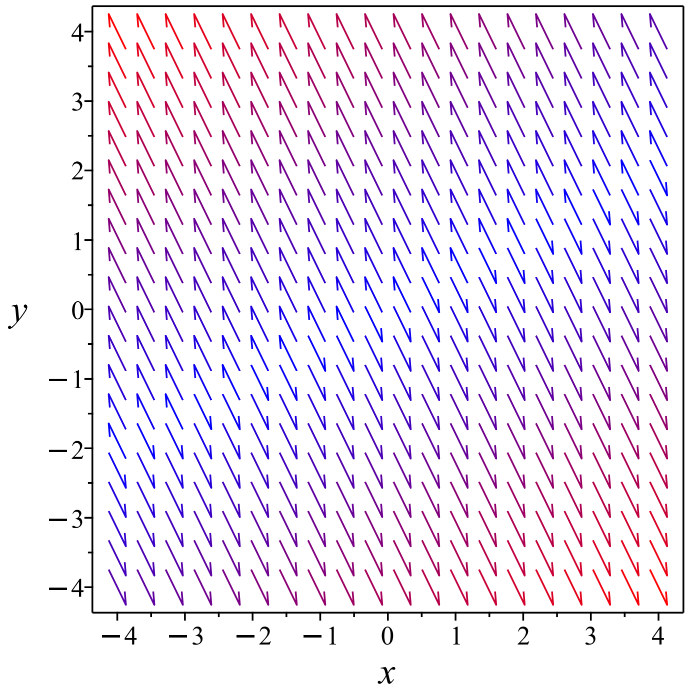


Figure 365: Phase plot

19.5.3 Maple step by step solution

Let's solve

$[x' = x - 2y(t), y'(t) = -2x + 4y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[5, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{5t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_2 \mathrm{e}^{5t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} 2c_1 \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_2 e^{5t}}{2} + 2c_1 \\ c_2 e^{5t} + c_1 \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x = -\frac{c_2 e^{5t}}{2} + 2c_1, y(t) = c_2 e^{5t} + c_1\right\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve([diff(x(t),t)=x(t)-2*y(t),diff(y(t),t)=-2*x(t)+4*y(t)],singsol=all)
```

$$x(t) = c_1 + c_2 e^{5t}$$
$$y(t) = -2c_2 e^{5t} + \frac{c_1}{2}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 62

```
DSolve[{x'[t]==x[t]-2*y[t],y'[t]==-2*x[t]+4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{1}{5}\left(c_1\left(e^{5t} + 4\right) - 2c_2\left(e^{5t} - 1\right)\right)$$
$$y(t) \to \frac{1}{5}\left(c_2\left(4e^{5t} + 1\right) - 2c_1\left(e^{5t} - 1\right)\right)$$

19.6 problem 1(f)

19.6.1 Solution using Matrix exponential method 2083
19.6.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2084
19.6.3 Maple step by step solution 2089

Internal problem ID [11544]
Internal file name [`OUTPUT/10526_Thursday_May_18_2023_04_22_04_AM_35912845/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 202
Problem number: 1(f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x' = -6y(t)$$
$$y'(t) = 6y(t)$$

19.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1 & -e^{6t}+1 \\ 0 & e^{6t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 1 & -\mathrm{e}^{6t}+1 \\ 0 & \mathrm{e}^{6t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} c_1 + (-\mathrm{e}^{6t}+1)\, c_2 \\ \mathrm{e}^{6t} c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det (A - \lambda I) = 0
$$

Expanding gives

$$
\det \left(\begin{bmatrix} 0 & -6 \\ 0 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det \left(\begin{bmatrix} -\lambda & -6 \\ 0 & 6-\lambda \end{bmatrix} \right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$
(-\lambda)(6-\lambda) = 0
$$

The roots of the above are the eigenvalues.

$$
\begin{aligned}
\lambda_1 &= 6 \\
\lambda_2 &= 0
\end{aligned}
$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$
\left(\begin{bmatrix} 0 & -6 \\ 0 & 6 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 0 & -6 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$
\left[\begin{array}{cc|c} 0 & -6 & 0 \\ 0 & 6 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 0 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 0 & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -6 \\ 0 & 6 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -6 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6 & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
6	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{6t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t} \end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^0 \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^0
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{6t} \\ \mathrm{e}^{6t} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{6t} + c_2 \\ c_1 \mathrm{e}^{6t} \end{bmatrix}
$$

The following is the phase plot of the system.

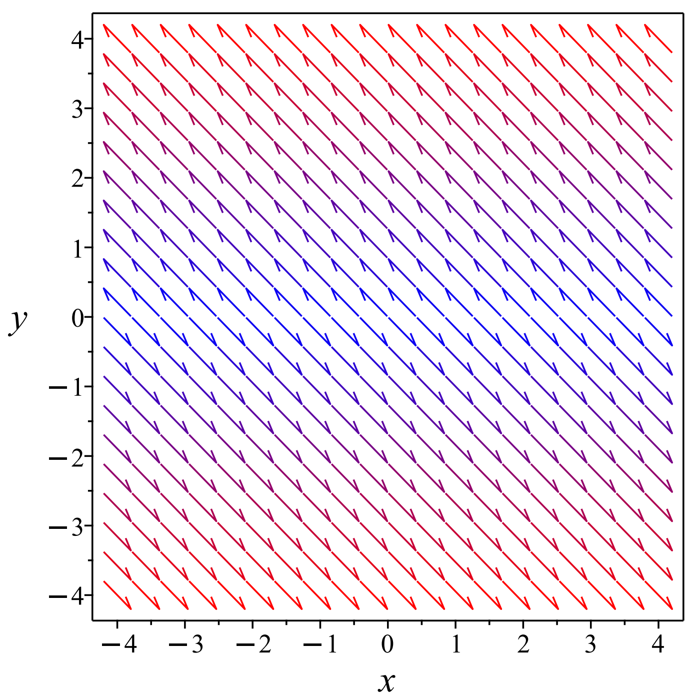


Figure 366: Phase plot

19.6.3 Maple step by step solution

Let's solve

$[x' = -6y(t), y'(t) = 6y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & -6 \\ 0 & 6 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & -6 \\ 0 & 6 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -6 \\ 0 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right], \left[6, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[6, \left[\begin{array}{c}-1\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{6t} \cdot \left[\begin{array}{c}-1\\ 1\end{array}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_2 e^{6t} \cdot \left[\begin{array}{c}-1\\ 1\end{array}\right] + \left[\begin{array}{c}c_1\\ 0\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x\\ y(t)\end{array}\right] = \left[\begin{array}{c}-c_2 e^{6t} + c_1\\ c_2 e^{6t}\end{array}\right]$$

- Solution to the system of ODEs

$$\{x = -c_2 e^{6t} + c_1, y(t) = c_2 e^{6t}\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([diff(x(t),t)=-6*y(t),diff(y(t),t)=6*y(t)],singsol=all)
```

$$x(t) = -c_2 e^{6t} + c_1$$
$$y(t) = c_2 e^{6t}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 30

```
DSolve[{x'[t]==-6*y[t],y'[t]==6*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to -c_2 e^{6t} + c_1 + c_2$$
$$y(t) \to c_2 e^{6t}$$

19.7 problem 3(a)

19.7.1 Solution using Matrix exponential method 2091
19.7.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2093
19.7.3 Maple step by step solution 2099

Internal problem ID [11545]
Internal file name [`OUTPUT/10527_Thursday_May_18_2023_04_22_06_AM_2091558/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 202
Problem number: 3(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= 2x + 3y(t) \\
y'(t) &= -x - 14
\end{aligned}
$$

19.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -14 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^t \cos\left(\sqrt{2}\, t\right) + \frac{e^t \sin\left(\sqrt{2}\, t\right)\sqrt{2}}{2} & \frac{3\, e^t \sin\left(\sqrt{2}\, t\right)\sqrt{2}}{2} \\ -\frac{e^t \sin\left(\sqrt{2}\, t\right)\sqrt{2}}{2} & e^t \cos\left(\sqrt{2}\, t\right) - \frac{e^t \sin\left(\sqrt{2}\, t\right)\sqrt{2}}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^t\left(\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)+2\cos\left(\sqrt{2}\, t\right)\right)}{2} & \frac{3\, e^t \sin\left(\sqrt{2}\, t\right)\sqrt{2}}{2} \\ -\frac{e^t \sin\left(\sqrt{2}\, t\right)\sqrt{2}}{2} & -\frac{e^t\left(\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)-2\cos\left(\sqrt{2}\, t\right)\right)}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^t\left(\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)+2\cos\left(\sqrt{2}\, t\right)\right)}{2} & \frac{3\, e^t \sin\left(\sqrt{2}\, t\right)\sqrt{2}}{2} \\ -\frac{e^t \sin\left(\sqrt{2}\, t\right)\sqrt{2}}{2} & -\frac{e^t\left(\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)-2\cos\left(\sqrt{2}\, t\right)\right)}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^t\left(\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)+2\cos\left(\sqrt{2}\, t\right)\right)c_1}{2} + \frac{3\, e^t \sin\left(\sqrt{2}\, t\right)\sqrt{2}\, c_2}{2} \\ -\frac{e^t \sin\left(\sqrt{2}\, t\right)\sqrt{2}\, c_1}{2} - \frac{e^t\left(\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)-2\cos\left(\sqrt{2}\, t\right)\right)c_2}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\left(\sqrt{2}\,(c_1+3c_2)\sin\left(\sqrt{2}\, t\right)+2\cos\left(\sqrt{2}\, t\right)c_1\right)e^t}{2} \\ -\frac{e^t\left((c_1+c_2)\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)-2\cos\left(\sqrt{2}\, t\right)c_2\right)}{2} \end{bmatrix}
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\, dt
$$

But

$$
\begin{aligned}
e^{-At} &= \left(e^{At}\right)^{-1} \\
&= \begin{bmatrix} -\frac{\left(\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)-2\cos\left(\sqrt{2}\, t\right)\right)e^{-t}}{2} & -\frac{3\sin\left(\sqrt{2}\, t\right)\sqrt{2}\, e^{-t}}{2} \\ \frac{\sin\left(\sqrt{2}\, t\right)\sqrt{2}\, e^{-t}}{2} & \frac{\left(\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)+2\cos\left(\sqrt{2}\, t\right)\right)e^{-t}}{2} \end{bmatrix}
\end{aligned}
$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{e^t\left(\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)+2\cos\left(\sqrt{2}\,t\right)\right)}{2} & \frac{3\,e^t\sin\left(\sqrt{2}\,t\right)\sqrt{2}}{2} \\ -\frac{e^t\sin\left(\sqrt{2}\,t\right)\sqrt{2}}{2} & -\frac{e^t\left(\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)-2\cos\left(\sqrt{2}\,t\right)\right)}{2} \end{bmatrix} \int \begin{bmatrix} -\frac{\left(\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)-2\cos\left(\sqrt{2}\,t\right)\right)e^{-t}}{2} & - \\ \frac{\sin\left(\sqrt{2}\,t\right)\sqrt{2}\,e^{-t}}{2} & \left(\sqrt{2}\,\sin\right. \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^t\left(\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)+2\cos\left(\sqrt{2}\,t\right)\right)}{2} & \frac{3\,e^t\sin\left(\sqrt{2}\,t\right)\sqrt{2}}{2} \\ -\frac{e^t\sin\left(\sqrt{2}\,t\right)\sqrt{2}}{2} & -\frac{e^t\left(\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)-2\cos\left(\sqrt{2}\,t\right)\right)}{2} \end{bmatrix} \begin{bmatrix} -7\,e^{-t}\left(\sqrt{2}\,\cos\left(\sqrt{2}\,t\right)+\sin\left(\sqrt{2}\,t\right)\right)\sqrt{} \\ -\frac{7\,e^{-t}\left(\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)-4\cos\left(\sqrt{2}\,t\right)\right)}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -14 \\ \frac{28}{3} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} \frac{e^t\sqrt{2}\,(c_1+3c_2)\sin\left(\sqrt{2}\,t\right)}{2} + e^t\cos\left(\sqrt{2}\,t\right)c_1 - 14 \\ -\frac{e^t\sqrt{2}\,(c_1+c_2)\sin\left(\sqrt{2}\,t\right)}{2} + e^t\cos\left(\sqrt{2}\,t\right)c_2 + \frac{28}{3} \end{bmatrix} \end{aligned}$$

19.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -14 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 3 \\ -1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 3 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 1 + i\sqrt{2} \\ \lambda_2 &= 1 - i\sqrt{2} \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1 + i\sqrt{2}$	1	complex eigenvalue
$1 - i\sqrt{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - i\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\begin{aligned} \left(\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} - \left(1 - i\sqrt{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 + i\sqrt{2} & 3 \\ -1 & i\sqrt{2} - 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 + i\sqrt{2} & 3 & 0 \\ -1 & i\sqrt{2} - 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{1+i\sqrt{2}} \Longrightarrow \left[\begin{array}{cc|c} 1+i\sqrt{2} & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i\sqrt{2} & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{3t}{1+i\sqrt{2}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{3t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3t}{1+i\sqrt{2}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{3t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{3t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{3t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + i\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} - \left(1+i\sqrt{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i\sqrt{2} & 3 \\ -1 & -1-i\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1-i\sqrt{2} & 3 & 0 \\ -1 & -1-i\sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{1-i\sqrt{2}} \Longrightarrow \left[\begin{array}{cc|c} 1-i\sqrt{2} & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i\sqrt{2} & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{i\sqrt{2}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{\mathrm{I}\sqrt{2}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{i\sqrt{2}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{\mathrm{I}\sqrt{2}-1} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{i\sqrt{2}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{\mathrm{I}\sqrt{2}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{i\sqrt{2}-1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{\mathrm{I}\sqrt{2}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{i\sqrt{2}-1} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$1+i\sqrt{2}$	1	1	No	$\left[\begin{array}{c} \frac{3}{i\sqrt{2}-1} \\ 1 \end{array}\right]$
$1-i\sqrt{2}$	1	1	No	$\left[\begin{array}{c} \frac{3}{-1-i\sqrt{2}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x \\ y(t) \end{array}\right] = c_1 \left[\begin{array}{c} \frac{3\,\mathrm{e}^{\left(1+i\sqrt{2}\right)t}}{i\sqrt{2}-1} \\ \mathrm{e}^{\left(1+i\sqrt{2}\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} \frac{3\,\mathrm{e}^{\left(1-i\sqrt{2}\right)t}}{-1-i\sqrt{2}} \\ \mathrm{e}^{\left(1-i\sqrt{2}\right)t} \end{array}\right]$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \left[\vec{x}_1 \quad \vec{x}_2 \quad \cdots\right]$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \left[\begin{array}{cc} \frac{3\,\mathrm{e}^{\left(1+i\sqrt{2}\right)t}}{i\sqrt{2}-1} & \frac{3\,\mathrm{e}^{\left(1-i\sqrt{2}\right)t}}{-1-i\sqrt{2}} \\ \mathrm{e}^{\left(1+i\sqrt{2}\right)t} & \mathrm{e}^{\left(1-i\sqrt{2}\right)t} \end{array}\right]$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \left[\begin{array}{cc} \frac{i\sqrt{2}\,\mathrm{e}^{-\left(1+i\sqrt{2}\right)t}}{4} & \frac{\sqrt{2}\,\mathrm{e}^{-\left(1+i\sqrt{2}\right)t}\left(i+\sqrt{2}\right)}{4} \\ -\frac{i\sqrt{2}\,\mathrm{e}^{\left(i\sqrt{2}-1\right)t}}{4} & -\frac{\mathrm{e}^{\left(i\sqrt{2}-1\right)t}\sqrt{2}\left(i-\sqrt{2}\right)}{4} \end{array}\right]$$

Hence

$$\vec{x}_p(t) = \left[\begin{array}{cc} \frac{3\,\mathrm{e}^{\left(1+i\sqrt{2}\right)t}}{i\sqrt{2}-1} & \frac{3\,\mathrm{e}^{\left(1-i\sqrt{2}\right)t}}{-1-i\sqrt{2}} \\ \mathrm{e}^{\left(1+i\sqrt{2}\right)t} & \mathrm{e}^{\left(1-i\sqrt{2}\right)t} \end{array}\right] \int \left[\begin{array}{cc} \frac{i\sqrt{2}\,\mathrm{e}^{-\left(1+i\sqrt{2}\right)t}}{4} & \frac{\sqrt{2}\,\mathrm{e}^{-\left(1+i\sqrt{2}\right)t}\left(i+\sqrt{2}\right)}{4} \\ -\frac{i\sqrt{2}\,\mathrm{e}^{\left(i\sqrt{2}-1\right)t}}{4} & -\frac{\mathrm{e}^{\left(i\sqrt{2}-1\right)t}\sqrt{2}\left(i-\sqrt{2}\right)}{4} \end{array}\right] \left[\begin{array}{c} 0 \\ -14 \end{array}\right] dt$$

$$= \left[\begin{array}{cc} \frac{3\,\mathrm{e}^{\left(1+i\sqrt{2}\right)t}}{i\sqrt{2}-1} & \frac{3\,\mathrm{e}^{\left(1-i\sqrt{2}\right)t}}{-1-i\sqrt{2}} \\ \mathrm{e}^{\left(1+i\sqrt{2}\right)t} & \mathrm{e}^{\left(1-i\sqrt{2}\right)t} \end{array}\right] \int \left[\begin{array}{c} -\frac{7\sqrt{2}\,\mathrm{e}^{-\left(1+i\sqrt{2}\right)t}\left(i+\sqrt{2}\right)}{2} \\ \frac{7\,\mathrm{e}^{\left(i\sqrt{2}-1\right)t}\sqrt{2}\left(i-\sqrt{2}\right)}{2} \end{array}\right] dt$$

$$= \left[\begin{array}{cc} \frac{3\,\mathrm{e}^{\left(1+i\sqrt{2}\right)t}}{i\sqrt{2}-1} & \frac{3\,\mathrm{e}^{\left(1-i\sqrt{2}\right)t}}{-1-i\sqrt{2}} \\ \mathrm{e}^{\left(1+i\sqrt{2}\right)t} & \mathrm{e}^{\left(1-i\sqrt{2}\right)t} \end{array}\right] \left[\begin{array}{c} -\frac{7i\sqrt{2}\,\mathrm{e}^{-\left(1+i\sqrt{2}\right)t}\left(i+\sqrt{2}\right)^2}{6} \\ \frac{7i\mathrm{e}^{\left(i\sqrt{2}-1\right)t}\sqrt{2}\left(i-\sqrt{2}\right)^2}{6} \end{array}\right]$$

$$= \left[\begin{array}{c} -14 \\ \frac{28}{3} \end{array}\right]$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\left[\begin{array}{c} x \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{3c_1\mathrm{e}^{\left(1+i\sqrt{2}\right)t}}{i\sqrt{2}-1} \\ c_1\mathrm{e}^{\left(1+i\sqrt{2}\right)t} \end{array}\right] + \left[\begin{array}{c} \frac{3c_2\mathrm{e}^{\left(1-i\sqrt{2}\right)t}}{-1-i\sqrt{2}} \\ c_2\mathrm{e}^{\left(1-i\sqrt{2}\right)t} \end{array}\right] + \left[\begin{array}{c} -14 \\ \frac{28}{3} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x \\ y(t) \end{array}\right] = \left[\begin{array}{c} -14 + c_2\left(i\sqrt{2}-1\right)\mathrm{e}^{-\left(i\sqrt{2}-1\right)t} + c_1\left(-1-i\sqrt{2}\right)\mathrm{e}^{\left(1+i\sqrt{2}\right)t} \\ c_1\mathrm{e}^{\left(1+i\sqrt{2}\right)t} + c_2\mathrm{e}^{-\left(i\sqrt{2}-1\right)t} + \frac{28}{3} \end{array}\right]$$

The following is the phase plot of the system.

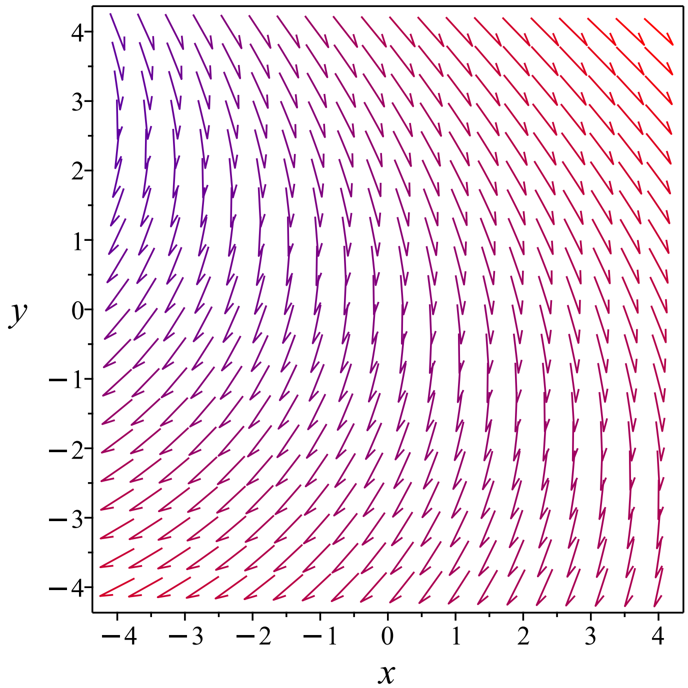


Figure 367: Phase plot

19.7.3 Maple step by step solution

Let's solve

$[x' = 2x + 3y(t), y'(t) = -x - 14]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ -14 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ -14 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t)=\left[\begin{array}{c}0\\-14\end{array}\right]$$

- Define the coefficient matrix

$$A=\left[\begin{array}{cc}2&3\\-1&0\end{array}\right]$$

- Rewrite the system as

$$\vec{x}'(t)=A\cdot\vec{x}(t)+\vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1-\mathrm{I}\sqrt{2},\left[\begin{array}{c}\frac{3}{-1-\mathrm{I}\sqrt{2}}\\1\end{array}\right]\right],\left[1+\mathrm{I}\sqrt{2},\left[\begin{array}{c}\frac{3}{\mathrm{I}\sqrt{2}-1}\\1\end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-\mathrm{I}\sqrt{2},\left[\begin{array}{c}\frac{3}{-1-\mathrm{I}\sqrt{2}}\\1\end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(1-\mathrm{I}\sqrt{2}\right)t}\cdot\left[\begin{array}{c}\frac{3}{-1-\mathrm{I}\sqrt{2}}\\1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{t}\cdot\left(\cos\left(\sqrt{2}\,t\right)-\mathrm{I}\sin\left(\sqrt{2}\,t\right)\right)\cdot\left[\begin{array}{c}\frac{3}{-1-\mathrm{I}\sqrt{2}}\\1\end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{t}\cdot\left[\begin{array}{c}\frac{3\left(\cos\left(\sqrt{2}\,t\right)-\mathrm{I}\sin\left(\sqrt{2}\,t\right)\right)}{-1-\mathrm{I}\sqrt{2}}\\\cos\left(\sqrt{2}\,t\right)-\mathrm{I}\sin\left(\sqrt{2}\,t\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_{1}(t)=\mathrm{e}^{t}\cdot\left[\begin{array}{c}\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)-\cos\left(\sqrt{2}\,t\right)\\\cos\left(\sqrt{2}\,t\right)\end{array}\right],\vec{x}_{2}(t)=\mathrm{e}^{t}\cdot\left[\begin{array}{c}\sqrt{2}\,\cos\left(\sqrt{2}\,t\right)+\sin\left(\sqrt{2}\,t\right)\\-\sin\left(\sqrt{2}\,t\right)\end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t) + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syste

$\phi(t) = \left[\begin{array}{cc} e^t\left(\sqrt{2}\sin\left(\sqrt{2}\,t\right) - \cos\left(\sqrt{2}\,t\right)\right) & e^t\left(\sqrt{2}\cos\left(\sqrt{2}\,t\right) + \sin\left(\sqrt{2}\,t\right)\right) \\ e^t\cos\left(\sqrt{2}\,t\right) & -e^t\sin\left(\sqrt{2}\,t\right) \end{array}\right]$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$\Phi(t) = \left[\begin{array}{cc} e^t\left(\sqrt{2}\sin\left(\sqrt{2}\,t\right) - \cos\left(\sqrt{2}\,t\right)\right) & e^t\left(\sqrt{2}\cos\left(\sqrt{2}\,t\right) + \sin\left(\sqrt{2}\,t\right)\right) \\ e^t\cos\left(\sqrt{2}\,t\right) & -e^t\sin\left(\sqrt{2}\,t\right) \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc} -1 & \sqrt{2} \\ 1 & 0 \end{array}\right]}$

- Evaluate and simplify to get the fundamental matrix

$\Phi(t) = \left[\begin{array}{cc} \frac{e^t\left(\sqrt{2}\cos\left(\sqrt{2}\,t\right)+\sin\left(\sqrt{2}\,t\right)\right)\sqrt{2}}{2} & \frac{3\,e^t\sin\left(\sqrt{2}\,t\right)\sqrt{2}}{2} \\ -\frac{e^t\sin\left(\sqrt{2}\,t\right)\sqrt{2}}{2} & -\frac{e^t\left(\sqrt{2}\sin\left(\sqrt{2}\,t\right)-2\cos\left(\sqrt{2}\,t\right)\right)}{2} \end{array}\right]$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

$\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$

- Take the derivative of the particular solution

$\overrightarrow{x}_p'(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- Cancel like terms

$\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$

- Multiply by the inverse of the fundamental matrix

$\overrightarrow{v}'(t) = \frac{1}{\Phi(t)} \cdot \overrightarrow{f}(t)$

- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{x}_p(t) = \left[\begin{array}{c} -7\,\mathrm{e}^t \sin\left(\sqrt{2}\,t\right)\sqrt{2} + 14\,\mathrm{e}^t \cos\left(\sqrt{2}\,t\right) - 14 \\ -\frac{7\,\mathrm{e}^t \sin\left(\sqrt{2}\,t\right)\sqrt{2}}{3} - \frac{28\,\mathrm{e}^t \cos\left(\sqrt{2}\,t\right)}{3} + \frac{28}{3} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{x}(t) = c_1 \overrightarrow{x}_1(t) + c_2 \overrightarrow{x}_2(t) + \left[\begin{array}{c} -7\,\mathrm{e}^t \sin\left(\sqrt{2}\,t\right)\sqrt{2} + 14\,\mathrm{e}^t \cos\left(\sqrt{2}\,t\right) - 14 \\ -\frac{7\,\mathrm{e}^t \sin\left(\sqrt{2}\,t\right)\sqrt{2}}{3} - \frac{28\,\mathrm{e}^t \cos\left(\sqrt{2}\,t\right)}{3} + \frac{28}{3} \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x \\ y(t) \end{array}\right] = \left[\begin{array}{c} -\mathrm{e}^t\left(-\sqrt{2}\,c_2 + c_1 - 14\right)\cos\left(\sqrt{2}\,t\right) - 14 + \mathrm{e}^t\left(\left(c_1 - 7\right)\sqrt{2} + c_2\right)\sin\left(\sqrt{2}\,t\right) \\ \frac{\mathrm{e}^t(3c_1 - 28)\cos\left(\sqrt{2}\,t\right)}{3} + \frac{28}{3} - \left(c_2 + \frac{7\sqrt{2}}{3}\right)\mathrm{e}^t \sin\left(\sqrt{2}\,t\right) \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x = -\mathrm{e}^t\left(-\sqrt{2}\,c_2 + c_1 - 14\right)\cos\left(\sqrt{2}\,t\right) - 14 + \mathrm{e}^t\left(\left(c_1 - 7\right)\sqrt{2} + c_2\right)\sin\left(\sqrt{2}\,t\right), y(t) = \frac{\mathrm{e}^t(3c_1 - 28)}{3}\right.$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 77

```
dsolve([diff(x(t),t)=2*x(t)+3*y(t),diff(y(t),t)=-x(t)-14],singsol=all)
```

$$x(t) = -14 + \mathrm{e}^t\left(\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)c_1 - \sqrt{2}\,\cos\left(\sqrt{2}\,t\right)c_2 - \sin\left(\sqrt{2}\,t\right)c_2 - \cos\left(\sqrt{2}\,t\right)c_1\right)$$
$$y(t) = \frac{28}{3} + \mathrm{e}^t\left(\sin\left(\sqrt{2}\,t\right)c_2 + \cos\left(\sqrt{2}\,t\right)c_1\right)$$

✓ Solution by Mathematica

Time used: 0.315 (sec). Leaf size: 89

```
DSolve[{x'[t]==2*x[t]+3*y[t],y'[t]==-x[t]-14},{x[t],y[t]},t,IncludeSingularSolutions -> True
```

$$
\begin{aligned}
x(t) &\to c_1 e^t \cos\left(\sqrt{2}t\right) + \frac{(c_1 + 3c_2)e^t \sin\left(\sqrt{2}t\right)}{\sqrt{2}} - 14 \\
y(t) &\to c_2 e^t \cos\left(\sqrt{2}t\right) - \frac{(c_1 + c_2)e^t \sin\left(\sqrt{2}t\right)}{\sqrt{2}} + \frac{28}{3}
\end{aligned}
$$

19.8 problem 3(b)

19.8.1 Solution using Matrix exponential method 2104
19.8.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2107
19.8.3 Maple step by step solution 2113

Internal problem ID [11546]
Internal file name [`OUTPUT/10528_Thursday_May_18_2023_04_22_09_AM_8866802/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 202
Problem number: 3(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x' = -3x + 3y(t)$$
$$y'(t) = x + 2y(t) - 1$$

19.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \left[\begin{array}{cc} \frac{\left(-5\sqrt{37}+37\right) e^{\frac{\left(-1+\sqrt{37}\right) t}{2}}}{74}+\frac{e^{-\frac{\left(1+\sqrt{37}\right) t}{2}} \left(5\sqrt{37}+37\right)}{74} & -\frac{3 \left(-e^{\frac{\left(-1+\sqrt{37}\right) t}{2}}+e^{-\frac{\left(1+\sqrt{37}\right) t}{2}}\right) \sqrt{37}}{37} \\ -\frac{\left(-e^{\frac{\left(-1+\sqrt{37}\right) t}{2}}+e^{-\frac{\left(1+\sqrt{37}\right) t}{2}}\right) \sqrt{37}}{37} & \frac{\left(5\sqrt{37}+37\right) e^{\frac{\left(-1+\sqrt{37}\right) t}{2}}}{74}+\frac{\left(-5\sqrt{37}+37\right) e^{-\frac{\left(1+\sqrt{37}\right) t}{2}}}{74} \end{array}\right]
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \left[\begin{array}{cc} \frac{\left(-5\sqrt{37}+37\right) e^{\frac{\left(-1+\sqrt{37}\right) t}{2}}}{74}+\frac{e^{-\frac{\left(1+\sqrt{37}\right) t}{2}} \left(5\sqrt{37}+37\right)}{74} & -\frac{3 \left(-e^{\frac{\left(-1+\sqrt{37}\right) t}{2}}+e^{-\frac{\left(1+\sqrt{37}\right) t}{2}}\right) \sqrt{37}}{37} \\ -\frac{\left(-e^{\frac{\left(-1+\sqrt{37}\right) t}{2}}+e^{-\frac{\left(1+\sqrt{37}\right) t}{2}}\right) \sqrt{37}}{37} & \frac{\left(5\sqrt{37}+37\right) e^{\frac{\left(-1+\sqrt{37}\right) t}{2}}}{74}+\frac{\left(-5\sqrt{37}+37\right) e^{-\frac{\left(1+\sqrt{37}\right) t}{2}}}{74} \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \end{array}\right] \\
&= \left[\begin{array}{c} \left(\frac{\left(-5\sqrt{37}+37\right) e^{\frac{\left(-1+\sqrt{37}\right) t}{2}}}{74}+\frac{e^{-\frac{\left(1+\sqrt{37}\right) t}{2}} \left(5\sqrt{37}+37\right)}{74}\right) c_1 -\frac{3 \left(-e^{\frac{\left(-1+\sqrt{37}\right) t}{2}}+e^{-\frac{\left(1+\sqrt{37}\right) t}{2}}\right) \sqrt{37}\, c_2}{37} \\ -\frac{\left(-e^{\frac{\left(-1+\sqrt{37}\right) t}{2}}+e^{-\frac{\left(1+\sqrt{37}\right) t}{2}}\right) \sqrt{37}\, c_1}{37}+\left(\frac{\left(5\sqrt{37}+37\right) e^{\frac{\left(-1+\sqrt{37}\right) t}{2}}}{74}+\frac{\left(-5\sqrt{37}+37\right) e^{-\frac{\left(1+\sqrt{37}\right) t}{2}}}{74}\right) c_2 \end{array}\right] \\
&= \left[\begin{array}{c} \frac{\left(\left(-5 c_1+6 c_2\right) \sqrt{37}+37 c_1\right) e^{\frac{\left(-1+\sqrt{37}\right) t}{2}}}{74}+\frac{5\, e^{-\frac{\left(1+\sqrt{37}\right) t}{2}} \left(\left(c_1-\frac{6 c_2}{5}\right) \sqrt{37}+\frac{37 c_1}{5}\right)}{74} \\ \frac{\left(\left(2 c_1+5 c_2\right) \sqrt{37}+37 c_2\right) e^{\frac{\left(-1+\sqrt{37}\right) t}{2}}}{74}-\frac{e^{-\frac{\left(1+\sqrt{37}\right) t}{2}} \left(\left(c_1+\frac{5 c_2}{2}\right) \sqrt{37}-\frac{37 c_2}{2}\right)}{37} \end{array}\right]
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\, dt
$$

But

$$
\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \left[\begin{array}{cc}
-\frac{e^{t}\left(5\sqrt{37}\,e^{-\frac{(1+\sqrt{37})t}{2}}-5\sqrt{37}\,e^{\frac{(-1+\sqrt{37})t}{2}}-37\,e^{-\frac{(1+\sqrt{37})t}{2}}-37\,e^{\frac{(-1+\sqrt{37})t}{2}}\right)}{74} & \frac{3\sqrt{37}\,e^{t}\left(-e^{\frac{(-1+\sqrt{37})t}{2}}+\cdots\right.}{37} \\
\frac{\sqrt{37}\,e^{t}\left(-e^{\frac{(-1+\sqrt{37})t}{2}}+e^{-\frac{(1+\sqrt{37})t}{2}}\right)}{37} & \frac{e^{t}\left(5\sqrt{37}\,e^{-\frac{(1+\sqrt{37})t}{2}}-5\sqrt{37}\,e^{\frac{(-1+\sqrt{37})t}{2}}+\cdots\right.}{74}
\end{array}\right.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \left[\begin{array}{cc}
\frac{\left(-5\sqrt{37}+37\right)e^{\frac{(-1+\sqrt{37})t}{2}}}{74}+\frac{e^{-\frac{(1+\sqrt{37})t}{2}}\left(5\sqrt{37}+37\right)}{74} & -\frac{3\left(-e^{\frac{(-1+\sqrt{37})t}{2}}+e^{-\frac{(1+\sqrt{37})t}{2}}\right)\sqrt{37}}{37} \\
-\frac{\left(-e^{\frac{(-1+\sqrt{37})t}{2}}+e^{-\frac{(1+\sqrt{37})t}{2}}\right)\sqrt{37}}{37} & \frac{\left(5\sqrt{37}+37\right)e^{\frac{(-1+\sqrt{37})t}{2}}}{74}+\frac{\left(-5\sqrt{37}+37\right)e^{-\frac{(1+\sqrt{37})t}{2}}}{74}
\end{array}\right]\int \left[\begin{array}{c} -\cdots \\ \cdots \end{array}\right. \\
&= \left[\begin{array}{cc}
\frac{\left(-5\sqrt{37}+37\right)e^{\frac{(-1+\sqrt{37})t}{2}}}{74}+\frac{e^{-\frac{(1+\sqrt{37})t}{2}}\left(5\sqrt{37}+37\right)}{74} & -\frac{3\left(-e^{\frac{(-1+\sqrt{37})t}{2}}+e^{-\frac{(1+\sqrt{37})t}{2}}\right)\sqrt{37}}{37} \\
-\frac{\left(-e^{\frac{(-1+\sqrt{37})t}{2}}+e^{-\frac{(1+\sqrt{37})t}{2}}\right)\sqrt{37}}{37} & \frac{\left(5\sqrt{37}+37\right)e^{\frac{(-1+\sqrt{37})t}{2}}}{74}+\frac{\left(-5\sqrt{37}+37\right)e^{-\frac{(1+\sqrt{37})t}{2}}}{74}
\end{array}\right]\left[\begin{array}{c} \frac{\left(\left(1\cdots\right.\right.}{\cdots} \\ \frac{\left(7\sqrt{\cdots}\right.}{\cdots} \end{array}\right. \\
&= \left[\begin{array}{c} \frac{1}{3} \\ \frac{1}{3} \end{array}\right]
\end{aligned}
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t)+\vec{x}_p(t) \\
&= \left[\begin{array}{c}
\frac{1}{3}+\frac{\left((-5c_1+6c_2)\sqrt{37}+37c_1\right)e^{\frac{(-1+\sqrt{37})t}{2}}}{74}+\frac{\left((5c_1-6c_2)\sqrt{37}+37c_1\right)e^{-\frac{(1+\sqrt{37})t}{2}}}{74} \\
\frac{1}{3}+\frac{\left((2c_1+5c_2)\sqrt{37}+37c_2\right)e^{\frac{(-1+\sqrt{37})t}{2}}}{74}+\frac{\left((-2c_1-5c_2)\sqrt{37}+37c_2\right)e^{-\frac{(1+\sqrt{37})t}{2}}}{74}
\end{array}\right]
\end{aligned}
$$

19.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 3 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 3 \\ 1 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \lambda - 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{2} + \frac{\sqrt{37}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{\sqrt{37}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{2}-\frac{\sqrt{37}}{2}$	1	real eigenvalue
$-\frac{1}{2}+\frac{\sqrt{37}}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{2}-\frac{\sqrt{37}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 3 \\ 1 & 2 \end{bmatrix} - \left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{5}{2}+\frac{\sqrt{37}}{2} & 3 \\ 1 & \frac{5}{2}+\frac{\sqrt{37}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{5}{2}+\frac{\sqrt{37}}{2} & 3 & 0 \\ 1 & \frac{5}{2}+\frac{\sqrt{37}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{-\frac{5}{2}+\frac{\sqrt{37}}{2}} \Longrightarrow \left[\begin{array}{cc|c} -\frac{5}{2}+\frac{\sqrt{37}}{2} & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{5}{2}+\frac{\sqrt{37}}{2} & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{6t}{-5+\sqrt{37}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{6t}{-5+\sqrt{37}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{6t}{-5+\sqrt{37}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{6t}{-5+\sqrt{37}} \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{6}{-5+\sqrt{37}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{6t}{-5+\sqrt{37}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{6}{-5+\sqrt{37}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{6t}{-5+\sqrt{37}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{6}{-5+\sqrt{37}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{2} + \frac{\sqrt{37}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 3 \\ 1 & 2 \end{bmatrix} - \left(-\frac{1}{2} + \frac{\sqrt{37}}{2} \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{5}{2} - \frac{\sqrt{37}}{2} & 3 \\ 1 & \frac{5}{2} - \frac{\sqrt{37}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{5}{2} - \frac{\sqrt{37}}{2} & 3 & 0 \\ 1 & \frac{5}{2} - \frac{\sqrt{37}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{-\frac{5}{2} - \frac{\sqrt{37}}{2}} \Longrightarrow \left[\begin{array}{cc|c} -\frac{5}{2} - \frac{\sqrt{37}}{2} & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{5}{2} - \frac{\sqrt{37}}{2} & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = \frac{6t}{5+\sqrt{37}} \right\}$

Hence the solution is

$$\begin{bmatrix} \frac{6t}{5+\sqrt{37}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{6t}{5+\sqrt{37}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{6t}{5+\sqrt{37}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{6}{5+\sqrt{37}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{6t}{5+\sqrt{37}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{6}{5+\sqrt{37}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{6t}{5+\sqrt{37}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{6}{5+\sqrt{37}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{1}{2} + \frac{\sqrt{37}}{2}$	1	1	No	$\begin{bmatrix} \frac{3}{\frac{5}{2}+\frac{\sqrt{37}}{2}} \\ 1 \end{bmatrix}$
$-\frac{1}{2} - \frac{\sqrt{37}}{2}$	1	1	No	$\begin{bmatrix} \frac{3}{\frac{5}{2}-\frac{\sqrt{37}}{2}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{1}{2} + \frac{\sqrt{37}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t} \\ &= \begin{bmatrix} \frac{3}{\frac{5}{2}+\frac{\sqrt{37}}{2}} \\ 1 \end{bmatrix} e^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t} \end{aligned}$$

Since eigenvalue $-\frac{1}{2}-\frac{\sqrt{37}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t}\\
&= \left[\begin{array}{c} \frac{3}{\frac{5}{2}-\frac{\sqrt{37}}{2}} \\ 1 \end{array}\right] e^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c} x \\ y(t) \end{array}\right] = c_1 \left[\begin{array}{c} \frac{3\,\mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}+\frac{\sqrt{37}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} \frac{3\,\mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}-\frac{\sqrt{37}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t} \end{array}\right]
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$
\Phi(t) = \left[\begin{array}{cc} \frac{3\,\mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}+\frac{\sqrt{37}}{2}} & \frac{3\,\mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}-\frac{\sqrt{37}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t} & \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t} \end{array}\right]
$$

The particular solution is then given by

$$
\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt
$$

But

$$
\Phi^{-1} = \left[\begin{array}{cc} \frac{\sqrt{37}\,\mathrm{e}^{-\frac{\left(-1+\sqrt{37}\right)t}{2}}}{37} & \frac{\sqrt{37}\left(5+\sqrt{37}\right)\mathrm{e}^{-\frac{\left(-1+\sqrt{37}\right)t}{2}}}{74} \\ -\frac{\sqrt{37}\,\mathrm{e}^{\frac{\left(1+\sqrt{37}\right)t}{2}}}{37} & \frac{\sqrt{37}\left(-5+\sqrt{37}\right)\mathrm{e}^{\frac{\left(1+\sqrt{37}\right)t}{2}}}{74} \end{array}\right]
$$

Hence

$$
\vec{x}_p(t) = \left[\begin{array}{cc} \frac{3\,\mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}+\frac{\sqrt{37}}{2}} & \frac{3\,\mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}-\frac{\sqrt{37}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t} & \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t} \end{array}\right] \int \left[\begin{array}{cc} \frac{\sqrt{37}\,\mathrm{e}^{-\frac{\left(-1+\sqrt{37}\right)t}{2}}}{37} & \frac{\sqrt{37}\left(5+\sqrt{37}\right)\mathrm{e}^{-\frac{\left(-1+\sqrt{37}\right)t}{2}}}{74} \\ -\frac{\sqrt{37}\,\mathrm{e}^{\frac{\left(1+\sqrt{37}\right)t}{2}}}{37} & \frac{\sqrt{37}\left(-5+\sqrt{37}\right)\mathrm{e}^{\frac{\left(1+\sqrt{37}\right)t}{2}}}{74} \end{array}\right] \left[\begin{array}{c} 0 \\ -1 \end{array}\right] dt
$$

$$
= \left[\begin{array}{cc} \frac{3\,\mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}+\frac{\sqrt{37}}{2}} & \frac{3\,\mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}-\frac{\sqrt{37}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t} & \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t} \end{array}\right] \int \left[\begin{array}{c} -\frac{\sqrt{37}\left(5+\sqrt{37}\right)\mathrm{e}^{-\frac{\left(-1+\sqrt{37}\right)t}{2}}}{74} \\ -\frac{\sqrt{37}\left(-5+\sqrt{37}\right)\mathrm{e}^{\frac{\left(1+\sqrt{37}\right)t}{2}}}{74} \end{array}\right] dt
$$

$$
= \left[\begin{array}{cc} \frac{3\,\mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}+\frac{\sqrt{37}}{2}} & \frac{3\,\mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}-\frac{\sqrt{37}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t} & \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t} \end{array}\right] \left[\begin{array}{c} \frac{\left(1+\sqrt{37}\right)\sqrt{37}\left(5+\sqrt{37}\right)\mathrm{e}^{-\frac{\left(-1+\sqrt{37}\right)t}{2}}}{1332} \\ -\frac{\left(-1+\sqrt{37}\right)\sqrt{37}\left(-5+\sqrt{37}\right)\mathrm{e}^{\frac{\left(1+\sqrt{37}\right)t}{2}}}{1332} \end{array}\right]
$$

$$
= \left[\begin{array}{c} \frac{1}{3} \\ \frac{1}{3} \end{array}\right]
$$

Now that we found particular solution, the final solution is

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

$$
\left[\begin{array}{c} x \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{3c_1\mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}+\frac{\sqrt{37}}{2}} \\ c_1\mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t} \end{array}\right] + \left[\begin{array}{c} \frac{3c_2\mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}-\frac{\sqrt{37}}{2}} \\ c_2\mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t} \end{array}\right] + \left[\begin{array}{c} \frac{1}{3} \\ \frac{1}{3} \end{array}\right]
$$

Which becomes

$$
\left[\begin{array}{c} x \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{c_1\left(-5+\sqrt{37}\right)\mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}}}{2} + \frac{1}{3} - \frac{c_2\left(5+\sqrt{37}\right)\mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}}}{2} \\ c_1\mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}} + c_2\mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}} + \frac{1}{3} \end{array}\right]
$$

The following is the phase plot of the system.

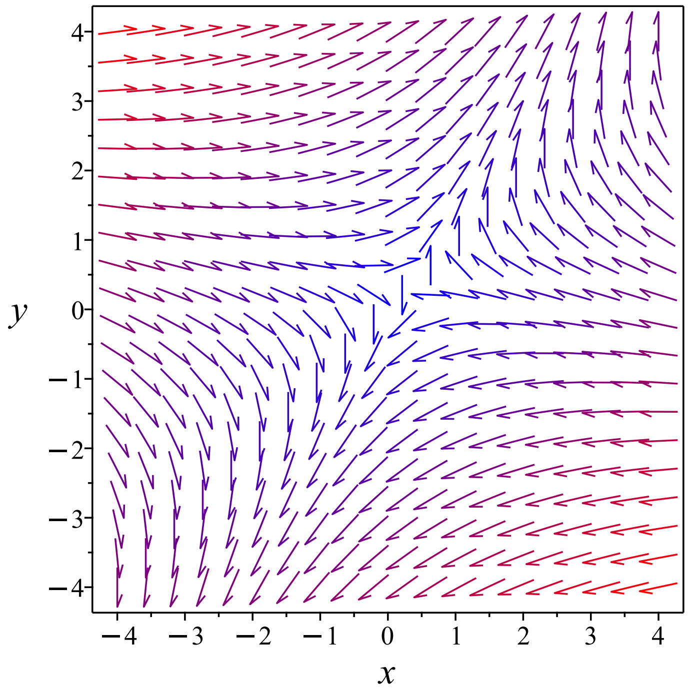


Figure 368: Phase plot

19.8.3 Maple step by step solution

Let's solve

$[x' = -3x + 3y(t), y'(t) = x + 2y(t) - 1]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -3 & 3 \\ 1 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -3 & 3 \\ 1 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & 3 \\ 1 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{2} - \frac{\sqrt{37}}{2}, \begin{bmatrix} \frac{3}{\frac{5}{2} - \frac{\sqrt{37}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} + \frac{\sqrt{37}}{2}, \begin{bmatrix} \frac{3}{\frac{5}{2} + \frac{\sqrt{37}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{1}{2} - \frac{\sqrt{37}}{2}, \begin{bmatrix} \frac{3}{\frac{5}{2} - \frac{\sqrt{37}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{\left(-\frac{1}{2} - \frac{\sqrt{37}}{2}\right)t} \cdot \begin{bmatrix} \frac{3}{\frac{5}{2} - \frac{\sqrt{37}}{2}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{1}{2} + \frac{\sqrt{37}}{2}, \begin{bmatrix} \frac{3}{\frac{5}{2} + \frac{\sqrt{37}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{\left(-\frac{1}{2} + \frac{\sqrt{37}}{2}\right)t} \cdot \begin{bmatrix} \frac{3}{\frac{5}{2} + \frac{\sqrt{37}}{2}} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \vec{x}_p(t)$$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \left[\begin{array}{cc} \frac{3\,\mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}-\frac{\sqrt{37}}{2}} & \frac{3\,\mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}+\frac{\sqrt{37}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t} & \mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t} \end{array}\right]$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \left[\begin{array}{cc} \frac{3\,\mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}-\frac{\sqrt{37}}{2}} & \frac{3\,\mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t}}{\frac{5}{2}+\frac{\sqrt{37}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{37}}{2}\right)t} & \mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{37}}{2}\right)t} \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc} \frac{3}{\frac{5}{2}-\frac{\sqrt{37}}{2}} & \frac{3}{\frac{5}{2}+\frac{\sqrt{37}}{2}} \\ 1 & 1 \end{array}\right]}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \left[\begin{array}{cc} \frac{\left(\left(-5+\sqrt{37}\right)\mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}}+\mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}}\left(5+\sqrt{37}\right)\right)\sqrt{37}}{74} & -\frac{3\left(-\mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}}+\mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}}\right)\sqrt{37}}{37} \\ -\frac{\left(-\mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}}+\mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}}\right)\sqrt{37}}{37} & \frac{\left(5\sqrt{37}+37\right)\mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}}}{74}+\frac{\left(-5\sqrt{37}+37\right)\mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}}}{74} \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

$\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \left[\begin{array}{c} -\frac{\left(\left(1+\sqrt{37}\right)\mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}}+\mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}}\left(-1+\sqrt{37}\right)-2\sqrt{37}\right)\sqrt{37}}{222} \\ \frac{\left(-7\sqrt{37}-37\right)\mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}}}{222}+\frac{1}{3}+\frac{\mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}}\left(7\sqrt{37}-37\right)}{222} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \left[\begin{array}{c} -\frac{\left(\left(1+\sqrt{37}\right)\mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}}+\mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}}\left(-1+\sqrt{37}\right)-2\sqrt{37}\right)\sqrt{37}}{222} \\ \frac{\left(-7\sqrt{37}-37\right)\mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}}}{222}+\frac{1}{3}+\frac{\mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}}\left(7\sqrt{37}-37\right)}{222} \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{\left((111c_2-1)\sqrt{37}-555c_2-37\right)\mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}}}{222}+\frac{1}{3}+\frac{\left((-111c_1+1)\sqrt{37}-555c_1-37\right)\mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}}}{222} \\ \frac{\left(222c_2-7\sqrt{37}-37\right)\mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}}}{222}+\frac{1}{3}+\frac{\left(222c_1+7\sqrt{37}-37\right)\mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}}}{222} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x = \frac{\left((111c_2-1)\sqrt{37}-555c_2-37\right)\mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}}}{222}+\frac{1}{3}+\frac{\left((-111c_1+1)\sqrt{37}-555c_1-37\right)\mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}}}{222}, y(t) = \frac{\left(222c_2-7\sqrt{37}-\right.}{2}\right.$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 88

```
dsolve([diff(x(t),t)=-3*x(t)+3*y(t),diff(y(t),t)=x(t)+2*y(t)-1],singsol=all)
```

$$x(t) = \mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}} c_2 + \mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}} c_1 + \frac{1}{3}$$

$$y(t) = \frac{\mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}} c_2\sqrt{37}}{6} - \frac{\mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}} c_1\sqrt{37}}{6} + \frac{5\,\mathrm{e}^{\frac{\left(-1+\sqrt{37}\right)t}{2}} c_2}{6} + \frac{5\,\mathrm{e}^{-\frac{\left(1+\sqrt{37}\right)t}{2}} c_1}{6} + \frac{1}{3}$$

✓ Solution by Mathematica

Time used: 0.67 (sec). Leaf size: 192

```
DSolve[{x'[t]==-3*x[t]+3*y[t],y'[t]==x[t]+2*y[t]-1},{x[t],y[t]},t,IncludeSingularSolutions -
```

$$x(t) \to \frac{1}{222} e^{-\frac{1}{2}\left(1+\sqrt{37}\right)t}\left(74e^{\frac{1}{2}\left(1+\sqrt{37}\right)t} - 3\left(\left(5\sqrt{37}-37\right)c_1 - 6\sqrt{37}c_2\right)e^{\sqrt{37}t} + 3\left(\left(37+5\sqrt{37}\right)c_1 - 6\sqrt{37}c_2\right)\right)$$

$$y(t) \to \frac{1}{222} e^{-\frac{1}{2}\left(1+\sqrt{37}\right)t}\left(74e^{\frac{1}{2}\left(1+\sqrt{37}\right)t} + 3\left(2\sqrt{37}c_1 + \left(37+5\sqrt{37}\right)c_2\right)e^{\sqrt{37}t} - 3\left(2\sqrt{37}c_1 + \left(5\sqrt{37}-37\right)c_2\right)\right)$$

20 Chapter 4, Linear Systems. Exercises page 218

20.1 problem 2(a) 2119
20.2 problem 2(b) 2128
20.3 problem 2(c) 2137
20.4 problem 2(d) 2146
20.5 problem 4 2155
20.6 problem 5 2163

20.1 problem 2(a)

20.1.1 Solution using Matrix exponential method 2119
20.1.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2120
20.1.3 Maple step by step solution 2125

Internal problem ID [11547]
Internal file name [`OUTPUT/10529_Thursday_May_18_2023_04_22_11_AM_96755911/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 218
Problem number: 2(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= -x + y(t) \\
y'(t) &= -3y(t)
\end{aligned}
$$

20.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t} & \frac{\mathrm{e}^{-t}}{2} - \frac{\mathrm{e}^{-3t}}{2} \\ 0 & \mathrm{e}^{-3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t} & \frac{e^{-t}}{2}-\frac{e^{-3t}}{2} \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}c_1+\left(\frac{e^{-t}}{2}-\frac{e^{-3t}}{2}\right)c_2 \\ e^{-3t}c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2c_1+c_2)e^{-t}}{2}-\frac{e^{-3t}c_2}{2} \\ e^{-3t}c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 1 \\ 0 & -3-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-1-\lambda)(-3-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \implies \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -3 is real and distinct then the

corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-3t} \\
&= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{-3t}
\end{aligned}
$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{\mathrm{e}^{-3t}}{2} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t} \\ 0 \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 \mathrm{e}^{-3t}}{2} + c_2 \mathrm{e}^{-t} \\ c_1 \mathrm{e}^{-3t} \end{bmatrix}
$$

The following is the phase plot of the system.

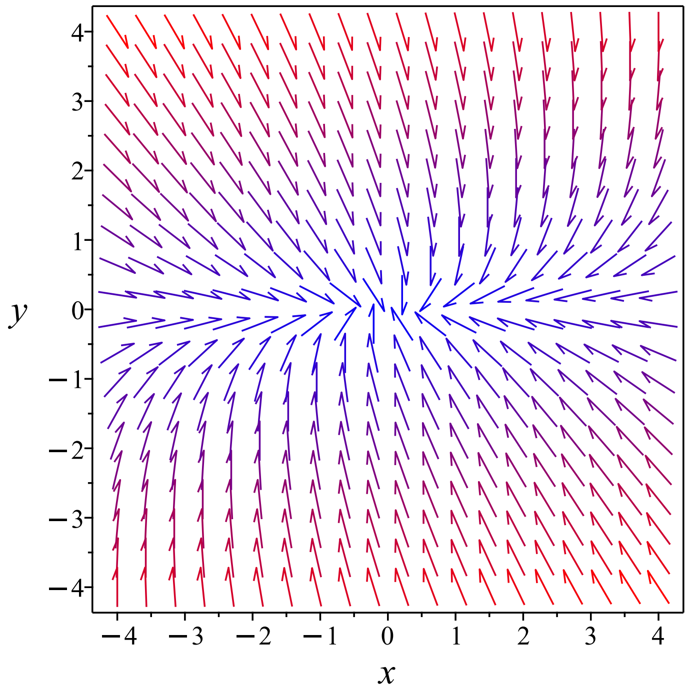


Figure 369: Phase plot

20.1.3 Maple step by step solution

Let's solve

$[x' = -x + y(t), y'(t) = -3y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}$$

- Rewrite the system as

  $\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-3, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-3, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_1 = \mathrm{e}^{-3t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[-1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_2 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

  $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

- Substitute solutions into the general solution

  $$\vec{x} = \mathrm{e}^{-3t} c_1 \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{e^{-3t}c_1}{2} + c_2 e^{-t} \\ e^{-3t}c_1 \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x = -\frac{e^{-3t}c_1}{2} + c_2 e^{-t}, y(t) = e^{-3t}c_1\right\}$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve([diff(x(t),t)=-x(t)+y(t),diff(y(t),t)=-3*y(t)],singsol=all)
```

$$x(t) = -\frac{c_2 e^{-3t}}{2} + e^{-t}c_1$$
$$y(t) = c_2 e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 43

```
DSolve[{x'[t]==-x[t]+y[t],y'[t]==-3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2}e^{-3t}\left((2c_1 + c_2)e^{2t} - c_2\right)$$
$$y(t) \to c_2 e^{-3t}$$

20.2 problem 2(b)

20.2.1 Solution using Matrix exponential method 2128
20.2.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2129
20.2.3 Maple step by step solution 2134

Internal problem ID [11548]
Internal file name [`OUTPUT/10530_Thursday_May_18_2023_04_22_13_AM_19836858/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 218
Problem number: 2(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x' = x$$
$$y'(t) = 3x - 4y(t)$$

20.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{t} & 0 \\ \frac{3(e^{5t}-1)e^{-4t}}{5} & e^{-4t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{t} & 0 \\ \frac{3(\mathrm{e}^{5t}-1)\mathrm{e}^{-4t}}{5} & \mathrm{e}^{-4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t} c_1 \\ \frac{3(\mathrm{e}^{5t}-1)\mathrm{e}^{-4t} c_1}{5} + \mathrm{e}^{-4t} c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t} c_1 \\ \frac{(3\,\mathrm{e}^{5t} c_1 - 3c_1 + 5c_2)\mathrm{e}^{-4t}}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & 0 \\ 3 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & 0 \\ 3 & -4-\lambda \end{bmatrix} \right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(1-\lambda)(-4-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -4$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
-4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 \\ 3 & -4 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 5 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 5 & 0 & 0 \\ 3 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{5} \Longrightarrow \left[\begin{array}{cc|c} 5 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 \\ 3 & -4 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 3 & -5 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 3 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{5t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{5t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{5t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{5t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{5t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-4	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-4t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-4t}\end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{t} \\ &= \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix} e^{t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{-4t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{5\,\mathrm{e}^{t}}{3} \\ \mathrm{e}^{t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{5 c_2 \mathrm{e}^{t}}{3} \\ \left(c_2 \mathrm{e}^{5t} + c_1\right) \mathrm{e}^{-4t} \end{bmatrix}$$

The following is the phase plot of the system.

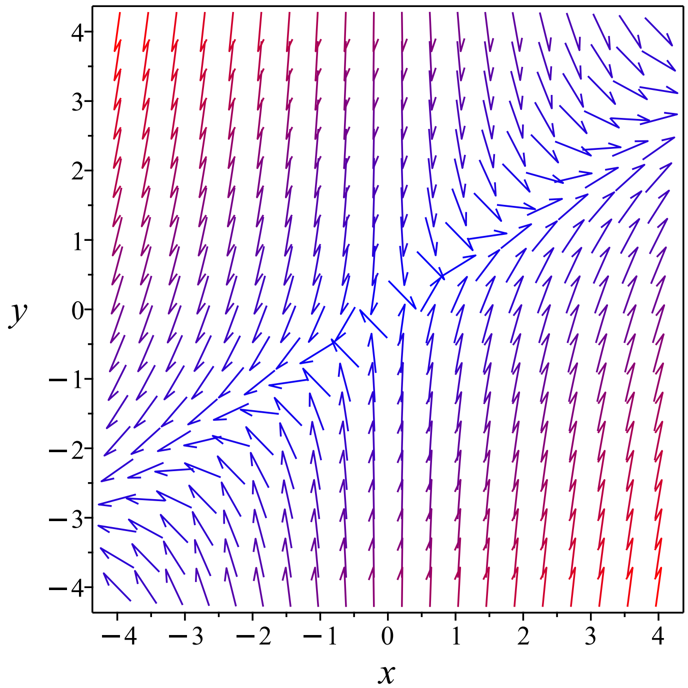


Figure 370: Phase plot

20.2.3 Maple step by step solution

Let's solve

$[x' = x, y'(t) = 3x - 4y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 3 & -4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 3 & -4 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 \\ 3 & -4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-4, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-4t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{t} \cdot \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{-4t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{t} \cdot \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{5c_2 e^t}{3} \\ (c_2 e^{5t} + c_1) e^{-4t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x = \frac{5c_2 e^t}{3}, y(t) = (c_2 e^{5t} + c_1) e^{-4t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve([diff(x(t),t)=x(t),diff(y(t),t)=3*x(t)-4*y(t)],singsol=all)
```

$$x(t) = c_2 e^t$$
$$y(t) = \frac{3c_2 e^t}{5} + e^{-4t} c_1$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 149

```
DSolve[{x'[t]==x[t]+y[t],y'[t]==3*x[t]-4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tru
```

$$x(t) \to \frac{1}{74} e^{-\frac{1}{2}\left(3+\sqrt{37}\right)t} \left(c_1\left(\left(37+5\sqrt{37}\right)e^{\sqrt{37}t} + 37 - 5\sqrt{37}\right) + 2\sqrt{37}c_2\left(e^{\sqrt{37}t} - 1\right)\right)$$
$$y(t) \to \frac{1}{74} e^{-\frac{1}{2}\left(3+\sqrt{37}\right)t} \left(6\sqrt{37}c_1\left(e^{\sqrt{37}t} - 1\right) - c_2\left(\left(5\sqrt{37} - 37\right)e^{\sqrt{37}t} - 37 - 5\sqrt{37}\right)\right)$$

20.3 problem 2(c)

20.3.1 Solution using Matrix exponential method 2137
20.3.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2138
20.3.3 Maple step by step solution 2143

Internal problem ID [11549]
Internal file name [`OUTPUT/10531_Thursday_May_18_2023_04_22_15_AM_5297857/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 218
Problem number: 2(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= -x + y(t) \\
y'(t) &= x - 2y(t)
\end{aligned}
$$

20.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{\left(5-\sqrt{5}\right) \mathrm{e}^{-\frac{\left(3+\sqrt{5}\right) t}{2}}}{10}+\frac{\mathrm{e}^{\frac{\left(\sqrt{5}-3\right) t}{2}} \left(5+\sqrt{5}\right)}{10} & -\frac{\left(-\mathrm{e}^{\frac{\left(\sqrt{5}-3\right) t}{2}}+\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right) t}{2}}\right) \sqrt{5}}{5} \\ -\frac{\left(-\mathrm{e}^{\frac{\left(\sqrt{5}-3\right) t}{2}}+\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right) t}{2}}\right) \sqrt{5}}{5} & \frac{\left(5+\sqrt{5}\right) \mathrm{e}^{-\frac{\left(3+\sqrt{5}\right) t}{2}}}{10}-\frac{\mathrm{e}^{\frac{\left(\sqrt{5}-3\right) t}{2}} \left(-5+\sqrt{5}\right)}{10} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{\left(5-\sqrt{5}\right)\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}}{10}+\frac{\mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}}\left(5+\sqrt{5}\right)}{10} & -\frac{\left(-\mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}}+\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}\right)\sqrt{5}}{5} \\ -\frac{\left(-\mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}}+\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}\right)\sqrt{5}}{5} & \frac{\left(5+\sqrt{5}\right)\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}}{10}-\frac{\mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}}\left(-5+\sqrt{5}\right)}{10} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\left(5-\sqrt{5}\right)\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}}{10}+\frac{\mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}}\left(5+\sqrt{5}\right)}{10}\right)c_1-\frac{\left(-\mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}}+\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}\right)\sqrt{5}\,c_2}{5} \\ -\frac{\left(-\mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}}+\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}\right)\sqrt{5}\,c_1}{5}+\left(\frac{\left(5+\sqrt{5}\right)\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}}{10}-\frac{\mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}}\left(-5+\sqrt{5}\right)}{10}\right)c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left((-c_1-2c_2)\sqrt{5}+5c_1\right)\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}}{10}+\frac{\mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}}\left((c_1+2c_2)\sqrt{5}+5c_1\right)}{10} \\ \frac{\left((-2c_1+c_2)\sqrt{5}+5c_2\right)\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}}{10}+\frac{\mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}}\left(\left(c_1-\frac{c_2}{2}\right)\sqrt{5}+\frac{5c_2}{2}\right)}{5} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}-1 & 1\\ 1 & -2\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-1-\lambda & 1\\ 1 & -2-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= \frac{\sqrt{5}}{2} - \frac{3}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{\sqrt{5}}{2}\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{3}{2} - \frac{\sqrt{5}}{2}$	1	real eigenvalue
$\frac{\sqrt{5}}{2} - \frac{3}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{3}{2} - \frac{\sqrt{5}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\begin{aligned}\left(\begin{bmatrix}-1 & 1\\ 1 & -2\end{bmatrix} - \left(-\frac{3}{2} - \frac{\sqrt{5}}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\\ \begin{bmatrix}\frac{1}{2} + \frac{\sqrt{5}}{2} & 1\\ 1 & \frac{\sqrt{5}}{2} - \frac{1}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\end{aligned}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}\frac{1}{2} + \frac{\sqrt{5}}{2} & 1 & 0\\ 1 & \frac{\sqrt{5}}{2} - \frac{1}{2} & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{\frac{1}{2}+\frac{\sqrt{5}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2}+\frac{\sqrt{5}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}+\frac{\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{\sqrt{5}+1}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{\sqrt{5}+1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}+1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}+1} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{\sqrt{5}}{2} - \frac{3}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} - \left(\frac{\sqrt{5}}{2} - \frac{3}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}-\frac{\sqrt{5}}{2} & 1 \\ 1 & -\frac{1}{2}-\frac{\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2}-\frac{\sqrt{5}}{2} & 1 & 0 \\ 1 & -\frac{1}{2}-\frac{\sqrt{5}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{\frac{1}{2}-\frac{\sqrt{5}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2}-\frac{\sqrt{5}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}-\frac{\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{\sqrt{5}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{\sqrt{5}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}-1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}-1} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$\frac{\sqrt{5}}{2}-\frac{3}{2}$	1	1	No	$\begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1 \end{bmatrix}$
$-\frac{3}{2}-\frac{\sqrt{5}}{2}$	1	1	No	$\begin{bmatrix} \frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{\sqrt{5}}{2}-\frac{3}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t} \\
&= \begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1 \end{bmatrix} e^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t}
\end{aligned}
$$

Since eigenvalue $-\frac{3}{2}-\frac{\sqrt{5}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \\
&= \begin{bmatrix} \frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix} e^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{\mathrm{e}^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t}}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ \mathrm{e}^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{\mathrm{e}^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ \mathrm{e}^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \end{bmatrix}
$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_2\left(\sqrt{5}-1\right)e^{-\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} + \frac{c_1 e^{\frac{\left(\sqrt{5}-3\right)t}{2}}\left(\sqrt{5}+1\right)}{2} \\ c_1 e^{\frac{\left(\sqrt{5}-3\right)t}{2}} + c_2 e^{-\frac{\left(3+\sqrt{5}\right)t}{2}} \end{bmatrix}$$

The following is the phase plot of the system.

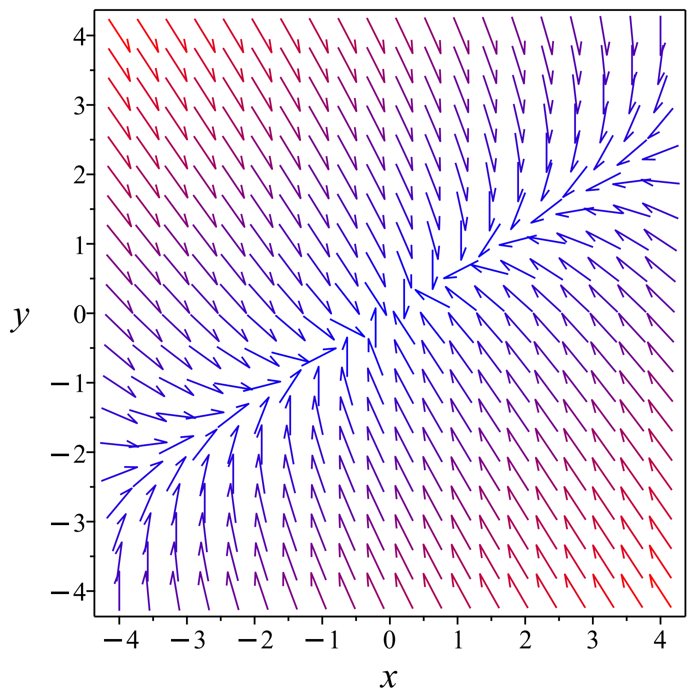


Figure 371: Phase plot

20.3.3 Maple step by step solution

Let's solve

$[x' = -x + y(t), y'(t) = x - 2y(t)]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{3}{2} - \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{-\frac{1}{2} - \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{\sqrt{5}}{2} - \frac{3}{2}, \begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2} - \frac{1}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{3}{2} - \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{-\frac{1}{2} - \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{\left(-\frac{3}{2} - \frac{\sqrt{5}}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{-\frac{1}{2} - \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\frac{\sqrt{5}}{2} - \frac{3}{2}, \begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2} - \frac{1}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{\left(\frac{\sqrt{5}}{2} - \frac{3}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2} - \frac{1}{2}} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs
$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$
- Substitute solutions into the general solution
$$\vec{x} = c_1 e^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix} + c_2 e^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1 \end{bmatrix}$$
- Substitute in vector of dependent variables
$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1\left(\sqrt{5}-1\right)e^{-\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} + \frac{c_2 e^{\frac{\left(\sqrt{5}-3\right)t}{2}}\left(\sqrt{5}+1\right)}{2} \\ c_1 e^{-\frac{\left(3+\sqrt{5}\right)t}{2}} + c_2 e^{\frac{\left(\sqrt{5}-3\right)t}{2}} \end{bmatrix}$$
- Solution to the system of ODEs
$$\left\{x = -\frac{c_1\left(\sqrt{5}-1\right)e^{-\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} + \frac{c_2 e^{\frac{\left(\sqrt{5}-3\right)t}{2}}\left(\sqrt{5}+1\right)}{2}, y(t) = c_1 e^{-\frac{\left(3+\sqrt{5}\right)t}{2}} + c_2 e^{\frac{\left(\sqrt{5}-3\right)t}{2}}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 86

```
dsolve([diff(x(t),t)=-x(t)+y(t),diff(y(t),t)=x(t)-2*y(t)],singsol=all)
```

$$x(t) = c_1 e^{\frac{\left(\sqrt{5}-3\right)t}{2}} + c_2 e^{-\frac{\left(3+\sqrt{5}\right)t}{2}}$$
$$y(t) = \frac{c_1 e^{\frac{\left(\sqrt{5}-3\right)t}{2}}\sqrt{5}}{2} - \frac{c_2 e^{-\frac{\left(3+\sqrt{5}\right)t}{2}}\sqrt{5}}{2} - \frac{c_1 e^{\frac{\left(\sqrt{5}-3\right)t}{2}}}{2} - \frac{c_2 e^{-\frac{\left(3+\sqrt{5}\right)t}{2}}}{2}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 145

```
DSolve[{x'[t]==-x[t]+y[t],y'[t]==x[t]-2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True
```

$$x(t) \to \frac{1}{10}e^{-\frac{1}{2}\left(3+\sqrt{5}\right)t}\left(c_1\left(\left(5+\sqrt{5}\right)e^{\sqrt{5}t} + 5 - \sqrt{5}\right) + 2\sqrt{5}c_2\left(e^{\sqrt{5}t} - 1\right)\right)$$
$$y(t) \to \frac{1}{10}e^{-\frac{1}{2}\left(3+\sqrt{5}\right)t}\left(2\sqrt{5}c_1\left(e^{\sqrt{5}t} - 1\right) - c_2\left(\left(\sqrt{5}-5\right)e^{\sqrt{5}t} - 5 - \sqrt{5}\right)\right)$$

20.4 problem 2(d)

20.4.1 Solution using Matrix exponential method 2146
20.4.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2147
20.4.3 Maple step by step solution 2152

Internal problem ID [11550]
Internal file name [`OUTPUT/10532_Thursday_May_18_2023_04_22_17_AM_84876972/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 218
Problem number: 2(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= x + y(t) \\
y'(t) &= -3x + 3y(t)
\end{aligned}
$$

20.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{2t}\cos\left(\sqrt{2}\,t\right)-\frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)e^{2t}}{2} & \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)e^{2t}}{2} \\ -\frac{3\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)e^{2t}}{2} & e^{2t}\cos\left(\sqrt{2}\,t\right)+\frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)e^{2t}}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{e^{2t}\left(\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)-2\cos\left(\sqrt{2}\,t\right)\right)}{2} & \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)e^{2t}}{2} \\ -\frac{3\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)e^{2t}}{2} & \frac{e^{2t}\left(\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)+2\cos\left(\sqrt{2}\,t\right)\right)}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{e^{2t}\left(\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)-2\cos\left(\sqrt{2}\,t\right)\right)}{2} & \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)e^{2t}}{2} \\ -\frac{3\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)e^{2t}}{2} & \frac{e^{2t}\left(\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)+2\cos\left(\sqrt{2}\,t\right)\right)}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{e^{2t}\left(\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)-2\cos\left(\sqrt{2}\,t\right)\right)c_1}{2}+\frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)e^{2t}c_2}{2} \\ -\frac{3\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)e^{2t}c_1}{2}+\frac{e^{2t}\left(\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)+2\cos\left(\sqrt{2}\,t\right)\right)c_2}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{e^{2t}\left(\sqrt{2}\,(c_1-c_2)\sin\left(\sqrt{2}\,t\right)-2\cos\left(\sqrt{2}\,t\right)c_1\right)}{2} \\ -\frac{3\left(\sqrt{2}\,\left(c_1-\frac{c_2}{3}\right)\sin\left(\sqrt{2}\,t\right)-\frac{2\cos\left(\sqrt{2}\,t\right)c_2}{3}\right)e^{2t}}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A-\lambda I\right)=0
$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ -3 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2 + i\sqrt{2}$$
$$\lambda_2 = 2 - i\sqrt{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2 - i\sqrt{2}$	1	complex eigenvalue
$2 + i\sqrt{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2 - i\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix} - \left(2 - i\sqrt{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} i\sqrt{2}-1 & 1 \\ -3 & 1+i\sqrt{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} i\sqrt{2}-1 & 1 & 0 \\ -3 & 1+i\sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{i\sqrt{2}-1} \implies \left[\begin{array}{cc|c} i\sqrt{2}-1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i\sqrt{2}-1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{i\sqrt{2}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{\mathrm{I}\sqrt{2}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{i\sqrt{2}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{\mathrm{I}\sqrt{2}-1} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{i\sqrt{2}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{\mathrm{I}\sqrt{2}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{i\sqrt{2}-1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{\mathrm{I}\sqrt{2}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{i\sqrt{2}-1} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 + i\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix} - \left(2 + i\sqrt{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 - i\sqrt{2} & 1 \\ -3 & 1 - i\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1-i\sqrt{2} & 1 & 0 \\ -3 & 1-i\sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{-1-i\sqrt{2}} \Longrightarrow \left[\begin{array}{cc|c} -1-i\sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-i\sqrt{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{1+i\sqrt{2}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{1+i\sqrt{2}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$2+i\sqrt{2}$	1	1	No	$\begin{bmatrix} \frac{1}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$
$2-i\sqrt{2}$	1	1	No	$\begin{bmatrix} \frac{1}{1-i\sqrt{2}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{\left(2+i\sqrt{2}\right)t}}{1+i\sqrt{2}} \\ e^{\left(2+i\sqrt{2}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^{\left(2-i\sqrt{2}\right)t}}{1-i\sqrt{2}} \\ e^{\left(2-i\sqrt{2}\right)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{i\left(i-\sqrt{2}\right)c_2 e^{-\left(i\sqrt{2}-2\right)t}}{3} - \frac{i e^{\left(2+i\sqrt{2}\right)t}\left(i+\sqrt{2}\right)c_1}{3} \\ c_1 e^{\left(2+i\sqrt{2}\right)t} + c_2 e^{-\left(i\sqrt{2}-2\right)t} \end{bmatrix}$$

The following is the phase plot of the system.

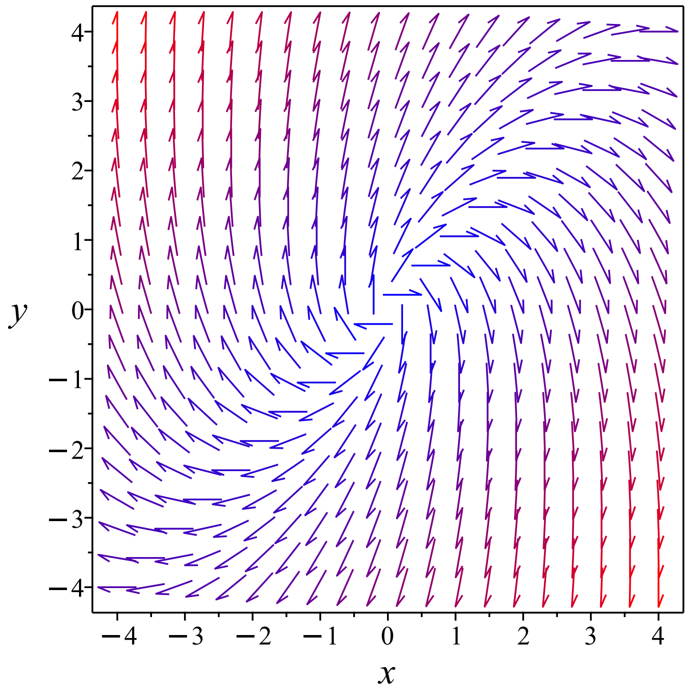


Figure 372: Phase plot

20.4.3 Maple step by step solution

Let's solve

$[x' = x + y(t), y'(t) = -3x + 3y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[2-\mathrm{I}\sqrt{2}, \begin{bmatrix} \frac{1}{1-\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}\right], \left[2+\mathrm{I}\sqrt{2}, \begin{bmatrix} \frac{1}{1+\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[2-\mathrm{I}\sqrt{2}, \begin{bmatrix} \frac{1}{1-\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

  $$\mathrm{e}^{\left(2-\mathrm{I}\sqrt{2}\right)t} \cdot \begin{bmatrix} \frac{1}{1-\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$\mathrm{e}^{2t} \cdot \left(\cos\left(\sqrt{2}\,t\right) - \mathrm{I}\sin\left(\sqrt{2}\,t\right)\right) \cdot \begin{bmatrix} \frac{1}{1-\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}$$

- Simplify expression

  $$\mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{\cos\left(\sqrt{2}\,t\right)-\mathrm{I}\sin\left(\sqrt{2}\,t\right)}{1-\mathrm{I}\sqrt{2}} \\ \cos\left(\sqrt{2}\,t\right) - \mathrm{I}\sin\left(\sqrt{2}\,t\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\overrightarrow{x}_1(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{3} \\ \cos\left(\sqrt{2}\,t\right) \end{bmatrix}, \overrightarrow{x}_2(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{\sqrt{2}\,\cos\left(\sqrt{2}\,t\right)}{3} - \frac{\sin\left(\sqrt{2}\,t\right)}{3} \\ -\sin\left(\sqrt{2}\,t\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t)$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{2t} \cdot \begin{bmatrix} \frac{\cos(\sqrt{2}\,t)}{3} + \frac{\sqrt{2}\,\sin(\sqrt{2}\,t)}{3} \\ \cos(\sqrt{2}\,t) \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} \frac{\sqrt{2}\,\cos(\sqrt{2}\,t)}{3} - \frac{\sin(\sqrt{2}\,t)}{3} \\ -\sin(\sqrt{2}\,t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{2t}\left(\left(\sqrt{2}\,c_2+c_1\right)\cos\left(\sqrt{2}\,t\right)+\sin\left(\sqrt{2}\,t\right)\left(\sqrt{2}\,c_1-c_2\right)\right)}{3} \\ e^{2t}\left(\cos(\sqrt{2}\,t)\,c_1 - c_2\sin(\sqrt{2}\,t)\right) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x = \frac{e^{2t}\left(\left(\sqrt{2}\,c_2+c_1\right)\cos\left(\sqrt{2}\,t\right)+\sin\left(\sqrt{2}\,t\right)\left(\sqrt{2}\,c_1-c_2\right)\right)}{3}, y(t) = e^{2t}\left(\cos(\sqrt{2}\,t)\,c_1 - c_2\sin(\sqrt{2}\,t)\right)\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 78

```
dsolve([diff(x(t),t)=x(t)+y(t),diff(y(t),t)=-3*x(t)+3*y(t)],singsol=all)
```

$$x(t) = e^{2t}\left(c_1\sin\left(\sqrt{2}\,t\right) + c_2\cos\left(\sqrt{2}\,t\right)\right)$$
$$y(t) = -e^{2t}\left(\sin\left(\sqrt{2}\,t\right)\sqrt{2}\,c_2 - \cos\left(\sqrt{2}\,t\right)\sqrt{2}\,c_1 - c_1\sin\left(\sqrt{2}\,t\right) - c_2\cos\left(\sqrt{2}\,t\right)\right)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 94

```
DSolve[{x'[t]==x[t]+y[t],y'[t]==-3*x[t]+3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tr
```

$$x(t) \to \frac{1}{2}e^{2t}\left(2c_1\cos\left(\sqrt{2}t\right) + \sqrt{2}(c_2 - c_1)\sin\left(\sqrt{2}t\right)\right)$$
$$y(t) \to \frac{1}{2}e^{2t}\left(2c_2\cos\left(\sqrt{2}t\right) + \sqrt{2}(c_2 - 3c_1)\sin\left(\sqrt{2}t\right)\right)$$

20.5 problem 4

20.5.1 Solution using Matrix exponential method 2155
20.5.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2156

Internal problem ID [11551]
Internal file name [`OUTPUT/10533_Thursday_May_18_2023_04_22_19_AM_85190726/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 218
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= x - 2y(t) \\
y'(t) &= 3x - 4y(t)
\end{aligned}
$$

With initial conditions

$$
[x(0) = 3, y(0) = 1]
$$

20.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} -2\,\mathrm{e}^{-2t} + 3\,\mathrm{e}^{-t} & -2\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t} \\ 3\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{-2t} & 3\,\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} -2\,\mathrm{e}^{-2t} + 3\,\mathrm{e}^{-t} & -2\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t} \\ 3\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{-2t} & 3\,\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -4\,\mathrm{e}^{-2t} + 7\,\mathrm{e}^{-t} \\ 7\,\mathrm{e}^{-t} - 6\,\mathrm{e}^{-2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 3\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -2 & 0 \\ 3 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix}\frac{2t}{3}\\ t\end{bmatrix}=\begin{bmatrix}\frac{2t}{3}\\ t\end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix}\frac{2t}{3}\\ t\end{bmatrix}=t\begin{bmatrix}\frac{2}{3}\\ 1\end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix}\frac{2t}{3}\\ t\end{bmatrix}=\begin{bmatrix}\frac{2}{3}\\ 1\end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix}\frac{2t}{3}\\ t\end{bmatrix}=\begin{bmatrix}2\\ 3\end{bmatrix}$$

Considering the eigenvalue $\lambda_2=-1$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}1&-2\\ 3&-4\end{bmatrix}-(-1)\begin{bmatrix}1&0\\ 0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}2&-2\\ 3&-3\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}2&-2&0\\ 3&-3&0\end{array}\right]$$

$$R_2=R_2-\frac{3R_1}{2}\implies\left[\begin{array}{cc|c}2&-2&0\\ 0&0&0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}2&-2\\ 0&0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-2t} \\ &= \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} e^{-2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{2\,\mathrm{e}^{-2t}}{3} \\ \mathrm{e}^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{-t} + \frac{2 c_2 \mathrm{e}^{-2t}}{3} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 3 \\ y(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + \frac{2c_2}{3} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 7 \\ c_2 = -6 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -4\,\mathrm{e}^{-2t} + 7\,\mathrm{e}^{-t} \\ 7\,\mathrm{e}^{-t} - 6\,\mathrm{e}^{-2t} \end{bmatrix}$$

The following is the phase plot of the system.

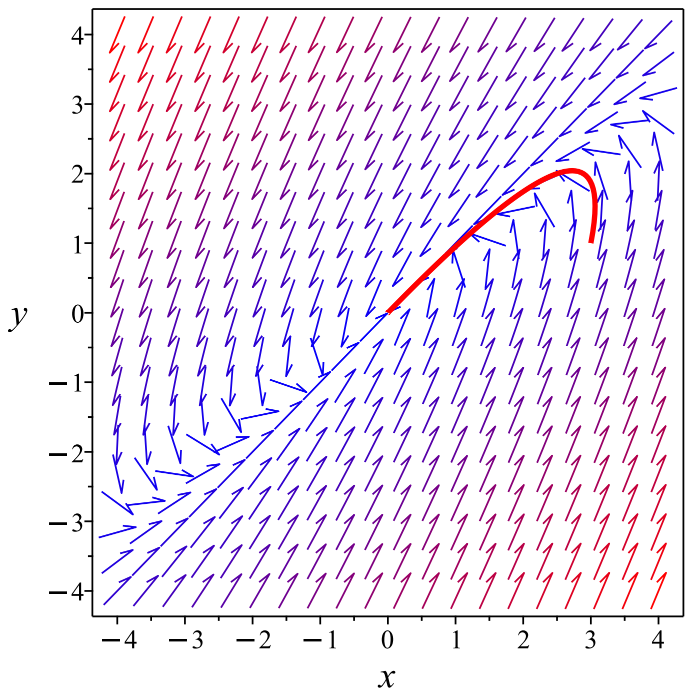
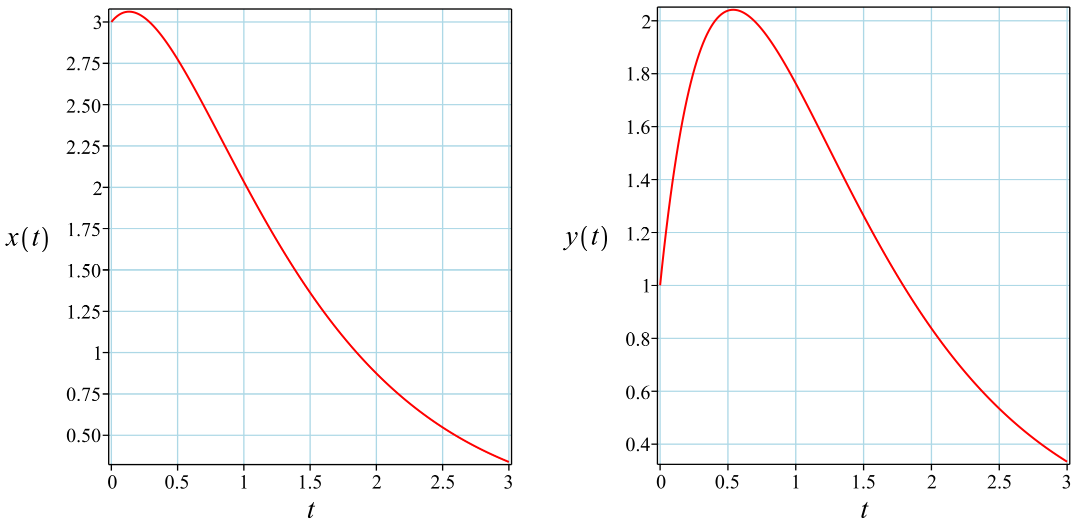


Figure 373: Phase plot

The following are plots of each solution.



✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve([diff(x(t),t) = x(t)-2*y(t), diff(y(t),t) = 3*x(t)-4*y(t), x(0) = 3, y(0) = 1], sings
```

$$x(t) = -4\,\mathrm{e}^{-2t} + 7\,\mathrm{e}^{-t}$$
$$y(t) = -6\,\mathrm{e}^{-2t} + 7\,\mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 34

```
DSolve[{x'[t]==x[t]-2*y[t],y'[t]==3*x[t]-4*y[t]},{x[0]==3,y[0]==1},{x[t],y[t]},t,IncludeSing
```

$$x(t) \to e^{-2t}(7e^t - 4)$$
$$y(t) \to e^{-2t}(7e^t - 6)$$

20.6 problem 5

20.6.1 Solution using Matrix exponential method 2163
20.6.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2164

Internal problem ID [11552]
Internal file name [`OUTPUT/10534_Thursday_May_18_2023_04_22_20_AM_89854620/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 218
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= 5x - y(t) \\
y'(t) &= 3x + y(t)
\end{aligned}
$$

With initial conditions

$$
[x(0) = 2, y(0) = -1]
$$

20.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} -\frac{e^{2t}}{2} + \frac{3\,e^{4t}}{2} & -\frac{e^{4t}}{2} + \frac{e^{2t}}{2} \\ \frac{3\,e^{4t}}{2} - \frac{3\,e^{2t}}{2} & \frac{3\,e^{2t}}{2} - \frac{e^{4t}}{2} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} -\frac{e^{2t}}{2} + \frac{3\,e^{4t}}{2} & -\frac{e^{4t}}{2} + \frac{e^{2t}}{2} \\ \frac{3\,e^{4t}}{2} - \frac{3\,e^{2t}}{2} & \frac{3\,e^{2t}}{2} - \frac{e^{4t}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{3\,e^{2t}}{2} + \frac{7\,e^{4t}}{2} \\ \frac{7\,e^{4t}}{2} - \frac{9\,e^{2t}}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 6\lambda + 8 = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
4	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 4 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{4t}\\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}\end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{2t}\\ &= \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^{2t}}{3} \\ e^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{4t} + \frac{c_2 e^{2t}}{3} \\ c_1 e^{4t} + c_2 e^{2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 2 \\ y(0) = -1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + \frac{c_2}{3} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{7}{2} \\ c_2 = -\frac{9}{2} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{3\,\mathrm{e}^{2t}}{2} + \frac{7\,\mathrm{e}^{4t}}{2} \\ \frac{7\,\mathrm{e}^{4t}}{2} - \frac{9\,\mathrm{e}^{2t}}{2} \end{bmatrix}$$

The following is the phase plot of the system.

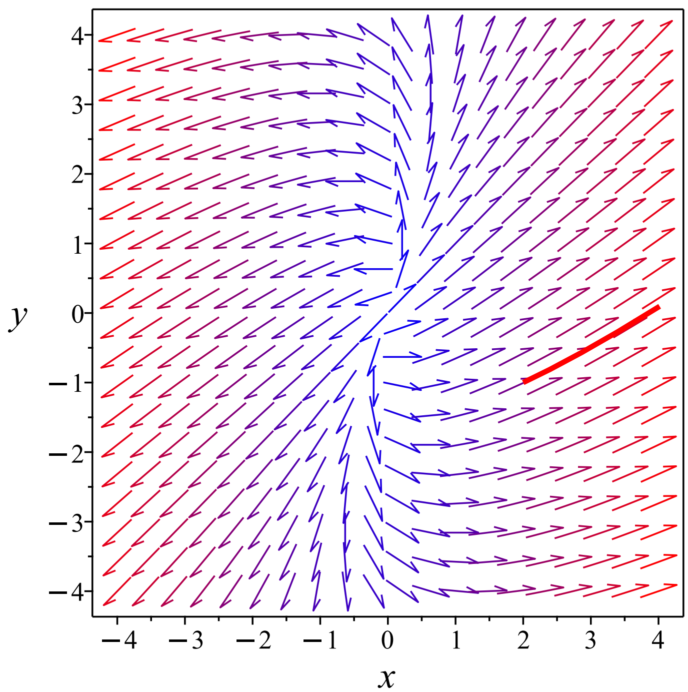
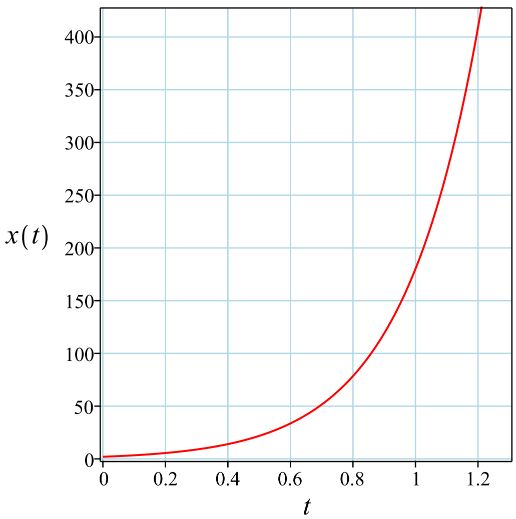
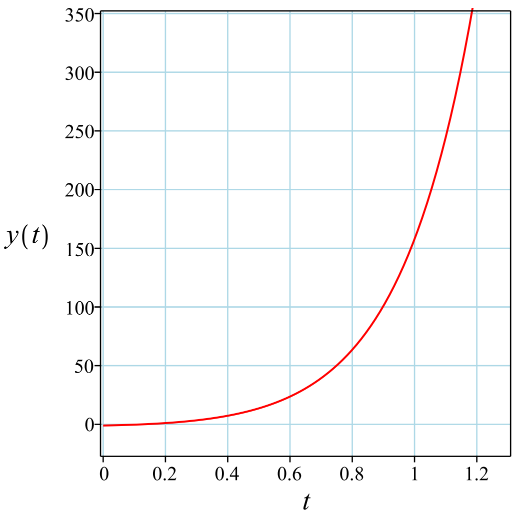


Figure 374: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve([diff(x(t),t) = 5*x(t)-y(t), diff(y(t),t) = 3*x(t)+y(t), x(0) = 2, y(0) = -1], singso
```

$$x(t) = \frac{7\,e^{4t}}{2} - \frac{3\,e^{2t}}{2}$$
$$y(t) = \frac{7\,e^{4t}}{2} - \frac{9\,e^{2t}}{2}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 44

```
DSolve[{x'[t]==5*x[t]-y[t],y'[t]==3*x[t]+y[t]},{x[0]==2,y[0]==-1},{x[t],y[t]},t,IncludeSingu
```

$$x(t) \to \frac{1}{2}e^{2t}\left(7e^{2t} - 3\right)$$
$$y(t) \to \frac{1}{2}e^{2t}\left(7e^{2t} - 9\right)$$

21 Chapter 4, Linear Systems. Exercises page 225

21.1 problem 1(a) 2172
21.2 problem 1(b) 2179

21.1 problem 1(a)

21.1.1 Solution using Matrix exponential method 2172
21.1.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2173

Internal problem ID [11553]
Internal file name [`OUTPUT/10535_Thursday_May_18_2023_04_22_22_AM_42491885/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 225
Problem number: 1(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= -3x + y(t) \\
y'(t) &= -3y(t)
\end{aligned}
$$

21.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{-3t} & t\,\mathrm{e}^{-3t} \\ 0 & \mathrm{e}^{-3t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-3t} & t\,e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-3t}c_1 + t\,e^{-3t}c_2 \\ e^{-3t}c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-3t}(c_2 t + c_1) \\ e^{-3t}c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -3-\lambda & 1 \\ 0 & -3-\lambda \end{bmatrix}\right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-3-\lambda)(-3-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-3$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-3 & 1 \\ 0 & -3\end{bmatrix}-(-3)\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1 \\ v_2\end{bmatrix}=\begin{bmatrix}0 \\ 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1 \\ v_2\end{bmatrix}=\begin{bmatrix}0 \\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}0 & 1 & 0 \\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1 \\ v_2\end{bmatrix}=\begin{bmatrix}0 \\ 0\end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1=t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2=0\}$

Hence the solution is

$$\begin{bmatrix}t \\ 0\end{bmatrix}=\begin{bmatrix}t \\ 0\end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	2	1	Yes	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

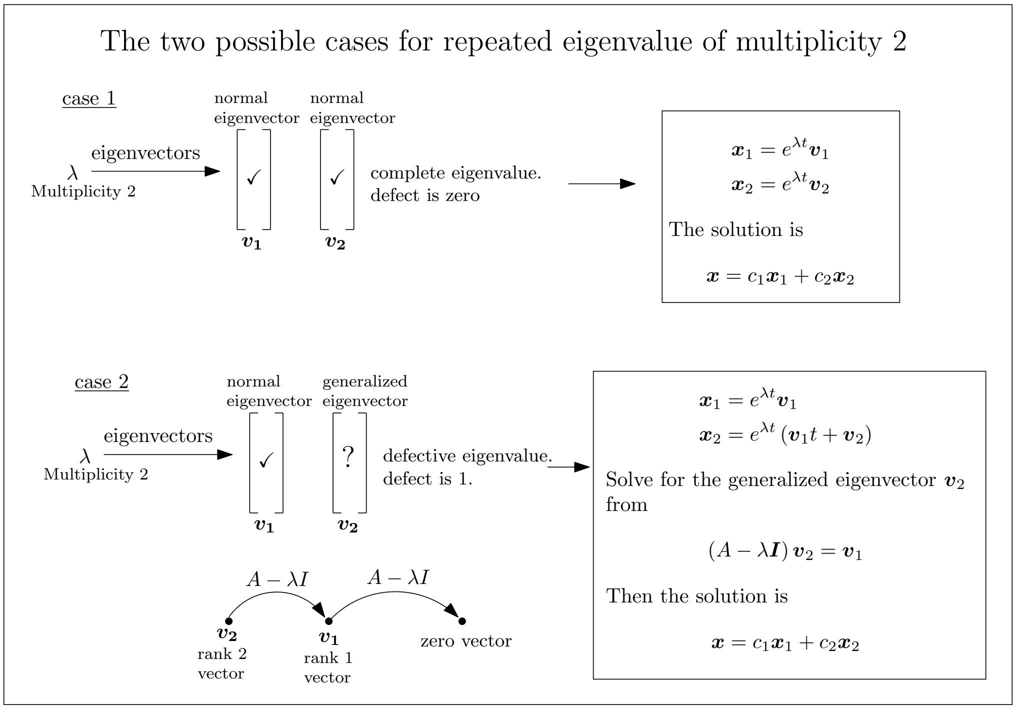


Figure 375: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -3. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathrm{e}^{-3t} \\
&= \begin{bmatrix} \mathrm{e}^{-3t} \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \mathrm{e}^{-3t} \\
&= \begin{bmatrix} \mathrm{e}^{-3t}(1+t) \\ \mathrm{e}^{-3t} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-3t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-3t}(1+t) \\ \mathrm{e}^{-3t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-3t}(c_2 t + c_1 + c_2) \\ c_2 \mathrm{e}^{-3t} \end{bmatrix}
$$

The following is the phase plot of the system.

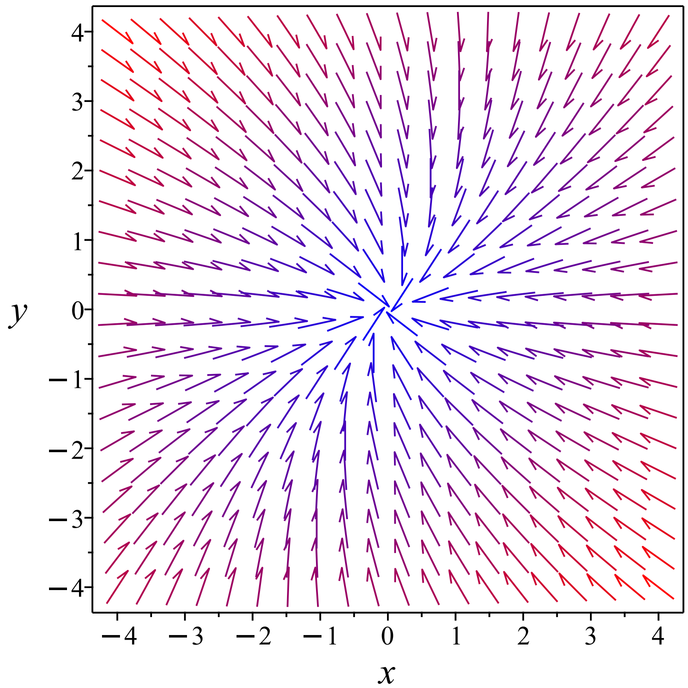


Figure 376: Phase plot

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 24

```
dsolve([diff(x(t),t)=-3*x(t)+y(t),diff(y(t),t)=-3*y(t)],singsol=all)
```

$$x(t) = (c_2t + c_1)\,\mathrm{e}^{-3t}$$
$$y(t) = c_2\mathrm{e}^{-3t}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 29

```
DSolve[{x'[t]==-3*x[t]+y[t],y'[t]==-3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-3t}(c_2t + c_1)$$
$$y(t) \to c_2e^{-3t}$$

21.2 problem 1(b)

21.2.1 Solution using Matrix exponential method 2179
21.2.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2180
21.2.3 Maple step by step solution 2185

Internal problem ID [11554]
Internal file name [`OUTPUT/10536_Thursday_May_18_2023_04_22_24_AM_43227928/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 225
Problem number: 1(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x' = x - y(t)$$
$$y'(t) = x + 3y(t)$$

21.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t}(1-t) & -e^{2t}t \\ e^{2t}t & e^{2t}(1+t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t}(1-t) & -e^{2t}t \\ e^{2t}t & e^{2t}(1+t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(1-t)\, c_1 - e^{2t}tc_2 \\ e^{2t}tc_1 + e^{2t}(1+t)\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} -e^{2t}(c_1(t-1) + c_2t) \\ e^{2t}(tc_1 + c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} \right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

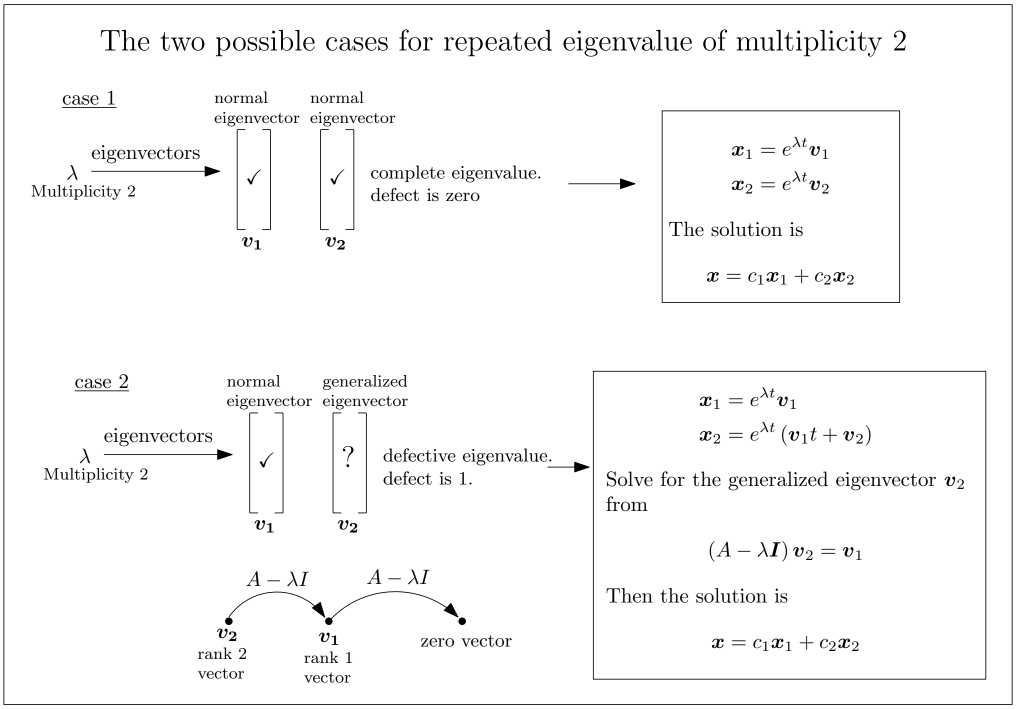


Figure 377: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \mathrm{e}^{2t} \\
&= \begin{bmatrix} -\mathrm{e}^{2t} t \\ \mathrm{e}^{2t}(1+t) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{2t} t \\ \mathrm{e}^{2t}(1+t) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{2t}(-t c_2 - c_1) \\ \mathrm{e}^{2t}(t c_2 + c_1 + c_2) \end{bmatrix}
$$

The following is the phase plot of the system.

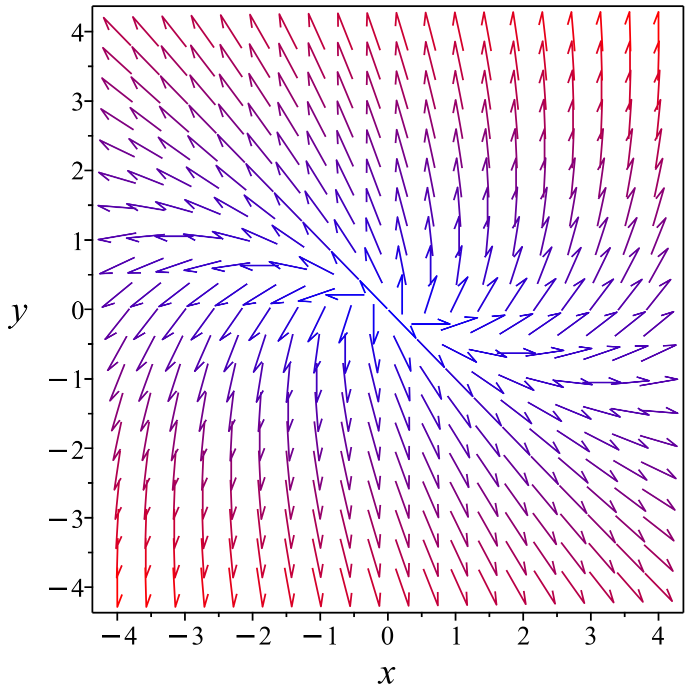


Figure 378: Phase plot

21.2.3 Maple step by step solution

Let's solve

$[x' = x - y(t), y'(t) = x + 3y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

  $$\overrightarrow{x}_1(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, and

  $\overrightarrow{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$

- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - 2\cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\cdot \vec{p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

$$\vec{x}_2(t) = e^{2t}\cdot \left(t\cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{2t}\cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t}\cdot \left(t\cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -((t-1)\, c_2 + c_1)\, e^{2t} \\ e^{2t}(c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x = -((t-1)\, c_2 + c_1)\, e^{2t}, y(t) = e^{2t}(c_2 t + c_1)\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve([diff(x(t),t)=x(t)-y(t),diff(y(t),t)=x(t)+3*y(t)],singsol=all)
```

$$x(t) = e^{2t}(c_2 t + c_1)$$
$$y(t) = -e^{2t}(c_2 t + c_1 + c_2)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 42

```
DSolve[{x'[t]==x[t]-y[t],y'[t]==x[t]+3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to -e^{2t}(c_1(t-1) + c_2 t)$$
$$y(t) \to e^{2t}((c_1 + c_2)t + c_2)$$

22 Chapter 4, Linear Systems. Exercises page 237

22.1 problem 4(a) 2190
22.2 problem 4(b) 2199
22.3 problem 4(c) 2206
22.4 problem 4(d) 2215
22.5 problem 4(e) 2224
22.6 problem 4(f) 2233
22.7 problem 4(g) 2242
22.8 problem 4(h) 2252
22.9 problem 5 2260
22.10 problem 6 2268

22.1 problem 4(a)

22.1.1 Solution using Matrix exponential method 2190
22.1.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2191
22.1.3 Maple step by step solution 2196

Internal problem ID [11555]
Internal file name [`OUTPUT/10537_Thursday_May_18_2023_04_22_26_AM_44003055/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 237
Problem number: 4(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= x + 2y(t) \\
y'(t) &= 3x + 2y(t)
\end{aligned}
$$

22.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{3\,\mathrm{e}^{-t}}{5} + \frac{2\,\mathrm{e}^{4t}}{5} & \frac{2\,\mathrm{e}^{4t}}{5} - \frac{2\,\mathrm{e}^{-t}}{5} \\ \frac{3\,\mathrm{e}^{4t}}{5} - \frac{3\,\mathrm{e}^{-t}}{5} & \frac{2\,\mathrm{e}^{-t}}{5} + \frac{3\,\mathrm{e}^{4t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{3\,\mathrm{e}^{-t}}{5}+\frac{2\,\mathrm{e}^{4t}}{5} & \frac{2\,\mathrm{e}^{4t}}{5}-\frac{2\,\mathrm{e}^{-t}}{5} \\ \frac{3\,\mathrm{e}^{4t}}{5}-\frac{3\,\mathrm{e}^{-t}}{5} & \frac{2\,\mathrm{e}^{-t}}{5}+\frac{3\,\mathrm{e}^{4t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{3\,\mathrm{e}^{-t}}{5}+\frac{2\,\mathrm{e}^{4t}}{5}\right)c_1+\left(\frac{2\,\mathrm{e}^{4t}}{5}-\frac{2\,\mathrm{e}^{-t}}{5}\right)c_2 \\ \left(\frac{3\,\mathrm{e}^{4t}}{5}-\frac{3\,\mathrm{e}^{-t}}{5}\right)c_1+\left(\frac{2\,\mathrm{e}^{-t}}{5}+\frac{3\,\mathrm{e}^{4t}}{5}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(3c_1-2c_2)\mathrm{e}^{-t}}{5}+\frac{2(c_1+c_2)\mathrm{e}^{4t}}{5} \\ \frac{(-3c_1+2c_2)\mathrm{e}^{-t}}{5}+\frac{3(c_1+c_2)\mathrm{e}^{4t}}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda - 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
4	1	1	No	$\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{4t} \\
&= \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} e^{4t}
\end{aligned}
$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{2\,\mathrm{e}^{4t}}{3} \\ \mathrm{e}^{4t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{2c_1\mathrm{e}^{4t}}{3} - c_2\mathrm{e}^{-t} \\ c_1\mathrm{e}^{4t} + c_2\mathrm{e}^{-t} \end{bmatrix}
$$

The following is the phase plot of the system.

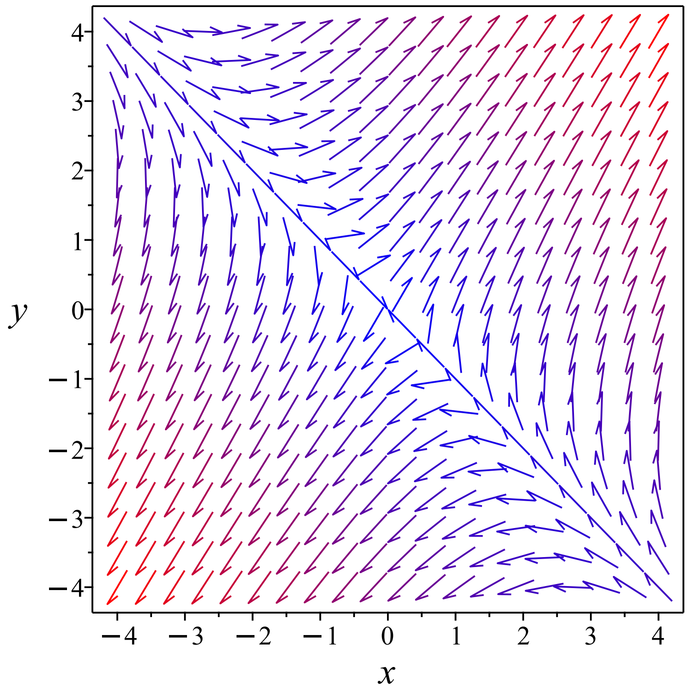


Figure 379: Phase plot

22.1.3 Maple step by step solution

Let's solve

$[x' = x + 2y(t), y'(t) = 3x + 2y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{4t} \cdot \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{4t} \cdot \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{-t} + \frac{2c_2 \mathrm{e}^{4t}}{3} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{4t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x = -c_1 \mathrm{e}^{-t} + \frac{2c_2 \mathrm{e}^{4t}}{3}, y(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{4t}\right\}$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=x(t)+2*y(t),diff(y(t),t)=3*x(t)+2*y(t)],singsol=all)
```

$$x(t) = c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{-t}$$
$$y(t) = \frac{3c_1 \mathrm{e}^{4t}}{2} - c_2 \mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 74

```
DSolve[{x'[t]==x[t]+2*y[t],y'[t]==3*x[t]+2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to \frac{1}{5}e^{-t}\left(c_1\left(2e^{5t}+3\right)+2c_2\left(e^{5t}-1\right)\right)$$
$$y(t) \to \frac{1}{5}e^{-t}\left(3c_1\left(e^{5t}-1\right)+c_2\left(3e^{5t}+2\right)\right)$$

22.2 problem 4(b)

22.2.1 Solution using Matrix exponential method 2199
22.2.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2200

Internal problem ID [11556]
Internal file name [`OUTPUT/10538_Thursday_May_18_2023_04_22_27_AM_7666863/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 237
Problem number: 4(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= -3x + 4y(t) \\
y'(t) &= -3y(t)
\end{aligned}
$$

22.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-3t} & 4t\,\mathrm{e}^{-3t} \\ 0 & \mathrm{e}^{-3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-3t} & 4t\,e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-3t}c_1 + 4t\,e^{-3t}c_2 \\ e^{-3t}c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-3t}(4c_2 t + c_1) \\ e^{-3t}c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -3 & 4 \\ 0 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -3-\lambda & 4 \\ 0 & -3-\lambda \end{bmatrix} \right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-3-\lambda)(-3-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-3$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-3 & 4\\ 0 & -3\end{bmatrix}-(-3)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}0 & 4\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}0 & 4 & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}0 & 4\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1=t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2=0\}$

Hence the solution is

$$\begin{bmatrix}t\\ 0\end{bmatrix}=\begin{bmatrix}t\\ 0\end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	2	1	Yes	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

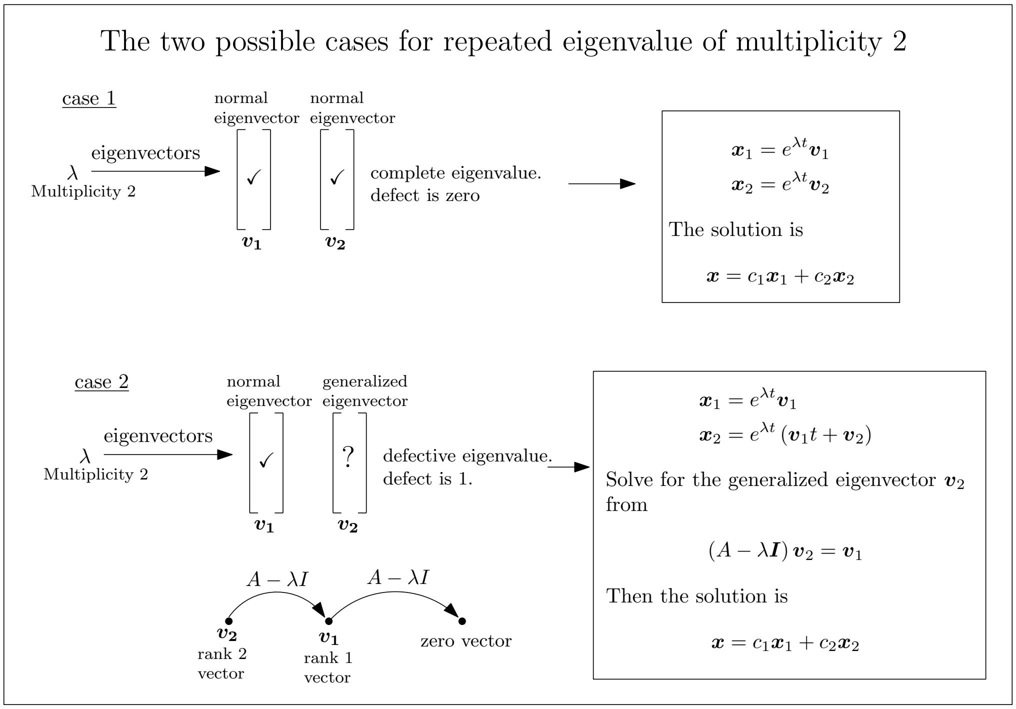


Figure 380: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -3 & 4 \\ 0 & -3 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -3. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathrm{e}^{-3t} \\
&= \begin{bmatrix} \mathrm{e}^{-3t} \\ 0 \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} \right) \mathrm{e}^{-3t} \\
&= \begin{bmatrix} \mathrm{e}^{-3t}(1+t) \\ \frac{\mathrm{e}^{-3t}}{4} \end{bmatrix}
\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-3t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-3t}(1+t) \\ \frac{\mathrm{e}^{-3t}}{4} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-3t}(c_2 t + c_1 + c_2) \\ \frac{c_2 \mathrm{e}^{-3t}}{4} \end{bmatrix}$$

The following is the phase plot of the system.

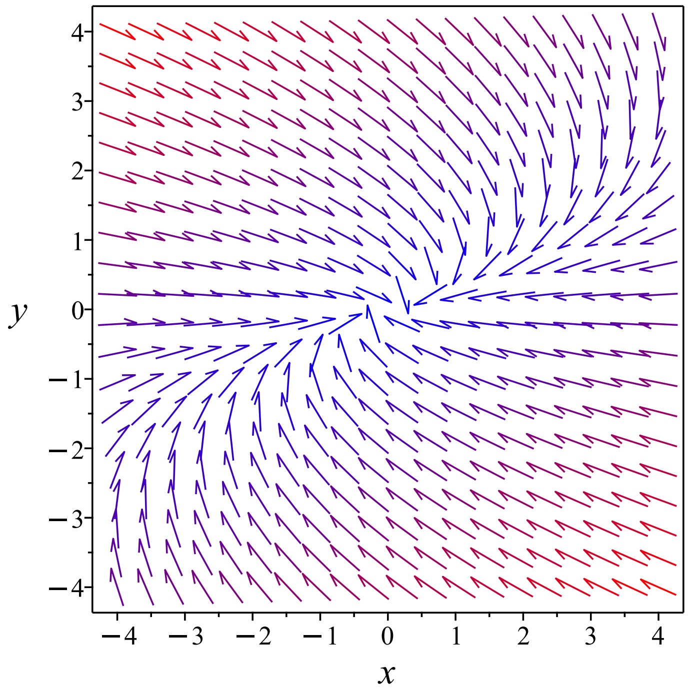


Figure 381: Phase plot

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve([diff(x(t),t)=-3*x(t)+4*y(t),diff(y(t),t)=-3*y(t)],singsol=all)
```

$$
\begin{aligned}
x(t) &= (4c_2t + c_1)\,\mathrm{e}^{-3t} \\
y(t) &= c_2\mathrm{e}^{-3t}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 30

```
DSolve[{x'[t]==-3*x[t]+4*y[t],y'[t]==-3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True
```

$$
\begin{aligned}
x(t) &\to e^{-3t}(4c_2t + c_1) \\
y(t) &\to c_2e^{-3t}
\end{aligned}
$$

22.3 problem 4(c)

22.3.1 Solution using Matrix exponential method 2206
22.3.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2207
22.3.3 Maple step by step solution 2212

Internal problem ID [11557]
Internal file name [`OUTPUT/10539_Thursday_May_18_2023_04_22_29_AM_55179/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 237
Problem number: 4(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= 2x + 2y(t) \\
y'(t) &= 6x + 3y(t)
\end{aligned}
$$

22.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{4\,\mathrm{e}^{-t}}{7} + \frac{3\,\mathrm{e}^{6t}}{7} & \frac{2\,\mathrm{e}^{6t}}{7} - \frac{2\,\mathrm{e}^{-t}}{7} \\ \frac{6\,\mathrm{e}^{6t}}{7} - \frac{6\,\mathrm{e}^{-t}}{7} & \frac{3\,\mathrm{e}^{-t}}{7} + \frac{4\,\mathrm{e}^{6t}}{7} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{4\,e^{-t}}{7}+\frac{3\,e^{6t}}{7} & \frac{2\,e^{6t}}{7}-\frac{2\,e^{-t}}{7} \\ \frac{6\,e^{6t}}{7}-\frac{6\,e^{-t}}{7} & \frac{3\,e^{-t}}{7}+\frac{4\,e^{6t}}{7} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{4\,e^{-t}}{7}+\frac{3\,e^{6t}}{7}\right)c_1+\left(\frac{2\,e^{6t}}{7}-\frac{2\,e^{-t}}{7}\right)c_2 \\ \left(\frac{6\,e^{6t}}{7}-\frac{6\,e^{-t}}{7}\right)c_1+\left(\frac{3\,e^{-t}}{7}+\frac{4\,e^{6t}}{7}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(4c_1-2c_2)\mathrm{e}^{-t}}{7}+\frac{3\,e^{6t}\left(c_1+\frac{2c_2}{3}\right)}{7} \\ \frac{(-6c_1+3c_2)\mathrm{e}^{-t}}{7}+\frac{6\,e^{6t}\left(c_1+\frac{2c_2}{3}\right)}{7} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A-\lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 2-\lambda & 2 \\ 6 & 3-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 5\lambda - 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 6$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & 2 & 0 \\ 6 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 6 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{2} \implies \left[\begin{array}{cc|c} -4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
6	1	1	No	$\left[\begin{array}{c}\frac{1}{2}\\ 1\end{array}\right]$
-1	1	1	No	$\left[\begin{array}{c}-\frac{2}{3}\\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{6t} \\
&= \left[\begin{array}{c}\frac{1}{2}\\ 1\end{array}\right] e^{6t}
\end{aligned}
$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\
&= \left[\begin{array}{c}-\frac{2}{3}\\ 1\end{array}\right] e^{-t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c}x\\ y(t)\end{array}\right] = c_1\left[\begin{array}{c}\frac{e^{6t}}{2}\\ e^{6t}\end{array}\right] + c_2\left[\begin{array}{c}-\frac{2\,e^{-t}}{3}\\ e^{-t}\end{array}\right]
$$

Which becomes

$$
\left[\begin{array}{c}x\\ y(t)\end{array}\right] = \left[\begin{array}{c}\frac{c_1 e^{6t}}{2} - \frac{2c_2 e^{-t}}{3}\\ c_1 e^{6t} + c_2 e^{-t}\end{array}\right]
$$

The following is the phase plot of the system.

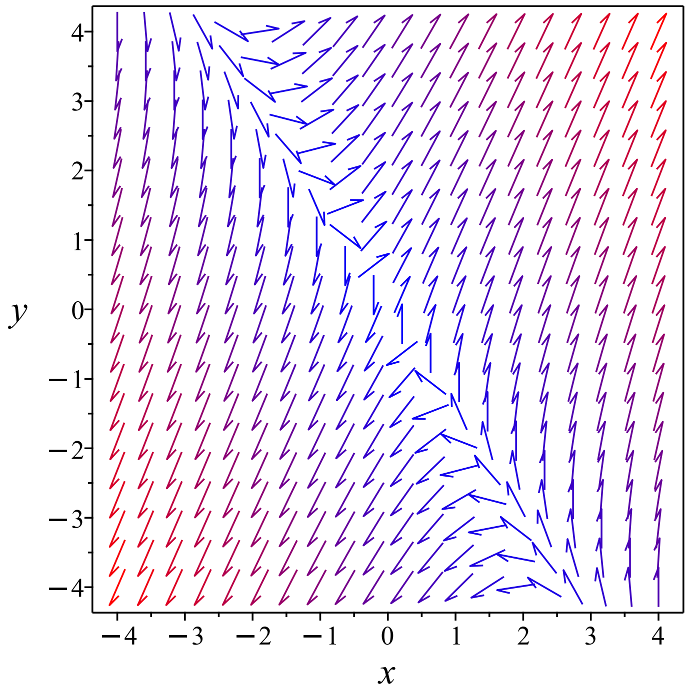


Figure 382: Phase plot

22.3.3 Maple step by step solution

Let's solve

$[x' = 2x + 2y(t), y'(t) = 6x + 3y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[6, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{6t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{6t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{2c_1 e^{-t}}{3} + \frac{c_2 e^{6t}}{2} \\ c_1 e^{-t} + c_2 e^{6t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x = -\frac{2c_1 e^{-t}}{3} + \frac{c_2 e^{6t}}{2}, y(t) = c_1 e^{-t} + c_2 e^{6t}\right\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=2*x(t)+2*y(t),diff(y(t),t)=6*x(t)+3*y(t)],singsol=all)
```

$$x(t) = e^{6t} c_1 + c_2 e^{-t}$$
$$y(t) = 2\, e^{6t} c_1 - \frac{3 c_2 e^{-t}}{2}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 74

```
DSolve[{x'[t]==2*x[t]+2*y[t],y'[t]==6*x[t]+3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{1}{7} e^{-t} \left(c_1 \left(3e^{7t} + 4\right) + 2c_2 \left(e^{7t} - 1\right)\right)$$
$$y(t) \to \frac{1}{7} e^{-t} \left(6c_1 \left(e^{7t} - 1\right) + c_2 \left(4e^{7t} + 3\right)\right)$$

22.4 problem 4(d)

22.4.1 Solution using Matrix exponential method 2215
22.4.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2216
22.4.3 Maple step by step solution 2221

Internal problem ID [11558]
Internal file name [`OUTPUT/10540_Thursday_May_18_2023_04_22_31_AM_75400280/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 237
Problem number: 4(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= -5x + 3y(t) \\
y'(t) &= 2x - 10y(t)
\end{aligned}
$$

22.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-11t}}{7} + \frac{6\,e^{-4t}}{7} & \frac{3\,e^{-4t}}{7} - \frac{3\,e^{-11t}}{7} \\ \frac{2\,e^{-4t}}{7} - \frac{2\,e^{-11t}}{7} & \frac{6\,e^{-11t}}{7} + \frac{e^{-4t}}{7} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-11t}}{7} + \frac{6\,e^{-4t}}{7} & \frac{3\,e^{-4t}}{7} - \frac{3\,e^{-11t}}{7} \\ \frac{2\,e^{-4t}}{7} - \frac{2\,e^{-11t}}{7} & \frac{6\,e^{-11t}}{7} + \frac{e^{-4t}}{7} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-11t}}{7} + \frac{6\,e^{-4t}}{7}\right) c_1 + \left(\frac{3\,e^{-4t}}{7} - \frac{3\,e^{-11t}}{7}\right) c_2 \\ \left(\frac{2\,e^{-4t}}{7} - \frac{2\,e^{-11t}}{7}\right) c_1 + \left(\frac{6\,e^{-11t}}{7} + \frac{e^{-4t}}{7}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-3c_2)e^{-11t}}{7} + \frac{6\,e^{-4t}\left(c_1+\frac{c_2}{2}\right)}{7} \\ \frac{(-2c_1+6c_2)e^{-11t}}{7} + \frac{2\,e^{-4t}\left(c_1+\frac{c_2}{2}\right)}{7} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -5-\lambda & 3 \\ 2 & -10-\lambda \end{bmatrix} \right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 15\lambda + 44 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= -4\\ \lambda_2 &= -11\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-4	1	real eigenvalue
-11	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -11$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\begin{aligned}\left(\begin{bmatrix}-5 & 3\\ 2 & -10\end{bmatrix} - (-11)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\\ \begin{bmatrix}6 & 3\\ 2 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\end{aligned}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}6 & 3 & 0\\ 2 & 1 & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c}6 & 3 & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}6 & 3\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix} - (-4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ 2 & -6 & 0 \end{array} \right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-4	1	1	No	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$
-11	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-4t} \\ &= \left[\begin{array}{c} 3 \\ 1 \end{array}\right] e^{-4t}\end{aligned}$$

Since eigenvalue -11 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-11t} \\ &= \left[\begin{array}{c} -\frac{1}{2} \\ 1 \end{array}\right] e^{-11t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x \\ y(t) \end{array}\right] = c_1 \left[\begin{array}{c} 3\,\mathrm{e}^{-4t} \\ \mathrm{e}^{-4t} \end{array}\right] + c_2 \left[\begin{array}{c} -\frac{\mathrm{e}^{-11t}}{2} \\ \mathrm{e}^{-11t} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x \\ y(t) \end{array}\right] = \left[\begin{array}{c} 3c_1 \mathrm{e}^{-4t} - \frac{c_2 \mathrm{e}^{-11t}}{2} \\ c_1 \mathrm{e}^{-4t} + c_2 \mathrm{e}^{-11t} \end{array}\right]$$

The following is the phase plot of the system.

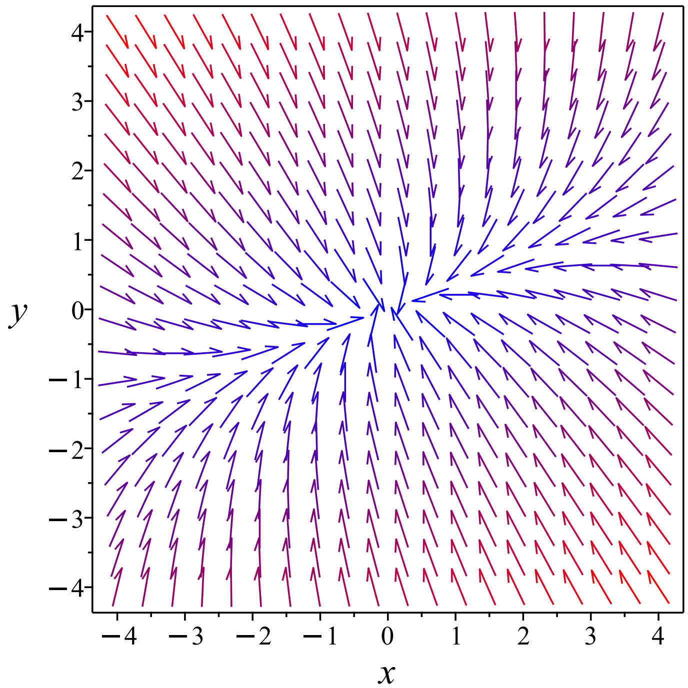


Figure 383: Phase plot

22.4.3 Maple step by step solution

Let's solve

$[x' = -5x + 3y(t)\,, y'(t) = 2x - 10y(t)]$

- Define vector

$$\vec{x}(t) = \left[\begin{array}{c} x \\ y(t) \end{array}\right]$$

- Convert system into a vector equation

$$\vec{x}'(t) = \left[\begin{array}{cc} -5 & 3 \\ 2 & -10 \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

- System to solve

$$\vec{x}'(t) = \left[\begin{array}{cc} -5 & 3 \\ 2 & -10 \end{array}\right] \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-11, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-4, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-11, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{-11t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-4, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{-4t} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{-11t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-4t} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{-11t}}{2} + 3c_2 e^{-4t} \\ c_1 e^{-11t} + c_2 e^{-4t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x = -\frac{c_1 e^{-11t}}{2} + 3c_2 e^{-4t}, y(t) = c_1 e^{-11t} + c_2 e^{-4t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=-5*x(t)+3*y(t),diff(y(t),t)=2*x(t)-10*y(t)],singsol=all)
```

$$x(t) = e^{-4t} c_1 + c_2 e^{-11t}$$
$$y(t) = \frac{e^{-4t} c_1}{3} - 2c_2 e^{-11t}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 72

```
DSolve[{x'[t]==-5*x[t]+3*y[t],y'[t]==2*x[t]-10*y[t]},{x[t],y[t]},t,IncludeSingularSolutions
```

$$x(t) \to \frac{1}{7} e^{-11t} \left(c_1 \left(6e^{7t} + 1\right) + 3c_2 \left(e^{7t} - 1\right)\right)$$
$$y(t) \to \frac{1}{7} e^{-11t} \left(2c_1 \left(e^{7t} - 1\right) + c_2 \left(e^{7t} + 6\right)\right)$$

22.5 problem 4(e)

22.5.1 Solution using Matrix exponential method 2224
22.5.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2225
22.5.3 Maple step by step solution 2229

Internal problem ID [11559]
Internal file name [`OUTPUT/10541_Thursday_May_18_2023_04_22_33_AM_6964693/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 237
Problem number: 4(e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= 2x \\
y'(t) &= 2y(t)
\end{aligned}
$$

22.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{2t} & 0 \\ 0 & \mathrm{e}^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{2t} c_1 \\ \mathrm{e}^{2t} c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(2-\lambda)(2-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1, v_2\}$ and there are no leading variables. Let $v_1 = t$. Let $v_2 = s$. Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ s \end{bmatrix}$$
$$= t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	2	No	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

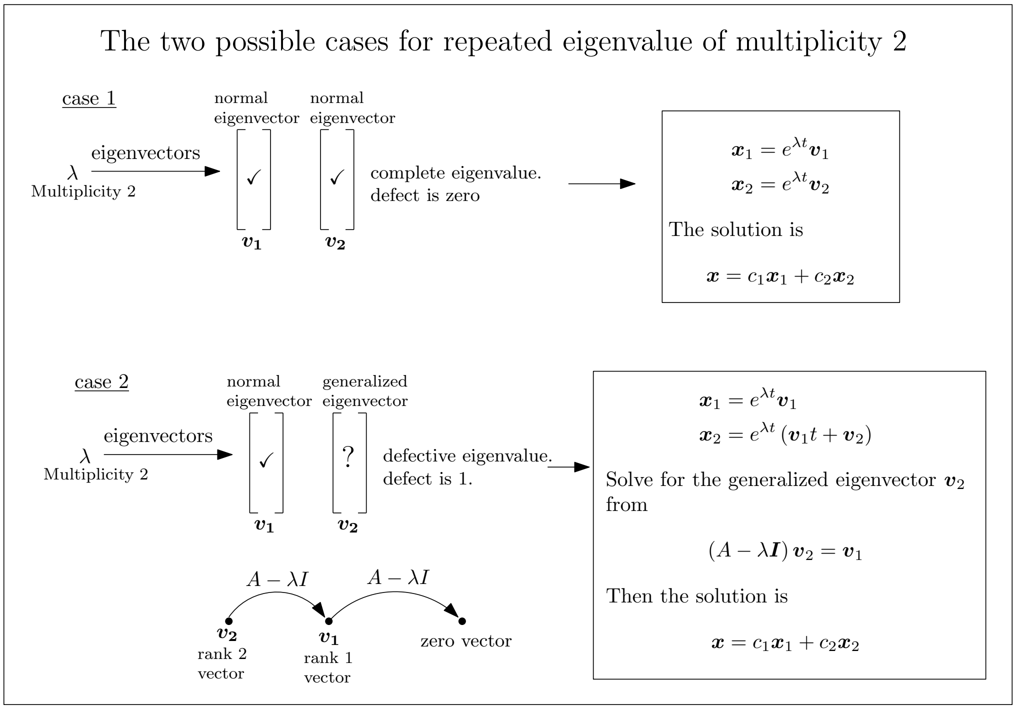


Figure 384: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2 e^{2t} \\ c_1 e^{2t} \end{bmatrix}$$

The following is the phase plot of the system.

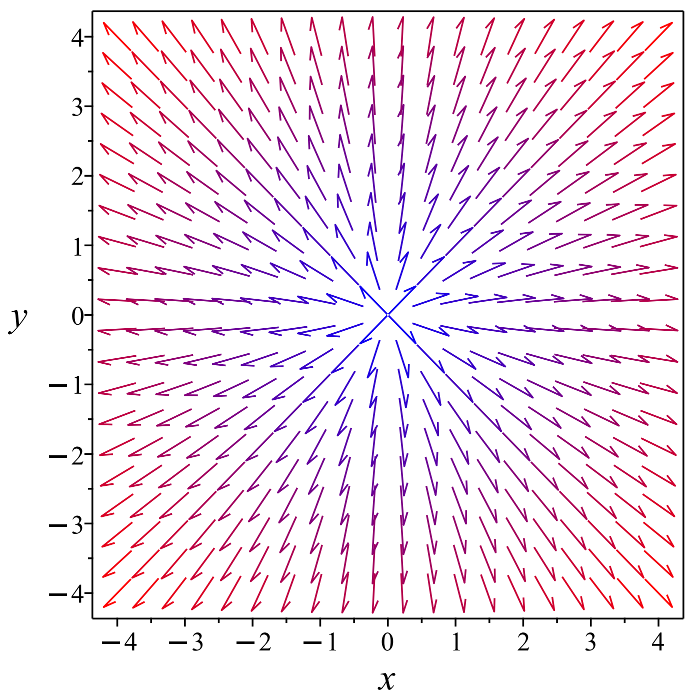


Figure 385: Phase plot

22.5.3 Maple step by step solution

Let's solve

$[x' = 2x, y'(t) = 2y(t)]$

- Define vector

$$\vec{x}(t)=\left[\begin{array}{c} x \\ y(t) \end{array}\right]$$

- Convert system into a vector equation

$$\vec{x}'(t)=\left[\begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array}\right]\cdot \vec{x}(t)+\left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

- System to solve

$$\vec{x}'(t)=\left[\begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array}\right]\cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A=\left[\begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array}\right]$$

- Rewrite the system as

$$\vec{x}'(t)=A\cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2,\left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right],\left[2,\left[\begin{array}{c} 1 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2,\left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 2

$$\vec{x}_1(t)=\mathrm{e}^{2t}\cdot \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda=2$ is the eigenvalue, and

$$\vec{x}_2(t)=\mathrm{e}^{\lambda t}\left(t\vec{v}+\vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_2(t)$ into the homogeneous system

$$\lambda \mathrm{e}^{\lambda t}\left(t\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda t}\vec{v}=\left(\mathrm{e}^{\lambda t}A\right)\cdot\left(t\vec{v}+\vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda \mathrm{e}^{\lambda t}\left(t\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda t}\vec{v}=\mathrm{e}^{\lambda t}\left(\lambda t\vec{v}+A\cdot\vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{x}_2(t)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}-2\cdot\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}0\\ 1\end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

- Second solution from eigenvalue 2

$$\vec{x}_2(t)=\mathrm{e}^{2t}\cdot\left(t\cdot\begin{bmatrix}0\\ 1\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{x}=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x}=c_1\mathrm{e}^{2t}\cdot\begin{bmatrix}0\\ 1\end{bmatrix}+c_2\mathrm{e}^{2t}\cdot\left(t\cdot\begin{bmatrix}0\\ 1\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix}x\\ y(t)\end{bmatrix}=\begin{bmatrix}0\\ \mathrm{e}^{2t}(c_2t+c_1)\end{bmatrix}$$

- Solution to the system of ODEs

$$\{x=0, y(t)=e^{2t}(c_2t+c_1)\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve([diff(x(t),t)=2*x(t)+0*y(t),diff(y(t),t)=0*x(t)+2*y(t)],singsol=all)
```

$$x(t)=c_2e^{2t}$$
$$y(t)=c_1e^{2t}$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 65

```
DSolve[{x'[t]==2*x[t]+0*y[t],y'[t]==0*x[t]+2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t)\to c_1e^{2t}$$
$$y(t)\to c_2e^{2t}$$
$$x(t)\to c_1e^{2t}$$
$$y(t)\to 0$$
$$x(t)\to 0$$
$$y(t)\to c_2e^{2t}$$
$$x(t)\to 0$$
$$y(t)\to 0$$

22.6 problem 4(f)

22.6.1 Solution using Matrix exponential method 2233
22.6.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2234
22.6.3 Maple step by step solution 2239

Internal problem ID [11560]
Internal file name [`OUTPUT/10542_Thursday_May_18_2023_04_22_34_AM_84749317/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 237
Problem number: 4(f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x' &= 3x - 2y(t) \\ y'(t) &= 4x - y(t) \end{aligned}$$

22.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$\begin{aligned} e^{At} &= \begin{bmatrix} e^t \cos(2t) + e^t \sin(2t) & -e^t \sin(2t) \\ 2\,e^t \sin(2t) & e^t \cos(2t) - e^t \sin(2t) \end{bmatrix} \\ &= \begin{bmatrix} e^t(\sin(2t) + \cos(2t)) & -e^t \sin(2t) \\ 2\,e^t \sin(2t) & e^t(\cos(2t) - \sin(2t)) \end{bmatrix} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t(\sin(2t)+\cos(2t)) & -e^t\sin(2t) \\ 2\,e^t\sin(2t) & e^t(\cos(2t)-\sin(2t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t(\sin(2t)+\cos(2t))\,c_1 - e^t\sin(2t)\,c_2 \\ 2\,e^t\sin(2t)\,c_1 + e^t(\cos(2t)-\sin(2t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t((c_1-c_2)\sin(2t)+c_1\cos(2t)) \\ (2c_1-c_2)\,e^t\sin(2t)+e^t\cos(2t)\,c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1+2i$	1	complex eigenvalue
$1-2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1-2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+2i & -2 \\ 4 & -2+2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+2i & -2 & 0 \\ 4 & -2+2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1+i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 2+2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+2i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 - i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1 + 2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - 2i & -2 \\ 4 & -2 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 - 2i & -2 & 0 \\ 4 & -2 - 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 - i) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 - 2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-2i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$1+2i$	1	1	No	$\left[\begin{array}{c} \frac{1}{2}+\frac{i}{2} \\ 1 \end{array}\right]$
$1-2i$	1	1	No	$\left[\begin{array}{c} \frac{1}{2}-\frac{i}{2} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x \\ y(t) \end{array}\right] = c_1\left[\begin{array}{c} \left(\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(1+2i)t} \\ \mathrm{e}^{(1+2i)t} \end{array}\right] + c_2\left[\begin{array}{c} \left(\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(1-2i)t} \\ \mathrm{e}^{(1-2i)t} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x \\ y(t) \end{array}\right] = \left[\begin{array}{c} \left(\frac{1}{2}+\frac{i}{2}\right) c_1\mathrm{e}^{(1+2i)t} + \left(\frac{1}{2}-\frac{i}{2}\right) c_2\mathrm{e}^{(1-2i)t} \\ c_1\mathrm{e}^{(1+2i)t} + c_2\mathrm{e}^{(1-2i)t} \end{array}\right]$$

The following is the phase plot of the system.

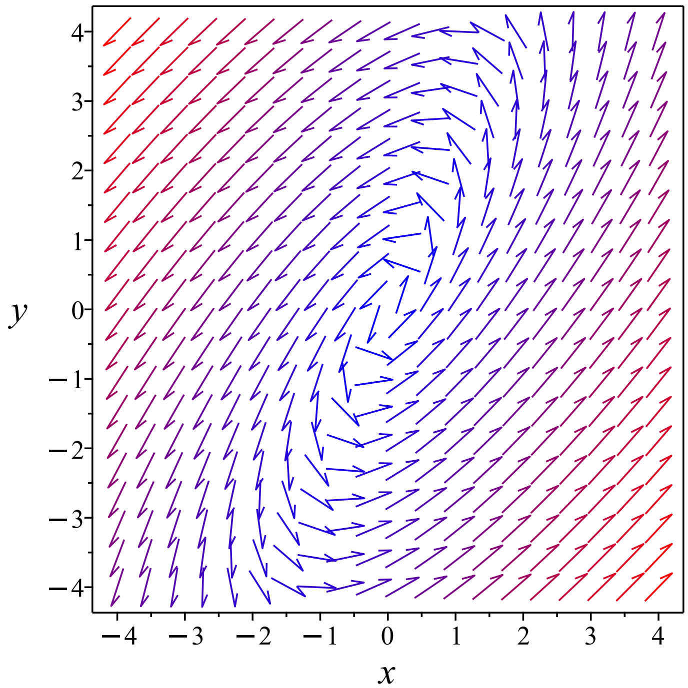


Figure 386: Phase plot

22.6.3 Maple step by step solution

Let's solve

$[x' = 3x - 2y(t), y'(t) = 4x - y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

- Rewrite the system as

  $$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1 - 2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1 + 2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[1 - 2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

  $$\mathrm{e}^{(1-2\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$\mathrm{e}^{t} \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} \frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

  $$\mathrm{e}^{t} \cdot \begin{bmatrix} \left(\frac{1}{2} - \frac{\mathrm{I}}{2}\right)(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\overrightarrow{x}_1(t) = \mathrm{e}^{t} \cdot \begin{bmatrix} \frac{\cos(2t)}{2} - \frac{\sin(2t)}{2} \\ \cos(2t) \end{bmatrix}, \overrightarrow{x}_2(t) = \mathrm{e}^{t} \cdot \begin{bmatrix} -\frac{\sin(2t)}{2} - \frac{\cos(2t)}{2} \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $$\overrightarrow{x} = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^t \cdot \begin{bmatrix} \frac{\cos(2t)}{2} - \frac{\sin(2t)}{2} \\ \cos(2t) \end{bmatrix} + c_2 e^t \cdot \begin{bmatrix} -\frac{\sin(2t)}{2} - \frac{\cos(2t)}{2} \\ -\sin(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{e^t((c_1-c_2)\cos(2t)-\sin(2t)(c_1+c_2))}{2} \\ e^t(c_1\cos(2t) - c_2\sin(2t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x = \frac{e^t((c_1-c_2)\cos(2t)-\sin(2t)(c_1+c_2))}{2}, y(t) = e^t(c_1\cos(2t) - c_2\sin(2t))\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 56

```
dsolve([diff(x(t),t)=3*x(t)-2*y(t),diff(y(t),t)=4*x(t)-y(t)],singsol=all)
```

$$x(t) = e^t(c_1\sin(2t) + c_2\cos(2t))$$
$$y(t) = -e^t(c_1\cos(2t) - c_2\cos(2t) - c_1\sin(2t) - c_2\sin(2t))$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 58

```
DSolve[{x'[t]==3*x[t]-2*y[t],y'[t]==4*x[t]-y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to e^t(c_1\cos(2t) + (c_1 - c_2)\sin(2t))$$
$$y(t) \to e^t(c_2\cos(2t) + (2c_1 - c_2)\sin(2t))$$

22.7 problem 4(g)

22.7.1 Solution using Matrix exponential method 2242
22.7.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2243
22.7.3 Maple step by step solution 2248

Internal problem ID [11561]
Internal file name [`OUTPUT/10543_Thursday_May_18_2023_04_22_36_AM_53792491/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 237
Problem number: 4(g).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= 5x - 4y(t) \\
y'(t) &= x + y(t)
\end{aligned}
$$

22.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} e^{3t}(2t+1) & -4t\,e^{3t} \\ t\,e^{3t} & e^{3t}(1-2t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{3t}(2t+1) & -4t\,\mathrm{e}^{3t} \\ t\,\mathrm{e}^{3t} & \mathrm{e}^{3t}(1-2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{3t}(2t+1)\,c_1 - 4t\,\mathrm{e}^{3t}c_2 \\ t\,\mathrm{e}^{3t}c_1 + \mathrm{e}^{3t}(1-2t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{3t}(2tc_1 - 4c_2t + c_1) \\ \mathrm{e}^{3t}(tc_1 - 2c_2t + c_2) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & -4 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	2	1	Yes	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

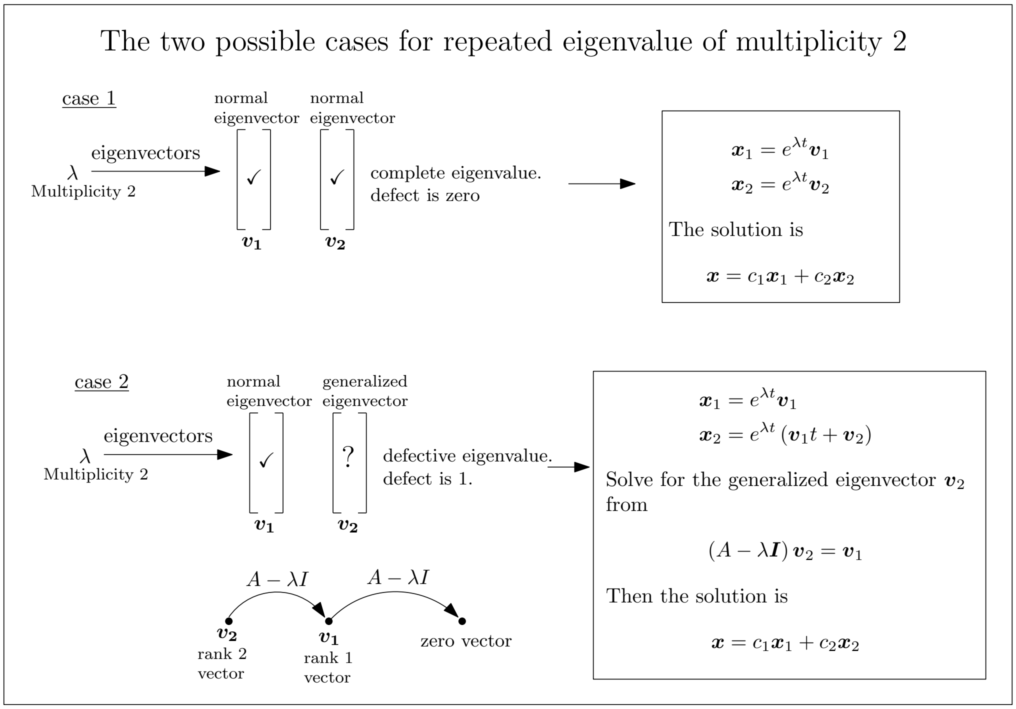


Figure 387: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 3. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathrm{e}^{3t} \\
&= \begin{bmatrix} 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) \mathrm{e}^{3t} \\
&= \begin{bmatrix} \mathrm{e}^{3t}(2t+3) \\ \mathrm{e}^{3t}(1+t) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{3t}(2t+3) \\ \mathrm{e}^{3t}(1+t) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} ((2t+3)\, c_2 + 2c_1)\, \mathrm{e}^{3t} \\ \mathrm{e}^{3t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

The following is the phase plot of the system.

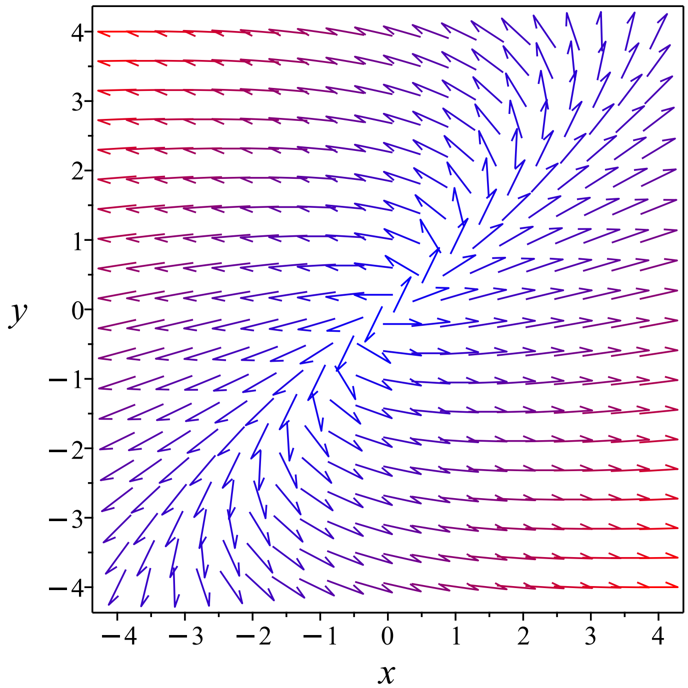


Figure 388: Phase plot

22.7.3 Maple step by step solution

Let's solve

$[x' = 5x - 4y(t), y'(t) = x + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $\left[\left[3, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $\left[3, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$

- First solution from eigenvalue 3

  $\overrightarrow{x}_1(t) = \mathrm{e}^{3t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, and

  $\overrightarrow{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$

- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

$$\left(\left[\begin{array}{cc} 5 & -4 \\ 1 & 1 \end{array}\right]-3\cdot \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right)\cdot \overrightarrow{p}=\left[\begin{array}{c} 2 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c} 1 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 3

$$\overrightarrow{x}_{2}(t)=\mathrm{e}^{3t}\cdot \left(t\cdot \left[\begin{array}{c} 2 \\ 1 \end{array}\right]+\left[\begin{array}{c} 1 \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

$$\overrightarrow{x}=c_{1}\overrightarrow{x}_{1}(t)+c_{2}\overrightarrow{x}_{2}(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{x}=c_{1}\mathrm{e}^{3t}\cdot \left[\begin{array}{c} 2 \\ 1 \end{array}\right]+c_{2}\mathrm{e}^{3t}\cdot \left(t\cdot \left[\begin{array}{c} 2 \\ 1 \end{array}\right]+\left[\begin{array}{c} 1 \\ 0 \end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x \\ y(t) \end{array}\right]=\left[\begin{array}{c} \mathrm{e}^{3t}(2c_{2}t+2c_{1}+c_{2}) \\ \mathrm{e}^{3t}(c_{2}t+c_{1}) \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x=\mathrm{e}^{3t}(2c_{2}t+2c_{1}+c_{2}),y(t)=\mathrm{e}^{3t}(c_{2}t+c_{1})\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve([diff(x(t),t)=5*x(t)-4*y(t),diff(y(t),t)=x(t)+y(t)],singsol=all)
```

$$x(t)=\mathrm{e}^{3t}(c_{2}t+c_{1})$$
$$y(t)=\frac{\mathrm{e}^{3t}(2c_{2}t+2c_{1}-c_{2})}{4}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 45

```
DSolve[{x'[t]==5*x[t]-4*y[t],y'[t]==x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tru
```

$$x(t) \to e^{3t}(2c_1t - 4c_2t + c_1)$$
$$y(t) \to e^{3t}((c_1 - 2c_2)t + c_2)$$

22.8 problem 4(h)

22.8.1 Solution using Matrix exponential method 2252
22.8.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2253
22.8.3 Maple step by step solution 2257

Internal problem ID [11562]
Internal file name [`OUTPUT/10544_Thursday_May_18_2023_04_22_38_AM_80579589/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 237
Problem number: 4(h).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x' &= 9y(t) \\ y'(t) &= -x \end{aligned}$$

22.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(3t) & 3\sin(3t) \\ -\frac{\sin(3t)}{3} & \cos(3t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(3t) & 3\sin(3t) \\ -\frac{\sin(3t)}{3} & \cos(3t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \cos(3t)\, c_1 + 3\sin(3t)\, c_2 \\ -\frac{\sin(3t)c_1}{3} + \cos(3t)\, c_2 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 9 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 9 \\ -1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3i$$
$$\lambda_2 = -3i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$3i$	1	complex eigenvalue
$-3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 9 \\ -1 & 0 \end{bmatrix} - (-3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3i & 9 \\ -1 & 3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3i & 9 & 0 \\ -1 & 3i & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{iR_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 3i & 9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3i & 9 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3it\}$

Hence the solution is

$$\begin{bmatrix} 3\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} 3it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3\,\mathrm{I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} 3i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} 3i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 9 \\ -1 & 0 \end{bmatrix} - (3i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3i & 9 \\ -1 & -3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3i & 9 & 0 \\ -1 & -3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{iR_1}{3} \Longrightarrow \left[\begin{array}{cc|c} -3i & 9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3i & 9 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -3it\}$

Hence the solution is

$$\begin{bmatrix} -3\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -3it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -3\,\mathrm{I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} -3i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -3\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -3i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$3i$	1	1	No	$\begin{bmatrix} -3i \\ 1 \end{bmatrix}$
$-3i$	1	1	No	$\begin{bmatrix} 3i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -3i\mathrm{e}^{3it} \\ \mathrm{e}^{3it} \end{bmatrix} + c_2 \begin{bmatrix} 3i\mathrm{e}^{-3it} \\ \mathrm{e}^{-3it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} 3i\left(c_2 \mathrm{e}^{-3it} - c_1 \mathrm{e}^{3it}\right) \\ c_1 \mathrm{e}^{3it} + c_2 \mathrm{e}^{-3it} \end{bmatrix}$$

The following is the phase plot of the system.

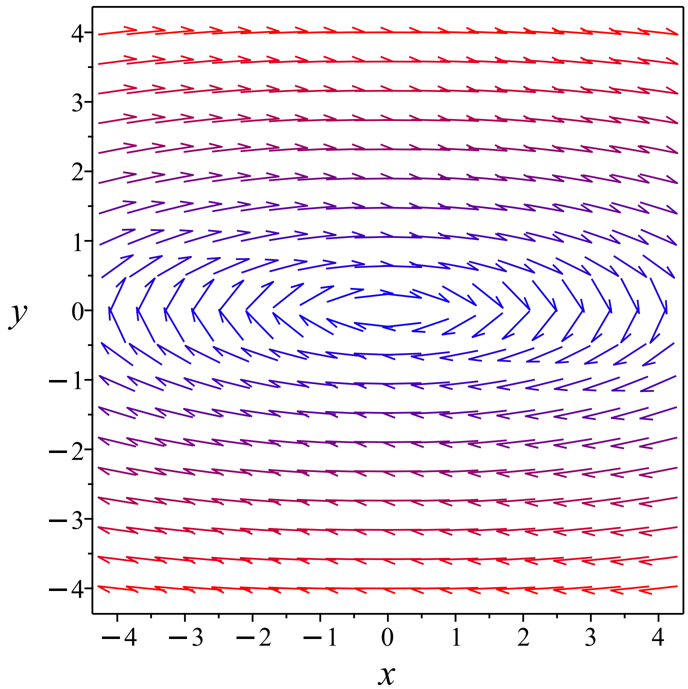


Figure 389: Phase plot

22.8.3 Maple step by step solution

Let's solve

$[x' = 9y(t)\,, y'(t) = -x]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & 9 \\ -1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 9 \\ -1 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 9 \\ -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3\,\mathrm{I}, \begin{bmatrix} 3\,\mathrm{I} \\ 1 \end{bmatrix}\right], \left[3\,\mathrm{I}, \begin{bmatrix} -3\,\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-3\,\mathrm{I}, \begin{bmatrix} 3\,\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-3\,\mathrm{I}t} \cdot \begin{bmatrix} 3\,\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(3t) - \mathrm{I}\sin(3t)) \cdot \begin{bmatrix} 3\,\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} 3\,\mathrm{I}(\cos(3t) - \mathrm{I}\sin(3t)) \\ \cos(3t) - \mathrm{I}\sin(3t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \begin{bmatrix} 3\sin(3t) \\ \cos(3t) \end{bmatrix}, \vec{x}_2(t) = \begin{bmatrix} 3\cos(3t) \\ -\sin(3t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$
- Substitute solutions into the general solution

  $$\vec{x} = \begin{bmatrix} 3c_2\cos(3t) + 3c_1\sin(3t) \\ -c_2\sin(3t) + c_1\cos(3t) \end{bmatrix}$$
- Substitute in vector of dependent variables

  $$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} 3c_2\cos(3t) + 3c_1\sin(3t) \\ -c_2\sin(3t) + c_1\cos(3t) \end{bmatrix}$$
- Solution to the system of ODEs

  $\{x = 3c_2\cos(3t) + 3c_1\sin(3t), y(t) = -c_2\sin(3t) + c_1\cos(3t)\}$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=0*x(t)+9*y(t),diff(y(t),t)=-x(t)+0*y(t)],singsol=all)
```

$$x(t) = c_1\sin(3t) + c_2\cos(3t)$$
$$y(t) = \frac{c_1\cos(3t)}{3} - \frac{c_2\sin(3t)}{3}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 42

```
DSolve[{x'[t]==0*x[t]+9*y[t],y'[t]==-x[t]+0*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to c_1\cos(3t) + 3c_2\sin(3t)$$
$$y(t) \to c_2\cos(3t) - \frac{1}{3}c_1\sin(3t)$$

22.9 problem 5

22.9.1 Solution using Matrix exponential method 2260
22.9.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2261

Internal problem ID [11563]
Internal file name [`OUTPUT/10545_Thursday_May_18_2023_04_22_40_AM_59283322/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 237
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= 2x + y(t) \\
y'(t) &= -x
\end{aligned}
$$

With initial conditions

$$
[x(0) = 1, y(0) = -1]
$$

22.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{t}(1+t) & t\,\mathrm{e}^{t} \\ -t\,\mathrm{e}^{t} & \mathrm{e}^{t}(1-t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^t(1+t) & t\,e^t \\ -t\,e^t & e^t(1-t) \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} e^t(1+t) - t\,e^t \\ -t\,e^t - e^t(1-t) \end{bmatrix} \\
&= \begin{bmatrix} e^t \\ -e^t \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

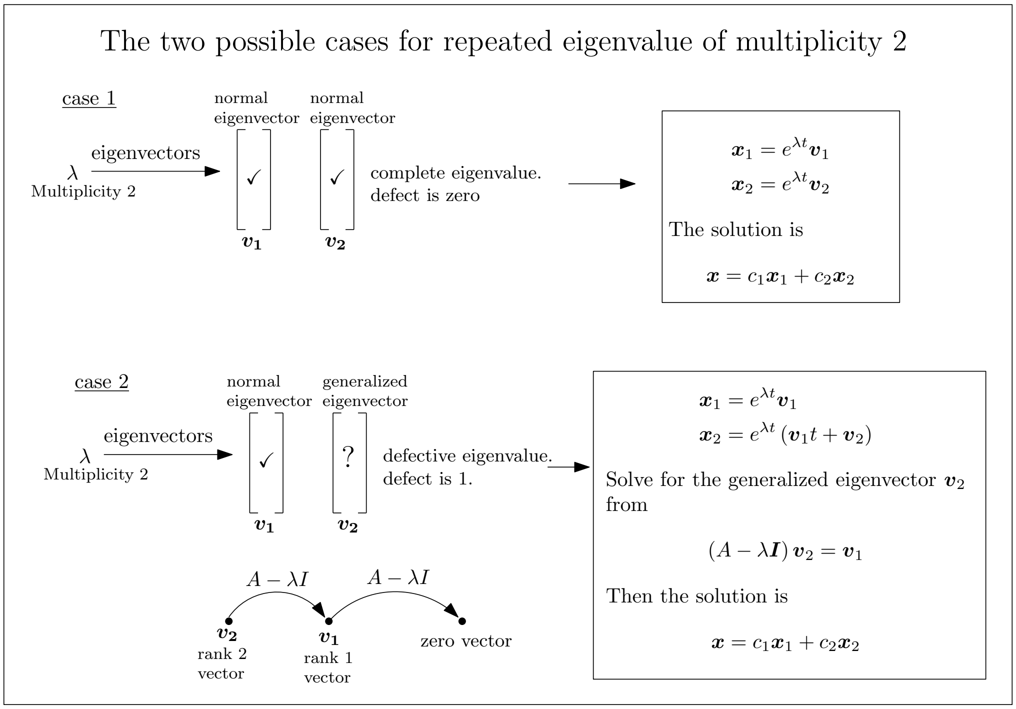


Figure 390: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 1. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{t} \\
&= \begin{bmatrix} -\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right) \mathrm{e}^{t} \\
&= \begin{bmatrix} -\mathrm{e}^{t}(t+2) \\ \mathrm{e}^{t}(1+t) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t}(-t-2) \\ \mathrm{e}^{t}(1+t) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -((t+2)\, c_2 + c_1)\, \mathrm{e}^{t} \\ \mathrm{e}^{t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x(0) = 1 \\ y(0) = -1 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t=0$ gives

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2c_2 - c_1 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -1 \\ c_2 = 0 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$

The following is the phase plot of the system.

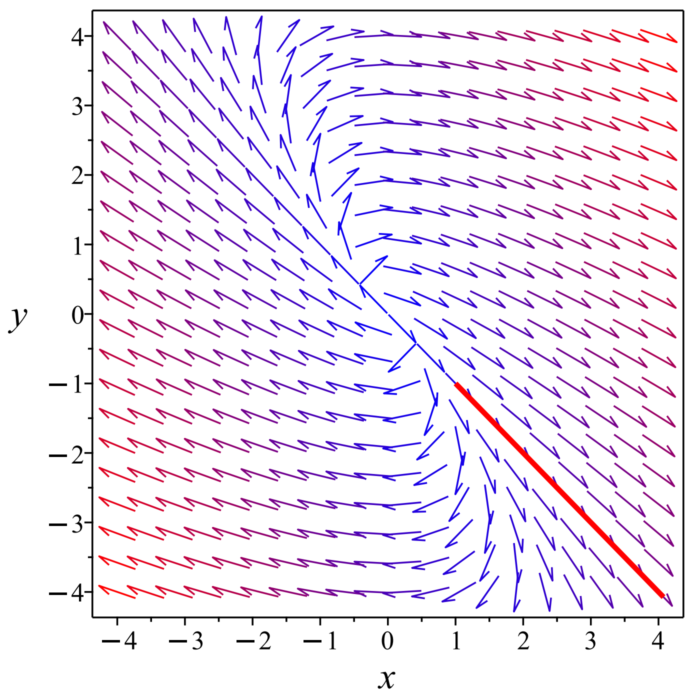
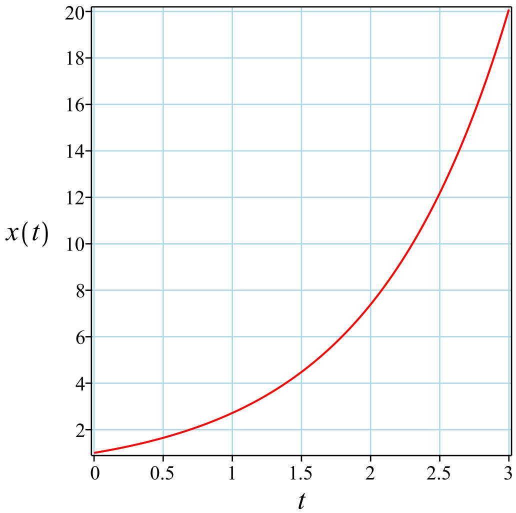
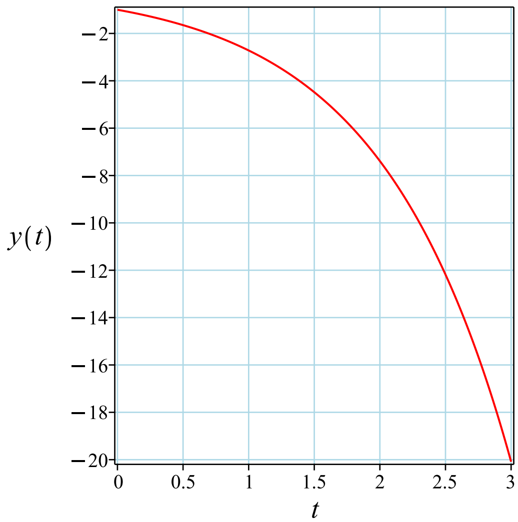


Figure 391: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(x(t),t) = 2*x(t)+y(t), diff(y(t),t) = -x(t), x(0) = 1, y(0) = -1], singsol=all)
```

$$x(t) = e^t$$
$$y(t) = -e^t$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 16

```
DSolve[{x'[t]==2*x[t]+y[t],y'[t]==-x[t]+0*y[t]},{x[0]==1,y[0]==-1},{x[t],y[t]},t,IncludeSing
```

$$x(t) \to e^t$$
$$y(t) \to -e^t$$

22.10 problem 6

22.10.1 Solution using Matrix exponential method 2268
22.10.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2269
22.10.3 Maple step by step solution 2274

Internal problem ID [11564]
Internal file name [`OUTPUT/10546_Thursday_May_18_2023_04_22_42_AM_72238829/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 237
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= x - 2y(t) \\
y'(t) &= -2x + 4y(t)
\end{aligned}
$$

22.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{4}{5} + \frac{e^{5t}}{5} & -\frac{2\,e^{5t}}{5} + \frac{2}{5} \\ -\frac{2\,e^{5t}}{5} + \frac{2}{5} & \frac{1}{5} + \frac{4\,e^{5t}}{5} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{4}{5}+\frac{e^{5t}}{5} & -\frac{2\,e^{5t}}{5}+\frac{2}{5} \\ -\frac{2\,e^{5t}}{5}+\frac{2}{5} & \frac{1}{5}+\frac{4\,e^{5t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{4}{5}+\frac{e^{5t}}{5}\right)c_1 + \left(-\frac{2\,e^{5t}}{5}+\frac{2}{5}\right)c_2 \\ \left(-\frac{2\,e^{5t}}{5}+\frac{2}{5}\right)c_1 + \left(\frac{1}{5}+\frac{4\,e^{5t}}{5}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-2c_2)e^{5t}}{5}+\frac{4c_1}{5}+\frac{2c_2}{5} \\ \frac{(-2c_1+4c_2)e^{5t}}{5}+\frac{2c_1}{5}+\frac{c_2}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 5\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$
$$\lambda_2 = 5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & -2 & 0 \\ -2 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
5	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{0} \\
&= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{0}
\end{aligned}
$$

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{5t} \\
&= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{5t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -\frac{e^{5t}}{2} \\ e^{5t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} 2c_1 - \frac{c_2 e^{5t}}{2} \\ c_1 + c_2 e^{5t} \end{bmatrix}
$$

The following is the phase plot of the system.

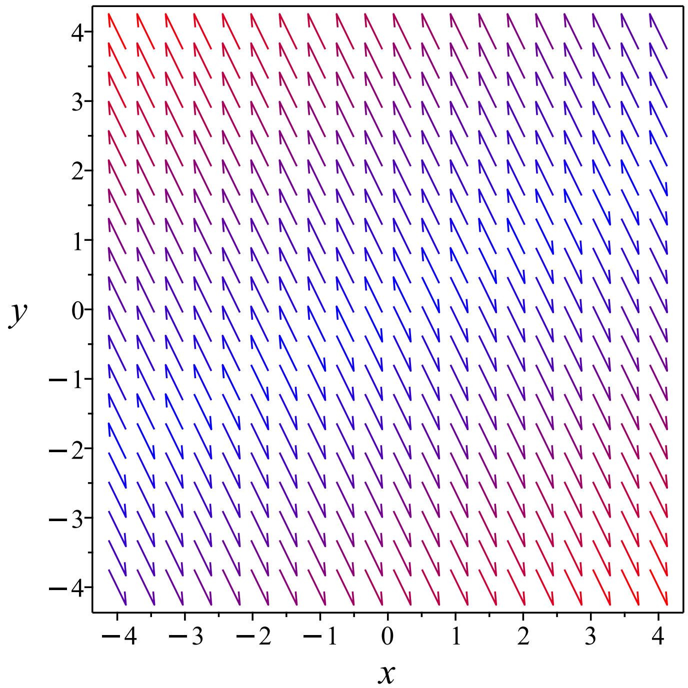


Figure 392: Phase plot

22.10.3 Maple step by step solution

Let's solve

$[x' = x - 2y(t), y'(t) = -2x + 4y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[5, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{5t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_2 \mathrm{e}^{5t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} 2c_1 \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_2 e^{5t}}{2} + 2c_1 \\ c_2 e^{5t} + c_1 \end{bmatrix}
$$

- Solution to the system of ODEs

  $\left\{x = -\frac{c_2 e^{5t}}{2} + 2c_1, y(t) = c_2 e^{5t} + c_1\right\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve([diff(x(t),t)=x(t)-2*y(t),diff(y(t),t)=-2*x(t)+4*y(t)],singsol=all)
```

$$
\begin{aligned}
x(t) &= c_1 + c_2 e^{5t} \\
y(t) &= -2c_2 e^{5t} + \frac{c_1}{2}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 62

```
DSolve[{x'[t]==x[t]-2*y[t],y'[t]==-2*x[t]+4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$
\begin{aligned}
x(t) &\to \frac{1}{5}\left(c_1\left(e^{5t} + 4\right) - 2c_2\left(e^{5t} - 1\right)\right) \\
y(t) &\to \frac{1}{5}\left(c_2\left(4e^{5t} + 1\right) - 2c_1\left(e^{5t} - 1\right)\right)
\end{aligned}
$$

23 Chapter 4, Linear Systems. Exercises page 244

23.1 problem 3 2278
23.2 problem 4 2288
23.3 problem 5 2298
23.4 problem 6 2309
23.5 problem 7 2322

23.1 problem 3

23.1.1 Solution using Matrix exponential method 2278
23.1.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2280

Internal problem ID [11565]
Internal file name [`OUTPUT/10547_Thursday_May_18_2023_04_22_43_AM_21741937/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 244
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x' &= 3x - y(t) + 1 \\ y'(t) &= x + y(t) + 2 \end{aligned}$$

With initial conditions

$$[x(0) = 1, y(0) = 2]$$

23.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{2t}(1+t) & -\mathrm{e}^{2t}t \\ \mathrm{e}^{2t}t & \mathrm{e}^{2t}(1-t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \mathrm{e}^{2t}(1+t) & -\mathrm{e}^{2t}t \\ \mathrm{e}^{2t}t & \mathrm{e}^{2t}(1-t) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{2t}(1+t) - 2\,\mathrm{e}^{2t}t \\ \mathrm{e}^{2t}t + 2\,\mathrm{e}^{2t}(1-t) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{2t}(1-t) \\ \mathrm{e}^{2t}(-t+2) \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} -\mathrm{e}^{-2t}(t-1) & t\,\mathrm{e}^{-2t} \\ -t\,\mathrm{e}^{-2t} & \mathrm{e}^{-2t}(1+t) \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \mathrm{e}^{2t}(1+t) & -\mathrm{e}^{2t}t \\ \mathrm{e}^{2t}t & \mathrm{e}^{2t}(1-t) \end{bmatrix} \int \begin{bmatrix} -\mathrm{e}^{-2t}(t-1) & t\,\mathrm{e}^{-2t} \\ -t\,\mathrm{e}^{-2t} & \mathrm{e}^{-2t}(1+t) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} dt \\
&= \begin{bmatrix} \mathrm{e}^{2t}(1+t) & -\mathrm{e}^{2t}t \\ \mathrm{e}^{2t}t & \mathrm{e}^{2t}(1-t) \end{bmatrix} \begin{bmatrix} -\frac{\mathrm{e}^{-2t}(2t+3)}{4} \\ -\frac{\mathrm{e}^{-2t}(2t+5)}{4} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{3}{4} \\ -\frac{5}{4} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} -e^{2t}t + e^{2t} - \frac{3}{4} \\ e^{2t}(-t+2) - \frac{5}{4} \end{bmatrix}
\end{aligned}
$$

23.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 4\lambda + 4 = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

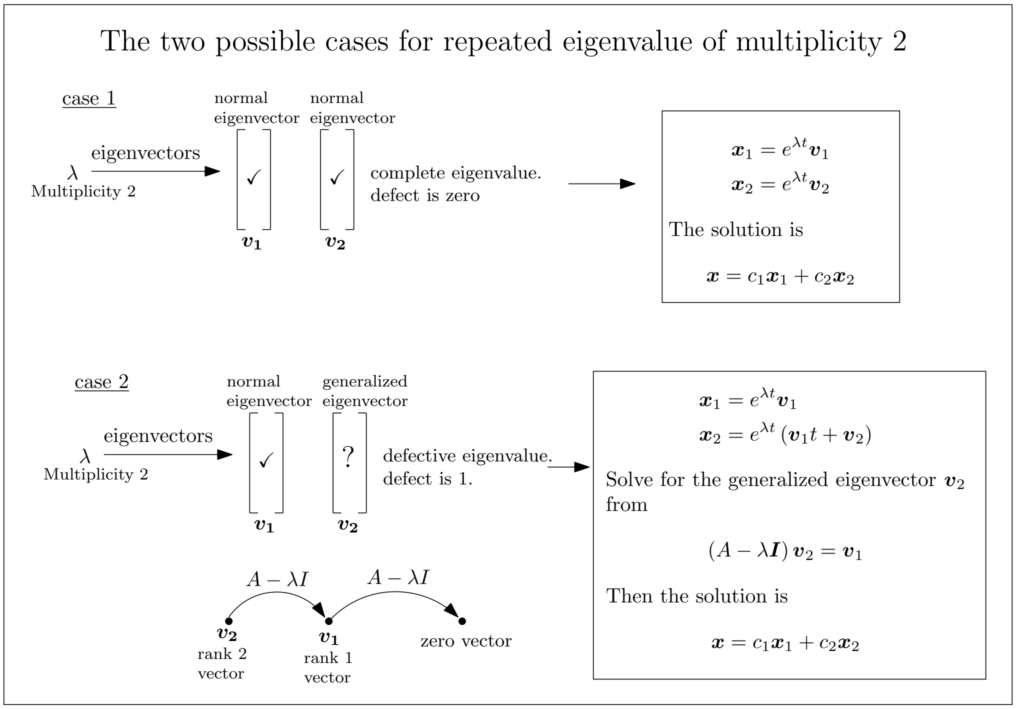


Figure 393: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\begin{aligned}\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}\end{aligned}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix}1\\1\end{bmatrix} \mathrm{e}^{2t}\\ &= \begin{bmatrix}\mathrm{e}^{2t}\\ \mathrm{e}^{2t}\end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}\\ &= \left(\begin{bmatrix}1\\1\end{bmatrix} t + \begin{bmatrix}2\\1\end{bmatrix}\right) \mathrm{e}^{2t}\\ &= \begin{bmatrix}\mathrm{e}^{2t}(t+2)\\ \mathrm{e}^{2t}(1+t)\end{bmatrix}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix}x\\ y(t)\end{bmatrix} = c_1 \begin{bmatrix}\mathrm{e}^{2t}\\ \mathrm{e}^{2t}\end{bmatrix} + c_2 \begin{bmatrix}\mathrm{e}^{2t}(t+2)\\ \mathrm{e}^{2t}(1+t)\end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix}\vec{x}_1 & \vec{x}_2 & \cdots\end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix}\mathrm{e}^{2t} & \mathrm{e}^{2t}(t+2)\\ \mathrm{e}^{2t} & \mathrm{e}^{2t}(1+t)\end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\mathrm{e}^{-2t}(1+t) & \mathrm{e}^{-2t}(t+2) \\ \mathrm{e}^{-2t} & -\mathrm{e}^{-2t} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \mathrm{e}^{2t} & \mathrm{e}^{2t}(t+2) \\ \mathrm{e}^{2t} & \mathrm{e}^{2t}(1+t) \end{bmatrix} \int \begin{bmatrix} -\mathrm{e}^{-2t}(1+t) & \mathrm{e}^{-2t}(t+2) \\ \mathrm{e}^{-2t} & -\mathrm{e}^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} dt \\
&= \begin{bmatrix} \mathrm{e}^{2t} & \mathrm{e}^{2t}(t+2) \\ \mathrm{e}^{2t} & \mathrm{e}^{2t}(1+t) \end{bmatrix} \int \begin{bmatrix} \mathrm{e}^{-2t}(3+t) \\ -\mathrm{e}^{-2t} \end{bmatrix} dt \\
&= \begin{bmatrix} \mathrm{e}^{2t} & \mathrm{e}^{2t}(t+2) \\ \mathrm{e}^{2t} & \mathrm{e}^{2t}(1+t) \end{bmatrix} \begin{bmatrix} -\frac{\mathrm{e}^{-2t}(2t+7)}{4} \\ \frac{\mathrm{e}^{-2t}}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{3}{4} \\ -\frac{5}{4} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x \\ y(t) \end{bmatrix} &= \begin{bmatrix} c_1\mathrm{e}^{2t} \\ c_1\mathrm{e}^{2t} \end{bmatrix} + \begin{bmatrix} c_2\mathrm{e}^{2t}(t+2) \\ c_2\mathrm{e}^{2t}(1+t) \end{bmatrix} + \begin{bmatrix} -\frac{3}{4} \\ -\frac{5}{4} \end{bmatrix}
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} + ((t+2)\,c_2 + c_1)\,\mathrm{e}^{2t} \\ -\frac{5}{4} + \mathrm{e}^{2t}((1+t)\,c_2 + c_1) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} + 2c_2 + c_1 \\ -\frac{5}{4} + c_2 + c_1 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{19}{4} \\ c_2 = -\frac{3}{2} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} + \left(-\frac{3t}{2} + \frac{7}{4}\right) \mathrm{e}^{2t} \\ -\frac{5}{4} + \mathrm{e}^{2t} \left(\frac{13}{4} - \frac{3t}{2}\right) \end{bmatrix}$$

The following is the phase plot of the system.

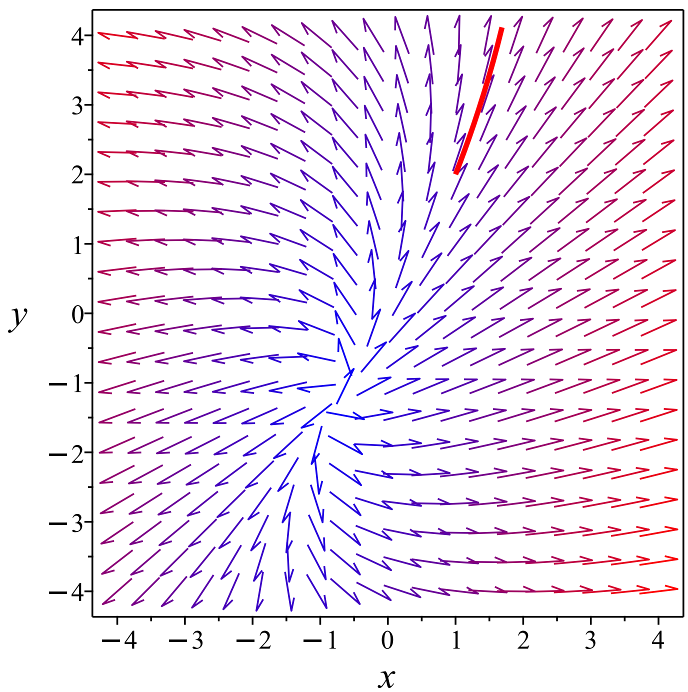
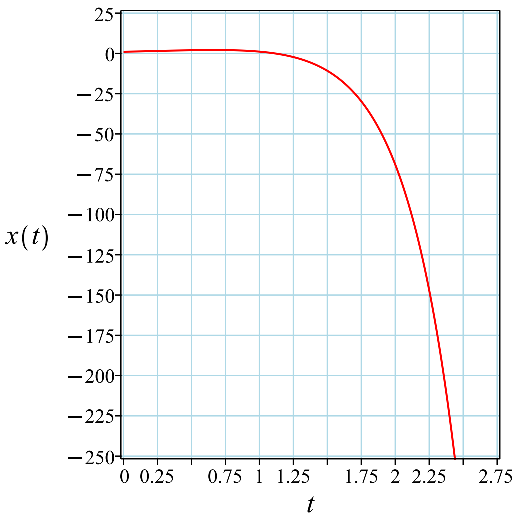
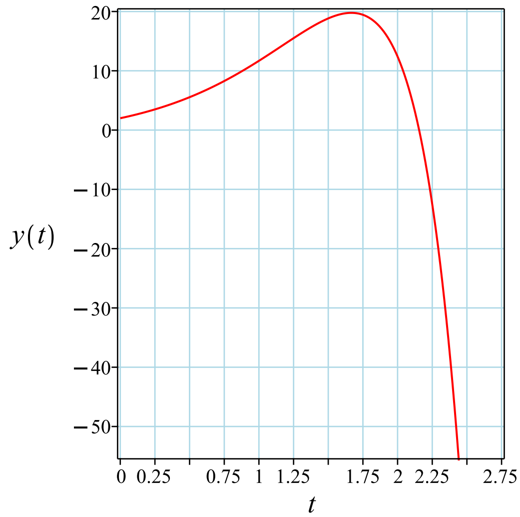


Figure 394: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 32

```
dsolve([diff(x(t),t) = 3*x(t)-y(t)+1, diff(y(t),t) = x(t)+y(t)+2, x(0) = 1, y(0) = 2], sings
```

$$x(t) = -\frac{3}{4} + \mathrm{e}^{2t}\left(-\frac{3t}{2} + \frac{7}{4}\right)$$
$$y(t) = -\frac{5}{4} + \mathrm{e}^{2t}\left(-\frac{3t}{2} + \frac{13}{4}\right)$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 42

```
DSolve[{x'[t]==3*x[t]-y[t]+1,y'[t]==x[t]+y[t]+2},{x[0]==1,y[0]==2},{x[t],y[t]},t,IncludeSing
```

$$x(t) \to \frac{1}{4}\left(e^{2t}(7-6t)-3\right)$$
$$y(t) \to \frac{1}{4}\left(e^{2t}(13-6t)-5\right)$$

23.2 problem 4

23.2.1 Solution using Matrix exponential method 2288
23.2.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2290
23.2.3 Maple step by step solution 2295

Internal problem ID [11566]
Internal file name [`OUTPUT/10548_Thursday_May_18_2023_04_22_45_AM_74645511/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 244
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= -5x + 3y(t) + \mathrm{e}^{-t} \\
y'(t) &= 2x - 10y(t)
\end{aligned}
$$

23.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{-t} \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-11t}}{7} + \frac{6\,e^{-4t}}{7} & \frac{3\,e^{-4t}}{7} - \frac{3\,e^{-11t}}{7} \\ \frac{2\,e^{-4t}}{7} - \frac{2\,e^{-11t}}{7} & \frac{6\,e^{-11t}}{7} + \frac{e^{-4t}}{7} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-11t}}{7} + \frac{6\,e^{-4t}}{7} & \frac{3\,e^{-4t}}{7} - \frac{3\,e^{-11t}}{7} \\ \frac{2\,e^{-4t}}{7} - \frac{2\,e^{-11t}}{7} & \frac{6\,e^{-11t}}{7} + \frac{e^{-4t}}{7} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-11t}}{7} + \frac{6\,e^{-4t}}{7}\right) c_1 + \left(\frac{3\,e^{-4t}}{7} - \frac{3\,e^{-11t}}{7}\right) c_2 \\ \left(\frac{2\,e^{-4t}}{7} - \frac{2\,e^{-11t}}{7}\right) c_1 + \left(\frac{6\,e^{-11t}}{7} + \frac{e^{-4t}}{7}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-3c_2)e^{-11t}}{7} + \frac{6\left(c_1+\frac{c_2}{2}\right)e^{-4t}}{7} \\ \frac{(-2c_1+6c_2)e^{-11t}}{7} + \frac{2\left(c_1+\frac{c_2}{2}\right)e^{-4t}}{7} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= \left(e^{At}\right)^{-1} \\
&= \begin{bmatrix} \frac{(e^{7t}+6)e^{4t}}{7} & -\frac{3(e^{7t}-1)e^{4t}}{7} \\ -\frac{2(e^{7t}-1)e^{4t}}{7} & \frac{(6\,e^{7t}+1)e^{4t}}{7} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{e^{-11t}}{7} + \frac{6\,e^{-4t}}{7} & \frac{3\,e^{-4t}}{7} - \frac{3\,e^{-11t}}{7} \\ \frac{2\,e^{-4t}}{7} - \frac{2\,e^{-11t}}{7} & \frac{6\,e^{-11t}}{7} + \frac{e^{-4t}}{7} \end{bmatrix} \int \begin{bmatrix} \frac{(e^{7t}+6)e^{4t}}{7} & -\frac{3(e^{7t}-1)e^{4t}}{7} \\ -\frac{2(e^{7t}-1)e^{4t}}{7} & \frac{(6\,e^{7t}+1)e^{4t}}{7} \end{bmatrix} \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{e^{-11t}}{7} + \frac{6\,e^{-4t}}{7} & \frac{3\,e^{-4t}}{7} - \frac{3\,e^{-11t}}{7} \\ \frac{2\,e^{-4t}}{7} - \frac{2\,e^{-11t}}{7} & \frac{6\,e^{-11t}}{7} + \frac{e^{-4t}}{7} \end{bmatrix} \begin{bmatrix} \frac{e^{10t}}{70} + \frac{2\,e^{3t}}{7} \\ -\frac{e^{10t}}{35} + \frac{2\,e^{3t}}{21} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3\,e^{-t}}{10} \\ \frac{e^{-t}}{15} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{(c_1-3c_2)e^{-11t}}{7} + \frac{3\,e^{-4t}(2c_1+c_2)}{7} + \frac{3\,e^{-t}}{10} \\ \frac{2(-c_1+3c_2)e^{-11t}}{7} + \frac{e^{-4t}(2c_1+c_2)}{7} + \frac{e^{-t}}{15} \end{bmatrix}$$

23.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix} + \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -5-\lambda & 3 \\ 2 & -10-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 15\lambda + 44 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -4$$
$$\lambda_2 = -11$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-4	1	real eigenvalue
-11	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -11$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix} - (-11)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & 3 & 0 \\ 2 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 6 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix} - (-4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ 2 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-4	1	1	No	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$
-11	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -4 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-4t}\\ &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-4t}\end{aligned}$$

Since eigenvalue -11 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-11t}\\ &= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{-11t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 3\,\mathrm{e}^{-4t} \\ \mathrm{e}^{-4t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{\mathrm{e}^{-11t}}{2} \\ \mathrm{e}^{-11t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 3\,\mathrm{e}^{-4t} & -\frac{\mathrm{e}^{-11t}}{2} \\ \mathrm{e}^{-4t} & \mathrm{e}^{-11t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{2\,\mathrm{e}^{4t}}{7} & \frac{\mathrm{e}^{4t}}{7} \\ -\frac{2\,\mathrm{e}^{11t}}{7} & \frac{6\,\mathrm{e}^{11t}}{7} \end{bmatrix}$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 3\,\mathrm{e}^{-4t} & -\frac{\mathrm{e}^{-11t}}{2} \\ \mathrm{e}^{-4t} & \mathrm{e}^{-11t} \end{bmatrix} \int \begin{bmatrix} \frac{2\,\mathrm{e}^{4t}}{7} & \frac{\mathrm{e}^{4t}}{7} \\ -\frac{2\,\mathrm{e}^{11t}}{7} & \frac{6\,\mathrm{e}^{11t}}{7} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{-t} \\ 0 \end{bmatrix} dt \\
&= \begin{bmatrix} 3\,\mathrm{e}^{-4t} & -\frac{\mathrm{e}^{-11t}}{2} \\ \mathrm{e}^{-4t} & \mathrm{e}^{-11t} \end{bmatrix} \int \begin{bmatrix} \frac{2\,\mathrm{e}^{3t}}{7} \\ -\frac{2\,\mathrm{e}^{10t}}{7} \end{bmatrix} dt \\
&= \begin{bmatrix} 3\,\mathrm{e}^{-4t} & -\frac{\mathrm{e}^{-11t}}{2} \\ \mathrm{e}^{-4t} & \mathrm{e}^{-11t} \end{bmatrix} \begin{bmatrix} \frac{2\,\mathrm{e}^{3t}}{21} \\ -\frac{\mathrm{e}^{10t}}{35} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3\,\mathrm{e}^{-t}}{10} \\ \frac{\mathrm{e}^{-t}}{15} \end{bmatrix}
\end{aligned}
$$

Now that we found particular solution, the final solution is

$$
\begin{gathered}
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} 3c_1\mathrm{e}^{-4t} \\ c_1\mathrm{e}^{-4t} \end{bmatrix} + \begin{bmatrix} -\frac{c_2\mathrm{e}^{-11t}}{2} \\ c_2\mathrm{e}^{-11t} \end{bmatrix} + \begin{bmatrix} \frac{3\,\mathrm{e}^{-t}}{10} \\ \frac{\mathrm{e}^{-t}}{15} \end{bmatrix}
\end{gathered}
$$

Which becomes

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} 3c_1\mathrm{e}^{-4t} - \frac{c_2\mathrm{e}^{-11t}}{2} + \frac{3\,\mathrm{e}^{-t}}{10} \\ c_1\mathrm{e}^{-4t} + c_2\mathrm{e}^{-11t} + \frac{\mathrm{e}^{-t}}{15} \end{bmatrix}
$$

23.2.3 Maple step by step solution

Let's solve

$\left[x' = -5x + 3y(t) + \frac{1}{\mathrm{e}^t}, y'(t) = 2x - 10y(t)\right]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -\frac{5x\,\mathrm{e}^t - 3y(t)\mathrm{e}^t - 1}{\mathrm{e}^t} + 5x - 3y(t) \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-11, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-4, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-11, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{-11t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-4, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{-4t} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-11t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{-4t} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{-11t}}{2} + 3c_2 e^{-4t} \\ c_1 e^{-11t} + c_2 e^{-4t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x = -\frac{c_1 e^{-11t}}{2} + 3c_2 e^{-4t}, y(t) = c_1 e^{-11t} + c_2 e^{-4t}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 48

```
dsolve([diff(x(t),t)=-5*x(t)+3*y(t)+exp(-t),diff(y(t),t)=2*x(t)-10*y(t)],singsol=all)
```

$$x(t) = 3c_2 e^{-4t} - \frac{e^{-11t} c_1}{2} + \frac{3\,e^{-t}}{10}$$
$$y(t) = c_2 e^{-4t} + e^{-11t} c_1 + \frac{e^{-t}}{15}$$

✓ Solution by Mathematica

Time used: 0.092 (sec). Leaf size: 88

```
DSolve[{x'[t]==-5*x[t]+3*y[t]+Exp[-t],y'[t]==2*x[t]-10*y[t]},{x[t],y[t]},t,IncludeSingularSo
```

$$x(t) \to \frac{1}{70} e^{-11t} \left(21e^{10t} + 30(2c_1 + c_2)e^{7t} + 10(c_1 - 3c_2)\right)$$
$$y(t) \to \frac{1}{105} e^{-11t} \left(7e^{10t} + 15(2c_1 + c_2)e^{7t} - 30(c_1 - 3c_2)\right)$$

23.3 problem 5

23.3.1 Solution using Matrix exponential method 2298
23.3.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2300
23.3.3 Maple step by step solution 2305

Internal problem ID [11567]
Internal file name [`OUTPUT/10549_Thursday_May_18_2023_04_22_47_AM_87115532/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 244
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= y(t) \\
y'(t) &= -x + \cos(wt)
\end{aligned}
$$

23.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \cos(wt) \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \cos(t)\, c_1 + \sin(t)\, c_2 \\ -\sin(t)\, c_1 + \cos(t)\, c_2 \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \int \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 0 \\ \cos(wt) \end{bmatrix} dt \\
&= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} -\frac{\cos(t(-1+w))}{-2+2w} + \frac{\cos(t(1+w))}{2+2w} \\ \frac{\sin(t(-1+w))}{-2+2w} + \frac{\sin(t(1+w))}{2+2w} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\cos(wt)}{w^2-1} \\ \frac{w\sin(wt)}{w^2-1} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} \cos(t)\, c_1 + \sin(t)\, c_2 - \frac{\cos(wt)}{w^2-1} \\ -\sin(t)\, c_1 + \cos(t)\, c_2 + \frac{w\sin(wt)}{w^2-1} \end{bmatrix}
\end{aligned}$$

23.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \cos(wt) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -i\mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} i\mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -i\mathrm{e}^{it} & i\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{i\mathrm{e}^{-it}}{2} & \frac{\mathrm{e}^{-it}}{2} \\ -\frac{i\mathrm{e}^{it}}{2} & \frac{\mathrm{e}^{it}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -i\mathrm{e}^{it} & i\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix} \int \begin{bmatrix} \frac{i\mathrm{e}^{-it}}{2} & \frac{\mathrm{e}^{-it}}{2} \\ -\frac{i\mathrm{e}^{it}}{2} & \frac{\mathrm{e}^{it}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ \cos(wt) \end{bmatrix} dt \\
&= \begin{bmatrix} -i\mathrm{e}^{it} & i\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix} \int \begin{bmatrix} \frac{\mathrm{e}^{-it}\cos(wt)}{2} \\ \frac{\mathrm{e}^{it}\cos(wt)}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} -i\mathrm{e}^{it} & i\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix} \begin{bmatrix} -\frac{\mathrm{e}^{-it}(i\cos(wt)-w\sin(wt))}{2w^2-2} \\ \frac{\mathrm{e}^{it}(i\cos(wt)+w\sin(wt))}{2w^2-2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\cos(wt)}{w^2-1} \\ \frac{w\sin(wt)}{w^2-1} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -ic_1\mathrm{e}^{it} \\ c_1\mathrm{e}^{it} \end{bmatrix} + \begin{bmatrix} ic_2\mathrm{e}^{-it} \\ c_2\mathrm{e}^{-it} \end{bmatrix} + \begin{bmatrix} -\frac{\cos(wt)}{w^2-1} \\ \frac{w\sin(wt)}{w^2-1} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{i(-1+w)c_2(1+w)\mathrm{e}^{-it}-ic_1(-1+w)(1+w)\mathrm{e}^{it}-\cos(wt)}{w^2-1} \\ \frac{\mathrm{e}^{-it}(c_1(w^2-1)\mathrm{e}^{2it}+w^2c_2+\mathrm{e}^{it}w\sin(wt)-c_2)}{w^2-1} \end{bmatrix}$$

23.3.3 Maple step by step solution

Let's solve

$[x' = y(t), y'(t) = -x + \cos(wt)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ \cos(wt) \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ \cos(wt) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ \cos(wt) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}t} \cdot \left[\begin{array}{c} \mathrm{I} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \left[\begin{array}{c} \mathrm{I} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} \mathrm{I}(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \left[\begin{array}{c} \sin(t) \\ \cos(t) \end{array}\right], \vec{x}_2(t) = \left[\begin{array}{c} \cos(t) \\ -\sin(t) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \vec{x}_p(t)$$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \left[\begin{array}{cc} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{array}\right]$$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \left[\begin{array}{cc} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]}$$

  - Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \left[\begin{array}{cc} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$
- Take the derivative of the particular solution

  $\overrightarrow{x}'_p(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(t) = \frac{1}{\Phi(t)} \cdot \overrightarrow{f}(t)$
- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{x}_p(t) = \left[\begin{array}{c} \frac{-\cos(wt)+\cos(t)}{w^2-1} \\ \frac{w\sin(wt)-\sin(t)}{w^2-1} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t) + \left[\begin{array}{c} \frac{-\cos(wt)+\cos(t)}{w^2-1} \\ \frac{w\sin(wt)-\sin(t)}{w^2-1} \end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{-\cos(wt)+\cos(t)}{w^2-1} + c_2\cos(t) + c_1\sin(t) \\ \frac{w\sin(wt)-\sin(t)}{w^2-1} - c_2\sin(t) + c_1\cos(t) \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x = \frac{-\cos(wt)+\cos(t)}{w^2-1} + c_2\cos(t) + c_1\sin(t), y(t) = \frac{w\sin(wt)-\sin(t)}{w^2-1} - c_2\sin(t) + c_1\cos(t)\right\}$$

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 72

```
dsolve([diff(x(t),t)=0*x(t)+y(t),diff(y(t),t)=-x(t)+cos(w*t)],singsol=all)
```

$$x(t) = c_2\sin(t) + c_1\cos(t) - \frac{\cos(wt)}{w^2-1}$$
$$y(t) = \frac{\cos(t)\, c_2 w^2 - \sin(t)\, c_1 w^2 + w\sin(wt) - c_2\cos(t) + c_1\sin(t)}{(-1+w)(1+w)}$$

✓ Solution by Mathematica

Time used: 0.119 (sec). Leaf size: 57

```
DSolve[{x'[t]==0*x[t]+y[t],y'[t]==-x[t]+Cos[w*t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to -\frac{\cos(tw)}{w^2-1} + c_1\cos(t) + c_2\sin(t)$$
$$y(t) \to \frac{w\sin(tw)}{w^2-1} + c_2\cos(t) - c_1\sin(t)$$

23.4 problem 6

23.4.1 Solution using Matrix exponential method 2309
23.4.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2311
23.4.3 Maple step by step solution 2317

Internal problem ID [11568]
Internal file name [`OUTPUT/10550_Thursday_May_18_2023_04_22_49_AM_35492760/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 244
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= 3x + 2y(t) + 3 \\
y'(t) &= 7x + 5y(t) + 2t
\end{aligned}
$$

23.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 2t \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \left[\begin{array}{cc} \frac{\left(\sqrt{15}+15\right) e^{-\left(-4+\sqrt{15}\right) t}}{30}-\frac{e^{\left(4+\sqrt{15}\right) t} \left(\sqrt{15}-15\right)}{30} & -\frac{\left(-e^{\left(4+\sqrt{15}\right) t}+e^{-\left(-4+\sqrt{15}\right) t}\right) \sqrt{15}}{15} \\ -\frac{7 \left(-e^{\left(4+\sqrt{15}\right) t}+e^{-\left(-4+\sqrt{15}\right) t}\right) \sqrt{15}}{30} & \frac{\left(-\sqrt{15}+15\right) e^{-\left(-4+\sqrt{15}\right) t}}{30}+\frac{e^{\left(4+\sqrt{15}\right) t} \left(\sqrt{15}+15\right)}{30} \end{array}\right]$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \left[\begin{array}{cc} \frac{\left(\sqrt{15}+15\right) e^{-\left(-4+\sqrt{15}\right) t}}{30}-\frac{e^{\left(4+\sqrt{15}\right) t} \left(\sqrt{15}-15\right)}{30} & -\frac{\left(-e^{\left(4+\sqrt{15}\right) t}+e^{-\left(-4+\sqrt{15}\right) t}\right) \sqrt{15}}{15} \\ -\frac{7 \left(-e^{\left(4+\sqrt{15}\right) t}+e^{-\left(-4+\sqrt{15}\right) t}\right) \sqrt{15}}{30} & \frac{\left(-\sqrt{15}+15\right) e^{-\left(-4+\sqrt{15}\right) t}}{30}+\frac{e^{\left(4+\sqrt{15}\right) t} \left(\sqrt{15}+15\right)}{30} \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \end{array}\right] \\ &= \left[\begin{array}{c} \left(\frac{\left(\sqrt{15}+15\right) e^{-\left(-4+\sqrt{15}\right) t}}{30}-\frac{e^{\left(4+\sqrt{15}\right) t} \left(\sqrt{15}-15\right)}{30}\right) c_1 - \frac{\left(-e^{\left(4+\sqrt{15}\right) t}+e^{-\left(-4+\sqrt{15}\right) t}\right) \sqrt{15}\, c_2}{15} \\ -\frac{7 \left(-e^{\left(4+\sqrt{15}\right) t}+e^{-\left(-4+\sqrt{15}\right) t}\right) \sqrt{15}\, c_1}{30} + \left(\frac{\left(-\sqrt{15}+15\right) e^{-\left(-4+\sqrt{15}\right) t}}{30}+\frac{e^{\left(4+\sqrt{15}\right) t} \left(\sqrt{15}+15\right)}{30}\right) c_2 \end{array}\right] \\ &= \left[\begin{array}{c} \frac{\left(\left(c_1-2c_2\right)\sqrt{15}+15c_1\right) e^{-\left(-4+\sqrt{15}\right) t}}{30}-\frac{e^{\left(4+\sqrt{15}\right) t} \left(\left(c_1-2c_2\right)\sqrt{15}-15c_1\right)}{30} \\ \frac{\left(\left(-7c_1-c_2\right)\sqrt{15}+15c_2\right) e^{-\left(-4+\sqrt{15}\right) t}}{30}+\frac{7\, e^{\left(4+\sqrt{15}\right) t} \left(\left(c_1+\frac{c_2}{7}\right)\sqrt{15}+\frac{15c_2}{7}\right)}{30} \end{array}\right] \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \left[\begin{array}{cc} \frac{e^{-8t}\left(\left(-\sqrt{15}+15\right) e^{-\left(-4+\sqrt{15}\right) t}+e^{\left(4+\sqrt{15}\right) t}\left(\sqrt{15}+15\right)\right)}{30} & \frac{\sqrt{15}\, e^{-8t}\left(-e^{\left(4+\sqrt{15}\right) t}+e^{-\left(-4+\sqrt{15}\right) t}\right)}{15} \\ \frac{7\sqrt{15}\, e^{-8t}\left(-e^{\left(4+\sqrt{15}\right) t}+e^{-\left(-4+\sqrt{15}\right) t}\right)}{30} & -\frac{e^{-8t}\left(\left(-\sqrt{15}-15\right) e^{-\left(-4+\sqrt{15}\right) t}+e^{\left(4+\sqrt{15}\right) t}\left(\sqrt{15}-15\right)\right)}{30} \end{array}\right] \end{aligned}$$

Hence

$$
\vec{x}_p(t) = \left[\begin{array}{cc} \frac{\left(\sqrt{15}+15\right)\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}}{30}-\frac{\mathrm{e}^{\left(4+\sqrt{15}\right)t}\left(\sqrt{15}-15\right)}{30} & -\frac{\left(-\mathrm{e}^{\left(4+\sqrt{15}\right)t}+\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}\right)\sqrt{15}}{15} \\ -\frac{7\left(-\mathrm{e}^{\left(4+\sqrt{15}\right)t}+\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}\right)\sqrt{15}}{30} & \frac{\left(-\sqrt{15}+15\right)\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}}{30}+\frac{\mathrm{e}^{\left(4+\sqrt{15}\right)t}\left(\sqrt{15}+15\right)}{30} \end{array}\right] \int \left[\begin{array}{c} \frac{\mathrm{e}^{-8t}\left(\left(-\cdots\right.\right.}{\cdots} \\ \frac{7\cdots}{\cdots} \end{array}\right]
$$

$$
= \left[\begin{array}{cc} \frac{\left(\sqrt{15}+15\right)\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}}{30}-\frac{\mathrm{e}^{\left(4+\sqrt{15}\right)t}\left(\sqrt{15}-15\right)}{30} & -\frac{\left(-\mathrm{e}^{\left(4+\sqrt{15}\right)t}+\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}\right)\sqrt{15}}{15} \\ -\frac{7\left(-\mathrm{e}^{\left(4+\sqrt{15}\right)t}+\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}\right)\sqrt{15}}{30} & \frac{\left(-\sqrt{15}+15\right)\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}}{30}+\frac{\mathrm{e}^{\left(4+\sqrt{15}\right)t}\left(\sqrt{15}+15\right)}{30} \end{array}\right] \left[\begin{array}{c} \frac{\left(\left(-16t-67\right)\sqrt{\cdots}\right.}{\cdots} \\ \frac{\left(\left(22t+94\right)\sqrt{1\cdots}\right.}{\cdots} \end{array}\right]
$$

$$
= \left[\begin{array}{c} 4t+17 \\ -6t-25 \end{array}\right]
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \left[\begin{array}{c} \frac{\left(\left(c_1-2c_2\right)\sqrt{15}+15c_1\right)\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}}{30}+\frac{\left(\left(-c_1+2c_2\right)\sqrt{15}+15c_1\right)\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{30}+4t+17 \\ \frac{\left(\left(-7c_1-c_2\right)\sqrt{15}+15c_2\right)\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}}{30}+\frac{\left(\left(7c_1+c_2\right)\sqrt{15}+15c_2\right)\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{30}-6t-25 \end{array}\right]
\end{aligned}
$$

23.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\left[\begin{array}{c} x' \\ y'(t) \end{array}\right] = \left[\begin{array}{cc} 3 & 2 \\ 7 & 5 \end{array}\right] \left[\begin{array}{c} x \\ y(t) \end{array}\right] + \left[\begin{array}{c} 3 \\ 2t \end{array}\right]
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 2 \\ 7 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 8\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4 + \sqrt{15}$$
$$\lambda_2 = 4 - \sqrt{15}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$4+\sqrt{15}$	1	real eigenvalue
$4-\sqrt{15}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 4 - \sqrt{15}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} - \left(4 - \sqrt{15}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1+\sqrt{15} & 2 \\ 7 & 1+\sqrt{15} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1+\sqrt{15} & 2 & 0 \\ 7 & 1+\sqrt{15} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{7R_1}{-1+\sqrt{15}} \Longrightarrow \left[\begin{array}{cc|c} -1+\sqrt{15} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+\sqrt{15} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{-1+\sqrt{15}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{-1+\sqrt{15}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{-1+\sqrt{15}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{-1+\sqrt{15}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{-1+\sqrt{15}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{-1+\sqrt{15}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{-1+\sqrt{15}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{-1+\sqrt{15}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{-1+\sqrt{15}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4 + \sqrt{15}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} - \left(4 + \sqrt{15}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 - \sqrt{15} & 2 \\ 7 & 1 - \sqrt{15} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 - \sqrt{15} & 2 & 0 \\ 7 & 1 - \sqrt{15} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{7R_1}{-1 - \sqrt{15}} \Longrightarrow \left[\begin{array}{cc|c} -1 - \sqrt{15} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 - \sqrt{15} & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{1+\sqrt{15}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{1+\sqrt{15}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{1+\sqrt{15}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{1+\sqrt{15}} \\ t \end{bmatrix} = t\begin{bmatrix} \frac{2}{1+\sqrt{15}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{1+\sqrt{15}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{1+\sqrt{15}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{1+\sqrt{15}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{1+\sqrt{15}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$4+\sqrt{15}$	1	1	No	$\begin{bmatrix} \frac{2}{1+\sqrt{15}} \\ 1 \end{bmatrix}$
$4-\sqrt{15}$	1	1	No	$\begin{bmatrix} \frac{2}{1-\sqrt{15}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $4+\sqrt{15}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\left(4+\sqrt{15}\right)t} \\ &= \begin{bmatrix} \frac{2}{1+\sqrt{15}} \\ 1 \end{bmatrix} e^{\left(4+\sqrt{15}\right)t} \end{aligned}$$

Since eigenvalue $4-\sqrt{15}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{\left(4-\sqrt{15}\right)t} \\ &= \begin{bmatrix} \frac{2}{1-\sqrt{15}} \\ 1 \end{bmatrix} e^{\left(4-\sqrt{15}\right)t} \end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x \\ y(t) \end{array}\right] = c_1 \left[\begin{array}{c} \frac{2\,\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{1+\sqrt{15}} \\ \mathrm{e}^{\left(4+\sqrt{15}\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} \frac{2\,\mathrm{e}^{\left(4-\sqrt{15}\right)t}}{1-\sqrt{15}} \\ \mathrm{e}^{\left(4-\sqrt{15}\right)t} \end{array}\right]$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \left[\vec{x}_1 \quad \vec{x}_2 \quad \cdots\right]$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \left[\begin{array}{cc} \frac{2\,\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{1+\sqrt{15}} & \frac{2\,\mathrm{e}^{\left(4-\sqrt{15}\right)t}}{1-\sqrt{15}} \\ \mathrm{e}^{\left(4+\sqrt{15}\right)t} & \mathrm{e}^{\left(4-\sqrt{15}\right)t} \end{array}\right]$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \left[\begin{array}{cc} \frac{7\sqrt{15}\,\mathrm{e}^{-\left(4+\sqrt{15}\right)t}}{30} & \frac{\sqrt{15}\left(1+\sqrt{15}\right)\mathrm{e}^{-\left(4+\sqrt{15}\right)t}}{30} \\ -\frac{7\sqrt{15}\,\mathrm{e}^{\left(-4+\sqrt{15}\right)t}}{30} & \frac{\mathrm{e}^{\left(-4+\sqrt{15}\right)t}\sqrt{15}\left(-1+\sqrt{15}\right)}{30} \end{array}\right]$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{2\,\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{1+\sqrt{15}} & \frac{2\,\mathrm{e}^{\left(4-\sqrt{15}\right)t}}{1-\sqrt{15}} \\ \mathrm{e}^{\left(4+\sqrt{15}\right)t} & \mathrm{e}^{\left(4-\sqrt{15}\right)t} \end{bmatrix} \int \begin{bmatrix} \frac{7\sqrt{15}\,\mathrm{e}^{-\left(4+\sqrt{15}\right)t}}{30} & \frac{\sqrt{15}\left(1+\sqrt{15}\right)\mathrm{e}^{-\left(4+\sqrt{15}\right)t}}{30} \\ -\frac{7\sqrt{15}\,\mathrm{e}^{\left(-4+\sqrt{15}\right)t}}{30} & \frac{\mathrm{e}^{\left(-4+\sqrt{15}\right)t}\sqrt{15}\left(-1+\sqrt{15}\right)}{30} \end{bmatrix} \begin{bmatrix} 3 \\ 2t \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{2\,\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{1+\sqrt{15}} & \frac{2\,\mathrm{e}^{\left(4-\sqrt{15}\right)t}}{1-\sqrt{15}} \\ \mathrm{e}^{\left(4+\sqrt{15}\right)t} & \mathrm{e}^{\left(4-\sqrt{15}\right)t} \end{bmatrix} \int \begin{bmatrix} \frac{\left((2t+21)\sqrt{15}+30t\right)\mathrm{e}^{-\left(4+\sqrt{15}\right)t}}{30} \\ \frac{\left((-2t-21)\sqrt{15}+30t\right)\mathrm{e}^{\left(-4+\sqrt{15}\right)t}}{30} \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{2\,\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{1+\sqrt{15}} & \frac{2\,\mathrm{e}^{\left(4-\sqrt{15}\right)t}}{1-\sqrt{15}} \\ \mathrm{e}^{\left(4+\sqrt{15}\right)t} & \mathrm{e}^{\left(4-\sqrt{15}\right)t} \end{bmatrix} \begin{bmatrix} \frac{\left(4\sqrt{15}\,t+17\sqrt{15}-16t-67\right)\sqrt{15}\,\mathrm{e}^{-\left(4+\sqrt{15}\right)t}\left(2\sqrt{15}\,t+2t+21\right)}{120t+90\sqrt{15}-90} \\ \frac{\left(4\sqrt{15}\,t+17\sqrt{15}+16t+67\right)\sqrt{15}\,\mathrm{e}^{\left(-4+\sqrt{15}\right)t}\left(2\sqrt{15}\,t-2t-21\right)}{-120t+90\sqrt{15}+90} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{32t^3+88t^2-456t-1071}{8t^2-12t-63} \\ \frac{-48t^3-128t^2+678t+1575}{8t^2-12t-63} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{2c_1\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{1+\sqrt{15}} \\ c_1\mathrm{e}^{\left(4+\sqrt{15}\right)t} \end{bmatrix} + \begin{bmatrix} \frac{2c_2\mathrm{e}^{\left(4-\sqrt{15}\right)t}}{1-\sqrt{15}} \\ c_2\mathrm{e}^{\left(4-\sqrt{15}\right)t} \end{bmatrix} + \begin{bmatrix} \frac{32t^3+88t^2-456t-1071}{8t^2-12t-63} \\ \frac{-48t^3-128t^2+678t+1575}{8t^2-12t-63} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_2\left(1+\sqrt{15}\right)\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}}{7} + \frac{c_1\left(-1+\sqrt{15}\right)\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{7} + 4t + 17 \\ c_1\mathrm{e}^{\left(4+\sqrt{15}\right)t} + c_2\mathrm{e}^{-\left(-4+\sqrt{15}\right)t} - 6t - 25 \end{bmatrix}$$

23.4.3 Maple step by step solution

Let's solve

$[x' = 3x + 2y(t) + 3, y'(t) = 7x + 5y(t) + 2t]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 3 \\ 2t \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 3 \\ 2t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 3 \\ 2t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[4-\sqrt{15}, \begin{bmatrix} \frac{2}{1-\sqrt{15}} \\ 1 \end{bmatrix}\right], \left[4+\sqrt{15}, \begin{bmatrix} \frac{2}{1+\sqrt{15}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[4-\sqrt{15}, \begin{bmatrix} \frac{2}{1-\sqrt{15}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{\left(4-\sqrt{15}\right)t} \cdot \begin{bmatrix} \frac{2}{1-\sqrt{15}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4+\sqrt{15}, \left[\begin{array}{c} \frac{2}{1+\sqrt{15}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{\left(4+\sqrt{15}\right)t} \cdot \left[\begin{array}{c} \frac{2}{1+\sqrt{15}} \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

$$\overrightarrow{x}(t) = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2 + \overrightarrow{x}_p(t)$$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \left[\begin{array}{cc} \frac{2\,\mathrm{e}^{\left(4-\sqrt{15}\right)t}}{1-\sqrt{15}} & \frac{2\,\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{1+\sqrt{15}} \\ \mathrm{e}^{\left(4-\sqrt{15}\right)t} & \mathrm{e}^{\left(4+\sqrt{15}\right)t} \end{array}\right]$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \left[\begin{array}{cc} \frac{2\,\mathrm{e}^{\left(4-\sqrt{15}\right)t}}{1-\sqrt{15}} & \frac{2\,\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{1+\sqrt{15}} \\ \mathrm{e}^{\left(4-\sqrt{15}\right)t} & \mathrm{e}^{\left(4+\sqrt{15}\right)t} \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc} \frac{2}{1-\sqrt{15}} & \frac{2}{1+\sqrt{15}} \\ 1 & 1 \end{array}\right]}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \left[\begin{array}{cc} \frac{\left(\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}\left(1+\sqrt{15}\right)+\mathrm{e}^{\left(4+\sqrt{15}\right)t}\left(-1+\sqrt{15}\right)\right)\sqrt{15}}{30} & -\frac{\left(-\mathrm{e}^{\left(4+\sqrt{15}\right)t}+\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}\right)\sqrt{15}}{15} \\ -\frac{7\left(-\mathrm{e}^{\left(4+\sqrt{15}\right)t}+\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}\right)\sqrt{15}}{30} & \frac{\left(-\sqrt{15}+15\right)\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}}{30}+\frac{\mathrm{e}^{\left(4+\sqrt{15}\right)t}\left(\sqrt{15}+15\right)}{30} \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

$$\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$$

- Take the derivative of the particular solution

$$\vec{x}_p'(t) = \Phi'(t)\cdot\vec{v}(t) + \Phi(t)\cdot\vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t)\cdot\vec{v}(t) + \Phi(t)\cdot\vec{v}'(t) = A\cdot\Phi(t)\cdot\vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A\cdot\Phi(t)\cdot\vec{v}(t) + \Phi(t)\cdot\vec{v}'(t) = A\cdot\Phi(t)\cdot\vec{v}(t) + \vec{f}(t)$$

- Cancel like terms

$$\Phi(t)\cdot\vec{v}'(t) = \vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)}\cdot\vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t)\cdot\left(\int_0^t \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \left[\begin{array}{c} -\frac{\sqrt{15}\left(17\sqrt{15}\,\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}+17\sqrt{15}\,\mathrm{e}^{\left(4+\sqrt{15}\right)t}-8\sqrt{15}\,t+67\,\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}-67\,\mathrm{e}^{\left(4+\sqrt{15}\right)t}-34\sqrt{15}\right)}{30\left(4+\sqrt{15}\right)^2\left(-4+\sqrt{15}\right)^2} \\ \frac{\left(94\sqrt{15}+375\right)\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}}{30}+\frac{\left(-94\sqrt{15}+375\right)\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{30}-6t-25 \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \left[\begin{array}{c} -\frac{\sqrt{15}\left(17\sqrt{15}\,\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}+17\sqrt{15}\,\mathrm{e}^{\left(4+\sqrt{15}\right)t}-8\sqrt{15}\,t+67\,\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}-67\,\mathrm{e}^{\left(4+\sqrt{15}\right)t}-34\sqrt{1}}{30\left(4+\sqrt{15}\right)^2\left(-4+\sqrt{15}\right)^2} \\ \frac{\left(94\sqrt{15}+375\right)\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}}{30}+\frac{\left(-94\sqrt{15}+375\right)\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{30}-6t-25 \end{array}\right.$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{\left(\left(-30c_1-469\right)\sqrt{15}-30c_1-1785\right)\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}}{210}+\frac{\left(\left(30c_2+469\right)\sqrt{15}-30c_2-1785\right)\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{210}+4t+17 \\ \frac{\left(30c_1+94\sqrt{15}+375\right)\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}}{30}+\frac{\left(30c_2-94\sqrt{15}+375\right)\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{30}-6t-25 \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x = \frac{\left((-30c_1-469)\sqrt{15}-30c_1-1785\right)\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}}{210} + \frac{\left((30c_2+469)\sqrt{15}-30c_2-1785\right)\mathrm{e}^{\left(4+\sqrt{15}\right)t}}{210} + 4t + 17, y(t) = \frac{(30c\right.$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 91

```
dsolve([diff(x(t),t)=3*x(t)+2*y(t)+3,diff(y(t),t)=7*x(t)+5*y(t)+2*t],singsol=all)
```

$$x(t) = \mathrm{e}^{\left(4+\sqrt{15}\right)t}c_2 + \mathrm{e}^{-\left(-4+\sqrt{15}\right)t}c_1 + 4t + 17$$

$$y(t) = \frac{\mathrm{e}^{\left(4+\sqrt{15}\right)t}c_2\sqrt{15}}{2} - \frac{\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}c_1\sqrt{15}}{2} + \frac{\mathrm{e}^{\left(4+\sqrt{15}\right)t}c_2}{2} + \frac{\mathrm{e}^{-\left(-4+\sqrt{15}\right)t}c_1}{2} - 6t - 25$$

✓ Solution by Mathematica

Time used: 2.783 (sec). Leaf size: 178

```
DSolve[{x'[t]==3*x[t]+2*y[t],y'[t]==7*x[t]+5*y[t]+2*t},{x[t],y[t]},t,IncludeSingularSolution
```

$$x(t) \to \frac{1}{30}e^{-\left(\left(\sqrt{15}-4\right)t\right)}\left(120e^{\left(\sqrt{15}-4\right)t}(t+8) + \left(2\sqrt{15}c_2 - \left(\sqrt{15}-15\right)c_1\right)e^{2\sqrt{15}t} + \left(15+\sqrt{15}\right)c_1 - 2\sqrt{15}c_2\right)$$

$$y(t) \to \frac{1}{30}e^{-\left(\left(\sqrt{15}-4\right)t\right)}\left(-60e^{\left(\sqrt{15}-4\right)t}(3t+23) + \left(7\sqrt{15}c_1 + \left(15+\sqrt{15}\right)c_2\right)e^{2\sqrt{15}t} - 7\sqrt{15}c_1 - \left(\sqrt{15}-15\right)c_2\right)$$

23.5 problem 7

23.5.1 Solution using Matrix exponential method 2322
23.5.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2323
23.5.3 Maple step by step solution 2328

Internal problem ID [11569]
Internal file name [`OUTPUT/10551_Thursday_May_18_2023_04_22_52_AM_54727567/index.tex`]

Book: A First Course in Differential Equations by J. David Logan. Third Edition. Springer-Verlag, NY. 2015.
Section: Chapter 4, Linear Systems. Exercises page 244
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x' = x - 3y(t)$$
$$y'(t) = 3x + 7y(t)$$

23.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{4t}(1-3t) & -3\,e^{4t}t \\ 3\,e^{4t}t & e^{4t}(3t+1) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{4t}(1-3t) & -3\,\mathrm{e}^{4t}t \\ 3\,\mathrm{e}^{4t}t & \mathrm{e}^{4t}(3t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{4t}(1-3t)\,c_1 - 3\,\mathrm{e}^{4t}tc_2 \\ 3\,\mathrm{e}^{4t}tc_1 + \mathrm{e}^{4t}(3t+1)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-3t) - 3c_2t)\,\mathrm{e}^{4t} \\ \mathrm{e}^{4t}(3tc_1 + 3c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

23.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 8\lambda + 16 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
4	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 4 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

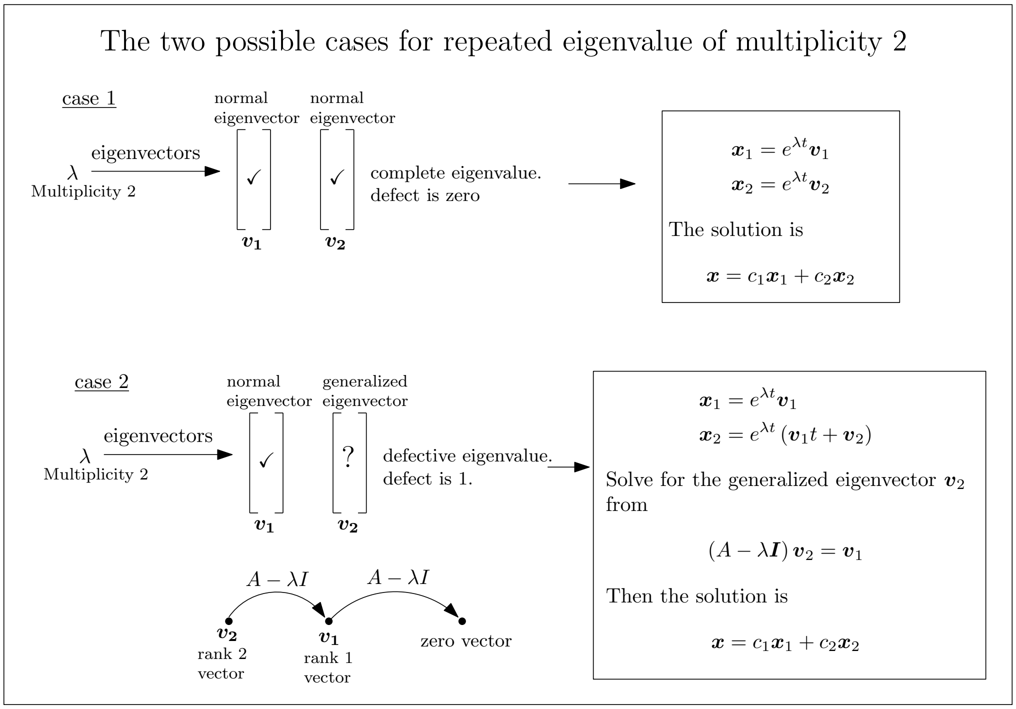


Figure 395: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\, \vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 4. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{4t} \\
&= \begin{bmatrix} -\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} \right) \mathrm{e}^{4t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{4t}(3t+2)}{3} \\ \mathrm{e}^{4t}(1+t) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{4t}\left(-t - \frac{2}{3}\right) \\ \mathrm{e}^{4t}(1+t) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{4t}\left(-c_1 - c_2 t - \frac{2}{3} c_2\right) \\ \mathrm{e}^{4t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

The following is the phase plot of the system.

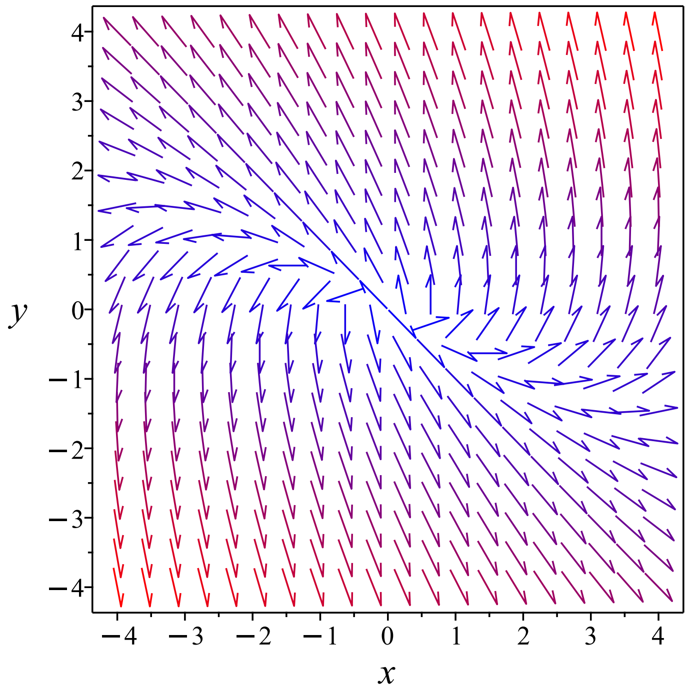


Figure 396: Phase plot

23.5.3 Maple step by step solution

Let's solve

$[x' = x - 3y(t)\,, y'(t) = 3x + 7y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[4, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$
- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[4, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$
- First solution from eigenvalue 4

  $$\overrightarrow{x}_1(t) = e^{4t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 4$ is the eigenvalue, and

  $\overrightarrow{x}_2(t) = e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$
- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

  $\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = \left(e^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$
- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = e^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$
- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$
- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$
- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 4

$$\left(\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} - 4 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 4

$$\vec{x}_2(t) = e^{4t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{4t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{4t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{4t}\left(-c_1 - c_2 t + \frac{1}{3} c_2\right) \\ e^{4t}(c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x = e^{4t}\left(-c_1 - c_2 t + \frac{1}{3} c_2\right), y(t) = e^{4t}(c_2 t + c_1)\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve([diff(x(t),t)=x(t)-3*y(t),diff(y(t),t)=3*x(t)+7*y(t)],singsol=all)
```

$$x(t) = e^{4t}(c_2 t + c_1)$$
$$y(t) = -\frac{e^{4t}(3c_2 t + 3c_1 + c_2)}{3}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 46

```
DSolve[{x'[t]==x[t]-3*y[t],y'[t]==3*x[t]+7*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$
\begin{aligned}
x(t) &\to -e^{4t}(c_1(3t-1)+3c_2t) \\
y(t) &\to e^{4t}(3(c_1+c_2)t+c_2)
\end{aligned}
$$