A Solution Manual For

Second order enumerated odes

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1.1 problem 1

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Internal problem ID [7390]
Internal file name [`OUTPUT/6357_Sunday_June_05_2022_04_41_34_PM_62088223/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y'' = 0}$$

1.1.1 Solving as second order ode quadrature ode

Integrating twice gives the solution

$$y = c_1 x + c_2$$

Summary
The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

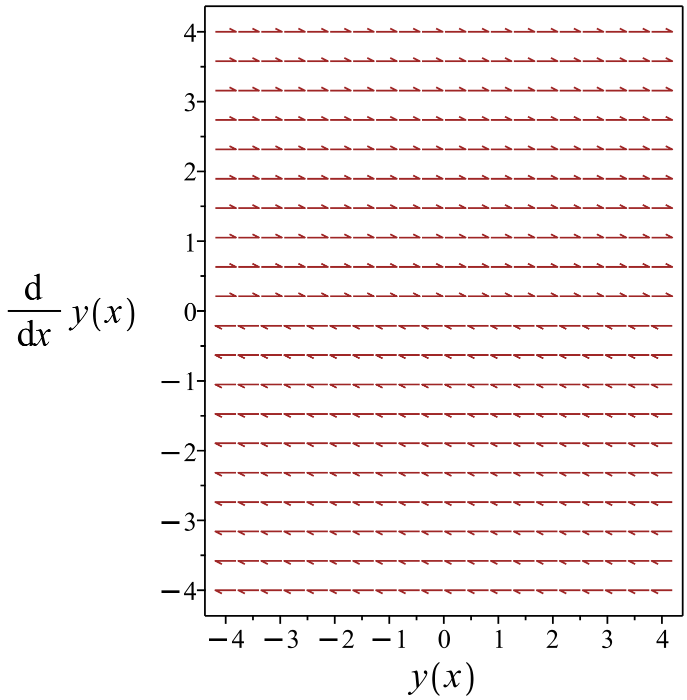

Figure 1: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 x + c_2$$

Verified OK.

1.1.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Summary
The solution(s) found are the following

$$y = c_2 x + c_1 \tag{1}$$

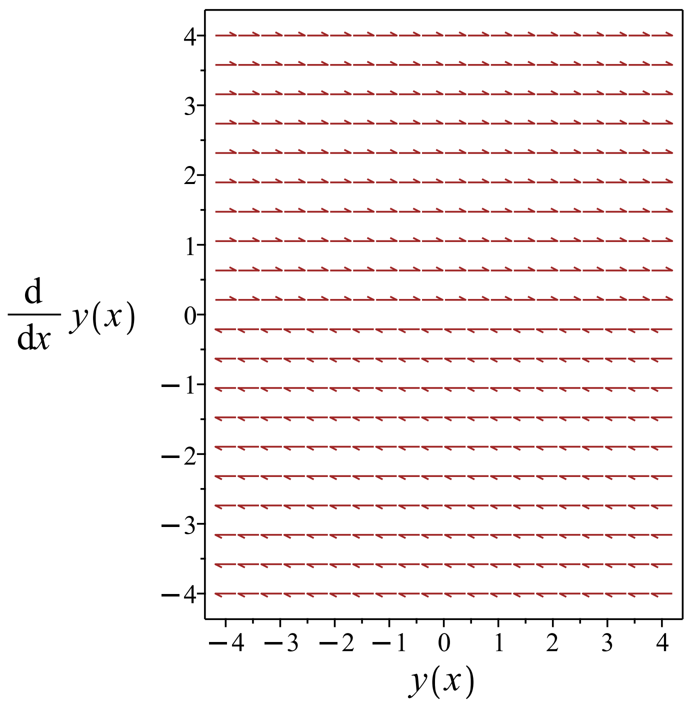

Figure 2: Slope field plot

Verification of solutions

$$y = c_2 x + c_1$$

Verified OK.

1.1.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' = 0$$

Integrating the above w.r.t x gives

$$\int y'y''dx = 0$$
$$\frac{y'^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{c_1}\,\sqrt{2} \tag{1}$$
$$y' = -\sqrt{c_1}\,\sqrt{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \sqrt{c_1}\,\sqrt{2}\, \mathrm{d}x \\ &= \sqrt{c_1}\,\sqrt{2}\,x + c_2 \end{aligned}$$

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\sqrt{c_1}\,\sqrt{2}\, \mathrm{d}x \\ &= -\sqrt{c_1}\,\sqrt{2}\,x + c_3 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{c_1}\,\sqrt{2}\,x + c_2 \tag{1}$$
$$y = -\sqrt{c_1}\,\sqrt{2}\,x + c_3 \tag{2}$$

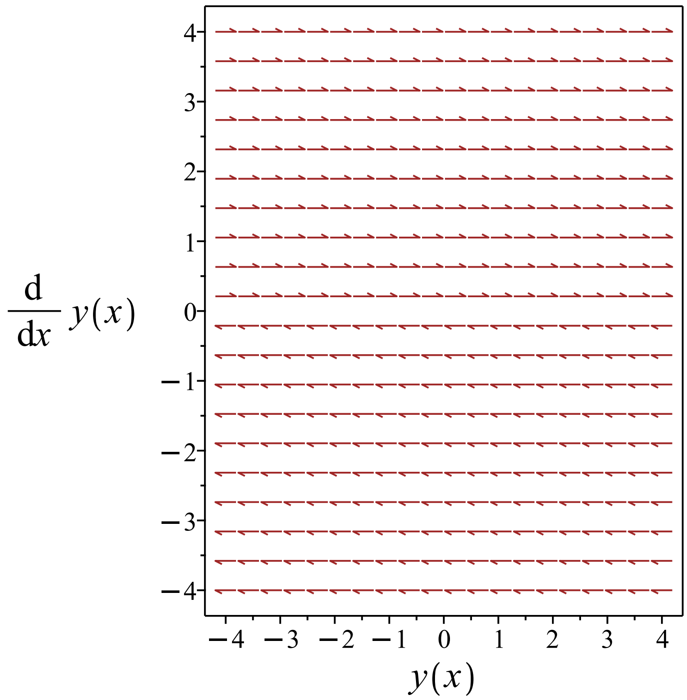

Figure 3: Slope field plot

Verification of solutions

$$y = \sqrt{c_1}\,\sqrt{2}\,x + c_2$$

Verified OK.

$$y = -\sqrt{c_1}\,\sqrt{2}\,x + c_3$$

Verified OK.

1.1.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = 0$$
$$y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$y = \int c_1 \, \mathrm{d}x$$
$$= c_1 x + c_2$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

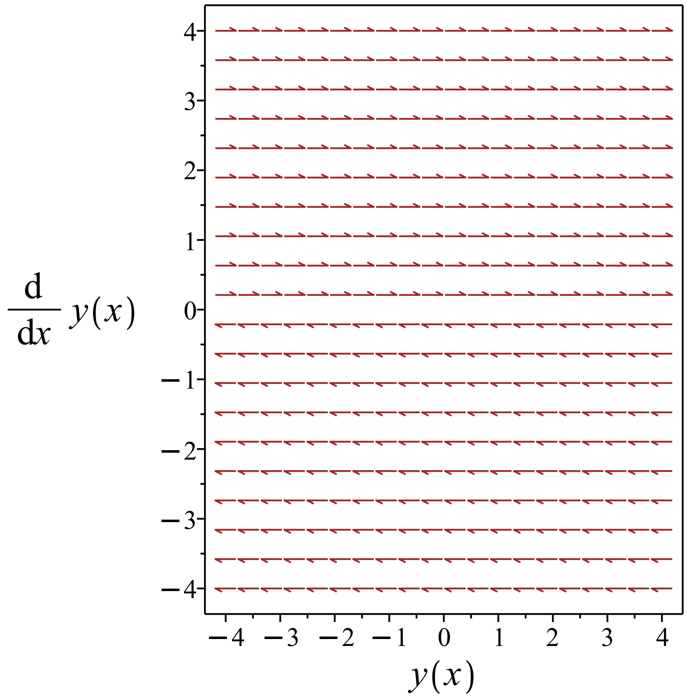

Figure 4: Slope field plot

Verification of solutions

$$y = c_1 x + c_2$$

Verified OK.

1.1.5 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 \, \mathrm{d}x \\ &= c_1 x + c_2 \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

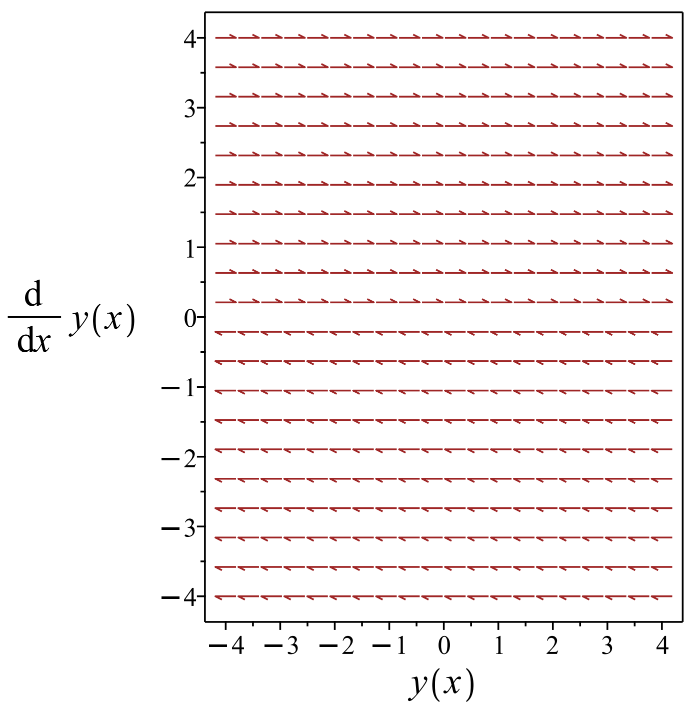

Figure 5: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 x + c_2$$

Verified OK.

1.1.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= 1 \int \frac{1}{1}\, dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 x + c_1 \tag{1}$$

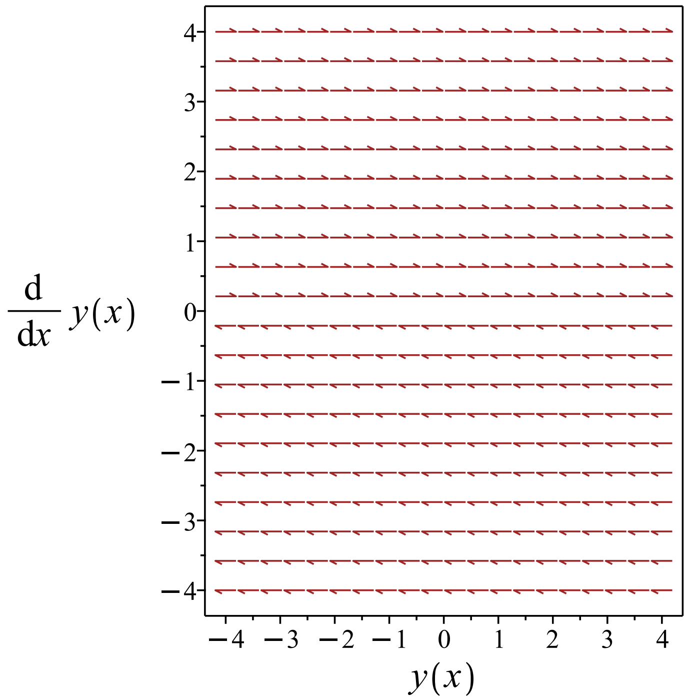

Figure 6: Slope field plot

<u>Verification of solutions</u>

$$y = c_2x + c_1$$

Verified OK.

1.1.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x) &= 1\\ q(x) &= 0\\ r(x) &= 0\\ s(x) &= 0\end{aligned}$$

Hence

$$p''(x) = 0$$
$$q'(x) = 0$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y' = c_1$$

We now have a first order ode to solve which is

$$y' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 \,\mathrm{d}x \\ &= c_1 x + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

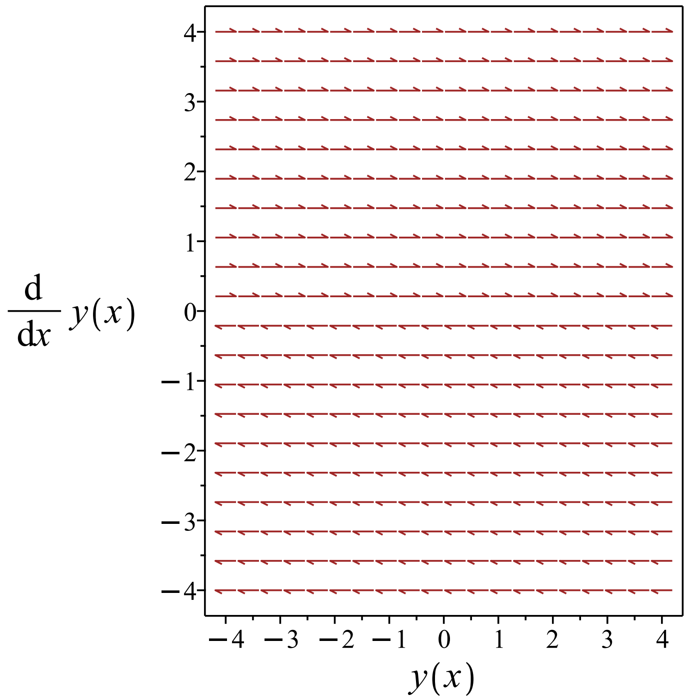

Figure 7: Slope field plot

Verification of solutions

$$y = c_1x + c_2$$

Verified OK.

1.1.8 Maple step by step solution

Let's solve

$y'' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

$r = 0$

- 1st solution of the ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_2 x + c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(y(x),x$2)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 12

```
DSolve[y''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x + c_1$$

1.2 problem 2

1.2.1 Solving as second order ode missing y ode 18
1.2.2 Solving as second order ode missing x ode 20
1.2.3 Solving using Kovacic algorithm 21
1.2.4 Maple step by step solution 22

Internal problem ID [7391]
Internal file name [`OUTPUT/6358_Sunday_June_05_2022_04_41_35_PM_38085196/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 2.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_high_degree", "second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y''^2 = 0}$$

1.2.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode. Solving the given ode for $p'(x)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$p'(x) = 0 \tag{1}$$
$$p'(x) = 0 \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} p(x) &= \int 0 \,\mathrm{d}x \\ &= c_1 \end{aligned}$$

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} p(x) &= \int 0 \,\mathrm{d}x \\ &= c_2 \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then the new first order ode to solve is

$$y' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 \,\mathrm{d}x \\ &= c_1 x + c_3 \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_2$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_2 \,\mathrm{d}x \\ &= c_2 x + c_4 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_3 \tag{1}$$
$$y = c_2 x + c_4 \tag{2}$$

Verification of solutions

$$y = c_1 x + c_3$$

Verified OK.

$$y = c_2 x + c_4$$

Verified OK.

1.2.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned}y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy}\end{aligned}$$

Hence the ode becomes

$$p(y)^2 \left(\frac{d}{dy}p(y)\right)^2 = 0$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\frac{d}{dy}p(y) = 0 \tag{1}$$

$$\frac{d}{dy}p(y) = 0 \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\begin{aligned}p(y) &= \int 0 \, \mathrm{d}y \\ &= c_1\end{aligned}$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\begin{aligned}p(y) &= \int 0 \, \mathrm{d}y \\ &= c_2\end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 \, \mathrm{d}x \\ &= c_1 x + c_3 \end{aligned}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_2$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_2 \, \mathrm{d}x \\ &= c_2 x + c_4 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_3 \tag{1}$$
$$y = c_2 x + c_4 \tag{2}$$

Verification of solutions

$$y = c_1 x + c_3$$

Verified OK.

$$y = c_2 x + c_4$$

Verified OK.

1.2.3 Solving using Kovacic algorithm

Solving for y'' from the ode gives

$$y'' = 0 \tag{1}$$

Now each ode is solved. Integrating twice gives the solution

$$y = c_1 x + c_2$$

Summary
The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2$$

Verified OK.

1.2.4 Maple step by step solution

Let's solve

$y''^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = 0$

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_2 x + c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(y(x),x$2)^2=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 12

```
DSolve[(y''[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x + c_1$$

1.3 problem 3

1.3.1 Solving as second order ode missing y ode 24
1.3.2 Solving as second order ode missing x ode 25
1.3.3 Maple step by step solution 26

Internal problem ID [7392]
Internal file name [`OUTPUT/6359_Sunday_June_05_2022_04_41_37_PM_65502618/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 3.
ODE order: 2.
ODE degree: 0.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y''^n = 0}$$

1.3.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)^{n-1} p'(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 \, \mathrm{d}x \\ &= c_1 x + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2$$

Verified OK.

1.3.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$\left(p(y)\left(\frac{d}{dy}p(y)\right)\right)^{n-1} p(y)\left(\frac{d}{dy}p(y)\right) = 0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned} p(y) &= \int 0 \, \mathrm{d}y \\ &= c_1 \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 \, \mathrm{d}x \\ &= c_1 x + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2$$

Verified OK.

1.3.3 Maple step by step solution

Let's solve

$y''^{n-1} y'' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = 0$

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_2 x + c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(y(x),x$2)^n=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 24

```
DSolve[(y''[x])^n==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} 0^{\frac{1}{n}} x^2 + c_2 x + c_1$$

1.4 problem 4

1.4.1 Solving as second order ode quadrature ode 28
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Internal problem ID [7393]
Internal file name [`OUTPUT/6360_Sunday_June_05_2022_04_41_38_PM_23698650/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{ay'' = 0}$$

1.4.1 Solving as second order ode quadrature ode

Integrating twice gives the solution

$$y = c_1 x + c_2$$

Summary
The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

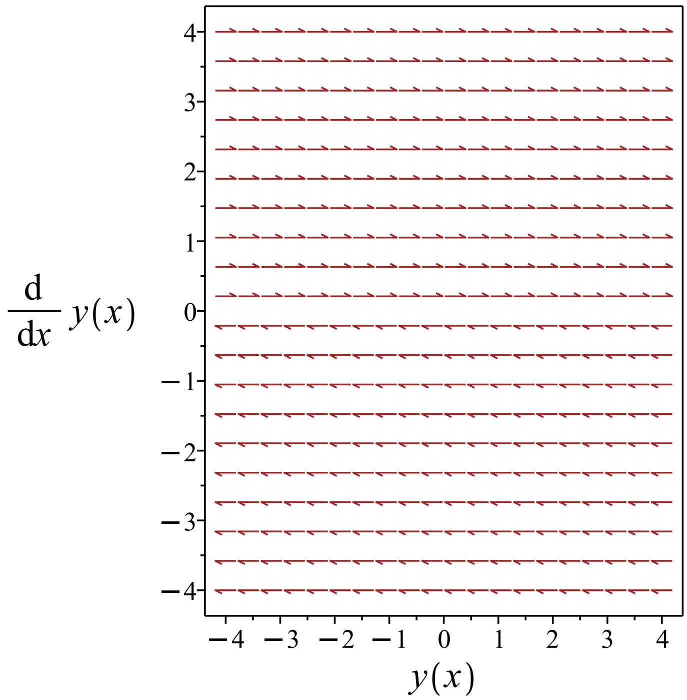

Figure 8: Slope field plot

Verification of solutions

$$y = c_1x + c_2$$

Verified OK.

1.4.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_2 x + c_1 \tag{1}$$

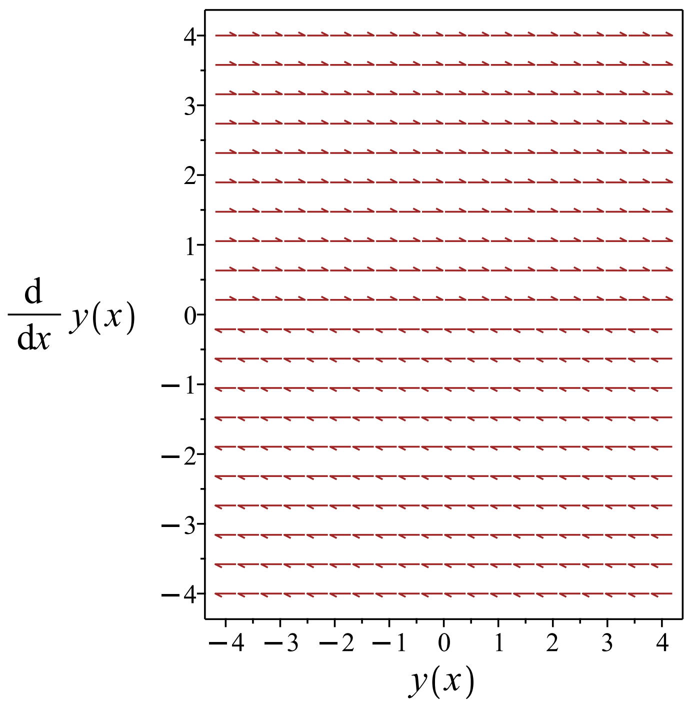

Figure 9: Slope field plot

<u>Verification of solutions</u>

$$y = c_2 x + c_1$$

Verified OK.

1.4.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' = 0$$

Integrating the above w.r.t x gives

$$\int y'y''dx = 0$$
$$\frac{y'^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{c_1}\,\sqrt{2} \tag{1}$$
$$y' = -\sqrt{c_1}\,\sqrt{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \sqrt{c_1}\,\sqrt{2}\,\mathrm{d}x \\ &= \sqrt{c_1}\,\sqrt{2}\,x + c_2 \end{aligned}$$

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\sqrt{c_1}\,\sqrt{2}\,\mathrm{d}x \\ &= -\sqrt{c_1}\,\sqrt{2}\,x + c_3 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{c_1}\,\sqrt{2}\,x + c_2 \tag{1}$$
$$y = -\sqrt{c_1}\,\sqrt{2}\,x + c_3 \tag{2}$$

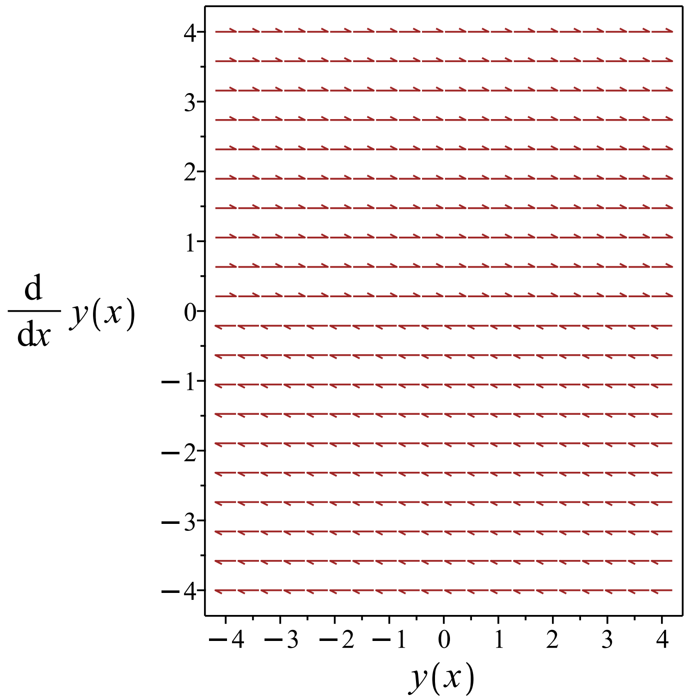

Figure 10: Slope field plot

Verification of solutions

$$y = \sqrt{c_1}\,\sqrt{2}\,x + c_2$$

Verified OK.

$$y = -\sqrt{c_1}\,\sqrt{2}\,x + c_3$$

Verified OK.

1.4.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = 0$$
$$y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$y = \int c_1 \, \mathrm{d}x$$
$$= c_1 x + c_2$$

Summary
The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

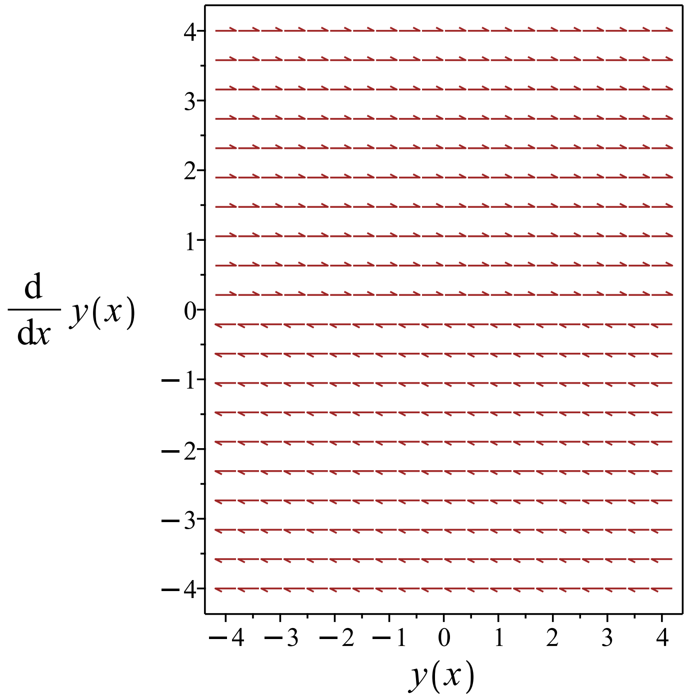

Figure 11: Slope field plot

Verification of solutions

$$y = c_1 x + c_2$$

Verified OK.

1.4.5 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$
\begin{aligned}
p(x) &= \int 0 \, \mathrm{d}x \\
&= c_1
\end{aligned}
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = c_1
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int c_1 \, \mathrm{d}x \\
&= c_1 x + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x + c_2 \tag{1}
$$

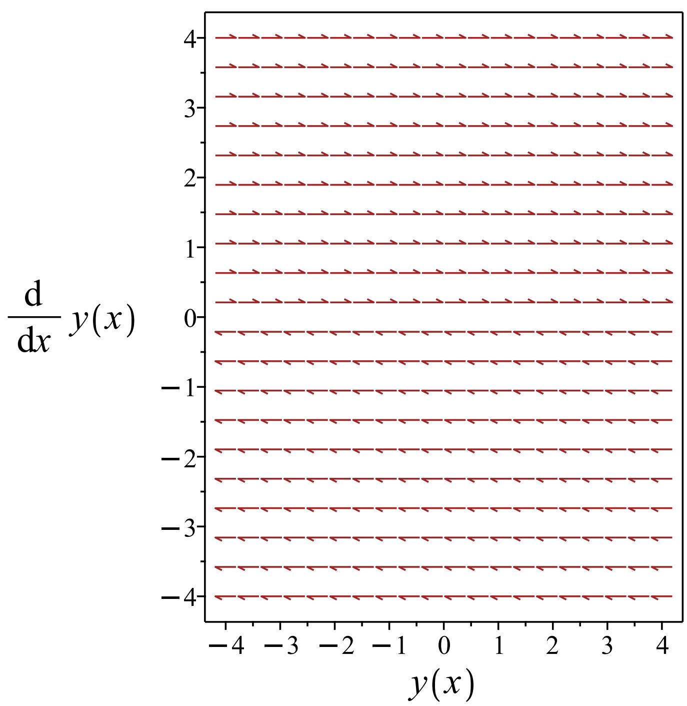

Figure 12: Slope field plot

Verification of solutions

$$
y = c_1 x + c_2
$$

Verified OK.

1.4.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 5: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= 1 \int \frac{1}{1} \, dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 x + c_1 \tag{1}$$

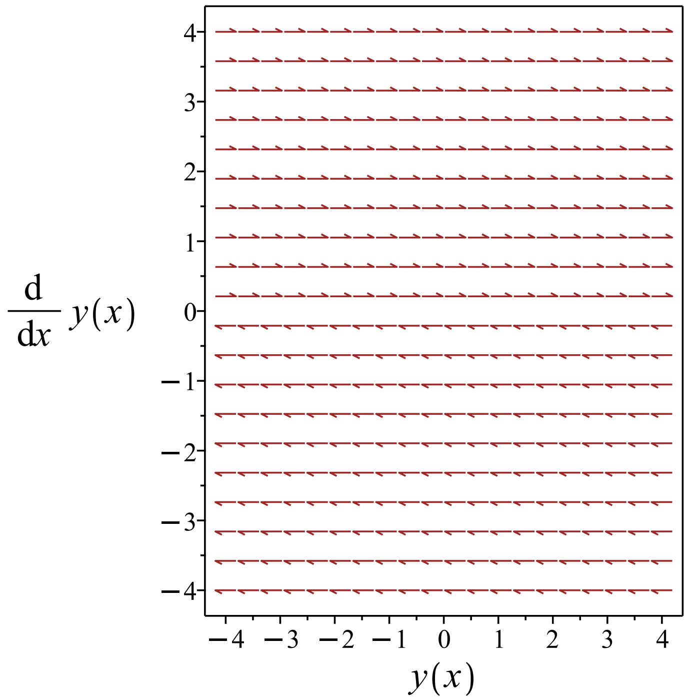

Figure 13: Slope field plot

<u>Verification of solutions</u>

$$y = c_2x + c_1$$

Verified OK.

1.4.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$p''(x) = 0$$
$$q'(x) = 0$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y' = c_1$$

We now have a first order ode to solve which is

$$y' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 \, \mathrm{d}x \\ &= c_1 x + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

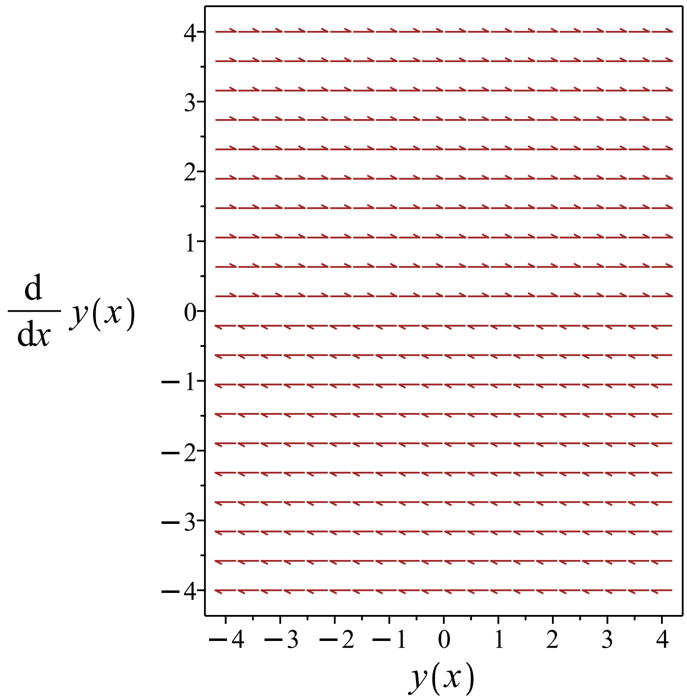

Figure 14: Slope field plot

Verification of solutions

$$y = c_1x + c_2$$

Verified OK.

1.4.8 Maple step by step solution

Let's solve

$y'' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

$r = 0$

- 1st solution of the ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_2 x + c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(a*diff(y(x),x$2)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 12

```
DSolve[a*y''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x + c_1$$

1.5 problem 5

1.5.1 Solving as second order ode missing y ode 42
1.5.2 Solving as second order ode missing x ode 44
1.5.3 Solving using Kovacic algorithm 45
1.5.4 Maple step by step solution 46

Internal problem ID [7394]
Internal file name [`OUTPUT/6361_Sunday_June_05_2022_04_41_40_PM_73947100/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 5.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_high_degree", "second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{a{y''}^2 = 0}$$

1.5.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode. Solving the given ode for $p'(x)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$p'(x) = 0 \tag{1}$$
$$p'(x) = 0 \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\begin{aligned}
p(x) &= \int 0 \, \mathrm{d}x \\
&= c_1
\end{aligned}
$$

Solving equation (2)

Integrating both sides gives

$$
\begin{aligned}
p(x) &= \int 0 \, \mathrm{d}x \\
&= c_2
\end{aligned}
$$

For solution (1) found earlier, since $p = y'$ then the new first order ode to solve is

$$
y' = c_1
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int c_1 \, \mathrm{d}x \\
&= c_1 x + c_3
\end{aligned}
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = c_2
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int c_2 \, \mathrm{d}x \\
&= c_2 x + c_4
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x + c_3 \tag{1}
$$

$$
y = c_2 x + c_4 \tag{2}
$$

Verification of solutions

$$
y = c_1 x + c_3
$$

Verified OK.

$$
y = c_2 x + c_4
$$

Verified OK.

1.5.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^2 \left(\frac{d}{dy}p(y)\right)^2 = 0$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\frac{d}{dy}p(y) = 0 \tag{1}$$

$$\frac{d}{dy}p(y) = 0 \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\begin{aligned} p(y) &= \int 0 \,\mathrm{d}y \\ &= c_1 \end{aligned}$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\begin{aligned} p(y) &= \int 0 \,\mathrm{d}y \\ &= c_2 \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 \, \mathrm{d}x \\ &= c_1 x + c_3 \end{aligned}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_2$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_2 \, \mathrm{d}x \\ &= c_2 x + c_4 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_3 \tag{1}$$
$$y = c_2 x + c_4 \tag{2}$$

Verification of solutions

$$y = c_1 x + c_3$$

Verified OK.

$$y = c_2 x + c_4$$

Verified OK.

1.5.3 Solving using Kovacic algorithm

Solving for y'' from the ode gives

$$y'' = 0 \tag{1}$$

Now each ode is solved. Integrating twice gives the solution

$$y = c_1 x + c_2$$

Summary
The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2$$

Verified OK.

1.5.4 Maple step by step solution

Let's solve

$y''^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = 0$

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_2 x + c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 9

```
dsolve(a*diff(y(x),x$2)^2=0,y(x), singsol=all)
```

$$y(x) = c_1x + c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 12

```
DSolve[a*(y''[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2x + c_1$$

1.6 problem 6

1.6.1 Solving as second order ode missing y ode 48
1.6.2 Solving as second order ode missing x ode 49
1.6.3 Maple step by step solution 50

Internal problem ID [7395]
Internal file name [`OUTPUT/6362_Sunday_June_05_2022_04_41_41_PM_62349154/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 6.
ODE order: 2.
ODE degree: 0.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{a{y''}^{n} = 0}$$

1.6.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)^{n-1} p'(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 \, \mathrm{d}x \\ &= c_1 x + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2$$

Verified OK.

1.6.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$\left(p(y)\left(\frac{d}{dy}p(y)\right)\right)^{n-1} p(y)\left(\frac{d}{dy}p(y)\right) = 0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned} p(y) &= \int 0 \, \mathrm{d}y \\ &= c_1 \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 \, \mathrm{d}x \\ &= c_1 x + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2$$

Verified OK.

1.6.3 Maple step by step solution

Let's solve

$y''^{n-1} y'' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = 0$

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_2 x + c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(a*diff(y(x),x$2)^n=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 24

```
DSolve[a*(y''[x])^n==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} 0^{\frac{1}{n}} x^2 + c_2 x + c_1$$

1.7 problem 7

1.7.1 Solving as second order ode quadrature ode 52
1.7.2 Solving as second order linear constant coeff ode 53
1.7.3 Solving as second order ode can be made integrable ode 56
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1.7.8 Maple step by step solution 68

Internal problem ID [7396]
Internal file name [`OUTPUT/6363_Sunday_June_05_2022_04_41_43_PM_73362624/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y'' = 1}$$

1.7.1 Solving as second order ode quadrature ode

The ODE can be written as

$$y'' = 1$$

Integrating once gives

$$y' = x + c_1$$

Integrating again gives

$$y = \frac{x^2}{2} + c_1 x + c_2$$

Summary
The solution(s) found are the following

$$y = \frac{1}{2}x^2 + c_1 x + c_2 \tag{1}$$

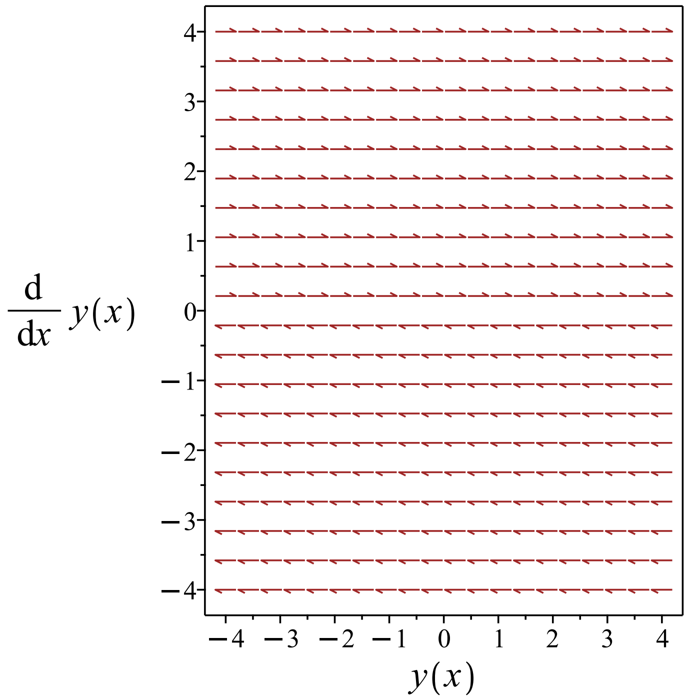

Figure 15: Slope field plot

Verification of solutions

$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

Verified OK.

1.7.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(\frac{x^2}{2}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_2 x + c_1 + \frac{1}{2}x^2 \tag{1}$$

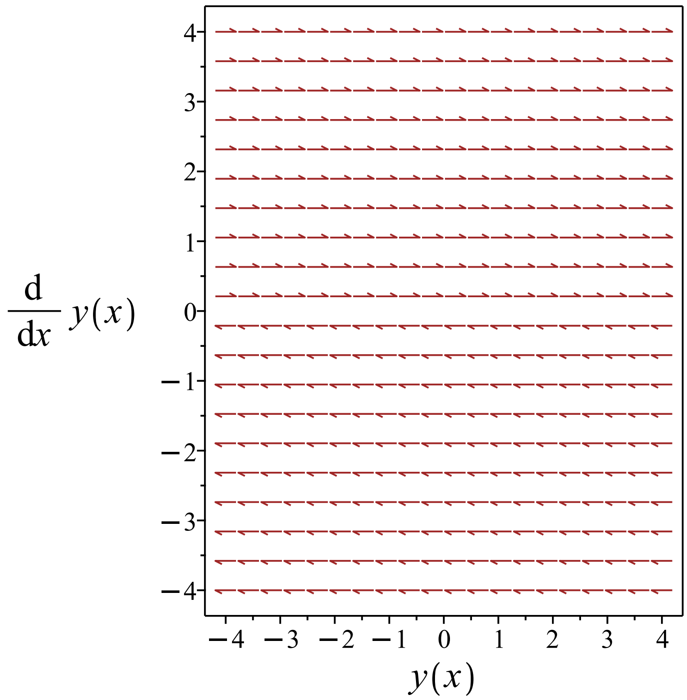

Figure 16: Slope field plot

Verification of solutions

$$y = c_2x + c_1 + \frac{1}{2}x^2$$

Verified OK.

1.7.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - y' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - y')\, dx = 0$$

$$\frac{y'^2}{2} - y = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{2y + 2c_1} \tag{1}$$

$$y' = -\sqrt{2y + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{2y+2c_1}}dy = \int dx$$
$$\sqrt{2y+2c_1} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{2y+2c_1}}dy = \int dx$$
$$-\sqrt{2y+2c_1} = x + c_3$$

Summary

The solution(s) found are the following

$$\sqrt{2y+2c_1} = x + c_2 \tag{1}$$
$$-\sqrt{2y+2c_1} = x + c_3 \tag{2}$$

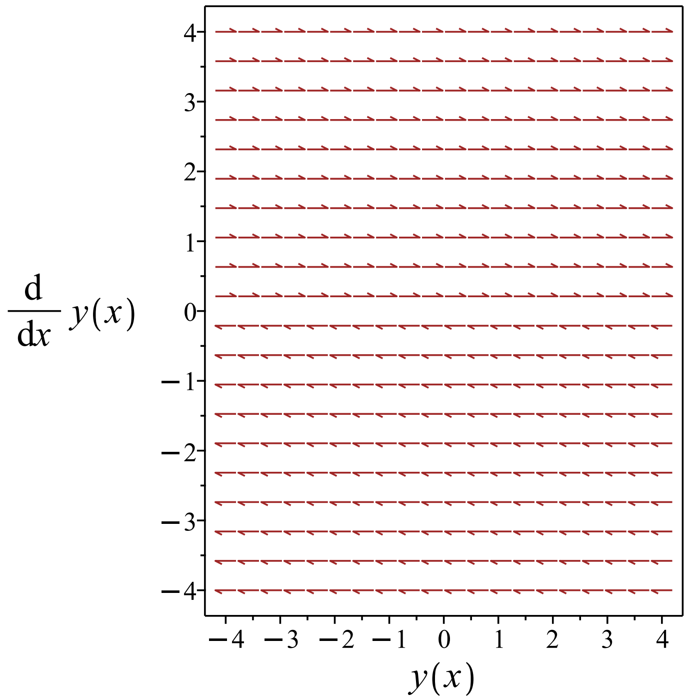

Figure 17: Slope field plot

Verification of solutions

$$\sqrt{2y+2c_1} = x + c_2$$

Verified OK.

$$-\sqrt{2y+2c_1} = x + c_3$$

Verified OK.

1.7.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int 1 dx$$
$$y' = x + c_1$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int x + c_1 \, \mathrm{d}x \\ &= \frac{1}{2}x^2 + c_1 x + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{2}x^2 + c_1 x + c_2 \tag{1}$$

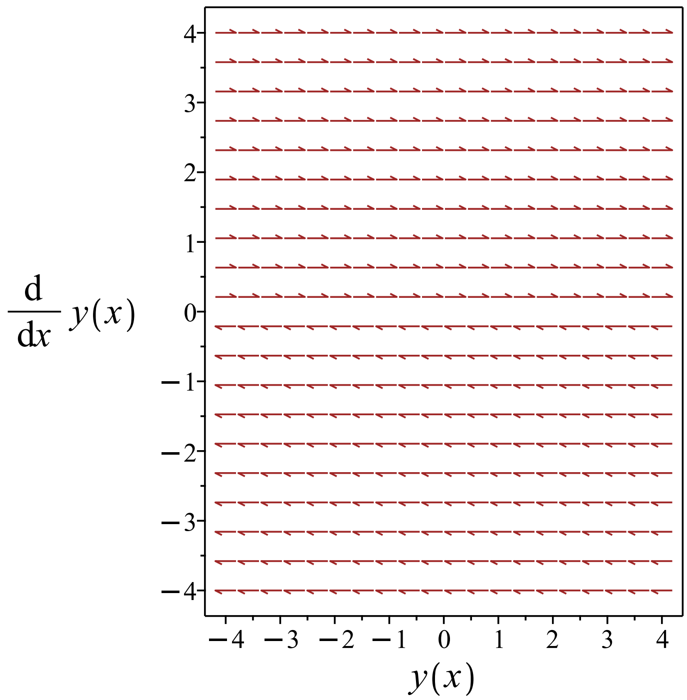

Figure 18: Slope field plot

Verification of solutions

$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

Verified OK.

1.7.5 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 1 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int 1 \, \mathrm{d}x \\ &= x + c_1 \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int x + c_1 \, \mathrm{d}x \\ &= \frac{1}{2}x^2 + c_1 x + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{2}x^2 + c_1 x + c_2 \tag{1}$$

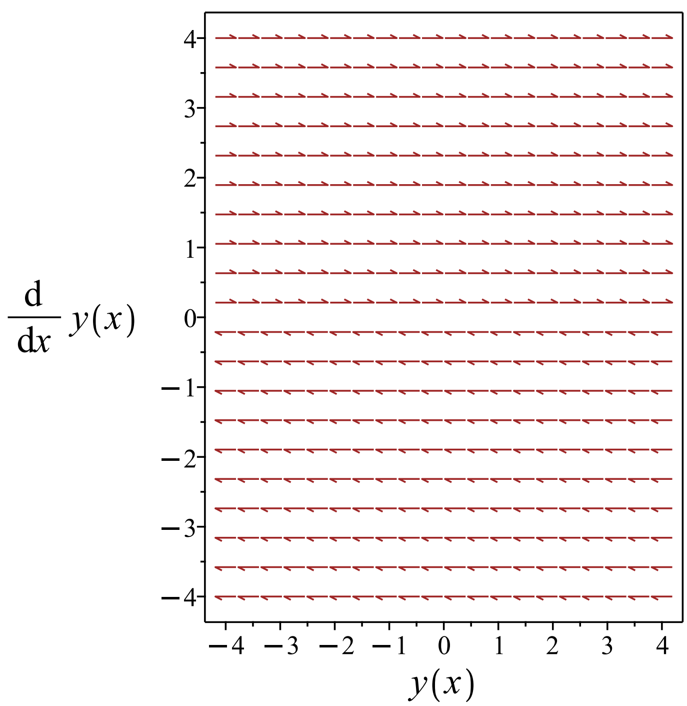

Figure 19: Slope field plot

Verification of solutions

$$y = \frac{1}{2}x^2 + c_1x + c_2$$

Verified OK.

1.7.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 9: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_2x + c_1) + \left(\frac{x^2}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2x + c_1 + \frac{1}{2}x^2 \tag{1}
$$

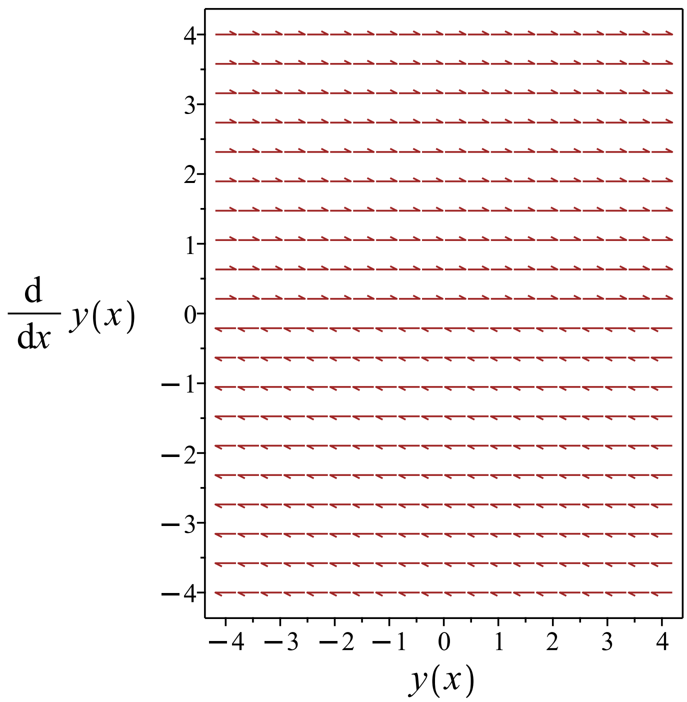

Figure 20: Slope field plot

Verification of solutions

$$
y = c_2x + c_1 + \frac{1}{2}x^2
$$

Verified OK.

1.7.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= 1 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\; dx$$

Substituting the above values for p, q, r, s gives

$$y' = \int 1\,dx$$

We now have a first order ode to solve which is

$$y' = x + c_1$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int x + c_1 \, \mathrm{d}x \\
&= \frac{1}{2}x^2 + c_1 x + c_2
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{1}{2}x^2 + c_1 x + c_2 \tag{1}
$$

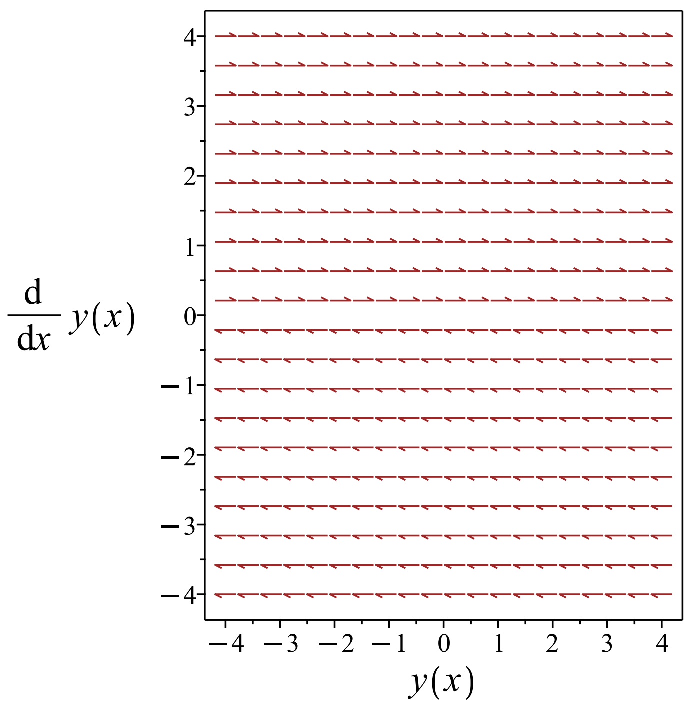

Figure 21: Slope field plot

Verification of solutions

$$
y = \frac{1}{2}x^2 + c_1 x + c_2
$$

Verified OK.

1.7.8 Maple step by step solution

Let's solve

$y'' = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = 1\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

$y_p(x) = -\left(\int x dx\right) + x\left(\int 1 dx\right)$

- Compute integrals

  $y_p(x) = \frac{x^2}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x + c_1 + \frac{1}{2}x^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)=1,y(x), singsol=all)
```

$$y(x) = \frac{1}{2}x^2 + c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 19

```
DSolve[y''[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2} + c_2 x + c_1$$

1.8 problem 8

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Internal problem ID [7397]
Internal file name [`OUTPUT/6364_Sunday_June_05_2022_04_41_44_PM_99458875/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 8.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_high_degree", "second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y''^2 = 1}$$

1.8.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)^2 - 1 = 0$$

Which is now solve for $p(x)$ as first order ode. Solving the given ode for $p'(x)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$p'(x) = 1 \tag{1}$$

$$p'(x) = -1 \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\begin{aligned}
p(x) &= \int 1 \, \mathrm{d}x \\
&= x + c_1
\end{aligned}
$$

Solving equation (2)

Integrating both sides gives

$$
\begin{aligned}
p(x) &= \int -1 \, \mathrm{d}x \\
&= -x + c_2
\end{aligned}
$$

For solution (1) found earlier, since $p = y'$ then the new first order ode to solve is

$$
y' = x + c_1
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int x + c_1 \, \mathrm{d}x \\
&= \frac{1}{2}x^2 + c_1 x + c_3
\end{aligned}
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = -x + c_2
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int -x + c_2 \, \mathrm{d}x \\
&= -\frac{1}{2}x^2 + c_2 x + c_4
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{1}{2}x^2 + c_1 x + c_3 \tag{1}
$$

$$
y = -\frac{1}{2}x^2 + c_2 x + c_4 \tag{2}
$$

Verification of solutions

$$y = \frac{1}{2}x^2 + c_1 x + c_3$$

Verified OK.

$$y = -\frac{1}{2}x^2 + c_2 x + c_4$$

Verified OK.

1.8.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^2 \left(\frac{d}{dy}p(y)\right)^2 = 1$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\frac{d}{dy}p(y) = \frac{1}{p(y)} \tag{1}$$

$$\frac{d}{dy}p(y) = -\frac{1}{p(y)} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int p dp = y + c_1$$
$$\frac{p^2}{2} = y + c_1$$

Solving for p gives these solutions

$$p_1 = \sqrt{2y + 2c_1}$$
$$p_2 = -\sqrt{2y + 2c_1}$$

Solving equation (2)

Integrating both sides gives

$$\int -p dp = y + c_2$$
$$-\frac{p^2}{2} = y + c_2$$

Solving for p gives these solutions

$$p_1 = \sqrt{-2c_2 - 2y}$$
$$p_2 = -\sqrt{-2c_2 - 2y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \sqrt{2y + 2c_1}$$

Integrating both sides gives

$$\int \frac{1}{\sqrt{2y + 2c_1}} dy = \int dx$$
$$\sqrt{2y + 2c_1} = x + c_3$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\sqrt{2y + 2c_1}$$

Integrating both sides gives

$$\int -\frac{1}{\sqrt{2y+2c_1}}dy = \int dx$$
$$-\sqrt{2y+2c_1} = x + c_4$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \sqrt{-2c_2 - 2y}$$

Integrating both sides gives

$$\int \frac{1}{\sqrt{-2c_2-2y}}dy = \int dx$$
$$\frac{2y+2c_2}{\sqrt{-2c_2-2y}} = x + c_5$$

For solution (4) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\sqrt{-2c_2 - 2y}$$

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-2c_2-2y}}dy = \int dx$$
$$-\frac{2(y+c_2)}{\sqrt{-2c_2-2y}} = x + c_6$$

Summary
The solution(s) found are the following

$$y = \frac{1}{2}c_3^2 + c_3x + \frac{1}{2}x^2 - c_1 \tag{1}$$
$$y = \frac{1}{2}c_4^2 + c_4x + \frac{1}{2}x^2 - c_1 \tag{2}$$
$$y = -\frac{1}{2}c_5^2 - xc_5 - \frac{1}{2}x^2 - c_2 \tag{3}$$
$$y = -\frac{1}{2}c_6^2 - xc_6 - \frac{1}{2}x^2 - c_2 \tag{4}$$

Verification of solutions

$$y = \frac{1}{2}c_3^2 + c_3x + \frac{1}{2}x^2 - c_1$$

Verified OK.

$$y = \frac{1}{2}c_4^2 + c_4x + \frac{1}{2}x^2 - c_1$$

Verified OK.

$$y = -\frac{1}{2}c_5^2 - xc_5 - \frac{1}{2}x^2 - c_2$$

Verified OK.

$$y = -\frac{1}{2}c_6^2 - xc_6 - \frac{1}{2}x^2 - c_2$$

Verified OK.

1.8.3 Solving using Kovacic algorithm

Solving for y'' from the ode gives

$$y'' = -1 \tag{1}$$

$$y'' = 1 \tag{2}$$

Now each ode is solved. The ODE can be written as

$$y'' = -1$$

Integrating once gives

$$y' = -x + c_1$$

Integrating again gives

$$y = -\frac{x^2}{2} + c_1x + c_2$$

The ODE can be written as

$$y'' = 1$$

Integrating once gives

$$y' = x + c_3$$

Integrating again gives

$$y = \frac{x^2}{2} + c_3x + c_4$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{2}x^2 + c_1 x + c_2 \tag{1}$$

$$y = \frac{1}{2}x^2 + c_3 x + c_4 \tag{2}$$

Verification of solutions

$$y = -\frac{1}{2}x^2 + c_1 x + c_2$$

Verified OK.

$$y = \frac{1}{2}x^2 + c_3 x + c_4$$

Verified OK.

1.8.4 Maple step by step solution

Let's solve

$y''^2 = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' = RootOf(\_Z^2 - 1)$

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = RootOf\left(\_Z^2 - 1\right)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = RootOf\left(\_Z^2 - 1\right)\left(-\left(\int x dx\right) + x\left(\int 1 dx\right)\right)$

  - Compute integrals

    $y_p(x) = \frac{RootOf\left(\_Z^2 - 1\right) x^2}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 x + \frac{RootOf\left(\_Z^2 - 1\right) x^2}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)^2=1,y(x), singsol=all)
```

$$y(x) = \frac{1}{2}x^2 + c_1x + c_2$$
$$y(x) = -\frac{1}{2}x^2 + c_1x + c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 37

```
DSolve[(y''[x])^2==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2}{2} + c_2x + c_1$$
$$y(x) \to \frac{x^2}{2} + c_2x + c_1$$

1.9 problem 9

1.9.1 Solving as second order ode quadrature ode 79
1.9.2 Solving as second order linear constant coeff ode 80
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Internal problem ID [7398]
Internal file name [`OUTPUT/6365_Sunday_June_05_2022_04_41_46_PM_46265585/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y'' = x}$$

1.9.1 Solving as second order ode quadrature ode

Integrating once gives

$$y' = \frac{x^2}{2} + c_1$$

Integrating again gives

$$y = \frac{x^3}{6} + c_1 x + c_2$$

Summary
The solution(s) found are the following

$$y = \frac{1}{6}x^3 + c_1 x + c_2 \tag{1}$$

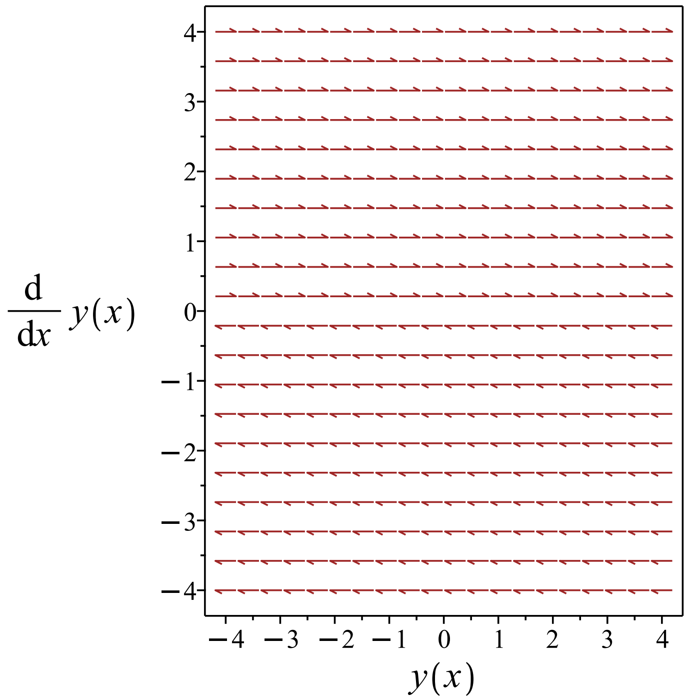

Figure 22: Slope field plot

Verification of solutions

$$y = \frac{1}{6}x^3 + c_1 x + c_2$$

Verified OK.

1.9.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)}\\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^3 + A_1 x^2$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6xA_2 + 2A_1 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^3}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(\frac{x^3}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x + c_1 + \frac{1}{6}x^3 \tag{1}$$

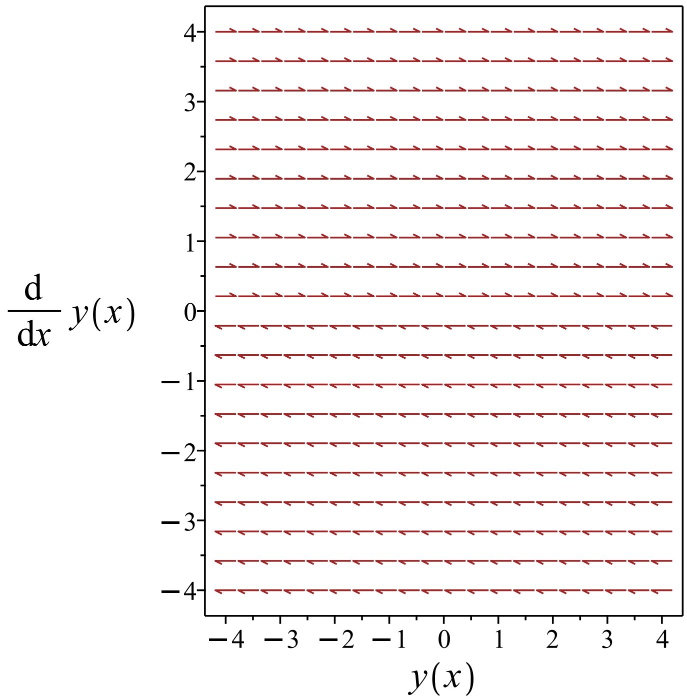

Figure 23: Slope field plot

Verification of solutions

$$y = c_2x + c_1 + \frac{1}{6}x^3$$

Verified OK.

1.9.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int x dx$$

$$y' = \frac{x^2}{2} + c_1$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int \frac{x^2}{2} + c_1 \, \mathrm{d}x \\ &= \frac{1}{6}x^3 + c_1x + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{6}x^3 + c_1 x + c_2 \tag{1}$$

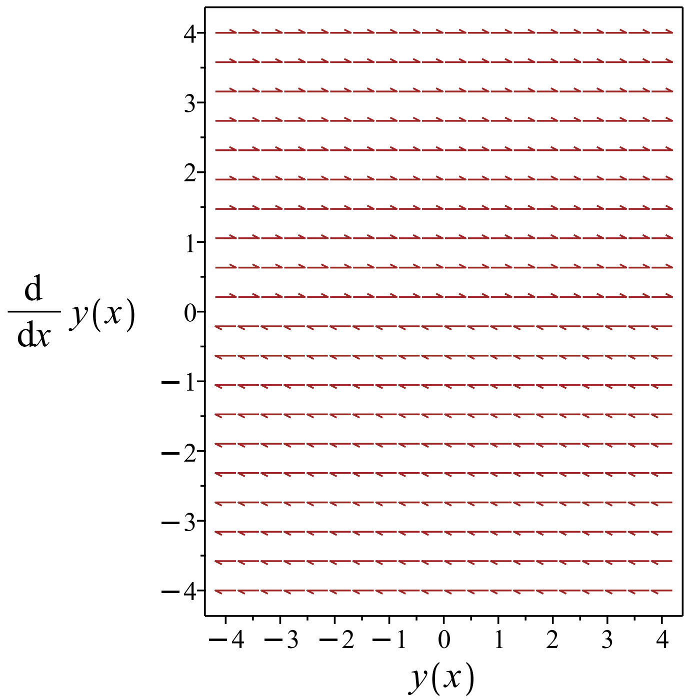

Figure 24: Slope field plot

Verification of solutions

$$y = \frac{1}{6}x^3 + c_1 x + c_2$$

Verified OK.

1.9.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - x = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int x \, \mathrm{d}x \\ &= \frac{x^2}{2} + c_1 \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{x^2}{2} + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{x^2}{2} + c_1 \, \mathrm{d}x \\ &= \frac{1}{6}x^3 + c_1 x + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{6}x^3 + c_1 x + c_2 \tag{1}$$

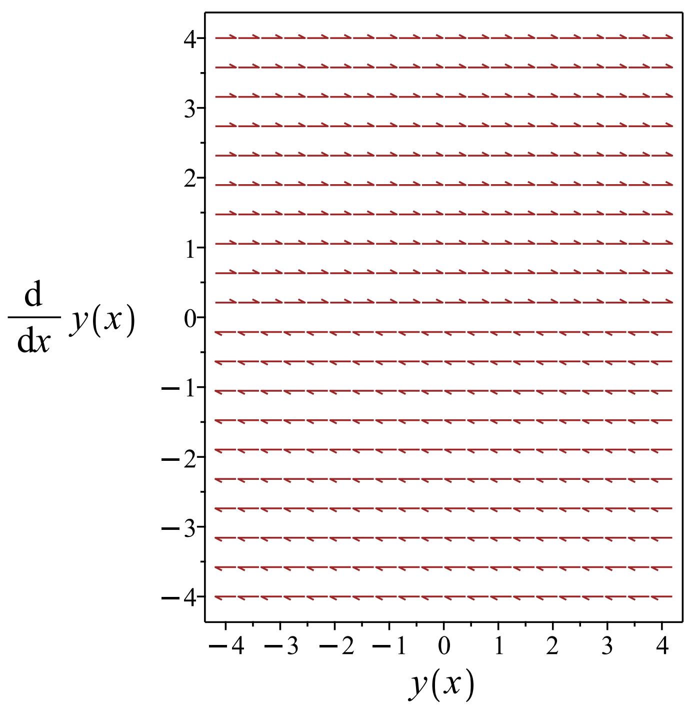

Figure 25: Slope field plot

Verification of solutions

$$y = \frac{1}{6}x^3 + c_1 x + c_2$$

Verified OK.

1.9.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 12: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2x^3 + A_1x^2$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6xA_2 + 2A_1 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^3}{6}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_2 x + c_1) + \left(\frac{x^3}{6}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2 x + c_1 + \frac{1}{6}x^3 \tag{1}
$$

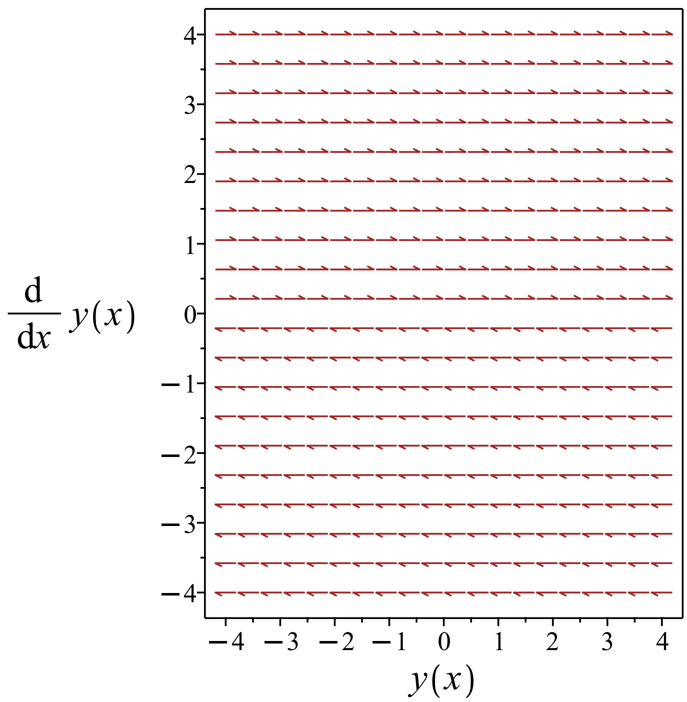

Figure 26: Slope field plot

Verification of solutions

$$
y = c_2 x + c_1 + \frac{1}{6}x^3
$$

Verified OK.

1.9.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$y' = \int x\,dx$$

We now have a first order ode to solve which is

$$y' = \frac{x^2}{2} + c_1$$

Integrating both sides gives

$$y = \int \frac{x^2}{2} + c_1 \,\mathrm{d}x$$
$$= \frac{1}{6}x^3 + c_1 x + c_2$$

Summary
The solution(s) found are the following

$$y = \frac{1}{6}x^3 + c_1 x + c_2 \tag{1}$$

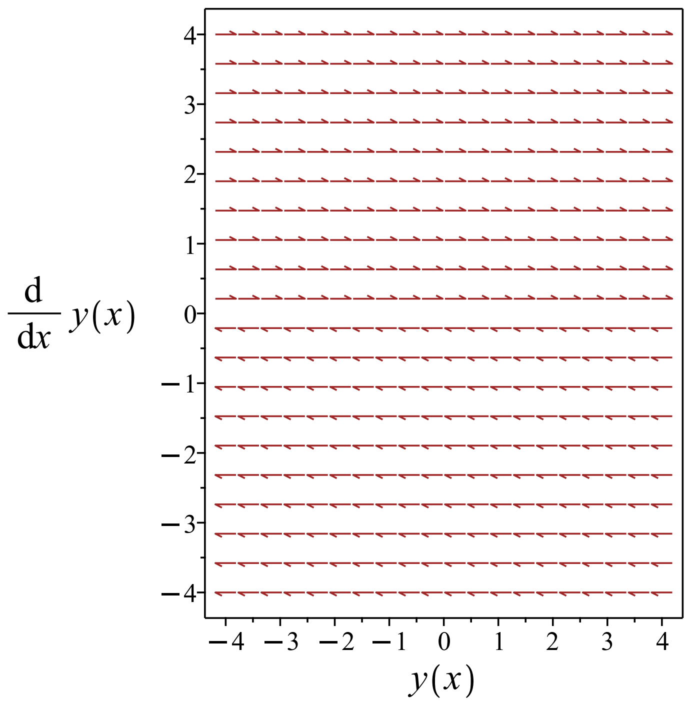

Figure 27: Slope field plot

Verification of solutions

$$y = \frac{1}{6}x^3 + c_1 x + c_2$$

Verified OK.

1.9.7 Maple step by step solution

Let's solve

$y'' = x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\left(\int x^2 dx\right) + x\left(\int x dx\right)$$

- Compute integrals

$$y_p(x) = \frac{x^3}{6}$$

- Substitute particular solution into general solution to ODE

$$y = c_2 x + c_1 + \frac{1}{6}x^3$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)=x,y(x), singsol=all)
```

$$y(x) = \frac{1}{6}x^3 + c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 19

```
DSolve[y''[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{6} + c_2 x + c_1$$

1.10 problem 10

1.10.1 Solving as second order ode missing y ode 95
1.10.2 Solving using Kovacic algorithm 97
1.10.3 Maple step by step solution 98

Internal problem ID [7399]
Internal file name [`OUTPUT/6366_Sunday_June_05_2022_04_41_48_PM_18474351/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 10.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_high_degree", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y''^2 = x}$$

1.10.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)^2 - x = 0$$

Which is now solve for $p(x)$ as first order ode. Solving the given ode for $p'(x)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$p'(x) = \sqrt{x} \tag{1}$$

$$p'(x) = -\sqrt{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\begin{aligned}
p(x) &= \int \sqrt{x}\, \mathrm{d}x \\
&= \frac{2x^{\frac{3}{2}}}{3} + c_1
\end{aligned}
$$

Solving equation (2)

Integrating both sides gives

$$
\begin{aligned}
p(x) &= \int -\sqrt{x}\, \mathrm{d}x \\
&= -\frac{2x^{\frac{3}{2}}}{3} + c_2
\end{aligned}
$$

For solution (1) found earlier, since $p = y'$ then the new first order ode to solve is

$$
y' = \frac{2x^{\frac{3}{2}}}{3} + c_1
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{2x^{\frac{3}{2}}}{3} + c_1\, \mathrm{d}x \\
&= c_1 x + \frac{4x^{\frac{5}{2}}}{15} + c_3
\end{aligned}
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = -\frac{2x^{\frac{3}{2}}}{3} + c_2
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int -\frac{2x^{\frac{3}{2}}}{3} + c_2\, \mathrm{d}x \\
&= c_2 x - \frac{4x^{\frac{5}{2}}}{15} + c_4
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x + \frac{4x^{\frac{5}{2}}}{15} + c_3 \tag{1}
$$

$$
y = c_2 x - \frac{4x^{\frac{5}{2}}}{15} + c_4 \tag{2}
$$

Verification of solutions

$$y = c_1x + \frac{4x^{\frac{5}{2}}}{15} + c_3$$

Verified OK.

$$y = c_2x - \frac{4x^{\frac{5}{2}}}{15} + c_4$$

Verified OK.

1.10.2 Solving using Kovacic algorithm

Solving for y'' from the ode gives

$$y'' = \sqrt{x} \tag{1}$$

$$y'' = -\sqrt{x} \tag{2}$$

Now each ode is solved. Integrating once gives

$$y' = \frac{2x^{\frac{3}{2}}}{3} + c_1$$

Integrating again gives

$$y = \frac{4x^{\frac{5}{2}}}{15} + c_1x + c_2$$

Integrating once gives

$$y' = -\frac{2x^{\frac{3}{2}}}{3} + c_3$$

Integrating again gives

$$y = -\frac{4x^{\frac{5}{2}}}{15} + c_3x + c_4$$

Summary

The solution(s) found are the following

$$y = \frac{4x^{\frac{5}{2}}}{15} + c_1x + c_2 \tag{1}$$

$$y = -\frac{4x^{\frac{5}{2}}}{15} + c_3x + c_4 \tag{2}$$

Verification of solutions

$$y = \frac{4x^{\frac{5}{2}}}{15} + c_1 x + c_2$$

Verified OK.

$$y = -\frac{4x^{\frac{5}{2}}}{15} + c_3 x + c_4$$

Verified OK.

1.10.3 Maple step by step solution

Let's solve

$y''^2 = x$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \sqrt{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' = \mathit{RootOf}\left(\_Z^2 - x\right)$

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 + c_2 x + y_p(x)$$

- □ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = \mathit{RootOf}\left(\_Z^2 - x\right)\right]$$
  - Wronskian of solutions of the homogeneous equation
    $$W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$
  - Compute Wronskian
    $$W(y_1(x), y_2(x)) = 1$$
  - Substitute functions into equation for $y_p(x)$
    $$y_p(x) = -\left(\int x \mathit{RootOf}\left(\_Z^2 - x\right) dx\right) + x\left(\int \mathit{RootOf}\left(\_Z^2 - x\right) dx\right)$$
  - Compute integrals
    $$y_p(x) = \frac{4x^2 \mathit{RootOf}\left(\_Z^2 - x\right)}{15}$$
- Substitute particular solution into general solution to ODE
  $$y = c_1 + c_2 x + \frac{4x^2 \mathit{RootOf}\left(\_Z^2 - x\right)}{15}$$

Maple trace

```
`Methods for second order ODEs:
Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting OD
   *** Sublevel 2 ***
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   <- quadrature successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   <- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)^2=x,y(x), singsol=all)
```

$$y(x) = \frac{4x^{\frac{5}{2}}}{15} + c_1 x + c_2$$
$$y(x) = -\frac{4x^{\frac{5}{2}}}{15} + c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 41

```
DSolve[(y''[x])^2==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{4x^{5/2}}{15} + c_2 x + c_1$$
$$y(x) \to \frac{4x^{5/2}}{15} + c_2 x + c_1$$

1.11 problem 11

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Internal problem ID [7400]
Internal file name [`OUTPUT/6367_Sunday_June_05_2022_04_41_49_PM_91902991/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 11.
ODE order: 2.
ODE degree: 3.

The type(s) of ODE detected by this program : **"second_order_ode_high_degree", "second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y''^3 = 0}$$

1.11.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)^3 = 0$$

Which is now solve for $p(x)$ as first order ode. Solving the given ode for $p'(x)$ results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$p'(x) = 0 \tag{1}$$
$$p'(x) = 0 \tag{2}$$
$$p'(x) = 0 \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} p(x) &= \int 0 \,\mathrm{d}x \\ &= c_1 \end{aligned}$$

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} p(x) &= \int 0 \,\mathrm{d}x \\ &= c_2 \end{aligned}$$

Solving equation (3)

Integrating both sides gives

$$\begin{aligned} p(x) &= \int 0 \,\mathrm{d}x \\ &= c_3 \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then the new first order ode to solve is

$$y' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 \,\mathrm{d}x \\ &= c_1 x + c_4 \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_2$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int c_2 \, \mathrm{d}x \\
&= c_2 x + c_5
\end{aligned}
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = c_3
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int c_3 \, \mathrm{d}x \\
&= c_3 x + c_6
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x + c_4 \tag{1}
$$
$$
y = c_2 x + c_5 \tag{2}
$$
$$
y = c_3 x + c_6 \tag{3}
$$

Verification of solutions

$$
y = c_1 x + c_4
$$

Verified OK.

$$
y = c_2 x + c_5
$$

Verified OK.

$$
y = c_3 x + c_6
$$

Verified OK.

1.11.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$
y' = p(y)
$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y)^3 \left(\frac{d}{dy}p(y)\right)^3 = 0
$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$
\frac{d}{dy}p(y) = 0 \tag{1}
$$

$$
\frac{d}{dy}p(y) = 0 \tag{2}
$$

$$
\frac{d}{dy}p(y) = 0 \tag{3}
$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\begin{aligned}
p(y) &= \int 0 \,\mathrm{d}y \\
&= c_1
\end{aligned}
$$

Solving equation (2)

Integrating both sides gives

$$
\begin{aligned}
p(y) &= \int 0 \,\mathrm{d}y \\
&= c_2
\end{aligned}
$$

Solving equation (3)

Integrating both sides gives

$$
\begin{aligned}
p(y) &= \int 0 \, \mathrm{d}y \\
&= c_3
\end{aligned}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = c_1
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int c_1 \, \mathrm{d}x \\
&= c_1 x + c_4
\end{aligned}
$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = c_2
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int c_2 \, \mathrm{d}x \\
&= c_2 x + c_5
\end{aligned}
$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = c_3
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int c_3 \, \mathrm{d}x \\
&= c_3 x + c_6
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x + c_4 \tag{1}
$$
$$
y = c_2 x + c_5 \tag{2}
$$
$$
y = c_3 x + c_6 \tag{3}
$$

Verification of solutions

$$y = c_1x + c_4$$

Verified OK.

$$y = c_2x + c_5$$

Verified OK.

$$y = c_3x + c_6$$

Verified OK.

1.11.3 Solving using Kovacic algorithm

Solving for y'' from the ode gives

$$y'' = 0 \tag{1}$$

Now each ode is solved. Integrating twice gives the solution

$$y = c_1x + c_2$$

Summary
The solution(s) found are the following

$$y = c_1x + c_2 \tag{1}$$

Verification of solutions

$$y = c_1x + c_2$$

Verified OK.

1.11.4 Maple step by step solution

Let's solve

$y''^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = 0$

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_2 x + c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 9

```
dsolve(diff(y(x),x$2)^3=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 12

```
DSolve[(y''[x])^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_2 x + c_1$$

1.12 problem 12

1.12.1 Solving as second order linear constant coeff ode 108
1.12.2 Solving as second order integrable as is ode 110
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Internal problem ID [7401]
Internal file name [`OUTPUT/6368_Sunday_June_05_2022_04_41_51_PM_26446548/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' = 0}$$

1.12.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(0)} \\ &= -\frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(0)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + c_2 \mathrm{e}^{-x}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 + c_2 \mathrm{e}^{-x} \tag{1}$$

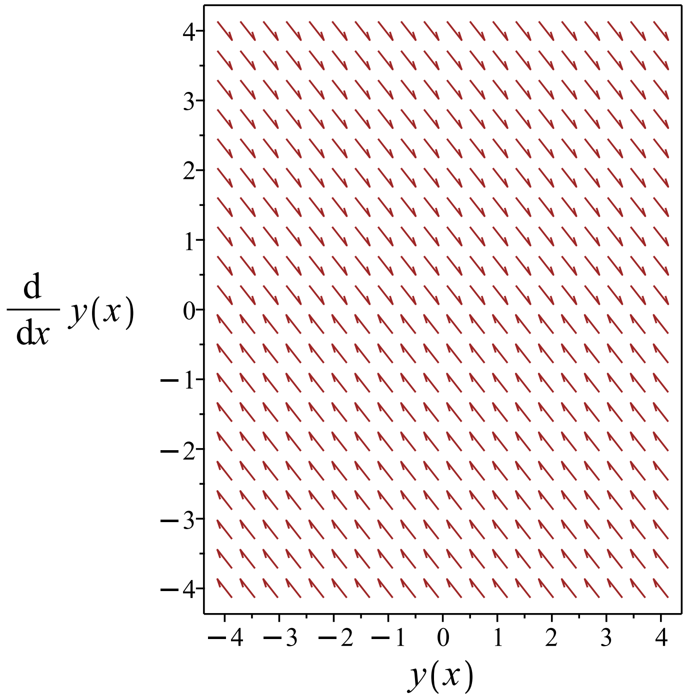

Figure 28: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 + c_2 e^{-x}$$

Verified OK.

1.12.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = 0$$
$$y' + y = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-y + c_1} dy = \int dx$$
$$-\ln(-y + c_1) = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{-y + c_1} = e^{x + c_2}$$

Which simplifies to

$$\frac{1}{-y+c_1} = c_3 e^x$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-x}}{c_3} + c_1 \tag{1}$$

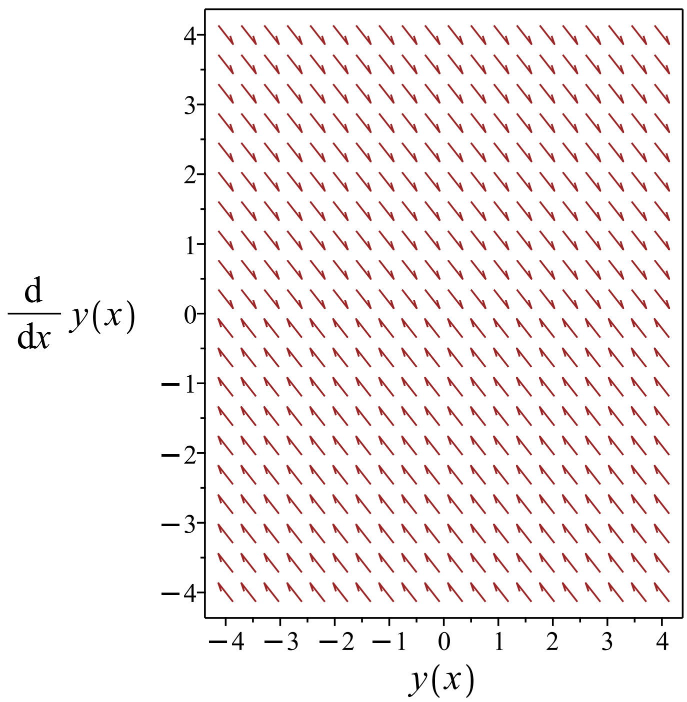

Figure 29: Slope field plot

Verification of solutions

$$y = -\frac{e^{-x}}{c_3} + c_1$$

Verified OK.

1.12.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned}\int -\frac{1}{p}dp &= \int dx\\ -\ln(p) &= x + c_1\end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{p} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{p} = c_2 e^x$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{e^{-x}}{c_2}$$

Integrating both sides gives

$$\begin{aligned}y &= \int \frac{e^{-x}}{c_2}\,dx\\ &= -\frac{e^{-x}}{c_2} + c_3\end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{-x}}{c_2} + c_3 \tag{1}$$

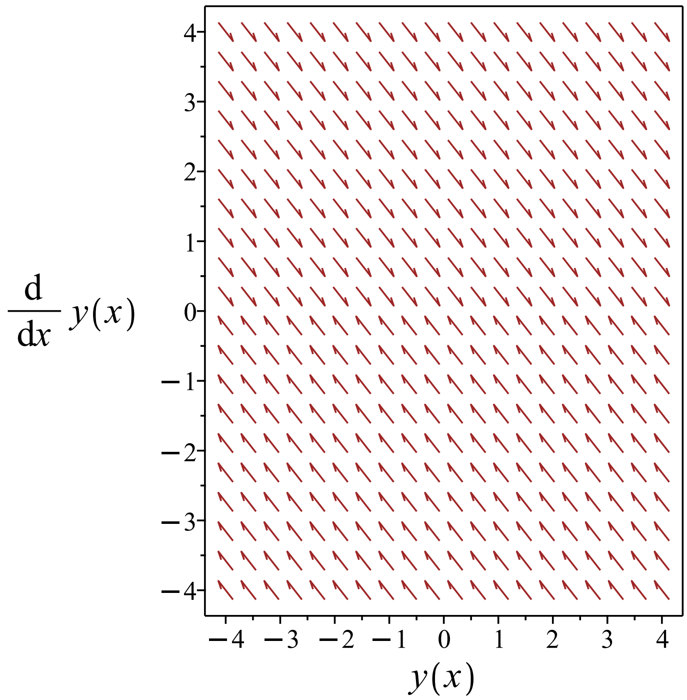

Figure 30: Slope field plot

Verification of solutions

$$y = -\frac{e^{-x}}{c_2} + c_3$$

Verified OK.

1.12.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = 0$$

$$y' + y = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-y+c_1} dy = \int dx$$
$$-\ln(-y+c_1) = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{-y+c_1} = e^{x+c_2}$$

Which simplifies to

$$\frac{1}{-y+c_1} = c_3 e^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-x}}{c_3} + c_1 \tag{1}$$

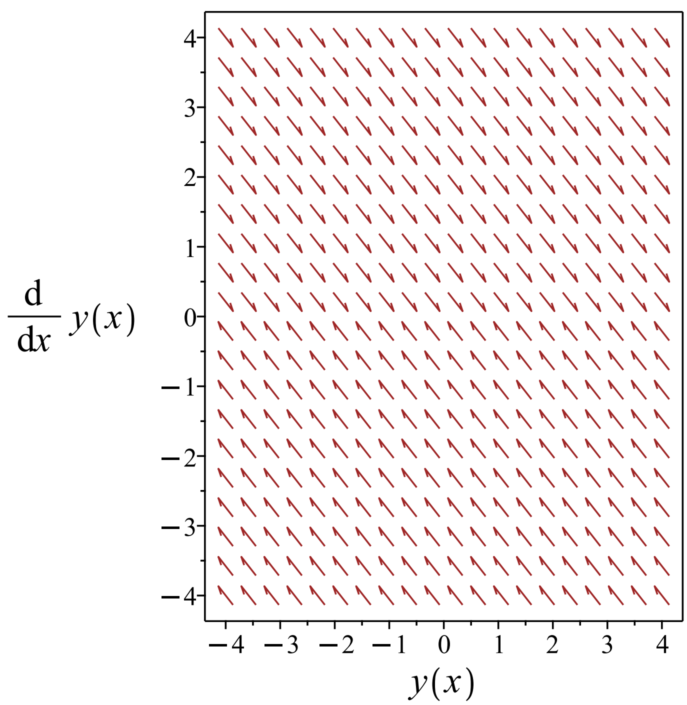

Figure 31: Slope field plot

Verification of solutions

$$y = -\frac{\mathrm{e}^{-x}}{c_3} + c_1$$

Verified OK.

1.12.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 16: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(e^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1(e^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{-x}\right) + c_2\left(e^{-x}(e^x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 \tag{1}$$

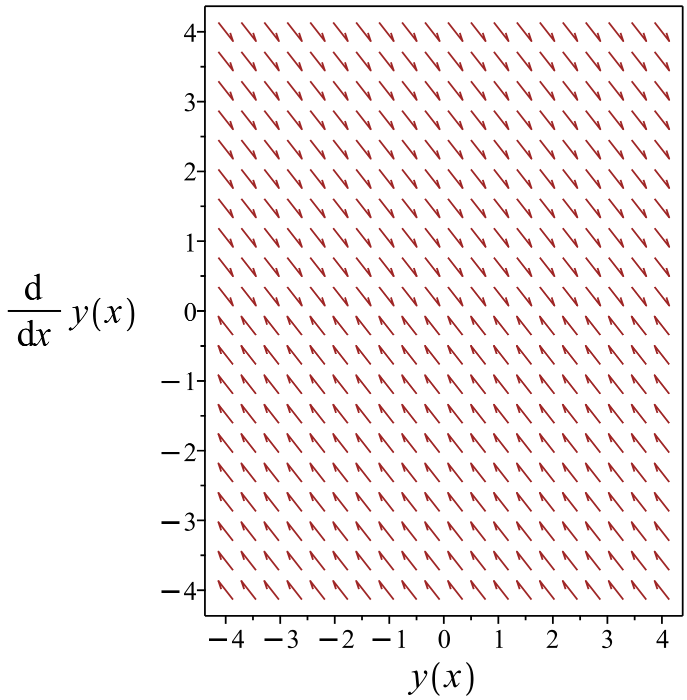

Figure 32: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2$$

Verified OK.

1.12.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx
$$

Substituting the above values for p, q, r, s gives

$$
y' + y = c_1
$$

We now have a first order ode to solve which is

$$
y' + y = c_1
$$

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{-y + c_1} dy &= \int dx \\
-\ln(-y + c_1) &= x + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\frac{1}{-y + c_1} = \mathrm{e}^{x + c_2}
$$

Which simplifies to

$$
\frac{1}{-y + c_1} = c_3 \mathrm{e}^{x}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = -\frac{\mathrm{e}^{-x}}{c_3} + c_1 \tag{1}
$$

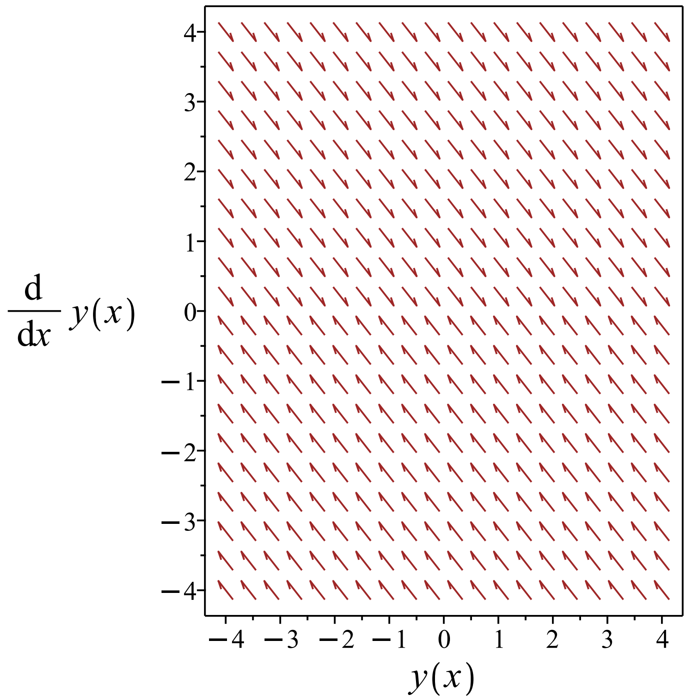

Figure 33: Slope field plot

Verification of solutions

$$y = -\frac{e^{-x}}{c_3} + c_1$$

Verified OK.

1.12.7 Maple step by step solution

Let's solve

$y'' + y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r + 1) = 0$

- Roots of the characteristic polynomial

$$r = (-1, 0)$$

- 1st solution of the ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the ODE

$$y_2(x) = 1$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{-x} + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + e^{-x} c_2$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 17

```
DSolve[y''[x]+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - c_1 e^{-x}$$

1.13 problem 13

1.13.1 Solving as second order ode missing y ode 122
1.13.2 Solving as second order ode missing x ode 124
1.13.3 Maple step by step solution 128

Internal problem ID [7402]
Internal file name [`OUTPUT/6369_Sunday_June_05_2022_04_41_52_PM_62970962/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 13.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y''^2 + y' = 0}$$

1.13.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)^2 + p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Solving the given ode for $p'(x)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$p'(x) = \sqrt{-p(x)} \tag{1}$$

$$p'(x) = -\sqrt{-p(x)} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-p}}dp = \int dx$$
$$-2\sqrt{-p\left(x\right)} = x+c_1$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-p}}dp = \int dx$$
$$2\sqrt{-p\left(x\right)} = x+c_2$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-2\sqrt{-y'} = x+c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{1}{4}x^2-\frac{1}{2}c_1x-\frac{1}{4}c_1^2\,\mathrm{d}x \\ &= -\frac{\left(x+c_1\right)^3}{12}+c_3 \end{aligned}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$2\sqrt{-y'} = x+c_2$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{1}{4}x^2-\frac{1}{2}c_2x-\frac{1}{4}c_2^2\,\mathrm{d}x \\ &= -\frac{\left(x+c_2\right)^3}{12}+c_4 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(x+c_1\right)^3}{12}+c_3 \tag{1}$$
$$y = -\frac{\left(x+c_2\right)^3}{12}+c_4 \tag{2}$$

Verification of solutions

$$y = -\frac{(x + c_1)^3}{12} + c_3$$

Verified OK.

$$y = -\frac{(x + c_2)^3}{12} + c_4$$

Verified OK.

1.13.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^2 \left(\frac{d}{dy}p(y)\right)^2 + p(y) = 0$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\frac{d}{dy}p(y) = -\frac{1}{\sqrt{-p\left(y\right)}} \tag{1}$$

$$\frac{d}{dy}p(y) = \frac{1}{\sqrt{-p\left(y\right)}} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int -\sqrt{-p}dp = \int dy$$

$$\frac{2(-p(y))^{\frac{3}{2}}}{3} = y + c_1$$

Solving equation (2)

Integrating both sides gives

$$\int \sqrt{-p}dp = \int dy$$

$$-\frac{2(-p(y))^{\frac{3}{2}}}{3} = y + c_2$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{2(-y')^{\frac{3}{2}}}{3} = y + c_1$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{(12y + 12c_1)^{\frac{2}{3}}}{4} \tag{1}$$

$$y' = -\left(-\frac{(12y + 12c_1)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\,(12y + 12c_1)^{\frac{1}{3}}}{4}\right)^2 \tag{2}$$

$$y' = -\left(-\frac{(12y + 12c_1)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\,(12y + 12c_1)^{\frac{1}{3}}}{4}\right)^2 \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int -\frac{4}{(12y + 12c_1)^{\frac{2}{3}}}dy = \int dx$$

$$-\frac{12(y + c_1)}{(12y + 12c_1)^{\frac{2}{3}}} = x + c_3$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{16}{(12y+12c_1)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)^2}dy = \int dx$$

$$-\frac{48(y+c_1)}{(12y+12c_1)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)^2} = x+c_4$$

Solving equation (3)

Integrating both sides gives

$$\int -\frac{16}{(12y+12c_1)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)^2}dy = \int dx$$

$$-\frac{48(y+c_1)}{(12y+12c_1)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)^2} = x+c_5$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{2(-y')^{\frac{3}{2}}}{3} = y+c_2$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{(-12y-12c_2)^{\frac{2}{3}}}{4} \tag{1}$$

$$y' = -\left(-\frac{(-12y-12c_2)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\,(-12y-12c_2)^{\frac{1}{3}}}{4}\right)^2 \tag{2}$$

$$y' = -\left(-\frac{(-12y-12c_2)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\,(-12y-12c_2)^{\frac{1}{3}}}{4}\right)^2 \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int -\frac{4}{(-12y-12c_2)^{\frac{2}{3}}}dy = \int dx$$

$$-\frac{12(y+c_2)}{(-12y-12c_2)^{\frac{2}{3}}} = x + c_6$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{16}{(-12y-12c_2)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)^2}dy = \int dx$$

$$-\frac{48(y+c_2)}{(-12y-12c_2)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)^2} = x + c_7$$

Solving equation (3)

Integrating both sides gives

$$\int -\frac{16}{(-12y-12c_2)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)^2}dy = \int dx$$

$$-\frac{48(y+c_2)}{(-12y-12c_2)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)^2} = x + c_8$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{12}c_3^3 - \frac{1}{4}c_3^2 x - \frac{1}{4}c_3 x^2 - \frac{1}{12}x^3 - c_1 \tag{1}$$

$$y = -c_1 + \frac{\left(2ic_4^2\sqrt{3} + 4ic_4 x\sqrt{3} + 2i\sqrt{3}\,x^2 - 2c_4^2 - 4c_4 x - 2x^2\right)^{\frac{3}{2}}}{96} \tag{2}$$

$$y = -c_1 + \frac{\left(-2ic_5^2\sqrt{3} - 4ic_5 x\sqrt{3} - 2i\sqrt{3}\,x^2 - 2c_5^2 - 4xc_5 - 2x^2\right)^{\frac{3}{2}}}{96} \tag{3}$$

$$y = -\frac{1}{12}c_6^3 - \frac{1}{4}c_6^2 x - \frac{1}{4}c_6 x^2 - \frac{1}{12}x^3 - c_2 \tag{4}$$

$$y = -c_2 - \frac{\left(-2ic_7^2\sqrt{3} - 4ic_7 x\sqrt{3} - 2i\sqrt{3}\,x^2 - 2c_7^2 - 4xc_7 - 2x^2\right)^{\frac{3}{2}}}{96} \tag{5}$$

$$y = -c_2 - \frac{\left(2ic_8^2\sqrt{3} + 4ic_8 x\sqrt{3} + 2i\sqrt{3}\,x^2 - 2c_8^2 - 4xc_8 - 2x^2\right)^{\frac{3}{2}}}{96} \tag{6}$$

Verification of solutions

$$y = -\frac{1}{12}c_3^3 - \frac{1}{4}c_3^2x - \frac{1}{4}c_3x^2 - \frac{1}{12}x^3 - c_1$$

Verified OK.

$$y = -c_1 + \frac{\left(2ic_4^2\sqrt{3} + 4ic_4x\sqrt{3} + 2i\sqrt{3}\,x^2 - 2c_4^2 - 4c_4x - 2x^2\right)^{\frac{3}{2}}}{96}$$

Verified OK.

$$y = -c_1 + \frac{\left(-2ic_5^2\sqrt{3} - 4ic_5x\sqrt{3} - 2i\sqrt{3}\,x^2 - 2c_5^2 - 4xc_5 - 2x^2\right)^{\frac{3}{2}}}{96}$$

Verified OK.

$$y = -\frac{1}{12}c_6^3 - \frac{1}{4}c_6^2x - \frac{1}{4}c_6x^2 - \frac{1}{12}x^3 - c_2$$

Verified OK.

$$y = -c_2 - \frac{\left(-2ic_7^2\sqrt{3} - 4ic_7x\sqrt{3} - 2i\sqrt{3}\,x^2 - 2c_7^2 - 4xc_7 - 2x^2\right)^{\frac{3}{2}}}{96}$$

Verified OK.

$$y = -c_2 - \frac{\left(2ic_8^2\sqrt{3} + 4ic_8x\sqrt{3} + 2i\sqrt{3}\,x^2 - 2c_8^2 - 4xc_8 - 2x^2\right)^{\frac{3}{2}}}{96}$$

Verified OK.

1.13.3 Maple step by step solution

Let's solve

$y''^2 + y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x)^2 + u(x) = 0$

- Separate variables

  $\frac{u'(x)}{\sqrt{-u(x)}} = 1$

- Integrate both sides with respect to x

$\int \frac{u'(x)}{\sqrt{-u(x)}}dx = \int 1dx + c_1$

- Evaluate integral

$-2\sqrt{-u\left(x\right)} = x + c_1$

- Solve for $u(x)$

$u(x) = -\frac{1}{4}x^2 - \frac{1}{2}c_1x - \frac{1}{4}c_1^2$

- Solve 1st ODE for $u(x)$

$u(x) = -\frac{1}{4}x^2 - \frac{1}{2}c_1x - \frac{1}{4}c_1^2$

- Make substitution $u = y'$

$y' = -\frac{1}{4}x^2 - \frac{1}{2}c_1x - \frac{1}{4}c_1^2$

- Integrate both sides to solve for y

$\int y'dx = \int \left(-\frac{1}{4}x^2 - \frac{1}{2}c_1x - \frac{1}{4}c_1^2\right) dx + c_2$

- Compute integrals

$y = -\frac{(x+c_1)^3}{12} + c_2$

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      -> Calling odsolve with the ODE`, diff(diff(diff(y(x), x), x), x)+1/2, y(x)`         *
         Methods for third order ODEs:
         --- Trying classification methods ---
         trying a quadrature
         <- quadrature successful
      <- 2nd order ODE linearizable_by_differentiation successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      <- 2nd order ODE linearizable_by_differentiation successful
-> Calling odsolve with the ODE`, diff(y(x), x) = 0, y(x), singsol = none`   *** Sublevel 2
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)^2+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1$$
$$y(x) = -\frac{1}{12}x^3 + \frac{1}{2}c_1x^2 - x\,c_1^2 + c_2$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 69

```
DSolve[(y''[x])^2+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^3}{12} - \frac{1}{4}ic_1x^2 + \frac{c_1{}^2x}{4} + c_2$$
$$y(x) \to -\frac{x^3}{12} + \frac{1}{4}ic_1x^2 + \frac{c_1{}^2x}{4} + c_2$$

1.14 problem 14

1.14.1 Solving as second order ode missing y ode 132
1.14.2 Solving as second order ode missing x ode 133
1.14.3 Maple step by step solution 135

Internal problem ID [7403]
Internal file name [`OUTPUT/6370_Sunday_June_05_2022_04_41_58_PM_92550606/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_xy]]
```

$$\boxed{y'' + y'^2 = 0}$$

1.14.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int -\frac{1}{p^2} dp = x + c_1$$
$$\frac{1}{p} = x + c_1$$

Solving for p gives these solutions

$$p_1 = \frac{1}{x + c_1}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{1}{x + c_1}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{x + c_1}\, \mathrm{d}x \\ &= \ln\left(x + c_1\right) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \ln\left(x + c_1\right) + c_2 \tag{1}$$

Verification of solutions

$$y = \ln\left(x + c_1\right) + c_2$$

Verified OK.

1.14.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + p(y)^2 = 0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int -\frac{1}{p}dp = \int dy$$
$$-\ln(p) = y + c_1$$

Raising both side to exponential gives

$$\frac{1}{p} = e^{y+c_1}$$

Which simplifies to

$$\frac{1}{p} = c_2 e^{y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{e^{-y}}{c_2}$$

Integrating both sides gives

$$\int c_2 e^{y} dy = x + c_3$$
$$c_2 e^{y} = x + c_3$$

Solving for y gives these solutions

$$y_1 = \ln\left(\frac{x + c_3}{c_2}\right)$$

Summary
The solution(s) found are the following

$$y = \ln\left(\frac{x + c_3}{c_2}\right) \tag{1}$$

Verification of solutions

$$y = \ln\left(\frac{x + c_3}{c_2}\right)$$

Verified OK.

1.14.3 Maple step by step solution

Let's solve

$y'' + y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) + u(x)^2 = 0$

- Separate variables

  $\frac{u'(x)}{u(x)^2} = -1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^2} dx = \int (-1)\, dx + c_1$

- Evaluate integral

  $-\frac{1}{u(x)} = -x + c_1$

- Solve for $u(x)$

  $u(x) = -\frac{1}{-x+c_1}$

- Solve 1st ODE for $u(x)$

  $u(x) = -\frac{1}{-x+c_1}$

- Make substitution $u = y'$

  $y' = -\frac{1}{-x+c_1}$

- Integrate both sides to solve for y

  $\int y' dx = \int -\frac{1}{-x+c_1} dx + c_2$

- Compute integrals

  $y = \ln(-x + c_1) + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x$2)+diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = \ln (c_1 x + c_2)$$

✓ Solution by Mathematica

Time used: 0.205 (sec). Leaf size: 15

```
DSolve[y''[x]+(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(x - c_1) + c_2$$

1.15 problem 15

1.15.1 Solving as second order linear constant coeff ode 137
1.15.2 Solving as second order integrable as is ode 141
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1.15.4 Solving as type second_order_integrable_as_is (not using ABC version) 144
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1.15.6 Solving as exact linear second order ode ode 151
1.15.7 Maple step by step solution 153

Internal problem ID [7404]
Internal file name [`OUTPUT/6371_Sunday_June_05_2022_04_42_00_PM_31127163/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' = 1}$$

1.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(0)} \\ &= -\frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(0)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 + c_2 e^{-x}) + (x)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} + x \tag{1}$$

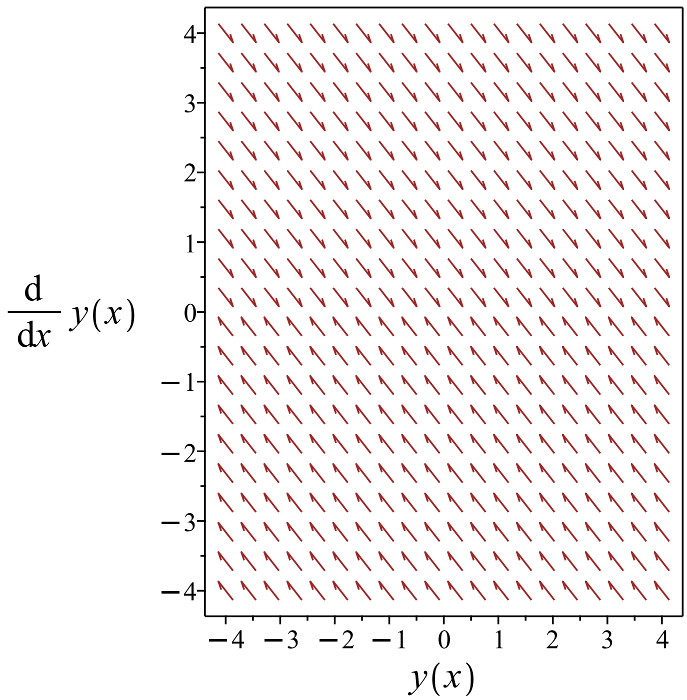

Figure 34: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 + c_2 e^{-x} + x$$

Verified OK.

1.15.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int 1 dx$$
$$y' + y = x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = x + c_1$$

Hence the ode is

$$y' + y = x + c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int 1 dx}$$
$$= \mathrm{e}^{x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)(x + c_1)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} y) = (\mathrm{e}^{x})(x + c_1)$$
$$\mathrm{d}(\mathrm{e}^{x} y) = ((x + c_1)\mathrm{e}^{x})\, \mathrm{d}x$$

Integrating gives

$$\mathrm{e}^{x} y = \int (x + c_1)\mathrm{e}^{x}\, \mathrm{d}x$$
$$\mathrm{e}^{x} y = (c_1 + x - 1)\mathrm{e}^{x} + c_2$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$y = \mathrm{e}^{-x}(c_1 + x - 1)\mathrm{e}^{x} + c_2\mathrm{e}^{-x}$$

which simplifies to

$$y = c_1 + x - 1 + c_2 e^{-x}$$

Summary
The solution(s) found are the following

$$y = c_1 + x - 1 + c_2 e^{-x} \tag{1}$$

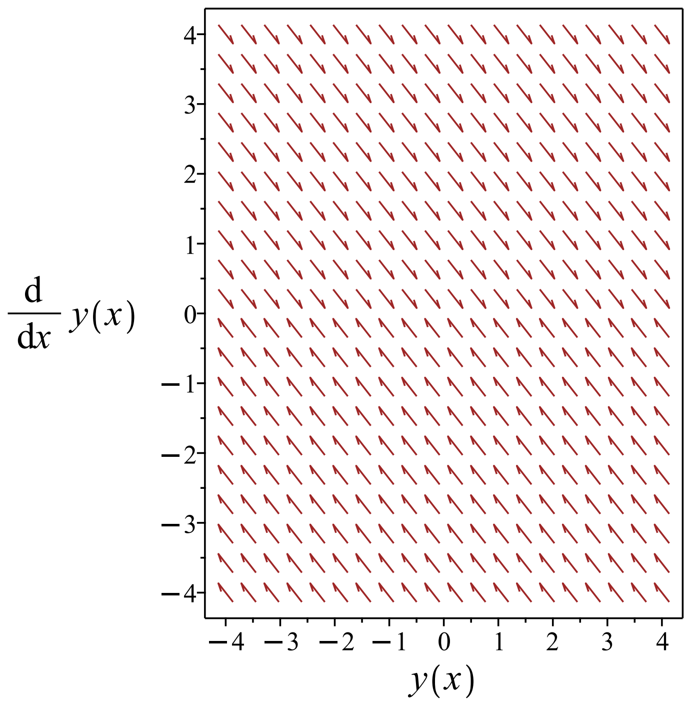

Figure 35: Slope field plot

Verification of solutions

$$y = c_1 + x - 1 + c_2 e^{-x}$$

Verified OK.

1.15.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - 1 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{-p+1} dp = \int dx$$
$$-\ln(-p+1) = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{-p+1} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{-p+1} = c_2 e^x$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{e^{-x}}{c_2} + 1$$

Integrating both sides gives

$$y = \int \frac{(c_2 e^x - 1) e^{-x}}{c_2} \, dx$$
$$= \frac{e^{-x}}{c_2} + \ln(e^x) + c_3$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-x}}{c_2} + \ln(e^x) + c_3 \tag{1}$$

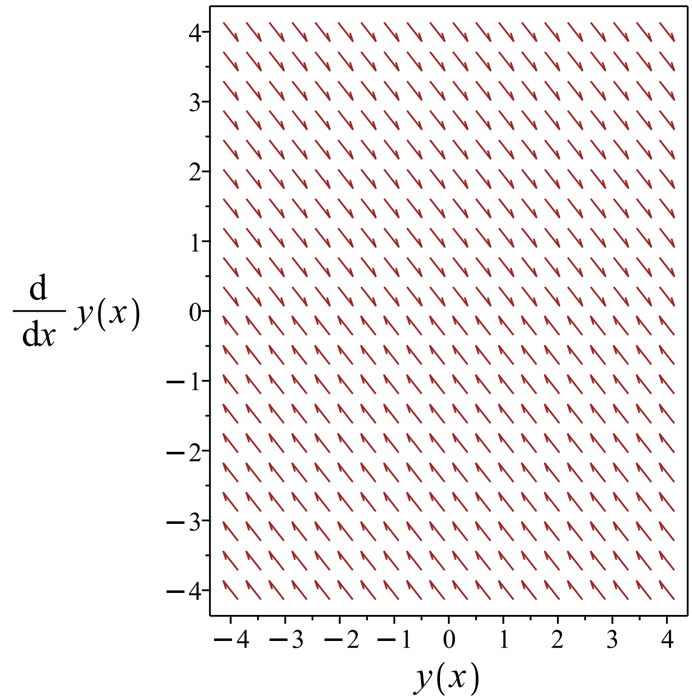

Figure 36: Slope field plot

Verification of solutions

$$y = \frac{e^{-x}}{c_2} + \ln(e^x) + c_3$$

Verified OK.

1.15.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = 1$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int 1 dx$$

$$y' + y = x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= x + c_1
\end{aligned}
$$

Hence the ode is

$$
y' + y = x + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1 dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(x + c_1) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} y) &= (\mathrm{e}^{x})(x + c_1) \\
\mathrm{d}(\mathrm{e}^{x} y) &= ((x + c_1)\,\mathrm{e}^{x})\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{x} y &= \int (x + c_1)\,\mathrm{e}^{x}\,\mathrm{d}x \\
\mathrm{e}^{x} y &= (c_1 + x - 1)\,\mathrm{e}^{x} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$
y = \mathrm{e}^{-x}(c_1 + x - 1)\,\mathrm{e}^{x} + c_2 \mathrm{e}^{-x}
$$

which simplifies to

$$
y = c_1 + x - 1 + c_2 \mathrm{e}^{-x}
$$

Summary
The solution(s) found are the following

$$
y = c_1 + x - 1 + c_2 \mathrm{e}^{-x} \tag{1}
$$

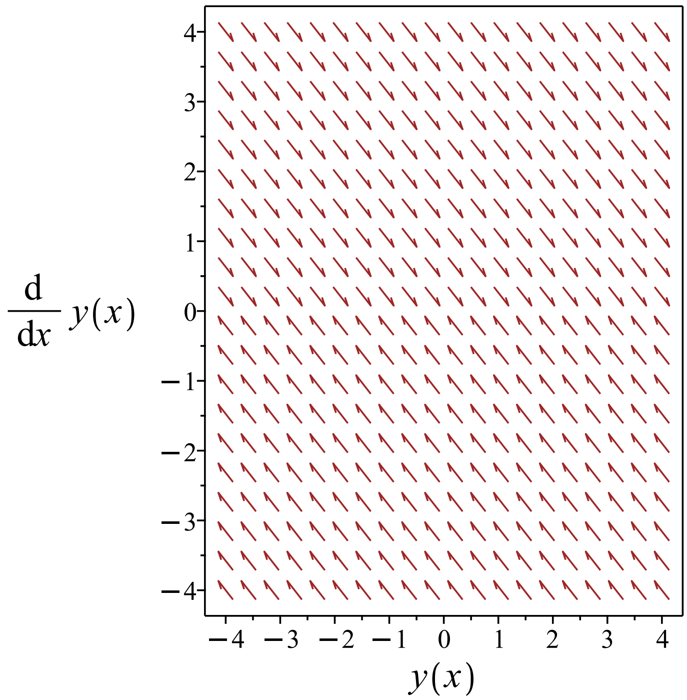

Figure 37: Slope field plot

Verification of solutions

$$y = c_1 + x - 1 + c_2 e^{-x}$$

Verified OK.

1.15.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 20: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x})\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} (\mathrm{e}^{x})\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, \mathrm{e}^{-x}\}
$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1 x
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2\right) + (x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 + x \tag{1}$$

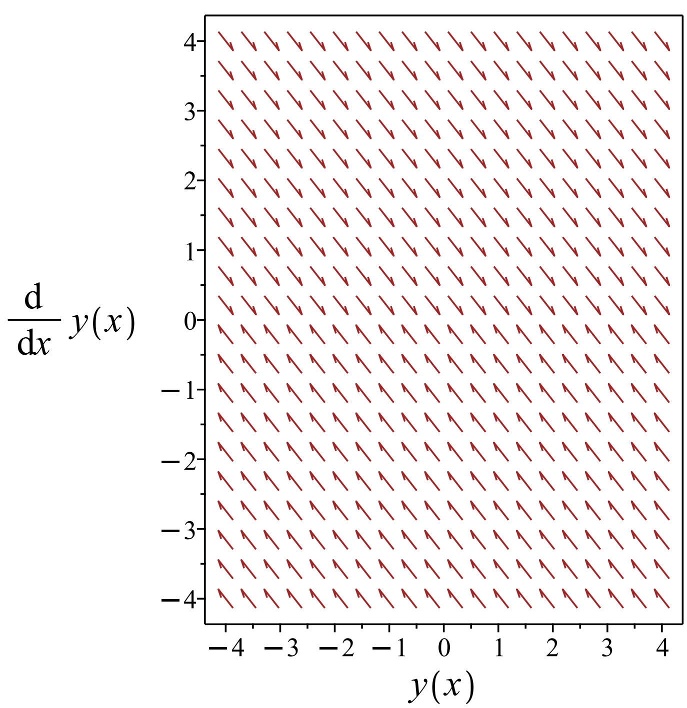

Figure 38: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 + x$$

Verified OK.

1.15.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= 1 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int 1\, dx$$

We now have a first order ode to solve which is

$$y' + y = x + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x + c_1 \end{aligned}$$

Hence the ode is

$$y' + y = x + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)(x + c_1) \\ \frac{d}{dx}(e^{x} y) &= (e^{x})(x + c_1) \\ d(e^{x} y) &= ((x + c_1) e^{x})\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{x} y &= \int (x + c_1) e^{x}\, dx \\ e^{x} y &= (c_1 + x - 1) e^{x} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$y = e^{-x}(c_1 + x - 1) e^{x} + c_2 e^{-x}$$

which simplifies to

$$y = c_1 + x - 1 + c_2 e^{-x}$$

Summary

The solution(s) found are the following

$$y = c_1 + x - 1 + c_2 e^{-x} \tag{1}$$

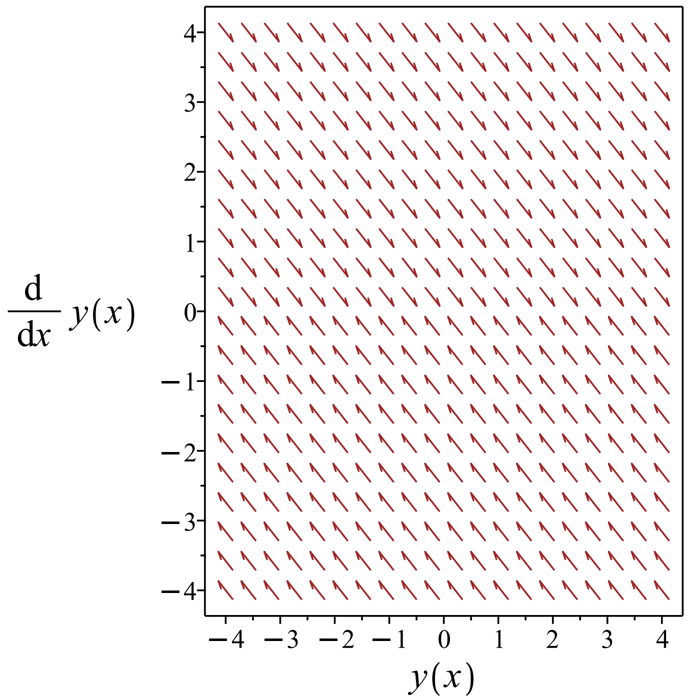

Figure 39: Slope field plot

Verification of solutions

$$y = c_1 + x - 1 + c_2 e^{-x}$$

Verified OK.

1.15.7 Maple step by step solution

Let's solve

$y'' + y' = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 0)$
- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 1\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-x}\left(\int \mathrm{e}^{x}dx\right) + \int 1dx$
  - Compute integrals

    $y_p(x) = x - 1$
- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + x - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)+1, _b(_a)`   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=1,y(x), singsol=all)
```

$$y(x) = -\mathrm{e}^{-x}c_1 + x + c_2$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 18

```
DSolve[y''[x]+y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - c_1e^{-x} + c_2$$

1.16 problem 16

1.16.1 Solving as second order ode missing y ode 156
1.16.2 Solving as second order ode missing x ode 158
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Internal problem ID [7405]
Internal file name [`OUTPUT/6372_Sunday_June_05_2022_04_42_02_PM_299138/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 16.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y''^2 + y' = 1}$$

1.16.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)^2 + p(x) - 1 = 0$$

Which is now solve for $p(x)$ as first order ode. Solving the given ode for $p'(x)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$p'(x) = \sqrt{1 - p(x)} \tag{1}$$

$$p'(x) = -\sqrt{1 - p(x)} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{1-p}}dp = \int dx$$
$$-2\sqrt{1-p(x)} = x + c_1$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{1-p}}dp = \int dx$$
$$2\sqrt{1-p(x)} = x + c_2$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-2\sqrt{-y'+1} = x + c_1$$

Integrating both sides gives

$$\begin{aligned}
y &= \int -\frac{1}{4}c_1^2 - \frac{1}{2}c_1 x - \frac{1}{4}x^2 + 1 \,\mathrm{d}x \\
&= -\frac{x^3}{12} - \frac{c_1 x^2}{4} - \frac{(c_1+2)(-2+c_1)x}{4} + c_3
\end{aligned}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$2\sqrt{-y'+1} = x + c_2$$

Integrating both sides gives

$$\begin{aligned}
y &= \int -\frac{1}{4}c_2^2 - \frac{1}{2}c_2 x - \frac{1}{4}x^2 + 1 \,\mathrm{d}x \\
&= -\frac{x^3}{12} - \frac{c_2 x^2}{4} - \frac{(c_2+2)(c_2-2)x}{4} + c_4
\end{aligned}$$

Summary
The solution(s) found are the following

$$
y = -\frac{x^3}{12} - \frac{c_1 x^2}{4} - \frac{(c_1+2)(-2+c_1)x}{4} + c_3 \tag{1}
$$

$$
y = -\frac{x^3}{12} - \frac{c_2 x^2}{4} - \frac{(c_2+2)(c_2-2)x}{4} + c_4 \tag{2}
$$

Verification of solutions

$$
y = -\frac{x^3}{12} - \frac{c_1 x^2}{4} - \frac{(c_1+2)(-2+c_1)x}{4} + c_3
$$

Verified OK.

$$
y = -\frac{x^3}{12} - \frac{c_2 x^2}{4} - \frac{(c_2+2)(c_2-2)x}{4} + c_4
$$

Verified OK.

1.16.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$
y' = p(y)
$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y)^2 \left(\frac{d}{dy}p(y)\right)^2 + p(y) = 1
$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\frac{d}{dy}p(y) = \frac{\sqrt{1-p(y)}}{p(y)} \tag{1}$$

$$\frac{d}{dy}p(y) = -\frac{\sqrt{1-p(y)}}{p(y)} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{p}{\sqrt{1-p}}dp = \int dy$$

$$-\frac{2(p(y)+2)\sqrt{1-p(y)}}{3} = y + c_1$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{p}{\sqrt{1-p}}dp = \int dy$$

$$\frac{2(p(y)+2)\sqrt{1-p(y)}}{3} = y + c_2$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{2(y'+2)\sqrt{-y'+1}}{3} = y + c_1$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these

will generate a solution. The equations generated are

$$y' = -\left(\frac{\left(6y+6c_1+2\sqrt{-16+9y^2+18c_1y+9c_1^2}\right)^{\frac{1}{3}}}{2}+\frac{2}{\left(6y+6c_1+2\sqrt{-16+9y^2+18c_1y+9c_1^2}\right)^{\frac{1}{3}}}\right)^2 + \tag{1}$$

$$y' = -\left(-\frac{\left(6y+6c_1+2\sqrt{-16+9y^2+18c_1y+9c_1^2}\right)^{\frac{1}{3}}}{4}-\frac{1}{\left(6y+6c_1+2\sqrt{-16+9y^2+18c_1y+9c_1^2}\right)^{\frac{1}{3}}} + \right. \tag{2}$$

$$y' = -\left(-\frac{\left(6y+6c_1+2\sqrt{-16+9y^2+18c_1y+9c_1^2}\right)^{\frac{1}{3}}}{4}-\frac{1}{\left(6y+6c_1+2\sqrt{-16+9y^2+18c_1y+9c_1^2}\right)^{\frac{1}{3}}} - \right. \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int -\frac{4\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{2}{3}}}{\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{4}{3}}+4\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{2}{3}}+16}dy = \int dx$$

$$-4\left(\int^y \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{2}{3}}}{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}+4\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16\right.}\right. \right. = x + c_3$$

Solving equation (2)

Integrating both sides gives

$$\int \frac{8\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{2}{3}}}{i\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{4}{3}}\sqrt{3}+\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{4}{3}}+16-8\left(\right.}$$
$$= \int dx$$

$$8\left(\int^{y} \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9\right.}{i\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}\sqrt{3}+\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-1\right.}\right.$$
$$= x+c_4$$

Solving equation (3)

Integrating both sides gives

$$\int -\frac{8\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{2}{3}}}{i\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{4}{3}}\sqrt{3}-\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{4}{3}}-16+8\left(\right.}$$
$$= \int dx$$

$$-8\left(\int^{y} \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+\right.}{i\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}\sqrt{3}-\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-\right.}\right.$$
$$= x+c_5$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{2(y'+2)\sqrt{-y'+1}}{3} = y+c_2$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\left(\frac{\left(-6y-6c_2+2\sqrt{-16+9y^2+18yc_2+9c_2^2}\right)^{\frac{1}{3}}}{2}+\frac{2}{\left(-6y-6c_2+2\sqrt{-16+9y^2+18yc_2+9c_2^2}\right)^{\frac{1}{3}}}\right) \tag{1}$$

$$y' = -\left(-\frac{\left(-6y-6c_2+2\sqrt{-16+9y^2+18yc_2+9c_2^2}\right)^{\frac{1}{3}}}{4}-\frac{1}{\left(-6y-6c_2+2\sqrt{-16+9y^2+18yc_2+9c_2^2}\right)^{\frac{1}{3}}} \tag{2}$$

$$y' = -\left(-\frac{\left(-6y-6c_2+2\sqrt{-16+9y^2+18yc_2+9c_2^2}\right)^{\frac{1}{3}}}{4}-\frac{1}{\left(-6y-6c_2+2\sqrt{-16+9y^2+18yc_2+9c_2^2}\right)^{\frac{1}{3}}} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int -\frac{4\left(-6y-6c_2+2\sqrt{9c_2^2+18c_2y+9y^2-16}\right)^{\frac{2}{3}}}{\left(-6y-6c_2+2\sqrt{9c_2^2+18c_2y+9y^2-16}\right)^{\frac{4}{3}}+4\left(-6y-6c_2+2\sqrt{9c_2^2+18c_2y+9y^2-16}\right)^{\frac{2}{3}}+16}dy = \int dx$$

$$-4\left(\int^{y}\frac{\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+9c_2^2-16}\right)^{\frac{2}{3}}}{\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+9c_2^2-16}\right)^{\frac{4}{3}}+4\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+9c_2^2-\right.}\right.$$
$$=x+c_6$$

Solving equation (2)

Integrating both sides gives

$$\int\frac{8\left(-6y-6c_2+2\sqrt{9c_2^2+18c_2y+9y^2-16}\right)^{\frac{2}{3}}}{i\left(-6y-6c_2+2\sqrt{9c_2^2+18c_2y+9y^2-16}\right)^{\frac{4}{3}}\sqrt{3}+16+\left(-6y-6c_2+2\sqrt{9c_2^2+18c_2y+9y^2-16}\right)^{\frac{4}{3}}-}$$
$$=\int dx$$

$$8\left(\int^{y}\frac{\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2\right.}}{i\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+9c_2^2-16}\right)^{\frac{4}{3}}\sqrt{3}+16+\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+\right.}\right.$$
$$=x+c_7$$

Solving equation (3)

Integrating both sides gives

$$\int -\frac{8\left(-6y-6c_2+2\sqrt{9c_2^2+18c_2y+9y^2-16}\right)^{\frac{2}{3}}}{i\left(-6y-6c_2+2\sqrt{9c_2^2+18c_2y+9y^2-16}\right)^{\frac{4}{3}}\sqrt{3}-16-\left(-6y-6c_2+2\sqrt{9c_2^2+18c_2y+9y^2-16}\right)^{\frac{4}{3}}+}$$
$$=\int dx$$

$$-8\left(\int^{y}\frac{\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac\right.}}{i\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+9c_2^2-16}\right)^{\frac{4}{3}}\sqrt{3}-16-\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac\right.}\right.$$
$$=x+c_8$$

Summary

The solution(s) found are the following

$$-4\left(\int^{y} \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{2}{3}}}{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}+4\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16\right.}}\right. \tag{1}$$
$$= x + c_3$$

$$8\left(\int^{y} \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9\right.}}{i\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}\sqrt{3}+\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-1\right.}}\right. \tag{2}$$
$$= x + c_4$$

$$-8\left(\int^{y} \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+\right.}}{i\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}\sqrt{3}-\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-\right.}}\right. \tag{3}$$
$$= x + c_5$$

$$-4\left(\int^{y} \frac{\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+9c_2^2-16}\right)^{\frac{2}{3}}}{\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+9c_2^2-16}\right)^{\frac{4}{3}}+4\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+9c_2^2-\right.}}\right. \tag{4}$$
$$= x + c_6$$

$$8\left(\int^{y} \frac{\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2\right.}}{i\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+9c_2^2-16}\right)^{\frac{4}{3}}\sqrt{3}+16+\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+\right.}}\right. \tag{5}$$
$$= x + c_7$$

$$-8\left(\int^{y} \frac{\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac\right.}}{i\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+9c_2^2-16}\right)^{\frac{4}{3}}\sqrt{3}-16-\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac\right.}}\right. \tag{6}$$
$$= x + c_8$$

Verification of solutions

$$-4\left(\int^{y} \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{2}{3}}}{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}+4\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16\right.}}\right.$$
$$= x + c_3$$

Verified OK.

$$8\left(\int^{y} \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9\right.}}{i\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}\sqrt{3}+\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-1\right.}}\right.$$
$$= x + c_4$$

Verified OK.

$$-8\left(\int^{y} \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+\right.}}{i\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}\sqrt{3}-\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-\right.}}\right.$$
$$= x + c_5$$

Verified OK.

$$-4\left(\int^{y} \frac{\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+9c_2^2-16}\right)^{\frac{2}{3}}}{\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+9c_2^2-16}\right)^{\frac{4}{3}}+4\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+9c_2^2-\right.}}\right.$$
$$= x + c_6$$

Verified OK.

$$8\left(\int^{y} \frac{\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2\right.}}{i\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+9c_2^2-16}\right)^{\frac{4}{3}}\sqrt{3}+16+\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+\right.}}\right.$$
$$= x + c_7$$

Verified OK.

$$-8\left(\int^{y} \frac{\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac\right.}}{i\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac_2+9c_2^2-16}\right)^{\frac{4}{3}}\sqrt{3}-16-\left(-6\_a-6c_2+2\sqrt{9\_a^2+18\_ac\right.}}\right.$$
$$= x + c_8$$

Verified OK.

1.16.3 Maple step by step solution

Let's solve

$y''^2 + y' = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x)^2 + u(x) = 1$

- Separate variables

  $\frac{u'(x)}{\sqrt{1-u(x)}} = 1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{\sqrt{1-u(x)}}dx = \int 1dx + c_1$

- Evaluate integral

  $-2\sqrt{1 - u(x)} = x + c_1$

- Solve for $u(x)$

  $u(x) = -\frac{1}{4}c_1^2 - \frac{1}{2}c_1 x - \frac{1}{4}x^2 + 1$

- Solve 1st ODE for $u(x)$

  $u(x) = -\frac{1}{4}c_1^2 - \frac{1}{2}c_1 x - \frac{1}{4}x^2 + 1$

- Make substitution $u = y'$

  $y' = -\frac{1}{4}c_1^2 - \frac{1}{2}c_1 x - \frac{1}{4}x^2 + 1$

- Integrate both sides to solve for y

  $\int y'dx = \int \left(-\frac{1}{4}c_1^2 - \frac{1}{2}c_1 x - \frac{1}{4}x^2 + 1\right) dx + c_2$

- Compute integrals

  $y = -\frac{x^3}{12} - \frac{c_1 x^2}{4} - \frac{(c_1+2)(-2+c_1)x}{4} + c_2$

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      <- 2nd order ODE linearizable_by_differentiation successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      <- 2nd order ODE linearizable_by_differentiation successful
-> Calling odsolve with the ODE`, diff(y(x), x) = 1, y(x), singsol = none`   *** Sublevel 2
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)^2+diff(y(x),x)=1,y(x), singsol=all)
```

$$y(x) = x + c_1$$
$$y(x) = -\frac{1}{12}x^3 + \frac{1}{2}c_1 x^2 - x\, c_1^2 + x + c_2$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 67

```
DSolve[(y''[x])^2+y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^3}{12} - \frac{c_1 x^2}{4} + x - \frac{{c_1}^2 x}{4} + c_2$$
$$y(x) \to -\frac{x^3}{12} + \frac{c_1 x^2}{4} + x - \frac{{c_1}^2 x}{4} + c_2$$

1.17 problem 17

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Internal problem ID [7406]
Internal file name [`OUTPUT/6373_Sunday_June_05_2022_04_42_06_PM_40607574/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{y'' + {y'}^2 = 1}$$

1.17.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x)^2 - 1 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{-p^2+1} dp = x + c_1$$
$$\operatorname{arctanh}(p) = x + c_1$$

Solving for p gives these solutions

$$p_1 = \tanh(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \tanh(x + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int \tanh(x + c_1)\, \mathrm{d}x \\ &= \ln(\cosh(x + c_1)) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \ln(\cosh(x + c_1)) + c_2 \tag{1}$$

Verification of solutions

$$y = \ln(\cosh(x + c_1)) + c_2$$

Verified OK.

1.17.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + p(y)^2 = 1$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int -\frac{p}{p^2-1}dp = \int dy$$
$$-\frac{\ln(p-1)}{2} - \frac{\ln(p+1)}{2} = y + c_1$$

The above can be written as

$$\left(-\frac{1}{2}\right)(\ln(p-1)+\ln(p+1)) = y + c_1$$
$$\ln(p-1)+\ln(p+1) = (-2)(y+c_1)$$
$$= -2y - 2c_1$$

Raising both side to exponential gives

$$e^{\ln(p-1)+\ln(p+1)} = -2c_1 e^{-2y}$$

Which simplifies to

$$p^2 - 1 = c_2 e^{-2y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \mathrm{RootOf}\left(\_Z^2 - c_2 e^{-2y} - 1\right)$$

Integrating both sides gives

$$\int \frac{1}{\mathrm{RootOf}\left(\_Z^2 - c_2 e^{-2y} - 1\right)}dy = \int dx$$
$$\int^y \frac{1}{\mathrm{RootOf}\left(\_Z^2 - c_2 e^{-2\_a} - 1\right)}d\_a = x + c_3$$

<u>Summary</u>

The solution(s) found are the following

$$\int^y \frac{1}{\mathrm{RootOf}\left(\_Z^2 - c_2 e^{-2\_a} - 1\right)}d\_a = x + c_3 \tag{1}$$

<u>Verification of solutions</u>

$$\int^y \frac{1}{\mathrm{RootOf}\left(\_Z^2 - c_2 e^{-2\_a} - 1\right)}d\_a = x + c_3$$

Verified OK.

1.17.3 Maple step by step solution

Let's solve

$y'' + y'^2 = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) + u(x)^2 = 1$

- Separate variables

  $\frac{u'(x)}{-u(x)^2+1} = 1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{-u(x)^2+1}dx = \int 1dx + c_1$

- Evaluate integral

  $\operatorname{arctanh}(u(x)) = x + c_1$

- Solve for $u(x)$

  $u(x) = \tanh(x + c_1)$

- Solve 1st ODE for $u(x)$

  $u(x) = \tanh(x + c_1)$

- Make substitution $u = y'$

  $y' = \tanh(x + c_1)$

- Integrate both sides to solve for y

  $\int y'dx = \int \tanh(x + c_1)\, dx + c_2$

- Compute integrals

  $y = \ln(\cosh(x + c_1)) + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+diff(y(x),x)^2=1,y(x), singsol=all)
```

$$y(x) = x - \ln(2) + \ln\left(\mathrm{e}^{-2x}c_1 - c_2\right)$$

✓ Solution by Mathematica

Time used: 0.333 (sec). Leaf size: 46

```
DSolve[y''[x]+(y'[x])^2==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\log\left(e^x\right) + \log\left(e^{2x} + e^{2c_1}\right) + c_2$$
$$y(x) \to -\log\left(e^x\right) + \log\left(e^{2x}\right) + c_2$$

1.18 problem 18

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Internal problem ID [7407]
Internal file name [`OUTPUT/6374_Sunday_June_05_2022_04_42_08_PM_74908298/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y' = x}$$

1.18.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.

y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(0)} \\ &= -\frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(0)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2xA_2 + A_1 + 2A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2}x^2 - x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 + c_2 e^{-x}) + \left(\frac{1}{2}x^2 - x\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + c_2 e^{-x} + \frac{x^2}{2} - x \tag{1}
$$

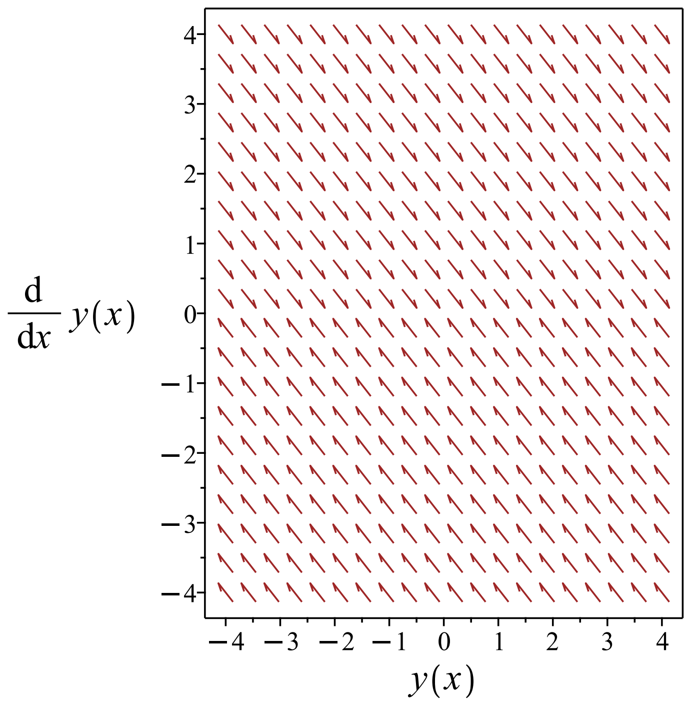

Figure 40: Slope field plot

Verification of solutions

$$
y = c_1 + c_2 e^{-x} + \frac{x^2}{2} - x
$$

Verified OK.

1.18.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int x dx$$
$$y' + y = \frac{x^2}{2} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = \frac{x^2}{2} + c_1$$

Hence the ode is

$$y' + y = \frac{x^2}{2} + c_1$$

The integrating factor μ is

$$\mu = e^{\int 1 dx}$$
$$= e^x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^2}{2} + c_1\right)$$
$$\frac{d}{dx}(e^x y) = (e^x)\left(\frac{x^2}{2} + c_1\right)$$
$$d(e^x y) = \left(\frac{(x^2 + 2c_1)\, e^x}{2}\right) dx$$

Integrating gives

$$e^x y = \int \frac{(x^2 + 2c_1)\, e^x}{2}\, dx$$
$$e^x y = \frac{(x^2 + 2c_1 - 2x + 2)\, e^x}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$y = \frac{\mathrm{e}^{-x}(x^2 + 2c_1 - 2x + 2)\,\mathrm{e}^{x}}{2} + c_2\mathrm{e}^{-x}$$

which simplifies to

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2\mathrm{e}^{-x}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2\mathrm{e}^{-x} \tag{1}$$

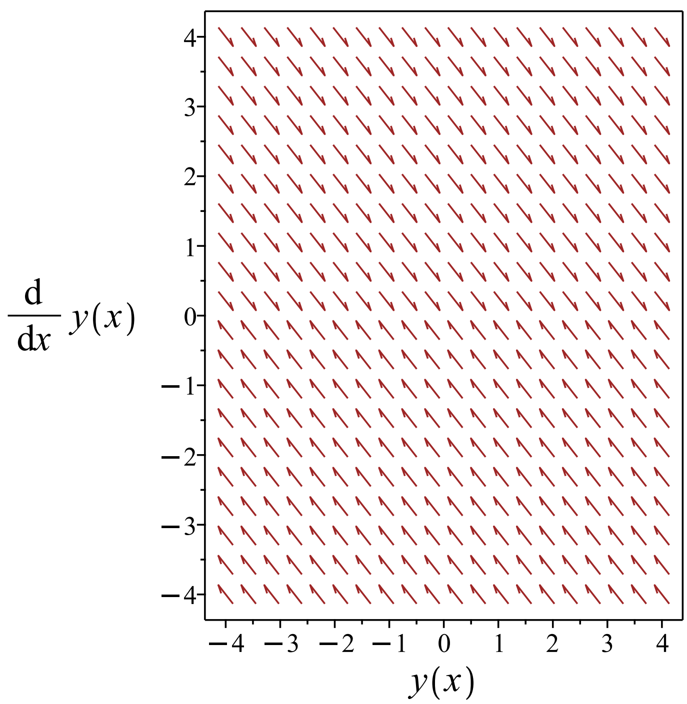

Figure 41: Slope field plot

Verification of solutions

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2\mathrm{e}^{-x}$$

Verified OK.

1.18.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - x = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x \end{aligned}$$

Hence the ode is

$$p'(x) + p(x) = x$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)(x) \\ \frac{d}{dx}(e^{x} p) &= (e^{x})(x) \\ d(e^{x} p) &= (x\, e^{x})\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{x} p &= \int x\, e^{x}\, dx \\ e^{x} p &= (x-1)\, e^{x} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$p(x) = e^{-x}(x-1)\,e^x + c_1 e^{-x}$$

which simplifies to

$$p(x) = x - 1 + c_1 e^{-x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x - 1 + c_1 e^{-x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int x - 1 + c_1 e^{-x}\,dx \\ &= -x + \frac{x^2}{2} - c_1 e^{-x} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -x + \frac{x^2}{2} - c_1 e^{-x} + c_2 \tag{1}$$

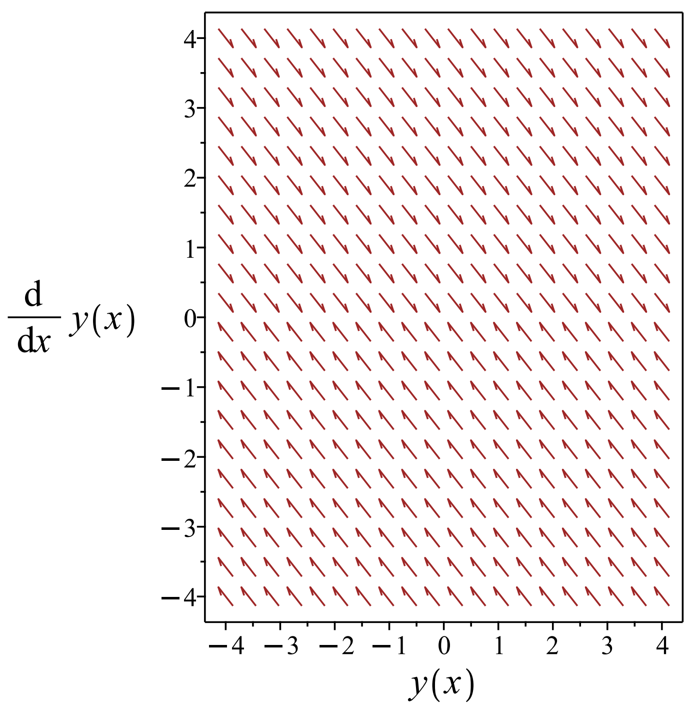

Figure 42: Slope field plot

Verification of solutions

$$y = -x + \frac{x^2}{2} - c_1 e^{-x} + c_2$$

Verified OK.

1.18.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = x$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\,dx = \int x dx$$
$$y' + y = \frac{x^2}{2} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = \frac{x^2}{2} + c_1$$

Hence the ode is

$$y' + y = \frac{x^2}{2} + c_1$$

The integrating factor μ is

$$\mu = e^{\int 1 dx}$$
$$= e^x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^2}{2} + c_1\right)$$
$$\frac{d}{dx}(e^x y) = (e^x)\left(\frac{x^2}{2} + c_1\right)$$
$$d(e^x y) = \left(\frac{(x^2 + 2c_1)\,e^x}{2}\right) dx$$

Integrating gives

$$e^x y = \int \frac{(x^2 + 2c_1)\, e^x}{2}\, dx$$

$$e^x y = \frac{(x^2 + 2c_1 - 2x + 2)\, e^x}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = \frac{e^{-x}(x^2 + 2c_1 - 2x + 2)\, e^x}{2} + c_2 e^{-x}$$

which simplifies to

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 e^{-x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 e^{-x} \tag{1}$$

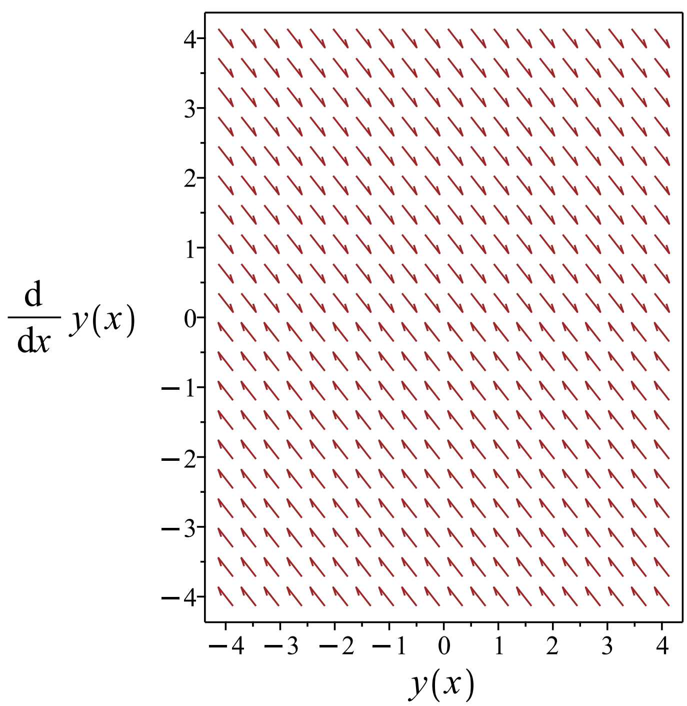

Figure 43: Slope field plot

Verification of solutions

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 e^{-x}$$

Verified OK.

1.18.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 24: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(\mathrm{e}^{x})\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2x^2 + A_1x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2xA_2 + A_1 + 2A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2}x^2 - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2) + \left(\frac{1}{2}x^2 - x\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + \frac{x^2}{2} - x \tag{1}$$

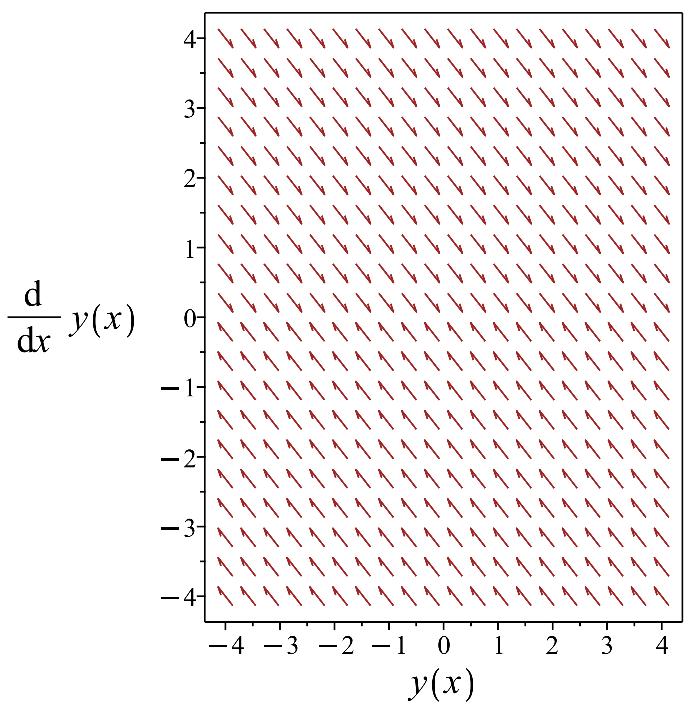

Figure 44: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-x} + c_2 + \frac{x^2}{2} - x$$

Verified OK.

1.18.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y''+q(x)\,y'+r(x)\,y=s(x)$$

is exact if

$$p''(x)-q'(x)+r(x)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x)&=1\\q(x)&=1\\r(x)&=0\\s(x)&=x\end{aligned}$$

Hence

$$\begin{aligned}p''(x)&=0\\q'(x)&=0\end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y'+(q(x)-p'(x))\,y\right)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$y'+y=\int x\,dx$$

We now have a first order ode to solve which is

$$y'+y=\frac{x^2}{2}+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \frac{x^2}{2} + c_1 \end{aligned}$$

Hence the ode is

$$y' + y = \frac{x^2}{2} + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^2}{2} + c_1\right) \\ \frac{d}{dx}(e^x y) &= (e^x)\left(\frac{x^2}{2} + c_1\right) \\ d(e^x y) &= \left(\frac{(x^2 + 2c_1)e^x}{2}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^x y &= \int \frac{(x^2 + 2c_1)e^x}{2} dx \\ e^x y &= \frac{(x^2 + 2c_1 - 2x + 2)e^x}{2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = \frac{e^{-x}(x^2 + 2c_1 - 2x + 2)e^x}{2} + c_2 e^{-x}$$

which simplifies to

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 e^{-x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 e^{-x} \tag{1}$$

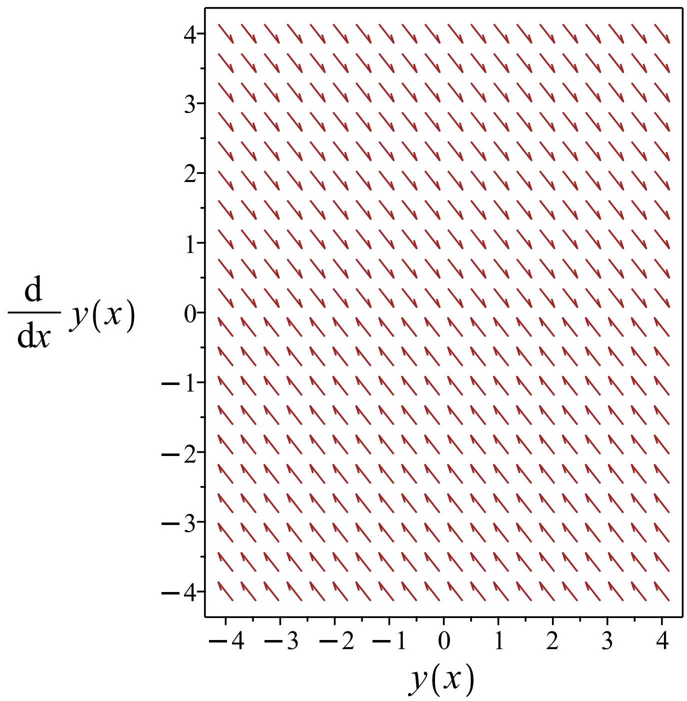

Figure 45: Slope field plot

Verification of solutions

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 e^{-x}$$

Verified OK.

1.18.7 Maple step by step solution

Let's solve

$y'' + y' = x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{array}\right]$$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-x}\left(\int x\, \mathrm{e}^{x} dx\right) + \int x dx$

  - Compute integrals

    $y_p(x) = 1 - x + \frac{1}{2}x^2$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + 1 - x + \frac{x^2}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)+_a, _b(_a)`   *** Sublevel 2 **
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=x,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} - e^{-x}c_1 - x + c_2$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 27

```
DSolve[y''[x]+y'[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2} - x - c_1e^{-x} + c_2$$

1.19 problem 19

1.19.1 Solving as second order ode missing y ode 194

Internal problem ID [7408]
Internal file name [`OUTPUT/6375_Sunday_June_05_2022_04_42_10_PM_194809/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 19.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y''^2 + y' = x}$$

1.19.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)^2 + p(x) - x = 0$$

Which is now solve for $p(x)$ as first order ode. The ode has the form

$$(p')^{\frac{n}{m}} = ax + bp + c \tag{1}$$

Where $n = 2, m = 1, a = 1, b = -1, c = 0$. Hence the ode is

$$(p')^2 = -p(x) + x$$

Let

$$u = ax + bp + c$$

Hence

$$\begin{aligned} u' &= a + bp' \\ p' &= \frac{u' - a}{b} \end{aligned}$$

Substituting the above in (1) gives

$$\begin{aligned} \left(\frac{u' - a}{b}\right)^{\frac{n}{m}} &= u \\ \left(\frac{u' - a}{b}\right)^{n} &= u^m \end{aligned}$$

Plugging in the above the values for n, m, a, b, c gives

$$(-u'(x) + 1)^2 = u$$

Therefore the solutions are

$$\begin{aligned} -u'(x) + 1 &= \sqrt{u} \\ -u'(x) + 1 &= -\sqrt{u} \end{aligned}$$

Rewriting the above gives

$$\begin{aligned} u'(x) &= -\sqrt{u} + 1 \\ u'(x) &= \sqrt{u} + 1 \end{aligned}$$

Each of the above is a separable ODE in $u(x)$. This results in

$$\begin{aligned} \frac{du}{-\sqrt{u} + 1} &= dx \\ \frac{du}{\sqrt{u} + 1} &= dx \end{aligned}$$

Integrating each of the above solutions gives

$$\begin{aligned} \int \frac{du}{-\sqrt{u} + 1} &= x + c_1 \\ \int \frac{du}{\sqrt{u} + 1} &= x + c_1 \end{aligned}$$

But since

$$
\begin{aligned}
u &= ax + bp + c \\
&= -p(x) + x
\end{aligned}
$$

Then the solutions can be written as

$$
\begin{aligned}
\int^{-p(x)+x} \frac{1}{-\sqrt{\tau}+1} d\tau &= x + c_1 \\
\int^{-p(x)+x} \frac{1}{\sqrt{\tau}+1} d\tau &= x + c_1
\end{aligned}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
\int^{-y'+x} \frac{1}{-\sqrt{\tau}+1} d\tau = x + c_1
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \text{RootOf}\left(-\left(\int^{x-\_Z} -\frac{1}{\sqrt{\tau}-1} d\tau\right) + x + c_1\right) \mathrm{d}x \\
&= \int \text{RootOf}\left(-\left(\int^{x-\_Z} -\frac{1}{\sqrt{\tau}-1} d\tau\right) + x + c_1\right) dx + c_2
\end{aligned}
$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
\int^{-y'+x} \frac{1}{\sqrt{\tau}+1} d\tau = x + c_1
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \text{RootOf}\left(-\left(\int^{x-\_Z} \frac{1}{\sqrt{\tau}+1} d\tau\right) + x + c_1\right) \mathrm{d}x \\
&= \int \text{RootOf}\left(-\left(\int^{x-\_Z} \frac{1}{\sqrt{\tau}+1} d\tau\right) + x + c_1\right) dx + c_3
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \int \text{RootOf}\left(-\left(\int^{x-\_Z} -\frac{1}{\sqrt{\tau}-1} d\tau\right) + x + c_1\right) dx + c_2 \tag{1}
$$

$$
y = \int \text{RootOf}\left(-\left(\int^{x-\_Z} \frac{1}{\sqrt{\tau}+1} d\tau\right) + x + c_1\right) dx + c_3 \tag{2}
$$

Verification of solutions

$$y=\int \mathrm{RootOf}\left(-\left(\int^{x-\_Z}-\frac{1}{\sqrt{\tau}-1}d\tau\right)+x+c_1\right)dx+c_2$$

Verified OK.

$$y=\int \mathrm{RootOf}\left(-\left(\int^{x-\_Z}\frac{1}{\sqrt{\tau}+1}d\tau\right)+x+c_1\right)dx+c_3$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting OD
   *** Sublevel 2 ***
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying 2nd order Liouville
   trying 2nd order WeierstrassP
   trying 2nd order JacobiSN
   differential order: 2; trying a linearization to 3rd order
   trying 2nd order ODE linearizable_by_differentiation
   trying 2nd order, 2 integrating factors of the form mu(x,y)
   trying differential order: 2; missing variables
   `, `-> Computing symmetries using: way = 3
   -> Calling odsolve with the ODE`, diff(_b(_a), _a) = (-_b(_a)+_a)^(1/2), _b(_a), HINT = [
      symmetry methods on request
   `, `1st order, trying reduction of order with given symmetries:`[1, 1]
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 122

```
dsolve(diff(y(x),x$2)^2+diff(y(x),x)=x,y(x), singsol=all)
```

$$y(x) = \int \left(-\mathrm{e}^{2\,\mathrm{RootOf}\left(\_Z - x - 2\,\mathrm{e}^{\_Z} + 2 + c_1 - \ln\left(\mathrm{e}^{\_Z}\left(\mathrm{e}^{\_Z} - 2\right)^2\right)\right)} + 2\,\mathrm{e}^{\mathrm{RootOf}\left(\_Z - x - 2\,\mathrm{e}^{\_Z} + 2 + c_1 - \ln\left(\mathrm{e}^{\_Z}\left(\mathrm{e}^{\_Z} - 2\right)^2\right)\right)} + x\right) dx - x + c_2$$

$$y(x) = \frac{2\,\mathrm{LambertW}\left(-c_1 \mathrm{e}^{-\frac{x}{2}-1}\right)^3}{3} + 3\,\mathrm{LambertW}\left(-c_1 \mathrm{e}^{-\frac{x}{2}-1}\right)^2 + 4\,\mathrm{LambertW}\left(-c_1 \mathrm{e}^{-\frac{x}{2}-1}\right) + \frac{x^2}{2} - x + c_2$$

✓ Solution by Mathematica

Time used: 24.995 (sec). Leaf size: 237

```
DSolve[(y''[x])^2+y'[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{3} W\left(e^{-\frac{x}{2}-1-\frac{c_1}{2}}\right)^3 + 3W\left(e^{-\frac{x}{2}-1-\frac{c_1}{2}}\right)^2 + 4W\left(e^{-\frac{x}{2}-1-\frac{c_1}{2}}\right) + \frac{x^2}{2} - x + c_2$$

$$y(x) \to \frac{2}{3} W\left(-e^{\frac{1}{2}(-x-2+c_1)}\right)^3 + 3W\left(-e^{\frac{1}{2}(-x-2+c_1)}\right)^2 + 4W\left(-e^{\frac{1}{2}(-x-2+c_1)}\right) + \frac{x^2}{2} - x + c_2$$

$$y(x) \to \frac{x^2}{2} - x + c_2$$

$$y(x) \to \frac{2}{3} W\left(-e^{-\frac{x}{2}-1}\right)^3 + 3W\left(-e^{-\frac{x}{2}-1}\right)^2 + 4W\left(-e^{-\frac{x}{2}-1}\right) + \frac{x^2}{2} - x + c_2$$

1.20 problem 20

1.20.1 Solving as second order ode missing y ode 199

Internal problem ID [7409]
Internal file name [`OUTPUT/6376_Sunday_June_05_2022_04_42_16_PM_97106534/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_xy]]`

$$\boxed{y'' + y'^2 = x}$$

1.20.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x)^2 - x = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x, p) \\ &= -p^2 + x \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$p' = -p^2 + x$$

With Riccati ODE standard form

$$p' = f_0(x) + f_1(x)p + f_2(x)p^2$$

Shows that $f_0(x) = x$, $f_1(x) = 0$ and $f_2(x) = -1$. Let

$$\begin{aligned} p &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= x \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) + xu(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \operatorname{AiryAi}(x) + c_2 \operatorname{AiryBi}(x)$$

The above shows that

$$u'(x) = c_1 \operatorname{AiryAi}(1, x) + c_2 \operatorname{AiryBi}(1, x)$$

Using the above in (1) gives the solution

$$p(x) = \frac{c_1 \operatorname{AiryAi}(1, x) + c_2 \operatorname{AiryBi}(1, x)}{c_1 \operatorname{AiryAi}(x) + c_2 \operatorname{AiryBi}(x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$p(x) = \frac{c_3 \operatorname{AiryAi}(1, x) + \operatorname{AiryBi}(1, x)}{c_3 \operatorname{AiryAi}(x) + \operatorname{AiryBi}(x)}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{c_3 \operatorname{AiryAi}(1, x) + \operatorname{AiryBi}(1, x)}{c_3 \operatorname{AiryAi}(x) + \operatorname{AiryBi}(x)}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_3 \operatorname{AiryAi}(1, x) + \operatorname{AiryBi}(1, x)}{c_3 \operatorname{AiryAi}(x) + \operatorname{AiryBi}(x)} \, \mathrm{d}x \\ &= \ln \left(c_3 \operatorname{AiryAi}(x) + \operatorname{AiryBi}(x)\right) + c_4 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \ln \left(c_3 \operatorname{AiryAi}(x) + \operatorname{AiryBi}(x)\right) + c_4 \tag{1}$$

Verification of solutions

$$y = \ln \left(c_3 \operatorname{AiryAi}(x) + \operatorname{AiryBi}(x)\right) + c_4$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)+diff(y(x),x)^2=x,y(x), singsol=all)
```

$$y(x) = \ln(\pi) + \ln(c_1 \operatorname{AiryAi}(x) - c_2 \operatorname{AiryBi}(x))$$

✓ Solution by Mathematica

Time used: 0.114 (sec). Leaf size: 15

```
DSolve[y''[x]+(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(x - c_1) + c_2$$

1.21 problem 21

1.21.1 Solving as second order linear constant coeff ode 203
1.21.2 Solving using Kovacic algorithm 205
1.21.3 Maple step by step solution 209

Internal problem ID [7410]
Internal file name [`OUTPUT/6377_Sunday_June_05_2022_04_42_19_PM_29542181/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' + y = 0}$$

1.21.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda \,\mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}
$$

Since roots are complex conjugate of each others, then let the roots be

$$
\lambda_{1,2} = \alpha \pm i\beta
$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$
y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))
$$

Which becomes

$$
y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \tag{1}
$$

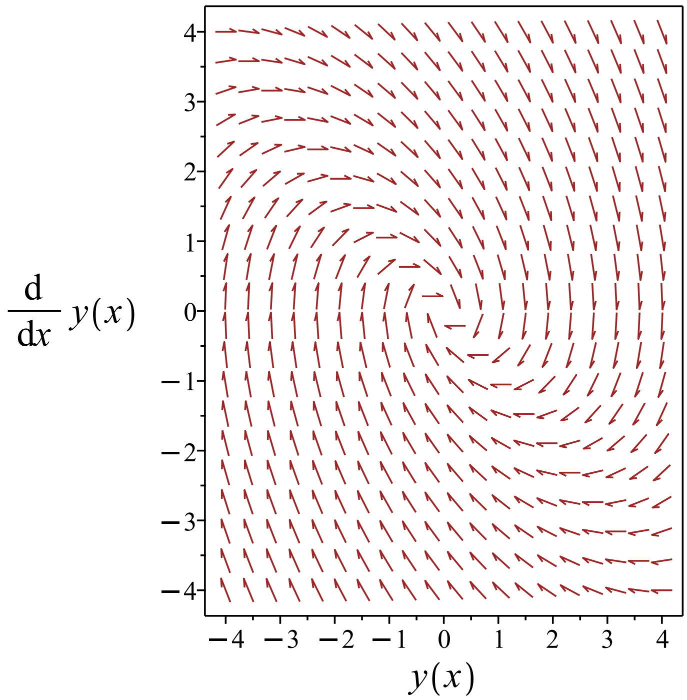

Figure 46: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Verified OK.

1.21.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 26: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-x}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3} \tag{1}
$$

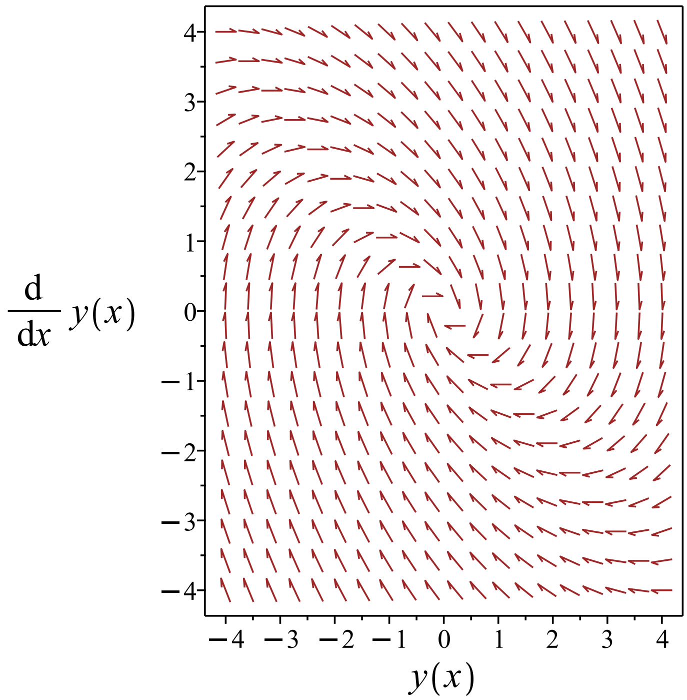

Figure 47: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3}$$

Verified OK.

1.21.3 Maple step by step solution

Let's solve

$y'' + y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the ODE

  $y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}} \left(c_1 \sin\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 42

```
DSolve[y''[x]+y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}\left(c_2\cos\left(\frac{\sqrt{3}x}{2}\right)+c_1\sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

1.22 problem 22

Internal problem ID [7411]
Internal file name [`OUTPUT/6378_Sunday_June_05_2022_04_42_21_PM_1760656/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 22.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

Unable to solve or complete the solution.

$$\boxed{y''^2 + y' + y = 0}$$

Does not support ODE with y''^n where $n \neq 1$ unless 1 is missing which is not the case here.

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      `, `-> Computing symmetries using: way = exp_sym
      -> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(-_b(_a)-_a)^(1/2) = 0, _b
         Methods for first order ODEs:
         --- Trying classification methods ---
         trying homogeneous types:
         trying exact
         Looking for potential symmetries
         trying an equivalence to an Abel ODE
         trying 1st order ODE linearizable_by_differentiation
      -> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integra
      trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cas
      trying differential order: 2; exact nonlinear
      trying 2nd order, integrating factor of the form mu(x,y)
      -> trying 2nd order, the S-function method
         -> trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for
         -> trying 2nd order, the S-function method
         -> trying 2nd order, No Point Symmetries Class V
         -> trying 2nd order, No Point Symmetries Class V
         -> trying 2nd order, No Point Symmetries Class V
      trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
      -> trying 2nd order, dynamical_symmetries, only a reduction of order through one integ
   solving 2nd order ODE of high degree, Lie methods
`, `2nd order, trying reduction of order with given symmetries:`[1, 0]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)^2+diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(y''[x])^2+y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

1.23 problem 23

1.23.1 Solving as second order ode missing x ode 215

Internal problem ID [7412]
Internal file name [`OUTPUT/6379_Sunday_June_05_2022_04_42_23_PM_71352758/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + {y'}^2 + y = 0}$$

1.23.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + p(y)^2 + y = 0$$

Which is now solved as first order ode for $p(y)$. Writing the ode as

$$\frac{d}{dy}p(y) = -\frac{p^2+y}{p}$$
$$\frac{d}{dy}p(y) = \omega(y,p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 28: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(y,p) &= 0 \\
\eta(y,p) &= \frac{\mathrm{e}^{-2y}}{p}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y,p) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial p}\right) S(y,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{\mathrm{e}^{-2y}}{p}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{p^2 \mathrm{e}^{2y}}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_y + \omega(y,p) S_p}{R_y + \omega(y,p) R_p} \tag{2}
$$

Where in the above R_y, R_p, S_y, S_p are all partial derivatives and $\omega(y,p)$ is the right hand side of the original ode given by

$$
\omega(y,p) = -\frac{p^2 + y}{p}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_y &= 1 \\
R_p &= 0 \\
S_y &= p^2 e^{2y} \\
S_p &= p\, e^{2y}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -e^{2y} y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for y, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -e^{2R} R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{(2R-1)\, e^{2R}}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to y, p coordinates. This results in

$$\frac{p(y)^2\, e^{2y}}{2} = -\frac{(2y-1)\, e^{2y}}{4} + c_1$$

Which simplifies to

$$\frac{(2p(y)^2 + 2y - 1)\, e^{2y}}{4} - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{(2y'^2 + 2y - 1)\, e^{2y}}{4} - c_1 = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{e^{-2y}\sqrt{-2\, e^{2y}\, (2y\, e^{2y} - e^{2y} - 4c_1)}}{2} \tag{1}$$

$$y' = -\frac{e^{-2y}\sqrt{-2\, e^{2y}\, (2y\, e^{2y} - e^{2y} - 4c_1)}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{2\,\mathrm{e}^{2y}}{\sqrt{-2\,\mathrm{e}^{2y}\left(2\,\mathrm{e}^{2y}y-\mathrm{e}^{2y}-4c_1\right)}}dy = \int dx$$

$$2\left(\int^{y}\frac{\mathrm{e}^{2\_a}}{\sqrt{-2\,\mathrm{e}^{2\_a}\left(2\,\mathrm{e}^{2\_a}\_a-\mathrm{e}^{2\_a}-4c_1\right)}}d\_a\right) = x+c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2\,\mathrm{e}^{2y}}{\sqrt{-2\,\mathrm{e}^{2y}\left(2\,\mathrm{e}^{2y}y-\mathrm{e}^{2y}-4c_1\right)}}dy = \int dx$$

$$-2\left(\int^{y}\frac{\mathrm{e}^{2\_a}}{\sqrt{-2\,\mathrm{e}^{2\_a}\left(2\,\mathrm{e}^{2\_a}\_a-\mathrm{e}^{2\_a}-4c_1\right)}}d\_a\right) = x+c_3$$

Summary

The solution(s) found are the following

$$2\left(\int^{y}\frac{\mathrm{e}^{2\_a}}{\sqrt{-2\,\mathrm{e}^{2\_a}\left(2\,\mathrm{e}^{2\_a}\_a-\mathrm{e}^{2\_a}-4c_1\right)}}d\_a\right) = x+c_2 \tag{1}$$

$$-2\left(\int^{y}\frac{\mathrm{e}^{2\_a}}{\sqrt{-2\,\mathrm{e}^{2\_a}\left(2\,\mathrm{e}^{2\_a}\_a-\mathrm{e}^{2\_a}-4c_1\right)}}d\_a\right) = x+c_3 \tag{2}$$

Verification of solutions

$$2\left(\int^{y}\frac{\mathrm{e}^{2\_a}}{\sqrt{-2\,\mathrm{e}^{2\_a}\left(2\,\mathrm{e}^{2\_a}\_a-\mathrm{e}^{2\_a}-4c_1\right)}}d\_a\right) = x+c_2$$

Verified OK.

$$-2\left(\int^{y}\frac{\mathrm{e}^{2\_a}}{\sqrt{-2\,\mathrm{e}^{2\_a}\left(2\,\mathrm{e}^{2\_a}\_a-\mathrm{e}^{2\_a}-4c_1\right)}}d\_a\right) = x+c_3$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)^2+_a = 0, _b(_a)`   *** S
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 61

```
dsolve(diff(y(x),x$2)+diff(y(x),x)^2+y(x)=0,y(x), singsol=all)
```

$$-2\left(\int^{y(x)} \frac{1}{\sqrt{2+4\,\mathrm{e}^{-2\_a}c_1-4\_a}}d\_a\right)-x-c_2=0$$
$$2\left(\int^{y(x)} \frac{1}{\sqrt{2+4\,\mathrm{e}^{-2\_a}c_1-4\_a}}d\_a\right)-x-c_2=0$$

✓ Solution by Mathematica

Time used: 0.786 (sec). Leaf size: 272

```
DSolve[y''[x]+(y'[x])^2+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} -\frac{\sqrt{2}}{\sqrt{2e^{-2K[1]}c_1 - 2K[1] + 1}} dK[1]\&\right][x + c_2]$$

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{\sqrt{2}}{\sqrt{2e^{-2K[2]}c_1 - 2K[2] + 1}} dK[2]\&\right][x + c_2]$$

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} -\frac{\sqrt{2}}{\sqrt{2e^{-2K[1]}(-c_1) - 2K[1] + 1}} dK[1]\&\right][x + c_2]$$

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} -\frac{\sqrt{2}}{\sqrt{2e^{-2K[1]}c_1 - 2K[1] + 1}} dK[1]\&\right][x + c_2]$$

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{\sqrt{2}}{\sqrt{2e^{-2K[2]}(-c_1) - 2K[2] + 1}} dK[2]\&\right][x + c_2]$$

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{\sqrt{2}}{\sqrt{2e^{-2K[2]}c_1 - 2K[2] + 1}} dK[2]\&\right][x + c_2]$$

1.24 problem 24

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Internal problem ID [7413]
Internal file name [`OUTPUT/6380_Sunday_June_05_2022_04_42_25_PM_51463080/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' + y = 1}$$

1.24.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda \, \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \right) \right) + (1) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right) + 1 \tag{1}$$

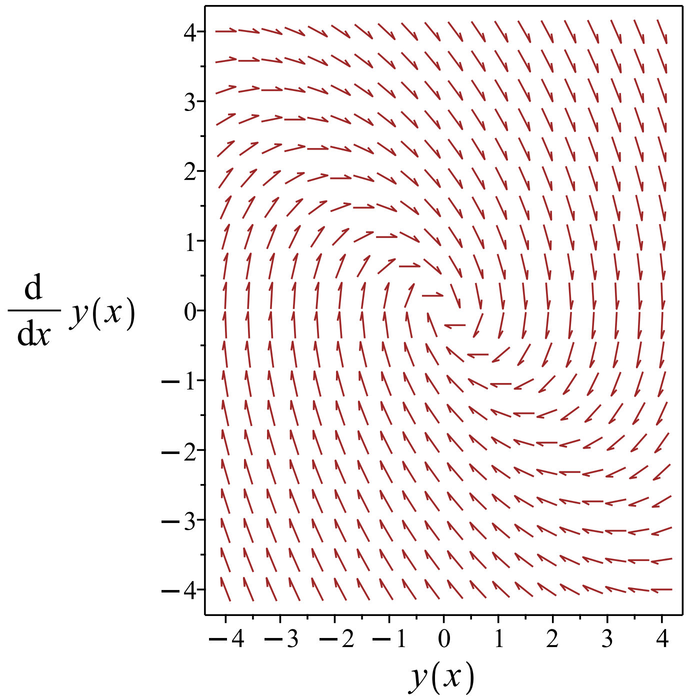

Figure 48: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right) + 1$$

Verified OK.

1.24.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 30: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), \frac{2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}\right) + (1)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3} + 1 \tag{1}
$$

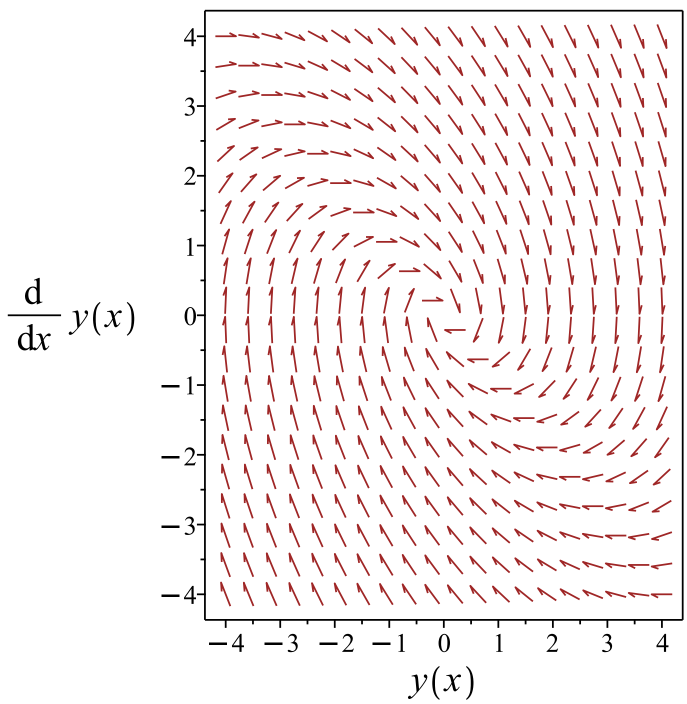

Figure 49: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3} + 1
$$

Verified OK.

1.24.3 Maple step by step solution

Let's solve

$y'' + y' + y = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 1\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \sqrt{3} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \end{bmatrix}$$

  - Compute Wronskian

$W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,e^{-x}}{2}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2\,e^{-\frac{x}{2}}\sqrt{3}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)\right)}{3}$

- Compute integrals

  $y_p(x) = 1$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}} + 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=1,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)c_2 + e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)c_1 + 1$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 49

```
DSolve[y''[x]+y'[x]+y[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}\left(e^{x/2} + c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 \sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

1.25 problem 25

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Internal problem ID [7414]
Internal file name [`OUTPUT/6381_Sunday_June_05_2022_04_42_27_PM_44417347/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' + y = x}$$

1.25.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=1, C=1$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}+\lambda \mathrm{e}^{\lambda x}+\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+\lambda+1=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2-4AC}$$

Substituting $A=1, B=1, C=1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{1^2-(4)(1)(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2}+\frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2}-\frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2}+\frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2}-\frac{i\sqrt{3}}{2}
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2}=\alpha \pm i\beta$$

Where $\alpha=-\frac{1}{2}$ and $\beta=\frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y=e^{\alpha x}\left(c_1 \cos(\beta x)+c_2 \sin(\beta x)\right)$$

Which becomes

$$y=e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right)+c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 + A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)\right) + (x - 1) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right) + x - 1 \tag{1}$$

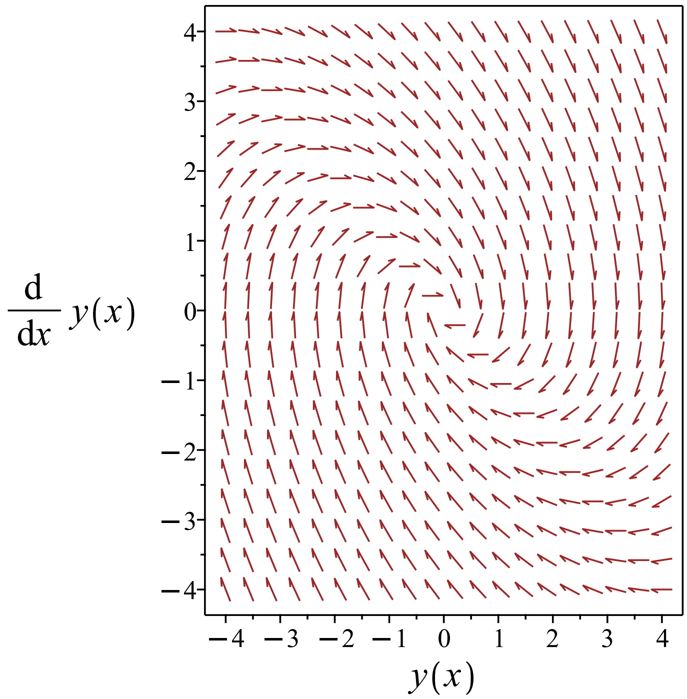

Figure 50: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right) + x - 1$$

Verified OK.

1.25.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 32: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \frac{2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 + A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x - 1$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}\right) + (x - 1)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3} + x - 1 \tag{1}
$$

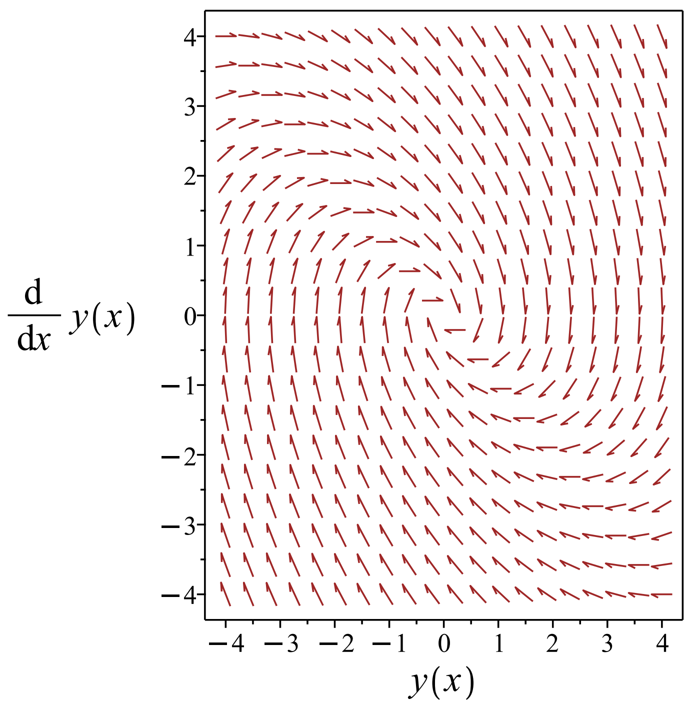

Figure 51: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3} + x - 1
$$

Verified OK.

1.25.3 Maple step by step solution

Let's solve

$y'' + y' + y = x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \sqrt{3} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \end{bmatrix}$

- ○ Compute Wronskian

$$W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,e^{-x}}{2}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{2\sqrt{3}\,e^{-\frac{x}{2}}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int x\,e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int x\,e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)\right)}{3}$$

- Compute integrals

$$y_p(x) = x - 1$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}} + x - 1$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=x,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)c_2 + e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)c_1 + x - 1$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 50

```
DSolve[y''[x]+y'[x]+y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right) - 1$$

1.26 problem 26

1.26.1 Solving as second order linear constant coeff ode 246
1.26.2 Solving using Kovacic algorithm 250
1.26.3 Maple step by step solution 255

Internal problem ID [7415]
Internal file name [`OUTPUT/6382_Sunday_June_05_2022_04_42_29_PM_61963261/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' + y = 1 + x}$$

1.26.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = 1 + x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)}\\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 + A_2 = 1 + x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \right) \right) + (x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \right) + x \tag{1}$$

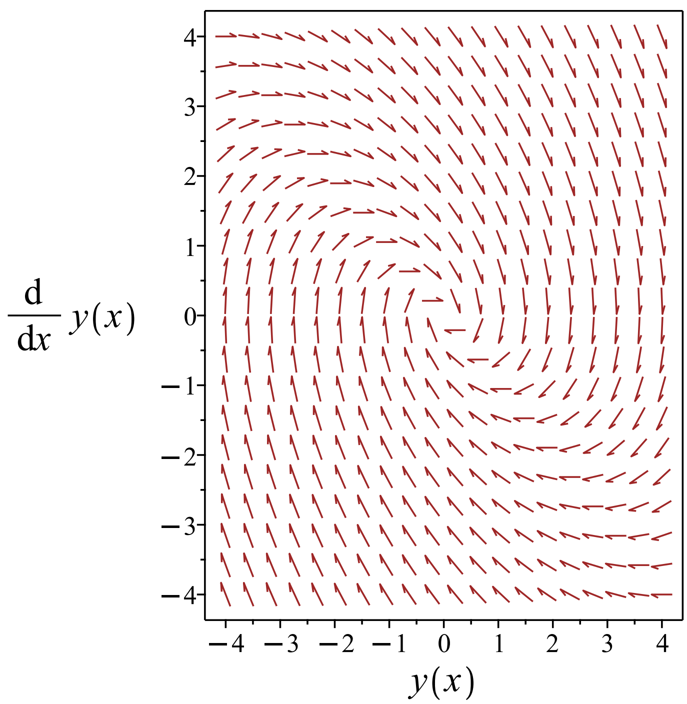

Figure 52: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \right) + x$$

Verified OK.

1.26.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 34: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+y'+y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=c_1\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\frac{2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1+x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1,x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right),\frac{2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p=A_2x+A_1$$

The unknowns $\{A_1,A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2x+A_1+A_2=1+x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1=0,A_2=1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p=x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3}\right) + (x)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3} + x \tag{1}
$$

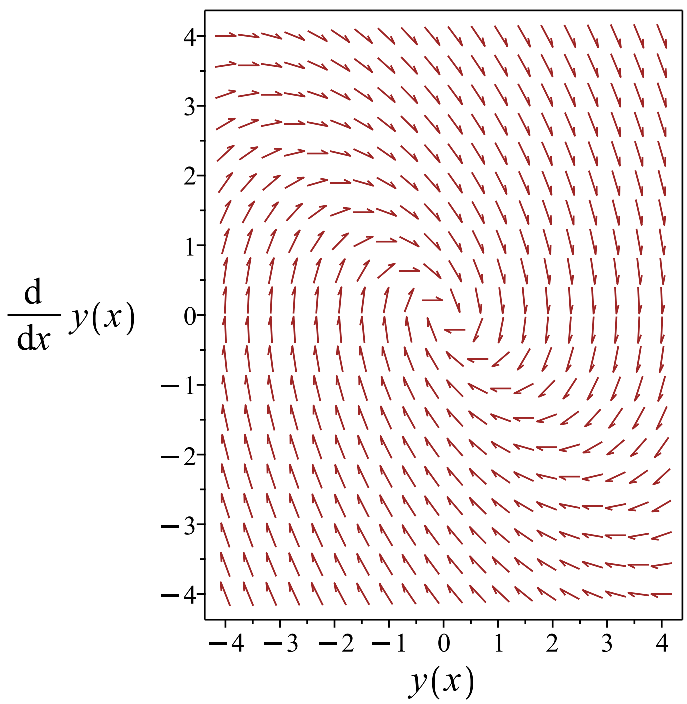

Figure 53: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3} + x
$$

Verified OK.

1.26.3 Maple step by step solution

Let's solve

$y'' + y' + y = 1 + x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 1 + x\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) & \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \\ -\frac{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{2} & -\frac{\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2} + \frac{\mathrm{e}^{-\frac{x}{2}} \sqrt{3} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2} \end{bmatrix}$

- ○ Compute Wronskian

$W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,e^{-x}}{2}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2\,e^{-\frac{x}{2}}\sqrt{3}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int (1+x)e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int (1+x)e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)\right)}{3}$

- Compute integrals

  $y_p(x) = x$

- Substitute particular solution into general solution to ODE

  $y = c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} + c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=1+x,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) c_2 + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_1 + x$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 49

```
DSolve[y''[x]+y'[x]+y[x]==1+x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right)$$

1.27 problem 27

1.27.1 Solving as second order linear constant coeff ode 258
1.27.2 Solving using Kovacic algorithm 262
1.27.3 Maple step by step solution 267

Internal problem ID [7416]
Internal file name [`OUTPUT/6383_Sunday_June_05_2022_04_42_33_PM_29690911/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' + y = x^2 + x + 1}$$

1.27.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = x^2 + x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda \, \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(1)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_3 x^2 + A_2 x + 2x A_3 + A_1 + A_2 + 2A_3 = x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)\right) + (x^2 - x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) + x^2 - x \tag{1}$$

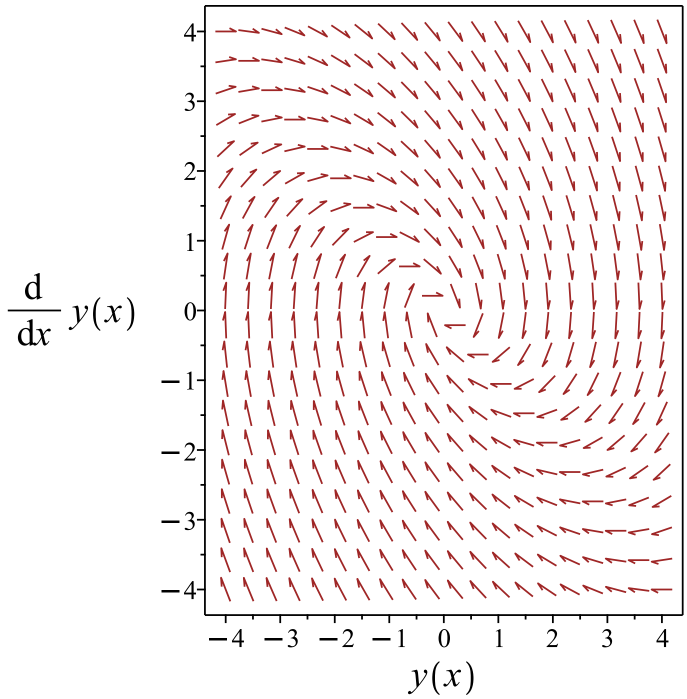

Figure 54: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) + x^2 - x$$

Verified OK.

1.27.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 36: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \frac{2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_3 x^2 + A_2 x + 2x A_3 + A_1 + A_2 + 2A_3 = x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3}\right) + (x^2 - x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3} + x^2 - x \tag{1}$$

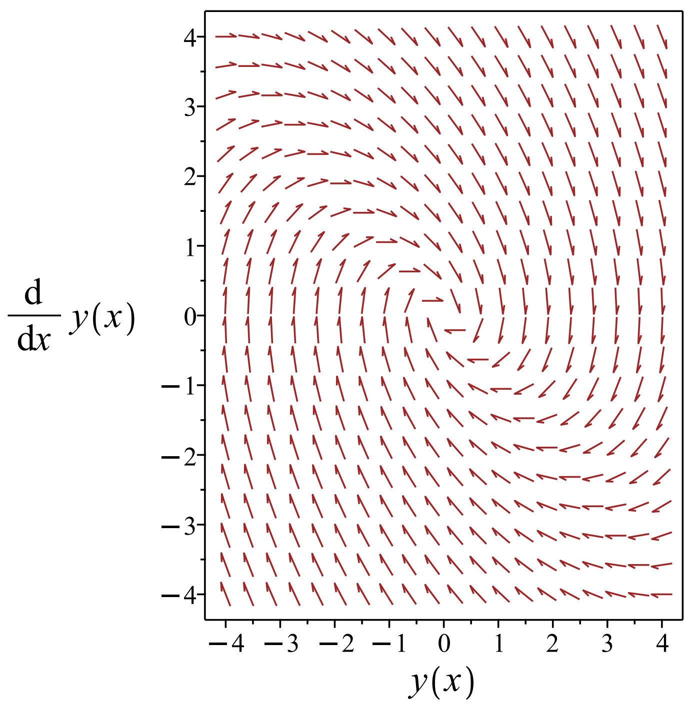

Figure 55: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3} + x^2 - x$$

Verified OK.

1.27.3 Maple step by step solution

Let's solve

$y'' + y' + y = x^2 + x + 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2 + x + 1\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \sqrt{3}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,e^{-x}}{2}$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\frac{2\sqrt{3}\,e^{-\frac{x}{2}}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}(x^2+x+1)\sin\left(\frac{\sqrt{3}\,x}{2}\right)dx\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}(x^2+x+1)\cos\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)\right)}{3}$$

- Compute integrals

  $y_p(x) = x(x-1)$

- Substitute particular solution into general solution to ODE

  $$y = c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} + x(x-1)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=1+x+x^2,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) c_2 + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_1 + x^2 - x$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 54

```
DSolve[y''[x]+y'[x]+y[x]==1+x+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}\left(e^{x/2}(x-1)x + c_2\cos\left(\frac{\sqrt{3}x}{2}\right) + c_1\sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

1.28 problem 28

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Internal problem ID [7417]
Internal file name [`OUTPUT/6384_Sunday_June_05_2022_04_42_35_PM_85484633/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' + y = x^3 + x^2 + x + 1}$$

1.28.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = x^3 + x^2 + x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda\, \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4 x^3 + A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_4 x^3 + A_3 x^2 + 3x^2 A_4 + A_2 x + 2x A_3 + 6x A_4 + A_1 + A_2 + 2A_3 = x^3 + x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 6, A_2 = -1, A_3 = -2, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^3 - 2x^2 - x + 6$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\right) + \left(x^3 - 2x^2 - x + 6\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) + x^3 - 2x^2 - x + 6 \tag{1}$$

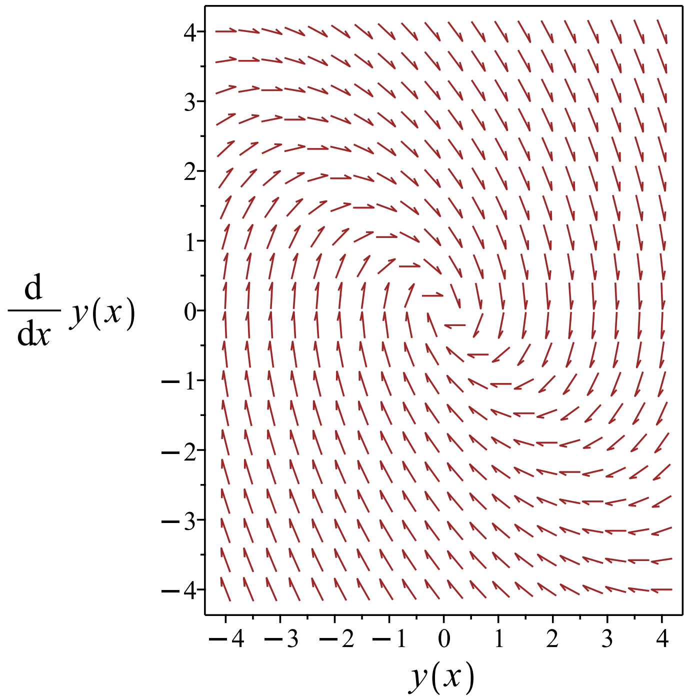

Figure 56: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) + x^3 - 2x^2 - x + 6$$

Verified OK.

1.28.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 38: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), \frac{2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4 x^3 + A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_4 x^3 + A_3 x^2 + 3x^2 A_4 + A_2 x + 2x A_3 + 6x A_4 + A_1 + A_2 + 2A_3 = x^3 + x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 6, A_2 = -1, A_3 = -2, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^3 - 2x^2 - x + 6$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3}\right) + \left(x^3 - 2x^2 - x + 6\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3} + x^3 - 2x^2 - x + 6 \tag{1}$$

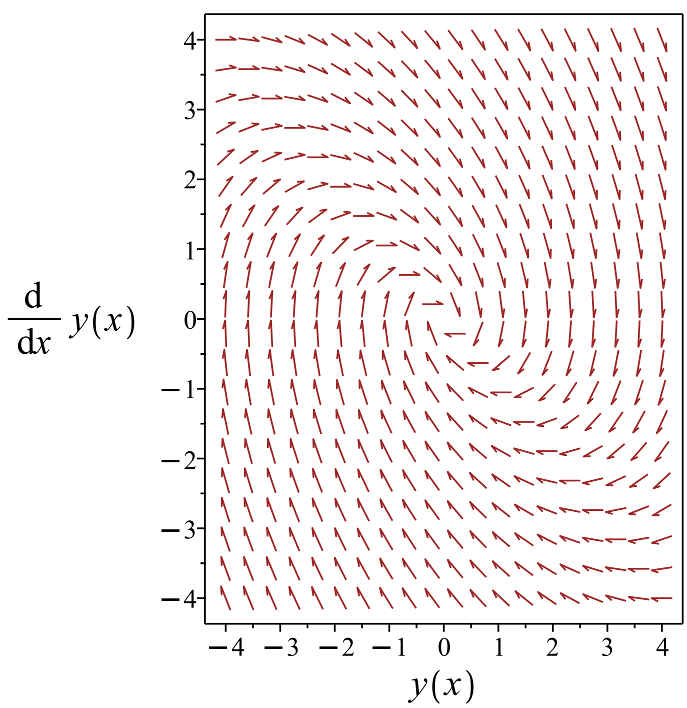

Figure 57: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3} + x^3 - 2x^2 - x + 6$$

Verified OK.

1.28.3 Maple step by step solution

Let's solve

$y'' + y' + y = x^3 + x^2 + x + 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^3 + x^2 + x + 1\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \sqrt{3} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,e^{-x}}{2}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = \frac{2\,e^{-\frac{x}{2}}\sqrt{3}\left(\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}(1+x)(x^2+1)\cos\left(\frac{\sqrt{3}\,x}{2}\right)dx\right) - \cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}(1+x)(x^2+1)\sin\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)\right)}{3}$$

- Compute integrals

  $$y_p(x) = x^3 - 2x^2 - x + 6$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} + x^3 - 2x^2 - x + 6$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=1+x+x^2+x^3,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) c_2 + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_1 + x^3 - 2x^2 - x + 6$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 60

```
DSolve[y''[x]+y'[x]+y[x]==1+x+x^2+x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^3 - 2x^2 - x + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right) + 6$$

1.29 problem 29

1.29.1 Solving as second order linear constant coeff ode 282
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1.29.3 Maple step by step solution 291

Internal problem ID [7418]
Internal file name [`OUTPUT/6385_Sunday_June_05_2022_04_42_37_PM_60772261/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' + y = \sin(x)}$$

1.29.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(x) + A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)\right) + (-\cos(x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \right) - \cos(x) \tag{1}$$

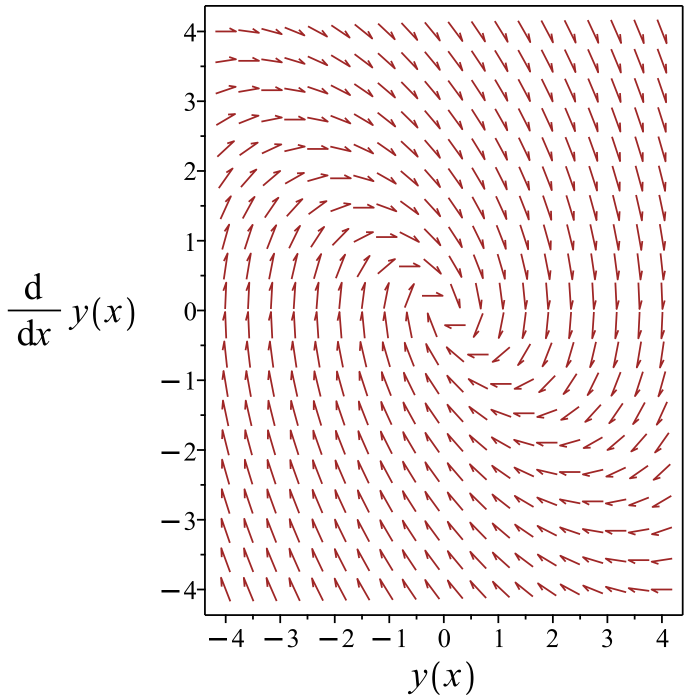

Figure 58: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \right) - \cos(x)$$

Verified OK.

1.29.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 40: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), \frac{2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(x) + A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}\right) + (-\cos(x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3} - \cos(x) \tag{1}
$$

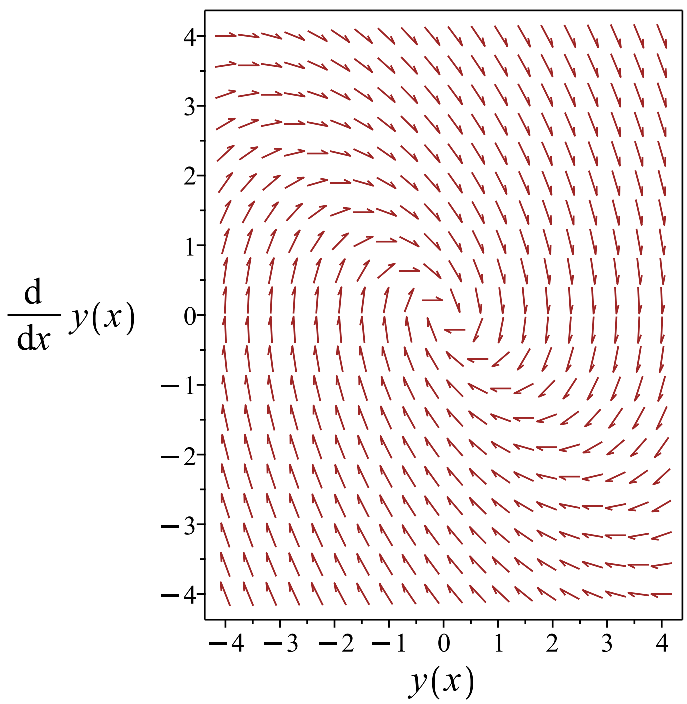

Figure 59: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3} - \cos(x)
$$

Verified OK.

1.29.3 Maple step by step solution

Let's solve

$y'' + y' + y = \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \sqrt{3} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \end{bmatrix}$

  ◦ Compute Wronskian

$W(y_1(x), y_2(x)) = \frac{\sqrt{3}\, e^{-x}}{2}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{2\, e^{-\frac{x}{2}} \sqrt{3} \left(-\cos\left(\frac{\sqrt{3}\, x}{2}\right) \left(\int e^{\frac{x}{2}} \sin(x) \sin\left(\frac{\sqrt{3}\, x}{2}\right) dx\right) + \sin\left(\frac{\sqrt{3}\, x}{2}\right) \left(\int e^{\frac{x}{2}} \sin(x) \cos\left(\frac{\sqrt{3}\, x}{2}\right) dx\right)\right)}{3}$

- Compute integrals

  $y_p(x) = -\cos(x)$

- Substitute particular solution into general solution to ODE

  $y = c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} + c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) - \cos(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=sin(x),y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) c_2 + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) c_1 - \cos(x)$$

✓ Solution by Mathematica

Time used: 0.309 (sec). Leaf size: 53

```
DSolve[y''[x]+y'[x]+y[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}\left(-e^{x/2}\cos(x) + c_2\cos\left(\frac{\sqrt{3}x}{2}\right) + c_1\sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

1.30 problem 30

1.30.1 Solving as second order linear constant coeff ode 294
1.30.2 Solving using Kovacic algorithm 298
1.30.3 Maple step by step solution 303

Internal problem ID [7419]
Internal file name [`OUTPUT/6386_Sunday_June_05_2022_04_42_40_PM_78911076/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' + y = \cos(x)}$$

1.30.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda \,\mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(1)}\\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(x) + A_2 \cos(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)\right) + (\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right) + \sin(x) \tag{1}$$

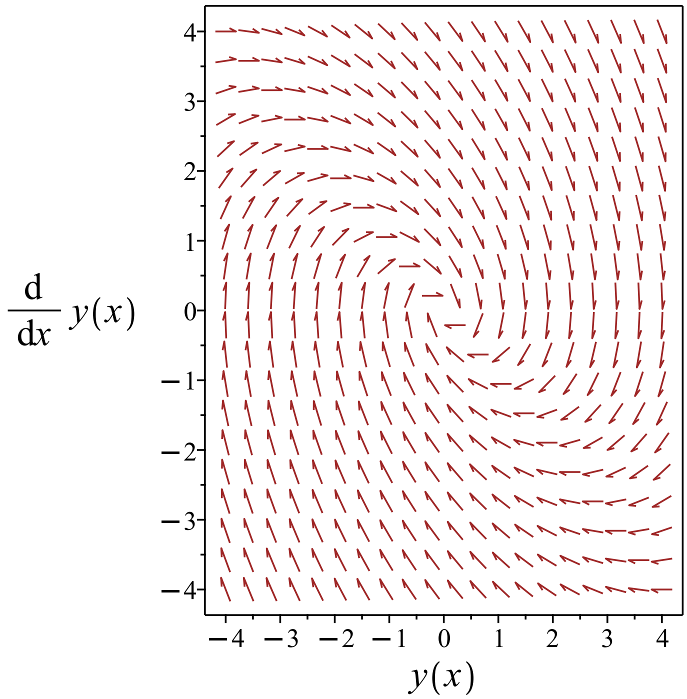

Figure 60: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right) + \sin(x)$$

Verified OK.

1.30.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 42: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+y'+y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)+\frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos\left(x\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos\left(x\right), \sin\left(x\right)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \frac{2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos\left(x\right)+A_2 \sin\left(x\right)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin\left(x\right)+A_2 \cos\left(x\right) = \cos\left(x\right)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \sin\left(x\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}\right) + (\sin(x))
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3} + \sin(x) \tag{1}
$$

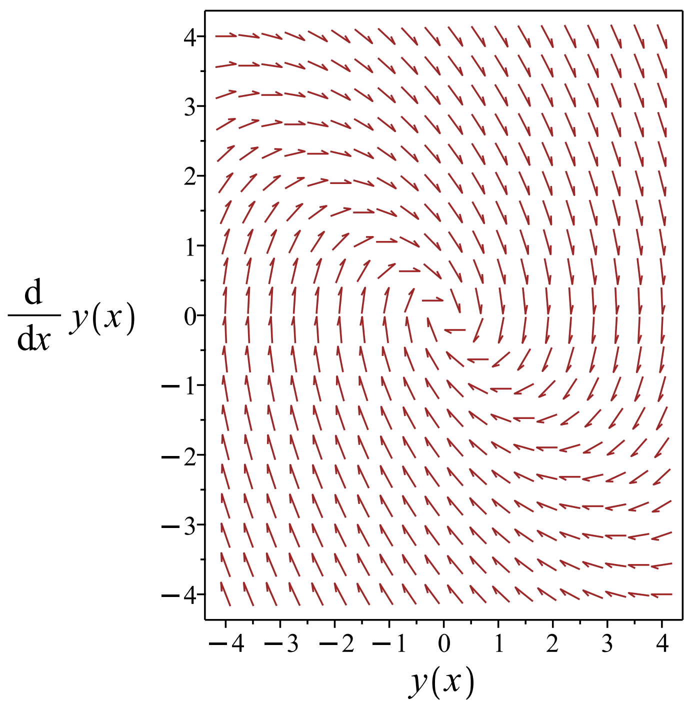

Figure 61: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3} + \sin(x)
$$

Verified OK.

1.30.3 Maple step by step solution

Let's solve

$y'' + y' + y = \cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \cos(x)\right]$

- Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}\sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}}\sqrt{3}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

$W(y_1(x), y_2(x)) = \frac{\sqrt{3}\, e^{-x}}{2}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{2\, e^{-\frac{x}{2}} \sqrt{3} \left(-\cos\left(\frac{\sqrt{3}\, x}{2}\right) \left(\int \cos(x) e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) dx\right) + \sin\left(\frac{\sqrt{3}\, x}{2}\right) \left(\int \cos(x) e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) dx\right)\right)}{3}$

- Compute integrals

  $y_p(x) = \sin(x)$

- Substitute particular solution into general solution to ODE

  $y = c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} + c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=cos(x),y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) c_2 + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) c_1 + \sin(x)$$

✓ Solution by Mathematica

Time used: 0.63 (sec). Leaf size: 50

```
DSolve[y''[x]+y'[x]+y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x) + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right)$$

1.31 problem 31

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Internal problem ID [7420]
Internal file name [`OUTPUT/6387_Sunday_June_05_2022_04_42_42_PM_56079143/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' = 1}$$

1.31.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(0)}\\ &= -\frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(0)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + c_2 \mathrm{e}^{-x}\right) + (x)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + c_2 \mathrm{e}^{-x} + x \tag{1}
$$

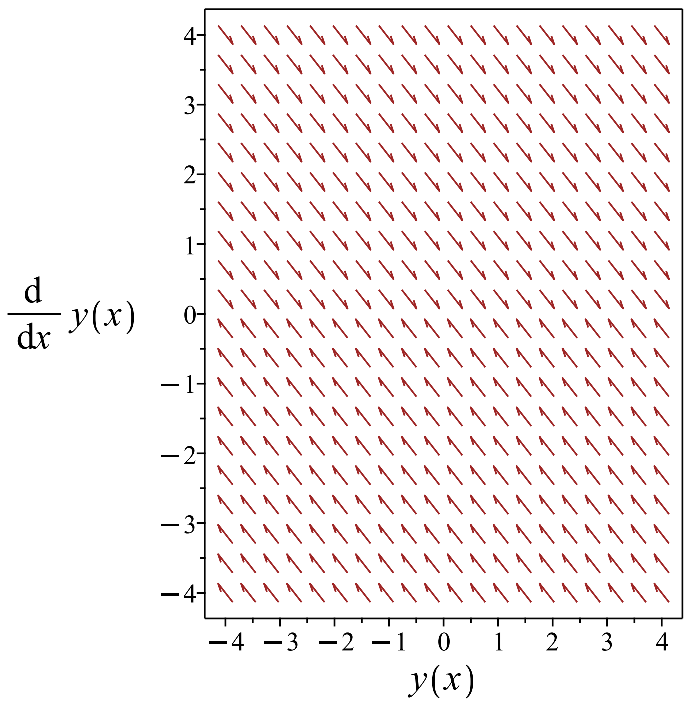

Figure 62: Slope field plot

Verification of solutions

$$
y = c_1 + c_2 \mathrm{e}^{-x} + x
$$

Verified OK.

1.31.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int 1 dx$$
$$y' + y = x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = x + c_1$$

Hence the ode is

$$y' + y = x + c_1$$

The integrating factor μ is

$$\mu = e^{\int 1 dx}$$
$$= e^{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(x + c_1)$$
$$\frac{d}{dx}(e^{x} y) = (e^{x})(x + c_1)$$
$$d(e^{x} y) = ((x + c_1) e^{x})\, dx$$

Integrating gives

$$e^{x} y = \int (x + c_1) e^{x}\, dx$$
$$e^{x} y = (c_1 + x - 1) e^{x} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$y = e^{-x}(c_1 + x - 1) e^{x} + c_2 e^{-x}$$

which simplifies to

$$y = c_1 + x - 1 + c_2 e^{-x}$$

Summary
The solution(s) found are the following

$$y = c_1 + x - 1 + c_2 e^{-x} \tag{1}$$

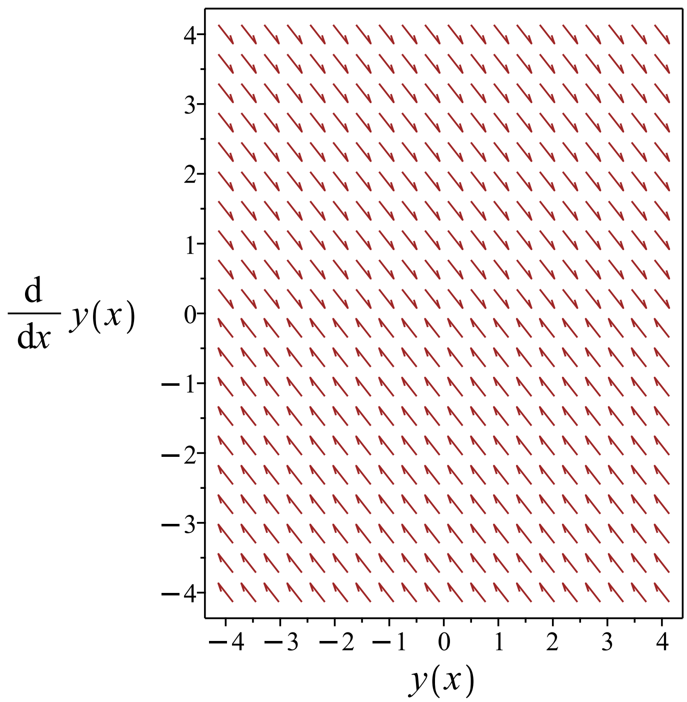

Figure 63: Slope field plot

Verification of solutions

$$y = c_1 + x - 1 + c_2 e^{-x}$$

Verified OK.

1.31.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - 1 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{-p+1} dp = \int dx$$
$$-\ln(-p+1) = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{-p+1} = \mathrm{e}^{x+c_1}$$

Which simplifies to

$$\frac{1}{-p+1} = c_2 \mathrm{e}^{x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{\mathrm{e}^{-x}}{c_2} + 1$$

Integrating both sides gives

$$y = \int \frac{(c_2 \mathrm{e}^{x} - 1)\,\mathrm{e}^{-x}}{c_2}\,\mathrm{d}x$$
$$= \frac{\mathrm{e}^{-x}}{c_2} + \ln(\mathrm{e}^{x}) + c_3$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-x}}{c_2} + \ln(\mathrm{e}^{x}) + c_3 \tag{1}$$

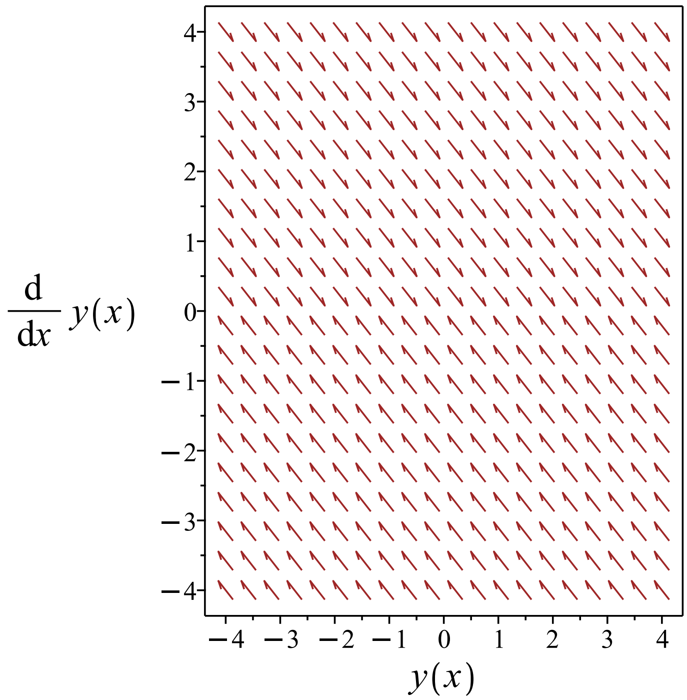

Figure 64: Slope field plot

Verification of solutions

$$y = \frac{e^{-x}}{c_2} + \ln\left(e^x\right) + c_3$$

Verified OK.

1.31.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = 1$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int 1 dx$$

$$y' + y = x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= x + c_1
\end{aligned}
$$

Hence the ode is

$$
y' + y = x + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1 dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(x + c_1) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} y) &= (\mathrm{e}^{x})\,(x + c_1) \\
\mathrm{d}(\mathrm{e}^{x} y) &= ((x + c_1)\,\mathrm{e}^{x})\;\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{x} y &= \int (x + c_1)\,\mathrm{e}^{x}\,\mathrm{d}x \\
\mathrm{e}^{x} y &= (c_1 + x - 1)\,\mathrm{e}^{x} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$
y = \mathrm{e}^{-x}(c_1 + x - 1)\,\mathrm{e}^{x} + c_2\mathrm{e}^{-x}
$$

which simplifies to

$$
y = c_1 + x - 1 + c_2\mathrm{e}^{-x}
$$

Summary
The solution(s) found are the following

$$
y = c_1 + x - 1 + c_2\mathrm{e}^{-x} \tag{1}
$$

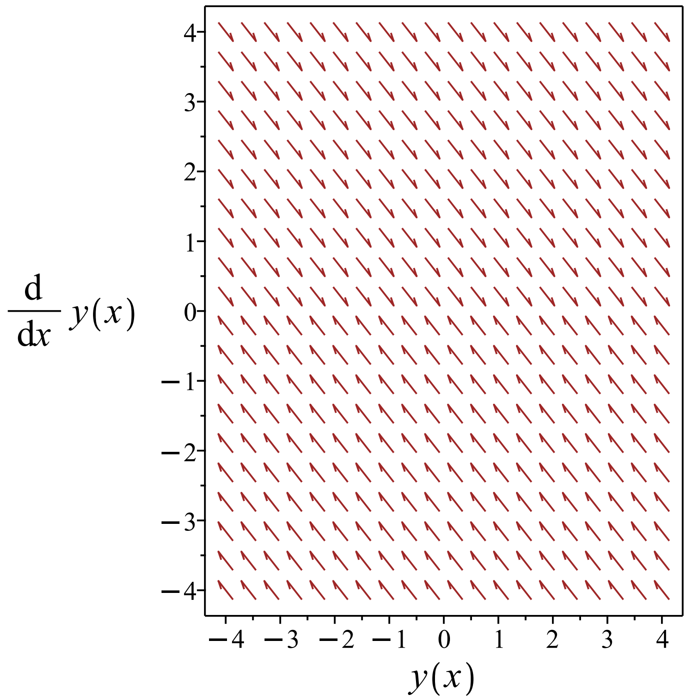

Figure 65: Slope field plot

Verification of solutions

$$y = c_1 + x - 1 + c_2 e^{-x}$$

Verified OK.

1.31.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 44: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx}\\ &= z_1 e^{-\frac{x}{2}}\\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx\\ &= y_1(\mathrm{e}^{x})\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\mathrm{e}^{x}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, \mathrm{e}^{-x}\}
$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1 x
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2\right) + (x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + x \tag{1}$$

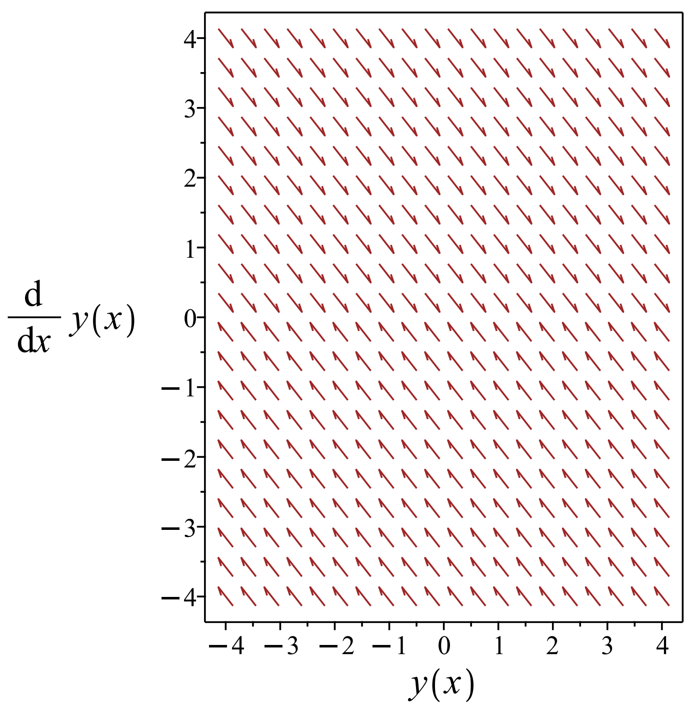

Figure 66: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 + x$$

Verified OK.

1.31.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x) y'' + q(x) y' + r(x) y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= 1 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x) y' + (q(x) - p'(x)) y)' = s(x)$$

Integrating gives

$$p(x) y' + (q(x) - p'(x)) y = \int s(x) \, dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int 1 \, dx$$

We now have a first order ode to solve which is

$$y' + y = x + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x + c_1 \end{aligned}$$

Hence the ode is

$$y' + y = x + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)(x + c_1) \\ \frac{d}{dx}(e^x y) &= (e^x)(x + c_1) \\ d(e^x y) &= ((x + c_1) e^x) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^x y &= \int (x + c_1) e^x dx \\ e^x y &= (c_1 + x - 1) e^x + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}(c_1 + x - 1) e^x + c_2 e^{-x}$$

which simplifies to

$$y = c_1 + x - 1 + c_2 e^{-x}$$

Summary
The solution(s) found are the following

$$y = c_1 + x - 1 + c_2 e^{-x} \tag{1}$$

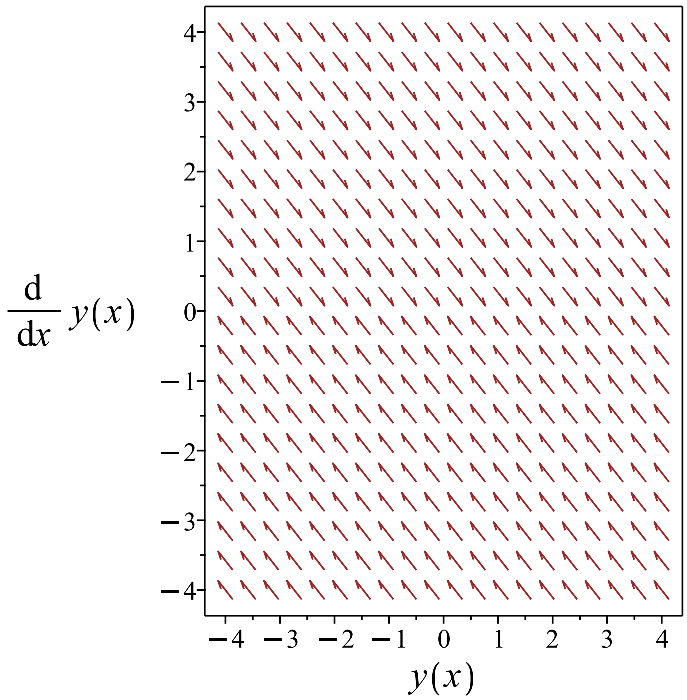

Figure 67: Slope field plot

Verification of solutions

$$y = c_1 + x - 1 + c_2 e^{-x}$$

Verified OK.

1.31.7 Maple step by step solution

Let's solve

$y'' + y' = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 0)$
- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 1\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-x}\left(\int \mathrm{e}^{x} dx\right) + \int 1 dx$
  - Compute integrals

    $y_p(x) = x - 1$
- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + x - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=1,y(x), singsol=all)
```

$$y(x) = -\mathrm{e}^{-x}c_1 + x + c_2$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 18

```
DSolve[y''[x]+y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - c_1 e^{-x} + c_2$$

1.32 problem 32

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Internal problem ID [7421]
Internal file name [`OUTPUT/6388_Sunday_June_05_2022_04_42_44_PM_8020723/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y' = x}$$

1.32.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda \, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(0)} \\
&= -\frac{1}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= -1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(0)x} + c_2 e^{(-1)x}
\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2xA_2 + A_1 + 2A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2}x^2 - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 e^{-x}) + \left(\frac{1}{2}x^2 - x\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} + \frac{x^2}{2} - x \tag{1}$$

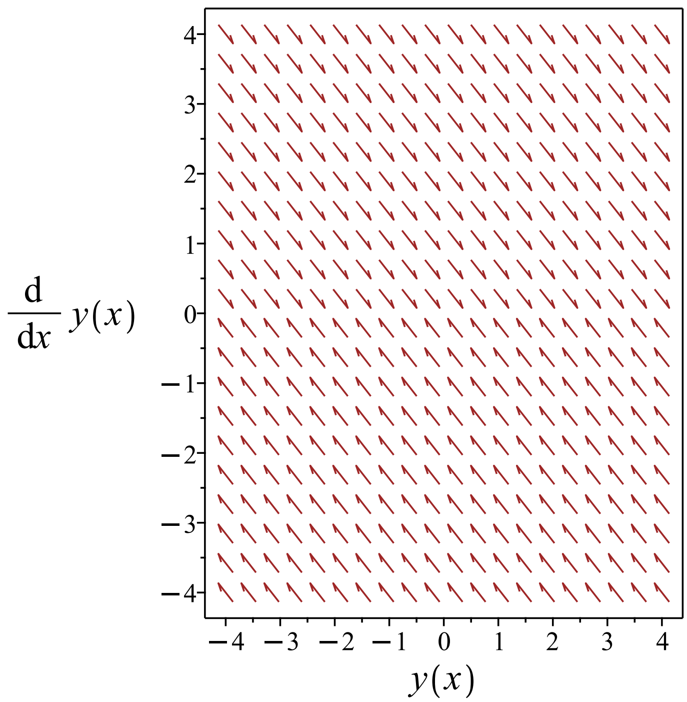

Figure 68: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} + \frac{x^2}{2} - x$$

Verified OK.

1.32.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (y'' + y')\, dx &= \int x dx \\
y' + y &= \frac{x^2}{2} + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= \frac{x^2}{2} + c_1
\end{aligned}
$$

Hence the ode is

$$
y' + y = \frac{x^2}{2} + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1 dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^2}{2} + c_1\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} y) &= (\mathrm{e}^{x})\left(\frac{x^2}{2} + c_1\right) \\
\mathrm{d}(\mathrm{e}^{x} y) &= \left(\frac{(x^2 + 2c_1)\,\mathrm{e}^{x}}{2}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{x} y &= \int \frac{(x^2 + 2c_1)\,\mathrm{e}^{x}}{2} \,\mathrm{d}x \\
\mathrm{e}^{x} y &= \frac{(x^2 + 2c_1 - 2x + 2)\,\mathrm{e}^{x}}{2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = \frac{e^{-x}(x^2 + 2c_1 - 2x + 2)\,e^x}{2} + c_2 e^{-x}$$

which simplifies to

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 e^{-x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 e^{-x} \tag{1}$$

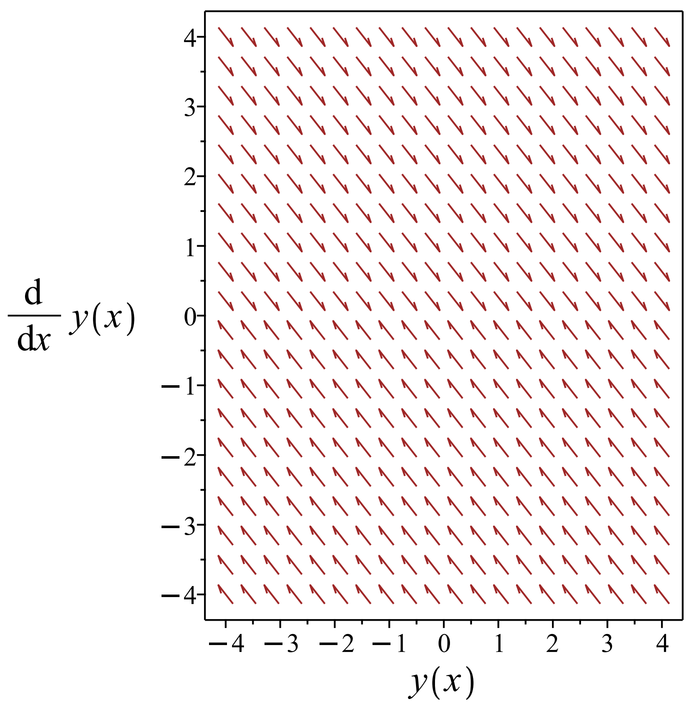

Figure 69: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 e^{-x}$$

Verified OK.

1.32.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - x = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x \end{aligned}$$

Hence the ode is

$$p'(x) + p(x) = x$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)(x) \\ \frac{d}{dx}(e^{x} p) &= (e^{x})(x) \\ d(e^{x} p) &= (x\, e^{x})\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{x} p &= \int x\, e^{x}\, dx \\ e^{x} p &= (x-1)\, e^{x} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^x$ results in

$$p(x) = \mathrm{e}^{-x}(x-1)\,\mathrm{e}^x + c_1\mathrm{e}^{-x}$$

which simplifies to

$$p(x) = x - 1 + c_1\mathrm{e}^{-x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x - 1 + c_1\mathrm{e}^{-x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int x - 1 + c_1\mathrm{e}^{-x}\,\mathrm{d}x \\ &= -x + \frac{x^2}{2} - c_1\mathrm{e}^{-x} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -x + \frac{x^2}{2} - c_1\mathrm{e}^{-x} + c_2 \tag{1}$$

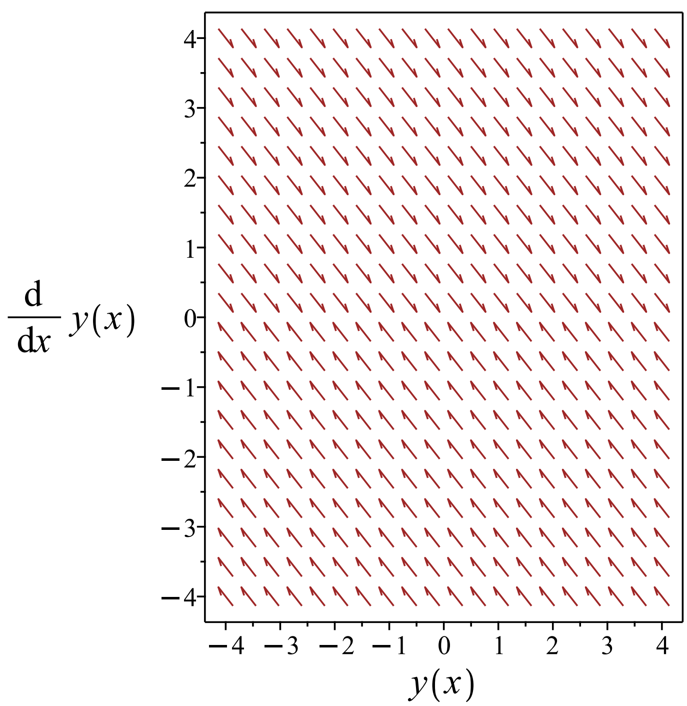

Figure 70: Slope field plot

Verification of solutions

$$
y = -x + \frac{x^2}{2} - c_1 \mathrm{e}^{-x} + c_2
$$

Verified OK.

1.32.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
y'' + y' = x
$$

Integrating both sides of the ODE w.r.t x gives

$$
\int (y'' + y')\, dx = \int x dx
$$

$$
y' + y = \frac{x^2}{2} + c_1
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= \frac{x^2}{2} + c_1
\end{aligned}
$$

Hence the ode is

$$
y' + y = \frac{x^2}{2} + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1 dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu) \left(\frac{x^2}{2} + c_1\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} y) &= (\mathrm{e}^{x}) \left(\frac{x^2}{2} + c_1\right) \\
\mathrm{d}(\mathrm{e}^{x} y) &= \left(\frac{(x^2 + 2c_1)\, \mathrm{e}^{x}}{2}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{x} y &= \int \frac{\left(x^{2}+2 c_{1}\right) e^{x}}{2}\, dx \\
e^{x} y &= \frac{\left(x^{2}+2 c_{1}-2 x+2\right) e^{x}}{2}+c_{2}
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$
y = \frac{e^{-x}\left(x^{2}+2 c_{1}-2 x+2\right) e^{x}}{2}+c_{2} e^{-x}
$$

which simplifies to

$$
y = \frac{x^{2}}{2}+c_{1}-x+1+c_{2} e^{-x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x^{2}}{2}+c_{1}-x+1+c_{2} e^{-x} \tag{1}
$$

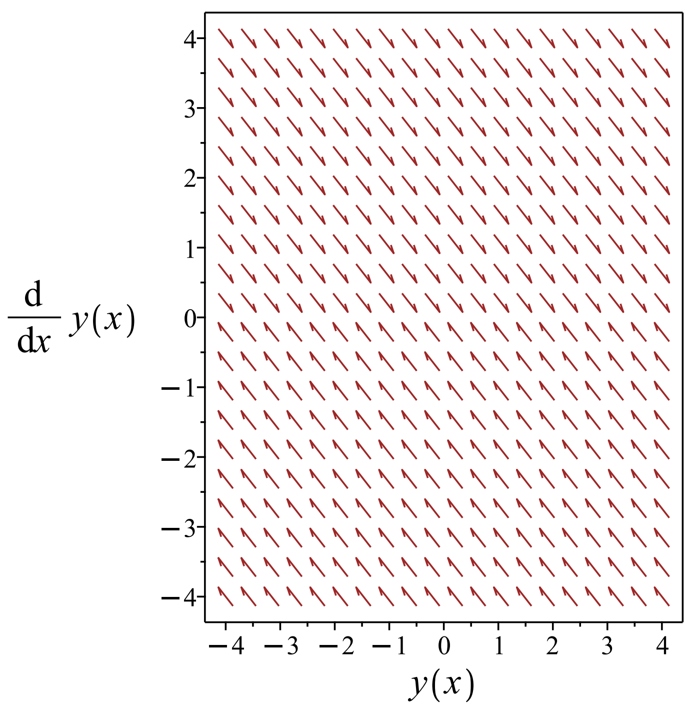

Figure 71: Slope field plot

Verification of solutions

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 \mathrm{e}^{-x}$$

Verified OK.

1.32.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 46: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(\mathrm{e}^{x})\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2xA_2 + A_1 + 2A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2}x^2 - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2) + \left(\frac{1}{2}x^2 - x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + \frac{x^2}{2} - x \tag{1}$$

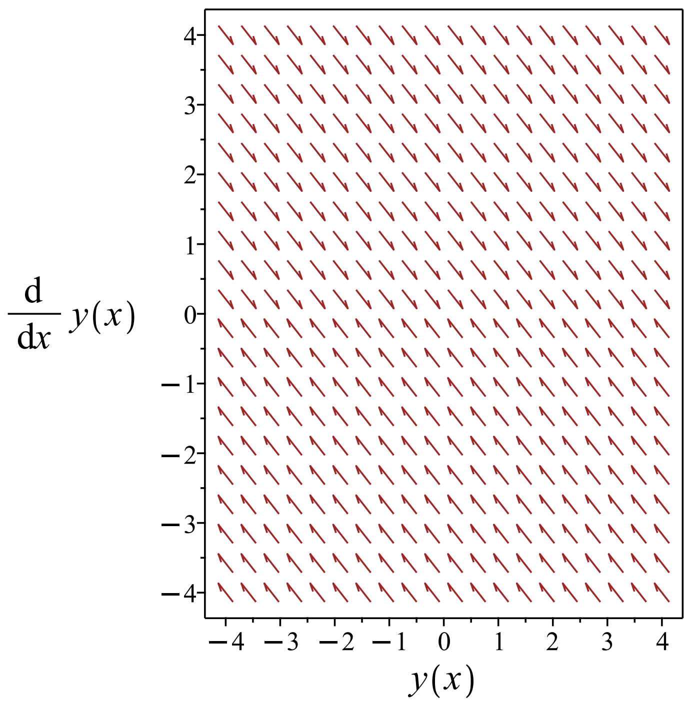

Figure 72: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 + \frac{x^2}{2} - x$$

Verified OK.

1.32.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int x\,dx$$

We now have a first order ode to solve which is

$$y' + y = \frac{x^2}{2} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \frac{x^2}{2} + c_1 \end{aligned}$$

Hence the ode is

$$y' + y = \frac{x^2}{2} + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 1 dx} \\ &= \mathrm{e}^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^2}{2} + c_1\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x y) &= (\mathrm{e}^x)\left(\frac{x^2}{2} + c_1\right) \\ \mathrm{d}(\mathrm{e}^x y) &= \left(\frac{(x^2 + 2c_1)\,\mathrm{e}^x}{2}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^x y &= \int \frac{(x^2 + 2c_1)\,\mathrm{e}^x}{2}\,\mathrm{d}x \\ \mathrm{e}^x y &= \frac{(x^2 + 2c_1 - 2x + 2)\,\mathrm{e}^x}{2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^x$ results in

$$y = \frac{\mathrm{e}^{-x}(x^2 + 2c_1 - 2x + 2)\,\mathrm{e}^x}{2} + c_2\mathrm{e}^{-x}$$

which simplifies to

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2\mathrm{e}^{-x}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 \mathrm{e}^{-x} \tag{1}$$

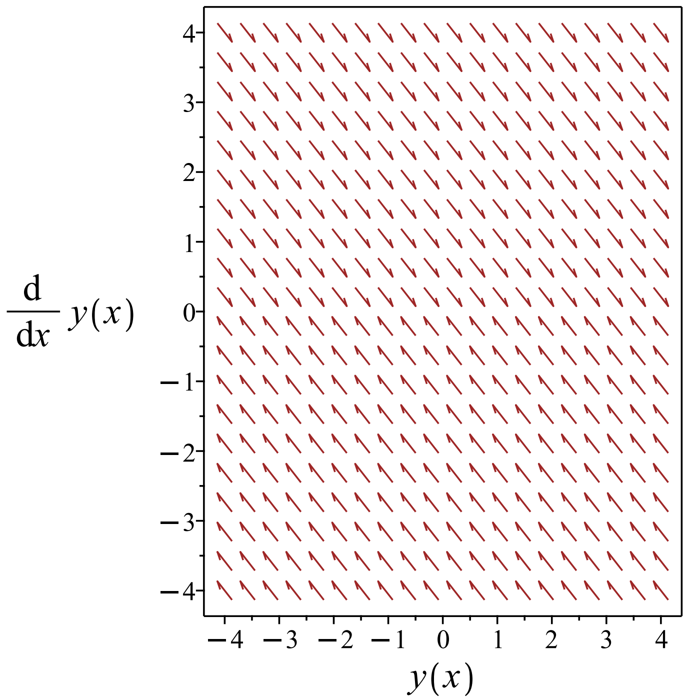

Figure 73: Slope field plot

Verification of solutions

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 \mathrm{e}^{-x}$$

Verified OK.

1.32.7 Maple step by step solution

Let's solve

$y'' + y' = x$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of homogeneous ODE

$r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-x} \left(\int x \,\mathrm{e}^{x} dx\right) + \int x dx$

  - Compute integrals

    $y_p(x) = 1 - x + \frac{1}{2} x^2$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + 1 - x + \frac{x^2}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=x,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} - \mathrm{e}^{-x} c_1 - x + c_2$$

✓ Solution by Mathematica
Time used: 0.036 (sec). Leaf size: 27

```
DSolve[y''[x]+y'[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2} - x - c_1 e^{-x} + c_2$$

1.33 problem 33

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Internal problem ID [7422]
Internal file name [`OUTPUT/6389_Sunday_June_05_2022_04_42_46_PM_8378803/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y' = 1 + x}$$

1.33.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = 1 + x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda \, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(0)} \\
&= -\frac{1}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= -1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(0)x} + c_2 e^{(-1)x}
\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2xA_2 + A_1 + 2A_2 = 1 + x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 + c_2 e^{-x}) + \left(\frac{x^2}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + c_2 e^{-x} + \frac{x^2}{2} \tag{1}
$$

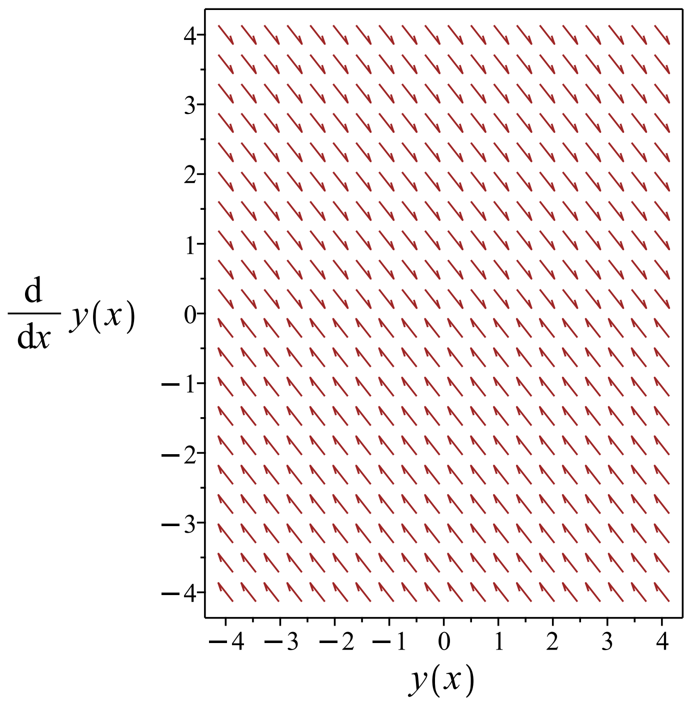

Figure 74: Slope field plot

Verification of solutions

$$
y = c_1 + c_2 e^{-x} + \frac{x^2}{2}
$$

Verified OK.

1.33.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int (1 + x)\, dx$$
$$y' + y = x + \frac{1}{2}x^2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = x + \frac{1}{2}x^2 + c_1$$

Hence the ode is

$$y' + y = x + \frac{1}{2}x^2 + c_1$$

The integrating factor μ is

$$\mu = e^{\int 1 dx}$$
$$= e^x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x + \frac{1}{2}x^2 + c_1\right)$$
$$\frac{d}{dx}(e^x y) = (e^x)\left(x + \frac{1}{2}x^2 + c_1\right)$$
$$d(e^x y) = \left(\frac{(x^2 + 2c_1 + 2x)\, e^x}{2}\right) dx$$

Integrating gives

$$e^x y = \int \frac{(x^2 + 2c_1 + 2x)\, e^x}{2}\, dx$$
$$e^x y = \frac{(x^2 + 2c_1)\, e^x}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = \frac{e^{-x}(x^2 + 2c_1)\, e^x}{2} + c_2 e^{-x}$$

which simplifies to

$$y = c_1 + c_2 e^{-x} + \frac{x^2}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} + \frac{x^2}{2} \tag{1}$$

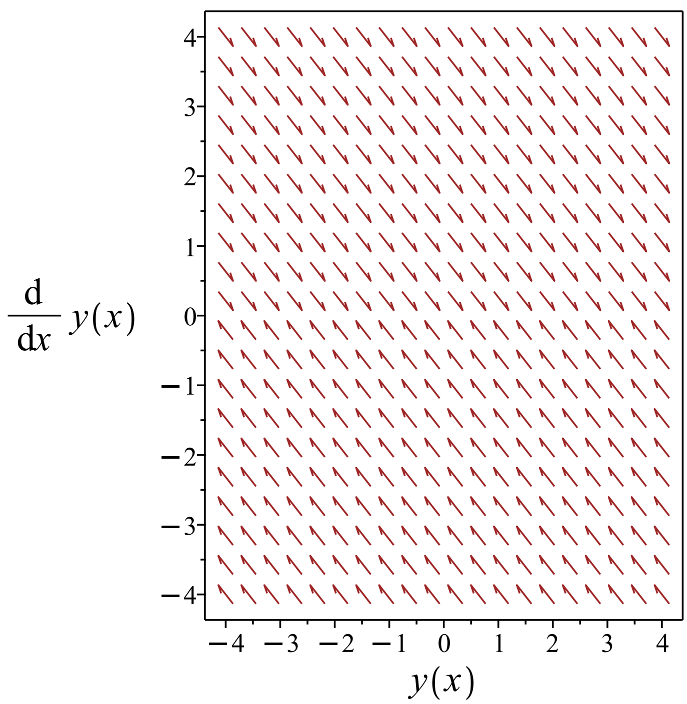

Figure 75: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 + c_2 e^{-x} + \frac{x^2}{2}$$

Verified OK.

1.33.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - 1 - x = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= 1 \\
q(x) &= 1 + x
\end{aligned}$$

Hence the ode is

$$p'(x) + p(x) = 1 + x$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int 1dx} \\
&= \mathrm{e}^{x}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\,(1 + x) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} p) &= (\mathrm{e}^{x})\,(1 + x) \\
\mathrm{d}(\mathrm{e}^{x} p) &= ((1 + x)\,\mathrm{e}^{x})\;\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\mathrm{e}^{x} p &= \int (1 + x)\,\mathrm{e}^{x}\,\mathrm{d}x \\
\mathrm{e}^{x} p &= x\,\mathrm{e}^{x} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$p(x) = e^{-x} x\, e^{x} + c_1 e^{-x}$$

which simplifies to

$$p(x) = x + c_1 e^{-x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x + c_1 e^{-x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int x + c_1 e^{-x}\, dx \\ &= \frac{x^2}{2} - c_1 e^{-x} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{2} - c_1 e^{-x} + c_2 \tag{1}$$

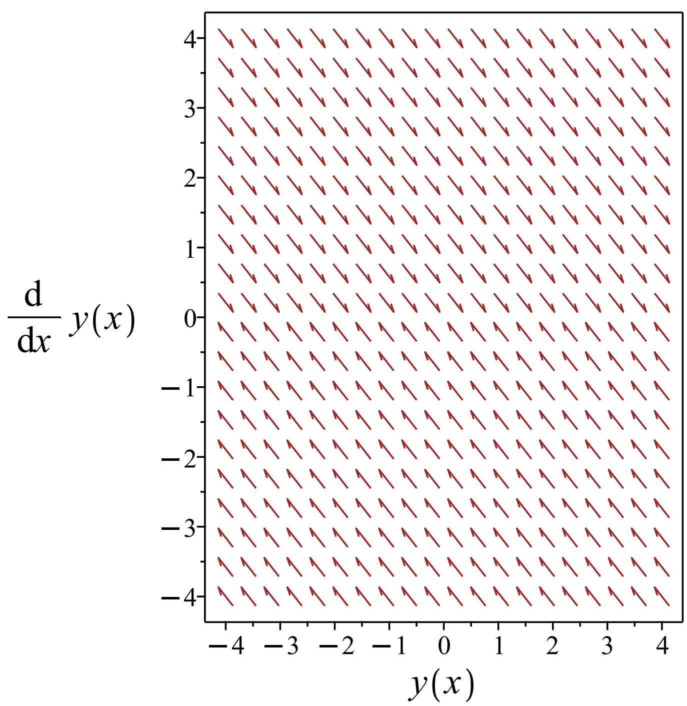

Figure 76: Slope field plot

Verification of solutions

$$y = \frac{x^2}{2} - c_1 \mathrm{e}^{-x} + c_2$$

Verified OK.

1.33.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = 1 + x$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\,dx = \int (1 + x)\,dx$$
$$y' + y = x + \frac{1}{2}x^2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = x + \frac{1}{2}x^2 + c_1$$

Hence the ode is

$$y' + y = x + \frac{1}{2}x^2 + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 1 dx} \\ &= \mathrm{e}^x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(x + \frac{1}{2}x^2 + c_1\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x y) &= (\mathrm{e}^x)\left(x + \frac{1}{2}x^2 + c_1\right) \\ \mathrm{d}(\mathrm{e}^x y) &= \left(\frac{(x^2 + 2c_1 + 2x)\,\mathrm{e}^x}{2}\right) \mathrm{d}x\end{aligned}$$

Integrating gives

$$e^x y = \int \frac{(x^2 + 2c_1 + 2x)\, e^x}{2}\, dx$$
$$e^x y = \frac{(x^2 + 2c_1)\, e^x}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = \frac{e^{-x}(x^2 + 2c_1)\, e^x}{2} + c_2 e^{-x}$$

which simplifies to

$$y = c_1 + c_2 e^{-x} + \frac{x^2}{2}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} + \frac{x^2}{2} \tag{1}$$

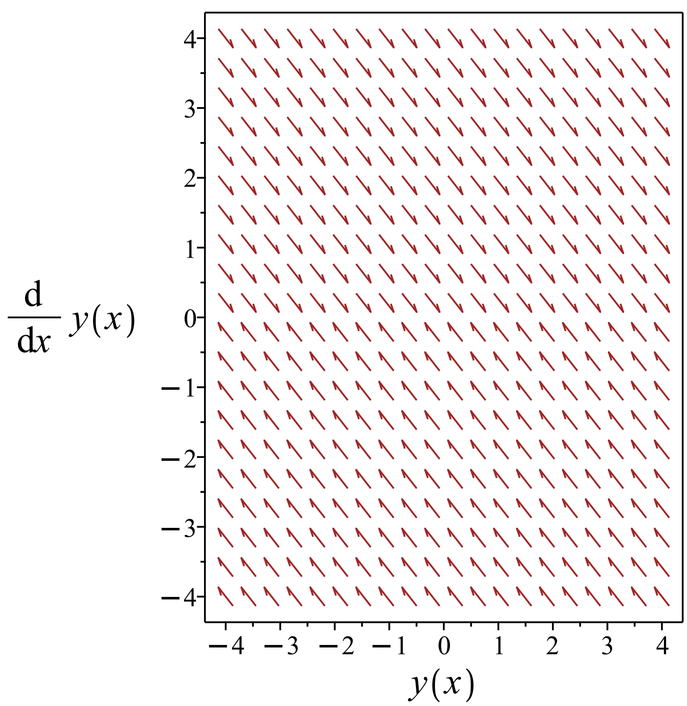

Figure 77: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} + \frac{x^2}{2}$$

Verified OK.

1.33.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 48: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(\mathrm{e}^{x})\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2xA_2 + A_1 + 2A_2 = 1 + x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2) + \left(\frac{x^2}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + \frac{x^2}{2} \tag{1}$$

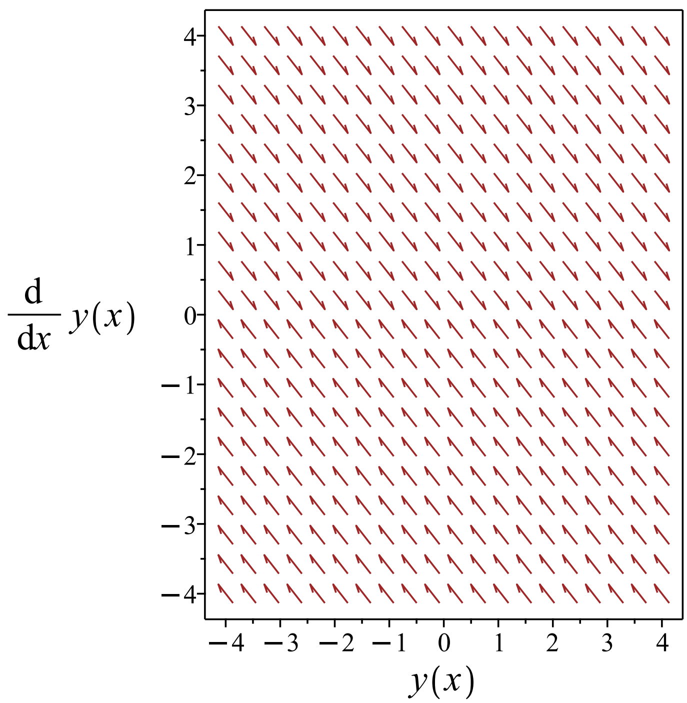

Figure 78: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 + \frac{x^2}{2}$$

Verified OK.

1.33.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= 1 + x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int 1 + x\, dx$$

We now have a first order ode to solve which is

$$y' + y = x + \frac{1}{2}x^2 + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x + \frac{1}{2}x^2 + c_1 \end{aligned}$$

Hence the ode is

$$y' + y = x + \frac{1}{2}x^2 + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 1dx} \\ &= \mathrm{e}^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(x + \frac{1}{2}x^2 + c_1\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x y) &= (\mathrm{e}^x)\left(x + \frac{1}{2}x^2 + c_1\right) \\ \mathrm{d}(\mathrm{e}^x y) &= \left(\frac{(x^2 + 2c_1 + 2x)\,\mathrm{e}^x}{2}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^x y &= \int \frac{(x^2 + 2c_1 + 2x)\,\mathrm{e}^x}{2}\,\mathrm{d}x \\ \mathrm{e}^x y &= \frac{(x^2 + 2c_1)\,\mathrm{e}^x}{2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^x$ results in

$$y = \frac{\mathrm{e}^{-x}(x^2 + 2c_1)\,\mathrm{e}^x}{2} + c_2\mathrm{e}^{-x}$$

which simplifies to

$$y = c_1 + c_2\mathrm{e}^{-x} + \frac{x^2}{2}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} + \frac{x^2}{2} \tag{1}$$

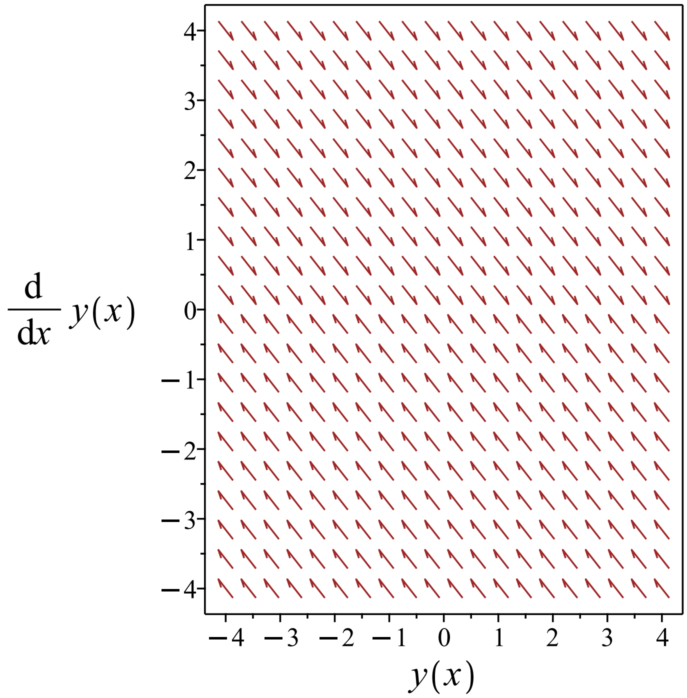

Figure 79: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} + \frac{x^2}{2}$$

Verified OK.

1.33.7 Maple step by step solution

Let's solve

$y'' + y' = 1 + x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 1 + x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-x}\left(\int (1+x)\,\mathrm{e}^{x} dx\right) + \int (1+x)\,dx$

  - Compute integrals

    $y_p(x) = \frac{x^2}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + \frac{x^2}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)+_a+1, _b(_a)`   *** Sublevel 2
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=1+x,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} - e^{-x}c_1 + c_2$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 24

```
DSolve[y''[x]+y'[x]==1+x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2} - c_1e^{-x} + c_2$$

1.34 problem 34

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Internal problem ID [7423]
Internal file name [`OUTPUT/6390_Sunday_June_05_2022_04_42_48_PM_69824905/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y' = x^2 + x + 1}$$

1.34.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = x^2 + x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda \,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(0)} \\
&= -\frac{1}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= -1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(0)x} + c_2 e^{(-1)x}
\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3 x^3 + A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3x^2 A_3 + 2x A_2 + 6x A_3 + A_1 + 2A_2 = x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 2, A_2 = -\frac{1}{2}, A_3 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 e^{-x}) + \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3} - \frac{x^2}{2} + 2x \tag{1}$$

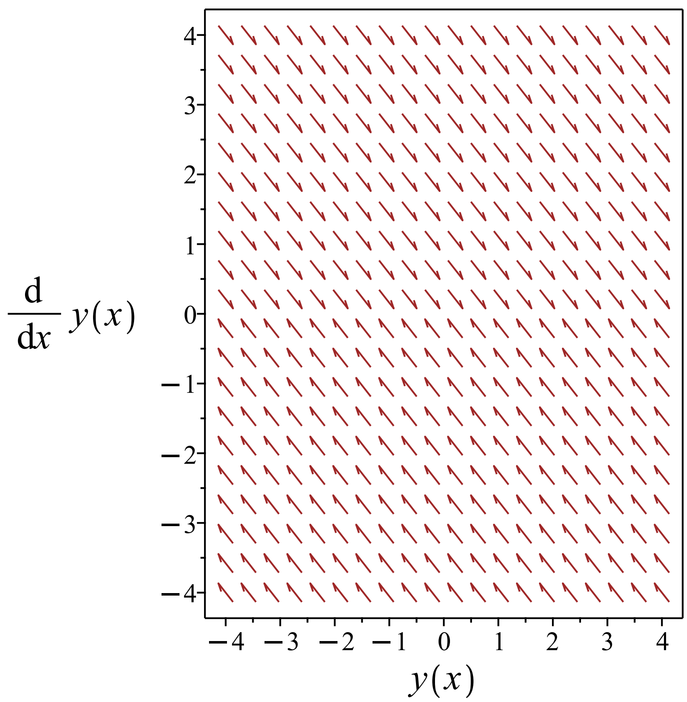

Figure 80: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3} - \frac{x^2}{2} + 2x$$

Verified OK.

1.34.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int (x^2 + x + 1)\, dx$$
$$y' + y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

Hence the ode is

$$y' + y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

The integrating factor μ is

$$\mu = e^{\int 1 dx}$$
$$= e^x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right)$$
$$\frac{d}{dx}(e^x y) = (e^x)\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right)$$
$$d(e^x y) = \left(\frac{(2x^3 + 3x^2 + 6c_1 + 6x)\, e^x}{6}\right) dx$$

Integrating gives

$$e^x y = \int \frac{(2x^3 + 3x^2 + 6c_1 + 6x)\, e^x}{6}\, dx$$
$$e^x y = \frac{(2x^3 - 3x^2 + 6c_1 + 12x - 12)\, e^x}{6} + c_2$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = \frac{e^{-x}(2x^3 - 3x^2 + 6c_1 + 12x - 12)\, e^x}{6} + c_2 e^{-x}$$

which simplifies to

$$y = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + 2x - 2 + c_2 e^{-x}$$

Summary
The solution(s) found are the following

$$y = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + 2x - 2 + c_2 e^{-x} \tag{1}$$

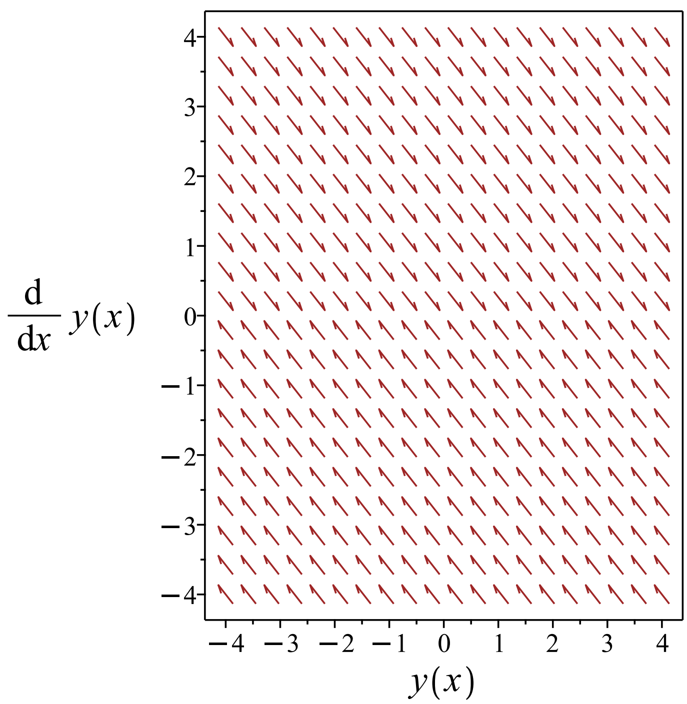

Figure 81: Slope field plot

Verification of solutions

$$y = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + 2x - 2 + c_2 e^{-x}$$

Verified OK.

1.34.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - x^2 - x - 1 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x^2 + x + 1 \end{aligned}$$

Hence the ode is

$$p'(x) + p(x) = x^2 + x + 1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 1dx} \\ &= \mathrm{e}^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(x^2 + x + 1\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x p) &= (\mathrm{e}^x)\left(x^2 + x + 1\right) \\ \mathrm{d}(\mathrm{e}^x p) &= \left(\left(x^2 + x + 1\right)\mathrm{e}^x\right)\, \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^x p &= \int \left(x^2 + x + 1\right)\mathrm{e}^x \,\mathrm{d}x \\ \mathrm{e}^x p &= \left(x^2 - x + 2\right)\mathrm{e}^x + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$p(x) = e^{-x}\left(x^2 - x + 2\right) e^x + c_1 e^{-x}$$

which simplifies to

$$p(x) = x^2 - x + 2 + c_1 e^{-x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x^2 - x + 2 + c_1 e^{-x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int x^2 - x + 2 + c_1 e^{-x} \, dx \\ &= 2x + \frac{x^3}{3} - c_1 e^{-x} - \frac{x^2}{2} + c_2 \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = 2x + \frac{x^3}{3} - c_1 e^{-x} - \frac{x^2}{2} + c_2 \tag{1}$$

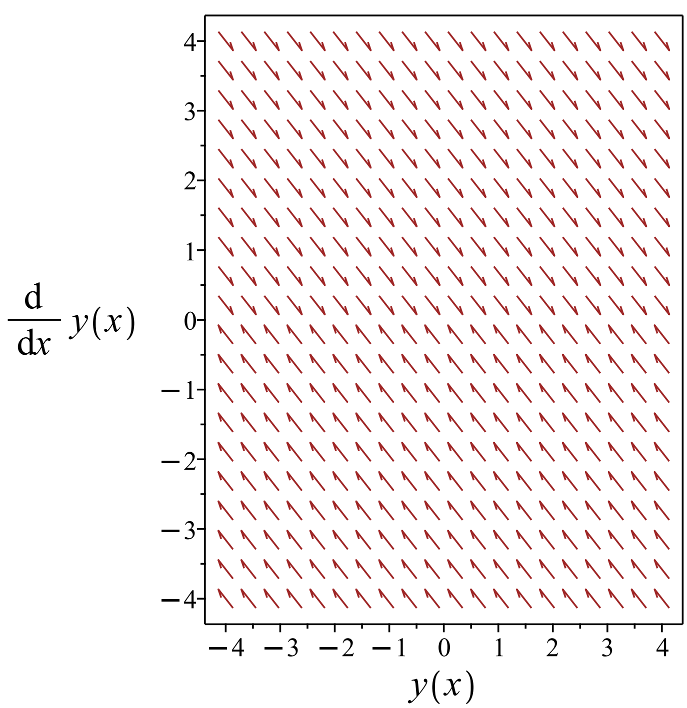

Figure 82: Slope field plot

Verification of solutions

$$y = 2x + \frac{x^3}{3} - c_1 e^{-x} - \frac{x^2}{2} + c_2$$

Verified OK.

1.34.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = x^2 + x + 1$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int (x^2 + x + 1)\, dx$$
$$y' + y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

Hence the ode is

$$y' + y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

The integrating factor μ is

$$\mu = e^{\int 1 dx}$$
$$= e^x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right)$$
$$\frac{d}{dx}(e^x y) = (e^x)\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right)$$
$$d(e^x y) = \left(\frac{(2x^3 + 3x^2 + 6c_1 + 6x)\, e^x}{6}\right) dx$$

Integrating gives

$$e^x y = \int \frac{(2x^3 + 3x^2 + 6c_1 + 6x)\,e^x}{6}\,dx$$
$$e^x y = \frac{(2x^3 - 3x^2 + 6c_1 + 12x - 12)\,e^x}{6} + c_2$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = \frac{e^{-x}(2x^3 - 3x^2 + 6c_1 + 12x - 12)\,e^x}{6} + c_2 e^{-x}$$

which simplifies to

$$y = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + 2x - 2 + c_2 e^{-x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + 2x - 2 + c_2 e^{-x} \tag{1}$$

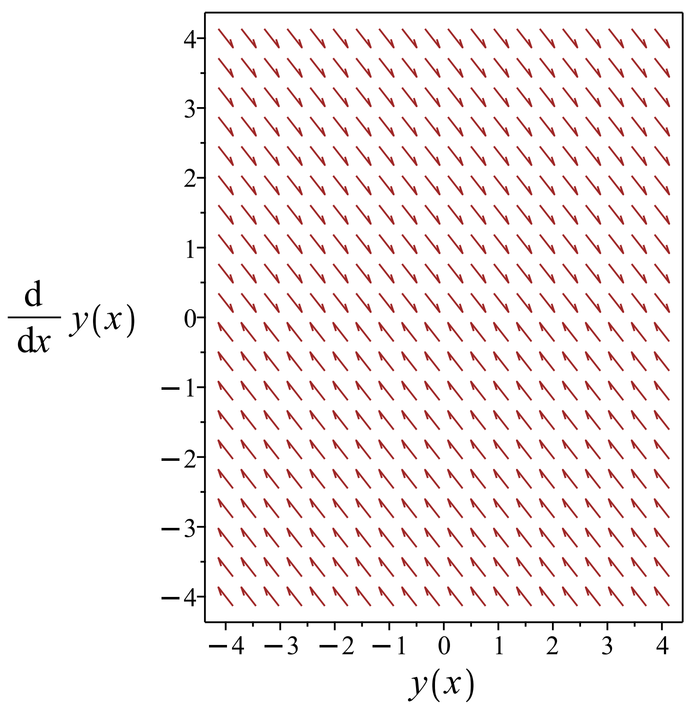

Figure 83: Slope field plot

Verification of solutions

$$y = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + 2x - 2 + c_2 e^{-x}$$

Verified OK.

1.34.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 50: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(\mathrm{e}^{x})\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3 x^3 + A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3x^2 A_3 + 2x A_2 + 6x A_3 + A_1 + 2A_2 = x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 2, A_2 = -\frac{1}{2}, A_3 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2\right) + \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 + \frac{x^3}{3} - \frac{x^2}{2} + 2x \tag{1}$$

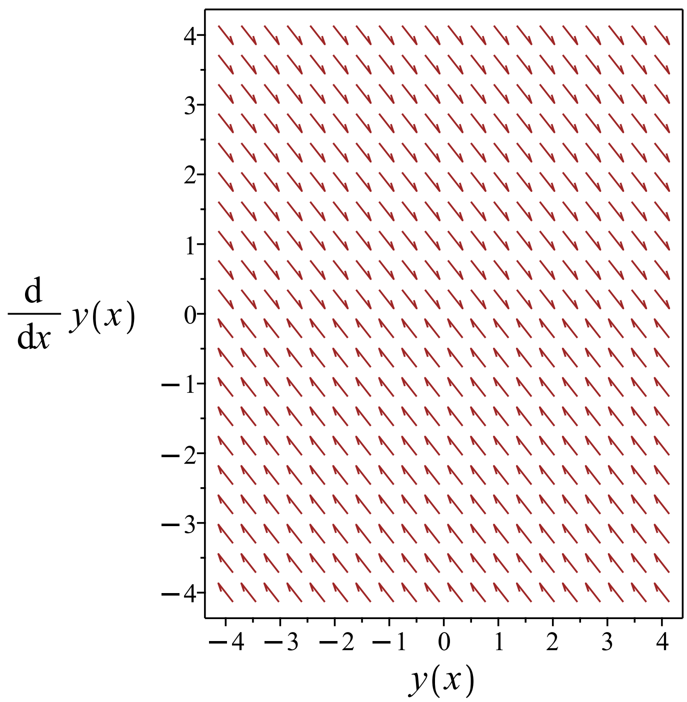

Figure 84: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \mathrm{e}^{-x} + c_2 + \frac{x^3}{3} - \frac{x^2}{2} + 2x$$

Verified OK.

1.34.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= x^2 + x + 1 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int x^2 + x + 1\,dx$$

We now have a first order ode to solve which is

$$y' + y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1 \end{aligned}$$

Hence the ode is

$$y' + y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 1 dx} \\ &= \mathrm{e}^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x y) &= (\mathrm{e}^x)\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) \\ \mathrm{d}(\mathrm{e}^x y) &= \left(\frac{(2x^3 + 3x^2 + 6c_1 + 6x)\,\mathrm{e}^x}{6}\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^x y &= \int \frac{(2x^3 + 3x^2 + 6c_1 + 6x)\,\mathrm{e}^x}{6} \,\mathrm{d}x \\ \mathrm{e}^x y &= \frac{(2x^3 - 3x^2 + 6c_1 + 12x - 12)\,\mathrm{e}^x}{6} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^x$ results in

$$y = \frac{\mathrm{e}^{-x}(2x^3 - 3x^2 + 6c_1 + 12x - 12)\,\mathrm{e}^x}{6} + c_2\mathrm{e}^{-x}$$

which simplifies to

$$y = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + 2x - 2 + c_2\mathrm{e}^{-x}$$

Summary
The solution(s) found are the following

$$y = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + 2x - 2 + c_2 e^{-x} \tag{1}$$

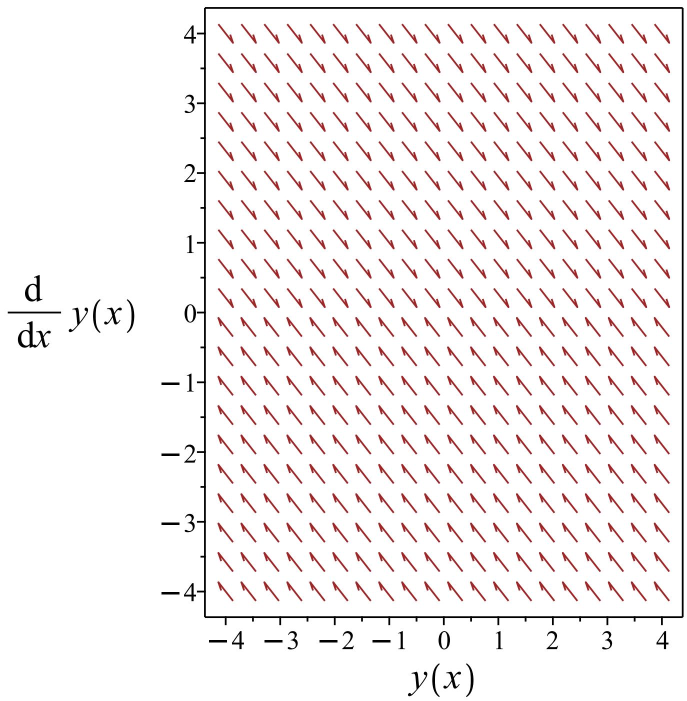

Figure 85: Slope field plot

Verification of solutions

$$y = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + 2x - 2 + c_2 e^{-x}$$

Verified OK.

1.34.7 Maple step by step solution

Let's solve

$y'' + y' = x^2 + x + 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x^2 + x + 1\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-x}\left(\int (x^2 + x + 1)\,\mathrm{e}^{x}dx\right) + \int (x^2 + x + 1)\,dx$

  ◦ Compute integrals

    $y_p(x) = -\frac{1}{2}x^2 + 2x - 2 + \frac{1}{3}x^3$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 - \frac{x^2}{2} + 2x - 2 + \frac{x^3}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a^2-_b(_a)+_a+1, _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=1+x+x^2,y(x), singsol=all)
```

$$y(x) = \frac{x^3}{3} - e^{-x}c_1 - \frac{x^2}{2} + 2x + c_2$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 34

```
DSolve[y''[x]+y'[x]==1+x+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{3} - \frac{x^2}{2} + 2x - c_1e^{-x} + c_2$$

1.35 problem 35

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Internal problem ID [7424]
Internal file name [`OUTPUT/6391_Sunday_June_05_2022_04_42_50_PM_79266519/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y' = x^3 + x^2 + x + 1}$$

1.35.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = x^3 + x^2 + x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.

y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(0)} \\ &= -\frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(0)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3, x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_4 x^4 + A_3 x^3 + A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4x^3 A_4 + 3x^2 A_3 + 12x^2 A_4 + 2x A_2 + 6x A_3 + A_1 + 2A_2 = x^3 + x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -4, A_2 = \frac{5}{2}, A_3 = -\frac{2}{3}, A_4 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 - 4x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 e^{-x}) + \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 - 4x\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} + \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} - 4x \tag{1}$$

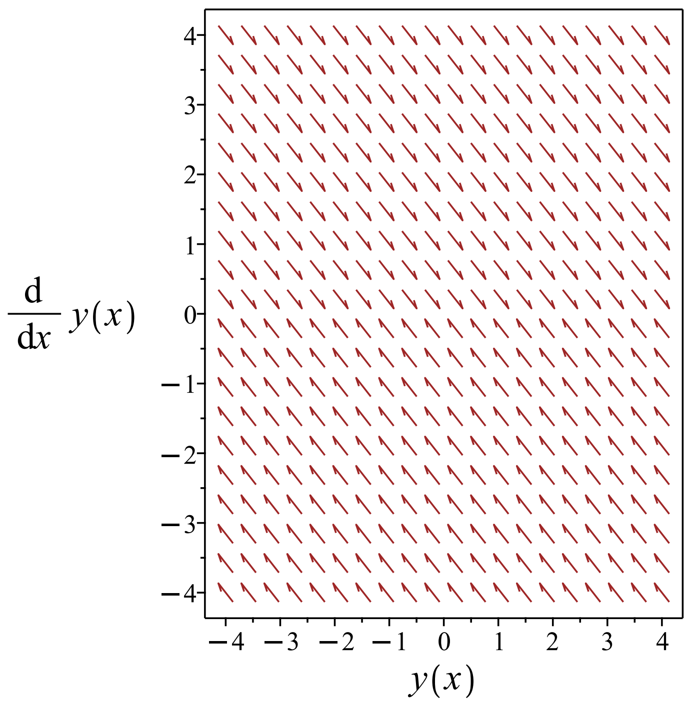

Figure 86: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} + \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} - 4x$$

Verified OK.

1.35.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int (x^3 + x^2 + x + 1)\, dx$$
$$y' + y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

Hence the ode is

$$y' + y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int 1 dx}$$
$$= \mathrm{e}^x$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x y) = (\mathrm{e}^x)\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right)$$
$$\mathrm{d}(\mathrm{e}^x y) = \left(\frac{(3x^4 + 4x^3 + 6x^2 + 12c_1 + 12x)\,\mathrm{e}^x}{12}\right) \mathrm{d}x$$

Integrating gives

$$\mathrm{e}^x y = \int \frac{(3x^4 + 4x^3 + 6x^2 + 12c_1 + 12x)\,\mathrm{e}^x}{12}\, \mathrm{d}x$$
$$\mathrm{e}^x y = \frac{(3x^4 - 8x^3 + 30x^2 + 12c_1 - 48x + 48)\,\mathrm{e}^x}{12} + c_2$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = \frac{e^{-x}(3x^4 - 8x^3 + 30x^2 + 12c_1 - 48x + 48)\,e^x}{12} + c_2 e^{-x}$$

which simplifies to

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + c_1 - 4x + 4 + c_2 e^{-x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + c_1 - 4x + 4 + c_2 e^{-x} \tag{1}$$

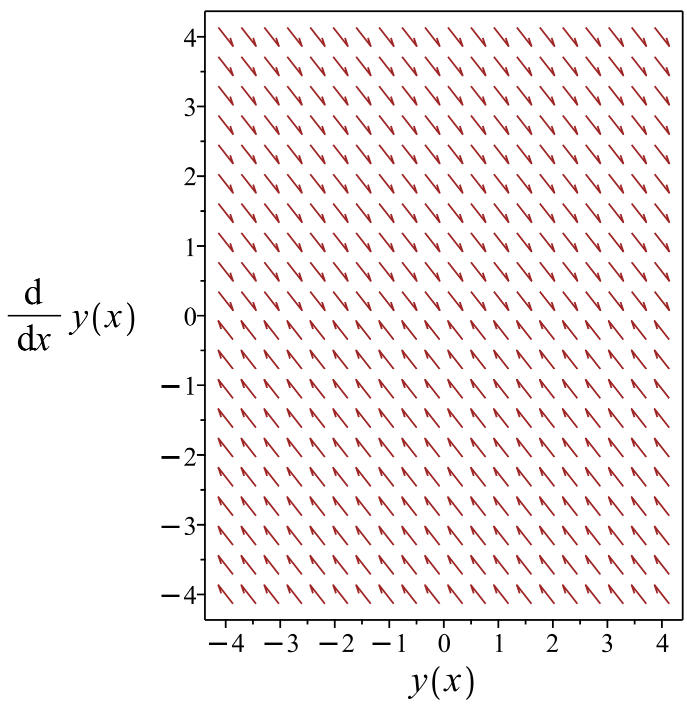

Figure 87: Slope field plot

Verification of solutions

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + c_1 - 4x + 4 + c_2 e^{-x}$$

Verified OK.

1.35.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - x^3 - x^2 - x - 1 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x^3 + x^2 + x + 1 \end{aligned}$$

Hence the ode is

$$p'(x) + p(x) = x^3 + x^2 + x + 1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 1dx} \\ &= \mathrm{e}^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(x^3 + x^2 + x + 1\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x p) &= (\mathrm{e}^x)\left(x^3 + x^2 + x + 1\right) \\ \mathrm{d}(\mathrm{e}^x p) &= \left((1 + x)\left(x^2 + 1\right)\mathrm{e}^x\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^x p &= \int (1 + x)\left(x^2 + 1\right)\mathrm{e}^x\,\mathrm{d}x \\ \mathrm{e}^x p &= \left(x^3 - 2x^2 + 5x - 4\right)\mathrm{e}^x + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$p(x) = \mathrm{e}^{-x}\left(x^{3}-2x^{2}+5x-4\right)\mathrm{e}^{x}+c_1\mathrm{e}^{-x}$$

which simplifies to

$$p(x) = x^{3}-2x^{2}+5x-4+c_1\mathrm{e}^{-x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x^{3}-2x^{2}+5x-4+c_1\mathrm{e}^{-x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int x^{3}-2x^{2}+5x-4+c_1\mathrm{e}^{-x}\,\mathrm{d}x \\ &= -4x+\frac{x^{4}}{4}-c_1\mathrm{e}^{-x}+\frac{5x^{2}}{2}-\frac{2x^{3}}{3}+c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -4x+\frac{x^{4}}{4}-c_1\mathrm{e}^{-x}+\frac{5x^{2}}{2}-\frac{2x^{3}}{3}+c_2 \tag{1}$$

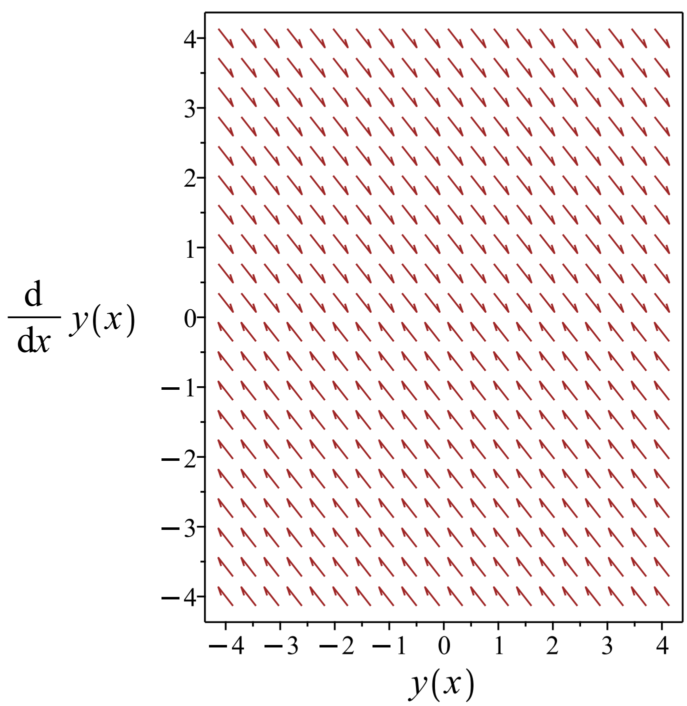

Figure 88: Slope field plot

Verification of solutions

$$y = -4x + \frac{x^4}{4} - c_1 \mathrm{e}^{-x} + \frac{5x^2}{2} - \frac{2x^3}{3} + c_2$$

Verified OK.

1.35.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = x^3 + x^2 + x + 1$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int (x^3 + x^2 + x + 1)\, dx$$

$$y' + y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1 \end{aligned}$$

Hence the ode is

$$y' + y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 1 dx} \\ &= \mathrm{e}^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x y) &= (\mathrm{e}^x)\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) \\ \mathrm{d}(\mathrm{e}^x y) &= \left(\frac{(3x^4 + 4x^3 + 6x^2 + 12c_1 + 12x)\,\mathrm{e}^x}{12}\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$e^{x}y = \int \frac{(3x^4 + 4x^3 + 6x^2 + 12c_1 + 12x)\, e^{x}}{12}\, dx$$
$$e^{x}y = \frac{(3x^4 - 8x^3 + 30x^2 + 12c_1 - 48x + 48)\, e^{x}}{12} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$y = \frac{e^{-x}(3x^4 - 8x^3 + 30x^2 + 12c_1 - 48x + 48)\, e^{x}}{12} + c_2 e^{-x}$$

which simplifies to

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + c_1 - 4x + 4 + c_2 e^{-x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + c_1 - 4x + 4 + c_2 e^{-x} \tag{1}$$

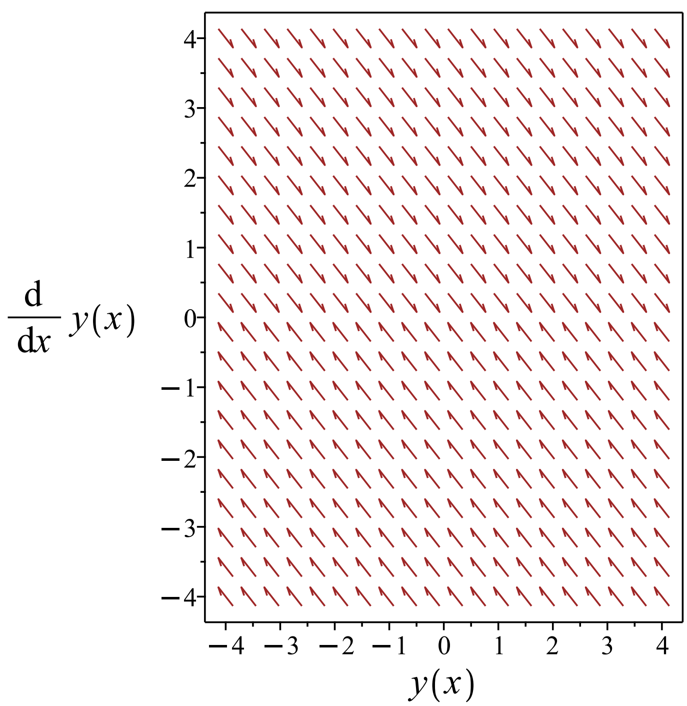

Figure 89: Slope field plot

Verification of solutions

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + c_1 - 4x + 4 + c_2 \mathrm{e}^{-x}$$

Verified OK.

1.35.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 52: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(\mathrm{e}^{x})\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+y'=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3+x^2+x+1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3, x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_4x^4 + A_3x^3 + A_2x^2 + A_1x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4x^3A_4 + 3x^2A_3 + 12x^2A_4 + 2xA_2 + 6xA_3 + A_1 + 2A_2 = x^3+x^2+x+1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -4, A_2 = \frac{5}{2}, A_3 = -\frac{2}{3}, A_4 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 - 4x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2) + \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 - 4x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} - 4x \tag{1}$$

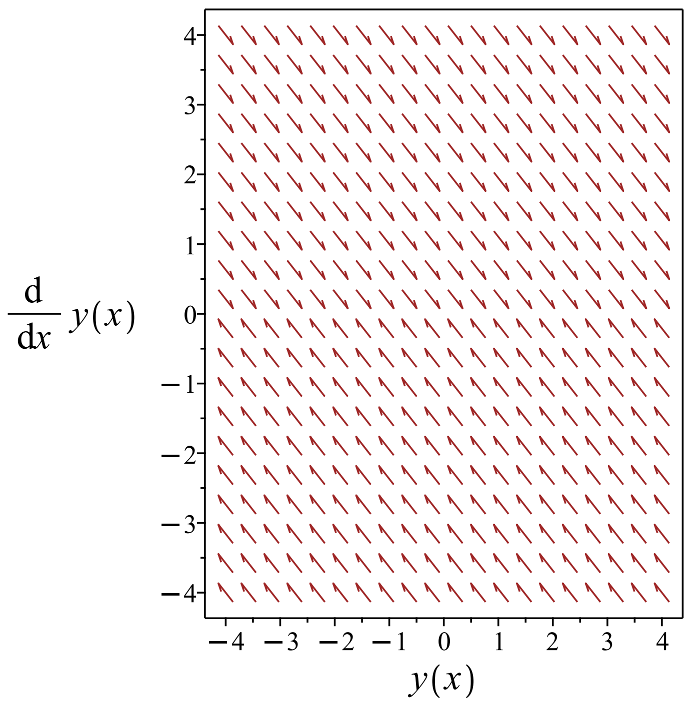

Figure 90: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 + \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} - 4x$$

Verified OK.

1.35.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= x^3 + x^2 + x + 1 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int x^3 + x^2 + x + 1\, dx$$

We now have a first order ode to solve which is

$$y' + y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1 \end{aligned}$$

Hence the ode is

$$y' + y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) \\ \frac{d}{dx}(e^x y) &= (e^x)\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) \\ d(e^x y) &= \left(\frac{(3x^4 + 4x^3 + 6x^2 + 12c_1 + 12x)\, e^x}{12}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^x y &= \int \frac{(3x^4 + 4x^3 + 6x^2 + 12c_1 + 12x)\, e^x}{12}\, dx \\ e^x y &= \frac{(3x^4 - 8x^3 + 30x^2 + 12c_1 - 48x + 48)\, e^x}{12} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = \frac{e^{-x}(3x^4 - 8x^3 + 30x^2 + 12c_1 - 48x + 48)\, e^x}{12} + c_2 e^{-x}$$

which simplifies to

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + c_1 - 4x + 4 + c_2 e^{-x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + c_1 - 4x + 4 + c_2 e^{-x} \tag{1}$$

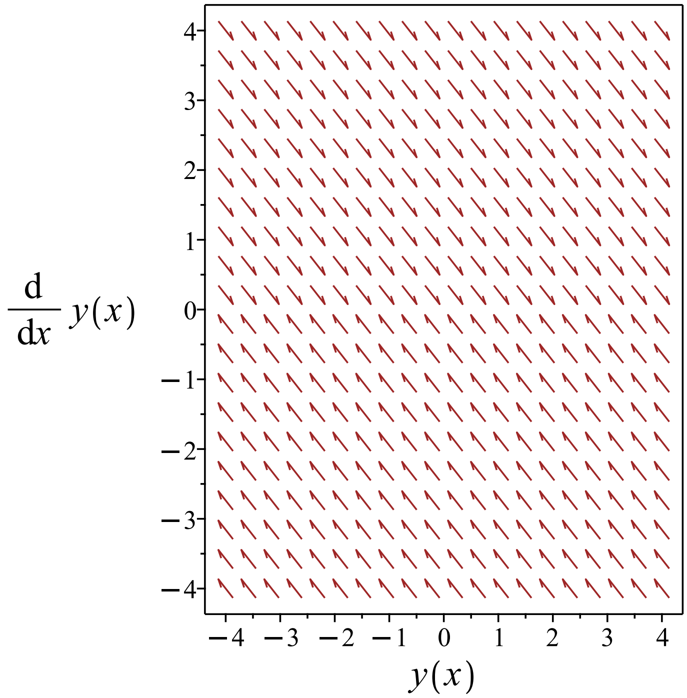

Figure 91: Slope field plot

Verification of solutions

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + c_1 - 4x + 4 + c_2 e^{-x}$$

Verified OK.

1.35.7 Maple step by step solution

Let's solve

$y'' + y' = x^3 + x^2 + x + 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x^3 + x^2 + x + 1\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\mathrm{e}^{-x}\left(\int (1+x)(x^2+1)\,\mathrm{e}^{x} dx\right) + \int (x^3 + x^2 + x + 1)\,dx$

  ◦ Compute integrals

  $y_p(x) = -\frac{2}{3}x^3 + \frac{5}{2}x^2 - 4x + 4 + \frac{1}{4}x^4$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 - \frac{2x^3}{3} + \frac{5x^2}{2} - 4x + 4 + \frac{x^4}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a^3+_a^2-_b(_a)+_a+1, _b(_a)`   *** Su
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=1+x+x^2+x^3,y(x), singsol=all)
```

$$y(x) = \frac{x^4}{4} - e^{-x}c_1 + \frac{5x^2}{2} - \frac{2x^3}{3} - 4x + c_2$$

✓ Solution by Mathematica

Time used: 0.131 (sec). Leaf size: 41

```
DSolve[y''[x]+y'[x]==1+x+x^2+x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} - 4x - c_1e^{-x} + c_2$$

1.36 problem 36

1.36.1 Solving as second order linear constant coeff ode 405
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Internal problem ID [7425]
Internal file name [`OUTPUT/6392_Sunday_June_05_2022_04_42_53_PM_82363521/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y' = \sin(x)}$$

1.36.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(0)} \\
&= -\frac{1}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= -1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(0)x} + c_2 e^{(-1)x}
\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \cos(x) - A_2 \sin(x) - A_1 \sin(x) + A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 + c_2 e^{-x}) + \left(-\frac{\cos(x)}{2} - \frac{\sin(x)}{2}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 + c_2 e^{-x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \tag{1}
$$

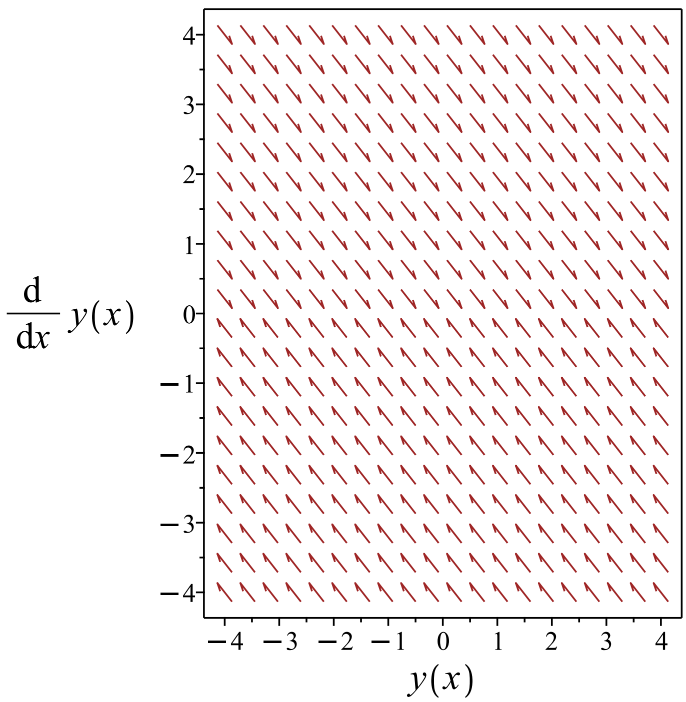

Figure 92: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 + c_2 e^{-x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2}
$$

Verified OK.

1.36.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (y'' + y')\, dx &= \int \sin(x)\, dx \\
y' + y &= -\cos(x) + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= -\cos(x) + c_1
\end{aligned}
$$

Hence the ode is

$$
y' + y = -\cos(x) + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 1 dx} \\
&= e^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(-\cos(x) + c_1) \\
\frac{d}{dx}(e^{x} y) &= (e^{x})(-\cos(x) + c_1) \\
d(e^{x} y) &= ((-\cos(x) + c_1) e^{x})\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{x} y &= \int (-\cos(x) + c_1) e^{x}\, dx \\
e^{x} y &= -\frac{e^{x} \cos(x)}{2} - \frac{\sin(x) e^{x}}{2} + c_1 e^{x} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$
y = e^{-x} \left(-\frac{e^{x} \cos(x)}{2} - \frac{\sin(x) e^{x}}{2} + c_1 e^{x} \right) + c_2 e^{-x}
$$

which simplifies to

$$y = c_1 + c_2 e^{-x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \tag{1}$$

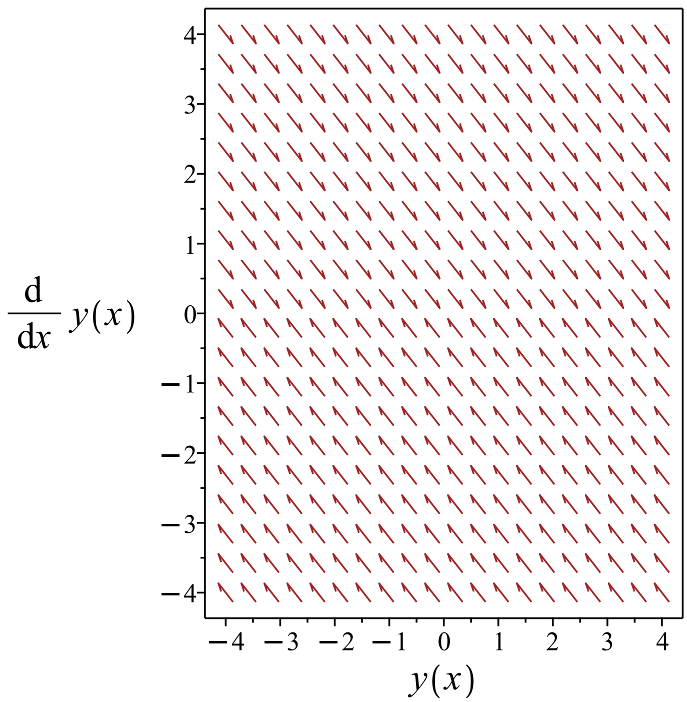

Figure 93: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Verified OK.

1.36.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - \sin(x) = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \sin(x) \end{aligned}$$

Hence the ode is

$$p'(x) + p(x) = \sin(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1 dx} \\ &= e^{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)(\sin(x)) \\ \frac{d}{dx}(e^{x} p) &= (e^{x})(\sin(x)) \\ d(e^{x} p) &= (\sin(x)\, e^{x})\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{x} p &= \int \sin(x)\, e^{x}\, dx \\ e^{x} p &= -\frac{e^{x} \cos(x)}{2} + \frac{\sin(x)\, e^{x}}{2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$p(x) = e^{-x}\left(-\frac{e^x \cos(x)}{2} + \frac{\sin(x) e^x}{2}\right) + c_1 e^{-x}$$

which simplifies to

$$p(x) = \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_1 e^{-x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_1 e^{-x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_1 e^{-x} \,dx \\ &= -c_1 e^{-x} - \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -c_1 e^{-x} - \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_2 \tag{1}$$

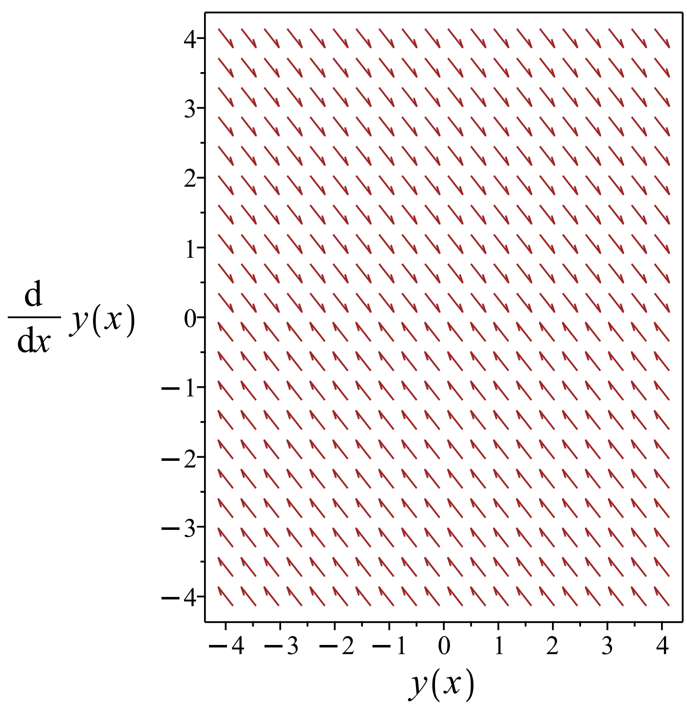

Figure 94: Slope field plot

Verification of solutions

$$y = -c_1 e^{-x} - \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_2$$

Verified OK.

1.36.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = \sin(x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int \sin(x)\, dx$$
$$y' + y = -\cos(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = -\cos(x) + c_1$$

Hence the ode is

$$y' + y = -\cos(x) + c_1$$

The integrating factor μ is

$$\mu = e^{\int 1 dx}$$
$$= e^{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(-\cos(x) + c_1)$$
$$\frac{d}{dx}(e^{x} y) = (e^{x})(-\cos(x) + c_1)$$
$$d(e^{x} y) = ((-\cos(x) + c_1) e^{x})\, dx$$

Integrating gives

$$e^x y = \int (-\cos(x) + c_1) e^x \, dx$$
$$e^x y = -\frac{e^x \cos(x)}{2} - \frac{\sin(x) e^x}{2} + c_1 e^x + c_2$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(-\frac{e^x \cos(x)}{2} - \frac{\sin(x) e^x}{2} + c_1 e^x\right) + c_2 e^{-x}$$

which simplifies to

$$y = c_1 + c_2 e^{-x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \tag{1}$$

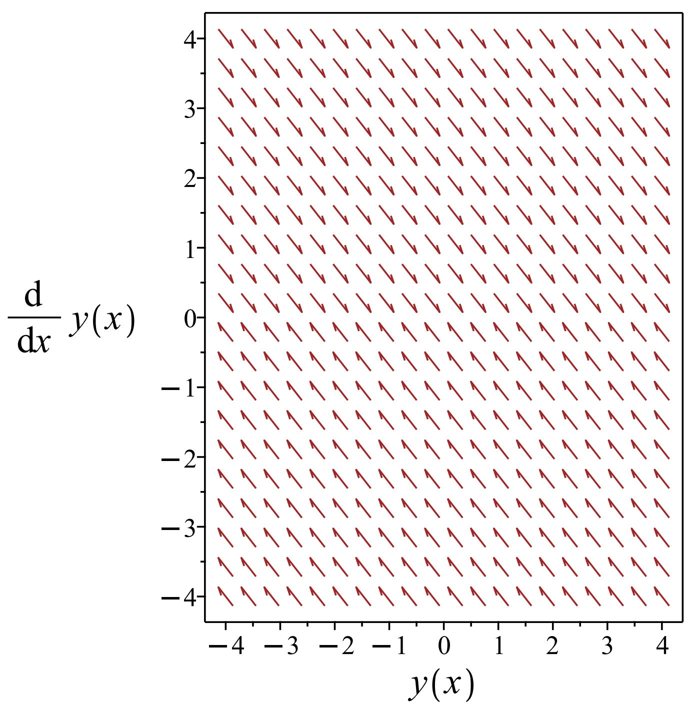

Figure 95: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Verified OK.

1.36.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 54: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(\mathrm{e}^{x})\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \cos(x) - A_2 \sin(x) - A_1 \sin(x) + A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 e^{-x} + c_2) + \left(-\frac{\cos(x)}{2} - \frac{\sin(x)}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \tag{1}$$

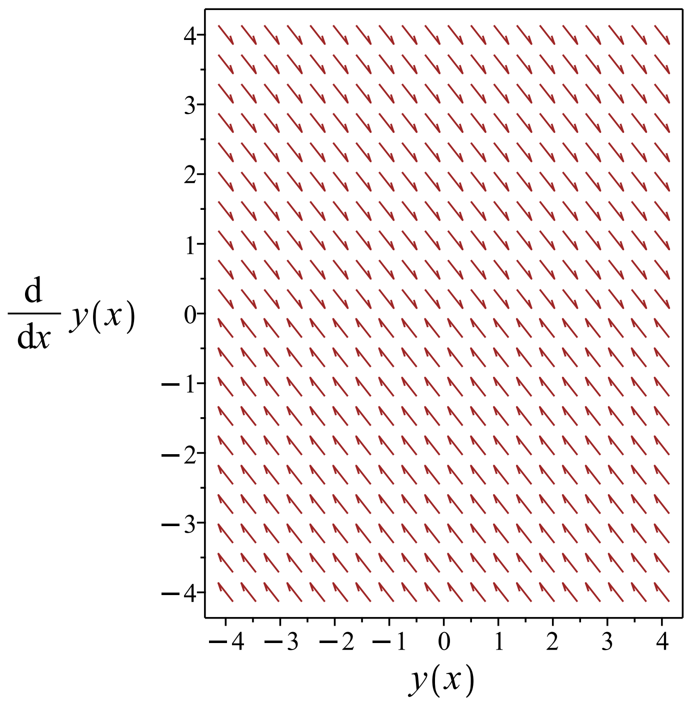

Figure 96: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 - \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Verified OK.

1.36.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= \sin(x) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int \sin(x)\,dx$$

We now have a first order ode to solve which is

$$y' + y = -\cos(x) + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= 1 \\ q(x) &= -\cos(x) + c_1\end{aligned}$$

Hence the ode is

$$y' + y = -\cos(x) + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1dx} \\ &= e^x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(-\cos(x) + c_1) \\ \frac{d}{dx}(e^x y) &= (e^x)(-\cos(x) + c_1) \\ d(e^x y) &= ((-\cos(x) + c_1)e^x)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^x y &= \int (-\cos(x) + c_1)e^x\, dx \\ e^x y &= -\frac{e^x \cos(x)}{2} - \frac{\sin(x)e^x}{2} + c_1 e^x + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(-\frac{e^x \cos(x)}{2} - \frac{\sin(x)e^x}{2} + c_1 e^x\right) + c_2 e^{-x}$$

which simplifies to

$$y = c_1 + c_2 e^{-x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \tag{1}$$

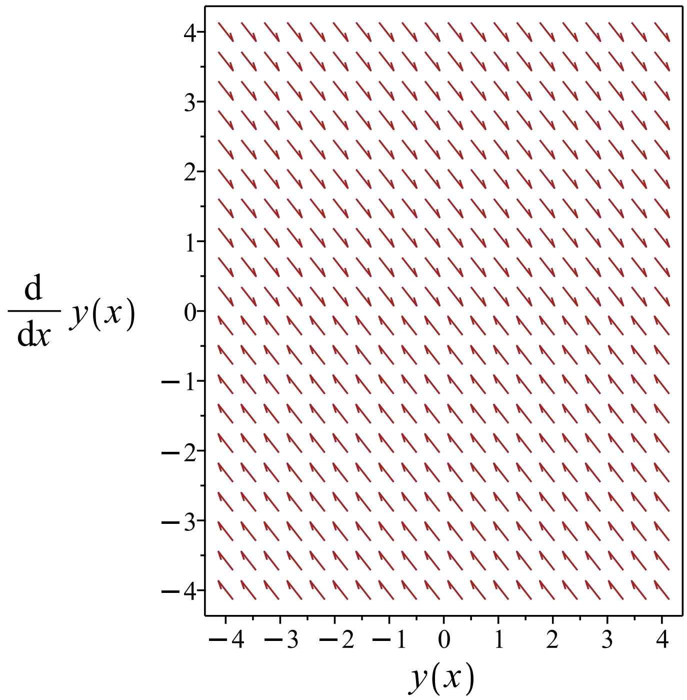

Figure 97: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Verified OK.

1.36.7 Maple step by step solution

Let's solve

$y'' + y' = \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r + 1) = 0$

- Roots of the characteristic polynomial

$r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\mathrm{e}^{-x} \left(\int \sin(x) \mathrm{e}^{x} dx\right) + \int \sin(x) dx$

  ○ Compute integrals

  $y_p(x) = -\frac{\cos(x)}{2} - \frac{\sin(x)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 - \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)+sin(_a), _b(_a)`   *** Sublevel
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=sin(x),y(x), singsol=all)
```

$$y(x) = -\mathrm{e}^{-x}c_1 - \frac{\sin (x)}{2} - \frac{\cos (x)}{2} + c_2$$

✓ Solution by Mathematica

Time used: 0.112 (sec). Leaf size: 29

```
DSolve[y''[x]+y'[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_1\left(-e^{-x}\right) + c_2$$

1.37 problem 37

1.37.1 Solving as second order linear constant coeff ode 425
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Internal problem ID [7426]
Internal file name [`OUTPUT/6393_Sunday_June_05_2022_04_42_55_PM_5839241/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y' = \cos(x)}$$

1.37.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(0)} \\
&= -\frac{1}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= -1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(0)x} + c_2 e^{(-1)x}
\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \cos(x) - A_2 \sin(x) - A_1 \sin(x) + A_2 \cos(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + c_2 \mathrm{e}^{-x}\right) + \left(\frac{\sin(x)}{2} - \frac{\cos(x)}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + c_2 \mathrm{e}^{-x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} \tag{1}
$$

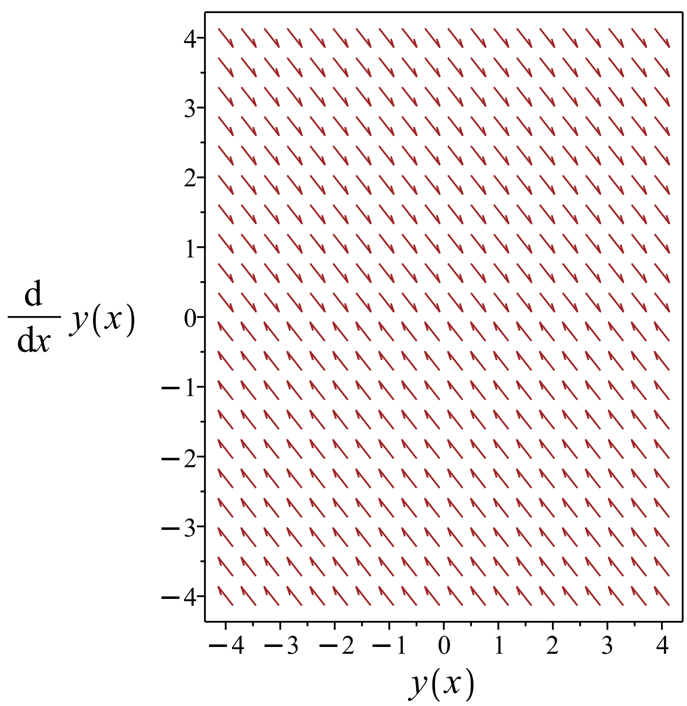

Figure 98: Slope field plot

Verification of solutions

$$
y = c_1 + c_2 \mathrm{e}^{-x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2}
$$

Verified OK.

1.37.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int \cos(x)\, dx$$
$$y' + y = \sin(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = \sin(x) + c_1$$

Hence the ode is

$$y' + y = \sin(x) + c_1$$

The integrating factor μ is

$$\mu = e^{\int 1 dx}$$
$$= e^{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(\sin(x) + c_1)$$
$$\frac{d}{dx}(e^{x} y) = (e^{x})(\sin(x) + c_1)$$
$$d(e^{x} y) = ((\sin(x) + c_1)e^{x})\, dx$$

Integrating gives

$$e^{x} y = \int (\sin(x) + c_1)e^{x}\, dx$$
$$e^{x} y = -\frac{e^{x}\cos(x)}{2} + \frac{\sin(x)e^{x}}{2} + c_1 e^{x} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$y = e^{-x}\left(-\frac{e^{x}\cos(x)}{2} + \frac{\sin(x)e^{x}}{2} + c_1 e^{x}\right) + c_2 e^{-x}$$

which simplifies to

$$y = c_1 + c_2 e^{-x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} \tag{1}$$

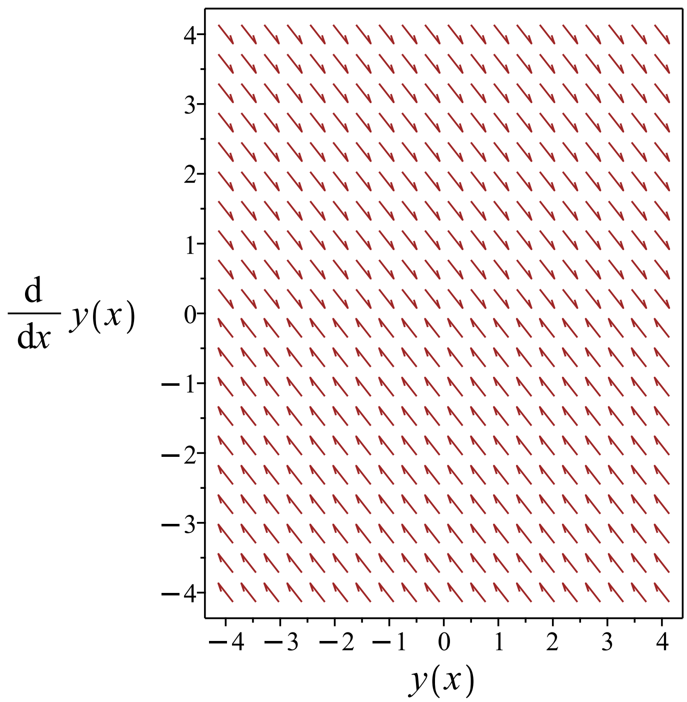

Figure 99: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$$

Verified OK.

1.37.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - \cos(x) = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \cos(x) \end{aligned}$$

Hence the ode is

$$p'(x) + p(x) = \cos(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)(\cos(x)) \\ \frac{d}{dx}(e^{x} p) &= (e^{x})(\cos(x)) \\ d(e^{x} p) &= (e^{x}\cos(x))\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{x} p &= \int e^{x}\cos(x)\, dx \\ e^{x} p &= \frac{e^{x}\cos(x)}{2} + \frac{\sin(x)\, e^{x}}{2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$p(x) = e^{-x}\left(\frac{e^x \cos(x)}{2} + \frac{\sin(x)\, e^x}{2}\right) + c_1 e^{-x}$$

which simplifies to

$$p(x) = \frac{\sin(x)}{2} + \frac{\cos(x)}{2} + c_1 e^{-x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{\sin(x)}{2} + \frac{\cos(x)}{2} + c_1 e^{-x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\sin(x)}{2} + \frac{\cos(x)}{2} + c_1 e^{-x}\, dx \\ &= -c_1 e^{-x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_2 \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -c_1 e^{-x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_2 \tag{1}$$

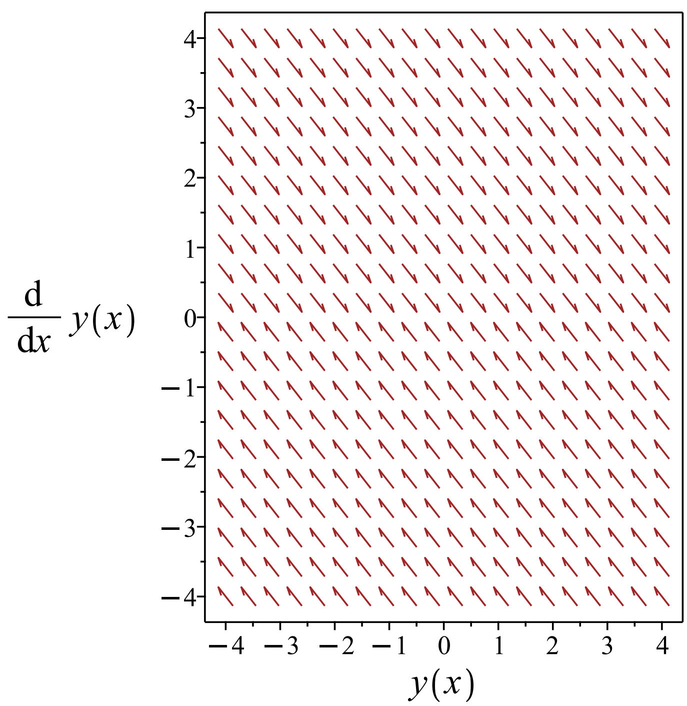

Figure 100: Slope field plot

Verification of solutions

$$y = -c_1 e^{-x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_2$$

Verified OK.

1.37.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = \cos(x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int \cos(x)\, dx$$
$$y' + y = \sin(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = \sin(x) + c_1$$

Hence the ode is

$$y' + y = \sin(x) + c_1$$

The integrating factor μ is

$$\mu = e^{\int 1 dx}$$
$$= e^x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(\sin(x) + c_1)$$
$$\frac{d}{dx}(e^x y) = (e^x)(\sin(x) + c_1)$$
$$d(e^x y) = ((\sin(x) + c_1) e^x)\, dx$$

Integrating gives

$$e^x y = \int (\sin(x) + c_1) e^x \, dx$$

$$e^x y = -\frac{e^x \cos(x)}{2} + \frac{\sin(x) e^x}{2} + c_1 e^x + c_2$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(-\frac{e^x \cos(x)}{2} + \frac{\sin(x) e^x}{2} + c_1 e^x\right) + c_2 e^{-x}$$

which simplifies to

$$y = c_1 + c_2 e^{-x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} \tag{1}$$

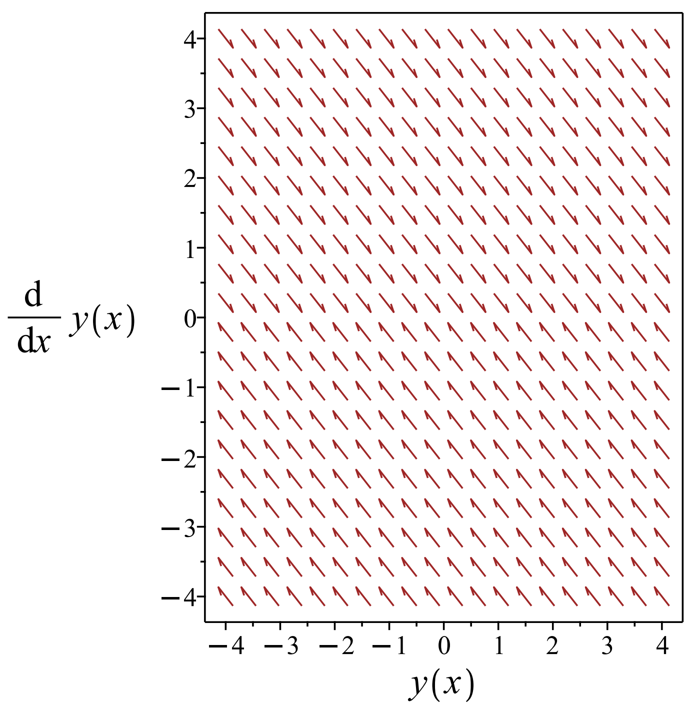

Figure 101: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$$

Verified OK.

1.37.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 56: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(\mathrm{e}^{x})\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \cos(x) - A_2 \sin(x) - A_1 \sin(x) + A_2 \cos(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2\right) + \left(\frac{\sin(x)}{2} - \frac{\cos(x)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} \tag{1}$$

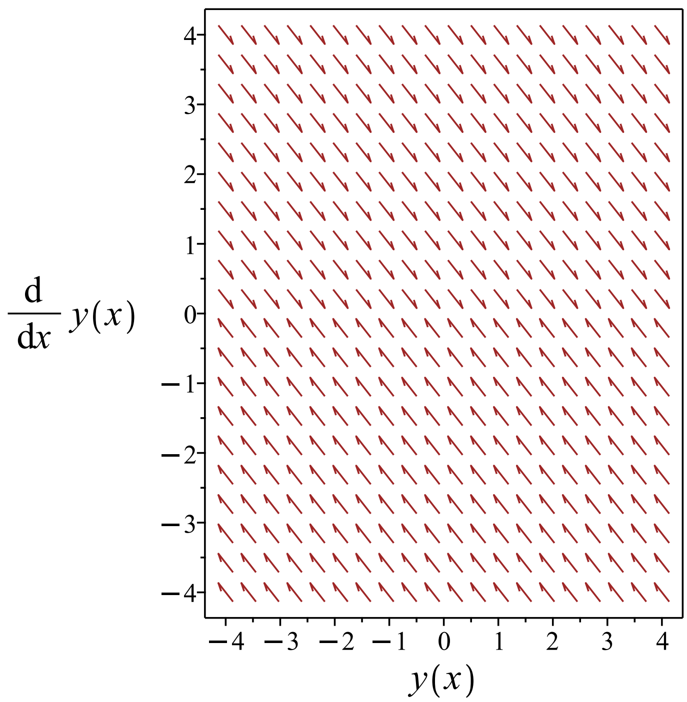

Figure 102: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 + \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$$

Verified OK.

1.37.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= \cos(x) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int \cos(x)\,dx$$

We now have a first order ode to solve which is

$$y' + y = \sin(x) + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \sin(x) + c_1 \end{aligned}$$

Hence the ode is

$$y' + y = \sin(x) + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 1dx} \\ &= \mathrm{e}^{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(\sin(x) + c_1) \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} y) &= (\mathrm{e}^{x})(\sin(x) + c_1) \\ \mathrm{d}(\mathrm{e}^{x} y) &= ((\sin(x) + c_1)\,\mathrm{e}^{x})\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{x} y &= \int (\sin(x) + c_1)\,\mathrm{e}^{x}\,\mathrm{d}x \\ \mathrm{e}^{x} y &= -\frac{\mathrm{e}^{x}\cos(x)}{2} + \frac{\sin(x)\,\mathrm{e}^{x}}{2} + c_1\mathrm{e}^{x} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$y = \mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{x}\cos(x)}{2} + \frac{\sin(x)\,\mathrm{e}^{x}}{2} + c_1\mathrm{e}^{x}\right) + c_2\mathrm{e}^{-x}$$

which simplifies to

$$y = c_1 + c_2\mathrm{e}^{-x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 + c_2\mathrm{e}^{-x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} \tag{1}$$

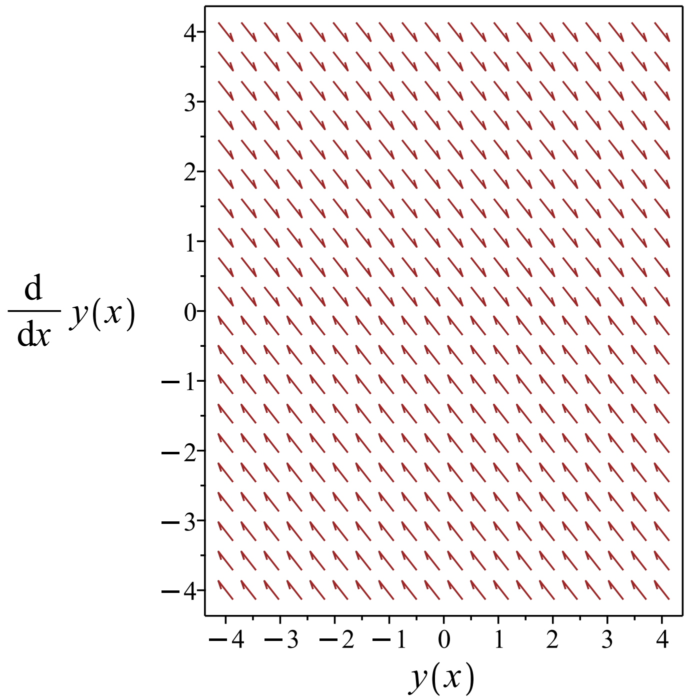

Figure 103: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$$

Verified OK.

1.37.7 Maple step by step solution

Let's solve

$y'' + y' = \cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r + 1) = 0$

- Roots of the characteristic polynomial

$r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cos(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-x} \left(\int \mathrm{e}^{x} \cos(x) \, dx\right) + \int \cos(x) \, dx$

  - Compute integrals

    $y_p(x) = \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)+cos(_a), _b(_a)`   *** Sublevel
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=cos(x),y(x), singsol=all)
```

$$y(x) = -e^{-x}c_1 + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_2$$

✓ Solution by Mathematica

Time used: 0.08 (sec). Leaf size: 28

```
DSolve[y''[x]+y'[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(\sin(x) - \cos(x) - 2c_1e^{-x}) + c_2$$

1.38 problem 38

1.38.1 Solving as second order linear constant coeff ode 445
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Internal problem ID [7427]
Internal file name [`OUTPUT/6394_Sunday_June_05_2022_04_42_57_PM_94134630/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' + y = 1}$$

1.38.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = \mathrm{e}^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (1) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + 1 \tag{1}$$

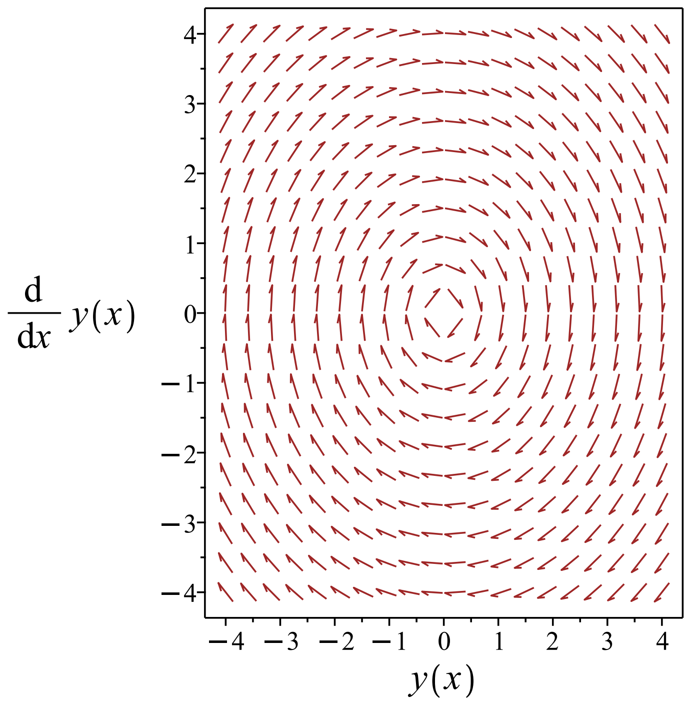

Figure 104: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + 1$$

Verified OK.

1.38.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + yy' - y' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + yy' - y')\,dx = 0$$

$$\frac{y'^2}{2} + \frac{y^2}{2} - y = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-y^2 + 2y + 2c_1} \tag{1}$$
$$y' = -\sqrt{-y^2 + 2y + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-y^2 + 2c_1 + 2y}} dy = \int dx$$
$$\arctan\left(\frac{-1 + y}{\sqrt{-y^2 + 2y + 2c_1}}\right) = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-y^2 + 2c_1 + 2y}} dy = \int dx$$
$$-\arctan\left(\frac{-1 + y}{\sqrt{-y^2 + 2y + 2c_1}}\right) = x + c_3$$

Summary

The solution(s) found are the following

$$\arctan\left(\frac{-1 + y}{\sqrt{-y^2 + 2y + 2c_1}}\right) = x + c_2 \tag{1}$$
$$-\arctan\left(\frac{-1 + y}{\sqrt{-y^2 + 2y + 2c_1}}\right) = x + c_3 \tag{2}$$

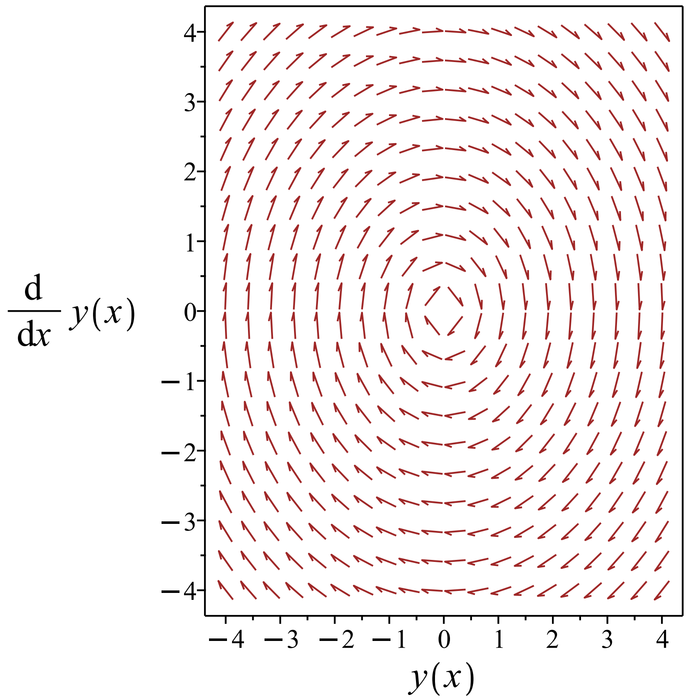

Figure 105: Slope field plot

Verification of solutions

$$\arctan\left(\frac{-1+y}{\sqrt{-y^2+2y+2c_1}}\right) = x + c_2$$

Verified OK.

$$-\arctan\left(\frac{-1+y}{\sqrt{-y^2+2y+2c_1}}\right) = x + c_3$$

Verified OK.

1.38.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 58: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (1) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + 1 \tag{1}$$

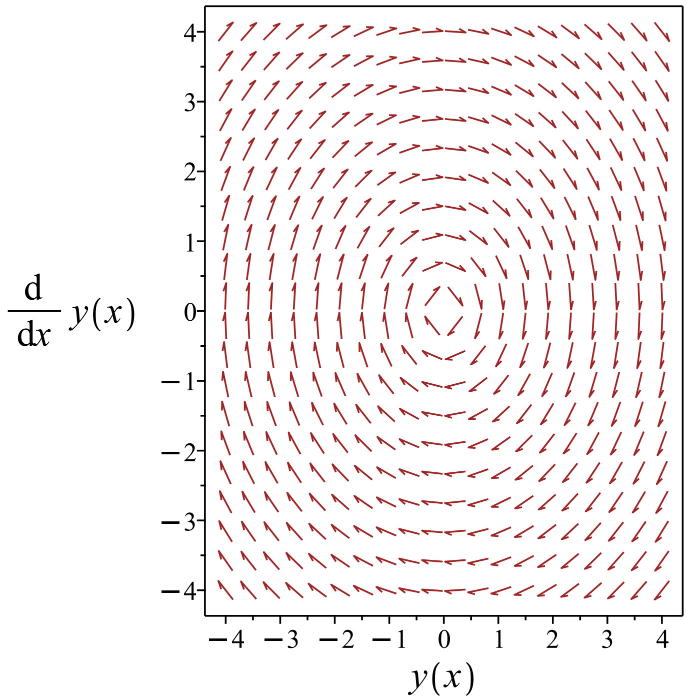

Figure 106: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + 1$$

Verified OK.

1.38.4 Maple step by step solution

Let's solve

$y'' + y = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 1\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x)\left(\int \sin(x)\, dx\right) + \sin(x)\left(\int \cos(x)\, dx\right)$

  ◦ Compute integrals

    $y_p(x) = 1$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)+y(x)=1,y(x), singsol=all)
```

$$y(x) = \sin(x) c_2 + \cos(x) c_1 + 1$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 17

```
DSolve[y''[x]+y[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(x) + c_2 \sin(x) + 1$$

1.39 problem 39

1.39.1 Solving as second order linear constant coeff ode 458
1.39.2 Solving using Kovacic algorithm 461
1.39.3 Maple step by step solution 466

Internal problem ID [7428]
Internal file name [`OUTPUT/6395_Sunday_June_05_2022_04_42_59_PM_64149738/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y = x}$$

1.39.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 = x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + x \tag{1}$$

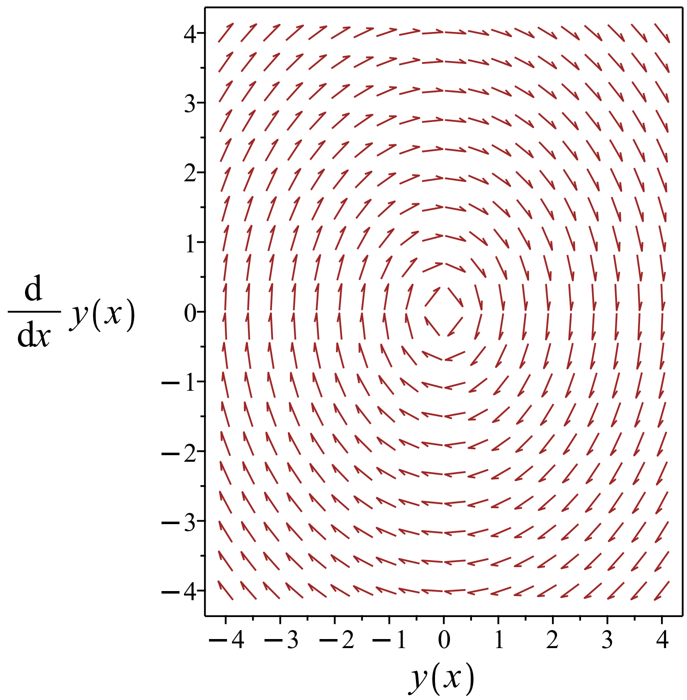

Figure 107: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + x$$

Verified OK.

1.39.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 60: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 = x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + x \tag{1}$$

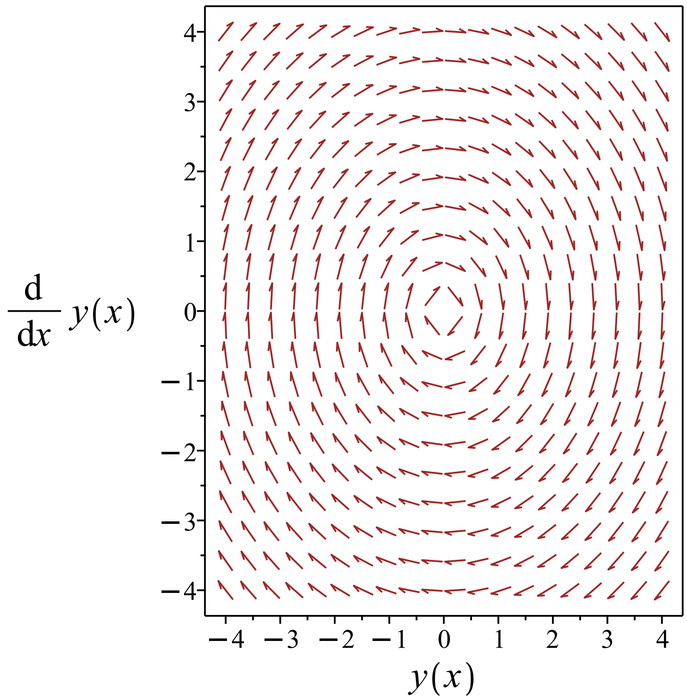

Figure 108: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + x$$

Verified OK.

1.39.3 Maple step by step solution

Let's solve

$y'' + y = x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x)\left(\int x \sin(x)\, dx\right) + \sin(x)\left(\int x \cos(x)\, dx\right)$

  - Compute integrals

    $y_p(x) = x$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)+y(x)=x,y(x), singsol=all)
```

$$y(x) = \sin(x) c_2 + \cos(x) c_1 + x$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 17

```
DSolve[y''[x]+y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_1 \cos(x) + c_2 \sin(x)$$

1.40 problem 40

1.40.1 Solving as second order linear constant coeff ode 469
1.40.2 Solving using Kovacic algorithm 472
1.40.3 Maple step by step solution 477

Internal problem ID [7429]
Internal file name [`OUTPUT/6396_Sunday_June_05_2022_04_43_01_PM_41964287/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y = 1 + x}$$

1.40.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 1 + x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 = 1 + x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1 + x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (1 + x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + 1 + x \tag{1}$$

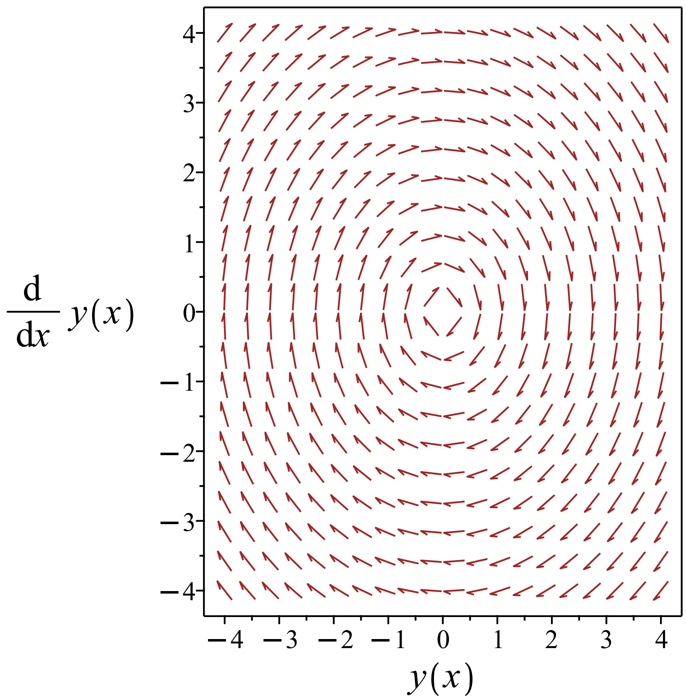

Figure 109: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + 1 + x$$

Verified OK.

1.40.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 62: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)\,(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x)))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 = 1 + x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1 + x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (1 + x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + 1 + x \tag{1}$$

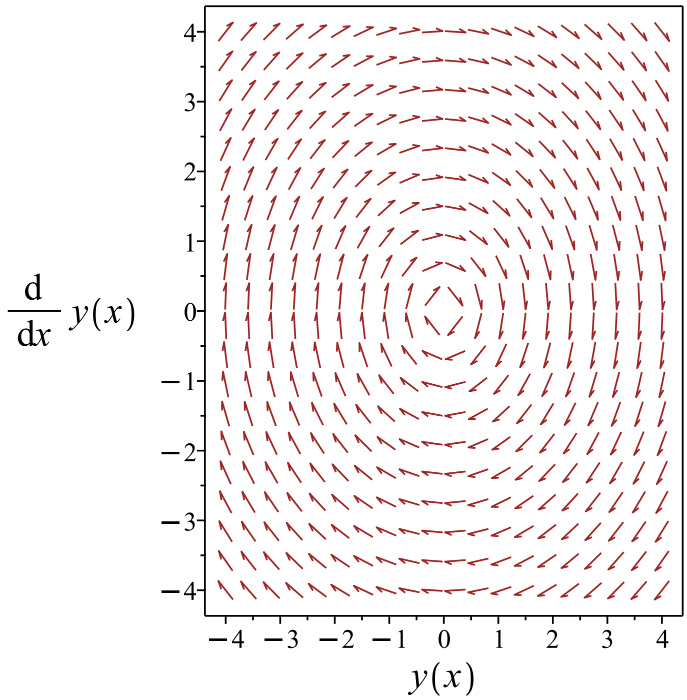

Figure 110: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + 1 + x$$

Verified OK.

1.40.3 Maple step by step solution

Let's solve

$y'' + y = 1 + x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 1 + x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x)\left(\int \sin(x)(1 + x) dx\right) + \sin(x)\left(\int \cos(x)(1 + x) dx\right)$

  - Compute integrals

    $y_p(x) = 1 + x$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + 1 + x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)+y(x)=1+x,y(x), singsol=all)
```

$$y(x) = \sin(x) c_2 + \cos(x) c_1 + x + 1$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 18

```
DSolve[y''[x]+y[x]==1+x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_1 \cos(x) + c_2 \sin(x) + 1$$

1.41 problem 41

1.41.1 Solving as second order linear constant coeff ode 480
1.41.2 Solving using Kovacic algorithm 483
1.41.3 Maple step by step solution 488

Internal problem ID [7430]
Internal file name [`OUTPUT/6397_Sunday_June_05_2022_04_43_03_PM_22837938/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y = x^2 + x + 1}$$

1.41.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = x^2 + x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_3x^2 + A_2x + A_1 + 2A_3 = x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 1, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 + x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (x^2 + x - 1) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + x^2 + x - 1 \tag{1}$$

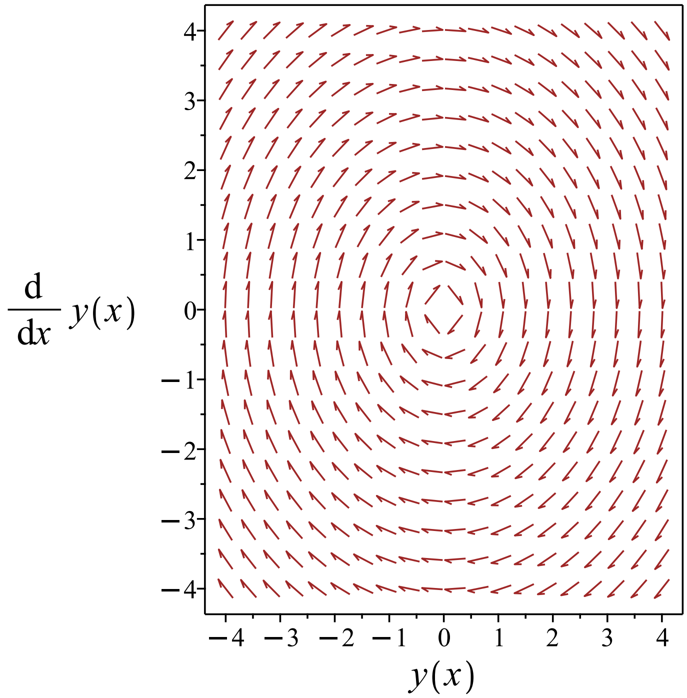

Figure 111: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + x^2 + x - 1$$

Verified OK.

1.41.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 64: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_3x^2 + A_2x + A_1 + 2A_3 = x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 1, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 + x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (x^2 + x - 1) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + x^2 + x - 1 \tag{1}$$

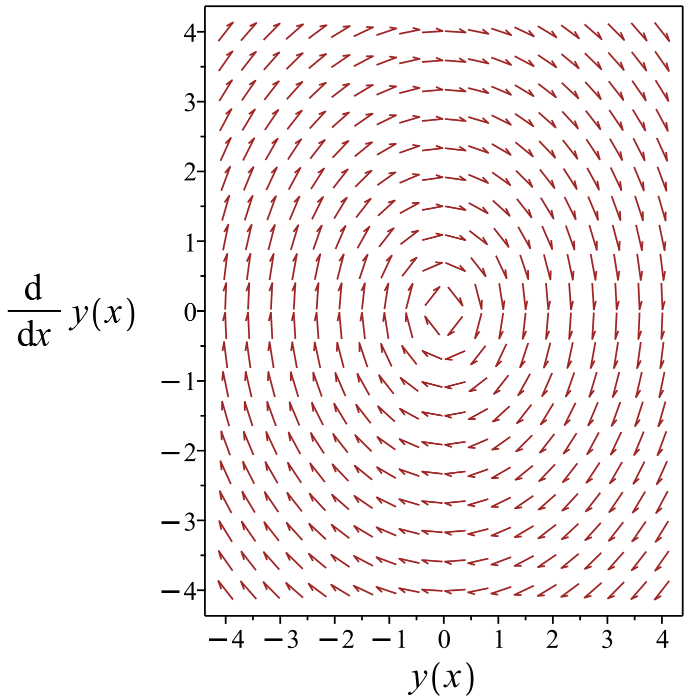

Figure 112: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + x^2 + x - 1$$

Verified OK.

1.41.3 Maple step by step solution

Let's solve

$y'' + y = x^2 + x + 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2 + x + 1\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x)\left(\int \sin(x)(x^2 + x + 1)\, dx\right) + \sin(x)\left(\int \cos(x)(x^2 + x + 1)\, dx\right)$

  ◦ Compute integrals

  $y_p(x) = x^2 + x - 1$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + x^2 + x - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)+y(x)=1+x+x^2,y(x), singsol=all)
```

$$y(x) = \sin(x) c_2 + \cos(x) c_1 + x^2 + x - 1$$

✓ Solution by Mathematica
Time used: 0.012 (sec). Leaf size: 21

```
DSolve[y''[x]+y[x]==1+x+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 + x + c_1 \cos(x) + c_2 \sin(x) - 1$$

1.42 problem 42

1.42.1 Solving as second order linear constant coeff ode 491
1.42.2 Solving using Kovacic algorithm 494
1.42.3 Maple step by step solution 499

Internal problem ID [7431]
Internal file name [`OUTPUT/6398_Sunday_June_05_2022_04_43_05_PM_91104413/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = x^3 + x^2 + x + 1}$$

1.42.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = x^3 + x^2 + x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4 x^3 + A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_4 x^3 + A_3 x^2 + A_2 x + 6x A_4 + A_1 + 2A_3 = x^3 + x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = -5, A_3 = 1, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^3 + x^2 - 5x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(x^3 + x^2 - 5x - 1\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + x^3 + x^2 - 5x - 1 \tag{1}$$

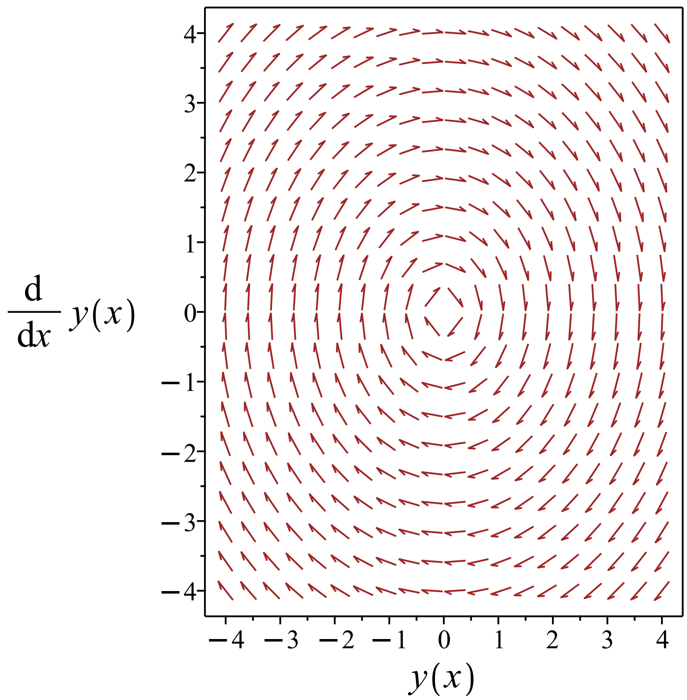

Figure 113: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + x^3 + x^2 - 5x - 1$$

Verified OK.

1.42.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 66: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2} dx \\
&= \cos(x) (\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x) (\tan(x)))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_4x^3 + A_3x^2 + A_2x + 6xA_4 + A_1 + 2A_3 = x^3 + x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = -5, A_3 = 1, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^3 + x^2 - 5x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(x^3 + x^2 - 5x - 1\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + x^3 + x^2 - 5x - 1 \tag{1}$$

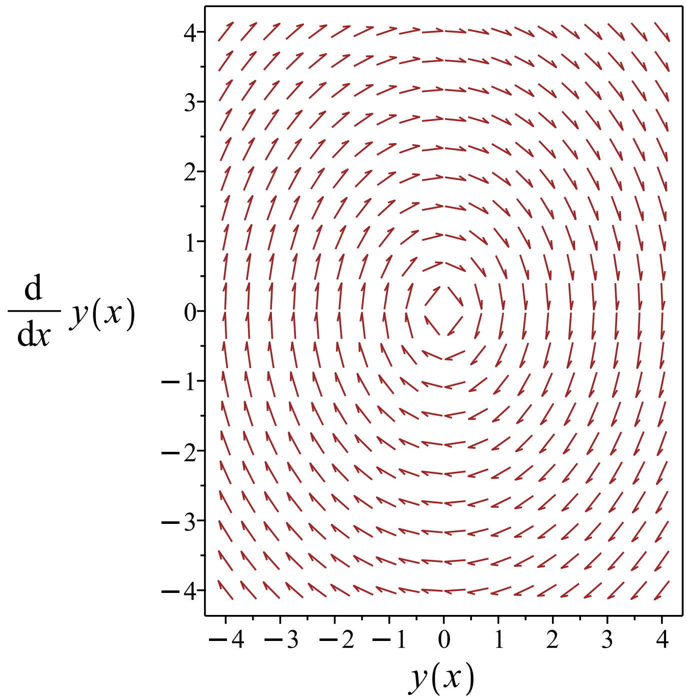

Figure 114: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + x^3 + x^2 - 5x - 1$$

Verified OK.

1.42.3 Maple step by step solution

Let's solve

$y'' + y = x^3 + x^2 + x + 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^3 + x^2 + x + 1\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int \sin(x) (1 + x) (x^2 + 1) dx\right) + \sin(x) \left(\int \cos(x) (1 + x) (x^2 + 1) dx\right)$

  - Compute integrals

    $y_p(x) = x^3 + x^2 - 5x - 1$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + x^3 + x^2 - 5x - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)=1+x+x^2+x^3,y(x), singsol=all)
```

$$y(x) = \sin(x)\, c_2 + \cos(x)\, c_1 + x^3 + x^2 - 5x - 1$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 26

```
DSolve[y''[x]+y[x]==1+x+x^2+x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^3 + x^2 - 5x + c_1 \cos(x) + c_2 \sin(x) - 1$$

1.43 problem 43

1.43.1 Solving as second order linear constant coeff ode 502
1.43.2 Solving using Kovacic algorithm 506
1.43.3 Maple step by step solution 510

Internal problem ID [7432]
Internal file name [`OUTPUT/6399_Sunday_June_05_2022_04_43_07_PM_44807988/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 43.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \sin(x)}$$

1.43.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(x), x\sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(x) + A_2 x \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\cos(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{x \cos(x)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{x \cos(x)}{2} \tag{1}$$

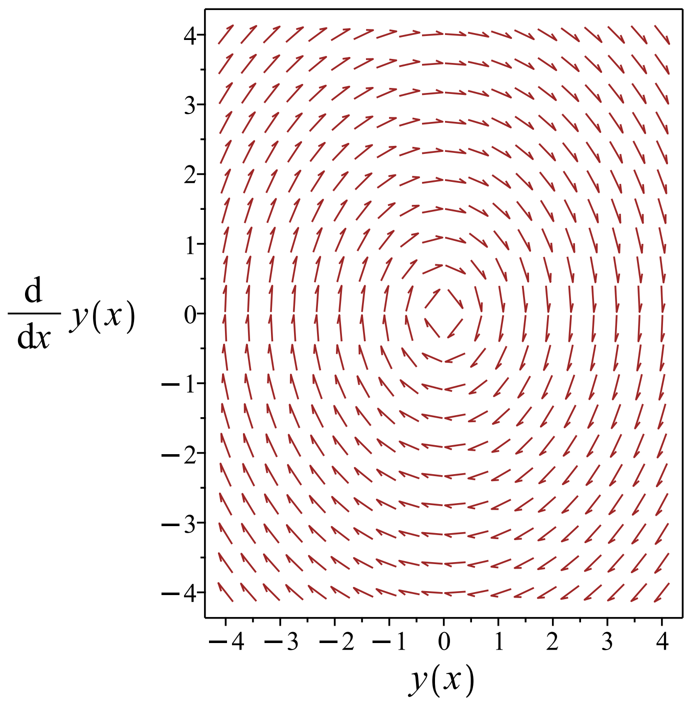

Figure 115: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{x \cos(x)}{2}$$

Verified OK.

1.43.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 68: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\
&= \cos(x)\,(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(x), x\sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\cos(x) + A_2 x\sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sin(x) + 2A_2\cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\cos(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + \left(-\frac{x\cos(x)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{x \cos(x)}{2} \tag{1}$$

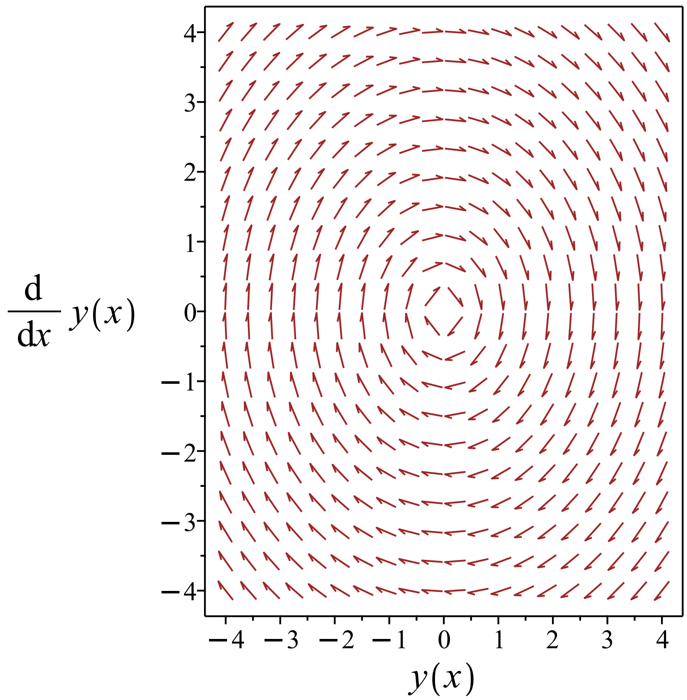

Figure 116: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{x \cos(x)}{2}$$

Verified OK.

1.43.3 Maple step by step solution

Let's solve

$y'' + y = \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int \sin(x)^2 dx\right) + \frac{\sin(x)\left(\int \sin(2x) dx\right)}{2}$

  ◦ Compute integrals

    $y_p(x) = \frac{\sin(x)}{4} - \frac{x \cos(x)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \frac{\sin(x)}{4} - \frac{x \cos(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)+y(x)=sin(x),y(x), singsol=all)
```

$$y(x) = \frac{(-x + 2c_1)\cos(x)}{2} + \frac{\sin(x)(2c_2 + 1)}{2}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 22

```
DSolve[y''[x]+y[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(-\frac{x}{2} + c_1\right)\cos(x) + c_2\sin(x)$$

1.44 problem 44

1.44.1 Solving as second order linear constant coeff ode 513
1.44.2 Solving using Kovacic algorithm 517
1.44.3 Maple step by step solution 521

Internal problem ID [7433]
Internal file name [`OUTPUT/6400_Sunday_June_05_2022_04_43_09_PM_83424516/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 44.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \cos(x)}$$

1.44.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(x), x\sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(x) + A_2 x \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\sin(x)}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{x \sin(x)}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) + \frac{x \sin(x)}{2} \tag{1}
$$

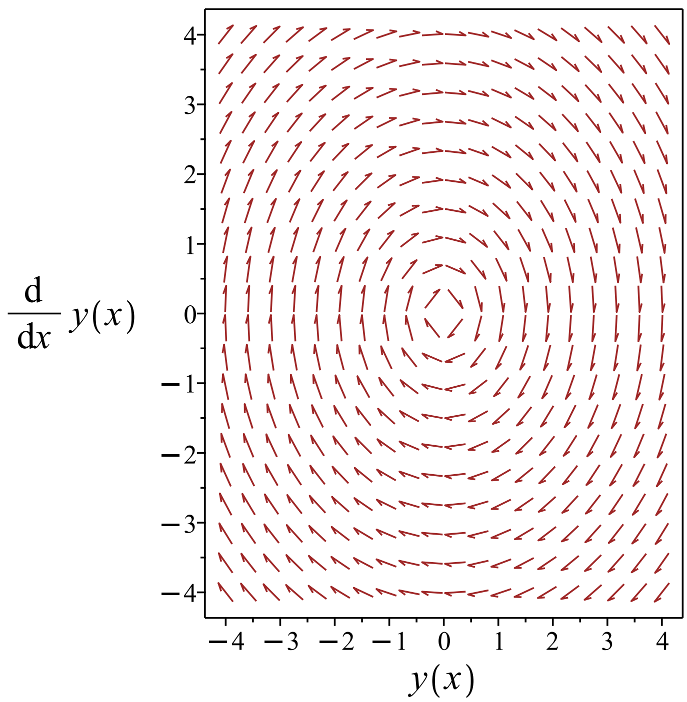

Figure 117: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) + \frac{x \sin(x)}{2}
$$

Verified OK.

1.44.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 70: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B=0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B=0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(x), x\sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(x) + A_2 x \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{x\sin(x)}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{x \sin(x)}{2} \tag{1}$$

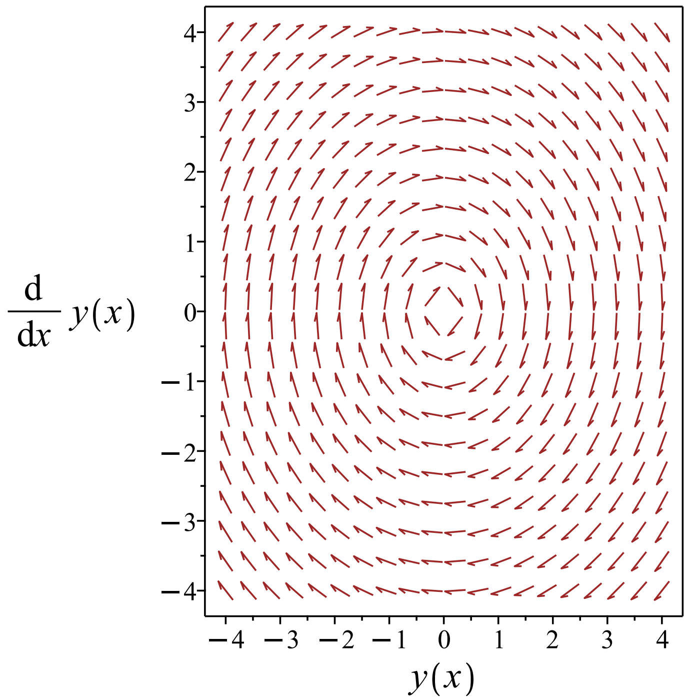

Figure 118: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{x \sin(x)}{2}$$

Verified OK.

1.44.3 Maple step by step solution

Let's solve

$y'' + y = \cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2+1=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r=(-\mathrm{I},\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x)=\cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\sin(x)$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(x)+c_2\sin(x)+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),f(x)=\cos(x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x),y_2(x))=\begin{bmatrix}\cos(x) & \sin(x)\\ -\sin(x) & \cos(x)\end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x),y_2(x))=1$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x)=-\frac{\cos(x)\left(\int\sin(2x)dx\right)}{2}+\sin(x)\left(\int\cos(x)^2\,dx\right)$

- ○ Compute integrals

  $y_p(x)=\frac{\cos(x)}{4}+\frac{x\sin(x)}{2}$

- Substitute particular solution into general solution to ODE

  $y=c_1\cos(x)+c_2\sin(x)+\frac{\cos(x)}{4}+\frac{x\sin(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)+y(x)=cos(x),y(x), singsol=all)
```

$$y(x) = \frac{(2c_2 + x)\sin(x)}{2} + \cos(x)\, c_1$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 28

```
DSolve[y''[x]+y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(x\sin(x) + \cos(x) + 2c_1\cos(x) + 2c_2\sin(x))$$

1.45 problem 45

1.45.1 Solving as second order ode missing x ode 524

Internal problem ID [7434]
Internal file name [`OUTPUT/6401_Sunday_June_05_2022_04_43_12_PM_499369/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 45.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y{y''}^{2}+y'=0}$$

1.45.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}$$

Hence the ode becomes

$$y p(y)^{2} \left(\frac{d}{dy}p(y)\right)^{2}+p(y)=0$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\frac{d}{dy}p(y) = -\frac{1}{\sqrt{-p(y)\, y}} \tag{1}$$

$$\frac{d}{dy}p(y) = \frac{1}{\sqrt{-p(y)\, y}} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$\begin{aligned}\frac{d}{dy}p(y) &= -\frac{1}{\sqrt{-py}}\\ \frac{d}{dy}p(y) &= \omega(y,p)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + ya_2 + a_1 \tag{1E}$$

$$\eta = pb_3 + yb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{b_3 - a_2}{\sqrt{-py}} + \frac{a_3}{py} + \frac{p(pa_3 + ya_2 + a_1)}{2\,(-py)^{\frac{3}{2}}} + \frac{y(pb_3 + yb_2 + b_1)}{2\,(-py)^{\frac{3}{2}}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2b_2(-py)^{\frac{3}{2}}\, py - p^2y^2a_2 + 3p^2y^2b_3 + p^3ya_3 + p\, y^3b_2 + 2a_3(-py)^{\frac{3}{2}} + p^2ya_1 + p\, y^2b_1}{2\,(-py)^{\frac{3}{2}}\, py} = 0$$

Setting the numerator to zero gives

$$2b_2(-py)^{\frac{3}{2}} py - p^2y^2a_2 + 3p^2y^2b_3 + p^3ya_3 + p\,y^3b_2 + 2a_3(-py)^{\frac{3}{2}} + p^2ya_1 + p\,y^2b_1 = 0 \quad \text{(6E)}$$

Since the PDE has radicals, simplifying gives

$$-py\left(2\sqrt{-py}\,pyb_2 - p^2a_3 + pya_2 - 3pyb_3 - y^2b_2 + 2\sqrt{-py}\,a_3 - pa_1 - yb_1\right) = 0$$

Looking at the above PDE shows the following are all the terms with $\{p, y\}$ in them.

$$\left\{p, y, \sqrt{-py}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{p, y\}$ in them

$$\left\{p = v_1, y = v_2, \sqrt{-py} = v_3\right\}$$

The above PDE (6E) now becomes

$$-v_1v_2\left(2v_3v_1v_2b_2 + v_1v_2a_2 - v_1^2a_3 - v_2^2b_2 - 3v_1v_2b_3 - v_1a_1 + 2v_3a_3 - v_2b_1\right) = 0 \quad \text{(7E)}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$v_2a_3v_1^3 - 2b_2v_3v_1^2v_2^2 + (-a_2 + 3b_3)\,v_1^2v_2^2 + a_1v_1^2v_2 + b_2v_1v_2^3 - 2v_3a_3v_1v_2 + b_1v_1v_2^2 = 0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ -2a_3 &= 0 \\ -2b_2 &= 0 \\ -a_2 + 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 3b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 3y \\
\eta &= p
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(y,p)\,\xi \\
&= p - \left(-\frac{1}{\sqrt{-py}}\right)(3y) \\
&= \frac{p\sqrt{-py} + 3y}{\sqrt{-py}} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y,p) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial p}\right) S(y,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{p\sqrt{-py}+3y}{\sqrt{-py}}} dy
\end{aligned}
$$

Which results in

$$S = \frac{2\ln\left(\left(-py\right)^{\frac{3}{2}} - 3y^{2}\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y,p)S_p}{R_y + \omega(y,p)R_p} \tag{2}$$

Where in the above R_y, R_p, S_y, S_p are all partial derivatives and $\omega(y,p)$ is the right hand side of the original ode given by

$$\omega(y,p) = -\frac{1}{\sqrt{-py}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_y &= 1 \\
R_p &= 0 \\
S_y &= \frac{i p^{\frac{3}{2}}\sqrt{y} + 4y}{y\left(i p^{\frac{3}{2}}\sqrt{y} + 3y\right)} \\
S_p &= \frac{\sqrt{p}\,\sqrt{y}}{p^{\frac{3}{2}}\sqrt{y} - 3iy}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\frac{-p^{\frac{3}{2}}\sqrt{y}+4iy}{y} + \frac{\sqrt{p}\,\sqrt{y}}{\sqrt{-py}}}{-p^{\frac{3}{2}}\sqrt{y} + 3iy} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for y, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to y, p coordinates. This results in

$$\frac{i\pi}{3}+\frac{2\ln\left(-p(y)^{\frac{3}{2}}\, y^{\frac{3}{2}}+3iy^{2}\right)}{3}=\ln\left(y\right)+c_{1}$$

Which simplifies to

$$\frac{i\pi}{3}+\frac{2\ln\left(-p(y)^{\frac{3}{2}}\, y^{\frac{3}{2}}+3iy^{2}\right)}{3}=\ln\left(y\right)+c_{1}$$

Solving equation (2)

Writing the ode as

$$\begin{aligned}\frac{d}{dy}p(y)&=\frac{1}{\sqrt{-py}}\\ \frac{d}{dy}p(y)&=\omega(y,p)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_{y}+\omega(\eta_{p}-\xi_{y})-\omega^{2}\xi_{p}-\omega_{y}\xi-\omega_{p}\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

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$$\eta=pb_{3}+yb_{2}+b_{1} \tag{2E}$$

Where the unknown coefficients are

$$\{a_{1},a_{2},a_{3},b_{1},b_{2},b_{3}\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_{2}+\frac{b_{3}-a_{2}}{\sqrt{-py}}+\frac{a_{3}}{py}-\frac{p(pa_{3}+ya_{2}+a_{1})}{2\left(-py\right)^{\frac{3}{2}}}-\frac{y(pb_{3}+yb_{2}+b_{1})}{2\left(-py\right)^{\frac{3}{2}}}=0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2b_{2}(-py)^{\frac{3}{2}}\, py+p^{2}y^{2}a_{2}-3p^{2}y^{2}b_{3}-p^{3}ya_{3}-p\, y^{3}b_{2}+2a_{3}(-py)^{\frac{3}{2}}-p^{2}ya_{1}-p\, y^{2}b_{1}}{2\left(-py\right)^{\frac{3}{2}}\, py}=0$$

Setting the numerator to zero gives

$$2b_2(-py)^{\frac{3}{2}}py+p^2y^2a_2-3p^2y^2b_3-p^3ya_3-py^3b_2+2a_3(-py)^{\frac{3}{2}}-p^2ya_1-py^2b_1=0 \quad \text{(6E)}$$

Since the PDE has radicals, simplifying gives

$$-py\left(2\sqrt{-py}\,pyb_2+p^2a_3-pya_2+3pyb_3+y^2b_2+2\sqrt{-py}\,a_3+pa_1+yb_1\right)=0$$

Looking at the above PDE shows the following are all the terms with $\{p, y\}$ in them.

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The above PDE (6E) now becomes

$$-v_1v_2\left(2v_3v_1v_2b_2-v_1v_2a_2+v_1^2a_3+v_2^2b_2+3v_1v_2b_3+v_1a_1+2v_3a_3+v_2b_1\right)=0 \quad \text{(7E)}$$

Collecting the above on the terms v_i introduced, and these are

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Equation (7E) now becomes

$$-v_2a_3v_1^3-2b_2v_3v_1^2v_2^2+(a_2-3b_3)\,v_1^2v_2^2-a_1v_1^2v_2-b_2v_1v_2^3-2v_3a_3v_1v_2-b_1v_1v_2^2=0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-a_1 &= 0\\
-2a_3 &= 0\\
-a_3 &= 0\\
-b_1 &= 0\\
-2b_2 &= 0\\
-b_2 &= 0\\
a_2-3b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 3b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

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Which results in

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Where in the above R_y, R_p, S_y, S_p are all partial derivatives and $\omega(y,p)$ is the right hand side of the original ode given by

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Evaluating all the partial derivatives gives

$$\begin{aligned}
R_y &= 1\\
R_p &= 0\\
S_y &= \frac{p^{\frac{3}{2}}\sqrt{y} + 4iy}{y\left(p^{\frac{3}{2}}\sqrt{y} + 3iy\right)}\\
S_p &= \frac{\sqrt{p}\,\sqrt{y}}{p^{\frac{3}{2}}\sqrt{y} + 3iy}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{p^{\frac{3}{2}}\sqrt{y}\,\sqrt{-py} + 4i\sqrt{-py}\,y + \sqrt{p}\,y^{\frac{3}{2}}}{y\left(p^{\frac{3}{2}}\sqrt{y} + 3iy\right)\sqrt{-py}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for y, p in terms of R, S from the result obtained earlier and simplifying. This gives

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To complete the solution, we just need to transform (4) back to y, p coordinates. This results in

$$-\frac{i\pi}{3}+\frac{2\ln\left(p(y)^{\frac{3}{2}}\, y^{\frac{3}{2}}+3iy^{2}\right)}{3}=\ln\left(y\right)+c_1$$

Which simplifies to

$$-\frac{i\pi}{3}+\frac{2\ln\left(p(y)^{\frac{3}{2}}\, y^{\frac{3}{2}}+3iy^{2}\right)}{3}=\ln\left(y\right)+c_1$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$\frac{i\pi}{3}+\frac{2\ln\left(-y'^{\frac{3}{2}}y^{\frac{3}{2}}+3iy^{2}\right)}{3}=\ln\left(y\right)+c_1$$

Solving the given ode for y' results in 6 differential equations to solve. Each one of these

will generate a solution. The equations generated are

$$y' = \frac{\left(\frac{(12y-4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{2}{3}}}{4} \tag{1}$$

$$y' = \left(-\frac{\left(\frac{(12y-4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\left(\frac{(12y-4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{1}{3}}}{4}\right)^{2} \tag{2}$$

$$y' = \left(-\frac{\left(\frac{(12y-4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\left(\frac{(12y-4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{1}{3}}}{4}\right)^{2} \tag{3}$$

$$y' = \frac{\left(\frac{(12y+4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{2}{3}}}{4} \tag{4}$$

$$y' = \left(-\frac{\left(\frac{(12y+4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\left(\frac{(12y+4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{1}{3}}}{4}\right)^{2} \tag{5}$$

$$y' = \left(-\frac{\left(\frac{(12y+4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\left(\frac{(12y+4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{1}{3}}}{4}\right)^{2} \tag{6}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{4}{\left(\frac{(12y-4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{2}{3}}} dy = \int dx$$

$$
4\left(\int^{y}\frac{1}{\left(\frac{(12\_a-4)\left(\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}}}d\_a\right)=x+c_3
$$

Solving equation (2)

Integrating both sides gives

$$
\int\frac{16}{\left(\frac{(12y-4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)^{2}}dy=\int dx
$$

$$
\frac{16\left(\int^{y}\frac{1}{\left(\frac{(12\_a-4)\left(\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}}=x+c_4
$$

Solving equation (3)

Integrating both sides gives

$$
\int\frac{16}{\left(\frac{(12y-4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)^{2}}dy=\int dx
$$

$$
\frac{16\left(\int^{y}\frac{1}{\left(\frac{(12\_a-4)\left(\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}}=x+c_5
$$

Solving equation (4)

Integrating both sides gives

$$\int \frac{4}{\left(\frac{(12y+4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{2}{3}}}dy = \int dx$$

$$\frac{3y^{\frac{4}{3}}\,\mathrm{hypergeom}\left(\left[\frac{2}{3},\frac{4}{3}\right],\left[\frac{7}{3}\right],-\frac{y\left(12\sqrt{-\frac{1}{y}}\,\sqrt{y}+12i\right)}{4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i}\right)}{\left(4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i\right)^{\frac{2}{3}}} = x + c_6$$

Solving equation (5)

Integrating both sides gives

$$\int \frac{16}{\left(\frac{(12y+4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)^2}dy = \int dx$$

$$\frac{12y^{\frac{4}{3}}\,\mathrm{hypergeom}\left(\left[\frac{2}{3},\frac{4}{3}\right],\left[\frac{7}{3}\right],-\frac{y\left(12\sqrt{-\frac{1}{y}}\,\sqrt{y}+12i\right)}{4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i}\right)}{\left(4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i\right)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)^2} = x + c_7$$

Solving equation (6)

Integrating both sides gives

$$\int \frac{16}{\left(\frac{(12y+4)\left(\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)^2}dy = \int dx$$

$$\frac{12y^{\frac{4}{3}}\,\mathrm{hypergeom}\left(\left[\frac{2}{3},\frac{4}{3}\right],\left[\frac{7}{3}\right],-\frac{y\left(12\sqrt{-\frac{1}{y}}\,\sqrt{y}+12i\right)}{4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i}\right)}{\left(4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i\right)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)^2} = x + c_8$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{i\pi}{3} + \frac{2\ln\left(y'^{\frac{3}{2}}y^{\frac{3}{2}} + 3iy^2\right)}{3} = \ln(y) + c_1$$

Solving the given ode for y' results in 6 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(\frac{(-12y-4)\left(-\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{2}{3}}}{4} \tag{1}$$

$$y' = \left(-\frac{\left(\frac{(-12y-4)\left(-\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\left(\frac{(-12y-4)\left(-\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{1}{3}}}{4}\right)^2 \tag{2}$$

$$y' = \left(-\frac{\left(\frac{(-12y-4)\left(-\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\left(\frac{(-12y-4)\left(-\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{1}{3}}}{4}\right)^2 \tag{3}$$

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Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{4}{\left(\frac{(-12y-4)\left(-\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{2}{3}}} dy = \int dx$$

$$\int^{y} \frac{4}{\left(\frac{(-12\_a-4)\left(-\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}}} d\_a = x + c_9$$

Solving equation (2)

Integrating both sides gives

$$\int \frac{16}{\left(\frac{(-12y-4)\left(-\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{2}{3}} \left(i\sqrt{3}-1\right)^{2}}dy = \int dx$$

$$\int^{y} \frac{16}{\left(\frac{(-12\_a-4)\left(-\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}} \left(i\sqrt{3}-1\right)^{2}}d\_a = x + \_C10$$

Solving equation (3)

Integrating both sides gives

$$\int \frac{16}{\left(\frac{(-12y-4)\left(-\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{2}{3}} \left(1+i\sqrt{3}\right)^{2}}dy = \int dx$$

$$\int^{y} \frac{16}{\left(\frac{(-12\_a-4)\left(-\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}} \left(1+i\sqrt{3}\right)^{2}}d\_a = x + \_C11$$

Solving equation (4)

Integrating both sides gives

$$\int \frac{4}{\left(\frac{(-12y+4)\left(-\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{2}{3}}}dy = \int dx$$

$$\int^{y} \frac{4}{\left(\frac{(-12\_a+4)\left(-\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}}}d\_a = x + \_C12$$

Solving equation (5)

Integrating both sides gives

$$\int \frac{16}{\left(\frac{(-12y+4)\left(-\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{2}{3}} \left(i\sqrt{3}-1\right)^{2}}dy = \int dx$$

$$\int^{y} \frac{16}{\left(\frac{(-12\_a+4)\left(-\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}} \left(i\sqrt{3}-1\right)^{2}}d\_a = x + \_C13$$

Solving equation (6)

Integrating both sides gives

$$\int \frac{16}{\left(\frac{(-12y+4)\left(-\sqrt{-\frac{1}{y}}\,\sqrt{y}+i\right)}{\sqrt{y}}\right)^{\frac{2}{3}} \left(1+i\sqrt{3}\right)^{2}}dy = \int dx$$

$$\int^{y} \frac{16}{\left(\frac{(-12\_a+4)\left(-\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}} \left(1+i\sqrt{3}\right)^{2}}d\_a = x + \_C14$$

Summary
The solution(s) found are the following

$$4\left(\int^{y} \frac{1}{\left(\frac{(12\_a-4)\left(\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}}}d\_a\right) = x+c_3 \tag{1}$$

$$\frac{16\left(\int^{y} \frac{1}{\left(\frac{(12\_a-4)\left(\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}} = x+c_4 \tag{2}$$

$$\frac{16\left(\int^{y} \frac{1}{\left(\frac{(12\_a-4)\left(\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}} = x+c_5 \tag{3}$$

$$\frac{3y^{\frac{4}{3}}\,\mathrm{hypergeom}\left(\left[\frac{2}{3},\frac{4}{3}\right],\left[\frac{7}{3}\right],-\frac{y\left(12\sqrt{-\frac{1}{y}}\,\sqrt{y}+12i\right)}{4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i}\right)}{\left(4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i\right)^{\frac{2}{3}}} = x+c_6 \tag{4}$$

$$\frac{12y^{\frac{4}{3}}\,\mathrm{hypergeom}\left(\left[\frac{2}{3},\frac{4}{3}\right],\left[\frac{7}{3}\right],-\frac{y\left(12\sqrt{-\frac{1}{y}}\,\sqrt{y}+12i\right)}{4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i}\right)}{\left(4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i\right)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)^{2}} = x+c_7 \tag{5}$$

$$\frac{12y^{\frac{4}{3}}\,\mathrm{hypergeom}\left(\left[\frac{2}{3},\frac{4}{3}\right],\left[\frac{7}{3}\right],-\frac{y\left(12\sqrt{-\frac{1}{y}}\,\sqrt{y}+12i\right)}{4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i}\right)}{\left(4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i\right)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)^{2}} = x+c_8 \tag{6}$$

$$\int^{y} \frac{4}{\left(\frac{(-12\_a-4)\left(-\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}}}d\_a = x+c_9 \tag{7}$$

$$\int^{y} \frac{16}{\left(\frac{(-12\_a-4)\left(-\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)^{2}}d\_a = x+\_C10 \tag{8}$$

$$\int^{y} \frac{16}{\left(\frac{(-12\_a-4)\left(-\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)^{2}}d\_a = x+\_C11 \tag{9}$$

$$\int^{y} \frac{4}{\left(\frac{(-12\_a+4)\left(-\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}}}d\_a = x+\_C12 \tag{10}$$

Verification of solutions

$$4\left(\int^{y}\frac{1}{\left(\frac{(12\_a-4)\left(\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}}}d\_a\right)=x+c_3$$

Verified OK.

$$\frac{16\left(\int^{y}\frac{1}{\left(\frac{(12\_a-4)\left(\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}}=x+c_4$$

Verified OK.

$$\frac{16\left(\int^{y}\frac{1}{\left(\frac{(12\_a-4)\left(\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\sqrt{\_a}}\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}}=x+c_5$$

Verified OK.

$$\frac{3y^{\frac{4}{3}}\operatorname{hypergeom}\left(\left[\frac{2}{3},\frac{4}{3}\right],\left[\frac{7}{3}\right],-\frac{y\left(12\sqrt{-\frac{1}{y}}\,\sqrt{y}+12i\right)}{4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i}\right)}{\left(4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i\right)^{\frac{2}{3}}}=x+c_6$$

Verified OK.

$$\frac{12y^{\frac{4}{3}}\operatorname{hypergeom}\left(\left[\frac{2}{3},\frac{4}{3}\right],\left[\frac{7}{3}\right],-\frac{y\left(12\sqrt{-\frac{1}{y}}\,\sqrt{y}+12i\right)}{4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i}\right)}{\left(4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i\right)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)^{2}}=x+c_7$$

Verified OK.

$$\frac{12y^{\frac{4}{3}}\operatorname{hypergeom}\left(\left[\frac{2}{3},\frac{4}{3}\right],\left[\frac{7}{3}\right],-\frac{y\left(12\sqrt{-\frac{1}{y}}\,\sqrt{y}+12i\right)}{4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i}\right)}{\left(4\sqrt{-\frac{1}{y}}\,\sqrt{y}+4i\right)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)^{2}}=x+c_8$$

Verified OK.

$$\int^{y}\frac{4}{\left(\frac{(-12\_a-4)\left(-\sqrt{-\frac{1}{\_a}}\,\sqrt{\_a}+i\right)}{\cdots}\right)^{\frac{2}{3}}}d\_a=x+c_9$$

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      -> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(-_b(_a)*_a)^(1/2)/_a = 0,
         symmetry methods on request
      `, `1st order, trying reduction of order with given symmetries:`[_a, 1/3*_b]
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 271

```
dsolve(y(x)*diff(y(x),x$2)^2+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1$$
$$y(x) = 0$$
$$-\left(\int^{y(x)}\frac{\_a}{\left(\_a^{\frac{3}{2}}\left(c_1-3\sqrt{\_a}\right)\right)^{\frac{2}{3}}}d\_a\right)-x-c_2 = 0$$
$$-\left(\int^{y(x)}\frac{\_a}{\left(\_a^{\frac{3}{2}}\left(c_1+3\sqrt{\_a}\right)\right)^{\frac{2}{3}}}d\_a\right)-x-c_2 = 0$$
$$\frac{-4\left(\int^{y(x)}\frac{\_a}{\left(\_a^{\frac{3}{2}}\left(c_1-3\sqrt{\_a}\right)\right)^{\frac{2}{3}}}d\_a\right)+2i(-x-c_2)\sqrt{3}+2x+2c_2}{\left(-i\sqrt{3}-1\right)^2} = 0$$
$$\frac{-4\left(\int^{y(x)}\frac{\_a}{\left(\_a^{\frac{3}{2}}\left(c_1-3\sqrt{\_a}\right)\right)^{\frac{2}{3}}}d\_a\right)+2i(x+c_2)\sqrt{3}+2x+2c_2}{\left(1-i\sqrt{3}\right)^2} = 0$$
$$\frac{-4\left(\int^{y(x)}\frac{\_a}{\left(\_a^{\frac{3}{2}}\left(c_1+3\sqrt{\_a}\right)\right)^{\frac{2}{3}}}d\_a\right)+2i(-x-c_2)\sqrt{3}+2x+2c_2}{\left(-i\sqrt{3}-1\right)^2} = 0$$
$$\frac{-4\left(\int^{y(x)}\frac{\_a}{\left(\_a^{\frac{3}{2}}\left(c_1+3\sqrt{\_a}\right)\right)^{\frac{2}{3}}}d\_a\right)+2i(x+c_2)\sqrt{3}+2x+2c_2}{\left(1-i\sqrt{3}\right)^2} = 0$$

✓ Solution by Mathematica

Time used: 61.116 (sec). Leaf size: 23861

```
DSolve[y[x]*y''[x]^2+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

1.46 problem 46

1.46.1 Solving as second order ode missing x ode 544

Internal problem ID [7435]
Internal file name [`OUTPUT/6402_Sunday_June_05_2022_04_43_22_PM_4687353/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 46.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{yy''^2 + y'^3 = 0}$$

1.46.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)^2 \left(\frac{d}{dy}p(y)\right)^2 + p(y)^3 = 0$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\frac{d}{dy}p(y) = \frac{\sqrt{-p(y)\, y}}{y} \tag{1}$$

$$\frac{d}{dy}p(y) = -\frac{\sqrt{-p(y)\, y}}{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$\begin{aligned}\frac{d}{dy}p(y) &= \frac{\sqrt{-py}}{y}\\ \frac{d}{dy}p(y) &= \omega(y,p)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + ya_2 + a_1 \tag{1E}$$

$$\eta = pb_3 + yb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2 + \frac{\sqrt{-py}\,(b_3 - a_2)}{y} + \frac{pa_3}{y}\\ &- \left(-\frac{\sqrt{-py}}{y^2} - \frac{p}{2y\sqrt{-py}}\right)(pa_3 + ya_2 + a_1) + \frac{pb_3 + yb_2 + b_1}{2\sqrt{-py}} = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2\sqrt{-py}\,pa_3+2\sqrt{-py}\,yb_2-p^2a_3+pya_2-pyb_3+y^2b_2-pa_1+yb_1}{2y\sqrt{-py}}=0$$

Setting the numerator to zero gives

$$2\sqrt{-py}\,pa_3+2\sqrt{-py}\,yb_2-p^2a_3+pya_2-pyb_3+y^2b_2-pa_1+yb_1=0 \tag{6E}$$

Simplifying the above gives

$$-p^2ya_3-p\,y^2b_3+2pa_3y\sqrt{-py}+2b_2y^2\sqrt{-py}+p\,y^2a_2+y^3b_2-pya_1+y^2b_1=0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{p,y\}$ in them.

$$\{p,y,\sqrt{-py}\}$$

The following substitution is now made to be able to collect on all terms with $\{p,y\}$ in them

$$\{p=v_1,y=v_2,\sqrt{-py}=v_3\}$$

The above PDE (6E) now becomes

$$v_1v_2^2a_2-v_1^2v_2a_3+2v_1a_3v_2v_3+v_2^3b_2+2b_2v_2^2v_3-v_1v_2^2b_3-v_1v_2a_1+v_2^2b_1=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2,v_3\}$$

Equation (7E) now becomes

$$-v_1^2v_2a_3+(-b_3+a_2)\,v_1v_2^2+2v_1a_3v_2v_3-v_1v_2a_1+v_2^3b_2+2b_2v_2^2v_3+v_2^2b_1=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1&=0\\ b_2&=0\\ -a_1&=0\\ -a_3&=0\\ 2a_3&=0\\ 2b_2&=0\\ -b_3+a_2&=0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= y \\
\eta &= p
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(y,p)\,\xi \\
&= p - \left(\frac{\sqrt{-py}}{y}\right)(y) \\
&= -\sqrt{-py} + p \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y,p) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial p}\right) S(y,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{-py} + p} dy
\end{aligned}
$$

Which results in

$$S = \ln(y+p) + \ln\left(\sqrt{-py}+y\right) - \ln\left(\sqrt{-py}-y\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y,p)S_p}{R_y + \omega(y,p)R_p} \tag{2}$$

Where in the above R_y, R_p, S_y, S_p are all partial derivatives and $\omega(y,p)$ is the right hand side of the original ode given by

$$\omega(y,p) = \frac{\sqrt{-py}}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_y &= 1 \\
R_p &= 0 \\
S_y &= \frac{\sqrt{y}\left(\sqrt{p}+i\sqrt{y}\right)}{\left(i\sqrt{p}\,\sqrt{y}+y\right)\left(\sqrt{p}\,\sqrt{y}+iy\right)} \\
S_p &= \frac{\left(i\sqrt{p}-\sqrt{y}\right)y}{\sqrt{p}\,\left(i\sqrt{p}\,\sqrt{y}+y\right)\left(\sqrt{p}\,\sqrt{y}+iy\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\left(\sqrt{-py}+y\right)\sqrt{p}+i(\sqrt{-py}-p)\sqrt{y}}{\sqrt{p}\,y\,(y+p)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for y, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to y, p coordinates. This results in

$$\ln\left(y+p(y)\right)-i\pi+\ln\left(iy-\sqrt{p\left(y\right)}\,\sqrt{y}\right)-\ln\left(\sqrt{p\left(y\right)}\,\sqrt{y}+iy\right)=c_1$$

Which simplifies to

$$\ln\left(y+p(y)\right)-i\pi+\ln\left(iy-\sqrt{p\left(y\right)}\,\sqrt{y}\right)-\ln\left(\sqrt{p\left(y\right)}\,\sqrt{y}+iy\right)=c_1$$

Solving equation (2)

Writing the ode as

$$\begin{aligned}\frac{d}{dy}p(y)&=-\frac{\sqrt{-py}}{y}\\ \frac{d}{dy}p(y)&=\omega(y,p)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y+\omega(\eta_p-\xi_y)-\omega^2\xi_p-\omega_y\xi-\omega_p\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=pa_3+ya_2+a_1 \tag{1E}$$

$$\eta=pb_3+yb_2+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2-\frac{\sqrt{-py}\,(b_3-a_2)}{y}+\frac{pa_3}{y}-\left(\frac{\sqrt{-py}}{y^2}+\frac{p}{2y\sqrt{-py}}\right)(pa_3+ya_2+a_1)-\frac{pb_3+yb_2+b_1}{2\sqrt{-py}}=0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2\sqrt{-py}\,pa_3+2\sqrt{-py}\,yb_2+p^2a_3-pya_2+pyb_3-y^2b_2+pa_1-yb_1}{2y\sqrt{-py}}=0$$

Setting the numerator to zero gives

$$2\sqrt{-py}\, pa_3 + 2\sqrt{-py}\, yb_2 + p^2 a_3 - pya_2 + pyb_3 - y^2 b_2 + pa_1 - yb_1 = 0 \tag{6E}$$

Simplifying the above gives

$$p^2 y a_3 + p\, y^2 b_3 + 2pa_3 y\sqrt{-py} + 2b_2 y^2 \sqrt{-py} - p\, y^2 a_2 - y^3 b_2 + pya_1 - y^2 b_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{p, y\}$ in them.

$$\{p, y, \sqrt{-py}\}$$

The following substitution is now made to be able to collect on all terms with $\{p, y\}$ in them

$$\{p = v_1, y = v_2, \sqrt{-py} = v_3\}$$

The above PDE (6E) now becomes

$$-v_1 v_2^2 a_2 + v_1^2 v_2 a_3 + 2v_1 a_3 v_2 v_3 - v_2^3 b_2 + 2b_2 v_2^2 v_3 + v_1 v_2^2 b_3 + v_1 v_2 a_1 - v_2^2 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$v_1^2 v_2 a_3 + (b_3 - a_2)\, v_1 v_2^2 + 2v_1 a_3 v_2 v_3 + v_1 v_2 a_1 - v_2^3 b_2 + 2b_2 v_2^2 v_3 - v_2^2 b_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_1 &= 0 \\
a_3 &= 0 \\
2a_3 &= 0 \\
-b_1 &= 0 \\
-b_2 &= 0 \\
2b_2 &= 0 \\
b_3 - a_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= y \\
\eta &= p
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(y,p)\,\xi \\
&= p - \left(-\frac{\sqrt{-py}}{y}\right)(y) \\
&= \sqrt{-py} + p \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y,p) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial p}\right) S(y,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\sqrt{-py} + p} dy
\end{aligned}
$$

Which results in

$$S = \ln(y+p) - \ln\left(\sqrt{-py}+y\right) + \ln\left(\sqrt{-py}-y\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y,p)S_p}{R_y + \omega(y,p)R_p} \tag{2}$$

Where in the above R_y, R_p, S_y, S_p are all partial derivatives and $\omega(y,p)$ is the right hand side of the original ode given by

$$\omega(y,p) = -\frac{\sqrt{-py}}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_y &= 1 \\
R_p &= 0 \\
S_y &= \frac{\left(i\sqrt{y}-\sqrt{p}\right)\sqrt{y}}{\left(i\sqrt{p}\,\sqrt{y}+y\right)\left(\sqrt{p}\,\sqrt{y}+iy\right)} \\
S_p &= \frac{y\left(i\sqrt{p}+\sqrt{y}\right)}{\sqrt{p}\,\left(i\sqrt{p}\,\sqrt{y}+y\right)\left(\sqrt{p}\,\sqrt{y}+iy\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\left(-\sqrt{-py}+y\right)\sqrt{p}+i\sqrt{y}\left(\sqrt{-py}+p\right)}{\sqrt{p}\,y\,(y+p)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for y, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to y, p coordinates. This results in

$$\ln(y+p(y))+i\pi-\ln\left(iy-\sqrt{p(y)}\,\sqrt{y}\right)+\ln\left(\sqrt{p(y)}\,\sqrt{y}+iy\right)=c_1$$

Which simplifies to

$$\ln(y+p(y))+i\pi-\ln\left(iy-\sqrt{p(y)}\,\sqrt{y}\right)+\ln\left(\sqrt{p(y)}\,\sqrt{y}+iy\right)=c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\ln(y+y')-i\pi+\ln\left(iy-\sqrt{y'}\,\sqrt{y}\right)-\ln\left(\sqrt{y'}\,\sqrt{y}+iy\right)=c_1$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \left(\frac{\left(18i\sqrt{y}\,e^{c_1}+8iy^{\frac{3}{2}}+3\sqrt{-48y^2e^{c_1}-24y\,e^{2c_1}-3\,e^{3c_1}}\right)^{\frac{1}{3}}}{3}-\frac{3\left(\frac{4y}{9}-\frac{e^{c_1}}{3}\right)}{\left(18i\sqrt{y}\,e^{c_1}+8iy^{\frac{3}{2}}+3\sqrt{-48y^2e^{c_1}-24y\right.} \tag{1}$$

$$y' = \left(-\frac{\left(18i\sqrt{y}\,e^{c_1}+8iy^{\frac{3}{2}}+3\sqrt{-48y^2e^{c_1}-24y\,e^{2c_1}-3\,e^{3c_1}}\right)^{\frac{1}{3}}}{6}+\frac{\frac{2y}{3}-\frac{e^{c_1}}{2}}{\left(18i\sqrt{y}\,e^{c_1}+8iy^{\frac{3}{2}}+3\sqrt{-48y^2e^{c_1}-2\right.} \tag{2}$$

$$y' = \left(-\frac{\left(18i\sqrt{y}\,e^{c_1}+8iy^{\frac{3}{2}}+3\sqrt{-48y^2e^{c_1}-24y\,e^{2c_1}-3\,e^{3c_1}}\right)^{\frac{1}{3}}}{6}+\frac{\frac{2y}{3}-\frac{e^{c_1}}{2}}{\left(18i\sqrt{y}\,e^{c_1}+8iy^{\frac{3}{2}}+3\sqrt{-48y^2e^{c_1}-2\right.} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{9\left(18i\sqrt{y}\,\mathrm{e}^{c_1}+8iy^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}y^2-24y\,\mathrm{e}\cdots}\right.}{\left(-i\sqrt{y}\,\left(18i\sqrt{y}\,\mathrm{e}^{c_1}+8iy^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}y^2-24y\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{1}{3}}-\left(18i\sqrt{y}\,\mathrm{e}^{c_1}+8i\cdots\right.\right.}$$

$$\int^y \frac{9\left(18i\sqrt{\_a}\,\mathrm{e}^{c_1}+8i\_a^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}\_a^2-24\_a\,\mathrm{e}^{2c_1}-3\cdots}\right.}{\left(-i\sqrt{\_a}\,\left(18i\sqrt{\_a}\,\mathrm{e}^{c_1}+8i\_a^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}\_a^2-24\_a\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{1}{3}}-\left(18i\sqrt{\_a}\,\mathrm{e}^{c_1}+8i\_a^{\frac{3}{2}}+3\cdots\right.\right.}$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\int \frac{}{\left(-i\sqrt{3}\,\left(18i\sqrt{y}\,\mathrm{e}^{c_1}+8iy^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}y^2-24y\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}}+3i\sqrt{3}\,\mathrm{e}^{c\cdots}\right.}$$

$$\int^y \frac{36\left(18i\sqrt{\_\cdots}\right.}{\left(-i\sqrt{3}\,\left(18i\sqrt{\_a}\,\mathrm{e}^{c_1}+8i\_a^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}\_a^2-24\_a\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}}+3i\sqrt{3}\,\mathrm{e}^{c_1}-4i\sqrt{3}\,\_a-2i\sqrt{\cdots}\right.}$$

<u>Solving equation (3)</u>

Integrating both sides gives

$$\int \frac{}{\left(-i\sqrt{3}\,\left(18i\sqrt{y}\,\mathrm{e}^{c_1}+8iy^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}y^2-24y\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}}+3i\sqrt{3}\,\mathrm{e}^{c\cdots}\right.}$$

$$\int^y \frac{36\left(18i\sqrt{\_\cdots}\right.}{\left(-i\sqrt{3}\,\left(18i\sqrt{\_a}\,\mathrm{e}^{c_1}+8i\_a^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}\_a^2-24\_a\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}}+3i\sqrt{3}\,\mathrm{e}^{c_1}-4i\sqrt{3}\,\_a+2i\sqrt{\cdots}\right.}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\ln(y+y')+i\pi-\ln\left(iy-\sqrt{y'}\,\sqrt{y}\right)+\ln\left(\sqrt{y'}\,\sqrt{y}+iy\right)=c_1$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \left(\frac{\left(-18i\sqrt{y}\,\mathrm{e}^{c_1} - 8iy^{\frac{3}{2}} + 3\sqrt{-48y^2\mathrm{e}^{c_1} - 24y\,\mathrm{e}^{2c_1} - 3\,\mathrm{e}^{3c_1}}\right)^{\frac{1}{3}}}{3} - \frac{3\left(\frac{4y}{9} - \frac{\mathrm{e}^{c_1}}{3}\right)}{\left(-18i\sqrt{y}\,\mathrm{e}^{c_1} - 8iy^{\frac{3}{2}} + 3\sqrt{-48y^2\mathrm{e}^{c_1} - \right.} \tag{1}$$

$$y' = \left(-\frac{\left(-18i\sqrt{y}\,\mathrm{e}^{c_1} - 8iy^{\frac{3}{2}} + 3\sqrt{-48y^2\mathrm{e}^{c_1} - 24y\,\mathrm{e}^{2c_1} - 3\,\mathrm{e}^{3c_1}}\right)^{\frac{1}{3}}}{6} + \frac{\frac{2y}{3} - \frac{\mathrm{e}^{c_1}}{2}}{\left(-18i\sqrt{y}\,\mathrm{e}^{c_1} - 8iy^{\frac{3}{2}} + 3\sqrt{-48y^2\mathrm{e}^{c_1}\right.} \tag{2}$$

$$y' = \left(-\frac{\left(-18i\sqrt{y}\,\mathrm{e}^{c_1} - 8iy^{\frac{3}{2}} + 3\sqrt{-48y^2\mathrm{e}^{c_1} - 24y\,\mathrm{e}^{2c_1} - 3\,\mathrm{e}^{3c_1}}\right)^{\frac{1}{3}}}{6} + \frac{\frac{2y}{3} - \frac{\mathrm{e}^{c_1}}{2}}{\left(-18i\sqrt{y}\,\mathrm{e}^{c_1} - 8iy^{\frac{3}{2}} + 3\sqrt{-48y^2\mathrm{e}^{c_1}\right.} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{9\left(-18i\sqrt{y}\,\mathrm{e}^{c_1} - 8iy^{\frac{3}{2}} + 3\sqrt{-48\,\mathrm{e}^{c_1}y^2 - 24\right.}{\left(i\left(-18i\sqrt{y}\,\mathrm{e}^{c_1} - 8iy^{\frac{3}{2}} + 3\sqrt{-48\,\mathrm{e}^{c_1}y^2 - 24y\,\mathrm{e}^{2c_1} - 3\,\mathrm{e}^{3c_1}}\right)^{\frac{1}{3}}\sqrt{y} - \left(-18i\sqrt{y}\,\mathrm{e}^{c_1} - \right.\right.}$$

$$\int^{y} \frac{9\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1} - 8i\_a^{\frac{3}{2}} + 3\sqrt{-48\,\mathrm{e}^{c_1}\_a^2 - 24\_a\,\mathrm{e}^{2c_1} - \right.}{\left(i\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1} - 8i\_a^{\frac{3}{2}} + 3\sqrt{-48\,\mathrm{e}^{c_1}\_a^2 - 24\_a\,\mathrm{e}^{2c_1} - 3\,\mathrm{e}^{3c_1}}\right)^{\frac{1}{3}}\sqrt{\_a} - \left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1} - 8i\_a^{\frac{3}{2}} + \right.\right.}$$

Solving equation (2)

Integrating both sides gives

$$\int \frac{}{\left(i\sqrt{3}\,\left(-18i\sqrt{y}\,\mathrm{e}^{c_1}-8iy^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}y^2-24y\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}}-2i\left(-18\right.\right.}$$

$$\int^{y} \frac{36\left(-18i\right.}{\left(i\sqrt{3}\,\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1}-8i\_a^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}\_a^2-24\_a\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}}-2i\sqrt{\_a}\,\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1}-8i\_\right.\right.}$$

Solving equation (3)

Integrating both sides gives

$$\int \frac{}{\left(i\sqrt{3}\,\left(-18i\sqrt{y}\,\mathrm{e}^{c_1}-8iy^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}y^2-24y\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}}+2i\left(-18\right.\right.}$$

$$\int^{y} \frac{36\left(-18i\right.}{\left(i\sqrt{3}\,\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1}-8i\_a^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}\_a^2-24\_a\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}}+2i\sqrt{\_a}\,\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1}-8i\_\right.\right.}$$

Summary

The solution(s) found are the following

$$\int^{y} \frac{9\left(18i\sqrt{\_a}\,\mathrm{e}^{c_1} + 8i\_a^{\frac{3}{2}} + 3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2} - 24\_a\,\mathrm{e}^{2c_1} - 3 \cdots\right.}{\left(-i\sqrt{\_a}\,\left(18i\sqrt{\_a}\,\mathrm{e}^{c_1} + 8i\_a^{\frac{3}{2}} + 3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2} - 24\_a\,\mathrm{e}^{2c_1} - 3\,\mathrm{e}^{3c_1}}\right)^{\frac{1}{3}} - \left(18i\sqrt{\_a}\,\mathrm{e}^{c_1} + 8i\_a^{\frac{3}{2}} + 3 \cdots\right.\right.} = x + c_3 \tag{1}$$

$$\int^{y} \frac{36\left(18i\sqrt{\cdots}\right.}{\left(-i\sqrt{3}\,\left(18i\sqrt{\_a}\,\mathrm{e}^{c_1} + 8i\_a^{\frac{3}{2}} + 3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2} - 24\_a\,\mathrm{e}^{2c_1} - 3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}} + 3i\sqrt{3}\,\mathrm{e}^{c_1} - 4i\sqrt{3}\_a - 2i\sqrt{\cdots}\right.} = x + c_4 \tag{2}$$

$$\int^{y} \frac{36\left(18i\sqrt{\cdots}\right.}{\left(-i\sqrt{3}\,\left(18i\sqrt{\_a}\,\mathrm{e}^{c_1} + 8i\_a^{\frac{3}{2}} + 3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2} - 24\_a\,\mathrm{e}^{2c_1} - 3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}} + 3i\sqrt{3}\,\mathrm{e}^{c_1} - 4i\sqrt{3}\_a + 2i\sqrt{\cdots}\right.} = x + c_5 \tag{3}$$

$$\int^{y} \frac{9\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1} - 8i\_a^{\frac{3}{2}} + 3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2} - 24\_a\,\mathrm{e}^{2c_1} - \cdots\right.}{\left(i\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1} - 8i\_a^{\frac{3}{2}} + 3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2} - 24\_a\,\mathrm{e}^{2c_1} - 3\,\mathrm{e}^{3c_1}}\right)^{\frac{1}{3}}\sqrt{\_a} - \left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1} - 8i\_a^{\frac{3}{2}} + \cdots\right.\right.} = x + c_6 \tag{4}$$

$$\int^{y} \frac{36\left(-18i\cdots\right.}{\left(i\sqrt{3}\,\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1} - 8i\_a^{\frac{3}{2}} + 3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2} - 24\_a\,\mathrm{e}^{2c_1} - 3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}} - 2i\sqrt{\_a}\,\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1} - 8i\_\cdots\right.\right.} = x + c_7 \tag{5}$$

$$\int^{y} \frac{36\left(-18i\cdots\right.}{\left(i\sqrt{3}\,\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1} - 8i\_a^{\frac{3}{2}} + 3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2} - 24\_a\,\mathrm{e}^{2c_1} - 3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}} + 2i\sqrt{\_a}\,\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1} - 8i\_\cdots\right.\right.} = x + c_8 \tag{6}$$

Verification of solutions

$$\int^{y}\frac{9\left(18i\sqrt{\_a}\,\mathrm{e}^{c_1}+8i\_a^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2}-24\_a\,\mathrm{e}^{2c_1}-3\cdots}\right.}{\left(-i\sqrt{\_a}\,\left(18i\sqrt{\_a}\,\mathrm{e}^{c_1}+8i\_a^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2}-24\_a\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{1}{3}}-\left(18i\sqrt{\_a}\,\mathrm{e}^{c_1}+8i\_a^{\frac{3}{2}}+3\cdots\right.\right.}$$
$$=x+c_3$$

Verified OK.

$$\int^{y}\frac{36\left(18i\sqrt{\_\cdots}\right.}{\left(-i\sqrt{3}\,\left(18i\sqrt{\_a}\,\mathrm{e}^{c_1}+8i\_a^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2}-24\_a\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}}+3i\sqrt{3}\,\mathrm{e}^{c_1}-4i\sqrt{3}\,\_a-2i\sqrt{\_\cdots}\right.}$$
$$=x+c_4$$

Verified OK.

$$\int^{y}\frac{36\left(18i\sqrt{\_\cdots}\right.}{\left(-i\sqrt{3}\,\left(18i\sqrt{\_a}\,\mathrm{e}^{c_1}+8i\_a^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2}-24\_a\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}}+3i\sqrt{3}\,\mathrm{e}^{c_1}-4i\sqrt{3}\,\_a+2i\sqrt{\_\cdots}\right.}$$
$$=x+c_5$$

Verified OK.

$$\int^{y}\frac{9\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1}-8i\_a^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2}-24\_a\,\mathrm{e}^{2c_1}-\cdots}\right.}{\left(i\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1}-8i\_a^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2}-24\_a\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{1}{3}}\sqrt{\_a}-\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1}-8i\_a^{\frac{3}{2}}+\cdots\right.\right.}$$
$$=x+c_6$$

Verified OK.

$$\int^{y}\frac{36\left(-18i\cdots\right.}{\left(i\sqrt{3}\,\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1}-8i\_a^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2}-24\_a\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}}-2i\sqrt{\_a}\,\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1}-8i\_\cdots\right.\right.}$$
$$=x+c_7$$

Verified OK.

$$\int^{y}\frac{36\left(-18i\cdots\right.}{\left(i\sqrt{3}\,\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1}-8i\_a^{\frac{3}{2}}+3\sqrt{-48\,\mathrm{e}^{c_1}\_a^{2}-24\_a\,\mathrm{e}^{2c_1}-3\,\mathrm{e}^{3c_1}}\right)^{\frac{2}{3}}+2i\sqrt{\_a}\,\left(-18i\sqrt{\_a}\,\mathrm{e}^{c_1}-8i\_\cdots\right.\right.}$$
$$=x+c_8$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      Try integration with the canonical coordinates of the symmetry [0, y]
      -> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(-_b(_a))^(3/2)-_b(_a)^2, _b(_a)
         symmetry methods on request
      `, `1st order, trying reduction of order with given symmetries:`[1, 0]
```

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 166

```
dsolve(y(x)*diff(y(x),x$2)^2+diff(y(x),x)^3=0,y(x), singsol=all)
```

$$y(x) = c_1$$
$$y(x) = 0$$
$$y(x) = \frac{c_2\left(\text{LambertW}\left(c_1 e^{-1+\frac{x}{2}}\right)+1\right)^2}{\text{LambertW}\left(c_1 e^{-1+\frac{x}{2}}\right)^2}$$
$$y(x) = \frac{c_2\left(\text{LambertW}\left(-c_1 e^{-1+\frac{x}{2}}\right)+1\right)^2}{\text{LambertW}\left(-c_1 e^{-1+\frac{x}{2}}\right)^2}$$
$$y(x) = e^{-\left(\int e^{2\,\text{RootOf}\left(e^{-Z}\ln\left(\left(e^{-Z}+1\right)^2\right)+c_1 e^{-Z}-2\,e^{-Z}\_Z+x\,e^{-Z}+\ln\left(\left(e^{-Z}+1\right)^2\right)+c_1-2\_Z+x-2\right)}dx\right)-2\left(\int e^{\text{RootOf}\left(e^{-Z}\ln\left(\left(e^{-Z}+1\right)^2\right)+c_1 e^{-Z}-2\,e^{-Z}\right.}\right.}$$

✓ Solution by Mathematica

Time used: 2.165 (sec). Leaf size: 361

```
DSolve[y[x]*y''[x]^2+y'[x]^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[-4\left(\frac{1}{2}\log\left(2\sqrt{\#1}-ic_1\right)-\frac{ic_1}{2\left(2\sqrt{\#1}-ic_1\right)}\right)\&\right][x+c_2]$$
$$y(x) \to \text{InverseFunction}\left[-4\left(\frac{ic_1}{2\left(2\sqrt{\#1}+ic_1\right)}+\frac{1}{2}\log\left(2\sqrt{\#1}+ic_1\right)\right)\&\right][x+c_2]$$
$$y(x) \to \text{InverseFunction}\left[-4\left(\frac{1}{2}\log\left(2\sqrt{\#1}-i(-c_1)\right)-\frac{i(-c_1)}{2\left(2\sqrt{\#1}-i(-c_1)\right)}\right)\&\right][x+c_2]$$
$$y(x) \to \text{InverseFunction}\left[-4\left(\frac{i(-c_1)}{2\left(2\sqrt{\#1}+i(-1)c_1\right)}+\frac{1}{2}\log\left(2\sqrt{\#1}+i(-1)c_1\right)\right)\&\right][x+c_2]$$
$$y(x) \to \text{InverseFunction}\left[-4\left(\frac{1}{2}\log\left(2\sqrt{\#1}-ic_1\right)-\frac{ic_1}{2\left(2\sqrt{\#1}-ic_1\right)}\right)\&\right][x+c_2]$$
$$y(x) \to \text{InverseFunction}\left[-4\left(\frac{ic_1}{2\left(2\sqrt{\#1}+ic_1\right)}+\frac{1}{2}\log\left(2\sqrt{\#1}+ic_1\right)\right)\&\right][x+c_2]$$

1.47 problem 47

1.47.1 Solving as second order ode missing x ode 561

Internal problem ID [7436]
Internal file name [`OUTPUT/6403_Sunday_June_05_2022_04_43_43_PM_35881863/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 47.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{y^2 {y''}^2 + y' = 0}$$

1.47.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$y^2 p(y)^2 \left(\frac{d}{dy}p(y)\right)^2 + p(y) = 0$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\frac{d}{dy}p(y) = -\frac{1}{\sqrt{-p(y)\, y}} \tag{1}$$

$$\frac{d}{dy}p(y) = \frac{1}{\sqrt{-p(y)\, y}} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= -\frac{1}{\sqrt{-p}\, y} \end{aligned}$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = \frac{1}{\sqrt{-p}}$. Integrating both sides gives

$$\frac{1}{\frac{1}{\sqrt{-p}}}\, dp = -\frac{1}{y}\, dy$$

$$\int \frac{1}{\frac{1}{\sqrt{-p}}}\, dp = \int -\frac{1}{y}\, dy$$

$$-\frac{2(-p)^{\frac{3}{2}}}{3} = -\ln(y) + c_1$$

The solution is

$$-\frac{2(-p(y))^{\frac{3}{2}}}{3} + \ln(y) - c_1 = 0$$

<u>Solving equation (2)</u>

In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{1}{\sqrt{-p}\, y} \end{aligned}$$

Where $f(y) = \frac{1}{y}$ and $g(p) = \frac{1}{\sqrt{-p}}$. Integrating both sides gives

$$\frac{1}{\frac{1}{\sqrt{-p}}}\,dp = \frac{1}{y}\,dy$$

$$\int \frac{1}{\frac{1}{\sqrt{-p}}}\,dp = \int \frac{1}{y}\,dy$$

$$-\frac{2(-p)^{\frac{3}{2}}}{3} = \ln(y) + c_2$$

The solution is

$$-\frac{2(-p(y))^{\frac{3}{2}}}{3} - \ln(y) - c_2 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{2(-y')^{\frac{3}{2}}}{3} + \ln(y) - c_1 = 0$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{(12\ln(y) - 12c_1)^{\frac{2}{3}}}{4} \tag{1}$$

$$y' = -\left(-\frac{(12\ln(y) - 12c_1)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\,(12\ln(y) - 12c_1)^{\frac{1}{3}}}{4}\right)^2 \tag{2}$$

$$y' = -\left(-\frac{(12\ln(y) - 12c_1)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\,(12\ln(y) - 12c_1)^{\frac{1}{3}}}{4}\right)^2 \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int -\frac{4}{(12\ln(y) - 12c_1)^{\frac{2}{3}}}dy = \int dx$$

$$-4\left(\int^y \frac{1}{(12\ln(\_a) - 12c_1)^{\frac{2}{3}}}d\_a\right) = x + c_3$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{16}{\left(12\ln\left(y\right)-12c_1\right)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)^{2}}dy = \int dx$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(12\ln\left(\_a\right)-12c_1\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}} = x + c_4$$

Solving equation (3)

Integrating both sides gives

$$\int -\frac{16}{\left(12\ln\left(y\right)-12c_1\right)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)^{2}}dy = \int dx$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(12\ln\left(\_a\right)-12c_1\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}} = x + c_5$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{2(-y')^{\frac{3}{2}}}{3} - \ln\left(y\right) - c_2 = 0$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{\left(-12\ln\left(y\right)-12c_2\right)^{\frac{2}{3}}}{4} \tag{1}$$

$$y' = -\left(-\frac{\left(-12\ln\left(y\right)-12c_2\right)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\left(-12\ln\left(y\right)-12c_2\right)^{\frac{1}{3}}}{4}\right)^{2} \tag{2}$$

$$y' = -\left(-\frac{\left(-12\ln\left(y\right)-12c_2\right)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\left(-12\ln\left(y\right)-12c_2\right)^{\frac{1}{3}}}{4}\right)^{2} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int -\frac{4}{\left(-12\ln\left(y\right)-12c_2\right)^{\frac{2}{3}}}dy = \int dx$$

$$-4\left(\int^{y}\frac{1}{\left(-12\ln\left(\_a\right)-12c_2\right)^{\frac{2}{3}}}d\_a\right) = x+c_6$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\int -\frac{16}{\left(-12\ln\left(y\right)-12c_2\right)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)^{2}}dy = \int dx$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(-12\ln\left(\_a\right)-12c_2\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}} = x+c_7$$

<u>Solving equation (3)</u>

Integrating both sides gives

$$\int -\frac{16}{\left(-12\ln\left(y\right)-12c_2\right)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)^{2}}dy = \int dx$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(-12\ln\left(\_a\right)-12c_2\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}} = x+c_8$$

Summary
The solution(s) found are the following

$$-4\left(\int^{y}\frac{1}{\left(12\ln\left(\_a\right)-12c_1\right)^{\frac{2}{3}}}d\_a\right)=x+c_3 \tag{1}$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(12\ln\left(\_a\right)-12c_1\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}}=x+c_4 \tag{2}$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(12\ln\left(\_a\right)-12c_1\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}}=x+c_5 \tag{3}$$

$$-4\left(\int^{y}\frac{1}{\left(-12\ln\left(\_a\right)-12c_2\right)^{\frac{2}{3}}}d\_a\right)=x+c_6 \tag{4}$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(-12\ln\left(\_a\right)-12c_2\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}}=x+c_7 \tag{5}$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(-12\ln\left(\_a\right)-12c_2\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}}=x+c_8 \tag{6}$$

Verification of solutions

$$-4\left(\int^{y}\frac{1}{\left(12\ln\left(\_a\right)-12c_1\right)^{\frac{2}{3}}}d\_a\right)=x+c_3$$

Verified OK.

$$-\frac{16\left(\int^{y}\frac{1}{\left(12\ln\left(\_a\right)-12c_1\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}}=x+c_4$$

Verified OK.

$$-\frac{16\left(\int^{y}\frac{1}{\left(12\ln\left(\_a\right)-12c_1\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}}=x+c_5$$

Verified OK.

$$-4\left(\int^{y}\frac{1}{\left(-12\ln\left(\_a\right)-12c_2\right)^{\frac{2}{3}}}d\_a\right)=x+c_6$$

Verified OK.

$$-\frac{16\left(\int^{y}\frac{1}{\left(-12\ln\left(\_a\right)-12c_2\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}}=x+c_7$$

Verified OK.

$$-\frac{16\left(\int^{y}\frac{1}{\left(-12\ln\left(\_a\right)-12c_2\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}}=x+c_8$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      -> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(-_b(_a))^(1/2)/_a = 0, _b
         symmetry methods on request
      `, `1st order, trying reduction of order with given symmetries:`[_a, 0]
```

✓ <u>Solution by Maple</u>

Time used: 0.063 (sec). Leaf size: 241

```
dsolve(y(x)^2*diff(y(x),x$2)^2+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1$$
$$y(x) = 0$$
$$-4\left(\int^{y(x)} \frac{1}{(-12\ln(\_a)+8c_1)^{\frac{2}{3}}}d\_a\right) - x - c_2 = 0$$
$$-4\left(\int^{y(x)} \frac{1}{(12\ln(\_a)-8c_1)^{\frac{2}{3}}}d\_a\right) - x - c_2 = 0$$
$$\frac{-16\left(\int^{y(x)} \frac{1}{(-12\ln(\_a)+8c_1)^{\frac{2}{3}}}d\_a\right) + 2i(-x-c_2)\sqrt{3} + 2x + 2c_2}{\left(-i\sqrt{3}-1\right)^2} = 0$$
$$\frac{-16\left(\int^{y(x)} \frac{1}{(-12\ln(\_a)+8c_1)^{\frac{2}{3}}}d\_a\right) + 2i(x+c_2)\sqrt{3} + 2x + 2c_2}{\left(1-i\sqrt{3}\right)^2} = 0$$
$$\frac{-16\left(\int^{y(x)} \frac{1}{(12\ln(\_a)-8c_1)^{\frac{2}{3}}}d\_a\right) + 2i(-x-c_2)\sqrt{3} + 2x + 2c_2}{\left(-i\sqrt{3}-1\right)^2} = 0$$
$$\frac{-16\left(\int^{y(x)} \frac{1}{(12\ln(\_a)-8c_1)^{\frac{2}{3}}}d\_a\right) + 2i(x+c_2)\sqrt{3} + 2x + 2c_2}{\left(1-i\sqrt{3}\right)^2} = 0$$

✓ Solution by Mathematica

Time used: 2.57 (sec). Leaf size: 449

```
DSolve[y[x]^2*y''[x]^2+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{-ic_1}(-\log(\#1)-ic_1)^{2/3}\Gamma\left(\frac{1}{3},-ic_1-\log(\#1)\right)}{(c_1-i\log(\#1))^{2/3}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{ic_1}(-\log(\#1)+ic_1)^{2/3}\Gamma\left(\frac{1}{3},ic_1-\log(\#1)\right)}{(i\log(\#1)+c_1)^{2/3}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{-i(-c_1)}(-\log(\#1)-i(-1)c_1)^{2/3}\Gamma\left(\frac{1}{3},-i(-1)c_1-\log(\#1)\right)}{(-i\log(\#1)-c_1)^{2/3}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{-ic_1}(-\log(\#1)-ic_1)^{2/3}\Gamma\left(\frac{1}{3},-ic_1-\log(\#1)\right)}{(c_1-i\log(\#1))^{2/3}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{i(-c_1)}(-\log(\#1)+i(-c_1))^{2/3}\Gamma\left(\frac{1}{3},i(-c_1)-\log(\#1)\right)}{(i\log(\#1)-c_1)^{2/3}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{ic_1}(-\log(\#1)+ic_1)^{2/3}\Gamma\left(\frac{1}{3},ic_1-\log(\#1)\right)}{(i\log(\#1)+c_1)^{2/3}}\&\right][x+c_2]$$

1.48 problem 48

1.48.1 Solving as second order ode missing x ode 571

Internal problem ID [7437]
Internal file name [`OUTPUT/6404_Sunday_June_05_2022_04_43_52_PM_91256229/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 48.
ODE order: 2.
ODE degree: 4.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{yy''^4 + y'^2 = 0}$$

1.48.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)^4 \left(\frac{d}{dy}p(y)\right)^4 + p(y)^2 = 0$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 4 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\frac{d}{dy}p(y) = \frac{\left(-p(y)^2 y^3\right)^{\frac{1}{4}}}{p\left(y\right) y} \tag{1}$$

$$\frac{d}{dy}p(y) = \frac{i\left(-p(y)^2 y^3\right)^{\frac{1}{4}}}{p\left(y\right) y} \tag{2}$$

$$\frac{d}{dy}p(y) = -\frac{\left(-p(y)^2 y^3\right)^{\frac{1}{4}}}{p\left(y\right) y} \tag{3}$$

$$\frac{d}{dy}p(y) = -\frac{i\left(-p(y)^2 y^3\right)^{\frac{1}{4}}}{p\left(y\right) y} \tag{4}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Writing the ode as

$$\frac{d}{dy}p(y) = \frac{(-p^2y^3)^{\frac{1}{4}}}{py}$$
$$\frac{d}{dy}p(y) = \omega(y,p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + ya_2 + a_1 \tag{1E}$$

$$\eta = pb_3 + yb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 &+ \frac{\left(-p^2 y^3\right)^{\frac{1}{4}} (b_3 - a_2)}{py} - \frac{\sqrt{-p^2 y^3}\, a_3}{p^2 y^2} \\
&- \left(-\frac{\left(-p^2 y^3\right)^{\frac{1}{4}}}{p\, y^2} - \frac{3py}{4 \left(-p^2 y^3\right)^{\frac{3}{4}}}\right) (p a_3 + y a_2 + a_1) \\
&- \left(-\frac{\left(-p^2 y^3\right)^{\frac{1}{4}}}{p^2 y} - \frac{y^2}{2 \left(-p^2 y^3\right)^{\frac{3}{4}}}\right) (p b_3 + y b_2 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-4 b_2 p^2 y^2 (-p^2 y^3)^{\frac{3}{4}} + p^4 y^3 a_3 - 3 p^3 y^4 a_2 + 6 p^3 y^4 b_3 + 2 p^2 y^5 b_2 + p^3 y^3 a_1 + 2 p^2 y^4 b_1 + 4 (-p^2 y^3)^{\frac{5}{4}} a_3}{4 p^2 y^2 \left(-p^2 y^3\right)^{\frac{3}{4}}} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
4 b_2 p^2 y^2 \left(-p^2 y^3\right)^{\frac{3}{4}} &- p^4 y^3 a_3 + 3 p^3 y^4 a_2 - 6 p^3 y^4 b_3 \\
&- 2 p^2 y^5 b_2 - p^3 y^3 a_1 - 2 p^2 y^4 b_1 - 4 \left(-p^2 y^3\right)^{\frac{5}{4}} a_3 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
p^2 y^2 \Big(4 \left(-p^2 y^3\right)^{\frac{3}{4}} b_2 &- p^2 y a_3 + 3 p\, y^2 a_2 - 6 p\, y^2 b_3 \\
&- 2 y^3 b_2 + 4 \left(-p^2 y^3\right)^{\frac{1}{4}} y a_3 - p y a_1 - 2 y^2 b_1\Big) = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{p, y\}$ in them.

$$
\left\{p, y, \left(-p^2 y^3\right)^{\frac{1}{4}}, \left(-p^2 y^3\right)^{\frac{3}{4}}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{p, y\}$ in them

$$
\left\{p = v_1, y = v_2, \left(-p^2 y^3\right)^{\frac{1}{4}} = v_3, \left(-p^2 y^3\right)^{\frac{3}{4}} = v_4\right\}
$$

The above PDE (6E) now becomes

$$v_1^2 v_2^2 \left(3 v_1 v_2^2 a_2 - v_1^2 v_2 a_3 - 2 v_2^3 b_2 - 6 v_1 v_2^2 b_3 - v_1 v_2 a_1 + 4 v_3 v_2 a_3 - 2 v_2^2 b_1 + 4 v_4 b_2\right) = 0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$-v_2^3 a_3 v_1^4 + (3a_2 - 6b_3) v_1^3 v_2^4 - a_1 v_1^3 v_2^3 - 2b_2 v_1^2 v_2^5 + 4a_3 v_3 v_1^2 v_2^3 - 2b_1 v_1^2 v_2^4 + 4v_4 b_2 v_1^2 v_2^2 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a_1 &= 0 \\ -a_3 &= 0 \\ 4a_3 &= 0 \\ -2b_1 &= 0 \\ -2b_2 &= 0 \\ 4b_2 &= 0 \\ 3a_2 - 6b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 2y \\ \eta &= p \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y, p) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial p}\right) S(y, p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dp}{dy} &= \frac{\eta}{\xi} \\ &= \frac{p}{2y} \\ &= \frac{p}{2y}\end{aligned}$$

This is easily solved to give

$$p(y) = c_1 \sqrt{y}$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{p}{\sqrt{y}}$$

And S is found from

$$\begin{aligned}dS &= \frac{dy}{\xi} \\ &= \frac{dy}{2y}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dy}{T} \\ &= \frac{\ln(y)}{2}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y, p) S_p}{R_y + \omega(y, p) R_p} \tag{2}$$

Where in the above R_y, R_p, S_y, S_p are all partial derivatives and $\omega(y,p)$ is the right hand side of the original ode given by

$$\omega(y,p) = \frac{(-p^2y^3)^{\frac{1}{4}}}{py}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_y &= -\frac{p}{2y^{\frac{3}{2}}} \\ R_p &= \frac{1}{\sqrt{y}} \\ S_y &= \frac{1}{2y} \\ S_p &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\sqrt{y}\,p}{p^2 - 2\,(-p^2y^3)^{\frac{1}{4}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for y, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{(1+i)\,\sqrt{2}\,\sqrt{R} - R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{2\ln\left(-R^{\frac{3}{2}} + i\sqrt{2} + \sqrt{2}\right)}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to y, p coordinates. This results in

$$\frac{\ln(y)}{2} = -\frac{2\ln\left(-\left(\frac{p(y)}{\sqrt{y}}\right)^{\frac{3}{2}} + i\sqrt{2} + \sqrt{2}\right)}{3} + c_1$$

Which simplifies to

$$\frac{\ln(y)}{2} = -\frac{2\ln\left(-\left(\frac{p(y)}{\sqrt{y}}\right)^{\frac{3}{2}} + i\sqrt{2} + \sqrt{2}\right)}{3} + c_1$$

Solving equation (2)

Writing the ode as

$$\frac{d}{dy}p(y) = \frac{i(-p^2y^3)^{\frac{1}{4}}}{py}$$
$$\frac{d}{dy}p(y) = \omega(y,p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + ya_2 + a_1 \tag{1E}$$
$$\eta = pb_3 + yb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{i(-p^2y^3)^{\frac{1}{4}}(b_3 - a_2)}{py} + \frac{\sqrt{-p^2y^3}\,a_3}{p^2y^2} \\ &- \left(-\frac{i(-p^2y^3)^{\frac{1}{4}}}{p\,y^2} - \frac{3ipy}{4\left(-p^2y^3\right)^{\frac{3}{4}}}\right)(pa_3 + ya_2 + a_1) \\ &- \left(-\frac{i(-p^2y^3)^{\frac{1}{4}}}{p^2y} - \frac{iy^2}{2\left(-p^2y^3\right)^{\frac{3}{4}}}\right)(pb_3 + yb_2 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{-ip^4y^3a_3 + 3ip^3y^4a_2 - 6ip^3y^4b_3 - 2ip^2y^5b_2 + 4b_2p^2y^2(-p^2y^3)^{\frac{3}{4}} - ip^3y^3a_1 - 2ip^2y^4b_1 + 4(-p^2y^3)^{\frac{5}{4}}\,a_3}{4p^2y^2\left(-p^2y^3\right)^{\frac{3}{4}}}$$
$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-ip^4y^3a_3 + 3ip^3y^4a_2 - 6ip^3y^4b_3 - 2ip^2y^5b_2 + 4b_2p^2y^2\left(-p^2y^3\right)^{\frac{3}{4}} \\
&- ip^3y^3a_1 - 2ip^2y^4b_1 + 4\left(-p^2y^3\right)^{\frac{5}{4}}a_3 = 0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&p^2y^2\Big(-ip^2ya_3 + 3ip\,y^2a_2 - 6ip\,y^2b_3 - 2iy^3b_2 \\
&+ 4\left(-p^2y^3\right)^{\frac{3}{4}}b_2 - ipya_1 - 2iy^2b_1 - 4\left(-p^2y^3\right)^{\frac{1}{4}}ya_3\Big) = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{p, y\}$ in them.

$$
\left\{p, y, \left(-p^2y^3\right)^{\frac{1}{4}}, \left(-p^2y^3\right)^{\frac{3}{4}}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{p, y\}$ in them

$$
\left\{p = v_1, y = v_2, \left(-p^2y^3\right)^{\frac{1}{4}} = v_3, \left(-p^2y^3\right)^{\frac{3}{4}} = v_4\right\}
$$

The above PDE (6E) now becomes

$$
v_1^2v_2^2\left(3iv_1v_2^2a_2 - iv_1^2v_2a_3 - 2iv_2^3b_2 - 6iv_1v_2^2b_3 - iv_1v_2a_1 - 2iv_2^2b_1 - 4v_3v_2a_3 + 4v_4b_2\right) = 0
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4\}
$$

Equation (7E) now becomes

$$
-iv_2^3a_3v_1^4 + (3ia_2 - 6ib_3)\,v_1^3v_2^4 - ia_1v_1^3v_2^3 - 2ib_2v_1^2v_2^5 - 2ib_1v_1^2v_2^4 - 4a_3v_3v_1^2v_2^3 + 4v_4b_2v_1^2v_2^2 = 0
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2ib_1 &= 0 \\
-2ib_2 &= 0 \\
-ia_1 &= 0 \\
-ia_3 &= 0 \\
-4a_3 &= 0 \\
4b_2 &= 0 \\
3ia_2 - 6ib_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= \frac{a_2}{2}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= y \\
\eta &= \frac{p}{2}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y, p) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial p}\right) S(y, p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$
\begin{aligned}
\frac{d}{dy}p(y) &= -\frac{(-p^2y^3)^{\frac{1}{4}}}{py} \\
\frac{d}{dy}p(y) &= \omega(y,p)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = pa_3 + ya_2 + a_1 \tag{1E}
$$

$$
\eta = pb_3 + yb_2 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 - \frac{(-p^2y^3)^{\frac{1}{4}}(b_3 - a_2)}{py} - \frac{\sqrt{-p^2y^3}\, a_3}{p^2y^2} \\
& - \left(\frac{(-p^2y^3)^{\frac{1}{4}}}{p\,y^2} + \frac{3py}{4\,(-p^2y^3)^{\frac{3}{4}}}\right)(pa_3 + ya_2 + a_1) \\
& - \left(\frac{(-p^2y^3)^{\frac{1}{4}}}{p^2y} + \frac{y^2}{2\,(-p^2y^3)^{\frac{3}{4}}}\right)(pb_3 + yb_2 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-4b_2p^2y^2(-p^2y^3)^{\frac{3}{4}} - p^4y^3a_3 + 3p^3y^4a_2 - 6p^3y^4b_3 - 2p^2y^5b_2 - p^3y^3a_1 - 2p^2y^4b_1 + 4(-p^2y^3)^{\frac{5}{4}}\, a_3}{4p^2y^2\,(-p^2y^3)^{\frac{3}{4}}} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&4b_2 p^2 y^2 \left(-p^2 y^3\right)^{\frac{3}{4}} + p^4 y^3 a_3 - 3p^3 y^4 a_2 + 6p^3 y^4 b_3 \\
&+ 2p^2 y^5 b_2 + p^3 y^3 a_1 + 2p^2 y^4 b_1 - 4\left(-p^2 y^3\right)^{\frac{5}{4}} a_3 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&p^2 y^2 \left(4\left(-p^2 y^3\right)^{\frac{3}{4}} b_2 + p^2 y a_3 - 3p\, y^2 a_2 + 6p\, y^2 b_3 \right. \\
&\left. + 2y^3 b_2 + 4\left(-p^2 y^3\right)^{\frac{1}{4}} y a_3 + p y a_1 + 2y^2 b_1\right) = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{p, y\}$ in them.

$$
\left\{p, y, \left(-p^2 y^3\right)^{\frac{1}{4}}, \left(-p^2 y^3\right)^{\frac{3}{4}}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{p, y\}$ in them

$$
\left\{p = v_1, y = v_2, \left(-p^2 y^3\right)^{\frac{1}{4}} = v_3, \left(-p^2 y^3\right)^{\frac{3}{4}} = v_4\right\}
$$

The above PDE (6E) now becomes

$$
v_1^2 v_2^2 \left(-3v_1 v_2^2 a_2 + v_1^2 v_2 a_3 + 2v_2^3 b_2 + 6v_1 v_2^2 b_3 + v_1 v_2 a_1 + 4v_3 v_2 a_3 + 2v_2^2 b_1 + 4v_4 b_2\right) = 0 \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4\}
$$

Equation (7E) now becomes

$$
v_2^3 a_3 v_1^4 + (-3a_2 + 6b_3)\, v_1^3 v_2^4 + a_1 v_1^3 v_2^3 + 2b_2 v_1^2 v_2^5 + 4a_3 v_3 v_1^2 v_2^3 + 2b_1 v_1^2 v_2^4 + 4v_4 b_2 v_1^2 v_2^2 = 0 \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
a_3 &= 0 \\
4a_3 &= 0 \\
2b_1 &= 0 \\
2b_2 &= 0 \\
4b_2 &= 0 \\
-3a_2 + 6b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= 2b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2y\\
\eta &= p
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y, p) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial p}\right) S(y, p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (4)

Writing the ode as

$$
\begin{aligned}
\frac{d}{dy}p(y) &= -\frac{i(-p^2y^3)^{\frac{1}{4}}}{py}\\
\frac{d}{dy}p(y) &= \omega(y, p)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + ya_2 + a_1 \tag{1E}$$

$$\eta = pb_3 + yb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{i(-p^2y^3)^{\frac{1}{4}}(b_3 - a_2)}{py} + \frac{\sqrt{-p^2y^3}\, a_3}{p^2y^2} \\
&- \left(\frac{i(-p^2y^3)^{\frac{1}{4}}}{p\,y^2} + \frac{3ipy}{4\,(-p^2y^3)^{\frac{3}{4}}}\right)(pa_3 + ya_2 + a_1) \\
&- \left(\frac{i(-p^2y^3)^{\frac{1}{4}}}{p^2y} + \frac{iy^2}{2\,(-p^2y^3)^{\frac{3}{4}}}\right)(pb_3 + yb_2 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{ip^4y^3a_3 - 3ip^3y^4a_2 + 6ip^3y^4b_3 + 2ip^2y^5b_2 + 4b_2p^2y^2(-p^2y^3)^{\frac{3}{4}} + ip^3y^3a_1 + 2ip^2y^4b_1 + 4(-p^2y^3)^{\frac{5}{4}}\,a_3}{4p^2y^2\,(-p^2y^3)^{\frac{3}{4}}} = 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&ip^4y^3a_3 - 3ip^3y^4a_2 + 6ip^3y^4b_3 + 2ip^2y^5b_2 + 4b_2p^2y^2\left(-p^2y^3\right)^{\frac{3}{4}} \\
&+ ip^3y^3a_1 + 2ip^2y^4b_1 + 4\left(-p^2y^3\right)^{\frac{5}{4}}\,a_3 = 0
\end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
&p^2y^2\Big(ip^2ya_3 - 3ip\,y^2a_2 + 6ip\,y^2b_3 + 2iy^3b_2 \\
&+ 4\left(-p^2y^3\right)^{\frac{3}{4}}\,b_2 + ipya_1 + 2iy^2b_1 - 4\left(-p^2y^3\right)^{\frac{1}{4}}\,ya_3\Big) = 0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{p, y\}$ in them.

$$\left\{p, y, \left(-p^2y^3\right)^{\frac{1}{4}}, \left(-p^2y^3\right)^{\frac{3}{4}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{p, y\}$ in them

$$\left\{p = v_1, y = v_2, \left(-p^2y^3\right)^{\frac{1}{4}} = v_3, \left(-p^2y^3\right)^{\frac{3}{4}} = v_4\right\}$$

The above PDE (6E) now becomes

$$v_1^2v_2^2\left(-3iv_1v_2^2a_2 + iv_1^2v_2a_3 + 2iv_2^3b_2 + 6iv_1v_2^2b_3 + iv_1v_2a_1 + 2iv_2^2b_1 - 4v_3v_2a_3 + 4v_4b_2\right) = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$iv_2^3a_3v_1^4 + (-3ia_2 + 6ib_3)\,v_1^3v_2^4 + ia_1v_1^3v_2^3 + 2ib_2v_1^2v_2^5 + 2ib_1v_1^2v_2^4 - 4a_3v_3v_1^2v_2^3 + 4v_4b_2v_1^2v_2^2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} ia_1 &= 0 \\ ia_3 &= 0 \\ 2ib_1 &= 0 \\ 2ib_2 &= 0 \\ -4a_3 &= 0 \\ 4b_2 &= 0 \\ -3ia_2 + 6ib_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= \frac{a_2}{2} \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = y$$
$$\eta = \frac{p}{2}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y, p) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial p}\right) S(y,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{\ln(y)}{2} = -\frac{2\ln\left(-\left(\frac{y'}{\sqrt{y}}\right)^{\frac{3}{2}} + i\sqrt{2} + \sqrt{2}\right)}{3} + c_1$$

Solving the given ode for y' results in 12 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 - (\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{2}{3}}}{y^{\frac{3}{2}}} \tag{1}$$

$$y' = \left(-\frac{\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 - (\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{1}{3}}}{2y} + \frac{i\sqrt{3}\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 - (\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{1}{3}}}{2y}\right)^2 \sqrt{y} \tag{2}$$

$$y' = \left(-\frac{\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 - (\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{1}{3}}}{2y} - \frac{i\sqrt{3}\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 - (\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{1}{3}}}{2y}\right)^2 \sqrt{y} \tag{3}$$

$$y' = \frac{\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 + i(\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{2}{3}}}{y^{\frac{3}{2}}} \tag{4}$$

$$y' = \left(-\frac{\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 + i(\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{1}{3}}}{2y} + \frac{i\sqrt{3}\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 + i(\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{1}{3}}}{2y}\right)^2 \sqrt{y} \tag{5}$$

$$y' = \left(-\frac{\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 + i(\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{1}{3}}}{2y} - \frac{i\sqrt{3}\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 + i(\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{1}{3}}}{2y}\right)^2 \sqrt{y} \tag{6}$$

$$y' = \frac{\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 + (\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{2}{3}}}{y^{\frac{3}{2}}} \tag{7}$$

$$y' = \left(-\frac{\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 + (\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{1}{3}}}{2y} + \frac{i\sqrt{3}\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 + (\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{1}{3}}}{2y}\right)^2 \sqrt{y} \tag{8}$$

$$y' = \left(-\frac{\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 + (\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{1}{3}}}{2y} - \frac{i\sqrt{3}\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 + (\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{1}{3}}}{2y}\right)^2 \sqrt{y} \tag{9}$$

$$y' = \frac{\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 - i(\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{2}{3}}}{y^{\frac{3}{2}}} \tag{10}$$

$$y' = \left(-\frac{\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 - i(\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{1}{3}}}{2y} - \frac{i\sqrt{3}\left(i\sqrt{2}\, y^3 + \sqrt{2}\, y^3 - i(\mathrm{e}^{2c_1} y^3)^{\frac{3}{4}}\right)^{\frac{1}{3}}}{2y}\right)^2 \sqrt{y}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{y^{\frac{3}{2}}}{\left(i\sqrt{2}\,y^{3}+\sqrt{2}\,y^{3}-\left(\mathrm{e}^{2c_1}y^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}dy=\int dx$$

$$\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\_a^{3}+\sqrt{2}\_a^{3}-\left(\mathrm{e}^{2c_1}\_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a=x+c_5$$

Solving equation (2)

Integrating both sides gives

$$\int \frac{4y^{\frac{3}{2}}}{\left(i\sqrt{2}\,y^{3}+\sqrt{2}\,y^{3}-\left(\mathrm{e}^{2c_1}y^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)^{2}}dy=\int dx$$

$$\frac{4\left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\_a^{3}+\sqrt{2}\_a^{3}-\left(\mathrm{e}^{2c_1}\_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}}=x+c_6$$

Solving equation (3)

Integrating both sides gives

$$\int \frac{4y^{\frac{3}{2}}}{\left(i\sqrt{2}\,y^{3}+\sqrt{2}\,y^{3}-\left(\mathrm{e}^{2c_1}y^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)^{2}}dy=\int dx$$

$$\frac{4\left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\_a^{3}+\sqrt{2}\_a^{3}-\left(\mathrm{e}^{2c_1}\_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}}=x+c_7$$

Solving equation (4)

Integrating both sides gives

$$\int \frac{y^{\frac{3}{2}}}{\left(i\sqrt{2}\, y^{3}+i \left({\mathrm e}^{2 c_{1}} y^{3}\right)^{\frac{3}{4}}+\sqrt{2}\, y^{3}\right)^{\frac{2}{3}}}d y = \int{d x}$$

$$\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\, \_a^{3}+i \left({\mathrm e}^{2 c_{1}} \_a^{3}\right)^{\frac{3}{4}}+\sqrt{2}\, \_a^{3}\right)^{\frac{2}{3}}}d\_a = x +c_{8}$$

Solving equation (5)

Integrating both sides gives

$$\int \frac{4 y^{\frac{3}{2}}}{\left(i\sqrt{2}\, y^{3}+i \left({\mathrm e}^{2 c_{1}} y^{3}\right)^{\frac{3}{4}}+\sqrt{2}\, y^{3}\right)^{\frac{2}{3}} \left(i \sqrt{3}-1\right)^{2}}d y = \int{d x}$$

$$\frac{4 \left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\, \_a^{3}+i \left({\mathrm e}^{2 c_{1}} \_a^{3}\right)^{\frac{3}{4}}+\sqrt{2}\, \_a^{3}\right)^{\frac{2}{3}}}d\_a \right)}{\left(i \sqrt{3}-1\right)^{2}} = x +c_{9}$$

Solving equation (6)

Integrating both sides gives

$$\int \frac{4 y^{\frac{3}{2}}}{\left(i\sqrt{2}\, y^{3}+i \left({\mathrm e}^{2 c_{1}} y^{3}\right)^{\frac{3}{4}}+\sqrt{2}\, y^{3}\right)^{\frac{2}{3}} \left(1+i \sqrt{3}\right)^{2}}d y = \int{d x}$$

$$\frac{4 \left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\, \_a^{3}+i \left({\mathrm e}^{2 c_{1}} \_a^{3}\right)^{\frac{3}{4}}+\sqrt{2}\, \_a^{3}\right)^{\frac{2}{3}}}d\_a \right)}{\left(1+i \sqrt{3}\right)^{2}} = x +c_{10}$$

Solving equation (7)

Integrating both sides gives

$$\int \frac{y^{\frac{3}{2}}}{\left(i\sqrt{2}\, y^{3}+\sqrt{2}\, y^{3}+\left({\mathrm e}^{2 c_{1}} y^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d y = \int d x$$

$$\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\, \_a^{3}+\sqrt{2}\, \_a^{3}+\left({\mathrm e}^{2 c_{1}} \_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a = x +c_{11}$$

Solving equation (8)

Integrating both sides gives

$$\int \frac{4 y^{\frac{3}{2}}}{\left(i\sqrt{2}\, y^{3}+\sqrt{2}\, y^{3}+\left({\mathrm e}^{2 c_{1}} y^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}} \left(i\sqrt{3}-1\right)^{2}}d y = \int d x$$

$$\frac{4 \left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\, \_a^{3}+\sqrt{2}\, \_a^{3}+\left({\mathrm e}^{2 c_{1}} \_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a \right)}{\left(i\sqrt{3}-1\right)^{2}} = x +c_{12}$$

Solving equation (9)

Integrating both sides gives

$$\int \frac{4 y^{\frac{3}{2}}}{\left(i\sqrt{2}\, y^{3}+\sqrt{2}\, y^{3}+\left({\mathrm e}^{2 c_{1}} y^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}} \left(1+i\sqrt{3}\right)^{2}}d y = \int d x$$

$$\frac{4 \left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\, \_a^{3}+\sqrt{2}\, \_a^{3}+\left({\mathrm e}^{2 c_{1}} \_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a \right)}{\left(1+i\sqrt{3}\right)^{2}} = x +c_{13}$$

Solving equation (10)

Integrating both sides gives

$$\int \frac{y^{\frac{3}{2}}}{\left(i\sqrt{2}\, y^{3}-i \left({\mathrm e}^{2 c_{1}} y^{3}\right)^{\frac{3}{4}}+\sqrt{2}\, y^{3}\right)^{\frac{2}{3}}}d y = \int d x$$

$$\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\, \_a^{3}-i \left({\mathrm e}^{2 c_{1}} \_a^{3}\right)^{\frac{3}{4}}+\sqrt{2}\, \_a^{3}\right)^{\frac{2}{3}}}d \_a = x +c_{14}$$

Solving equation (11)

Integrating both sides gives

$$\int \frac{4 y^{\frac{3}{2}}}{\left(i\sqrt{2}\, y^{3}-i \left({\mathrm e}^{2 c_{1}} y^{3}\right)^{\frac{3}{4}}+\sqrt{2}\, y^{3}\right)^{\frac{2}{3}} \left(1+i \sqrt{3}\right)^{2}}d y = \int d x$$

$$\frac{4 \left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\, \_a^{3}-i \left({\mathrm e}^{2 c_{1}} \_a^{3}\right)^{\frac{3}{4}}+\sqrt{2}\, \_a^{3}\right)^{\frac{2}{3}}}d \_a \right)}{\left(1+i \sqrt{3}\right)^{2}} = x +\_C15$$

Solving equation (12)

Integrating both sides gives

$$\int \frac{4 y^{\frac{3}{2}}}{\left(i\sqrt{2}\, y^{3}-i \left({\mathrm e}^{2 c_{1}} y^{3}\right)^{\frac{3}{4}}+\sqrt{2}\, y^{3}\right)^{\frac{2}{3}} \left(i \sqrt{3}-1\right)^{2}}d y = \int d x$$

$$\frac{4 \left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\, \_a^{3}-i \left({\mathrm e}^{2 c_{1}} \_a^{3}\right)^{\frac{3}{4}}+\sqrt{2}\, \_a^{3}\right)^{\frac{2}{3}}}d \_a \right)}{\left(i \sqrt{3}-1\right)^{2}} = x +\_C16$$

Summary

The solution(s) found are the following

$$\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+\sqrt{2}\,\_a^{3}-\left({\mathrm e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a = x+c_{5} \tag{1}$$

$$\frac{4\left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+\sqrt{2}\,\_a^{3}-\left({\mathrm e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}} = x+c_{6} \tag{2}$$

$$\frac{4\left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+\sqrt{2}\,\_a^{3}-\left({\mathrm e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}} = x+c_{7} \tag{3}$$

$$\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+i\left({\mathrm e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}+\sqrt{2}\,\_a^{3}\right)^{\frac{2}{3}}}d\_a = x+c_{8} \tag{4}$$

$$\frac{4\left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+i\left({\mathrm e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}+\sqrt{2}\,\_a^{3}\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}} = x+c_{9} \tag{5}$$

$$\frac{4\left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+i\left({\mathrm e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}+\sqrt{2}\,\_a^{3}\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}} = x+c_{10} \tag{6}$$

$$\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+\sqrt{2}\,\_a^{3}+\left({\mathrm e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a = x+c_{11} \tag{7}$$

$$\frac{4\left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+\sqrt{2}\,\_a^{3}+\left({\mathrm e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}} = x+c_{12} \tag{8}$$

$$\frac{4\left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+\sqrt{2}\,\_a^{3}+\left({\mathrm e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}} = x+c_{13} \tag{9}$$

$$\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}-i\left({\mathrm e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}+\sqrt{2}\,\_a^{3}\right)^{\frac{2}{3}}}d\_a = x+c_{14} \tag{10}$$

$$\frac{4\left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}-i\left({\mathrm e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}+\sqrt{2}\,\_a^{3}\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}} = x+\_C15 \tag{11}$$

Verification of solutions

$$\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+\sqrt{2}\,\_a^{3}-\left(\mathrm{e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a=x+c_{5}$$

Verified OK.

$$\frac{4\left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+\sqrt{2}\,\_a^{3}-\left(\mathrm{e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}}=x+c_{6}$$

Verified OK.

$$\frac{4\left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+\sqrt{2}\,\_a^{3}-\left(\mathrm{e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}}=x+c_{7}$$

Verified OK.

$$\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+i\left(\mathrm{e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}+\sqrt{2}\,\_a^{3}\right)^{\frac{2}{3}}}d\_a=x+c_{8}$$

Verified OK.

$$\frac{4\left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+i\left(\mathrm{e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}+\sqrt{2}\,\_a^{3}\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}}=x+c_{9}$$

Verified OK.

$$\frac{4\left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+i\left(\mathrm{e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}+\sqrt{2}\,\_a^{3}\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}}=x+c_{10}$$

Verified OK.

$$\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+\sqrt{2}\,\_a^{3}+\left(\mathrm{e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a=x+c_{11}$$

Verified OK.

$$\frac{4\left(\int^{y}\frac{\_a^{\frac{3}{2}}}{\left(i\sqrt{2}\,\_a^{3}+\sqrt{2}\,\_a^{3}+\left(\mathrm{e}^{2c_{1}}\_a^{3}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}}=x+c_{12}$$

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 4 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      -> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(-_b(_a)^2*_a^3)^(1/4)/_a
         symmetry methods on request
      `, `1st order, trying reduction of order with given symmetries:`[_a, 1/2*_b]
```

✓ Solution by Maple

Time used: 0.157 (sec). Leaf size: 2926

```
dsolve(y(x)*diff(y(x),x$2)^4+diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = c_1$$
$$y(x) = 0$$
$$\int^{y(x)} \frac{\_a^2}{\sqrt{\_a^3 \left(2\_a - (c_1\_a)^{\frac{1}{4}}\right) \left(-2\_a^3 + \_a^2 (c_1\_a)^{\frac{1}{4}}\right)^{\frac{1}{3}}}} d\_a - x - c_2 = 0$$
$$\int^{y(x)} \frac{\_a^2}{\sqrt{-\_a^3 \left(i (c_1\_a)^{\frac{1}{4}} - 2\_a\right) \left(\left(i (c_1\_a)^{\frac{1}{4}} - 2\_a\right) \_a^2\right)^{\frac{1}{3}}}} d\_a - x - c_2 = 0$$
$$\int^{y(x)} \frac{\_a^2}{\sqrt{\_a^3 \left(2\_a + (c_1\_a)^{\frac{1}{4}}\right) \left(-2\_a^3 - \_a^2 (c_1\_a)^{\frac{1}{4}}\right)^{\frac{1}{3}}}} d\_a - x - c_2 = 0$$
$$\int^{y(x)} \frac{\_a^2}{\sqrt{\_a^3 \left(i (c_1\_a)^{\frac{1}{4}} + 2\_a\right) \left(-\left(i (c_1\_a)^{\frac{1}{4}} + 2\_a\right) \_a^2\right)^{\frac{1}{3}}}} d\_a - x - c_2 = 0$$
$$\sqrt{2} \left(\int^{y(x)} \frac{\_a^2}{\sqrt{\left(-2\_a + (c_1\_a)^{\frac{1}{4}}\right) (1 + i\sqrt{3}) \_a^3 \left(-2\_a^3 + \_a^2 (c_1\_a)^{\frac{1}{4}}\right)^{\frac{1}{3}}}} d\_a\right) - x - c_2 = 0$$
$$\sqrt{2} \left(\int^{y(x)} \frac{\_a^2}{\sqrt{(i - \sqrt{3}) \_a^3 \left(\left(i (c_1\_a)^{\frac{1}{4}} - 2\_a\right) \_a^2\right)^{\frac{1}{3}} \left((c_1\_a)^{\frac{1}{4}} + 2i\_a\right)}} d\_a\right) - x - c_2 = 0$$
$$\sqrt{2} \left(\int^{y(x)} \frac{\_a^2}{\sqrt{-2 (1 + i\sqrt{3}) \left(-2\_a^3 - \_a^2 (c_1\_a)^{\frac{1}{4}}\right)^{\frac{1}{3}} \left(\_a + \frac{(c_1\_a)^{\frac{1}{4}}}{2}\right) \_a^3}} d\_a\right) - x - c_2 = 0$$
$$\sqrt{2} \left(\int^{y(x)} \frac{\_a^2}{\sqrt{-\_a^3 \left(i (c_1\_a)^{\frac{1}{4}} + 2\_a\right) \left(-\left(i (c_1\_a)^{\frac{1}{4}} + 2\_a\right) \_a^2\right)^{\frac{1}{3}} (1 + i\sqrt{3})}} d\_a\right) - x - c_2 = 0$$
$$-\left(\int^{y(x)} \frac{\_a^2}{\sqrt{\_a^3 \left(2\_a - (c_1\_a)^{\frac{1}{4}}\right) \left(-2\_a^3 + \_a^2 (c_1\_a)^{\frac{1}{4}}\right)^{\frac{1}{3}}}} d\_a\right) - x - c_2 = 0$$

✓ Solution by Mathematica

Time used: 4.322 (sec). Leaf size: 1237

```
DSolve[y[x]*y''[x]^4+y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

1.49 problem 49

Internal problem ID [7438]
Internal file name [`OUTPUT/6405_Sunday_June_05_2022_04_46_32_PM_41712938/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 49.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"algebraic", "second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{y^3 y''^2 + y y' = 0}$$

The ode

$$y^3 y''^2 + y y' = 0$$

is factored to

$$y\left(y^2 y''^2 + y'\right) = 0$$

Which gives the following equations

$$y = 0 \tag{1}$$
$$y^2 y''^2 + y' = 0 \tag{2}$$

Each of the above equations is now solved.

Solving ODE (1) Since $y = 0$, is missing derivative in y then it is an algebraic equation. Solving for y.

$$y = 0$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$

Verification of solutions

$$y = 0$$

Verified OK.

Solving ODE (2) This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$y^2 p(y)^2 \left(\frac{d}{dy}p(y)\right)^2 + p(y) = 0$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\frac{d}{dy}p(y) = -\frac{1}{\sqrt{-p(y)\,y}} \tag{1}$$

$$\frac{d}{dy}p(y) = \frac{1}{\sqrt{-p(y)\,y}} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= -\frac{1}{\sqrt{-p\,y}} \end{aligned}$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = \frac{1}{\sqrt{-p}}$. Integrating both sides gives

$$\frac{1}{\frac{1}{\sqrt{-p}}}\, dp = -\frac{1}{y}\, dy$$

$$\int \frac{1}{\frac{1}{\sqrt{-p}}}\, dp = \int -\frac{1}{y}\, dy$$

$$-\frac{2(-p)^{\frac{3}{2}}}{3} = -\ln(y) + c_1$$

The solution is

$$-\frac{2(-p(y))^{\frac{3}{2}}}{3} + \ln(y) - c_1 = 0$$

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{1}{\sqrt{-p}\, y} \end{aligned}$$

Where $f(y) = \frac{1}{y}$ and $g(p) = \frac{1}{\sqrt{-p}}$. Integrating both sides gives

$$\frac{1}{\frac{1}{\sqrt{-p}}}\, dp = \frac{1}{y}\, dy$$

$$\int \frac{1}{\frac{1}{\sqrt{-p}}}\, dp = \int \frac{1}{y}\, dy$$

$$-\frac{2(-p)^{\frac{3}{2}}}{3} = \ln(y) + c_2$$

The solution is

$$-\frac{2(-p(y))^{\frac{3}{2}}}{3} - \ln(y) - c_2 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{2(-y')^{\frac{3}{2}}}{3} + \ln(y) - c_1 = 0$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{(12\ln(y) - 12c_1)^{\frac{2}{3}}}{4} \tag{1}$$

$$y' = -\left(-\frac{(12\ln(y) - 12c_1)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\,(12\ln(y) - 12c_1)^{\frac{1}{3}}}{4}\right)^2 \tag{2}$$

$$y' = -\left(-\frac{(12\ln(y) - 12c_1)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\,(12\ln(y) - 12c_1)^{\frac{1}{3}}}{4}\right)^2 \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int -\frac{4}{(12\ln(y) - 12c_1)^{\frac{2}{3}}} dy = \int dx$$

$$-4\left(\int^y \frac{1}{(12\ln(\_a) - 12c_1)^{\frac{2}{3}}} d\_a\right) = x + c_3$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{16}{(12\ln(y) - 12c_1)^{\frac{2}{3}}\,(i\sqrt{3} - 1)^2} dy = \int dx$$

$$-\frac{16\left(\int^y \frac{1}{(12\ln(\_a) - 12c_1)^{\frac{2}{3}}} d\_a\right)}{(i\sqrt{3} - 1)^2} = x + c_4$$

Solving equation (3)

Integrating both sides gives

$$\int -\frac{16}{(12\ln(y) - 12c_1)^{\frac{2}{3}}\,(1 + i\sqrt{3})^2} dy = \int dx$$

$$-\frac{16\left(\int^y \frac{1}{(12\ln(\_a) - 12c_1)^{\frac{2}{3}}} d\_a\right)}{(1 + i\sqrt{3})^2} = x + c_5$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{2(-y')^{\frac{3}{2}}}{3} - \ln(y) - c_2 = 0$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{(-12\ln(y) - 12c_2)^{\frac{2}{3}}}{4} \tag{1}$$

$$y' = -\left(-\frac{(-12\ln(y) - 12c_2)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\,(-12\ln(y) - 12c_2)^{\frac{1}{3}}}{4}\right)^2 \tag{2}$$

$$y' = -\left(-\frac{(-12\ln(y) - 12c_2)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\,(-12\ln(y) - 12c_2)^{\frac{1}{3}}}{4}\right)^2 \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int -\frac{4}{(-12\ln(y) - 12c_2)^{\frac{2}{3}}}dy = \int dx$$

$$-4\left(\int^y \frac{1}{(-12\ln(\_a) - 12c_2)^{\frac{2}{3}}}d\_a\right) = x + c_6$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{16}{(-12\ln(y) - 12c_2)^{\frac{2}{3}}\left(1 + i\sqrt{3}\right)^2}dy = \int dx$$

$$-\frac{16\left(\int^y \frac{1}{(-12\ln(\_a) - 12c_2)^{\frac{2}{3}}}d\_a\right)}{\left(1 + i\sqrt{3}\right)^2} = x + c_7$$

Solving equation (3)

Integrating both sides gives

$$\int -\frac{16}{\left(-12\ln \left(y\right)-12c_2\right)^{\frac{2}{3}} \left(i\sqrt{3}-1\right)^{2}}dy = \int dx$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(-12\ln \left(\_a\right)-12c_2\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}} = x+c_8$$

Summary

The solution(s) found are the following

$$-4\left(\int^{y}\frac{1}{\left(12\ln \left(\_a\right)-12c_1\right)^{\frac{2}{3}}}d\_a\right) = x+c_3 \tag{1}$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(12\ln \left(\_a\right)-12c_1\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}} = x+c_4 \tag{2}$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(12\ln \left(\_a\right)-12c_1\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}} = x+c_5 \tag{3}$$

$$-4\left(\int^{y}\frac{1}{\left(-12\ln \left(\_a\right)-12c_2\right)^{\frac{2}{3}}}d\_a\right) = x+c_6 \tag{4}$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(-12\ln \left(\_a\right)-12c_2\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}} = x+c_7 \tag{5}$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(-12\ln \left(\_a\right)-12c_2\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}} = x+c_8 \tag{6}$$

Verification of solutions

$$-4\left(\int^{y}\frac{1}{\left(12\ln\left(\_a\right)-12c_1\right)^{\frac{2}{3}}}d\_a\right)=x+c_3$$

Verified OK.

$$-\frac{16\left(\int^{y}\frac{1}{\left(12\ln(\_a)-12c_1\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}}=x+c_4$$

Verified OK.

$$-\frac{16\left(\int^{y}\frac{1}{\left(12\ln(\_a)-12c_1\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}}=x+c_5$$

Verified OK.

$$-4\left(\int^{y}\frac{1}{\left(-12\ln\left(\_a\right)-12c_2\right)^{\frac{2}{3}}}d\_a\right)=x+c_6$$

Verified OK.

$$-\frac{16\left(\int^{y}\frac{1}{\left(-12\ln(\_a)-12c_2\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^{2}}=x+c_7$$

Verified OK.

$$-\frac{16\left(\int^{y}\frac{1}{\left(-12\ln(\_a)-12c_2\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^{2}}=x+c_8$$

Verified OK.

Summary

The solution(s) found are the following

$$-4\left(\int^{y}\frac{1}{\left(12\ln\left(\_a\right)-12c_1\right)^{\frac{2}{3}}}d\_a\right)=x+c_3 \tag{1}$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(12\ln(\_a)-12c_1\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^2}=x+c_4 \tag{2}$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(12\ln(\_a)-12c_1\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^2}=x+c_5 \tag{3}$$

$$-4\left(\int^{y}\frac{1}{\left(-12\ln\left(\_a\right)-12c_2\right)^{\frac{2}{3}}}d\_a\right)=x+c_6 \tag{4}$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(-12\ln(\_a)-12c_2\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i\sqrt{3}\right)^2}=x+c_7 \tag{5}$$

$$-\frac{16\left(\int^{y}\frac{1}{\left(-12\ln(\_a)-12c_2\right)^{\frac{2}{3}}}d\_a\right)}{\left(i\sqrt{3}-1\right)^2}=x+c_8 \tag{6}$$

Verification of solutions

$$-4\left(\int^{y} \frac{1}{\left(12 \ln \left(\_a\right)-12 c_{1}\right)^{\frac{2}{3}}}d\_a\right) = x+c_{3}$$

Verified OK.

$$-\frac{16\left(\int^{y} \frac{1}{\left(12 \ln (\_a)-12 c_{1}\right)^{\frac{2}{3}}}d\_a\right)}{\left(i \sqrt{3}-1\right)^{2}} = x+c_{4}$$

Verified OK.

$$-\frac{16\left(\int^{y} \frac{1}{\left(12 \ln (\_a)-12 c_{1}\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i \sqrt{3}\right)^{2}} = x+c_{5}$$

Verified OK.

$$-4\left(\int^{y} \frac{1}{\left(-12 \ln \left(\_a\right)-12 c_{2}\right)^{\frac{2}{3}}}d\_a\right) = x+c_{6}$$

Verified OK.

$$-\frac{16\left(\int^{y} \frac{1}{\left(-12 \ln (\_a)-12 c_{2}\right)^{\frac{2}{3}}}d\_a\right)}{\left(1+i \sqrt{3}\right)^{2}} = x+c_{7}$$

Verified OK.

$$-\frac{16\left(\int^{y} \frac{1}{\left(-12 \ln (\_a)-12 c_{2}\right)^{\frac{2}{3}}}d\_a\right)}{\left(i \sqrt{3}-1\right)^{2}} = x+c_{8}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      <- differential order: 2; canonical coordinates successful
      <- differential order 2; missing variables successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      <- differential order: 2; canonical coordinates successful
      <- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 241

```
dsolve(y(x)^3*diff(y(x),x$2)^2+y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1$$
$$y(x) = 0$$
$$-4\left(\int^{y(x)}\frac{1}{\left(-12\ln\left(\_a\right)+8c_1\right)^{\frac{2}{3}}}d\_a\right)-x-c_2 = 0$$
$$-4\left(\int^{y(x)}\frac{1}{\left(12\ln\left(\_a\right)-8c_1\right)^{\frac{2}{3}}}d\_a\right)-x-c_2 = 0$$
$$\frac{-16\left(\int^{y(x)}\frac{1}{(-12\ln(\_a)+8c_1)^{\frac{2}{3}}}d\_a\right)+2i(-x-c_2)\sqrt{3}+2x+2c_2}{\left(-i\sqrt{3}-1\right)^2} = 0$$
$$\frac{-16\left(\int^{y(x)}\frac{1}{(-12\ln(\_a)+8c_1)^{\frac{2}{3}}}d\_a\right)+2i(x+c_2)\sqrt{3}+2x+2c_2}{\left(1-i\sqrt{3}\right)^2} = 0$$
$$\frac{-16\left(\int^{y(x)}\frac{1}{(12\ln(\_a)-8c_1)^{\frac{2}{3}}}d\_a\right)+2i(-x-c_2)\sqrt{3}+2x+2c_2}{\left(-i\sqrt{3}-1\right)^2} = 0$$
$$\frac{-16\left(\int^{y(x)}\frac{1}{(12\ln(\_a)-8c_1)^{\frac{2}{3}}}d\_a\right)+2i(x+c_2)\sqrt{3}+2x+2c_2}{\left(1-i\sqrt{3}\right)^2} = 0$$

✓ Solution by Mathematica

Time used: 2.526 (sec). Leaf size: 459

```
DSolve[y[x]^3*y''[x]^2+y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{-ic_1}(-\log(\#1) - ic_1)^{2/3}\Gamma\left(\frac{1}{3}, -ic_1 - \log(\#1)\right)}{(c_1 - i\log(\#1))^{2/3}}\&\right][x + c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{ic_1}(-\log(\#1) + ic_1)^{2/3}\Gamma\left(\frac{1}{3}, ic_1 - \log(\#1)\right)}{(i\log(\#1) + c_1)^{2/3}}\&\right][x + c_2]$$

$$y(x) \to 0$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{-i(-c_1)}(-\log(\#1) - i(-1)c_1)^{2/3}\Gamma\left(\frac{1}{3}, -i(-1)c_1 - \log(\#1)\right)}{(-i\log(\#1) - c_1)^{2/3}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{-ic_1}(-\log(\#1) - ic_1)^{2/3}\Gamma\left(\frac{1}{3}, -ic_1 - \log(\#1)\right)}{(c_1 - i\log(\#1))^{2/3}}\&\right][x + c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{i(-c_1)}(-\log(\#1) + i(-c_1))^{2/3}\Gamma\left(\frac{1}{3}, i(-c_1) - \log(\#1)\right)}{(i\log(\#1) - c_1)^{2/3}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{ic_1}(-\log(\#1) + ic_1)^{2/3}\Gamma\left(\frac{1}{3}, ic_1 - \log(\#1)\right)}{(i\log(\#1) + c_1)^{2/3}}\&\right][x + c_2]$$

1.50 problem 50

1.50.1 Solving as second order ode missing x ode 608
1.50.2 Maple step by step solution 610

Internal problem ID [7439]
Internal file name [`OUTPUT/6406_Sunday_June_05_2022_04_46_41_PM_5439210/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 50.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1],
  [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{yy'' + y'^3 = 0}$$

1.50.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) + p(y)^3 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= -\frac{p^2}{y}
\end{aligned}$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = p^2$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p^2}\,dp &= -\frac{1}{y}\,dy \\
\int \frac{1}{p^2}\,dp &= \int -\frac{1}{y}\,dy \\
-\frac{1}{p} &= -\ln(y) + c_1
\end{aligned}$$

The solution is

$$-\frac{1}{p(y)} + \ln(y) - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{1}{y'} + \ln(y) - c_1 = 0$$

Integrating both sides gives

$$\begin{aligned}
\int (\ln(y) - c_1)\,dy &= x + c_2 \\
-c_1 y + y\ln(y) - y &= x + c_2
\end{aligned}$$

Solving for y gives these solutions

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left((x+c_2)\mathrm{e}^{-c_1-1}\right)+c_1+1} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{\mathrm{LambertW}\left((x+c_2)\mathrm{e}^{-c_1-1}\right)+c_1+1}$$

Verified OK.

1.50.2 Maple step by step solution

Let's solve

$yy'' + y'^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y), y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $yu(y)\left(\frac{d}{dy}u(y)\right) + u(y)^3 = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)^2} = -\frac{1}{y}$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy}u(y)}{u(y)^2}dy = \int -\frac{1}{y}dy + c_1$

- Evaluate integral

  $-\frac{1}{u(y)} = -\ln(y) + c_1$

- Solve for $u(y)$

  $u(y) = \frac{1}{\ln(y)-c_1}$

- Solve 1st ODE for $u(y)$

  $u(y) = \frac{1}{\ln(y)-c_1}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = \frac{1}{\ln(y)-c_1}$

- Separate variables

  $y'(\ln(y) - c_1) = 1$

- Integrate both sides with respect to x

  $\int y'(\ln(y) - c_1)\, dx = \int 1dx + c_2$

- Evaluate integral

  $-c_1 y + y\ln(y) - y = x + c_2$

- Solve for y

  $y = e^{\mathit{LambertW}\left((x+c_2)e^{-c_1-1}\right)+c_1+1}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)^3/_a = 0, _b(_a)`   *** S
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 27

```
dsolve(y(x)*diff(y(x),x$2)+diff(y(x),x)^3=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1$$
$$y(x) = \frac{x + c_2}{\text{LambertW}\left((x + c_2)\, \mathrm{e}^{c_1 - 1}\right)}$$

✓ Solution by Mathematica

Time used: 60.106 (sec). Leaf size: 26

```
DSolve[y[x]*y''[x]+y'[x]^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x + c_2}{W\left(e^{-1-c_1}(x + c_2)\right)}$$

1.51 problem 51

Internal problem ID [7440]
Internal file name [`OUTPUT/6407_Sunday_June_05_2022_04_46_47_PM_1899037/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 51.
ODE order: 2.
ODE degree: 3.

The type(s) of ODE detected by this program : **"algebraic", "second_order_ode_missing_x"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{yy''^3 + y^3y' = 0}$$

The ode

$$yy''^3 + y^3y' = 0$$

is factored to

$$y\left(y^2y' + y''^3\right) = 0$$

Which gives the following equations

$$y = 0 \tag{1}$$

$$y^2y' + y''^3 = 0 \tag{2}$$

Each of the above equations is now solved.

Solving ODE (1) Since $y = 0$, is missing derivative in y then it is an algebraic equation. Solving for y.

$$y = 0$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$

Verification of solutions

$$y = 0$$

Verified OK.

Solving ODE (2) This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$y^2 p(y) + p(y)^3 \left(\frac{d}{dy}p(y)\right)^3 = 0$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\frac{d}{dy}p(y) = \frac{(-y^2 p(y))^{\frac{1}{3}}}{p(y)} \tag{1}$$

$$\frac{d}{dy}p(y) = -\frac{(-y^2 p(y))^{\frac{1}{3}}}{2p(y)} - \frac{i\sqrt{3}\,(-y^2 p(y))^{\frac{1}{3}}}{2p(y)} \tag{2}$$

$$\frac{d}{dy}p(y) = -\frac{(-y^2 p(y))^{\frac{1}{3}}}{2p(y)} + \frac{i\sqrt{3}\,(-y^2 p(y))^{\frac{1}{3}}}{2p(y)} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$\frac{d}{dy}p(y) = \frac{(-y^2p)^{\frac{1}{3}}}{p}$$
$$\frac{d}{dy}p(y) = \omega(y, p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + ya_2 + a_1 \tag{1E}$$
$$\eta = pb_3 + yb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(-y^2p)^{\frac{1}{3}}(b_3 - a_2)}{p} - \frac{(-y^2p)^{\frac{2}{3}}a_3}{p^2} + \frac{2y(pa_3 + ya_2 + a_1)}{3(-y^2p)^{\frac{2}{3}}} - \left(-\frac{(-y^2p)^{\frac{1}{3}}}{p^2} - \frac{y^2}{3p(-y^2p)^{\frac{2}{3}}}\right)(pb_3 + yb_2 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{3(-y^2p)^{\frac{4}{3}}a_3 - 5p^2y^2a_2 + 5p^2y^2b_3 + 2p\,y^3b_2 - 3b_2p^2(-y^2p)^{\frac{2}{3}} - 2p^3ya_3 + 2p\,y^2b_1 - 2p^2ya_1}{3p^2(-y^2p)^{\frac{2}{3}}} = 0$$

Setting the numerator to zero gives

$$-3(-y^2p)^{\frac{4}{3}}a_3 + 3b_2p^2(-y^2p)^{\frac{2}{3}} + 2p^3ya_3 + 5p^2y^2a_2 - 5p^2y^2b_3 - 2p\,y^3b_2 + 2p^2ya_1 - 2p\,y^2b_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
p\Big(3\left(-y^2p\right)^{\frac{2}{3}} p b_2 + 3\left(-y^2p\right)^{\frac{1}{3}} y^2 a_3 + 2p^2 y a_3 \\
+ 5p\,y^2 a_2 - 5p\,y^2 b_3 - 2y^3 b_2 + 2pya_1 - 2y^2 b_1\Big) = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{p, y\}$ in them.

$$
\left\{p, y, \left(-y^2p\right)^{\frac{1}{3}}, \left(-y^2p\right)^{\frac{2}{3}}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{p, y\}$ in them

$$
\left\{p = v_1, y = v_2, \left(-y^2p\right)^{\frac{1}{3}} = v_3, \left(-y^2p\right)^{\frac{2}{3}} = v_4\right\}
$$

The above PDE (6E) now becomes

$$
v_1\left(5v_1v_2^2a_2 + 2v_1^2v_2a_3 + 3v_3v_2^2a_3 - 2v_2^3b_2 - 5v_1v_2^2b_3 + 2v_1v_2a_1 - 2v_2^2b_1 + 3v_4v_1b_2\right) = 0 \quad \text{(7E)}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4\}
$$

Equation (7E) now becomes

$$
2a_3v_2v_1^3 + (5a_2 - 5b_3)\, v_2^2v_1^2 + 2a_1v_2v_1^2 + 3b_2v_4v_1^2 + 3a_3v_2^2v_3v_1 - 2b_2v_2^3v_1 - 2b_1v_2^2v_1 = 0 \quad \text{(8E)}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
2a_1 &= 0 \\
2a_3 &= 0 \\
3a_3 &= 0 \\
-2b_1 &= 0 \\
-2b_2 &= 0 \\
3b_2 &= 0 \\
5a_2 - 5b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= y \\
\eta &= p
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(y,p)\,\xi \\
&= p - \left(\frac{(-y^2 p)^{\frac{1}{3}}}{p}\right)(y) \\
&= \frac{p^2 - y(-y^2 p)^{\frac{1}{3}}}{p} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y,p) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial p}\right) S(y,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{p^2 - y(-y^2 p)^{\frac{1}{3}}}{p}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{3 \ln \left((-y^2 p)^{\frac{5}{3}} - y^5 \right)}{5}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_y + \omega(y,p) S_p}{R_y + \omega(y,p) R_p} \tag{2}
$$

Where in the above R_y, R_p, S_y, S_p are all partial derivatives and $\omega(y,p)$ is the right hand side of the original ode given by

$$
\omega(y,p) = \frac{(-y^2 p)^{\frac{1}{3}}}{p}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_y &= 1 \\
R_p &= 0 \\
S_y &= \frac{\left(2i\sqrt{3} - 2\right) y^{\frac{1}{3}} p^{\frac{5}{3}} + 6y^2}{y \left(y^{\frac{1}{3}} \left(i\sqrt{3} - 1\right) p^{\frac{5}{3}} + 2y^2 \right)} \\
S_p &= \frac{\left(i\sqrt{3} - 1\right) p^{\frac{2}{3}} y^{\frac{1}{3}}}{y^{\frac{1}{3}} \left(i\sqrt{3} - 1\right) p^{\frac{5}{3}} + 2y^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{y^{\frac{4}{3}} \left(i\sqrt{3} - 1\right) (-y^2 p)^{\frac{1}{3}} + 2p^2 \left(i\sqrt{3} - 1\right) y^{\frac{1}{3}} + 6p^{\frac{1}{3}} y^2}{p^{\frac{1}{3}} \left(y^{\frac{1}{3}} \left(i\sqrt{3} - 1\right) p^{\frac{5}{3}} + 2y^2 \right) y} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for y, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{2}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to y, p coordinates. This results in

$$-\frac{3\ln(2)}{5} - \frac{3i\pi}{10} + \frac{3\ln\left(p(y)^{\frac{5}{3}}\left(\sqrt{3}+i\right)y^{\frac{10}{3}} - 2iy^5\right)}{5} = 2\ln(y) + c_1$$

Which simplifies to

$$-\frac{3\ln(2)}{5} - \frac{3i\pi}{10} + \frac{3\ln\left(p(y)^{\frac{5}{3}}\left(\sqrt{3}+i\right)y^{\frac{10}{3}} - 2iy^5\right)}{5} = 2\ln(y) + c_1$$

Solving equation (2)

Writing the ode as

$$\frac{d}{dy}p(y) = -\frac{(-y^2p)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{2p}$$
$$\frac{d}{dy}p(y) = \omega(y, p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + ya_2 + a_1 \tag{1E}$$
$$\eta = pb_3 + yb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2-\frac{\left(-y^2 p\right)^{\frac{1}{3}}\left(1+i \sqrt{3}\right)\left(b_3-a_2\right)}{2 p} \\
& -\frac{\left(-y^2 p\right)^{\frac{2}{3}}\left(1+i \sqrt{3}\right)^2 a_3}{4 p^2}-\frac{\left(1+i \sqrt{3}\right) y\left(p a_3+y a_2+a_1\right)}{3\left(-y^2 p\right)^{\frac{2}{3}}} \\
& -\left(\frac{y^2\left(1+i \sqrt{3}\right)}{6 p\left(-y^2 p\right)^{\frac{2}{3}}}+\frac{\left(-y^2 p\right)^{\frac{1}{3}}\left(1+i \sqrt{3}\right)}{2 p^2}\right)\left(p b_3+y b_2+b_1\right)=0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{3 i \sqrt{3}\left(-y^2 p\right)^{\frac{4}{3}} a_3+2 i \sqrt{3}\, p^3 y a_3+5 i \sqrt{3}\, p^2 y^2 a_2-5 i \sqrt{3}\, p^2 y^2 b_3-2 i \sqrt{3}\, p\, y^3 b_2+2 i \sqrt{3}\, p^2 y a_1-2 i \sqrt{3}\, p\, y^2 b_1 \cdots}{6 p^2\left(-y^2 p\right)^{\frac{2}{3}}}
$$

$$
=0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& -3 i \sqrt{3}\left(-y^2 p\right)^{\frac{4}{3}} a_3-2 i \sqrt{3}\, p^3 y a_3-5 i \sqrt{3}\, p^2 y^2 a_2+5 i \sqrt{3}\, p^2 y^2 b_3 \\
& +2 i \sqrt{3}\, p\, y^3 b_2-2 i \sqrt{3}\, p^2 y a_1+2 i \sqrt{3}\, p\, y^2 b_1+3\left(-y^2 p\right)^{\frac{4}{3}} a_3+6 b_2 p^2\left(-y^2 p\right)^{\frac{2}{3}} \\
& -2 p^3 y a_3-5 p^2 y^2 a_2+5 p^2 y^2 b_3+2 p\, y^3 b_2-2 p^2 y a_1+2 p\, y^2 b_1=0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
& 6\left(-y^2 p\right)^{\frac{4}{3}} a_3+4 i \sqrt{3}\, p\, y^2 b_1-4 i \sqrt{3}\, p^2 y a_1-6 i \sqrt{3}\left(-y^2 p\right)^{\frac{4}{3}} a_3-10 i \sqrt{3}\, p^2 y^2 a_2 \\
& -4 i \sqrt{3}\, p^3 y a_3+10 i \sqrt{3}\, p^2 y^2 b_3+4 i \sqrt{3}\, p\, y^3 b_2-10 p^2 y^2 a_2+10 p^2 y^2 b_3 \\
& +4 p\, y^3 b_2+12 b_2 p^2\left(-y^2 p\right)^{\frac{2}{3}}-4 p^3 y a_3+4 p\, y^2 b_1-4 p^2 y a_1=0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
& 2 p\Big(3 i \sqrt{3}\left(-y^2 p\right)^{\frac{1}{3}} y^2 a_3-2 i \sqrt{3}\, p^2 y a_3-5 i \sqrt{3}\, p\, y^2 a_2+5 i \sqrt{3}\, p\, y^2 b_3 \\
& +2 i \sqrt{3}\, y^3 b_2-2 i \sqrt{3}\, p y a_1+2 i \sqrt{3}\, y^2 b_1+6\left(-y^2 p\right)^{\frac{2}{3}} p b_2-3\left(-y^2 p\right)^{\frac{1}{3}} y^2 a_3 \\
& -2 p^2 y a_3-5 p\, y^2 a_2+5 p\, y^2 b_3+2 y^3 b_2-2 p y a_1+2 y^2 b_1\Big)=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{p, y\}$ in them.

$$\left\{p, y, \left(-y^2 p\right)^{\frac{1}{3}}, \left(-y^2 p\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{p, y\}$ in them

$$\left\{p = v_1, y = v_2, \left(-y^2 p\right)^{\frac{1}{3}} = v_3, \left(-y^2 p\right)^{\frac{2}{3}} = v_4\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 2v_1\Big(3i\sqrt{3}\, v_3 v_2^2 a_3 - 2i\sqrt{3}\, v_1^2 v_2 a_3 - 5i\sqrt{3}\, v_1 v_2^2 a_2 + 5i\sqrt{3}\, v_1 v_2^2 b_3 \\ & + 2i\sqrt{3}\, v_2^3 b_2 - 2i\sqrt{3}\, v_1 v_2 a_1 + 2i\sqrt{3}\, v_2^2 b_1 + 6 v_4 v_1 b_2 - 3 v_3 v_2^2 a_3 \\ & - 2v_1^2 v_2 a_3 - 5 v_1 v_2^2 a_2 + 5 v_1 v_2^2 b_3 + 2 v_2^3 b_2 - 2 v_1 v_2 a_1 + 2 v_2^2 b_1\Big) = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned} & \left(-4i\sqrt{3}\, a_3 - 4a_3\right) v_2 v_1^3 + \left(-10i\sqrt{3}\, a_2 + 10i\sqrt{3}\, b_3 - 10a_2 + 10b_3\right) v_2^2 v_1^2 \\ & + \left(-4i\sqrt{3}\, a_1 - 4a_1\right) v_2 v_1^2 + 12 b_2 v_4 v_1^2 + \left(4i\sqrt{3}\, b_2 + 4b_2\right) v_2^3 v_1 \\ & + \left(6i\sqrt{3}\, a_3 - 6a_3\right) v_2^2 v_3 v_1 + \left(4i\sqrt{3}\, b_1 + 4b_1\right) v_2^2 v_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 12b_2 &= 0 \\ -4i\sqrt{3}\, a_1 - 4a_1 &= 0 \\ -4i\sqrt{3}\, a_3 - 4a_3 &= 0 \\ 4i\sqrt{3}\, b_1 + 4b_1 &= 0 \\ 4i\sqrt{3}\, b_2 + 4b_2 &= 0 \\ 6i\sqrt{3}\, a_3 - 6a_3 &= 0 \\ -10i\sqrt{3}\, a_2 + 10i\sqrt{3}\, b_3 - 10a_2 + 10b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= y \\
\eta &= p
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(y,p)\,\xi \\
&= p - \left(-\frac{(-y^2p)^{\frac{1}{3}}\,(1+i\sqrt{3})}{2p}\right)(y) \\
&= \frac{i\sqrt{3}\,(-y^2p)^{\frac{1}{3}}\,y + y(-y^2p)^{\frac{1}{3}} + 2p^2}{2p} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y,p) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial p}\right) S(y,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{i\sqrt{3}\,(-y^2p)^{\frac{1}{3}}y+y(-y^2p)^{\frac{1}{3}}+2p^2}{2p}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{3\ln\left(i\sqrt{3}\,y^5 + 2(-y^2p)^{\frac{5}{3}} + y^5\right)}{5}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_y + \omega(y,p)S_p}{R_y + \omega(y,p)R_p} \tag{2}
$$

Where in the above R_y, R_p, S_y, S_p are all partial derivatives and $\omega(y,p)$ is the right hand side of the original ode given by

$$
\omega(y,p) = -\frac{(-y^2p)^{\frac{1}{3}}\,(1+i\sqrt{3})}{2p}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_y &= 1 \\
R_p &= 0 \\
S_y &= \frac{2\left(\sqrt{3}+i\right)p^{\frac{5}{3}}y^{\frac{1}{3}} + 3y^2\left(i-\sqrt{3}\right)}{y\left(\left(\sqrt{3}+i\right)p^{\frac{5}{3}}y^{\frac{1}{3}} + y^2\left(i-\sqrt{3}\right)\right)} \\
S_p &= \frac{p^{\frac{2}{3}}\left(\sqrt{3}+i\right)y^{\frac{1}{3}}}{\left(\sqrt{3}+i\right)p^{\frac{5}{3}}y^{\frac{1}{3}} + y^2\left(i-\sqrt{3}\right)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{-2iy^{\frac{4}{3}}(-y^2p)^{\frac{1}{3}} + 2p^2y^{\frac{1}{3}}\left(\sqrt{3}+i\right) + 3y^2p^{\frac{1}{3}}\left(i-\sqrt{3}\right)}{\left(\left(\sqrt{3}+i\right)p^{\frac{5}{3}}y^{\frac{1}{3}} + y^2\left(i-\sqrt{3}\right)\right)p^{\frac{1}{3}}y} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for y, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{2}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to y, p coordinates. This results in

$$-\frac{3i\pi}{10} + \frac{3\ln\left(p(y)^{\frac{5}{3}}\left(\sqrt{3}+i\right)y^{\frac{10}{3}} + y^5\left(i-\sqrt{3}\right)\right)}{5} = 2\ln(y) + c_1$$

Which simplifies to

$$-\frac{3i\pi}{10} + \frac{3\ln\left(p(y)^{\frac{5}{3}}\left(\sqrt{3}+i\right)y^{\frac{10}{3}} + y^5\left(i-\sqrt{3}\right)\right)}{5} = 2\ln(y) + c_1$$

Solving equation (3)

Writing the ode as

$$\begin{aligned}\frac{d}{dy}p(y) &= \frac{\left(-y^2p\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{2p} \\ \frac{d}{dy}p(y) &= \omega(y,p)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + ya_2 + a_1 \tag{1E}$$

$$\eta = pb_3 + yb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2+\frac{\left(-y^2p\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)\left(b_3-a_2\right)}{2p}\\
&-\frac{\left(-y^2p\right)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)^2a_3}{4p^2}+\frac{\left(i\sqrt{3}-1\right)y(pa_3+ya_2+a_1)}{3\left(-y^2p\right)^{\frac{2}{3}}}\\
&-\left(-\frac{y^2\left(i\sqrt{3}-1\right)}{6p\left(-y^2p\right)^{\frac{2}{3}}}-\frac{\left(-y^2p\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{2p^2}\right)(pb_3+yb_2+b_1)=0
\end{aligned}\tag{5E}
$$

Putting the above in normal form gives

$$
\frac{3i\sqrt{3}\left(-y^2p\right)^{\frac{4}{3}}a_3+2i\sqrt{3}\,p^3ya_3+5i\sqrt{3}\,p^2y^2a_2-5i\sqrt{3}\,p^2y^2b_3-2i\sqrt{3}\,p\,y^3b_2+2i\sqrt{3}\,p^2ya_1-2i\sqrt{3}\,p\,y^2b_1+}{6p^2\left(-y^2p\right)^{\frac{2}{3}}}
$$

$$
=0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&3i\sqrt{3}\left(-y^2p\right)^{\frac{4}{3}}a_3+2i\sqrt{3}\,p^3ya_3+5i\sqrt{3}\,p^2y^2a_2-5i\sqrt{3}\,p^2y^2b_3\\
&-2i\sqrt{3}\,p\,y^3b_2+2i\sqrt{3}\,p^2ya_1-2i\sqrt{3}\,p\,y^2b_1+3\left(-y^2p\right)^{\frac{4}{3}}a_3+6b_2p^2\left(-y^2p\right)^{\frac{2}{3}}\\
&-2p^3ya_3-5p^2y^2a_2+5p^2y^2b_3+2p\,y^3b_2-2p^2ya_1+2p\,y^2b_1=0
\end{aligned}\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&6\left(-y^2p\right)^{\frac{4}{3}}a_3+4i\sqrt{3}\,p^3ya_3-4i\sqrt{3}\,p\,y^3b_2+6i\sqrt{3}\left(-y^2p\right)^{\frac{4}{3}}a_3-4i\sqrt{3}\,p\,y^2b_1\\
&+4i\sqrt{3}\,p^2ya_1-10i\sqrt{3}\,p^2y^2b_3+10i\sqrt{3}\,p^2y^2a_2-10p^2y^2a_2+10p^2y^2b_3\\
&+4p\,y^3b_2+12b_2p^2\left(-y^2p\right)^{\frac{2}{3}}-4p^3ya_3+4p\,y^2b_1-4p^2ya_1=0
\end{aligned}\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-2p\Big(3i\sqrt{3}\left(-y^2p\right)^{\frac{1}{3}}y^2a_3-2i\sqrt{3}\,p^2ya_3-5i\sqrt{3}\,p\,y^2a_2+5i\sqrt{3}\,p\,y^2b_3\\
&+2i\sqrt{3}\,y^3b_2-2i\sqrt{3}\,pya_1+2i\sqrt{3}\,y^2b_1-6\left(-y^2p\right)^{\frac{2}{3}}pb_2+3\left(-y^2p\right)^{\frac{1}{3}}y^2a_3\\
&+2p^2ya_3+5p\,y^2a_2-5p\,y^2b_3-2y^3b_2+2pya_1-2y^2b_1\Big)=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{p, y\}$ in them.

$$\left\{p, y, \left(-y^{2} p\right)^{\frac{1}{3}}, \left(-y^{2} p\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{p, y\}$ in them

$$\left\{p = v_{1}, y = v_{2}, \left(-y^{2} p\right)^{\frac{1}{3}} = v_{3}, \left(-y^{2} p\right)^{\frac{2}{3}} = v_{4}\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-2 v_{1}\Big(3 i \sqrt{3}\, v_{3} v_{2}^{2} a_{3}-2 i \sqrt{3}\, v_{1}^{2} v_{2} a_{3}-5 i \sqrt{3}\, v_{1} v_{2}^{2} a_{2}+5 i \sqrt{3}\, v_{1} v_{2}^{2} b_{3} \\
&+2 i \sqrt{3}\, v_{2}^{3} b_{2}-2 i \sqrt{3}\, v_{1} v_{2} a_{1}+2 i \sqrt{3}\, v_{2}^{2} b_{1}-6 v_{4} v_{1} b_{2}+3 v_{3} v_{2}^{2} a_{3} \\
&+2 v_{1}^{2} v_{2} a_{3}+5 v_{1} v_{2}^{2} a_{2}-5 v_{1} v_{2}^{2} b_{3}-2 v_{2}^{3} b_{2}+2 v_{1} v_{2} a_{1}-2 v_{2}^{2} b_{1}\Big)=0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_{1}, v_{2}, v_{3}, v_{4}\}$$

Equation (7E) now becomes

$$\begin{aligned}
&\left(4 i \sqrt{3}\, a_{3}-4 a_{3}\right) v_{2} v_{1}^{3}+\left(10 i \sqrt{3}\, a_{2}-10 i \sqrt{3}\, b_{3}-10 a_{2}+10 b_{3}\right) v_{2}^{2} v_{1}^{2} \\
&+\left(4 i \sqrt{3}\, a_{1}-4 a_{1}\right) v_{2} v_{1}^{2}+12 b_{2} v_{4} v_{1}^{2}+\left(-4 i \sqrt{3}\, b_{2}+4 b_{2}\right) v_{2}^{3} v_{1} \\
&+\left(-6 i \sqrt{3}\, a_{3}-6 a_{3}\right) v_{2}^{2} v_{3} v_{1}+\left(-4 i \sqrt{3}\, b_{1}+4 b_{1}\right) v_{2}^{2} v_{1}=0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
12 b_{2} &= 0 \\
-6 i \sqrt{3}\, a_{3}-6 a_{3} &= 0 \\
-4 i \sqrt{3}\, b_{1}+4 b_{1} &= 0 \\
-4 i \sqrt{3}\, b_{2}+4 b_{2} &= 0 \\
4 i \sqrt{3}\, a_{1}-4 a_{1} &= 0 \\
4 i \sqrt{3}\, a_{3}-4 a_{3} &= 0 \\
10 i \sqrt{3}\, a_{2}-10 i \sqrt{3}\, b_{3}-10 a_{2}+10 b_{3} &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= y \\
\eta &= p
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(y,p)\,\xi \\
&= p - \left(\frac{(-y^2p)^{\frac{1}{3}}\,(i\sqrt{3}-1)}{2p}\right)(y) \\
&= \frac{-i\sqrt{3}\,(-y^2p)^{\frac{1}{3}}\,y + y(-y^2p)^{\frac{1}{3}} + 2p^2}{2p} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y,p) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial p}\right) S(y,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-i\sqrt{3}\,(-y^2 p)^{\frac{1}{3}} y + y(-y^2 p)^{\frac{1}{3}} + 2p^2}{2p}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{3 \ln \left(i\sqrt{3}\, y^5 - 2(-y^2 p)^{\frac{5}{3}} - y^5\right)}{5}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_y + \omega(y,p) S_p}{R_y + \omega(y,p) R_p} \tag{2}
$$

Where in the above R_y, R_p, S_y, S_p are all partial derivatives and $\omega(y,p)$ is the right hand side of the original ode given by

$$
\omega(y,p) = \frac{(-y^2 p)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2p}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_y &= 1 \\
R_p &= 0 \\
S_y &= \frac{y^{\frac{1}{3}} \left(2p^{\frac{5}{3}} + 3y^{\frac{5}{3}}\right)}{\left(y^{\frac{2}{3}} p^{\frac{1}{3}} + y\right) \left(p^{\frac{4}{3}} y^{\frac{2}{3}} + p^{\frac{2}{3}} y^{\frac{4}{3}} - p^{\frac{1}{3}} y^{\frac{5}{3}} - yp + y^2\right)} \\
S_p &= \frac{y^{\frac{4}{3}} p^{\frac{2}{3}}}{\left(y^{\frac{2}{3}} p^{\frac{1}{3}} + y\right) \left(p^{\frac{4}{3}} y^{\frac{2}{3}} + p^{\frac{2}{3}} y^{\frac{4}{3}} - p^{\frac{1}{3}} y^{\frac{5}{3}} - yp + y^2\right)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{y^{\frac{1}{3}} \left(y(-y^2 p)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right) + 4p^2 + 6p^{\frac{1}{3}} y^{\frac{5}{3}}\right)}{2p^{\frac{1}{3}} \left(-p^{\frac{4}{3}} y^{\frac{2}{3}} + p^{\frac{1}{3}} y^{\frac{5}{3}} - p^{\frac{2}{3}} y^{\frac{4}{3}} + y\,(p - y)\right) \left(y^{\frac{2}{3}} p^{\frac{1}{3}} + y\right)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for y, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{2}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to y, p coordinates. This results in

$$\frac{2i\pi}{5} + \frac{3\ln(2)}{5} + \frac{3\ln\left(y^{\frac{2}{3}}p(y)^{\frac{1}{3}} + y\right)}{5} + \frac{3\ln\left(y^{\frac{8}{3}}p(y)^{\frac{4}{3}} - p(y)\,y^3 + y^{\frac{10}{3}}p(y)^{\frac{2}{3}} - y^{\frac{11}{3}}p(y)^{\frac{1}{3}} + y^4\right)}{5} = 2\ln(y) +$$

Which simplifies to

$$\frac{2i\pi}{5} + \frac{3\ln(2)}{5} + \frac{3\ln\left(y^{\frac{2}{3}}p(y)^{\frac{1}{3}} + y\right)}{5} + \frac{3\ln\left(y^{\frac{8}{3}}p(y)^{\frac{4}{3}} - p(y)\,y^3 + y^{\frac{10}{3}}p(y)^{\frac{2}{3}} - y^{\frac{11}{3}}p(y)^{\frac{1}{3}} + y^4\right)}{5} = 2\ln(y) +$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{3\ln(2)}{5} - \frac{3i\pi}{10} + \frac{3\ln\left(y'^{\frac{5}{3}}\left(\sqrt{3} + i\right)y^{\frac{10}{3}} - 2iy^5\right)}{5} = 2\ln(y) + c_1$$

Solving the given ode for y' results in 5 differential equations to solve. Each one of these

will generate a solution. The equations generated are

$$y' = \frac{2^{\frac{3}{5}} \left(-\frac{\left((-y)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_1}{3}}-i y^{5}\right) \left(\sqrt{3}+i\right)^{4}}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}{\left(\sqrt{3}+i\right)^{3}} \tag{1}$$

$$y' = \frac{\left(\frac{\sqrt{5}}{4}-\frac{1}{4}+\frac{i \sqrt{2}\, \sqrt{5+\sqrt{5}}}{4}\right)^{3} 2^{\frac{3}{5}} \left(-\frac{\left((-y)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_1}{3}}-i y^{5}\right) \left(\sqrt{3}+i\right)^{4}}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}{\left(\sqrt{3}+i\right)^{3}} \tag{2}$$

$$y' = \frac{\left(-\frac{\sqrt{5}}{4}-\frac{1}{4}+\frac{i \sqrt{2}\, \sqrt{5-\sqrt{5}}}{4}\right)^{3} 2^{\frac{3}{5}} \left(-\frac{\left((-y)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_1}{3}}-i y^{5}\right) \left(\sqrt{3}+i\right)^{4}}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}{\left(\sqrt{3}+i\right)^{3}} \tag{3}$$

$$y' = \frac{\left(-\frac{\sqrt{5}}{4}-\frac{1}{4}-\frac{i \sqrt{2}\, \sqrt{5-\sqrt{5}}}{4}\right)^{3} 2^{\frac{3}{5}} \left(-\frac{\left((-y)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_1}{3}}-i y^{5}\right) \left(\sqrt{3}+i\right)^{4}}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}{\left(\sqrt{3}+i\right)^{3}} \tag{4}$$

$$y' = \frac{\left(\frac{\sqrt{5}}{4}-\frac{1}{4}-\frac{i \sqrt{2}\, \sqrt{5+\sqrt{5}}}{4}\right)^{3} 2^{\frac{3}{5}} \left(-\frac{\left((-y)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_1}{3}}-i y^{5}\right) \left(\sqrt{3}+i\right)^{4}}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}{\left(\sqrt{3}+i\right)^{3}} \tag{5}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\int \frac{2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3}}{2 \left(-\frac{\left((-y)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_1}{3}}-i y^{5}\right) \left(\sqrt{3}+i\right)^{4}}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}dy = \int dx$$

$$\frac{2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3} \left(\int^{y}\frac{1}{\left(-\frac{\left((-\_a)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_1}{3}}-i \_a^{5}\right) \left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{2} = x + c_4$$

Solving equation (2)

Integrating both sides gives

$$\int \frac{32\, 2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3}}{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}}+\sqrt{5}-1\right)^{3} \left(-\frac{\left(\left(-y\right)^{\frac{10}{3}} \mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i y^{5}\right)\left(\sqrt{3}+i\right)^{4}}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}dy = \int dx$$

$$\frac{32\, 2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3} \left(\int^{y} \frac{1}{\left(-\frac{\left(\left(-\_a\right)^{\frac{10}{3}} \mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}}+\sqrt{5}-1\right)^{3}} = x+c_5$$

Solving equation (3)

Integrating both sides gives

$$\int \frac{32\, 2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3}}{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}}-\sqrt{5}-1\right)^{3} \left(-\frac{\left(\left(-y\right)^{\frac{10}{3}} \mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i y^{5}\right)\left(\sqrt{3}+i\right)^{4}}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}dy = \int dx$$

$$\frac{32\, 2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3} \left(\int^{y} \frac{1}{\left(-\frac{\left(\left(-\_a\right)^{\frac{10}{3}} \mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}}-\sqrt{5}-1\right)^{3}} = x+c_6$$

Solving equation (4)

Integrating both sides gives

$$\int -\frac{32\, 2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3}}{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}}+\sqrt{5}+1\right)^{3} \left(-\frac{\left(\left(-y\right)^{\frac{10}{3}} \mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i y^{5}\right)\left(\sqrt{3}+i\right)^{4}}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}dy = \int dx$$

$$
-\frac{32\,2^{\frac{2}{5}}\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(-\frac{\left((-\_a)^{\frac{10}{3}}\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}}+\sqrt{5}+1\right)^{3}}=x+c_7
$$

Solving equation (5)

Integrating both sides gives

$$
\int -\frac{32\,2^{\frac{2}{5}}\left(\sqrt{3}+i\right)^{3}}{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}}-\sqrt{5}+1\right)^{3}\left(-\frac{\left((-y)^{\frac{10}{3}}\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-iy^{5}\right)\left(\sqrt{3}+i\right)^{4}}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}dy=\int dx
$$

$$
-\frac{32\,2^{\frac{2}{5}}\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(-\frac{\left((-\_a)^{\frac{10}{3}}\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}}-\sqrt{5}+1\right)^{3}}=x+c_8
$$

For solution (2) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$
-\frac{3i\pi}{10}+\frac{3\ln\left(y'^{\frac{5}{3}}\left(\sqrt{3}+i\right)y^{\frac{10}{3}}+y^{5}\left(i-\sqrt{3}\right)\right)}{5}=2\ln\left(y\right)+c_1
$$

Solving the given ode for y' results in 5 differential equations to solve. Each one of these

will generate a solution. The equations generated are

$$y' = \frac{\left(\frac{\left(-e^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln(-y)}{3}}-iy^5+y^5\sqrt{3}\right)\left(\sqrt{3}+i\right)^4}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}{\left(\sqrt{3}+i\right)^3} \tag{1}$$

$$y' = \frac{\left(\frac{\sqrt{5}}{4}-\frac{1}{4}+\frac{i\sqrt{2}\,\sqrt{5+\sqrt{5}}}{4}\right)^3\left(\frac{\left(-e^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln(-y)}{3}}-iy^5+y^5\sqrt{3}\right)\left(\sqrt{3}+i\right)^4}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}{\left(\sqrt{3}+i\right)^3} \tag{2}$$

$$y' = \frac{\left(-\frac{\sqrt{5}}{4}-\frac{1}{4}+\frac{i\sqrt{2}\,\sqrt{5-\sqrt{5}}}{4}\right)^3\left(\frac{\left(-e^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln(-y)}{3}}-iy^5+y^5\sqrt{3}\right)\left(\sqrt{3}+i\right)^4}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}{\left(\sqrt{3}+i\right)^3} \tag{3}$$

$$y' = \frac{\left(-\frac{\sqrt{5}}{4}-\frac{1}{4}-\frac{i\sqrt{2}\,\sqrt{5-\sqrt{5}}}{4}\right)^3\left(\frac{\left(-e^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln(-y)}{3}}-iy^5+y^5\sqrt{3}\right)\left(\sqrt{3}+i\right)^4}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}{\left(\sqrt{3}+i\right)^3} \tag{4}$$

$$y' = \frac{\left(\frac{\sqrt{5}}{4}-\frac{1}{4}-\frac{i\sqrt{2}\,\sqrt{5+\sqrt{5}}}{4}\right)^3\left(\frac{\left(-e^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln(-y)}{3}}-iy^5+y^5\sqrt{3}\right)\left(\sqrt{3}+i\right)^4}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}{\left(\sqrt{3}+i\right)^3} \tag{5}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\int \frac{\left(\sqrt{3}+i\right)^3}{\left(\frac{\left(-e^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln(-y)}{3}}+\sqrt{3}\,y^5-iy^5\right)\left(\sqrt{3}+i\right)^4}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}dy = \int dx$$

$$\left(\sqrt{3}+i\right)^3\left(\int^y \frac{1}{\left(\frac{\left(-e^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln(-\_a)}{3}}+\sqrt{3}\,\_a^5-i\_a^5\right)\left(\sqrt{3}+i\right)^4}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right) = x+c_9$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\int \frac{64\left(\sqrt{3}+i\right)^{3}}{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}}+\sqrt{5}-1\right)^{3}\left(\frac{\left(-\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln(-y)}{3}}+\sqrt{3}\,y^{5}-iy^{5}\right)\left(\sqrt{3}+i\right)^{4}}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}dy = \int dx$$

$$\frac{64\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(\frac{\left(-\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln(-\_a)}{3}}+\sqrt{3}\,\_a^{5}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}}+\sqrt{5}-1\right)^{3}} = x+\_C10$$

Solving equation (3)

Integrating both sides gives

$$\int \frac{64\left(\sqrt{3}+i\right)^{3}}{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}}-\sqrt{5}-1\right)^{3}\left(\frac{\left(-\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln(-y)}{3}}+\sqrt{3}\,y^{5}-iy^{5}\right)\left(\sqrt{3}+i\right)^{4}}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}dy = \int dx$$

$$\frac{64\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(\frac{\left(-\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln(-\_a)}{3}}+\sqrt{3}\,\_a^{5}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}}-\sqrt{5}-1\right)^{3}} = x+\_C11$$

Solving equation (4)

Integrating both sides gives

$$\int -\frac{64\left(\sqrt{3}+i\right)^{3}}{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}}+\sqrt{5}+1\right)^{3}\left(\frac{\left(-\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln(-y)}{3}}+\sqrt{3}\,y^{5}-iy^{5}\right)\left(\sqrt{3}+i\right)^{4}}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}dy = \int dx$$

$$
-\frac{64\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(\frac{\left(-\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln(-\_a)}{3}}+\sqrt{3}\,\_a^{5}-i\,\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}}+\sqrt{5}+1\right)^{3}}=x+\_C12
$$

Solving equation (5)

Integrating both sides gives

$$
\int-\frac{64\left(\sqrt{3}+i\right)^{3}}{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}}-\sqrt{5}+1\right)^{3}\left(\frac{\left(-\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln(-y)}{3}}+\sqrt{3}\,y^{5}-iy^{5}\right)\left(\sqrt{3}+i\right)^{4}}{y^{\frac{10}{3}}}\right)^{\frac{3}{5}}}dy=\int dx
$$

$$
-\frac{64\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(\frac{\left(-\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln(-\_a)}{3}}+\sqrt{3}\,\_a^{5}-i\,\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}}-\sqrt{5}+1\right)^{3}}=x+\_C13
$$

For solution (3) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$
\frac{2i\pi}{5}+\frac{3\ln(2)}{5}+\frac{3\ln\left(y^{\frac{2}{3}}y'^{\frac{1}{3}}+y\right)}{5}+\frac{3\ln\left(y^{\frac{8}{3}}y'^{\frac{4}{3}}-y^{3}y'+y^{\frac{10}{3}}y'^{\frac{2}{3}}-y^{\frac{11}{3}}y'^{\frac{1}{3}}+y^{4}\right)}{5}=2\ln(y)+c_1
$$

Integrating both sides gives

$$
\int-\frac{y}{\left(-64\,\mathrm{RootOf}\left(16777216\_Z^{5}\mathrm{e}^{-\frac{20c_1}{3}-\frac{20\ln\left(-\frac{8}{y}\right)}{3}}+40y^{6}\_Z^{4}\mathrm{e}^{-5c_1}+40960y^{2}\_Z^{3}\mathrm{e}^{-\frac{10c_1}{3}-\cdots}\right.\right.}
$$

$$
\int^{y}-\frac{\_a^{2}}{\left(-64\,\mathrm{RootOf}\left(16777216\_Z^{5}\mathrm{e}^{-\frac{20c_1}{3}-\frac{20\ln\left(-\frac{8}{\_a}\right)}{3}}+40\_a^{6}\_Z^{4}\mathrm{e}^{-5c_1}+40960\_a^{2}\_Z^{3}\mathrm{e}^{-\frac{10c_1}{3}-\frac{10\ln\left(-\frac{8}{\_a}\right)}{3}}-\cdots\right.\right.}
$$

Summary

The solution(s) found are the following

$$\frac{2^{\frac{2}{5}}\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(-\frac{\left(\left(-\_a\right)^{\frac{10}{3}}{\mathrm e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{2}=x+c_4 \tag{1}$$

$$\frac{32\,2^{\frac{2}{5}}\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(-\frac{\left(\left(-\_a\right)^{\frac{10}{3}}{\mathrm e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}}+\sqrt{5}-1\right)^{3}}=x+c_5 \tag{2}$$

$$\frac{32\,2^{\frac{2}{5}}\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(-\frac{\left(\left(-\_a\right)^{\frac{10}{3}}{\mathrm e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}}-\sqrt{5}-1\right)^{3}}=x+c_6 \tag{3}$$

$$-\frac{32\,2^{\frac{2}{5}}\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(-\frac{\left(\left(-\_a\right)^{\frac{10}{3}}{\mathrm e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}}+\sqrt{5}+1\right)^{3}}=x+c_7 \tag{4}$$

$$-\frac{32\,2^{\frac{2}{5}}\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(-\frac{\left(\left(-\_a\right)^{\frac{10}{3}}{\mathrm e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}}-\sqrt{5}+1\right)^{3}}=x+c_8 \tag{5}$$

$$\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(\frac{\left(-{\mathrm e}^{\frac{i\pi}{6}+\frac{5c_1}{3}+\frac{10\ln\left(-\_a\right)}{3}}+\sqrt{3}\,\_a^{5}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)=x+c_9 \tag{6}$$

Verification of solutions

$$\frac{2^{\frac{2}{5}}\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(-\frac{\left((-\_a)^{\frac{10}{3}}\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{2}=x+c_4$$

Verified OK.

$$\frac{32\,2^{\frac{2}{5}}\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(-\frac{\left((-\_a)^{\frac{10}{3}}\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}}+\sqrt{5}-1\right)^{3}}=x+c_5$$

Verified OK.

$$\frac{32\,2^{\frac{2}{5}}\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(-\frac{\left((-\_a)^{\frac{10}{3}}\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}}-\sqrt{5}-1\right)^{3}}=x+c_6$$

Verified OK.

$$-\frac{32\,2^{\frac{2}{5}}\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(-\frac{\left((-\_a)^{\frac{10}{3}}\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}}+\sqrt{5}+1\right)^{3}}=x+c_7$$

Verified OK.

$$-\frac{32\,2^{\frac{2}{5}}\left(\sqrt{3}+i\right)^{3}\left(\int^{y}\frac{1}{\left(-\frac{\left((-\_a)^{\frac{10}{3}}\mathrm{e}^{\frac{i\pi}{6}+\frac{5c_1}{3}}-i\_a^{5}\right)\left(\sqrt{3}+i\right)^{4}}{637\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a\right)}{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}}-\sqrt{5}+1\right)^{3}}=x+c_8$$

Summary

The solution(s) found are the following

$$\frac{2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3} \left(\int^{y} \frac{1}{\left(-\frac{\left(\left(-\_a\right)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_1}{3}}-i \_a^{5}\right) \left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a \right)}{2} = x + c_4 \tag{1}$$

$$\frac{32\, 2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3} \left(\int^{y} \frac{1}{\left(-\frac{\left(\left(-\_a\right)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_1}{3}}-i \_a^{5}\right) \left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a \right)}{\left(i \sqrt{2}\, \sqrt{5+\sqrt{5}}+\sqrt{5}-1\right)^{3}} = x + c_5 \tag{2}$$

$$\frac{32\, 2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3} \left(\int^{y} \frac{1}{\left(-\frac{\left(\left(-\_a\right)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_1}{3}}-i \_a^{5}\right) \left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a \right)}{\left(i \sqrt{2}\, \sqrt{5-\sqrt{5}}-\sqrt{5}-1\right)^{3}} = x + c_6 \tag{3}$$

$$-\frac{32\, 2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3} \left(\int^{y} \frac{1}{\left(-\frac{\left(\left(-\_a\right)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_1}{3}}-i \_a^{5}\right) \left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a \right)}{\left(i \sqrt{2}\, \sqrt{5-\sqrt{5}}+\sqrt{5}+1\right)^{3}} = x + c_7 \tag{4}$$

$$-\frac{32\, 2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3} \left(\int^{y} \frac{1}{\left(-\frac{\left(\left(-\_a\right)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_1}{3}}-i \_a^{5}\right) \left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a \right)}{\left(i \sqrt{2}\, \sqrt{5+\sqrt{5}}-\sqrt{5}+1\right)^{3}} = x + c_8 \tag{5}$$

$$\left(\sqrt{3}+i\right)^{3} \left(\int^{y} \frac{1}{\left(\frac{\left(-{\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_1}{3}+\frac{10 \ln\left(-\_a\right)}{3}}+\sqrt{3}\, \_a^{5}-i \_a^{5}\right) \left(\sqrt{3}+i\right)^{4}}{\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_a \right) = x + c_9 \tag{6}$$

Verification of solutions

$$\frac{2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3} \left(\int^{y} \frac{1}{\left(-\frac{\left(\left(-\_\_a\right)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_{1}}{3}}-i \_\_a^{5}\right) \left(\sqrt{3}+i\right)^{4}}{\_\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_\_a \right)}{2} = x +c_{4}$$

Verified OK.

$$\frac{32\, 2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3} \left(\int^{y} \frac{1}{\left(-\frac{\left(\left(-\_\_a\right)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_{1}}{3}}-i \_\_a^{5}\right) \left(\sqrt{3}+i\right)^{4}}{\_\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_\_a \right)}{\left(i \sqrt{2}\, \sqrt{5+\sqrt{5}}+\sqrt{5}-1\right)^{3}} = x +c_{5}$$

Verified OK.

$$\frac{32\, 2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3} \left(\int^{y} \frac{1}{\left(-\frac{\left(\left(-\_\_a\right)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_{1}}{3}}-i \_\_a^{5}\right) \left(\sqrt{3}+i\right)^{4}}{\_\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_\_a \right)}{\left(i \sqrt{2}\, \sqrt{5-\sqrt{5}}-\sqrt{5}-1\right)^{3}} = x +c_{6}$$

Verified OK.

$$-\frac{32\, 2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3} \left(\int^{y} \frac{1}{\left(-\frac{\left(\left(-\_\_a\right)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_{1}}{3}}-i \_\_a^{5}\right) \left(\sqrt{3}+i\right)^{4}}{\_\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_\_a \right)}{\left(i \sqrt{2}\, \sqrt{5-\sqrt{5}}+\sqrt{5}+1\right)^{3}} = x +c_{7}$$

Verified OK.

$$-\frac{32\, 2^{\frac{2}{5}} \left(\sqrt{3}+i\right)^{3} \left(\int^{y} \frac{1}{\left(-\frac{\left(\left(-\_\_a\right)^{\frac{10}{3}} {\mathrm e}^{\frac{i \pi}{6}+\frac{5 c_{1}}{3}}-i \_\_a^{5}\right) \left(\sqrt{3}+i\right)^{4}}{\_\_a^{\frac{10}{3}}}\right)^{\frac{3}{5}}}d\_\_a \right)}{\left(i \sqrt{2}\, \sqrt{5+\sqrt{5}}-\sqrt{5}+1\right)^{3}} = x +c_{8}$$

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 3 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      Try integration with the canonical coordinates of the symmetry [0, y]
      -> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)^2+(-_b(_a))^(1/3), _b(_a)
         symmetry methods on request
      `, `1st order, trying reduction of order with given symmetries:`[1, 0]
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 128

```
dsolve(y(x)*diff(y(x),x$2)^3+y(x)^3*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1$$
$$y(x) = e^{\int \mathrm{RootOf}\left(x - \left(\int^{\_Z} -\frac{1}{\_f^2 - (-\_f)^{\frac{1}{3}}} d\_f\right) + c_1\right) dx + c_2}$$
$$y(x) = e^{\int \mathrm{RootOf}\left(x + 2\left(\int^{\_Z} \frac{1}{i\sqrt{3}\,(-\_f)^{\frac{1}{3}} + 2\_f^2 + (-\_f)^{\frac{1}{3}}} d\_f\right) + c_1\right) dx + c_2}$$
$$y(x) = e^{\int \mathrm{RootOf}\left(x - 2\left(\int^{\_Z} \frac{1}{i\sqrt{3}\,(-\_f)^{\frac{1}{3}} - 2\_f^2 - (-\_f)^{\frac{1}{3}}} d\_f\right) + c_1\right) dx + c_2}$$

✓ Solution by Mathematica

Time used: 3.023 (sec). Leaf size: 800

```
DSolve[y[x]*y''[x]^3+y[x]^3*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

$$y(x) \to \text{InverseFunction}\left[\frac{\#1\left(1-\frac{3\#1^{5/3}}{5c_1}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},\frac{3\#1^{5/3}}{5c_1}\right)}{\left(-\#1^{5/3}+\frac{5c_1}{3}\right)^{3/5}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\#1\left(1+\frac{3\sqrt[3]{-1}\#1^{5/3}}{5c_1}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},-\frac{3\sqrt[3]{-1}\#1^{5/3}}{5c_1}\right)}{\left(\sqrt[3]{-1}\#1^{5/3}+\frac{5c_1}{3}\right)^{3/5}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\#1\left(1-\frac{3(-1)^{2/3}\#1^{5/3}}{5c_1}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},\frac{3(-1)^{2/3}\#1^{5/3}}{5c_1}\right)}{\left(-(-1)^{2/3}\#1^{5/3}+\frac{5c_1}{3}\right)^{3/5}}\&\right][x+c_2]$$

$$y(x) \to 0$$

$$y(x) \to \text{InverseFunction}\left[\frac{\#1\left(1-\frac{3\#1^{5/3}}{5(-c_1)}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},\frac{3\#1^{5/3}}{5(-c_1)}\right)}{\left(-\#1^{5/3}+\frac{5(-c_1)}{3}\right)^{3/5}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\#1\left(1+\frac{3\sqrt[3]{-1}\#1^{5/3}}{5(-c_1)}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},-\frac{3\sqrt[3]{-1}\#1^{5/3}}{5(-c_1)}\right)}{\left(\sqrt[3]{-1}\#1^{5/3}+\frac{5}{3}(-1)c_1\right)^{3/5}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\#1\left(1-\frac{3(-1)^{2/3}\#1^{5/3}}{5(-c_1)}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},\frac{3(-1)^{2/3}\#1^{5/3}}{5(-c_1)}\right)}{\left(-(-1)^{2/3}\#1^{5/3}+\frac{5(-c_1)}{3}\right)^{3/5}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\#1\left(1-\frac{3\#1^{5/3}}{5c_1}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},\frac{3\#1^{5/3}}{5c_1}\right)}{\left(-\#1^{5/3}+\frac{5c_1}{3}\right)^{3/5}}\&\right][x+c_2]$$

$$y(x)$$

$$\left[\#1\left(1+\frac{3\sqrt[3]{-1}\#1^{5/3}}{5c_1}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},-\frac{3\sqrt[3]{-1}\#1^{5/3}}{5c_1}\right)\right.$$

1.52 problem 52

Internal problem ID [7441]
Internal file name [`OUTPUT/6408_Sunday_June_05_2022_04_46_57_PM_71426058/index.tex`]

Book: Second order enumerated odes
Section: section 1
Problem number: 52.
ODE order: 2.
ODE degree: 3.

The type(s) of ODE detected by this program : **"algebraic", "second_order_ode_missing_x"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{yy''^3 + y^3y'^5 = 0}$$

The ode

$$yy''^3 + y^3y'^5 = 0$$

is factored to

$$y\left(y'^5y^2 + y''^3\right) = 0$$

Which gives the following equations

$$y = 0 \tag{1}$$

$$y'^5y^2 + y''^3 = 0 \tag{2}$$

Each of the above equations is now solved.

Solving ODE (1) Since $y = 0$, is missing derivative in y then it is an algebraic equation. Solving for y.

$$y = 0$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$

Verification of solutions

$$y = 0$$

Verified OK.

Solving ODE (2) This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^5 y^2 + p(y)^3 \left(\frac{d}{dy}p(y)\right)^3 = 0$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\frac{d}{dy}p(y) = \left(-p(y)^2 y^2\right)^{\frac{1}{3}} \tag{1}$$

$$\frac{d}{dy}p(y) = -\frac{\left(-p(y)^2 y^2\right)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\left(-p(y)^2 y^2\right)^{\frac{1}{3}}}{2} \tag{2}$$

$$\frac{d}{dy}p(y) = -\frac{\left(-p(y)^2 y^2\right)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\left(-p(y)^2 y^2\right)^{\frac{1}{3}}}{2} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$\frac{d}{dy}p(y) = \left(-p^2y^2\right)^{\frac{1}{3}}$$
$$\frac{d}{dy}p(y) = \omega(y,p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + ya_2 + a_1 \tag{1E}$$
$$\eta = pb_3 + yb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \left(-p^2y^2\right)^{\frac{1}{3}}(b_3 - a_2) - \left(-p^2y^2\right)^{\frac{2}{3}}a_3 + \frac{2p^2y(pa_3 + ya_2 + a_1)}{3\left(-p^2y^2\right)^{\frac{2}{3}}} + \frac{2p\,y^2(pb_3 + yb_2 + b_1)}{3\left(-p^2y^2\right)^{\frac{2}{3}}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{3(-p^2y^2)^{\frac{4}{3}}a_3 - 2p^3ya_3 - 5p^2y^2a_2 + p^2y^2b_3 - 2p\,y^3b_2 - 2p^2ya_1 - 2p\,y^2b_1 - 3b_2(-p^2y^2)^{\frac{2}{3}}}{3\left(-p^2y^2\right)^{\frac{2}{3}}} = 0$$

Setting the numerator to zero gives

$$-3\left(-p^2y^2\right)^{\frac{4}{3}}a_3 + 2p^3ya_3 + 5p^2y^2a_2 - p^2y^2b_3 + 2p\,y^3b_2 + 2p^2ya_1 + 2p\,y^2b_1 + 3b_2\left(-p^2y^2\right)^{\frac{2}{3}} = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&3p^2y^2\left(-p^2y^2\right)^{\frac{1}{3}} a_3 + 2p^3ya_3 + 5p^2y^2a_2 - p^2y^2b_3 \\
&+ 2p\,y^3b_2 + 2p^2ya_1 + 2p\,y^2b_1 + 3b_2\left(-p^2y^2\right)^{\frac{2}{3}} = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{p, y\}$ in them.

$$
\left\{p, y, \left(-p^2y^2\right)^{\frac{1}{3}}, \left(-p^2y^2\right)^{\frac{2}{3}}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{p, y\}$ in them

$$
\left\{p = v_1, y = v_2, \left(-p^2y^2\right)^{\frac{1}{3}} = v_3, \left(-p^2y^2\right)^{\frac{2}{3}} = v_4\right\}
$$

The above PDE (6E) now becomes

$$
3v_1^2v_2^2v_3a_3 + 5v_1^2v_2^2a_2 + 2v_1^3v_2a_3 + 2v_1v_2^3b_2 - v_1^2v_2^2b_3 + 2v_1^2v_2a_1 + 2v_1v_2^2b_1 + 3b_2v_4 = 0 \quad (7E)
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4\}
$$

Equation (7E) now becomes

$$
2v_1^3v_2a_3 + 3v_1^2v_2^2v_3a_3 + (5a_2 - b_3)\,v_1^2v_2^2 + 2v_1^2v_2a_1 + 2v_1v_2^3b_2 + 2v_1v_2^2b_1 + 3b_2v_4 = 0 \quad (8E)
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
2a_1 &= 0 \\
2a_3 &= 0 \\
3a_3 &= 0 \\
2b_1 &= 0 \\
2b_2 &= 0 \\
3b_2 &= 0 \\
5a_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 5a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= y \\
\eta &= 5p
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y, p) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial p}\right) S(y, p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dp}{dy} &= \frac{\eta}{\xi} \\
&= \frac{5p}{y} \\
&= \frac{5p}{y}
\end{aligned}
$$

This is easily solved to give

$$
p(y) = c_1 y^5
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{p}{y^5}
$$

And S is found from

$$\begin{aligned} dS &= \frac{dy}{\xi} \\ &= \frac{dy}{y} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dy}{T} \\ &= \ln{(y)} \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y,p)S_p}{R_y + \omega(y,p)R_p} \tag{2}$$

Where in the above R_y, R_p, S_y, S_p are all partial derivatives and $\omega(y,p)$ is the right hand side of the original ode given by

$$\omega(y,p) = \left(-p^2 y^2\right)^{\frac{1}{3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_y &= -\frac{5p}{y^6} \\ R_p &= \frac{1}{y^5} \\ S_y &= \frac{1}{y} \\ S_p &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y^5}{y\left(-p^2 y^2\right)^{\frac{1}{3}} - 5p} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for y, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2}{i R^{\frac{2}{3}}\sqrt{3} + R^{\frac{2}{3}} - 10R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{3i \arctan\left(\frac{\left(50R^{\frac{1}{3}}-5\right)\sqrt{3}}{15}\right)}{5} - \frac{\ln\left(1+125R\right)}{5} + \frac{\ln\left(5R^{\frac{1}{3}}+1\right)}{5} - \frac{\ln\left(25R^{\frac{2}{3}}-5R^{\frac{1}{3}}+1\right)}{10} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to y, p coordinates. This results in

$$\ln(y) = -\frac{3i \arctan\left(\frac{\left(50\left(\frac{p(y)}{y^5}\right)^{\frac{1}{3}}-5\right)\sqrt{3}}{15}\right)}{5} - \frac{\ln\left(1+\frac{125p(y)}{y^5}\right)}{5} + \frac{\ln\left(5\left(\frac{p(y)}{y^5}\right)^{\frac{1}{3}}+1\right)}{5} - \frac{\ln\left(25\left(\frac{p(y)}{y^5}\right)^{\frac{2}{3}} - 5\left(\frac{p}{\cdot}\right.\right.}{10}$$

Which simplifies to

$$\ln(y) = -\frac{3i \arctan\left(\frac{\left(50\left(\frac{p(y)}{y^5}\right)^{\frac{1}{3}}-5\right)\sqrt{3}}{15}\right)}{5} - \frac{\ln\left(1+\frac{125p(y)}{y^5}\right)}{5} + \frac{\ln\left(5\left(\frac{p(y)}{y^5}\right)^{\frac{1}{3}}+1\right)}{5} - \frac{\ln\left(25\left(\frac{p(y)}{y^5}\right)^{\frac{2}{3}} - 5\left(\frac{p}{\cdot}\right.\right.}{10}$$

Solving equation (2)

Writing the ode as

$$\frac{d}{dy}p(y) = -\frac{(-p^2y^2)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,(-p^2y^2)^{\frac{1}{3}}}{2}$$
$$\frac{d}{dy}p(y) = \omega(y,p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + ya_2 + a_1 \tag{1E}$$
$$\eta = pb_3 + yb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \left(-\frac{(-p^2y^2)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,(-p^2y^2)^{\frac{1}{3}}}{2}\right)(b_3 - a_2) \\
&- \left(-\frac{(-p^2y^2)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,(-p^2y^2)^{\frac{1}{3}}}{2}\right)^2 a_3 \\
&- \left(\frac{p^2y}{3\,(-p^2y^2)^{\frac{2}{3}}} + \frac{i\sqrt{3}\,p^2y}{3\,(-p^2y^2)^{\frac{2}{3}}}\right)(pa_3 + ya_2 + a_1) \\
&- \left(\frac{p\,y^2}{3\,(-p^2y^2)^{\frac{2}{3}}} + \frac{i\sqrt{3}\,p\,y^2}{3\,(-p^2y^2)^{\frac{2}{3}}}\right)(pb_3 + yb_2 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{3i\sqrt{3}\,(-p^2y^2)^{\frac{4}{3}}\,a_3 + 2i\sqrt{3}\,p^3ya_3 + 5i\sqrt{3}\,p^2y^2a_2 - i\sqrt{3}\,p^2y^2b_3 + 2i\sqrt{3}\,p\,y^3b_2 + 2i\sqrt{3}\,p^2ya_1 + 2i\sqrt{3}\,p\,y^2b_1}{6\,(-p^2y^2)^{\frac{2}{3}}} = 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-3i\sqrt{3}\,\left(-p^2y^2\right)^{\frac{4}{3}} a_3 - 2i\sqrt{3}\,p^3ya_3 - 5i\sqrt{3}\,p^2y^2a_2 + i\sqrt{3}\,p^2y^2b_3 \\
&- 2i\sqrt{3}\,p\,y^3b_2 - 2i\sqrt{3}\,p^2ya_1 - 2i\sqrt{3}\,p\,y^2b_1 + 3\left(-p^2y^2\right)^{\frac{4}{3}} a_3 - 2p^3ya_3 \\
&- 5p^2y^2a_2 + p^2y^2b_3 - 2p\,y^3b_2 - 2p^2ya_1 - 2p\,y^2b_1 + 6b_2\left(-p^2y^2\right)^{\frac{2}{3}} = 0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}
&6\left(-p^2y^2\right)^{\frac{4}{3}} a_3 - 4i\sqrt{3}\,p\,y^2b_1 - 4i\sqrt{3}\,p^2ya_1 - 4i\sqrt{3}\,p\,y^3b_2 - 10i\sqrt{3}\,p^2y^2a_2 \\
&- 6i\sqrt{3}\,\left(-p^2y^2\right)^{\frac{4}{3}} a_3 + 2i\sqrt{3}\,p^2y^2b_3 - 4i\sqrt{3}\,p^3ya_3 - 4p^3ya_3 - 10p^2y^2a_2 \\
&+ 2p^2y^2b_3 - 4p\,y^3b_2 - 4p^2ya_1 - 4p\,y^2b_1 + 12b_2\left(-p^2y^2\right)^{\frac{2}{3}} = 0
\end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-6p^2y^2\left(-p^2y^2\right)^{\frac{1}{3}} a_3 - 4i\sqrt{3}\, p\, y^2 b_1 - 4i\sqrt{3}\, p^2 y a_1 - 4i\sqrt{3}\, p\, y^3 b_2 - 10i\sqrt{3}\, p^2 y^2 a_2 \\
&+ 6i\sqrt{3}\, p^2 y^2\left(-p^2y^2\right)^{\frac{1}{3}} a_3 + 2i\sqrt{3}\, p^2 y^2 b_3 - 4i\sqrt{3}\, p^3 y a_3 - 4p^3 y a_3 \\
&- 10p^2y^2a_2 + 2p^2y^2b_3 - 4p\, y^3 b_2 - 4p^2 y a_1 - 4p\, y^2 b_1 + 12b_2\left(-p^2y^2\right)^{\frac{2}{3}} = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{p, y\}$ in them.

$$
\left\{p, y, \left(-p^2y^2\right)^{\frac{1}{3}}, \left(-p^2y^2\right)^{\frac{2}{3}}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{p, y\}$ in them

$$
\left\{p = v_1, y = v_2, \left(-p^2y^2\right)^{\frac{1}{3}} = v_3, \left(-p^2y^2\right)^{\frac{2}{3}} = v_4\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-6v_1^2v_2^2v_3a_3 - 4i\sqrt{3}\, v_1v_2^2b_1 - 4i\sqrt{3}\, v_1^2v_2a_1 - 4i\sqrt{3}\, v_1v_2^3b_2 - 10i\sqrt{3}\, v_1^2v_2^2a_2 \\
&+ 6i\sqrt{3}\, v_1^2v_2^2v_3a_3 + 2i\sqrt{3}\, v_1^2v_2^2b_3 - 4i\sqrt{3}\, v_1^3v_2a_3 - 4v_1^3v_2a_3 \\
&- 10v_1^2v_2^2a_2 + 2v_1^2v_2^2b_3 - 4v_1v_2^3b_2 - 4v_1^2v_2a_1 - 4v_1v_2^2b_1 + 12b_2v_4 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(-4i\sqrt{3}\, a_3 - 4a_3\right) v_1^3v_2 + \left(6i\sqrt{3}\, a_3 - 6a_3\right) v_1^2v_2^2v_3 \\
&+ \left(-10i\sqrt{3}\, a_2 + 2i\sqrt{3}\, b_3 - 10a_2 + 2b_3\right) v_1^2v_2^2 + \left(-4i\sqrt{3}\, a_1 - 4a_1\right) v_1^2v_2 \\
&+ \left(-4i\sqrt{3}\, b_2 - 4b_2\right) v_1v_2^3 + \left(-4i\sqrt{3}\, b_1 - 4b_1\right) v_1v_2^2 + 12b_2v_4 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
12b_2 &= 0 \\
-4i\sqrt{3}\, a_1 - 4a_1 &= 0 \\
-4i\sqrt{3}\, a_3 - 4a_3 &= 0 \\
-4i\sqrt{3}\, b_1 - 4b_1 &= 0 \\
-4i\sqrt{3}\, b_2 - 4b_2 &= 0 \\
6i\sqrt{3}\, a_3 - 6a_3 &= 0 \\
-10i\sqrt{3}\, a_2 + 2i\sqrt{3}\, b_3 - 10a_2 + 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 5a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= y \\
\eta &= 5p
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y, p) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial p}\right) S(y, p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$\frac{d}{dy}p(y) = -\frac{(-p^2y^2)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,(-p^2y^2)^{\frac{1}{3}}}{2}$$
$$\frac{d}{dy}p(y) = \omega(y,p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + ya_2 + a_1 \tag{1E}$$
$$\eta = pb_3 + yb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &+ \left(-\frac{(-p^2y^2)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,(-p^2y^2)^{\frac{1}{3}}}{2}\right)(b_3 - a_2) \\ &- \left(-\frac{(-p^2y^2)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,(-p^2y^2)^{\frac{1}{3}}}{2}\right)^2 a_3 \\ &- \left(\frac{p^2y}{3\,(-p^2y^2)^{\frac{2}{3}}} - \frac{i\sqrt{3}\,p^2y}{3\,(-p^2y^2)^{\frac{2}{3}}}\right)(pa_3 + ya_2 + a_1) \\ &- \left(\frac{p\,y^2}{3\,(-p^2y^2)^{\frac{2}{3}}} - \frac{i\sqrt{3}\,p\,y^2}{3\,(-p^2y^2)^{\frac{2}{3}}}\right)(pb_3 + yb_2 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{3i\sqrt{3}\,(-p^2y^2)^{\frac{4}{3}}\,a_3 + 2i\sqrt{3}\,p^3ya_3 + 5i\sqrt{3}\,p^2y^2a_2 - i\sqrt{3}\,p^2y^2b_3 + 2i\sqrt{3}\,p\,y^3b_2 + 2i\sqrt{3}\,p^2ya_1 + 2i\sqrt{3}\,p\,y^2b_1 + \cdots}{6\,(-p^2y^2)^{\frac{2}{3}}}$$
$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&3i\sqrt{3}\left(-p^{2}y^{2}\right)^{\frac{4}{3}} a_{3}+2i\sqrt{3}\,p^{3}y a_{3}+5i\sqrt{3}\,p^{2}y^{2}a_{2}-i\sqrt{3}\,p^{2}y^{2}b_{3}\\
&+2i\sqrt{3}\,p\,y^{3}b_{2}+2i\sqrt{3}\,p^{2}y a_{1}+2i\sqrt{3}\,p\,y^{2}b_{1}+3\left(-p^{2}y^{2}\right)^{\frac{4}{3}} a_{3}-2p^{3}y a_{3}\\
&-5p^{2}y^{2}a_{2}+p^{2}y^{2}b_{3}-2p\,y^{3}b_{2}-2p^{2}y a_{1}-2p\,y^{2}b_{1}+6b_{2}\left(-p^{2}y^{2}\right)^{\frac{2}{3}}=0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&6\left(-p^{2}y^{2}\right)^{\frac{4}{3}} a_{3}+4i\sqrt{3}\,p\,y^{3}b_{2}-2i\sqrt{3}\,p^{2}y^{2}b_{3}+10i\sqrt{3}\,p^{2}y^{2}a_{2}+4i\sqrt{3}\,p^{3}y a_{3}\\
&+6i\sqrt{3}\left(-p^{2}y^{2}\right)^{\frac{4}{3}} a_{3}+4i\sqrt{3}\,p\,y^{2}b_{1}+4i\sqrt{3}\,p^{2}y a_{1}-4p^{3}y a_{3}-10p^{2}y^{2}a_{2}\\
&+2p^{2}y^{2}b_{3}-4p\,y^{3}b_{2}-4p^{2}y a_{1}-4p\,y^{2}b_{1}+12b_{2}\left(-p^{2}y^{2}\right)^{\frac{2}{3}}=0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-6p^{2}y^{2}\left(-p^{2}y^{2}\right)^{\frac{1}{3}} a_{3}+4i\sqrt{3}\,p\,y^{3}b_{2}-2i\sqrt{3}\,p^{2}y^{2}b_{3}+10i\sqrt{3}\,p^{2}y^{2}a_{2}\\
&+4i\sqrt{3}\,p^{3}y a_{3}-6i\sqrt{3}\,p^{2}y^{2}\left(-p^{2}y^{2}\right)^{\frac{1}{3}} a_{3}+4i\sqrt{3}\,p\,y^{2}b_{1}+4i\sqrt{3}\,p^{2}y a_{1}-4p^{3}y a_{3}\\
&-10p^{2}y^{2}a_{2}+2p^{2}y^{2}b_{3}-4p\,y^{3}b_{2}-4p^{2}y a_{1}-4p\,y^{2}b_{1}+12b_{2}\left(-p^{2}y^{2}\right)^{\frac{2}{3}}=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{p, y\}$ in them.

$$
\left\{p, y, \left(-p^{2}y^{2}\right)^{\frac{1}{3}}, \left(-p^{2}y^{2}\right)^{\frac{2}{3}}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{p, y\}$ in them

$$
\left\{p = v_{1}, y = v_{2}, \left(-p^{2}y^{2}\right)^{\frac{1}{3}} = v_{3}, \left(-p^{2}y^{2}\right)^{\frac{2}{3}} = v_{4}\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-6v_{1}^{2}v_{2}^{2}v_{3}a_{3}+4i\sqrt{3}\,v_{1}v_{2}^{3}b_{2}-2i\sqrt{3}\,v_{1}^{2}v_{2}^{2}b_{3}+10i\sqrt{3}\,v_{1}^{2}v_{2}^{2}a_{2}+4i\sqrt{3}\,v_{1}^{3}v_{2}a_{3}\\
&-6i\sqrt{3}\,v_{1}^{2}v_{2}^{2}v_{3}a_{3}+4i\sqrt{3}\,v_{1}v_{2}^{2}b_{1}+4i\sqrt{3}\,v_{1}^{2}v_{2}a_{1}-4v_{1}^{3}v_{2}a_{3}\\
&-10v_{1}^{2}v_{2}^{2}a_{2}+2v_{1}^{2}v_{2}^{2}b_{3}-4v_{1}v_{2}^{3}b_{2}-4v_{1}^{2}v_{2}a_{1}-4v_{1}v_{2}^{2}b_{1}+12b_{2}v_{4}=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_{1}, v_{2}, v_{3}, v_{4}\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(4i\sqrt{3}\,a_3-4a_3\right)v_1^3v_2+\left(-6i\sqrt{3}\,a_3-6a_3\right)v_1^2v_2^2v_3\\
&+\left(10i\sqrt{3}\,a_2-2i\sqrt{3}\,b_3-10a_2+2b_3\right)v_1^2v_2^2+\left(4i\sqrt{3}\,a_1-4a_1\right)v_1^2v_2\\
&+\left(4i\sqrt{3}\,b_2-4b_2\right)v_1v_2^3+\left(4i\sqrt{3}\,b_1-4b_1\right)v_1v_2^2+12b_2v_4=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
12b_2&=0\\
-6i\sqrt{3}\,a_3-6a_3&=0\\
4i\sqrt{3}\,a_1-4a_1&=0\\
4i\sqrt{3}\,a_3-4a_3&=0\\
4i\sqrt{3}\,b_1-4b_1&=0\\
4i\sqrt{3}\,b_2-4b_2&=0\\
10i\sqrt{3}\,a_2-2i\sqrt{3}\,b_3-10a_2+2b_3&=0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1&=0\\
a_2&=a_2\\
a_3&=0\\
b_1&=0\\
b_2&=0\\
b_3&=5a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi&=y\\
\eta&=5p
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y,p)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi}=\frac{dp}{\eta}=dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial p}\right) S(y,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\ln(y) = -\frac{3i \arctan\left(\frac{\left(50\left(\frac{y'}{y^5}\right)^{\frac{1}{3}}-5\right)\sqrt{3}}{15}\right)}{5} - \frac{\ln\left(1+\frac{125y'}{y^5}\right)}{5} + \frac{\ln\left(5\left(\frac{y'}{y^5}\right)^{\frac{1}{3}}+1\right)}{5} - \frac{\ln\left(25\left(\frac{y'}{y^5}\right)^{\frac{2}{3}} - 5\left(\frac{y'}{y^5}\right)^{\frac{1}{3}} +\right.}{10}$$

Integrating both sides gives

$$\int \frac{}{y^5\left(3\tan\left(\operatorname{RootOf}\left(2i\ln\left(5\left(\frac{3\tan(\_Z)^3\sqrt{3}}{1000}+\frac{3\sqrt{3}\tan(\_Z)}{1000}+\frac{9\tan(\_Z)^2}{1000}+\frac{1}{1000}\right)^{\frac{1}{3}}+1\right)-\right.\right.\right.\right.}$$

$$\int^y \frac{}{\_a^5\left(3\tan\left(\operatorname{RootOf}\left(2i\ln\left(5\left(\frac{3\tan(\_Z)^3\sqrt{3}}{1000}+\frac{3\sqrt{3}\tan(\_Z)}{1000}+\frac{9\tan(\_Z)^2}{1000}+\frac{1}{1000}\right)^{\frac{1}{3}}+1\right)-i\ln\left(\frac{9\left(\right.}{}\right.\right.\right.\right.\right.}$$

Summary

The solution(s) found are the following

$$\int^y \frac{}{\_a^5\left(3\tan\left(\operatorname{RootOf}\left(2i\ln\left(5\left(\frac{3\tan(\_Z)^3\sqrt{3}}{1000}+\frac{3\sqrt{3}\tan(\_Z)}{1000}+\frac{9\tan(\_Z)^2}{1000}+\frac{1}{1000}\right)^{\frac{1}{3}}+1\right)-i\ln\left(\frac{9\left(\right.}{}\right.\right.\right.\right.\right.} \tag{1}$$

$= x + c_4$

Verification of solutions

$$\int^{y} \frac{}{\_a^5 \left(3\tan\left(\operatorname{RootOf}\left(2i\ln\left(5\left(\frac{3\tan(\_Z)^3\sqrt{3}}{1000}+\frac{3\sqrt{3}\,\tan(\_Z)}{1000}+\frac{9\tan(\_Z)^2}{1000}+\frac{1}{1000}\right)^{\frac{1}{3}}+1\right)-i\ln\left(\frac{9\left(\right.}{}\right.\right.\right.\right.}$$
$$= x + c_4$$

Warning, solution could not be verified

Summary

The solution(s) found are the following

$$\int^{y} \frac{}{\_a^5 \left(3\tan\left(\operatorname{RootOf}\left(2i\ln\left(5\left(\frac{3\tan(\_Z)^3\sqrt{3}}{1000}+\frac{3\sqrt{3}\,\tan(\_Z)}{1000}+\frac{9\tan(\_Z)^2}{1000}+\frac{1}{1000}\right)^{\frac{1}{3}}+1\right)-i\ln\left(\frac{9\left(\right.}{}\right.\right.\right.\right.} \tag{1}$$
$$= x + c_4$$

Verification of solutions

$$\int^{y} \frac{}{\_a^5 \left(3\tan\left(\operatorname{RootOf}\left(2i\ln\left(5\left(\frac{3\tan(\_Z)^3\sqrt{3}}{1000}+\frac{3\sqrt{3}\,\tan(\_Z)}{1000}+\frac{9\tan(\_Z)^2}{1000}+\frac{1}{1000}\right)^{\frac{1}{3}}+1\right)-i\ln\left(\frac{9\left(\right.}{}\right.\right.\right.\right.}$$
$$= x + c_4$$

Warning, solution could not be verified

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 3 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      -> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(-_a^2*_b(_a)^2)^(1/3)*_b(
         symmetry methods on request
      `, `1st order, trying reduction of order with given symmetries:`[_a, 5*_b]
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 214

```
dsolve(y(x)*diff(y(x),x$2)^3+y(x)^3*diff(y(x),x)^5=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1$$
$$\int^{y(x)} \frac{1}{\mathrm{RootOf}\left(5\left(\int_{\_g}^{\_Z} \frac{1}{\_a\left(-\_f^2\_a^2\right)^{\frac{1}{3}}-5\_f}d\_f\right)-\ln\left(\_a^5+125\right)+5c_1\right)}d\_a$$
$$-x-c_2 = 0$$
$$\int^{y(x)} \frac{1}{\mathrm{RootOf}\left(-i\ln\left(\_a^5+125\right)+\sqrt{3}\,\ln\left(\_a^5+125\right)+20\left(\int_{\_g}^{\_Z} \frac{1}{2i\_a\left(-\_f^2\_a^2\right)^{\frac{1}{3}}+5i\_f+5\sqrt{3}\,\_f}d\_f\right)-2\right.}$$
$$-x-c_2 = 0$$
$$\int^{y(x)} \frac{1}{\mathrm{RootOf}\left(20\left(\int_{\_g}^{\_Z} \frac{1}{-2i\_a\left(-\_f^2\_a^2\right)^{\frac{1}{3}}-5i\_f+5\sqrt{3}\,\_f}d\_f\right)+i\ln\left(\_a^5+125\right)+\sqrt{3}\,\ln\left(\_a^5+125\right)+2\right.}$$
$$-x-c_2 = 0$$

✓ Solution by Mathematica

Time used: 24.581 (sec). Leaf size: 449

```
DSolve[y[x]*y''[x]^3+y[x]^3*y'[x]^5==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},\frac{3\#1^{5/3}}{5c_1}\right)}{c_1{}^3}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},-\frac{3i\left(-i+\sqrt{3}\right)\#1^{5/3}}{10c_1}\right)}{c_1{}^3}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},\frac{3i\left(i+\sqrt{3}\right)\#1^{5/3}}{10c_1}\right)}{c_1{}^3}\&\right][x+c_2]$$

$$y(x) \to 0$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},\frac{3\#1^{5/3}}{5(-c_1)}\right)}{(-c_1)^3}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},-\frac{3i\left(-i+\sqrt{3}\right)\#1^{5/3}}{10(-c_1)}\right)}{(-c_1)^3}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},\frac{3i\left(i+\sqrt{3}\right)\#1^{5/3}}{10(-c_1)}\right)}{(-c_1)^3}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},\frac{3\#1^{5/3}}{5c_1}\right)}{c_1{}^3}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},-\frac{3i\left(-i+\sqrt{3}\right)\#1^{5/3}}{10c_1}\right)}{c_1{}^3}\&\right][x$$

2 section 2

2.1 problem 1 662
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2.1 problem 1

2.1.1 Solving as second order nonlinear solved by mainardi lioville method ode 662

Internal problem ID [7442]
Internal file name [OUTPUT/6409_Sunday_June_05_2022_04_47_15_PM_70061536/index.tex]

Book: Second order enumerated odes
Section: section 2
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__nonlinear__solved__by__mainardi__l oville__method"**

Maple gives the following as the ode type

```
[_Liouville, [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{y'' + xy' + yy'^2 = 0}$$

2.1.1 Solving as second order nonlinear solved by mainardi lioville method ode

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y)y'^2 = 0 \tag{1A}$$

Where in this problem

$$f(x) = x$$
$$g(y) = y$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx}\ln(y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$
\begin{aligned}
g\frac{dy}{dx} &= \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx} \\
&= \frac{d}{dx}\int g dy
\end{aligned} \tag{4A}
$$

Substituting (3A,4A) back into (2A) gives

$$
\frac{d}{dx}\ln\left(y'\right) + \frac{d}{dx}\int g dy = -f \tag{5A}
$$

Integrating the above w.r.t. x gives

$$
\ln\left(y'\right) + \int g dy = -\int f dx + c_1
$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$
y' = c_2 e^{\int -g dy} e^{\int -f dx} \tag{6A}
$$

Where c_2 is a new arbitrary constant. But since $g = y$ and $f = x$, then

$$
\begin{aligned}
\int -g dy &= \int -y dy \\
&= -\frac{y^2}{2} \\
\int -f dx &= \int -x dx \\
&= -\frac{x^2}{2}
\end{aligned}
$$

Substituting the above into Eq(6A) gives

$$
y' = c_2 \mathrm{e}^{-\frac{y^2}{2}} \mathrm{e}^{-\frac{x^2}{2}}
$$

Which is now solved as first order separable ode. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= c_2 \mathrm{e}^{-\frac{y^2}{2}} \mathrm{e}^{-\frac{x^2}{2}}
\end{aligned}
$$

Where $f(x) = c_2 e^{-\frac{x^2}{2}}$ and $g(y) = e^{-\frac{y^2}{2}}$. Integrating both sides gives

$$\frac{1}{e^{-\frac{y^2}{2}}}\, dy = c_2 e^{-\frac{x^2}{2}}\, dx$$

$$\int \frac{1}{e^{-\frac{y^2}{2}}}\, dy = \int c_2 e^{-\frac{x^2}{2}}\, dx$$

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} = \frac{c_2\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} + c_3$$

The solution is

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} - \frac{c_2\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} - c_3 = 0$$

Summary

The solution(s) found are the following

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} - \frac{c_2\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} - c_3 = 0 \tag{1}$$

Verification of solutions

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} - \frac{c_2\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} - c_3 = 0$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+y(x)*diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = -i \operatorname{RootOf}\left(i\sqrt{\pi}\, \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) c_1 + i\sqrt{2}\, c_2 - \operatorname{erf}(\_Z)\sqrt{\pi}\right)\sqrt{2}$$

✓ Solution by Mathematica

Time used: 0.125 (sec). Leaf size: 44

```
DSolve[y''[x]+x*y'[x]+y[x]*(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -i\sqrt{2}\operatorname{erf}^{-1}\left(i\left(\sqrt{\frac{2}{\pi}}c_2 - c_1\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right)\right)$$

2.2 problem 2

2.2.1 Solving as second order nonlinear solved by mainardi lioville method ode 666

Internal problem ID [7443]
Internal file name [OUTPUT/6410_Sunday_June_05_2022_04_47_18_PM_19823218/index.tex]

Book: Second order enumerated odes
Section: section 2
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_nonlinear_solved_by_mainardi_lioville_method"**

Maple gives the following as the ode type

```
[_Liouville, [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{y'' + y' \sin(x) + y y'^2 = 0}$$

2.2.1 Solving as second order nonlinear solved by mainardi lioville method ode

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y)y'^2 = 0 \tag{1A}$$

Where in this problem

$$f(x) = \sin(x)$$
$$g(y) = y$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx}\ln(y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$g\frac{dy}{dx} = \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx}$$
$$= \frac{d}{dx}\int g dy \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx}\ln(y') + \frac{d}{dx}\int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln(y') + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy}\, e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = y$ and $f = \sin(x)$, then

$$\int -g dy = \int -y dy$$
$$= -\frac{y^2}{2}$$
$$\int -f dx = \int -\sin(x)\, dx$$
$$= \cos(x)$$

Substituting the above into Eq(6A) gives

$$y' = c_2 e^{-\frac{y^2}{2}} e^{\cos(x)}$$

Which is now solved as first order separable ode. In canonical form the ODE is

$$y' = F(x, y)$$
$$= f(x)g(y)$$
$$= c_2 e^{-\frac{y^2}{2}} e^{\cos(x)}$$

Where $f(x) = c_2 \mathrm{e}^{\cos(x)}$ and $g(y) = \mathrm{e}^{-\frac{y^2}{2}}$. Integrating both sides gives

$$\frac{1}{\mathrm{e}^{-\frac{y^2}{2}}}\, dy = c_2 \mathrm{e}^{\cos(x)}\, dx$$

$$\int \frac{1}{\mathrm{e}^{-\frac{y^2}{2}}}\, dy = \int c_2 \mathrm{e}^{\cos(x)}\, dx$$

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} = \int c_2 \mathrm{e}^{\cos(x)} dx + c_3$$

The solution is

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} - \left(\int c_2 \mathrm{e}^{\cos(x)} dx\right) - c_3 = 0$$

Summary

The solution(s) found are the following

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} - \left(\int c_2 \mathrm{e}^{\cos(x)} dx\right) - c_3 = 0 \tag{1}$$

Verification of solutions

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} - \left(\int c_2 \mathrm{e}^{\cos(x)} dx\right) - c_3 = 0$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)+sin(x)*diff(y(x),x)+y(x)*diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = -i \operatorname{RootOf}\left(i\sqrt{2}\, c_1 \left(\int \mathrm{e}^{\cos(x)} dx\right) + i\sqrt{2}\, c_2 - \operatorname{erf}\left(\_Z\right) \sqrt{\pi}\right) \sqrt{2}$$

✓ Solution by Mathematica

Time used: 0.329 (sec). Leaf size: 47

```
DSolve[y''[x]+Sin[x]*y'[x]+y[x]*(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -i\sqrt{2}\text{erf}^{-1}\left(i\sqrt{\frac{2}{\pi}}\left(\int_1^x -e^{\cos(K[1])} c_1 dK[1] + c_2\right)\right)$$

2.3 problem 3

Internal problem ID [7444]
Internal file name [`OUTPUT/6411_Sunday_June_05_2022_04_47_20_PM_43215893/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + (1-x)\, y' + y^2 {y'}^2 = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 62

```
dsolve(diff(y(x),x$2)+(1-x)*diff(y(x),x)+y(x)^2*diff(y(x),x)^2=0,y(x), singsol=all)
```

$$c_1 \operatorname{erf}\left(\frac{i\sqrt{2}\,(x-1)}{2}\right) - c_2 + \frac{2\, 3^{\frac{5}{6}} y(x)\, \pi}{9\Gamma\left(\frac{2}{3}\right) \left(-y\left(x\right)^{3}\right)^{\frac{1}{3}}} - \frac{y(x)\, \Gamma\left(\frac{1}{3}, -\frac{y(x)^3}{3}\right) 3^{\frac{1}{3}}}{3\left(-y\left(x\right)^{3}\right)^{\frac{1}{3}}} = 0$$

✓ Solution by Mathematica

Time used: 0.374 (sec). Leaf size: 67

```
DSolve[y''[x]+(1-x)*y'[x]+y[x]^2*(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[-\frac{\#1\Gamma\left(\frac{1}{3},-\frac{\#1^3}{3}\right)}{3^{2/3}\sqrt[3]{-\#1^3}}\&\right]\left[c_2-\sqrt{\frac{\pi}{2e}}c_1\text{erfi}\left(\frac{x-1}{\sqrt{2}}\right)\right]$$

2.4 problem 4

Internal problem ID [7445]
Internal file name [`OUTPUT/6412_Sunday_June_05_2022_04_47_26_PM_44867493/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

Unable to solve or complete the solution.

$$y'' + (\sin(x) + 2x)\, y' + \cos(y)\, y{y'}^2 = 0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$2)+(sin(x)+2*x)*diff(y(x),x)+cos(y(x))*y(x)*diff(y(x),x)^2=0,y(x), singso
```

$$\int^{y(x)} e^{\cos(\_a)+\sin(\_a)\_a} d\_a - c_1 \left(\int e^{-x^2+\cos(x)} dx\right) - c_2 = 0$$

✓ Solution by Mathematica

Time used: 1.16 (sec). Leaf size: 53

```
DSolve[y''[x]+(Sin[x]+2*x)*y'[x]+Cos[y[x]]*y[x]*(y'[x])^2==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} e^{\cos(K[1])+K[1]\sin(K[1])}dK[1]\&\right]\left[\int_1^x -e^{\cos(K[2])-K[2]^2}c_1dK[2]+c_2\right]$$

2.5 problem 5

2.5.1 Solving as second order ode missing x ode 674

Internal problem ID [7446]
Internal file name [`OUTPUT/6413_Sunday_June_05_2022_04_47_37_PM_44932305/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{y''y' + y^2 = 0}$$

2.5.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^2 \left(\frac{d}{dy}p(y)\right) + y^2 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= -\frac{y^2}{p^2}
\end{aligned}$$

Where $f(y) = -y^2$ and $g(p) = \frac{1}{p^2}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{1}{p^2}}\,dp &= -y^2\,dy \\
\int \frac{1}{\frac{1}{p^2}}\,dp &= \int -y^2\,dy \\
\frac{p^3}{3} &= -\frac{y^3}{3} + c_1
\end{aligned}$$

The solution is

$$\frac{p(y)^3}{3} + \frac{y^3}{3} - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{y'^3}{3} + \frac{y^3}{3} - c_1 = 0$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \left(-y^3 + 3c_1\right)^{\frac{1}{3}} \tag{1}$$

$$y' = -\frac{\left(-y^3 + 3c_1\right)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\left(-y^3 + 3c_1\right)^{\frac{1}{3}}}{2} \tag{2}$$

$$y' = -\frac{\left(-y^3 + 3c_1\right)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\left(-y^3 + 3c_1\right)^{\frac{1}{3}}}{2} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\left(-y^3+3c_1\right)^{\frac{1}{3}}}dy = \int dx$$

$$\int^{y} \frac{1}{\left(-\_a^3+3c_1\right)^{\frac{1}{3}}}d\_a = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int \frac{1}{-\frac{\left(-y^3+3c_1\right)^{\frac{1}{3}}}{2}-\frac{i\sqrt{3}\left(-y^3+3c_1\right)^{\frac{1}{3}}}{2}}dy = \int dx$$

$$\int^{y} \frac{1}{-\frac{\left(-\_a^3+3c_1\right)^{\frac{1}{3}}}{2}-\frac{i\sqrt{3}\left(-\_a^3+3c_1\right)^{\frac{1}{3}}}{2}}d\_a = x + c_3$$

Solving equation (3)

Integrating both sides gives

$$\int \frac{1}{-\frac{\left(-y^3+3c_1\right)^{\frac{1}{3}}}{2}+\frac{i\sqrt{3}\left(-y^3+3c_1\right)^{\frac{1}{3}}}{2}}dy = \int dx$$

$$\int^{y} \frac{1}{-\frac{\left(-\_a^3+3c_1\right)^{\frac{1}{3}}}{2}+\frac{i\sqrt{3}\left(-\_a^3+3c_1\right)^{\frac{1}{3}}}{2}}d\_a = x + c_4$$

Summary

The solution(s) found are the following

$$\int^{y} \frac{1}{\left(-\_a^3+3c_1\right)^{\frac{1}{3}}}d\_a = x + c_2 \tag{1}$$

$$\int^{y} \frac{1}{-\frac{\left(-\_a^3+3c_1\right)^{\frac{1}{3}}}{2}-\frac{i\sqrt{3}\left(-\_a^3+3c_1\right)^{\frac{1}{3}}}{2}}d\_a = x + c_3 \tag{2}$$

$$\int^{y} \frac{1}{-\frac{\left(-\_a^3+3c_1\right)^{\frac{1}{3}}}{2}+\frac{i\sqrt{3}\left(-\_a^3+3c_1\right)^{\frac{1}{3}}}{2}}d\_a = x + c_4 \tag{3}$$

Verification of solutions

$$\int^{y} \frac{1}{(-\_a^3 + 3c_1)^{\frac{1}{3}}} d\_a = x + c_2$$

Verified OK.

$$\int^{y} \frac{1}{-\frac{(-\_a^3+3c_1)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,(-\_a^3+3c_1)^{\frac{1}{3}}}{2}} d\_a = x + c_3$$

Verified OK.

$$\int^{y} \frac{1}{-\frac{(-\_a^3+3c_1)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,(-\_a^3+3c_1)^{\frac{1}{3}}}{2}} d\_a = x + c_4$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
Try integration with the canonical coordinates of the symmetry [0, y]
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(_b(_a)^3+1)/_b(_a), _b(_a), explicit,
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0]
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 61

```
dsolve(diff(y(x),x$2)*diff(y(x),x)+y(x)^2=0,y(x), singsol=all)
```

$$y(x) = 0$$

$$y(x) = \mathrm{e}^{\frac{\sqrt{3}\left(\int \tan\left(\operatorname{RootOf}\left(-\sqrt{3}\ln\left(\cos\left(\_Z\right)^2\right)-2\sqrt{3}\ln\left(\tan\left(\_Z\right)+\sqrt{3}\right)+6\sqrt{3}\,c_1+6\sqrt{3}\,x+6\_Z\right)\right)dx\right)}{2}+c_2+\frac{x}{2}}$$

✓ Solution by Mathematica

Time used: 1.356 (sec). Leaf size: 180

```
DSolve[y''[x]*y'[x]+y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{}{\sqrt[3]{1 + \text{InverseFunction}\left[\frac{1}{6}\log\left(\#1^2 - \#1 + 1\right) + \frac{\arctan\left(\frac{2\#1-1}{\sqrt{3}}\right)}{\sqrt{3}} - \frac{1}{3}\log(\#1+1)\&\right][-x + c_1]}\sqrt[3]{\text{Inv}}}$$

2.6 problem 6

2.6.1 Solving as second order ode missing x ode 679

Internal problem ID [7447]
Internal file name [`OUTPUT/6414_Sunday_June_05_2022_04_47_42_PM_82581/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{y''y' + y^n = 0}$$

2.6.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^2 \left(\frac{d}{dy}p(y)\right) + y^{n-1}y = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= -\frac{y^{n-1}y}{p^2}
\end{aligned}$$

Where $f(y) = -y^{n-1}y$ and $g(p) = \frac{1}{p^2}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{1}{p^2}}\,dp &= -y^{n-1}y\,dy \\
\int \frac{1}{\frac{1}{p^2}}\,dp &= \int -y^{n-1}y\,dy \\
\frac{p^3}{3} &= -\frac{y^{n+1}}{n+1} + c_1
\end{aligned}$$

The solution is

$$\frac{p(y)^3}{3} + \frac{y^{n+1}}{n+1} - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{y'^3}{3} + \frac{y^{n+1}}{n+1} - c_1 = 0$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(\left(3c_1n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{\frac{1}{3}}}{n+1} \tag{1}$$

$$y' = -\frac{\left(\left(3c_1n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{\frac{1}{3}}}{2\,(n+1)} - \frac{i\sqrt{3}\left(\left(3c_1n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{\frac{1}{3}}}{2\,(n+1)} \tag{2}$$

$$y' = -\frac{\left(\left(3c_1n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{\frac{1}{3}}}{2\,(n+1)} + \frac{i\sqrt{3}\left(\left(3c_1n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{\frac{1}{3}}}{2n+2} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{n+1}{\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{\frac{1}{3}}} dy = \int dx$$

$$\int^y \frac{n+1}{\left(\left(3c_1 n - 3\_a^{n+1} + 3c_1\right)(n+1)^2\right)^{\frac{1}{3}}} d\_a = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2(n+1)}{\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)} dy = \int dx$$

$$-\frac{2\left(\int^y \frac{n+1}{\left(\left(3c_1 n - 3\_a^{n+1} + 3c_1\right)(n+1)^2\right)^{\frac{1}{3}}} d\_a\right)}{1 + i\sqrt{3}} = x + c_3$$

Solving equation (3)

Integrating both sides gives

$$\int \frac{2n+2}{\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)} dy = \int dx$$

$$\frac{2\left(\int^y \frac{n+1}{\left(\left(3c_1 n - 3\_a^{n+1} + 3c_1\right)(n+1)^2\right)^{\frac{1}{3}}} d\_a\right)}{i\sqrt{3} - 1} = x + c_4$$

Summary

The solution(s) found are the following

$$\int^y \frac{n+1}{\left(\left(3c_1 n - 3\_a^{n+1} + 3c_1\right)(n+1)^2\right)^{\frac{1}{3}}} d\_a = x + c_2 \tag{1}$$

$$-\frac{2\left(\int^y \frac{n+1}{\left(\left(3c_1 n - 3\_a^{n+1} + 3c_1\right)(n+1)^2\right)^{\frac{1}{3}}} d\_a\right)}{1 + i\sqrt{3}} = x + c_3 \tag{2}$$

$$\frac{2\left(\int^y \frac{n+1}{\left(\left(3c_1 n - 3\_a^{n+1} + 3c_1\right)(n+1)^2\right)^{\frac{1}{3}}} d\_a\right)}{i\sqrt{3} - 1} = x + c_4 \tag{3}$$

Verification of solutions

$$\int^{y}\frac{n+1}{\left(\left(3c_1n-3\_a^{n+1}+3c_1\right)\left(n+1\right)^2\right)^{\frac{1}{3}}}d\_a=x+c_2$$

Verified OK.

$$-\frac{2\left(\int^{y}\frac{n+1}{\left(\left(3c_1n-3\_a^{n+1}+3c_1\right)\left(n+1\right)^2\right)^{\frac{1}{3}}}d\_a\right)}{1+i\sqrt{3}}=x+c_3$$

Verified OK.

$$\frac{2\left(\int^{y}\frac{n+1}{\left(\left(3c_1n-3\_a^{n+1}+3c_1\right)\left(n+1\right)^2\right)^{\frac{1}{3}}}d\_a\right)}{i\sqrt{3}-1}=x+c_4$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_a^n/_b(_a) = 0, _b(_a), HINT =
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[-3/(n-2)*_a, -_b*(1+n)/(n-2
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 174

```
dsolve(diff(y(x),x$2)*diff(y(x),x)+y(x)^n=0,y(x), singsol=all)
```

$$\frac{(-2-2n)\left(\int^{y(x)}\frac{1}{\left(-(3\_a^{1+n}-c_1)(1+n)^2\right)^{\frac{1}{3}}}d\_a\right)-\left(1+i\sqrt{3}\right)(x+c_2)}{1+i\sqrt{3}}=0$$

$$-\frac{2i(1+n)\left(\int^{y(x)}\frac{1}{\left(-(3\_a^{1+n}-c_1)(1+n)^2\right)^{\frac{1}{3}}}d\_a\right)+(x+c_2)\left(\sqrt{3}+i\right)}{\sqrt{3}+i}=0$$

$$\left(\int^{y(x)}\frac{1}{\left(-\left(3\_a^{1+n}-c_1\right)\left(1+n\right)^2\right)^{\frac{1}{3}}}d\_a\right)n+\int^{y(x)}\frac{1}{\left(-\left(3\_a^{1+n}-c_1\right)\left(1+n\right)^2\right)^{\frac{1}{3}}}d\_a-c_2-x=0$$

✓ Solution by Mathematica

Time used: 2.4 (sec). Leaf size: 910

```
DSolve[y''[x]*y'[x]+y[x]^n==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\frac{\#1\sqrt[3]{n+1}\sqrt[3]{1-\frac{\#1^{n+1}}{c_1(n+1)}}\,\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{n+1},1+\frac{1}{n+1},\frac{\#1^{n+1}}{(n+1)c_1}\right)}{\sqrt[3]{-3\#1^{n+1}+3c_1(n+1)}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{(-1)^{2/3}\#1\sqrt[3]{n+1}\sqrt[3]{1-\frac{\#1^{n+1}}{c_1(n+1)}}\,\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{n+1},1+\frac{1}{n+1},\frac{\#1^{n+1}}{(n+1)c_1}\right)}{\sqrt[3]{-3\#1^{n+1}+3c_1(n+1)}}\&\right] + c_2]$$

$$y(x) \to \text{InverseFunction}\left[-\frac{\sqrt[3]{-\frac{1}{3}}\#1\sqrt[3]{n+1}\sqrt[3]{1-\frac{\#1^{n+1}}{c_1(n+1)}}\,\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{n+1},1+\frac{1}{n+1},\frac{\#1^{n+1}}{(n+1)c_1}\right)}{\sqrt[3]{-\#1^{n+1}+c_1(n+1)}}\&\right] + c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\#1\sqrt[3]{n+1}\sqrt[3]{1-\frac{\#1^{n+1}}{(-c_1)(n+1)}}\,\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{n+1},1+\frac{1}{n+1},\frac{\#1^{n+1}}{(n+1)(-c_1)}\right)}{\sqrt[3]{-3\#1^{n+1}+3(-c_1)(n+1)}}\&\right] + c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{(-1)^{2/3}\#1\sqrt[3]{n+1}\sqrt[3]{1-\frac{\#1^{n+1}}{(-c_1)(n+1)}}\,\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{n+1},1+\frac{1}{n+1},\frac{\#1^{n+1}}{(n+1)(-c_1)}\right)}{\sqrt[3]{-3\#1^{n+1}+3(-c_1)(n+1)}} + c_2]$$

$$y(x) \to \text{InverseFunction}\left[-\frac{\sqrt[3]{-\frac{1}{3}}\#1\sqrt[3]{n+1}\sqrt[3]{1-\frac{\#1^{n+1}}{(-c_1)(n+1)}}\,\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{n+1},1+\frac{1}{n+1},\frac{\#1^{n+1}}{(n+1)(-c_1)}\right)}{\sqrt[3]{-\#1^{n+1}+(-c_1)(n+1)}}\right.$$

2.7 problem 8

2.7.1 Solving as homogeneousTypeC ode 685
2.7.2 Solving as first order ode lie symmetry lookup ode 687

Internal problem ID [7448]
Internal file name [`OUTPUT/6415_Sunday_June_05_2022_04_47_48_PM_81109528/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - (x + y)^4 = 0}$$

2.7.1 Solving as homogeneousTypeC ode

Let

$$z = x + y \tag{1}$$

Then

$$z'(x) = 1 + y'$$

Therefore

$$y' = z'(x) - 1$$

Hence the given ode can now be written as

$$z'(x) - 1 = z^4$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{z^4+1} dz$$

$$x + c_1 = \frac{\sqrt{2}\left(\ln\left(\frac{z^2+\sqrt{2}\,z+1}{z^2-\sqrt{2}\,z+1}\right) + 2\arctan\left(\sqrt{2}\,z+1\right) + 2\arctan\left(\sqrt{2}\,z-1\right)\right)}{8}$$

Replacing z back by its value from (1) then the above gives the solution as

$$\frac{\sqrt{2}\left(\ln\left(\frac{(x+y)^2+\sqrt{2}\,(x+y)+1}{(x+y)^2-\sqrt{2}\,(x+y)+1}\right) + 2\arctan\left(\sqrt{2}\,(x+y)+1\right) + 2\arctan\left(\sqrt{2}\,(x+y)-1\right)\right)}{8} = x + c_1$$

Summary

The solution(s) found are the following

$$\frac{\sqrt{2}\left(\ln\left(\frac{(x+y)^2+\sqrt{2}\,(x+y)+1}{(x+y)^2-\sqrt{2}\,(x+y)+1}\right) + 2\arctan\left(\sqrt{2}\,(x+y)+1\right) + 2\arctan\left(\sqrt{2}\,(x+y)-1\right)\right)}{8} = x + c_1 \tag{1}$$

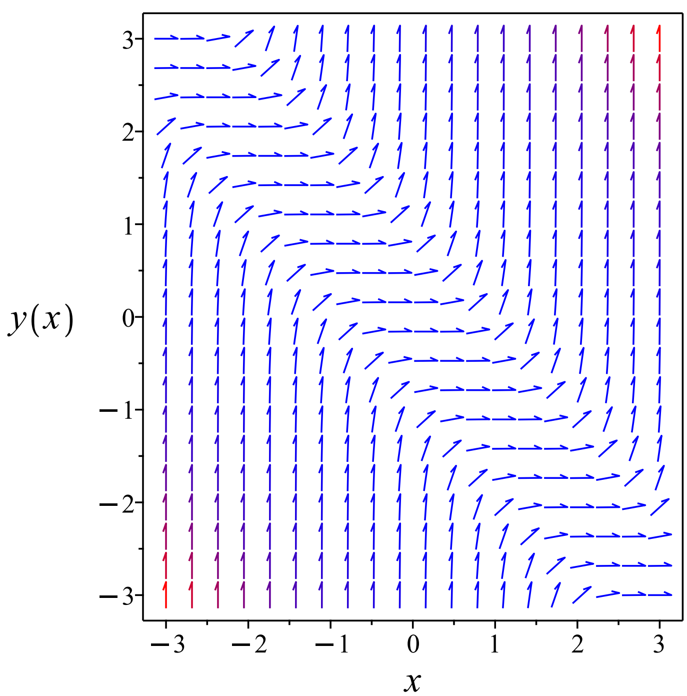


Figure 119: Slope field plot

Verification of solutions

$$\frac{\sqrt{2}\left(\ln\left(\frac{(x+y)^2+\sqrt{2}\,(x+y)+1}{(x+y)^2-\sqrt{2}\,(x+y)+1}\right)+2\arctan\left(\sqrt{2}\,(x+y)+1\right)+2\arctan\left(\sqrt{2}\,(x+y)-1\right)\right)}{8}$$
$$= x + c_1$$

Verified OK.

2.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= (x+y)^4 \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 73: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 1 \\
\eta(x, y) &= -1
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{-1}{1} \\ &= -1\end{aligned}$$

This is easily solved to give

$$y = -x + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = x + y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (x+y)^4$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 1 \\ S_x &= 1 \\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{1 + (x + y)^4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^4 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\sqrt{2}\left(\ln\left(\frac{R^2+\sqrt{2}\,R+1}{R^2-\sqrt{2}\,R+1}\right) + 2\arctan\left(\sqrt{2}\,R+1\right) + 2\arctan\left(\sqrt{2}\,R-1\right)\right)}{8} + c_1 \tag{4}$$

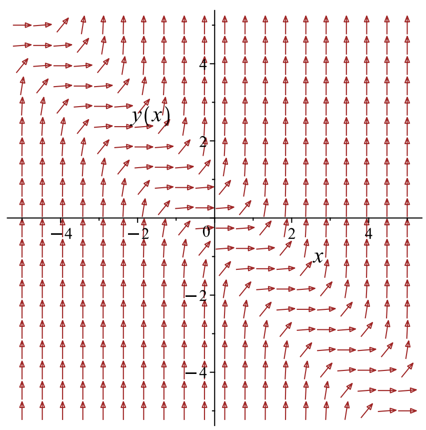
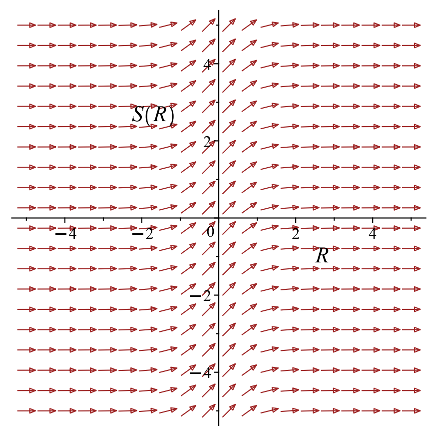
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = \frac{\sqrt{2}\left(\ln\left(\frac{(x+y)^2+\sqrt{2}\,(x+y)+1}{(x+y)^2-\sqrt{2}\,(x+y)+1}\right) + 2\arctan\left(\sqrt{2}\,(x+y)+1\right) + 2\arctan\left(\sqrt{2}\,(x+y)-1\right)\right)}{8} + c_1$$

Which simplifies to

$$x = \frac{\sqrt{2}\left(\ln\left(\frac{(x+y)^2+\sqrt{2}\,(x+y)+1}{(x+y)^2-\sqrt{2}\,(x+y)+1}\right) + 2\arctan\left(\sqrt{2}\,(x+y)+1\right) + 2\arctan\left(\sqrt{2}\,(x+y)-1\right)\right)}{8} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (x+y)^4$  	$R = x + y$ $S = x$	$\frac{dS}{dR} = \frac{1}{R^4+1}$ 

Summary

The solution(s) found are the following

$$x = \frac{\sqrt{2}\left(\ln\left(\frac{(x+y)^2+\sqrt{2}\,(x+y)+1}{(x+y)^2-\sqrt{2}\,(x+y)+1}\right) + 2\arctan\left(\sqrt{2}\,(x+y)+1\right) + 2\arctan\left(\sqrt{2}\,(x+y)-1\right)\right)}{8} + c_1 \tag{1}$$

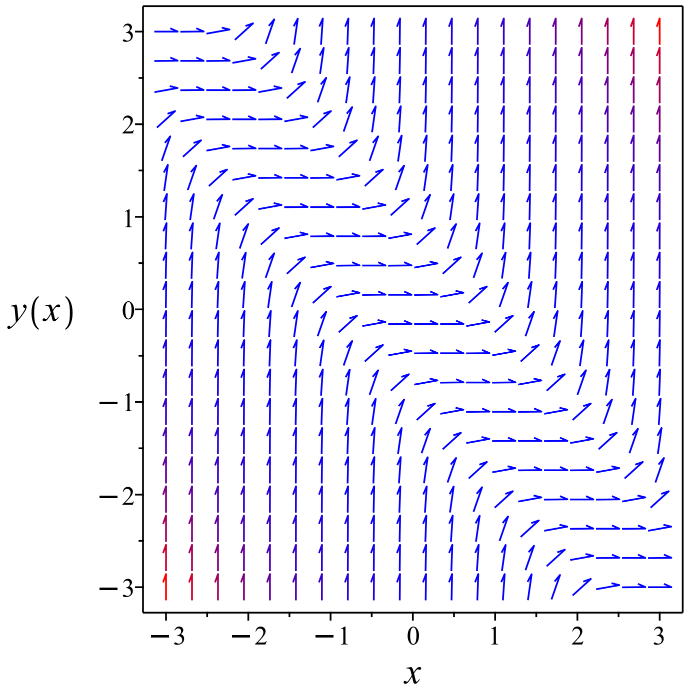


Figure 120: Slope field plot

Verification of solutions

$$x = \frac{\sqrt{2}\left(\ln\left(\frac{(x+y)^2+\sqrt{2}\,(x+y)+1}{(x+y)^2-\sqrt{2}\,(x+y)+1}\right)+2\arctan\left(\sqrt{2}\,(x+y)+1\right)+2\arctan\left(\sqrt{2}\,(x+y)-1\right)\right)}{8} + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.391 (sec). Leaf size: 882

```
dsolve(diff(y(x), x) = (x + y(x))^4,y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 0.115 (sec). Leaf size: 88

```
DSolve[y'[x] == (x + y[x])^4,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{4}\text{RootSum}\left[\#1^4+4\#1^3y(x)+6\#1^2y(x)^2+4\#1y(x)^3+y(x)^4+1\&,\frac{\log(x-\#1)}{\#1^3+3\#1^2y(x)+3\#1y(x)^2+y(x)^3}\&\right]-x=c_1,y(x)\right]$$

2.8 problem 9

Internal problem ID [7449]
Internal file name [`OUTPUT/6416_Sunday_June_05_2022_04_51_28_PM_59827055/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

Unable to solve or complete the solution.

$$y'' + (x+3)\,y' + (3+y^2)\,{y'}^2 = 0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)+(3+x)*diff(y(x),x)+(3+y(x)^2)*(diff(y(x),x))^2=0,y(x), singsol=all)
```

$$c_1 \operatorname{erf}\left(\frac{\sqrt{2}\,(x+3)}{2}\right) - c_2 + \int^{y(x)} \mathrm{e}^{\frac{\_a(\_a^2+9)}{3}} d\_a = 0$$

✓ Solution by Mathematica

Time used: 0.359 (sec). Leaf size: 61

```
DSolve[y''[x]+(3+x)*y'[x]+(3+y[x]^2)*(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} e^{\frac{K[1]^3}{3}+3K[1]}dK[1]\&\right]\left[c_2 - e^{9/2}\sqrt{\frac{\pi}{2}}c_1\text{erf}\left(\frac{x+3}{\sqrt{2}}\right)\right]$$

2.9 problem 10

2.9.1 Solving as second order nonlinear solved by mainardi lioville method ode 696

Internal problem ID [7450]
Internal file name [`OUTPUT/6417_Sunday_June_05_2022_04_51_32_PM_3199463/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_nonlinear_solved_by_mainardi_lioville_method"**

Maple gives the following as the ode type

```
[_Liouville, [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{y'' + xy' + yy'^2 = 0}$$

2.9.1 Solving as second order nonlinear solved by mainardi lioville method ode

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y)y'^2 = 0 \tag{1A}$$

Where in this problem

$$f(x) = x$$
$$g(y) = y$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx} \ln(y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$
\begin{aligned}
g\frac{dy}{dx} &= \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx} \\
&= \frac{d}{dx}\int g dy \qquad (4A)
\end{aligned}
$$

Substituting (3A,4A) back into (2A) gives

$$
\frac{d}{dx}\ln(y') + \frac{d}{dx}\int g dy = -f \qquad (5A)
$$

Integrating the above w.r.t. x gives

$$
\ln(y') + \int g dy = -\int f dx + c_1
$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$
y' = c_2 e^{\int -g dy} e^{\int -f dx} \qquad (6A)
$$

Where c_2 is a new arbitrary constant. But since $g = y$ and $f = x$, then

$$
\begin{aligned}
\int -g dy &= \int -y dy \\
&= -\frac{y^2}{2} \\
\int -f dx &= \int -x dx \\
&= -\frac{x^2}{2}
\end{aligned}
$$

Substituting the above into Eq(6A) gives

$$
y' = c_2 e^{-\frac{y^2}{2}} e^{-\frac{x^2}{2}}
$$

Which is now solved as first order separable ode. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= c_2 e^{-\frac{y^2}{2}} e^{-\frac{x^2}{2}}
\end{aligned}
$$

Where $f(x) = c_2 \mathrm{e}^{-\frac{x^2}{2}}$ and $g(y) = \mathrm{e}^{-\frac{y^2}{2}}$. Integrating both sides gives

$$\frac{1}{\mathrm{e}^{-\frac{y^2}{2}}}\,dy = c_2 \mathrm{e}^{-\frac{x^2}{2}}\,dx$$

$$\int \frac{1}{\mathrm{e}^{-\frac{y^2}{2}}}\,dy = \int c_2 \mathrm{e}^{-\frac{x^2}{2}}\,dx$$

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} = \frac{c_2\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} + c_3$$

The solution is

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} - \frac{c_2\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} - c_3 = 0$$

Summary

The solution(s) found are the following

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} - \frac{c_2\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} - c_3 = 0 \tag{1}$$

Verification of solutions

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} - \frac{c_2\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} - c_3 = 0$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+y(x)*(diff(y(x),x))^2=0,y(x), singsol=all)
```

$$y(x) = -i \operatorname{RootOf}\left(i\sqrt{\pi}\, \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) c_1 + i\sqrt{2}\, c_2 - \operatorname{erf}\left(\_Z\right) \sqrt{\pi}\right) \sqrt{2}$$

✓ Solution by Mathematica

Time used: 0.088 (sec). Leaf size: 44

```
DSolve[y''[x]+x*y'[x]+y[x]*(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -i\sqrt{2}\operatorname{erf}^{-1}\left(i\left(\sqrt{\frac{2}{\pi}}c_2 - c_1\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right)\right)$$

2.10 problem 11

2.10.1 Solving as second order ode missing y ode 700
2.10.2 Solving as second order nonlinear solved by mainardi lioville method ode 708

Internal problem ID [7451]
Internal file name [`OUTPUT/6418_Sunday_June_05_2022_04_51_34_PM_21417269/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y", "second_order_nonlinear_solved_by_mainardi_lioville_method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], _Liouville, [_2nd_order, _reducible,
    _mu_xy]]
```

$$\boxed{y'' + y' \sin(x) + {y'}^2 = 0}$$

2.10.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + (\sin(x) + p(x))\, p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Writing the ode as

$$\begin{aligned} p'(x) &= -(\sin(x) + p)\, p \\ p'(x) &= \omega(x, p) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_p - \xi_x) - \omega^2 \xi_p - \omega_x \xi - \omega_p \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 75: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,p) &= 0 \\ \eta(x,p) &= p^2 e^{-\cos(x)}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,p) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dp}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial p}\right) S(x,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{p^2 \mathrm{e}^{-\cos(x)}} dy \end{aligned}$$

Which results in

$$S = -\frac{\mathrm{e}^{\cos(x)}}{p}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,p) S_p}{R_x + \omega(x,p) R_p} \tag{2}$$

Where in the above R_x, R_p, S_x, S_p are all partial derivatives and $\omega(x,p)$ is the right hand side of the original ode given by

$$\omega(x,p) = -(\sin(x) + p)\, p$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_p &= 0 \\ S_x &= \frac{\sin(x)\, \mathrm{e}^{\cos(x)}}{p} \\ S_p &= \frac{\mathrm{e}^{\cos(x)}}{p^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\mathrm{e}^{\cos(x)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -e^{\cos(R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int -e^{\cos(R)} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, p coordinates. This results in

$$-\frac{e^{\cos(x)}}{p(x)} = \int -e^{\cos(x)} dx + c_1$$

Which simplifies to

$$-\frac{e^{\cos(x)}}{p(x)} = \int -e^{\cos(x)} dx + c_1$$

Which gives

$$p(x) = -\frac{e^{\cos(x)}}{\int -e^{\cos(x)} dx + c_1}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{e^{\cos(x)}}{\int -e^{\cos(x)} dx + c_1}$$

Writing the ode as

$$y' = -\frac{e^{\cos(x)}}{\int -e^{\cos(x)} dx + c_1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{e^{\cos(x)}(b_3 - a_2)}{\int -e^{\cos(x)}dx + c_1} - \frac{e^{2\cos(x)} a_3}{\left(\int -e^{\cos(x)}dx + c_1\right)^2} - \left(\frac{\sin(x)\, e^{\cos(x)}}{\int -e^{\cos(x)}dx + c_1} - \frac{e^{2\cos(x)}}{\left(\int -e^{\cos(x)}dx + c_1\right)^2}\right)(x a_2 + y a_3 + a_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\sin(x)\left(\int -e^{\cos(x)}dx\right) e^{\cos(x)} x a_2 + \sin(x)\left(\int -e^{\cos(x)}dx\right) e^{\cos(x)} y a_3 + \sin(x)\, e^{\cos(x)} c_1 x a_2 + \sin(x)\, e^{\cos(x)} c_1 y}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-\sin(x)\left(\int -e^{\cos(x)}dx\right) e^{\cos(x)} x a_2 - \sin(x)\left(\int -e^{\cos(x)}dx\right) e^{\cos(x)} y a_3 \\ &- \sin(x)\, e^{\cos(x)} c_1 x a_2 - \sin(x)\, e^{\cos(x)} c_1 y a_3 \\ &- \sin(x)\left(\int -e^{\cos(x)}dx\right) e^{\cos(x)} a_1 - \sin(x)\, e^{\cos(x)} c_1 a_1 + e^{2\cos(x)} x a_2 \\ &+ e^{2\cos(x)} y a_3 + \left(\int -e^{\cos(x)}dx\right)^2 b_2 + \left(\int -e^{\cos(x)}dx\right) e^{\cos(x)} a_2 \\ &- \left(\int -e^{\cos(x)}dx\right) e^{\cos(x)} b_3 + 2\left(\int -e^{\cos(x)}dx\right) c_1 b_2 \\ &+ e^{2\cos(x)} a_1 - e^{2\cos(x)} a_3 + e^{\cos(x)} c_1 a_2 - e^{\cos(x)} c_1 b_3 + c_1^2 b_2 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}&\left(-\left(\int -\mathrm{e}^{\cos(x)}dx\right) x a_2-\left(\int -\mathrm{e}^{\cos(x)}dx\right) y a_3-c_1 x a_2-c_1 y a_3\right.\\&\left.-\left(\int -\mathrm{e}^{\cos(x)}dx\right) a_1-c_1 a_1\right) \sin(x)\, \mathrm{e}^{\cos(x)}-\mathrm{e}^{\cos(x)} c_1 b_3\\&+\mathrm{e}^{2\cos(x)} x a_2+\mathrm{e}^{2\cos(x)} y a_3-\left(\int -\mathrm{e}^{\cos(x)}dx\right) \mathrm{e}^{\cos(x)} b_3\\&-\mathrm{e}^{2\cos(x)} a_3+\mathrm{e}^{\cos(x)} c_1 a_2+\left(\int -\mathrm{e}^{\cos(x)}dx\right) \mathrm{e}^{\cos(x)} a_2+\mathrm{e}^{2\cos(x)} a_1\\&+2\left(\int -\mathrm{e}^{\cos(x)}dx\right) c_1 b_2+c_1^2 b_2+\left(\int -\mathrm{e}^{\cos(x)}dx\right)^2 b_2=0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \int -\mathrm{e}^{\cos(x)}dx, \cos(x), \mathrm{e}^{2\cos(x)}, \mathrm{e}^{\cos(x)}, \sin(x)\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \int -\mathrm{e}^{\cos(x)}dx=v_3, \cos(x)=v_4, \mathrm{e}^{2\cos(x)}=v_5, \mathrm{e}^{\cos(x)}=v_6, \sin(x)=v_7\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&(-c_1 a_2 v_1-c_1 a_3 v_2-v_3 v_1 a_2-v_3 v_2 a_3-c_1 a_1-v_3 a_1) v_7 v_6-v_6 c_1 b_3+v_5 v_1 a_2\\&+v_5 v_2 a_3-v_3 v_6 b_3-v_5 a_3+v_6 c_1 a_2+v_3 v_6 a_2+v_5 a_1+2 v_3 c_1 b_2+c_1^2 b_2+v_3^2 b_2=0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$$

Equation (7E) now becomes

$$\begin{aligned}&-a_2 c_1 v_6 v_7 v_1-a_2 v_7 v_6 v_1 v_3+v_5 v_1 a_2-c_1 a_3 v_6 v_7 v_2-a_3 v_7 v_6 v_2 v_3\\&+v_5 v_2 a_3+v_3^2 b_2-a_1 v_6 v_7 v_3+(-b_3+a_2) v_6 v_3+2 v_3 c_1 b_2\\&+(a_1-a_3) v_5-c_1 a_1 v_7 v_6+(c_1 a_2-c_1 b_3) v_6+c_1^2 b_2=0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_2 &= 0 \\
a_3 &= 0 \\
b_2 &= 0 \\
c_1^2 b_2 &= 0 \\
-a_1 &= 0 \\
-a_2 &= 0 \\
-a_3 &= 0 \\
-c_1 a_1 &= 0 \\
-c_1 a_3 &= 0 \\
-c_1 a_2 &= 0 \\
2b_2 c_1 &= 0 \\
a_1 - a_3 &= 0 \\
-b_3 + a_2 &= 0 \\
c_1 a_2 - c_1 b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= 1
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{1} dy \end{aligned}$$

Which results in

$$S = y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{e^{\cos(x)}}{\int -e^{\cos(x)}dx + c_1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 0 \\ S_y &= 1 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{\cos(x)}}{\int e^{\cos(x)}dx - c_1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^{\cos(R)}}{\int e^{\cos(R)}dR - c_1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln\left(\int e^{\cos(R)}dR - c_1\right) + c_2 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y = \ln\left(\int e^{\cos(x)}dx - c_1\right) + c_2$$

Which simplifies to

$$y = \ln\left(\int e^{\cos(x)}dx - c_1\right) + c_2$$

Which gives

$$y = \ln\left(\int e^{\cos(x)}dx - c_1\right) + c_2$$

Summary
The solution(s) found are the following

$$y = \ln\left(\int e^{\cos(x)}dx - c_1\right) + c_2 \tag{1}$$

Verification of solutions

$$y = \ln\left(\int e^{\cos(x)}dx - c_1\right) + c_2$$

Verified OK.

2.10.2 Solving as second order nonlinear solved by mainardi lioville method ode

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y){y'}^2 = 0 \tag{1A}$$

Where in this problem

$$f(x) = \sin(x)$$
$$g(y) = 1$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx} \ln(y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned} g\frac{dy}{dx} &= \left(\frac{d}{dy} \int g dy\right) \frac{dy}{dx} \\ &= \frac{d}{dx} \int g dy \end{aligned} \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx} \ln(y') + \frac{d}{dx} \int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln(y') + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy} e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = 1$ and $f = \sin(x)$, then

$$\begin{aligned} \int -g dy &= \int (-1) \, dy \\ &= -y \\ \int -f dx &= \int -\sin(x) \, dx \\ &= \cos(x) \end{aligned}$$

Substituting the above into Eq(6A) gives

$$y' = c_2 e^{-y} e^{\cos(x)}$$

Which is now solved as first order separable ode. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= c_2 \mathrm{e}^{-y} \mathrm{e}^{\cos(x)}
\end{aligned}
$$

Where $f(x) = c_2 \mathrm{e}^{\cos(x)}$ and $g(y) = \mathrm{e}^{-y}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\mathrm{e}^{-y}}\, dy &= c_2 \mathrm{e}^{\cos(x)}\, dx \\
\int \frac{1}{\mathrm{e}^{-y}}\, dy &= \int c_2 \mathrm{e}^{\cos(x)}\, dx \\
\mathrm{e}^{y} &= \int c_2 \mathrm{e}^{\cos(x)} dx + c_3
\end{aligned}
$$

The solution is

$$
\mathrm{e}^{y} - \left(\int c_2 \mathrm{e}^{\cos(x)} dx\right) - c_3 = 0
$$

Summary

The solution(s) found are the following

$$
\mathrm{e}^{y} - \left(\int c_2 \mathrm{e}^{\cos(x)} dx\right) - c_3 = 0 \tag{1}
$$

Verification of solutions

$$
\mathrm{e}^{y} - \left(\int c_2 \mathrm{e}^{\cos(x)} dx\right) - c_3 = 0
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)+sin(x)*diff(y(x),x)+(diff(y(x),x))^2=0,y(x), singsol=all)
```

$$y(x) = \ln \left(c_1 \left(\int \mathrm{e}^{\cos(x)} dx\right) + c_2\right)$$

✓ Solution by Mathematica

Time used: 60.089 (sec). Leaf size: 43

```
DSolve[y''[x]+Sin[x]*y'[x]+(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \int_1^x \frac{e^{\cos(K[2])}}{c_1 - \int_1^{K[2]} -e^{\cos(K[1])} dK[1]} dK[2] + c_2$$

2.11 problem 12

2.11.1 Solving as second order nonlinear solved by mainardi lioville method ode 712

Internal problem ID [7452]
Internal file name [OUTPUT/6419_Sunday_June_05_2022_04_51_38_PM_58438087/index.tex]

Book: Second order enumerated odes
Section: section 2
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_nonlinear_solved_by_mainardi_lioville_method"**

Maple gives the following as the ode type

```
[_Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

$$\boxed{3y'' + y' \cos(x) + \sin(y) {y'}^2 = 0}$$

2.11.1 Solving as second order nonlinear solved by mainardi lioville method ode

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y){y'}^2 = 0 \tag{1A}$$

Where in this problem

$$f(x) = \frac{\cos(x)}{3}$$
$$g(y) = \frac{\sin(y)}{3}$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx}\ln(y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned} g\frac{dy}{dx} &= \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx} \\ &= \frac{d}{dx}\int g dy \end{aligned} \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx}\ln(y') + \frac{d}{dx}\int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln(y') + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy} e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = \frac{\sin(y)}{3}$ and $f = \frac{\cos(x)}{3}$, then

$$\begin{aligned} \int -g dy &= \int -\frac{\sin(y)}{3} dy \\ &= \frac{\cos(y)}{3} \\ \int -f dx &= \int -\frac{\cos(x)}{3} dx \\ &= -\frac{\sin(x)}{3} \end{aligned}$$

Substituting the above into Eq(6A) gives

$$y' = c_2 e^{\frac{\cos(y)}{3}} e^{-\frac{\sin(x)}{3}}$$

Which is now solved as first order separable ode. In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= c_2 e^{\frac{\cos(y)}{3}} e^{-\frac{\sin(x)}{3}} \end{aligned}$$

Where $f(x) = c_2 \mathrm{e}^{-\frac{\sin(x)}{3}}$ and $g(y) = \mathrm{e}^{\frac{\cos(y)}{3}}$. Integrating both sides gives

$$\frac{1}{\mathrm{e}^{\frac{\cos(y)}{3}}}\, dy = c_2 \mathrm{e}^{-\frac{\sin(x)}{3}}\, dx$$
$$\int \frac{1}{\mathrm{e}^{\frac{\cos(y)}{3}}}\, dy = \int c_2 \mathrm{e}^{-\frac{\sin(x)}{3}}\, dx$$
$$\int^{y} \mathrm{e}^{-\frac{\cos(\_a)}{3}} d\_a = \int c_2 \mathrm{e}^{-\frac{\sin(x)}{3}} dx + c_3$$

Which results in

$$\int^{y} \mathrm{e}^{-\frac{\cos(\_a)}{3}} d\_a = \int c_2 \mathrm{e}^{-\frac{\sin(x)}{3}} dx + c_3$$

The solution is

$$\int^{y} \mathrm{e}^{-\frac{\cos(\_a)}{3}} d\_a - \left(\int c_2 \mathrm{e}^{-\frac{\sin(x)}{3}} dx\right) - c_3 = 0$$

Summary

The solution(s) found are the following

$$\int^{y} \mathrm{e}^{-\frac{\cos(\_a)}{3}} d\_a - \left(\int c_2 \mathrm{e}^{-\frac{\sin(x)}{3}} dx\right) - c_3 = 0 \tag{1}$$

Verification of solutions

$$\int^{y} \mathrm{e}^{-\frac{\cos(\_a)}{3}} d\_a - \left(\int c_2 \mathrm{e}^{-\frac{\sin(x)}{3}} dx\right) - c_3 = 0$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(3*diff(y(x),x$2)+cos(x)*diff(y(x),x)+sin(y(x))*(diff(y(x),x))^2=0,y(x), singsol=all)
```

$$\int^{y(x)} \mathrm{e}^{-\frac{\cos(\_a)}{3}} d\_a - c_1 \left(\int \mathrm{e}^{-\frac{\sin(x)}{3}} dx \right) - c_2 = 0$$

✓ Solution by Mathematica

Time used: 0.601 (sec). Leaf size: 47

```
DSolve[3*y''[x]+Cos[x]*y'[x]+Sin[y[x]]*(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} e^{-\frac{1}{3}\cos(K[1])} dK[1] \& \right] \left[\int_1^x -e^{-\frac{1}{3}\sin(K[2])} c_1 dK[2] + c_2 \right]$$

2.12 problem 13

2.12.1 Solving as second order nonlinear solved by mainardi lioville method ode 716

Internal problem ID [7453]
Internal file name [`OUTPUT/6420_Sunday_June_05_2022_04_51_42_PM_54690149/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_nonlinear_solved_by_mainardi_lioville_method"**

Maple gives the following as the ode type

```
[_Liouville, [_2nd_order, _with_linear_symmetries], [_2nd_order
   , _reducible, _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{10y'' + x^2y' + \frac{3{y'}^2}{y} = 0}$$

2.12.1 Solving as second order nonlinear solved by mainardi lioville method ode

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y){y'}^2 = 0 \tag{1A}$$

Where in this problem

$$\begin{aligned} f(x) &= \frac{x^2}{10} \\ g(y) &= \frac{3}{10y} \end{aligned}$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx}\ln(y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned} g\frac{dy}{dx} &= \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx} \\ &= \frac{d}{dx}\int g dy \end{aligned} \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx}\ln(y') + \frac{d}{dx}\int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln(y') + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy}\, e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = \frac{3}{10y}$ and $f = \frac{x^2}{10}$, then

$$\begin{aligned} \int -g dy &= \int -\frac{3}{10y} dy \\ &= -\frac{3\ln(y)}{10} \\ \int -f dx &= \int -\frac{x^2}{10} dx \\ &= -\frac{x^3}{30} \end{aligned}$$

Substituting the above into Eq(6A) gives

$$y' = \frac{c_2 \mathrm{e}^{-\frac{x^3}{30}}}{y^{\frac{3}{10}}}$$

Which is now solved as first order separable ode. In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{c_2 \mathrm{e}^{-\frac{x^3}{30}}}{y^{\frac{3}{10}}} \end{aligned}$$

Where $f(x) = c_2 e^{-\frac{x^3}{30}}$ and $g(y) = \frac{1}{y^{\frac{3}{10}}}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y^{\frac{3}{10}}}}\,dy = c_2 e^{-\frac{x^3}{30}}\,dx$$

$$\int \frac{1}{\frac{1}{y^{\frac{3}{10}}}}\,dy = \int c_2 e^{-\frac{x^3}{30}}\,dx$$

$$\frac{10y^{\frac{13}{10}}}{13} = \frac{10^{\frac{1}{3}} 9^{\frac{2}{3}} c_2 \left(\frac{3\,243^{\frac{1}{6}}\,10^{\frac{5}{6}}\,x\,e^{-\frac{x^3}{60}}\,\mathrm{WhittakerM}\left(\frac{1}{6},\frac{2}{3},\frac{x^3}{30}\right)}{40(x^3)^{\frac{1}{6}}} + \frac{3\,30^{\frac{5}{6}}\,e^{-\frac{x^3}{60}}\,\mathrm{WhittakerM}\left(\frac{7}{6},\frac{2}{3},\frac{x^3}{30}\right)}{x^2(x^3)^{\frac{1}{6}}}\right)}{9} + c_3$$

The solution is

$$\frac{10y^{\frac{13}{10}}}{13} - \frac{10^{\frac{1}{3}} 9^{\frac{2}{3}} c_2 \left(\frac{3\,243^{\frac{1}{6}}\,10^{\frac{5}{6}}\,x\,e^{-\frac{x^3}{60}}\,\mathrm{WhittakerM}\left(\frac{1}{6},\frac{2}{3},\frac{x^3}{30}\right)}{40(x^3)^{\frac{1}{6}}} + \frac{3\,30^{\frac{5}{6}}\,e^{-\frac{x^3}{60}}\,\mathrm{WhittakerM}\left(\frac{7}{6},\frac{2}{3},\frac{x^3}{30}\right)}{x^2(x^3)^{\frac{1}{6}}}\right)}{9} - c_3 = 0$$

Summary

The solution(s) found are the following

$$\frac{10y^{\frac{13}{10}}}{13} - \frac{10^{\frac{1}{3}} 9^{\frac{2}{3}} c_2 \left(\frac{3\,243^{\frac{1}{6}}\,10^{\frac{5}{6}}\,x\,e^{-\frac{x^3}{60}}\,\mathrm{WhittakerM}\left(\frac{1}{6},\frac{2}{3},\frac{x^3}{30}\right)}{40(x^3)^{\frac{1}{6}}} + \frac{3\,30^{\frac{5}{6}}\,e^{-\frac{x^3}{60}}\,\mathrm{WhittakerM}\left(\frac{7}{6},\frac{2}{3},\frac{x^3}{30}\right)}{x^2(x^3)^{\frac{1}{6}}}\right)}{9} - c_3 = 0 \quad (1)$$

Verification of solutions

$$\frac{10y^{\frac{13}{10}}}{13} - \frac{10^{\frac{1}{3}} 9^{\frac{2}{3}} c_2 \left(\frac{3\,243^{\frac{1}{6}}\,10^{\frac{5}{6}}\,x\,e^{-\frac{x^3}{60}}\,\mathrm{WhittakerM}\left(\frac{1}{6},\frac{2}{3},\frac{x^3}{30}\right)}{40(x^3)^{\frac{1}{6}}} + \frac{3\,30^{\frac{5}{6}}\,e^{-\frac{x^3}{60}}\,\mathrm{WhittakerM}\left(\frac{7}{6},\frac{2}{3},\frac{x^3}{30}\right)}{x^2(x^3)^{\frac{1}{6}}}\right)}{9} - c_3 = 0$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 55

```
dsolve(10*diff(y(x),x$2)+x^2*diff(y(x),x)+3/y(x)*(diff(y(x),x))^2=0,y(x), singsol=all)
```

$$-\frac{3\left(c_1 x \,\mathrm{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{x^3}{30}\right) \mathrm{e}^{-\frac{x^3}{60}} 30^{\frac{1}{6}}+\frac{4\left(x^3\right)^{\frac{1}{6}}\left(c_1 x \,\mathrm{e}^{-\frac{x^3}{30}}+c_2-\frac{10 y(x)^{\frac{13}{10}}}{13}\right)}{3}\right)}{4\left(x^3\right)^{\frac{1}{6}}}=0$$

✓ Solution by Mathematica

Time used: 66.444 (sec). Leaf size: 73

```
DSolve[10*y''[x]+x^2*y'[x]+3/y[x]*(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \exp\left(\int_1^x \frac{30 e^{-\frac{1}{30}K[1]^3}\sqrt[3]{K[1]^3}}{30 c_1 \sqrt[3]{K[1]^3}-13\sqrt[3]{30}\Gamma\left(\frac{1}{3}, \frac{K[1]^3}{30}\right)K[1]}dK[1]\right)$$

2.13 problem 14

Internal problem ID [7454]
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Book: Second order enumerated odes
Section: section 2
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

Unable to solve or complete the solution.

$$10y'' + (e^x + 3x)\, y' + \frac{3\, e^y {y'}^2}{\sin(y)} = 0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 38

```
dsolve(10*diff(y(x),x$2)+(exp(x)+3*x)*diff(y(x),x)+3/sin(y(x))*exp(y(x))*(diff(y(x),x))^2=0,
```

$$\int^{y(x)} e^{\frac{3\left(\int \csc(\_b)e^{\_b}d\_b\right)}{10}} d\_b - c_1 \left(\int e^{-\frac{3x^2}{20}-\frac{e^x}{10}} dx\right) - c_2 = 0$$

✓ Solution by Mathematica

Time used: 0.234 (sec). Leaf size: 90

```
DSolve[10*y''[x]+(Exp[x]+3*x)*y'[x]+3/Sin[y[x]]*Exp[y[x]]*(y'[x])^2==0,y[x],x,IncludeSingula
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \exp\left(\left(-\frac{3}{10}-\frac{3i}{10}\right)e^{(1+i)K[1]}\,\text{Hypergeometric2F1}\left(\frac{1}{2}-\frac{i}{2},1,\frac{3}{2}-\frac{i}{2},e^{2iK[1]}\right)\right)dK[\right.$$
$$\left. -e^{\frac{1}{20}\left(-3K[2]^2-2e^{K[2]}\right)}c_1dK[2]+c_2\right]$$

2.14 problem 15

2.14.1 Solving as second order euler ode ode 722
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Internal problem ID [7455]
Internal file name [`OUTPUT/6422_Sunday_June_05_2022_04_51_52_PM_88946529/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - \frac{2y}{x^2} = x\,\mathrm{e}^{-\sqrt{x}}}$$

The ode can be written as

$$x^2 y'' - 2y = x^3 \mathrm{e}^{-\sqrt{x}}$$

Which shows it is a Euler ODE.

2.14.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 0, C = -2, f(x) = x^3 \mathrm{e}^{-\sqrt{x}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$x^2y''-2y=0$$

This is Euler second order ODE. Let the solution be $y=x^r$, then $y'=rx^{r-1}$ and $y''=r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2}+0rx^{r-1}-2x^r=0$$

Simplifying gives

$$r(r-1)\,x^r+0\,x^r-2x^r=0$$

Since $x^r\neq 0$ then dividing throughout by x^r gives

$$r(r-1)+0-2=0$$

Or

$$r^2-r-2=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned}r_1&=-1\\ r_2&=2\end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y=c_1y_1+c_2y_2$$

Where $y_1=x^{r_1}$ and $y_2=x^{r_2}$. Hence

$$y=\frac{c_1}{x}+c_2x^2$$

Next, we find the particular solution to the ODE

$$x^2y''-2y=x^3\mathrm{e}^{-\sqrt{x}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x^2 \\ -\frac{1}{x^2} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)(2x) - (x^2)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = 3$$

Which simplifies to

$$W = 3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5 e^{-\sqrt{x}}}{3x^2}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^3 \mathrm{e}^{-\sqrt{x}}}{3} dx$$

Hence

$$u_1 = \frac{2\,\mathrm{e}^{-\sqrt{x}} x^{\frac{7}{2}}}{3} + \frac{14x^3 \mathrm{e}^{-\sqrt{x}}}{3} + 28x^{\frac{5}{2}} \mathrm{e}^{-\sqrt{x}} + 140x^2 \mathrm{e}^{-\sqrt{x}} + 560x^{\frac{3}{2}} \mathrm{e}^{-\sqrt{x}} + 1680x\,\mathrm{e}^{-\sqrt{x}} + 3360\sqrt{x}\,\mathrm{e}^{-\sqrt{x}} + 3360\,\mathrm{e}^{-\sqrt{x}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2 \mathrm{e}^{-\sqrt{x}}}{3x^2}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\mathrm{e}^{-\sqrt{x}}}{3} dx$$

Hence

$$u_2 = -\frac{2\sqrt{x}\,\mathrm{e}^{-\sqrt{x}}}{3} - \frac{2\,\mathrm{e}^{-\sqrt{x}}}{3}$$

Which simplifies to

$$u_1 = \frac{2\,\mathrm{e}^{-\sqrt{x}} \left(x^{\frac{7}{2}} + 42x^{\frac{5}{2}} + 840x^{\frac{3}{2}} + 7x^3 + 210x^2 + 5040\sqrt{x} + 2520x + 5040\right)}{3}$$

$$u_2 = -\frac{2\,\mathrm{e}^{-\sqrt{x}} \left(\sqrt{x} + 1\right)}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2\,\mathrm{e}^{-\sqrt{x}} \left(x^{\frac{7}{2}} + 42x^{\frac{5}{2}} + 840x^{\frac{3}{2}} + 7x^3 + 210x^2 + 5040\sqrt{x} + 2520x + 5040\right)}{3x} - \frac{2\,\mathrm{e}^{-\sqrt{x}} \left(\sqrt{x} + 1\right) x^2}{3}$$

Which simplifies to

$$y_p(x) = \frac{4\,\mathrm{e}^{-\sqrt{x}} \left(7x^{\frac{5}{2}} + 140x^{\frac{3}{2}} + x^3 + 35x^2 + 840\sqrt{x} + 420x + 840\right)}{x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{4\,\mathrm{e}^{-\sqrt{x}}\left(7x^{\frac{5}{2}} + 140x^{\frac{3}{2}} + x^3 + 35x^2 + 840\sqrt{x} + 420x + 840\right) + c_2x^3 + c_1}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{4\,\mathrm{e}^{-\sqrt{x}}\left(7x^{\frac{5}{2}} + 140x^{\frac{3}{2}} + x^3 + 35x^2 + 840\sqrt{x} + 420x + 840\right) + c_2x^3 + c_1}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{4\,\mathrm{e}^{-\sqrt{x}}\left(7x^{\frac{5}{2}} + 140x^{\frac{3}{2}} + x^3 + 35x^2 + 840\sqrt{x} + 420x + 840\right) + c_2x^3 + c_1}{x}
$$

Verified OK.

2.14.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
x^2y' - 2yx = -2\,\mathrm{e}^{-\sqrt{x}}x^{\frac{7}{2}} - 14x^3\mathrm{e}^{-\sqrt{x}} - 84x^{\frac{5}{2}}\mathrm{e}^{-\sqrt{x}} - 420x^2\mathrm{e}^{-\sqrt{x}} - 1680x^{\frac{3}{2}}\mathrm{e}^{-\sqrt{x}} - 5040x\,\mathrm{e}^{-\sqrt{x}} - 10080\sqrt{x}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{\left(-14x^3 - 420x^2 - 5040x - 10080\sqrt{x} - 1680x^{\frac{3}{2}} - 84x^{\frac{5}{2}} - 2x^{\frac{7}{2}} - 10080\right)\mathrm{e}^{-\sqrt{x}} + c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{2y}{x} = \frac{\left(-14x^3 - 420x^2 - 5040x - 10080\sqrt{x} - 1680x^{\frac{3}{2}} - 84x^{\frac{5}{2}} - 2x^{\frac{7}{2}} - 10080\right)\mathrm{e}^{-\sqrt{x}} + c_1}{x^2}
$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x}dx}$$
$$= \frac{1}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\left(-14x^3 - 420x^2 - 5040x - 10080\sqrt{x} - 1680x^{\frac{3}{2}} - 84x^{\frac{5}{2}} - 2x^{\frac{7}{2}} - 10080\right)e^{-\sqrt{x}} + c_1}{x^2}\right)$$

$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \left(\frac{1}{x^2}\right)\left(\frac{\left(-14x^3 - 420x^2 - 5040x - 10080\sqrt{x} - 1680x^{\frac{3}{2}} - 84x^{\frac{5}{2}} - 2x^{\frac{7}{2}} - 10080\right)e^{-\sqrt{x}} + c_1}{x^2}\right)$$

$$d\left(\frac{y}{x^2}\right) = \left(\frac{\left(-14x^3 - 420x^2 - 5040x - 10080\sqrt{x} - 1680x^{\frac{3}{2}} - 84x^{\frac{5}{2}} - 2x^{\frac{7}{2}} - 10080\right)e^{-\sqrt{x}} + c_1}{x^4}\right) dx$$

Integrating gives

$$\frac{y}{x^2} = \int \frac{\left(-14x^3 - 420x^2 - 5040x - 10080\sqrt{x} - 1680x^{\frac{3}{2}} - 84x^{\frac{5}{2}} - 2x^{\frac{7}{2}} - 10080\right)e^{-\sqrt{x}} + c_1}{x^4} dx$$

$$\frac{y}{x^2} = -\frac{c_1}{3x^3} + \frac{3360\,e^{-\sqrt{x}}}{x^3} + \frac{3360\,e^{-\sqrt{x}}}{x^{\frac{5}{2}}} + \frac{1680\,e^{-\sqrt{x}}}{x^2} + \frac{560\,e^{-\sqrt{x}}}{x^{\frac{3}{2}}} + \frac{140\,e^{-\sqrt{x}}}{x} + \frac{28\,e^{-\sqrt{x}}}{\sqrt{x}} + 4\,e^{-\sqrt{x}} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(-\frac{c_1}{3x^3} + \frac{3360\,e^{-\sqrt{x}}}{x^3} + \frac{3360\,e^{-\sqrt{x}}}{x^{\frac{5}{2}}} + \frac{1680\,e^{-\sqrt{x}}}{x^2} + \frac{560\,e^{-\sqrt{x}}}{x^{\frac{3}{2}}} + \frac{140\,e^{-\sqrt{x}}}{x} + \frac{28\,e^{-\sqrt{x}}}{\sqrt{x}} + 4\,e^{-\sqrt{x}}\right) + c_2$$

which simplifies to

$$y = \frac{28\left(x^3 + 20x^2 + 120x + 120\sqrt{x} + 60x^{\frac{3}{2}} + 5x^{\frac{5}{2}} + \frac{x^{\frac{7}{2}}}{7}\right)e^{-\sqrt{x}} - \frac{c_1\sqrt{x}}{3} + c_2x^{\frac{7}{2}}}{x^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$y = \frac{28\left(x^3 + 20x^2 + 120x + 120\sqrt{x} + 60x^{\frac{3}{2}} + 5x^{\frac{5}{2}} + \frac{x^{\frac{7}{2}}}{7}\right)e^{-\sqrt{x}} - \frac{c_1\sqrt{x}}{3} + c_2x^{\frac{7}{2}}}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{28\left(x^3 + 20x^2 + 120x + 120\sqrt{x} + 60x^{\frac{3}{2}} + 5x^{\frac{5}{2}} + \frac{x^{\frac{7}{2}}}{7}\right)e^{-\sqrt{x}} - \frac{c_1\sqrt{x}}{3} + c_2x^{\frac{7}{2}}}{x^{\frac{3}{2}}}$$

Verified OK.

2.14.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' - 2y = x^3 e^{-\sqrt{x}}$$

Integrating both sides of the ODE w.r.t x gives

$$x^2y' - 2yx = -2\,e^{-\sqrt{x}} x^{\frac{7}{2}} - 14x^3 e^{-\sqrt{x}} - 84x^{\frac{5}{2}} e^{-\sqrt{x}} - 420x^2 e^{-\sqrt{x}} - 1680x^{\frac{3}{2}} e^{-\sqrt{x}} - 5040x\,e^{-\sqrt{x}} - 10080\sqrt{x}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$

$$q(x) = \frac{\left(-14x^3 - 420x^2 - 5040x - 10080\sqrt{x} - 1680x^{\frac{3}{2}} - 84x^{\frac{5}{2}} - 2x^{\frac{7}{2}} - 10080\right) e^{-\sqrt{x}} + c_1}{x^2}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{\left(-14x^3 - 420x^2 - 5040x - 10080\sqrt{x} - 1680x^{\frac{3}{2}} - 84x^{\frac{5}{2}} - 2x^{\frac{7}{2}} - 10080\right) e^{-\sqrt{x}} + c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x} dx} = \frac{1}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\left(-14x^3 - 420x^2 - 5040x - 10080\sqrt{x} - 1680x^{\frac{3}{2}} - 84x^{\frac{5}{2}} - 2x^{\frac{7}{2}} - 10080\right) e^{-\sqrt{x}} + c_1}{x^2}\right)$$

$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \left(\frac{1}{x^2}\right)\left(\frac{\left(-14x^3 - 420x^2 - 5040x - 10080\sqrt{x} - 1680x^{\frac{3}{2}} - 84x^{\frac{5}{2}} - 2x^{\frac{7}{2}} - 10080\right) e^{-\sqrt{x}} + c_1}{x^2}\right)$$

$$d\left(\frac{y}{x^2}\right) = \left(\frac{\left(-14x^3 - 420x^2 - 5040x - 10080\sqrt{x} - 1680x^{\frac{3}{2}} - 84x^{\frac{5}{2}} - 2x^{\frac{7}{2}} - 10080\right) e^{-\sqrt{x}} + c_1}{x^4}\right) dx$$

Integrating gives

$$\frac{y}{x^2} = \int \frac{\left(-14x^3 - 420x^2 - 5040x - 10080\sqrt{x} - 1680x^{\frac{3}{2}} - 84x^{\frac{5}{2}} - 2x^{\frac{7}{2}} - 10080\right) e^{-\sqrt{x}} + c_1}{x^4} dx$$

$$\frac{y}{x^2} = -\frac{c_1}{3x^3} + \frac{3360\, e^{-\sqrt{x}}}{x^3} + \frac{3360\, e^{-\sqrt{x}}}{x^{\frac{5}{2}}} + \frac{1680\, e^{-\sqrt{x}}}{x^2} + \frac{560\, e^{-\sqrt{x}}}{x^{\frac{3}{2}}} + \frac{140\, e^{-\sqrt{x}}}{x} + \frac{28\, e^{-\sqrt{x}}}{\sqrt{x}} + 4\, e^{-\sqrt{x}} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(-\frac{c_1}{3x^3} + \frac{3360\, e^{-\sqrt{x}}}{x^3} + \frac{3360\, e^{-\sqrt{x}}}{x^{\frac{5}{2}}} + \frac{1680\, e^{-\sqrt{x}}}{x^2} + \frac{560\, e^{-\sqrt{x}}}{x^{\frac{3}{2}}} + \frac{140\, e^{-\sqrt{x}}}{x} + \frac{28\, e^{-\sqrt{x}}}{\sqrt{x}} + 4\, e^{-\sqrt{x}}\right) + c_2$$

which simplifies to

$$y = \frac{28\left(x^3 + 20x^2 + 120x + 120\sqrt{x} + 60x^{\frac{3}{2}} + 5x^{\frac{5}{2}} + \frac{x^{\frac{7}{2}}}{7}\right) e^{-\sqrt{x}} - \frac{c_1\sqrt{x}}{3} + c_2 x^{\frac{7}{2}}}{x^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$y = \frac{28\left(x^3 + 20x^2 + 120x + 120\sqrt{x} + 60x^{\frac{3}{2}} + 5x^{\frac{5}{2}} + \frac{x^{\frac{7}{2}}}{7}\right) e^{-\sqrt{x}} - \frac{c_1\sqrt{x}}{3} + c_2 x^{\frac{7}{2}}}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{28\left(x^3 + 20x^2 + 120x + 120\sqrt{x} + 60x^{\frac{3}{2}} + 5x^{\frac{5}{2}} + \frac{x^{\frac{7}{2}}}{7}\right) e^{-\sqrt{x}} - \frac{c_1\sqrt{x}}{3} + c_2 x^{\frac{7}{2}}}{x^{\frac{3}{2}}}$$

Verified OK.

2.14.4 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 77: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -1$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \frac{1}{x} \int \frac{1}{\frac{1}{x^2}}\,dx \\ &= \frac{1}{x}\left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\frac{x^3}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x} + \frac{c_2 x^2}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$

$$y_2 = \frac{x^2}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x^2}{3} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{x^2}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x^2}{3} \\ -\frac{1}{x^2} & \frac{2x}{3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{2x}{3}\right) - \left(\frac{x^2}{3}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^5 \mathrm{e}^{-\sqrt{x}}}{3}}{x^2}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^3 \mathrm{e}^{-\sqrt{x}}}{3} dx$$

Hence

$$u_1 = \frac{2\,\mathrm{e}^{-\sqrt{x}} x^{\frac{7}{2}}}{3} + \frac{14x^3 \mathrm{e}^{-\sqrt{x}}}{3} + 28x^{\frac{5}{2}} \mathrm{e}^{-\sqrt{x}} + 140x^2 \mathrm{e}^{-\sqrt{x}} \\ + 560x^{\frac{3}{2}} \mathrm{e}^{-\sqrt{x}} + 1680x\,\mathrm{e}^{-\sqrt{x}} + 3360\sqrt{x}\,\mathrm{e}^{-\sqrt{x}} + 3360\,\mathrm{e}^{-\sqrt{x}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2 \mathrm{e}^{-\sqrt{x}}}{x^2}\, dx$$

Which simplifies to

$$u_2 = \int \mathrm{e}^{-\sqrt{x}} dx$$

Hence

$$u_2 = -2\sqrt{x}\,\mathrm{e}^{-\sqrt{x}} - 2\,\mathrm{e}^{-\sqrt{x}}$$

Which simplifies to

$$u_1 = \frac{2\,\mathrm{e}^{-\sqrt{x}}\left(x^{\frac{7}{2}}+42x^{\frac{5}{2}}+840x^{\frac{3}{2}}+7x^3+210x^2+5040\sqrt{x}+2520x+5040\right)}{3}$$
$$u_2 = -2\,\mathrm{e}^{-\sqrt{x}}\left(\sqrt{x}+1\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2\,\mathrm{e}^{-\sqrt{x}}\left(x^{\frac{7}{2}}+42x^{\frac{5}{2}}+840x^{\frac{3}{2}}+7x^3+210x^2+5040\sqrt{x}+2520x+5040\right)}{3x} - \frac{2\,\mathrm{e}^{-\sqrt{x}}\left(\sqrt{x}+1\right)x^2}{3}$$

Which simplifies to

$$y_p(x) = \frac{4\,\mathrm{e}^{-\sqrt{x}}\left(7x^{\frac{5}{2}}+140x^{\frac{3}{2}}+x^3+35x^2+840\sqrt{x}+420x+840\right)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x}+\frac{c_2x^2}{3}\right) + \left(\frac{4\,\mathrm{e}^{-\sqrt{x}}\left(7x^{\frac{5}{2}}+140x^{\frac{3}{2}}+x^3+35x^2+840\sqrt{x}+420x+840\right)}{x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x}+\frac{c_2x^2}{3}+\frac{4\,\mathrm{e}^{-\sqrt{x}}\left(7x^{\frac{5}{2}}+140x^{\frac{3}{2}}+x^3+35x^2+840\sqrt{x}+420x+840\right)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x}+\frac{c_2x^2}{3}+\frac{4\,\mathrm{e}^{-\sqrt{x}}\left(7x^{\frac{5}{2}}+140x^{\frac{3}{2}}+x^3+35x^2+840\sqrt{x}+420x+840\right)}{x}$$

Verified OK.

2.14.5 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 \\ q(x) &= 0 \\ r(x) &= -2 \\ s(x) &= x^3 \mathrm{e}^{-\sqrt{x}} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$2 - (0) + (-2) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$x^2 y' - 2yx = \int x^3 \mathrm{e}^{-\sqrt{x}}\, dx$$

We now have a first order ode to solve which is

$$x^2 y' - 2yx = -2\,\mathrm{e}^{-\sqrt{x}} x^{\frac{7}{2}} - 14x^3 \mathrm{e}^{-\sqrt{x}} - 84x^{\frac{5}{2}} \mathrm{e}^{-\sqrt{x}} - 420x^2 \mathrm{e}^{-\sqrt{x}} - 1680x^{\frac{3}{2}} \mathrm{e}^{-\sqrt{x}} - 5040x\,\mathrm{e}^{-\sqrt{x}} - 10080\sqrt{x}$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$

$$q(x) = \frac{\left(-14x^{3}-420x^{2}-5040x-10080\sqrt{x}-1680x^{\frac{3}{2}}-84x^{\frac{5}{2}}-2x^{\frac{7}{2}}-10080\right) \mathrm{e}^{-\sqrt{x}}+c_{1}}{x^{2}}$$

Hence the ode is

$$y'-\frac{2y}{x} = \frac{\left(-14x^{3}-420x^{2}-5040x-10080\sqrt{x}-1680x^{\frac{3}{2}}-84x^{\frac{5}{2}}-2x^{\frac{7}{2}}-10080\right) \mathrm{e}^{-\sqrt{x}}+c_{1}}{x^{2}}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{2}{x}dx}\\ &= \frac{1}{x^{2}}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{\left(-14x^{3}-420x^{2}-5040x-10080\sqrt{x}-1680x^{\frac{3}{2}}-84x^{\frac{5}{2}}-2x^{\frac{7}{2}}-10080\right) \mathrm{e}^{-\sqrt{x}}+c_{1}}{x^{2}}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^{2}}\right) = \left(\frac{1}{x^{2}}\right)\left(\frac{\left(-14x^{3}-420x^{2}-5040x-10080\sqrt{x}-1680x^{\frac{3}{2}}-84x^{\frac{5}{2}}-2x^{\frac{7}{2}}-10080\right) \mathrm{e}^{-\sqrt{x}}+c_{1}}{x^{2}}\right)$$

$$\mathrm{d}\left(\frac{y}{x^{2}}\right) = \left(\frac{\left(-14x^{3}-420x^{2}-5040x-10080\sqrt{x}-1680x^{\frac{3}{2}}-84x^{\frac{5}{2}}-2x^{\frac{7}{2}}-10080\right) \mathrm{e}^{-\sqrt{x}}+c_{1}}{x^{4}}\right) \mathrm{d}x$$

Integrating gives

$$\frac{y}{x^{2}} = \int \frac{\left(-14x^{3}-420x^{2}-5040x-10080\sqrt{x}-1680x^{\frac{3}{2}}-84x^{\frac{5}{2}}-2x^{\frac{7}{2}}-10080\right) \mathrm{e}^{-\sqrt{x}}+c_{1}}{x^{4}} \mathrm{d}x$$

$$\frac{y}{x^{2}} = -\frac{c_{1}}{3x^{3}}+\frac{3360\,\mathrm{e}^{-\sqrt{x}}}{x^{3}}+\frac{3360\,\mathrm{e}^{-\sqrt{x}}}{x^{\frac{5}{2}}}+\frac{1680\,\mathrm{e}^{-\sqrt{x}}}{x^{2}}+\frac{560\,\mathrm{e}^{-\sqrt{x}}}{x^{\frac{3}{2}}}+\frac{140\,\mathrm{e}^{-\sqrt{x}}}{x}+\frac{28\,\mathrm{e}^{-\sqrt{x}}}{\sqrt{x}}+4\,\mathrm{e}^{-\sqrt{x}}+c_{2}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^{2}}$ results in

$$y = x^{2}\left(-\frac{c_{1}}{3x^{3}}+\frac{3360\,\mathrm{e}^{-\sqrt{x}}}{x^{3}}+\frac{3360\,\mathrm{e}^{-\sqrt{x}}}{x^{\frac{5}{2}}}+\frac{1680\,\mathrm{e}^{-\sqrt{x}}}{x^{2}}+\frac{560\,\mathrm{e}^{-\sqrt{x}}}{x^{\frac{3}{2}}}+\frac{140\,\mathrm{e}^{-\sqrt{x}}}{x}+\frac{28\,\mathrm{e}^{-\sqrt{x}}}{\sqrt{x}}+4\,\mathrm{e}^{-\sqrt{x}}\right)+c_{2}$$

which simplifies to

$$y = \frac{28\left(x^{3}+20x^{2}+120x+120\sqrt{x}+60x^{\frac{3}{2}}+5x^{\frac{5}{2}}+\frac{x^{\frac{7}{2}}}{7}\right) \mathrm{e}^{-\sqrt{x}}-\frac{c_{1}\sqrt{x}}{3}+c_{2}x^{\frac{7}{2}}}{x^{\frac{3}{2}}}$$

Summary
The solution(s) found are the following

$$y = \frac{28\left(x^3+20x^2+120x+120\sqrt{x}+60x^{\frac{3}{2}}+5x^{\frac{5}{2}}+\frac{x^{\frac{7}{2}}}{7}\right)e^{-\sqrt{x}}-\frac{c_1\sqrt{x}}{3}+c_2x^{\frac{7}{2}}}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{28\left(x^3+20x^2+120x+120\sqrt{x}+60x^{\frac{3}{2}}+5x^{\frac{5}{2}}+\frac{x^{\frac{7}{2}}}{7}\right)e^{-\sqrt{x}}-\frac{c_1\sqrt{x}}{3}+c_2x^{\frac{7}{2}}}{x^{\frac{3}{2}}}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 51

```
dsolve(diff(diff(y(x),x),x)-2/x^2*y(x) = x*exp(-x^(1/2)),y(x), singsol=all)
```

$$y(x) = \frac{4\,e^{-\sqrt{x}}\left(7x^{\frac{5}{2}}+140x^{\frac{3}{2}}+x^3+35x^2+840\sqrt{x}+420x+840\right)+c_1x^3+c_2}{x}$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 54

```
DSolve[y''[x]-2/x^2*y[x] == x*Exp[-x^(1/2)],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-2e^{-\sqrt{x}}\left(\sqrt{x}+1\right)x^3 + 3(c_2x^3 + c_1) + 2\Gamma\left(8, \sqrt{x}\right)}{3x}$$

2.15 problem 16

2.15.1 Solving using Kovacic algorithm 742

Internal problem ID [7456]
Internal file name [OUTPUT/6423_Sunday_June_05_2022_04_51_54_PM_60518487/index.tex]

Book: Second order enumerated odes
Section: section 2
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' - \frac{y'}{\sqrt{x}} + \frac{\left(x + \sqrt{x} - 8\right) y}{4x^2} = x$$

2.15.1 Solving using Kovacic algorithm

Writing the ode as

$$y'' - \frac{y'}{\sqrt{x}} + \left(\frac{1}{4x} + \frac{1}{4x^{\frac{3}{2}}} - \frac{2}{x^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{1}{\sqrt{x}} \\ C &= \frac{1}{4x} + \frac{1}{4x^{\frac{3}{2}}} - \frac{2}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 78: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)(0) \\ &= -\frac{1}{x} \\ &= -\frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-\frac{1}{\sqrt{x}}}{1} \, dx} \\
&= z_1 e^{\sqrt{x}} \\
&= z_1 \left(\mathrm{e}^{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\sqrt{x}}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{1}{\sqrt{x}}}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{2\sqrt{x}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x^3}{3}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^{\sqrt{x}}}{x}\right) + c_2\left(\frac{e^{\sqrt{x}}}{x}\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - \frac{y'}{\sqrt{x}} + \left(\frac{1}{4x} + \frac{1}{4x^{\frac{3}{2}}} - \frac{2}{x^2}\right) y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1 e^{\sqrt{x}}}{x} + \frac{c_2 x^2 e^{\sqrt{x}}}{3}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{e^{\sqrt{x}}}{x} \\
y_2 &= \frac{x^2 e^{\sqrt{x}}}{3}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{e^{\sqrt{x}}}{x} & \frac{x^2 e^{\sqrt{x}}}{3} \\ \frac{d}{dx}\left(\frac{e^{\sqrt{x}}}{x}\right) & \frac{d}{dx}\left(\frac{x^2 e^{\sqrt{x}}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{e^{\sqrt{x}}}{x} & \frac{x^2 e^{\sqrt{x}}}{3} \\ -\frac{e^{\sqrt{x}}}{x^2} + \frac{e^{\sqrt{x}}}{2x^{\frac{3}{2}}} & \frac{2x\, e^{\sqrt{x}}}{3} + \frac{x^{\frac{3}{2}} e^{\sqrt{x}}}{6} \end{vmatrix}$$

Therefore

$$W = \left(\frac{e^{\sqrt{x}}}{x}\right)\left(\frac{2x\, e^{\sqrt{x}}}{3} + \frac{x^{\frac{3}{2}} e^{\sqrt{x}}}{6}\right) - \left(\frac{x^2 e^{\sqrt{x}}}{3}\right)\left(-\frac{e^{\sqrt{x}}}{x^2} + \frac{e^{\sqrt{x}}}{2x^{\frac{3}{2}}}\right)$$

Which simplifies to

$$W = e^{2\sqrt{x}}$$

Which simplifies to

$$W = e^{2\sqrt{x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^3 e^{\sqrt{x}}}{3}}{e^{2\sqrt{x}}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^3 e^{-\sqrt{x}}}{3} dx$$

Hence

$$
\begin{aligned}
u_1 = {} & \frac{2\,\mathrm{e}^{-\sqrt{x}} x^{\frac{7}{2}}}{3} + \frac{14x^3 \mathrm{e}^{-\sqrt{x}}}{3} + 28x^{\frac{5}{2}} \mathrm{e}^{-\sqrt{x}} + 140x^2 \mathrm{e}^{-\sqrt{x}} \\
& + 560x^{\frac{3}{2}} \mathrm{e}^{-\sqrt{x}} + 1680x\,\mathrm{e}^{-\sqrt{x}} + 3360\sqrt{x}\,\mathrm{e}^{-\sqrt{x}} + 3360\,\mathrm{e}^{-\sqrt{x}}
\end{aligned}
$$

And Eq. (3) becomes

$$
u_2 = \int \frac{\mathrm{e}^{\sqrt{x}}}{\mathrm{e}^{2\sqrt{x}}}\, dx
$$

Which simplifies to

$$
u_2 = \int \mathrm{e}^{-\sqrt{x}} dx
$$

Hence

$$
u_2 = -2\sqrt{x}\,\mathrm{e}^{-\sqrt{x}} - 2\,\mathrm{e}^{-\sqrt{x}}
$$

Which simplifies to

$$
\begin{aligned}
u_1 &= \frac{2\,\mathrm{e}^{-\sqrt{x}} \left(x^{\frac{7}{2}} + 42x^{\frac{5}{2}} + 840x^{\frac{3}{2}} + 7x^3 + 210x^2 + 5040\sqrt{x} + 2520x + 5040\right)}{3} \\
u_2 &= -2\,\mathrm{e}^{-\sqrt{x}} \left(\sqrt{x} + 1\right)
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = {} & \frac{2\,\mathrm{e}^{-\sqrt{x}} \left(x^{\frac{7}{2}} + 42x^{\frac{5}{2}} + 840x^{\frac{3}{2}} + 7x^3 + 210x^2 + 5040\sqrt{x} + 2520x + 5040\right) \mathrm{e}^{\sqrt{x}}}{3x} \\
& - \frac{2\,\mathrm{e}^{-\sqrt{x}} \left(\sqrt{x} + 1\right) x^2 \mathrm{e}^{\sqrt{x}}}{3}
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = \frac{28x^{\frac{5}{2}} + 560x^{\frac{3}{2}} + 4x^3 + 140x^2 + 3360\sqrt{x} + 1680x + 3360}{x}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1 \mathrm{e}^{\sqrt{x}}}{x} + \frac{c_2 x^2 \mathrm{e}^{\sqrt{x}}}{3}\right) + \left(\frac{28x^{\frac{5}{2}} + 560x^{\frac{3}{2}} + 4x^3 + 140x^2 + 3360\sqrt{x} + 1680x + 3360}{x}\right)
\end{aligned}
$$

Which simplifies to

$$y = \frac{e^{\sqrt{x}}(c_2x^3+3c_1)}{3x} + \frac{28x^{\frac{5}{2}}+560x^{\frac{3}{2}}+4x^3+140x^2+3360\sqrt{x}+1680x+3360}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{\sqrt{x}}(c_2x^3+3c_1)}{3x} + \frac{28x^{\frac{5}{2}}+560x^{\frac{3}{2}}+4x^3+140x^2+3360\sqrt{x}+1680x+3360}{x} \quad (1)$$

Verification of solutions

$$y = \frac{e^{\sqrt{x}}(c_2x^3+3c_1)}{3x} + \frac{28x^{\frac{5}{2}}+560x^{\frac{3}{2}}+4x^3+140x^2+3360\sqrt{x}+1680x+3360}{x}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$2)-1/sqrt(x)*diff(y(x),x)+1/(4*x^2)*(x+sqrt(x)-8)*y(x)=x,y(x), singsol=al
```

$$y(x) = \frac{560x^{\frac{3}{2}}+28x^{\frac{5}{2}}+(c_1x^3+c_2)e^{\sqrt{x}}+4x^3+140x^2+1680x+3360\sqrt{x}+3360}{x}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 63

```
DSolve[y''[x]-1/Sqrt[x]*y'[x]+1/(4*x^2)*(x+Sqrt[x]-8)*y[x]==x,y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{-2x^{7/2} + x^3\left(-2 + c_2 e^{\sqrt{x}}\right) + 2e^{\sqrt{x}}\Gamma\left(8, \sqrt{x}\right) + 3c_1 e^{\sqrt{x}}}{3x}$$

2.16 problem 17

2.16.1 Solving as second order change of variable on x method 2 ode . 752
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Internal problem ID [7457]
Internal file name [`OUTPUT/6424_Sunday_June_19_2022_05_02_01_PM_9550685/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{y'' + \frac{2y'}{x} + \frac{a^2 y}{x^4} = 0}$$

2.16.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^4 y'' + 2y' x^3 + a^2 y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{a^2}{x^4}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{2}{x}dx\right)}dx \\
&= \int e^{-2\ln(x)}\,dx \\
&= \int \frac{1}{x^2}dx \\
&= -\frac{1}{x}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{a^2}{x^4}}{\frac{1}{x^4}} \\
&= a^2
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + a^2 y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = a^2$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + a^2 e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$a^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = a^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(a^2)}\\ &= \pm\sqrt{-a^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-a^2}\\ \lambda_2 &= -\sqrt{-a^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-a^2}\\ \lambda_2 &= -\sqrt{-a^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1\tau} + c_2 e^{\lambda_2\tau}\\ y(\tau) &= c_1 e^{\left(\sqrt{-a^2}\right)\tau} + c_2 e^{\left(-\sqrt{-a^2}\right)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{\sqrt{-a^2}\,\tau} + c_2 e^{-\sqrt{-a^2}\,\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \mathrm{e}^{-\frac{\sqrt{-a^2}}{x}} + c_2 \mathrm{e}^{\frac{\sqrt{-a^2}}{x}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{\sqrt{-a^2}}{x}} + c_2 \mathrm{e}^{\frac{\sqrt{-a^2}}{x}} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{\sqrt{-a^2}}{x}} + c_2 \mathrm{e}^{\frac{\sqrt{-a^2}}{x}}$$

Verified OK.

2.16.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^4 y'' + 2y' x^3 + a^2 y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= \frac{a^2}{x^4} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{\frac{a^2}{x^4}}}{c} \\
\tau'' &= -\frac{2a^2}{c\sqrt{\frac{a^2}{x^4}}\, x^5}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2a^2}{c\sqrt{\frac{a^2}{x^4}}\, x^5} + \frac{2}{x}\frac{\sqrt{\frac{a^2}{x^4}}}{c}}{\left(\frac{\sqrt{\frac{a^2}{x^4}}}{c}\right)^2} \\
&= 0
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{\frac{a^2}{x^4}}\, dx}{c} \\
&= -\frac{x\sqrt{\frac{a^2}{x^4}}}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(\frac{a}{x}\right) - c_2 \sin\left(\frac{a}{x}\right)$$

Summary
The solution(s) found are the following

$$y = c_1 \cos\left(\frac{a}{x}\right) - c_2 \sin\left(\frac{a}{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(\frac{a}{x}\right) - c_2 \sin\left(\frac{a}{x}\right)$$

Verified OK.

2.16.3 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + 2xy' + \frac{a^2 y}{x^2} = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) xy' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= -\frac{1}{2} \\ \beta &= a \\ n &= \frac{1}{2} \\ \gamma &= -1 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \sqrt{2}\, \sin\left(\frac{a}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{a}{x}}} - \frac{c_2 \sqrt{2}\, \cos\left(\frac{a}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{a}{x}}}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1\sqrt{2}\,\sin\left(\frac{a}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{a}{x}}} - \frac{c_2\sqrt{2}\,\cos\left(\frac{a}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{a}{x}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{2}\,\sin\left(\frac{a}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{a}{x}}} - \frac{c_2\sqrt{2}\,\cos\left(\frac{a}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{a}{x}}}$$

Verified OK.

2.16.4 Solving using Kovacic algorithm

Writing the ode as

$$x^4y'' + 2y'x^3 + a^2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= 2x^3 \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2}{x^4} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -a^2 \\
t &= x^4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{a^2}{x^4}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 79: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\alpha_c^{+} = \frac{1}{2}\left(\frac{b}{a} + v\right)$$
$$\alpha_c^{-} = \frac{1}{2}\left(-\frac{b}{a} + v\right)$$

The partial fraction decomposition of r is

$$r = -\frac{a^2}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{ia}{x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{ia}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = ia$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum

$[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{ia}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{0}{ia} + 2\right) &= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{0}{ia} + 2\right) &= 1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{a^2}{x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{ia}{x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{ia}{x^2} + \frac{1}{x} + (-)\,(0) \\
&= -\frac{ia}{x^2} + \frac{1}{x} \\
&= \frac{-ia + x}{x^2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{ia}{x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{2ia}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{ia}{x^2} + \frac{1}{x}\right)^2 - \left(-\frac{a^2}{x^4}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{ia}{x^2} + \frac{1}{x}\right) dx} \\
&= x \, \mathrm{e}^{\frac{ia}{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3}{x^4}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{ia}{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3}{x^4}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{i\mathrm{e}^{-\frac{2ia}{x}}}{2a}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{ia}{x}}\right) + c_2\left(\mathrm{e}^{\frac{ia}{x}}\left(-\frac{i\mathrm{e}^{-\frac{2ia}{x}}}{2a}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{\frac{ia}{x}} - \frac{ic_2\mathrm{e}^{-\frac{ia}{x}}}{2a} \tag{1}
$$

Verification of solutions

$$y = c_1 e^{\frac{ia}{x}} - \frac{i c_2 e^{-\frac{ia}{x}}}{2a}$$

Verified OK.

2.16.5 Maple step by step solution

Let's solve

$x^4 y'' + 2y' x^3 + a^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - \frac{a^2 y}{x^4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + \frac{a^2 y}{x^4} = 0$

- Multiply by denominators of the ODE

  $x^4 y'' + 2y' x^3 + a^2 y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$x^4\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)+2\left(\frac{d}{dt}y(t)\right)x^2+a^2y(t)=0$$

- Simplify

$$x^2\left(\frac{d^2}{dt^2}y(t)\right)+\left(\frac{d}{dt}y(t)\right)x^2+a^2y(t)=0$$

- Isolate 2nd derivative

$$\frac{d^2}{dt^2}y(t)=-\frac{a^2y(t)}{x^2}-\frac{d}{dt}y(t)$$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^2}{dt^2}y(t)+\frac{d}{dt}y(t)+\frac{a^2y(t)}{x^2}=0$$

- Characteristic polynomial of ODE

$$r^2+r+\frac{a^2}{x^2}=0$$

- Factor the characteristic polynomial

$$\frac{r^2x^2+r\,x^2+a^2}{x^2}=0$$

- Roots of the characteristic polynomial

$$r=\left(\frac{-\frac{x}{2}+\frac{\sqrt{-4a^2+x^2}}{2}}{x},\frac{-\frac{x}{2}-\frac{\sqrt{-4a^2+x^2}}{2}}{x}\right)$$

- 1st solution of the ODE

$$y_1(t)=\mathrm{e}^{\frac{\left(-\frac{x}{2}+\frac{\sqrt{-4a^2+x^2}}{2}\right)t}{x}}$$

- 2nd solution of the ODE

$$y_2(t)=\mathrm{e}^{\frac{\left(-\frac{x}{2}-\frac{\sqrt{-4a^2+x^2}}{2}\right)t}{x}}$$

- General solution of the ODE

$$y(t)=c_1y_1(t)+c_2y_2(t)$$

- Substitute in solutions

$$y(t)=c_1\mathrm{e}^{\frac{\left(-\frac{x}{2}+\frac{\sqrt{-4a^2+x^2}}{2}\right)t}{x}}+c_2\mathrm{e}^{\frac{\left(-\frac{x}{2}-\frac{\sqrt{-4a^2+x^2}}{2}\right)t}{x}}$$

- Change variables back using $t=\ln(x)$

$$y=c_1\mathrm{e}^{\frac{\left(-\frac{x}{2}+\frac{\sqrt{-4a^2+x^2}}{2}\right)\ln(x)}{x}}+c_2\mathrm{e}^{\frac{\left(-\frac{x}{2}-\frac{\sqrt{-4a^2+x^2}}{2}\right)\ln(x)}{x}}$$

- Simplify

$$y=c_1x^{\frac{-x+\sqrt{-4a^2+x^2}}{2x}}+c_2x^{-\frac{x+\sqrt{-4a^2+x^2}}{2x}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+2/x*diff(y(x),x)+a^2/x^4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin\left(\frac{a}{x}\right) + c_2 \cos\left(\frac{a}{x}\right)$$

✓ Solution by Mathematica
Time used: 0.022 (sec). Leaf size: 25

```
DSolve[y''[x]+2/x*y'[x]+a^2/x^4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos\left(\frac{a}{x}\right) - c_2 \sin\left(\frac{a}{x}\right)$$

2.17 problem 18

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Internal problem ID [7458]
Internal file name [`OUTPUT/6425_Sunday_June_19_2022_05_02_02_PM_44998391/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[_Gegenbauer, [_2nd_order, _linear, `_with_symmetry_[0,F(x)]`]]
```

$$\boxed{(1-x^2)\,y''-xy'-c^2y=0}$$

2.17.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$(1-x^2)\,y''-xy'-c^2y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=\frac{x}{x^2-1}$$
$$q(x)=\frac{c^2}{x^2-1}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{x}{x^2-1}dx\right)} dx \\
&= \int e^{-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2}} dx \\
&= \int \frac{1}{\sqrt{x-1}\,\sqrt{1+x}} dx \\
&= \frac{\sqrt{(x-1)(1+x)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{1+x}}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{c^2}{x^2-1}}{\frac{1}{(x-1)(1+x)}} \\
&= c^2
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = c^2$. Let the solution be $y(\tau) = e^{\lambda \tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda \tau} + c^2 e^{\lambda \tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda \tau}$ gives

$$c^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = c^2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(c^2)} \\
&= \pm \sqrt{-c^2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +\sqrt{-c^2} \\
\lambda_2 &= -\sqrt{-c^2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= \sqrt{-c^2} \\
\lambda_2 &= -\sqrt{-c^2}
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\
y(\tau) &= c_1 e^{\left(\sqrt{-c^2}\right)\tau} + c_2 e^{\left(-\sqrt{-c^2}\right)\tau}
\end{aligned}$$

Or

$$y(\tau) = c_1 e^{\sqrt{-c^2}\,\tau} + c_2 e^{-\sqrt{-c^2}\,\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{-c^2}\,\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{-c^2}\,\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{-c^2}\,\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{-c^2}\,\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} \tag{1}$$

Verification of solutions

$$y = c_1 \left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{-c^2}\,\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{-c^2}\,\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}}$$

Verified OK.

2.17.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$\left(1-x^2\right) y'' - xy' - c^2 y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x}{x^2-1} \\ q(x) &= \frac{c^2}{x^2-1} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{c^2}{x^2-1}}}{c} \\ \tau'' &= -\frac{c^2 x}{c\sqrt{\frac{c^2}{x^2-1}}\,(x^2-1)^2} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{c^2 x}{c\sqrt{\frac{c^2}{x^2-1}}\,(x^2-1)^2} + \frac{x}{x^2-1}\frac{\sqrt{\frac{c^2}{x^2-1}}}{c}}{\left(\frac{\sqrt{\frac{c^2}{x^2-1}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\, dx \\ &= \frac{\int \sqrt{\frac{c^2}{x^2-1}}dx}{c} \\ &= \frac{\sqrt{\frac{c^2}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{c} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(c\sqrt{\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) + c_2 \sin\left(c\sqrt{\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(c\sqrt{\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) + c_2 \sin\left(c\sqrt{\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(c\sqrt{\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) + c_2 \sin\left(c\sqrt{\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)$$

Verified OK.

2.17.3 Solving using Kovacic algorithm

Writing the ode as

$$\left(1-x^2\right)y'' - xy' - c^2 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x^2 \\ B &= -x \\ C &= -c^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4c^2x^2 + 4c^2 - x^2 - 2}{4\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4c^2x^2 + 4c^2 - x^2 - 2 \\ t &= 4\left(x^2 - 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4c^2x^2 + 4c^2 - x^2 - 2}{4\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 81: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(x-1)^2} + \frac{\frac{1}{16} - \frac{c^2}{2}}{x-1} - \frac{3}{16(1+x)^2} + \frac{-\frac{1}{16} + \frac{c^2}{2}}{1+x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4c^2x^2 + 4c^2 - x^2 - 2}{4(x^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-1$. Hence

$$\begin{aligned}E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\ &= \{2\}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1,2,3\}$
-1	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\ &= \frac{1}{2}(2-(1+(1)))\\ &= 0\end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{1}{(x-(1))}+\frac{1}{(x-(-1))}\right)\\ &= \frac{1}{2x-2}+\frac{1}{2x+2}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x-2} + \frac{1}{2x+2}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)w + \frac{4c^2x^2 - 4c^2 + x^2}{4\left(x^2-1\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{x + 2c\sqrt{1-x^2}}{2\left(x-1\right)\left(1+x\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{x+2c\sqrt{1-x^2}}{2(x-1)(1+x)}dx} \\ &= \left(x^2-1\right)^{\frac{1}{4}} \mathrm{e}^{-c \arcsin(x)}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1-x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x-1)}{4} - \frac{\ln(1+x)}{4}} \\ &= z_1 \left(\frac{1}{(x-1)^{\frac{1}{4}}\left(1+x\right)^{\frac{1}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2-1)^{\frac{1}{4}} \,\mathrm{e}^{-c \arcsin(x)}}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1-x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{2c \arcsin(x)}}{\sqrt{x^2-1}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2-1)^{\frac{1}{4}} \,\mathrm{e}^{-c \arcsin(x)}}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}\right) + c_2 \left(\frac{(x^2-1)^{\frac{1}{4}} \,\mathrm{e}^{-c \arcsin(x)}}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}} \left(\int \frac{\mathrm{e}^{2c \arcsin(x)}}{\sqrt{x^2-1}} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 (x^2-1)^{\frac{1}{4}} \,\mathrm{e}^{-c \arcsin(x)}}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}} + \frac{c_2 (x^2-1)^{\frac{1}{4}} \,\mathrm{e}^{-c \arcsin(x)} \left(\int \frac{\mathrm{e}^{2c \arcsin(x)}}{\sqrt{x^2-1}} dx\right)}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 (x^2-1)^{\frac{1}{4}} \,\mathrm{e}^{-c \arcsin(x)}}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}} + \frac{c_2 (x^2-1)^{\frac{1}{4}} \,\mathrm{e}^{-c \arcsin(x)} \left(\int \frac{\mathrm{e}^{2c \arcsin(x)}}{\sqrt{x^2-1}} dx\right)}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}$$

Verified OK.

2.17.4 Maple step by step solution

Let's solve

$\left(1-x^2\right) y''-x y'-c^2 y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{x y'}{x^2-1}-\frac{c^2 y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{x y'}{x^2-1}+\frac{c^2 y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x}{x^2-1}, P_3(x)=\frac{c^2}{x^2-1}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=\frac{1}{2}$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$y''(x^2-1)+x y'+c^2 y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)+c^2 y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(-1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(c^2+k^2+2kr+r^2)\right) u^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + a_k(c^2+k^2+2kr+r^2) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k\,(c^2+k^2+2kr+r^2)}{(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k\,(c^2+k^2)}{(2k+1)(k+1)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k\,(c^2+k^2)}{(2k+1)(k+1)}\right]$$

- Revert the change of variables $u = 1+x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{a_k\,(c^2+k^2)}{(2k+1)(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = \frac{a_k\left(c^2+k^2+k+\frac{1}{4}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k\left(c^2+k^2+k+\frac{1}{4}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k\left(c^2+k^2+k+\frac{1}{4}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+\frac{1}{2}}\right), a_{k+1} = \frac{a_k\left(c^2+k^2\right)}{(2k+1)(k+1)}, b_{k+1} = \frac{b_k\left(c^2+k^2+k+\frac{1}{4}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve((1-x^2)*diff(y(x),x$2)-x*diff(y(x),x)-c^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(x+\sqrt{x^2-1}\right)^{ic} + c_2\left(x+\sqrt{x^2-1}\right)^{-ic}$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 89

```
DSolve[(1-x^2)*y''[x]-x*y'[x]-c^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos\left(\frac{1}{2}c\left(\log\left(1-\frac{x}{\sqrt{x^2-1}}\right)-\log\left(\frac{x}{\sqrt{x^2-1}}+1\right)\right)\right) - c_2 \sin\left(\frac{1}{2}c\left(\log\left(1-\frac{x}{\sqrt{x^2-1}}\right)-\log\left(\frac{x}{\sqrt{x^2-1}}+1\right)\right)\right)$$

2.18 problem 19

2.18.1 Solving as second order change of variable on x method 2 ode . 782
2.18.2 Solving as second order change of variable on x method 1 ode . 788
2.18.3 Solving as second order bessel ode ode 793

Internal problem ID [7459]
Internal file name [`OUTPUT/6426_Sunday_June_19_2022_05_02_04_PM_23103942/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x^6y'' + 3y'x^5 + a^2y = \frac{1}{x^2}}$$

2.18.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^6y'' + 3y'x^5 + a^2y = 0$$

In normal form the ode

$$x^6y'' + 3y'x^5 + a^2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{a^2}{x^6} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\left(\int p(x) dx\right)} dx \\ &= \int e^{-\left(\int \frac{3}{x} dx\right)} dx \\ &= \int e^{-3 \ln(x)}\, dx \\ &= \int \frac{1}{x^3} dx \\ &= -\frac{1}{2x^2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{a^2}{x^6}}{\frac{1}{x^6}} \\ &= a^2 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2} y(\tau) + a^2 y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = a^2$. Let the solution be $y(\tau) = e^{\lambda \tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda \tau} + a^2 e^{\lambda \tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda \tau}$ gives

$$a^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = a^2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(a^2)} \\
&= \pm \sqrt{-a^2}
\end{aligned}$$

Hence

$$\lambda_1 = +\sqrt{-a^2}$$
$$\lambda_2 = -\sqrt{-a^2}$$

Which simplifies to

$$\lambda_1 = \sqrt{-a^2}$$
$$\lambda_2 = -\sqrt{-a^2}$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{\left(\sqrt{-a^2}\right)\tau} + c_2 e^{\left(-\sqrt{-a^2}\right)\tau}$$

Or

$$
y(\tau) = c_1 e^{\sqrt{-a^2}\,\tau} + c_2 e^{-\sqrt{-a^2}\,\tau}
$$

The above solution is now transformed back to y using (6) which results in

$$
y = c_1 e^{-\frac{\sqrt{-a^2}}{2x^2}} + c_2 e^{\frac{\sqrt{-a^2}}{2x^2}}
$$

Therefore the homogeneous solution y_h is

$$
y_h = c_1 e^{-\frac{\sqrt{-a^2}}{2x^2}} + c_2 e^{\frac{\sqrt{-a^2}}{2x^2}}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= e^{-\frac{\sqrt{-a^2}}{2x^2}} \\
y_2 &= e^{\frac{\sqrt{-a^2}}{2x^2}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} e^{-\frac{\sqrt{-a^2}}{2x^2}} & e^{\frac{\sqrt{-a^2}}{2x^2}} \\ \frac{d}{dx}\left(e^{-\frac{\sqrt{-a^2}}{2x^2}}\right) & \frac{d}{dx}\left(e^{\frac{\sqrt{-a^2}}{2x^2}}\right) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} e^{-\frac{\sqrt{-a^2}}{2x^2}} & e^{\frac{\sqrt{-a^2}}{2x^2}} \\ \frac{\sqrt{-a^2}\, e^{-\frac{\sqrt{-a^2}}{2x^2}}}{x^3} & -\frac{\sqrt{-a^2}\, e^{\frac{\sqrt{-a^2}}{2x^2}}}{x^3} \end{vmatrix}$$

Therefore

$$W = \left(e^{-\frac{\sqrt{-a^2}}{2x^2}}\right)\left(-\frac{\sqrt{-a^2}\, e^{\frac{\sqrt{-a^2}}{2x^2}}}{x^3}\right) - \left(e^{\frac{\sqrt{-a^2}}{2x^2}}\right)\left(\frac{\sqrt{-a^2}\, e^{-\frac{\sqrt{-a^2}}{2x^2}}}{x^3}\right)$$

Which simplifies to

$$W = -\frac{2\, e^{-\frac{\sqrt{-a^2}}{2x^2}} \sqrt{-a^2}\, e^{\frac{\sqrt{-a^2}}{2x^2}}}{x^3}$$

Which simplifies to

$$W = -\frac{2\sqrt{-a^2}}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{\frac{\sqrt{-a^2}}{2x^2}}}{x^2}}{-2x^3\sqrt{-a^2}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{\frac{\sqrt{-a^2}}{2x^2}}}{2x^5\sqrt{-a^2}} dx$$

Hence

$$u_1 = \frac{-\frac{e^{\frac{\sqrt{-a^2}}{2x^2}}}{x^2\sqrt{-a^2}} - \frac{2\, e^{\frac{\sqrt{-a^2}}{2x^2}}}{a^2}}{2\sqrt{-a^2}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-\frac{\sqrt{-a^2}}{2x^2}}}{x^2}}{-2x^3\sqrt{-a^2}}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{-\frac{\sqrt{-a^2}}{2x^2}}}{2x^5\sqrt{-a^2}}dx$$

Hence

$$u_2 = -\frac{\frac{e^{-\frac{\sqrt{-a^2}}{2x^2}}}{x^2\sqrt{-a^2}} - \frac{2\,e^{-\frac{\sqrt{-a^2}}{2x^2}}}{a^2}}{2\sqrt{-a^2}}$$

Which simplifies to

$$u_1 = \frac{e^{\frac{\sqrt{-a^2}}{2x^2}}\left(2x^2\sqrt{-a^2}+a^2\right)}{2a^4x^2}$$

$$u_2 = \frac{e^{-\frac{\sqrt{-a^2}}{2x^2}}\left(-2x^2\sqrt{-a^2}+a^2\right)}{2a^4x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^{\frac{\sqrt{-a^2}}{2x^2}}\left(2x^2\sqrt{-a^2}+a^2\right)e^{-\frac{\sqrt{-a^2}}{2x^2}}}{2a^4x^2} + \frac{e^{-\frac{\sqrt{-a^2}}{2x^2}}\left(-2x^2\sqrt{-a^2}+a^2\right)e^{\frac{\sqrt{-a^2}}{2x^2}}}{2a^4x^2}$$

Which simplifies to

$$y_p(x) = \frac{1}{a^2x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-\frac{\sqrt{-a^2}}{2x^2}} + c_2 e^{\frac{\sqrt{-a^2}}{2x^2}}\right) + \left(\frac{1}{a^2x^2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-\frac{\sqrt{-a^2}}{2x^2}} + c_2 e^{\frac{\sqrt{-a^2}}{2x^2}} + \frac{1}{a^2x^2} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-\frac{\sqrt{-a^2}}{2x^2}} + c_2 e^{\frac{\sqrt{-a^2}}{2x^2}} + \frac{1}{a^2x^2}$$

Verified OK.

2.18.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^6, B = 3x^5, C = a^2, f(x) = \frac{1}{x^2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^6y'' + 3y'x^5 + a^2y = 0$$

In normal form the ode

$$x^6y'' + 3y'x^5 + a^2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{a^2}{x^6} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{a^2}{x^6}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{3a^2}{c\sqrt{\frac{a^2}{x^6}}\,x^7}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{3a^2}{c\sqrt{\frac{a^2}{x^6}}\,x^7} + \frac{3}{x}\frac{\sqrt{\frac{a^2}{x^6}}}{c}}{\left(\frac{\sqrt{\frac{a^2}{x^6}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{\frac{a^2}{x^6}}dx}{c} \\
&= -\frac{x\sqrt{\frac{a^2}{x^6}}}{2c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos\left(\frac{a}{2x^2}\right) - c_2 \sin\left(\frac{a}{2x^2}\right)
$$

Now the particular solution to this ODE is found

$$
x^6 y'' + 3y'x^5 + a^2 y = \frac{1}{x^2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-\frac{\sqrt{-a^2}}{2x^2}}$$
$$y_2 = e^{\frac{\sqrt{-a^2}}{2x^2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-\frac{\sqrt{-a^2}}{2x^2}} & e^{\frac{\sqrt{-a^2}}{2x^2}} \\ \frac{d}{dx}\left(e^{-\frac{\sqrt{-a^2}}{2x^2}}\right) & \frac{d}{dx}\left(e^{\frac{\sqrt{-a^2}}{2x^2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-\frac{\sqrt{-a^2}}{2x^2}} & e^{\frac{\sqrt{-a^2}}{2x^2}} \\ \frac{\sqrt{-a^2}\, e^{-\frac{\sqrt{-a^2}}{2x^2}}}{x^3} & -\frac{\sqrt{-a^2}\, e^{\frac{\sqrt{-a^2}}{2x^2}}}{x^3} \end{vmatrix}$$

Therefore

$$W = \left(e^{-\frac{\sqrt{-a^2}}{2x^2}}\right)\left(-\frac{\sqrt{-a^2}\, e^{\frac{\sqrt{-a^2}}{2x^2}}}{x^3}\right) - \left(e^{\frac{\sqrt{-a^2}}{2x^2}}\right)\left(\frac{\sqrt{-a^2}\, e^{-\frac{\sqrt{-a^2}}{2x^2}}}{x^3}\right)$$

Which simplifies to

$$W = -\frac{2\,\mathrm{e}^{-\frac{\sqrt{-a^2}}{2x^2}}\sqrt{-a^2}\,\mathrm{e}^{\frac{\sqrt{-a^2}}{2x^2}}}{x^3}$$

Which simplifies to

$$W = -\frac{2\sqrt{-a^2}}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{\frac{\sqrt{-a^2}}{2x^2}}}{x^2}}{-2x^3\sqrt{-a^2}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\mathrm{e}^{\frac{\sqrt{-a^2}}{2x^2}}}{2x^5\sqrt{-a^2}}dx$$

Hence

$$u_1 = \frac{-\frac{\mathrm{e}^{\frac{\sqrt{-a^2}}{2x^2}}}{x^2\sqrt{-a^2}} - \frac{2\,\mathrm{e}^{\frac{\sqrt{-a^2}}{2x^2}}}{a^2}}{2\sqrt{-a^2}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{-\frac{\sqrt{-a^2}}{2x^2}}}{x^2}}{-2x^3\sqrt{-a^2}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\mathrm{e}^{-\frac{\sqrt{-a^2}}{2x^2}}}{2x^5\sqrt{-a^2}}dx$$

Hence

$$u_2 = -\frac{\frac{\mathrm{e}^{-\frac{\sqrt{-a^2}}{2x^2}}}{x^2\sqrt{-a^2}} - \frac{2\,\mathrm{e}^{-\frac{\sqrt{-a^2}}{2x^2}}}{a^2}}{2\sqrt{-a^2}}$$

Which simplifies to

$$u_1 = \frac{e^{\frac{\sqrt{-a^2}}{2x^2}} \left(2x^2\sqrt{-a^2} + a^2\right)}{2a^4x^2}$$

$$u_2 = \frac{e^{-\frac{\sqrt{-a^2}}{2x^2}} \left(-2x^2\sqrt{-a^2} + a^2\right)}{2a^4x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^{\frac{\sqrt{-a^2}}{2x^2}} \left(2x^2\sqrt{-a^2} + a^2\right) e^{-\frac{\sqrt{-a^2}}{2x^2}}}{2a^4x^2} + \frac{e^{-\frac{\sqrt{-a^2}}{2x^2}} \left(-2x^2\sqrt{-a^2} + a^2\right) e^{\frac{\sqrt{-a^2}}{2x^2}}}{2a^4x^2}$$

Which simplifies to

$$y_p(x) = \frac{1}{a^2x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\frac{a}{2x^2}\right) - c_2 \sin\left(\frac{a}{2x^2}\right)\right) + \left(\frac{1}{a^2x^2}\right) \\ &= \frac{1}{a^2x^2} + c_1 \cos\left(\frac{a}{2x^2}\right) - c_2 \sin\left(\frac{a}{2x^2}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{1}{a^2x^2} + c_1 \cos\left(\frac{a}{2x^2}\right) - c_2 \sin\left(\frac{a}{2x^2}\right)$$

Summary

The solution(s) found are the following

$$y = \frac{1}{a^2x^2} + c_1 \cos\left(\frac{a}{2x^2}\right) - c_2 \sin\left(\frac{a}{2x^2}\right) \tag{1}$$

Verification of solutions

$$y = \frac{1}{a^2x^2} + c_1 \cos\left(\frac{a}{2x^2}\right) - c_2 \sin\left(\frac{a}{2x^2}\right)$$

Verified OK.

2.18.3 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + 3xy' + \frac{a^2y}{x^4} = \frac{1}{x^6} \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= -1 \\
\beta &= \frac{a}{2} \\
n &= \frac{1}{2} \\
\gamma &= -2
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{2c_1 \sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}} - \frac{2c_2 \cos\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{2c_1 \sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}} - \frac{2c_2 \cos\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-\frac{\sqrt{-a^2}}{2x^2}}$$
$$y_2 = e^{\frac{\sqrt{-a^2}}{2x^2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-\frac{\sqrt{-a^2}}{2x^2}} & e^{\frac{\sqrt{-a^2}}{2x^2}} \\ \frac{d}{dx}\left(e^{-\frac{\sqrt{-a^2}}{2x^2}}\right) & \frac{d}{dx}\left(e^{\frac{\sqrt{-a^2}}{2x^2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-\frac{\sqrt{-a^2}}{2x^2}} & e^{\frac{\sqrt{-a^2}}{2x^2}} \\ \frac{\sqrt{-a^2}\, e^{-\frac{\sqrt{-a^2}}{2x^2}}}{x^3} & -\frac{\sqrt{-a^2}\, e^{\frac{\sqrt{-a^2}}{2x^2}}}{x^3} \end{vmatrix}$$

Therefore

$$W = \left(e^{-\frac{\sqrt{-a^2}}{2x^2}}\right)\left(-\frac{\sqrt{-a^2}\, e^{\frac{\sqrt{-a^2}}{2x^2}}}{x^3}\right) - \left(e^{\frac{\sqrt{-a^2}}{2x^2}}\right)\left(\frac{\sqrt{-a^2}\, e^{-\frac{\sqrt{-a^2}}{2x^2}}}{x^3}\right)$$

Which simplifies to

$$W = -\frac{2\,\mathrm{e}^{-\frac{\sqrt{-a^2}}{2x^2}}\sqrt{-a^2}\,\mathrm{e}^{\frac{\sqrt{-a^2}}{2x^2}}}{x^3}$$

Which simplifies to

$$W = -\frac{2\sqrt{-a^2}}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{\frac{\sqrt{-a^2}}{2x^2}}}{x^6}}{-\frac{2\sqrt{-a^2}}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\mathrm{e}^{\frac{\sqrt{-a^2}}{2x^2}}}{2x^5\sqrt{-a^2}}dx$$

Hence

$$u_1 = \frac{-\frac{\mathrm{e}^{\frac{\sqrt{-a^2}}{2x^2}}}{x^2\sqrt{-a^2}} - \frac{2\,\mathrm{e}^{\frac{\sqrt{-a^2}}{2x^2}}}{a^2}}{2\sqrt{-a^2}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{-\frac{\sqrt{-a^2}}{2x^2}}}{x^6}}{-\frac{2\sqrt{-a^2}}{x}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\mathrm{e}^{-\frac{\sqrt{-a^2}}{2x^2}}}{2x^5\sqrt{-a^2}}dx$$

Hence

$$u_2 = -\frac{\frac{\mathrm{e}^{-\frac{\sqrt{-a^2}}{2x^2}}}{x^2\sqrt{-a^2}} - \frac{2\,\mathrm{e}^{-\frac{\sqrt{-a^2}}{2x^2}}}{a^2}}{2\sqrt{-a^2}}$$

Which simplifies to

$$u_1 = \frac{e^{\frac{\sqrt{-a^2}}{2x^2}} \left(2x^2\sqrt{-a^2} + a^2\right)}{2a^4x^2}$$

$$u_2 = \frac{e^{-\frac{\sqrt{-a^2}}{2x^2}} \left(-2x^2\sqrt{-a^2} + a^2\right)}{2a^4x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^{\frac{\sqrt{-a^2}}{2x^2}} \left(2x^2\sqrt{-a^2} + a^2\right) e^{-\frac{\sqrt{-a^2}}{2x^2}}}{2a^4x^2} + \frac{e^{-\frac{\sqrt{-a^2}}{2x^2}} \left(-2x^2\sqrt{-a^2} + a^2\right) e^{\frac{\sqrt{-a^2}}{2x^2}}}{2a^4x^2}$$

Which simplifies to

$$y_p(x) = \frac{1}{a^2x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{2c_1 \sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}} - \frac{2c_2 \cos\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}}\right) + \left(\frac{1}{a^2x^2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{2c_1 \sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}} - \frac{2c_2 \cos\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}} + \frac{1}{a^2x^2} \tag{1}$$

Verification of solutions

$$y = \frac{2c_1 \sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}} - \frac{2c_2 \cos\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}} + \frac{1}{a^2x^2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(x^6*diff(y(x),x$2)+3*x^5*diff(y(x),x)+a^2*y(x)=1/x^2,y(x), singsol=all)
```

$$y(x) = \sin\left(\frac{a}{2x^2}\right)c_2 + \cos\left(\frac{a}{2x^2}\right)c_1 + \frac{1}{a^2x^2}$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 38

```
DSolve[x^6*y''[x]+3*x^5*y'[x]+a^2*y[x]==1/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{a^2x^2} + c_1\cos\left(\frac{a}{2x^2}\right) - c_2\sin\left(\frac{a}{2x^2}\right)$$

2.19 problem 20

2.19.1 Solving as second order euler ode ode 798
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Internal problem ID [7460]
Internal file name [`OUTPUT/6427_Sunday_June_19_2022_05_02_06_PM_96573020/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 3xy' + 3y = 2x^3 - x^2}$$

2.19.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 3, f(x) = 2x^3 - x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 3y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 3x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 3 = 0$$

Or

$$r^2 - 4r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^3 + c_1x$$

Next, we find the particular solution to the ODE

$$x^2y'' - 3xy' + 3y = 2x^3 - x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^3 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x)\left(3x^2\right) - \left(x^3\right)(1)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3(2x^3 - x^2)}{2x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \left(x - \frac{1}{2}\right) dx$$

Hence

$$u_1 = -\frac{1}{2}x^2 + \frac{1}{2}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(2x^3 - x^2)}{2x^5} \, dx$$

Which simplifies to

$$u_2 = \int \frac{2x - 1}{2x^2} dx$$

Hence

$$u_2 = \frac{1}{2x} + \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{1}{2}x^2 + \frac{1}{2}x\right) x + \left(\frac{1}{2x} + \ln(x)\right) x^3$$

Which simplifies to

$$y_p(x) = x^3 \ln(x) - \frac{x^3}{2} + x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\ln(x) x^2 + \left(c_2 - \frac{1}{2}\right) x^2 + x + c_1\right) x \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(\ln(x) x^2 + \left(c_2 - \frac{1}{2}\right) x^2 + x + c_1\right) x \tag{1}$$

Verification of solutions

$$
y = \left(\ln\left(x\right) x^{2}+\left(c_{2}-\frac{1}{2}\right) x^{2}+x +c_{1}\right) x
$$

Verified OK.

2.19.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
x^{2} y''-3 x y'+3 y = 0
$$

In normal form the ode

$$
x^{2} y''-3 x y'+3 y = 0 \tag{1}
$$

Becomes

$$
y''+p(x) y'+q(x) y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{3}{x}\\
q(x) &= \frac{3}{x^{2}}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$
\frac{d^{2}}{d\tau^{2}}y(\tau)+p_{1}\left(\frac{d}{d\tau}y(\tau)\right)+q_{1}y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_{1}(\tau) = \frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'\left(x\right)^{2}} \tag{4}
$$

$$
q_{1}(\tau) = \frac{q(x)}{\tau'\left(x\right)^{2}} \tag{5}
$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}\,dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{3}{x}dx\right)}\,dx \\
&= \int e^{3\ln(x)}\,dx \\
&= \int x^3 dx \\
&= \frac{x^4}{4}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{3}{x^2}}{x^6} \\
&= \frac{3}{x^8}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{x^8} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{3}{x^8} = \frac{3}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 3\tau^r = 0$$

Simplifying gives

$$16r(r-1)\,\tau^r + 0\,\tau^r + 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$16r(r-1) + 0 + 3 = 0$$

Or

$$16r^2 - 16r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{4}$$
$$r_2 = \frac{3}{4}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{4}} + c_2\tau^{\frac{3}{4}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,(x^4)^{\frac{1}{4}}\left(c_2\sqrt{x^4} + 2c_1\right)}{4}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}\,(x^4)^{\frac{1}{4}}\left(c_2\sqrt{x^4} + 2c_1\right)}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^4\right)^{\frac{1}{4}}$$
$$y_2 = \left(x^4\right)^{\frac{3}{4}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^4\right)^{\frac{1}{4}} & \left(x^4\right)^{\frac{3}{4}} \\ \frac{d}{dx}\left(\left(x^4\right)^{\frac{1}{4}}\right) & \frac{d}{dx}\left(\left(x^4\right)^{\frac{3}{4}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^4\right)^{\frac{1}{4}} & \left(x^4\right)^{\frac{3}{4}} \\ \frac{x^3}{\left(x^4\right)^{\frac{3}{4}}} & \frac{3x^3}{\left(x^4\right)^{\frac{1}{4}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^4\right)^{\frac{1}{4}}\right)\left(\frac{3x^3}{\left(x^4\right)^{\frac{1}{4}}}\right) - \left(\left(x^4\right)^{\frac{3}{4}}\right)\left(\frac{x^3}{\left(x^4\right)^{\frac{3}{4}}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x^4\right)^{\frac{3}{4}} \left(2x^3 - x^2\right)}{2x^5} dx$$

Which simplifies to

$$u_1 = -\int \frac{\left(x^4\right)^{\frac{3}{4}} \left(2x - 1\right)}{2x^3} dx$$

Hence

$$u_1 = -\frac{(x-1)\left(x^4\right)^{\frac{3}{4}}}{2x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(x^4\right)^{\frac{1}{4}} \left(2x^3 - x^2\right)}{2x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{\left(x^4\right)^{\frac{1}{4}} \left(2x - 1\right)}{2x^3} dx$$

Hence

$$u_2 = \frac{\left(x^4\right)^{\frac{1}{4}}}{2x^2} + \frac{\left(x^4\right)^{\frac{1}{4}} \ln(x)}{x}$$

Which simplifies to

$$u_1 = -\frac{(x-1)\left(x^4\right)^{\frac{3}{4}}}{2x^2}$$

$$u_2 = \frac{\left(x^4\right)^{\frac{1}{4}} \left(2x \ln(x) + 1\right)}{2x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2(x-1)}{2} + \frac{x^2(2x \ln(x) + 1)}{2}$$

Which simplifies to

$$y_p(x) = x^3 \ln(x) - \frac{x^3}{2} + x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\sqrt{2}\,(x^4)^{\frac{1}{4}}\left(c_2\sqrt{x^4} + 2c_1\right)}{4}\right) + \left(x^3 \ln(x) - \frac{x^3}{2} + x^2\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\sqrt{2}\,(x^4)^{\frac{1}{4}}\left(c_2\sqrt{x^4} + 2c_1\right)}{4} + x^3 \ln(x) - \frac{x^3}{2} + x^2 \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{\sqrt{2}\,(x^4)^{\frac{1}{4}}\left(c_2\sqrt{x^4} + 2c_1\right)}{4} + x^3 \ln(x) - \frac{x^3}{2} + x^2$$

Verified OK.

2.19.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 3, f(x) = 2x^3 - x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 3y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\, x^3} - \frac{3}{x}\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{4c\sqrt{3}}{3} \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) - \frac{4c\sqrt{3}\left(\frac{d}{d\tau} y(\tau)\right)}{3} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{\frac{2\sqrt{3}\,c\tau}{3}} \left(c_1 \cosh\left(\frac{\sqrt{3}\,c\tau}{3}\right) + i c_2 \sinh\left(\frac{\sqrt{3}\,c\tau}{3}\right) \right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q}\, dx \\
&= \frac{\int \sqrt{3}\,\sqrt{\frac{1}{x^2}} dx}{c} \\
&= \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{\left((i c_2 + c_1)\, x^2 - i c_2 + c_1\right) x}{2}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - 3 x y' + 3 y = 2x^3 - x^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \left(x^4\right)^{\frac{1}{4}} \\
y_2 &= \left(x^4\right)^{\frac{3}{4}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^4)^{\frac{1}{4}} & (x^4)^{\frac{3}{4}} \\ \frac{d}{dx}\left((x^4)^{\frac{1}{4}}\right) & \frac{d}{dx}\left((x^4)^{\frac{3}{4}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^4)^{\frac{1}{4}} & (x^4)^{\frac{3}{4}} \\ \frac{x^3}{(x^4)^{\frac{3}{4}}} & \frac{3x^3}{(x^4)^{\frac{1}{4}}} \end{vmatrix}$$

Therefore

$$W = \left((x^4)^{\frac{1}{4}}\right)\left(\frac{3x^3}{(x^4)^{\frac{1}{4}}}\right) - \left((x^4)^{\frac{3}{4}}\right)\left(\frac{x^3}{(x^4)^{\frac{3}{4}}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^4)^{\frac{3}{4}}\,(2x^3 - x^2)}{2x^5}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^4)^{\frac{3}{4}}\,(2x - 1)}{2x^3} dx$$

Hence

$$u_1 = -\frac{(x-1)\left(x^4\right)^{\frac{3}{4}}}{2x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(x^4\right)^{\frac{1}{4}}\left(2x^3 - x^2\right)}{2x^5}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\left(x^4\right)^{\frac{1}{4}}(2x-1)}{2x^3}dx$$

Hence

$$u_2 = \frac{\left(x^4\right)^{\frac{1}{4}}}{2x^2} + \frac{\left(x^4\right)^{\frac{1}{4}}\ln(x)}{x}$$

Which simplifies to

$$u_1 = -\frac{(x-1)\left(x^4\right)^{\frac{3}{4}}}{2x^2}$$

$$u_2 = \frac{\left(x^4\right)^{\frac{1}{4}}(2x\ln(x)+1)}{2x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2(x-1)}{2} + \frac{x^2(2x\ln(x)+1)}{2}$$

Which simplifies to

$$y_p(x) = x^3\ln(x) - \frac{x^3}{2} + x^2$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{((ic_2+c_1)\,x^2 - ic_2 + c_1)\,x}{2}\right) + \left(x^3\ln(x) - \frac{x^3}{2} + x^2\right) \\
&= x^3\ln(x) - \frac{x^3}{2} + x^2 + \frac{((ic_2+c_1)\,x^2 - ic_2 + c_1)\,x}{2}
\end{aligned}$$

Which simplifies to

$$y = \frac{(2\ln(x)\, x^2 + (ic_2 + c_1 - 1)\, x^2 + 2x - ic_2 + c_1)\, x}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{(2\ln(x)\, x^2 + (ic_2 + c_1 - 1)\, x^2 + 2x - ic_2 + c_1)\, x}{2} \tag{1}$$

Verification of solutions

$$y = \frac{(2\ln(x)\, x^2 + (ic_2 + c_1 - 1)\, x^2 + 2x - ic_2 + c_1)\, x}{2}$$

Verified OK.

2.19.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 3, f(x) = 2x^3 - x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 3y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{3}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{3}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{3}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\
\ln(u) &= -3\ln(x) + c_1 \\
u &= \mathrm{e}^{-3\ln(x)+c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x^3 \\
&= c_2 x^3 - \frac{1}{2} c_1 x
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - 3xy' + 3y = 2x^3 - x^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= x^3
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^3 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x)\left(3x^2\right) - \left(x^3\right)(1)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3(2x^3 - x^2)}{2x^5}\,dx$$

Which simplifies to

$$u_1 = -\int \left(x - \frac{1}{2}\right) dx$$

Hence

$$u_1 = -\frac{1}{2}x^2 + \frac{1}{2}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(2x^3 - x^2)}{2x^5} \, dx$$

Which simplifies to

$$u_2 = \int \frac{2x - 1}{2x^2} dx$$

Hence

$$u_2 = \frac{1}{2x} + \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{1}{2}x^2 + \frac{1}{2}x\right)x + \left(\frac{1}{2x} + \ln(x)\right)x^3$$

Which simplifies to

$$y_p(x) = x^3 \ln(x) - \frac{x^3}{2} + x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right)x^3\right) + \left(x^3 \ln(x) - \frac{x^3}{2} + x^2\right) \\ &= x^3 \ln(x) - \frac{x^3}{2} + x^2 + \left(-\frac{c_1}{2x^2} + c_2\right)x^3 \end{aligned}$$

Which simplifies to

$$y = \frac{x(2\ln(x)\,x^2 + 2c_2x^2 - x^2 - c_1 + 2x)}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{x(2 \ln(x) x^2 + 2c_2 x^2 - x^2 - c_1 + 2x)}{2} \tag{1}$$

Verification of solutions

$$y = \frac{x(2 \ln(x) x^2 + 2c_2 x^2 - x^2 - c_1 + 2x)}{2}$$

Verified OK.

2.19.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + (2AB' + B^2)\, v' + (AB'' + BB' + CB)\, v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)\, v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)\, u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x^2 \\
B &= -3x \\
C &= 3 \\
F &= 2x^3 - x^2
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(x^2\right)(0) + (-3x)(-3) + (3)(-3x) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-3x^3v'' + \left(3x^2\right)v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
-3x^2(u'(x)\,x - u(x)) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= \frac{u}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{1}{x}\,dx \\
\ln(u) &= \ln(x) + c_1 \\
u &= \mathrm{e}^{\ln(x)+c_1} \\
&= c_1 x
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1 x
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$v(x) = \int c_1 x \, \mathrm{d}x$$
$$= \frac{c_1 x^2}{2} + c_2$$

Therefore the homogeneous solution is

$$y_h(x) = Bv$$
$$= (-3x)\left(\frac{c_1 x^2}{2} + c_2\right)$$
$$= -\frac{3x(c_1 x^2 + 2c_2)}{2}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^3 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x)\left(3x^2\right) - \left(x^3\right)(1)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3(2x^3 - x^2)}{2x^5}\,dx$$

Which simplifies to

$$u_1 = -\int \left(x - \frac{1}{2}\right) dx$$

Hence

$$u_1 = -\frac{1}{2}x^2 + \frac{1}{2}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(2x^3 - x^2)}{2x^5}\,dx$$

Which simplifies to

$$u_2 = \int \frac{2x - 1}{2x^2} dx$$

Hence

$$u_2 = \frac{1}{2x} + \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{1}{2}x^2 + \frac{1}{2}x\right)x + \left(\frac{1}{2x} + \ln(x)\right)x^3$$

Which simplifies to

$$y_p(x) = x^3 \ln(x) - \frac{x^3}{2} + x^2$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-\frac{3x(c_1x^2 + 2c_2)}{2}\right) + \left(x^3 \ln(x) - \frac{x^3}{2} + x^2\right) \\ &= x^3 \ln(x) + \frac{(-3c_1 - 1)\, x^3}{2} + x^2 - 3c_2x \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x^3 \ln(x) + \frac{(-3c_1 - 1)\, x^3}{2} + x^2 - 3c_2x \tag{1}$$

Verification of solutions

$$y = x^3 \ln(x) + \frac{(-3c_1 - 1)\, x^3}{2} + x^2 - 3c_2x$$

Verified OK.

2.19.6 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 3xy' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 83: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\,(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}} \\
&= z_1\left(x^{\frac{3}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - 3xy' + 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x + \frac{1}{2} c_2 x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \frac{x^3}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{x^3}{2} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{x^3}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{x^3}{2} \\ 1 & \frac{3x^2}{2} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{3x^2}{2}\right) - \left(\frac{x^3}{2}\right)(1)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^3(2x^3 - x^2)}{2}}{x^5} \, dx$$

Which simplifies to

$$u_1 = -\int \left(x - \frac{1}{2}\right) dx$$

Hence

$$u_1 = -\frac{1}{2}x^2 + \frac{1}{2}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(2x^3 - x^2)}{x^5} \, dx$$

Which simplifies to

$$u_2 = \int \frac{2x - 1}{x^2} dx$$

Hence

$$u_2 = \frac{1}{x} + 2\ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{1}{2}x^2 + \frac{1}{2}x\right)x + \frac{\left(\frac{1}{x} + 2\ln(x)\right)x^3}{2}$$

Which simplifies to

$$y_p(x) = x^3 \ln(x) - \frac{x^3}{2} + x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x + \frac{1}{2} c_2 x^3\right) + \left(x^3 \ln(x) - \frac{x^3}{2} + x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + \frac{c_2 x^3}{2} + x^3 \ln(x) - \frac{x^3}{2} + x^2 \tag{1}$$

Verification of solutions

$$y = c_1 x + \frac{c_2 x^3}{2} + x^3 \ln(x) - \frac{x^3}{2} + x^2$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+3*y(x)=2*x^3-x^2,y(x), singsol=all)
```

$$y(x) = \frac{x(2x^2 \ln(x) + (c_1 - 1) x^2 + 2x + 2c_2)}{2}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 27

```
DSolve[x^2*y''[x]-3*x*y'[x]+3*y[x]==2*x^3-x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\left(x^2 \log(x) + \left(-\frac{1}{2} + c_2\right) x^2 + x + c_1\right)$$

2.20 problem 21

2.20.1 Solving as second order change of variable on x method 2 ode . 831
2.20.2 Solving as second order change of variable on x method 1 ode . 834

Internal problem ID [7461]
Internal file name [`OUTPUT/6428_Sunday_June_19_2022_05_02_07_PM_94047950/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{y'' + \cot(x) y' + 4y \csc(x)^2 = 0}$$

2.20.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$y'' + \cot(x) y' + 4y \csc(x)^2 = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \cot(x) \\ q(x) &= 4 \csc(x)^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \cot(x)dx\right)}dx \\ &= \int e^{-\ln(\sin(x))}\,dx \\ &= \int \csc(x)\,dx \\ &= -\ln(\csc(x) + \cot(x))\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{4\csc(x)^2}{\csc(x)^2} \\ &= 4\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + 4y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + 4\,e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(2\ln(\csc(x) + \cot(x))) - c_2 \sin(2\ln(\csc(x) + \cot(x)))$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2\ln(\csc(x) + \cot(x))) - c_2 \sin(2\ln(\csc(x) + \cot(x))) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(2\ln(\csc(x) + \cot(x))) - c_2 \sin(2\ln(\csc(x) + \cot(x)))$$

Verified OK.

2.20.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$y'' + \cot(x)\, y' + 4y \csc(x)^2 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \cot(x) \\ q(x) &= 4\csc(x)^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\csc(x)^2}}{c} \qquad (6) \\ \tau'' &= -\frac{2\csc(x)^2 \cot(x)}{c\sqrt{\csc(x)^2}} \end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2\csc(x)^2\cot(x)}{c\sqrt{\csc(x)^2}} + \cot(x)\,\frac{2\sqrt{\csc(x)^2}}{c}}{\left(\frac{2\sqrt{\csc(x)^2}}{c}\right)^2} \\
&= 0
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\csc(x)^2}dx}{c} \\
&= \frac{2\sqrt{\csc(x)^2}\,\ln(-\cot(x) + \csc(x))\sin(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos(2\ln(-\cot(x) + \csc(x))) + c_2 \sin(2\ln(-\cot(x) + \csc(x)))$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2\ln(-\cot(x) + \csc(x))) + c_2 \sin(2\ln(-\cot(x) + \csc(x))) \qquad (1)$$

Verification of solutions

$$y = c_1 \cos(2\ln(-\cot(x) + \csc(x))) + c_2 \sin(2\ln(-\cot(x) + \csc(x)))$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)+cot(x)*diff(y(x),x)+4*y(x)*csc(x)^2=0,y(x), singsol=all)
```

$$y(x) = c_1(\csc(x) + \cot(x))^{-2i} + c_2(\csc(x) + \cot(x))^{2i}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 25

```
DSolve[y''[x]+Cot[x]*y'[x]+4*y[x]*Csc[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(2\text{arctanh}(\cos(x))) - c_2 \sin(2\text{arctanh}(\cos(x)))$$

2.21 problem 22

Internal problem ID [7462]
Internal file name [`OUTPUT/6429_Sunday_June_19_2022_05_02_08_PM_66157276/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

Unable to solve or complete the solution.

$$\boxed{(x^2+1)\,y''+(1+x)\,y'+y=4\cos(\ln(1+x))}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 408

```
dsolve((1+x^2)*diff(y(x),x$2)+(1+x)*diff(y(x),x)+y(x)=4*cos(ln(1+x)),y(x), singsol=all)
```

$$y(x) = \operatorname{hypergeom}\left([i,-i],\left[\frac{1}{2}+\frac{i}{2}\right],\frac{1}{2}-\frac{ix}{2}\right)c_2$$
$$+(x+i)^{\frac{1}{2}-\frac{i}{2}}\operatorname{hypergeom}\left(\left[\frac{1}{2}+\frac{i}{2},\frac{1}{2}-\frac{3i}{2}\right],\left[\frac{3}{2}-\frac{i}{2}\right],\frac{1}{2}-\frac{ix}{2}\right)c_1$$
$$+80\left(\int \frac{}{(x^2+1)\left(10\operatorname{hypergeom}\left(\left[\frac{1}{2}+\frac{i}{2},\frac{1}{2}-\frac{3i}{2}\right],\left[\frac{3}{2}-\frac{i}{2}\right],\frac{1}{2}-\frac{ix}{2}\right)\left((-1-i+(-1+i)x)\operatorname{hypergeom}\left(\right.\right.\right.}\right.$$
$$-i],\left[\frac{1}{2}+\frac{i}{2}\right],\frac{1}{2}-\frac{ix}{2}\right)$$
$$-80\left(\int \frac{\cos(\ln(x+1}{7\left(\frac{10((1-i+(-1-i)x)\operatorname{hypergeom}([1-i,1+i],[\frac{3}{2}+\frac{i}{2}],\frac{1}{2}-\frac{ix}{2})+(-1+i)\operatorname{hypergeom}([i,-i],[\frac{1}{2}+\frac{i}{2}],\frac{1}{2}-\frac{ix}{2}))\operatorname{hypergeom}([\frac{1}{2}+\frac{i}{2},\frac{1}{2}-}{7}\right.}\right.$$
$$+i)^{\frac{1}{2}-\frac{i}{2}}\operatorname{hypergeom}\left(\left[\frac{1}{2}+\frac{i}{2},\frac{1}{2}-\frac{3i}{2}\right],\left[\frac{3}{2}-\frac{i}{2}\right],\frac{1}{2}-\frac{ix}{2}\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(1+x^2)*y''[x]+(1+x)*y'[x]+y[x]==4*Cos[Log[1+x]],y[x],x,IncludeSingularSolutions -> T
```

Not solved

2.22 problem 23

2.22.1 Solving as second order change of variable on x method 2 ode . 840
2.22.2 Solving as second order change of variable on x method 1 ode . 843

Internal problem ID [7463]
Internal file name [`OUTPUT/6430_Sunday_June_19_2022_05_02_12_PM_6804064/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + \tan(x) y' + y \cos(x)^2 = 0}$$

2.22.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$y'' + \tan(x) y' + y \cos(x)^2 = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= \cos(x)^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \tan(x)dx\right)} dx \\
&= \int e^{\ln(\cos(x))}\, dx \\
&= \int \cos(x)\, dx \\
&= \sin(x) \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\cos(x)^2}{\cos(x)^2} \\
&= 1 \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} + \mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = \mathrm{e}^0(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$y(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(\sin(x)) + c_2 \sin(\sin(x))$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(\sin(x)) + c_2 \sin(\sin(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(\sin(x)) + c_2 \sin(\sin(x))$$

Verified OK.

2.22.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$y'' + \tan(x)\, y' + y \cos(x)^2 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= \cos(x)^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{\cos(2x)}{2} + \frac{1}{2}}}{c} \\ \tau'' &= -\frac{2\sin(x)\cos(x)}{c\sqrt{2\cos(2x) + 2}} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2\sin(x)\cos(x)}{c\sqrt{2\cos(2x)+2}} + \tan(x)\,\frac{\sqrt{\frac{\cos(2x)}{2}+\frac{1}{2}}}{c}}{\left(\frac{\sqrt{\frac{\cos(2x)}{2}+\frac{1}{2}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{\frac{\cos(2x)}{2}+\frac{1}{2}}dx}{c} \\
&= \frac{\sqrt{\frac{\cos(2x)}{2}+\frac{1}{2}}\,\sin(x)}{c\cos(x)}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos(\sin(x)) + c_2 \sin(\tan(x)\,|\cos(x)|)
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(\sin(x)) + c_2 \sin(\tan(x)\,|\cos(x)|) \tag{1}
$$

Verification of solutions

$$
y = c_1 \cos(\sin(x)) + c_2 \sin(\tan(x)\,|\cos(x)|)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   <- constant coefficients successful
   Change of variables used:
      [x = arcsin(t)]
   Linear ODE actually solved:
      (-2*t^2+2)*u(t)+(-2*t^2+2)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)+tan(x)*diff(y(x),x)+cos(x)^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(\sin(x)) + c_2 \cos(\sin(x))$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 18

```
DSolve[y''[x]+Tan[x]*y'[x]+Cos[x]^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \sin(\sin(x)) + c_1 \cos(\sin(x))$$

2.23 problem 24

2.23.1 Solving as second order change of variable on x method 2 ode . 846
2.23.2 Solving as second order change of variable on x method 1 ode . 854
2.23.3 Solving as second order bessel ode ode 861
2.23.4 Solving using Kovacic algorithm 866

Internal problem ID [7464]
Internal file name [`OUTPUT/6431_Sunday_June_19_2022_05_02_13_PM_82068666/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{xy'' - y' + 4yx^3 = 8x^3 \sin(x)^2}$$

2.23.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$xy'' - y' + 4yx^3 = 0$$

In normal form the ode

$$xy'' - y' + 4yx^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 4x^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)}dx \\ &= \int e^{\ln(x)}\,dx \\ &= \int x dx \\ &= \frac{x^2}{2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{4x^2}{x^2} \\ &= 4 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0\\ \frac{d^2}{d\tau^2}y(\tau) + 4y(\tau) &= 0\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + 4\,e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x^2) + c_2 \sin(x^2)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x^2)$$
$$y_2 = \sin(x^2)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x^2) & \sin(x^2) \\ \frac{d}{dx}(\cos(x^2)) & \frac{d}{dx}(\sin(x^2)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x^2) & \sin(x^2) \\ -2x\sin(x^2) & 2x\cos(x^2) \end{vmatrix}$$

Therefore

$$W = \left(\cos(x^2)\right)\left(2x\cos(x^2)\right) - \left(\sin(x^2)\right)\left(-2x\sin(x^2)\right)$$

Which simplifies to

$$W = 2\cos\left(x^2\right)^2 x + 2\sin\left(x^2\right)^2 x$$

Which simplifies to

$$W = 2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8\sin(x^2)\,x^3\sin(x)^2}{2x^2}\,dx$$

Which simplifies to

$$u_1 = -\int 4\sin\left(x^2\right)x\sin(x)^2\,dx$$

Hence

$$\begin{aligned} u_1 = \cos\left(x^2\right) &- \frac{\cos(x^2 - 2x)}{2} \\ &+ \frac{\sqrt{\pi}\,\sqrt{2}\left(\cos(1)\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) - \sin(1)\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{2} \\ &- \frac{\cos(x^2 + 2x)}{2} \\ &- \frac{\sqrt{\pi}\,\sqrt{2}\left(\cos(1)\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right) - \sin(1)\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)\right)}{2} \end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\cos\left(x^2\right) x^3 \sin\left(x\right)^2}{2x^2}\, dx$$

Which simplifies to

$$u_2 = \int 4\cos\left(x^2\right) x \sin\left(x\right)^2\, dx$$

Hence

$$\begin{aligned}u_2 = \sin\left(x^2\right) - \frac{\sin\left(x^2-2x\right)}{2} \\ - \frac{\sqrt{\pi}\,\sqrt{2}\left(\cos\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) + \sin\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{2} \\ - \frac{\sin\left(x^2+2x\right)}{2} \\ + \frac{\sqrt{\pi}\,\sqrt{2}\left(\cos\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right) + \sin\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)\right)}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}u_1 = \cos\left(x^2\right) - \cos\left(x^2\right)\cos\left(2x\right) + \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\ - \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} \\ + \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2}\end{aligned}$$

$$\begin{aligned}u_2 = \sin\left(x^2\right) - \sin\left(x^2\right)\cos\left(2x\right) - \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\ - \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\sin\left(1\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} \\ + \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2}\end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\cos\left(x^2\right) - \cos\left(x^2\right)\cos(2x) + \frac{\sqrt{2}\,\sqrt{\pi}\,\cos(1)\,\text{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\sin(1)\,\text{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\cos(1)\,\text{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\sin(1)\,\text{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2}\right)\cos\left(x^2\right) + \left(\sin\left(x^2\right) - \sin\left(x^2\right)\cos(2x) - \frac{\sqrt{2}\,\sqrt{\pi}\,\cos(1)\,\text{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\text{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\sin(1)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\cos(1)\,\text{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\sin(1)\,\text{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2}\right)\sin\left(x^2\right)$$

Which simplifies to

$$y_p(x) = -\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1) + \cos(x^2)\sin(1))\,\text{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1) + \cos(x^2)\sin(1))\,\text{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1) - \cos(x^2)\cos(1))\,\text{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1) - \cos(x^2)\cos(1))\,\text{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} - \cos(2x) + 1$$

Therefore the general solution is

$$y = y_h + y_p$$

$$
\begin{aligned}
&= \left(c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right)\right) \\
&+ \left(-\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1) + \cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \right.\\
&+ \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1) + \cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} \\
&- \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1) - \cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\
&\left. + \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1) - \cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} - \cos(2x) + 1\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right) \\
&- \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1) + \cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\
&+ \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1) + \cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} \\
&- \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1) - \cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\
&+ \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1) - \cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} - \cos(2x) + 1
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right) \\
&- \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1) + \cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\
&+ \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1) + \cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} \\
&- \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1) - \cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\
&+ \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1) - \cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} - \cos(2x) + 1
\end{aligned}
$$

Verified OK.

2.23.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x, B = -1, C = 4x^3, f(x) = 8x^3 \sin(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$xy'' - y' + 4yx^3 = 0$$

In normal form the ode

$$xy'' - y' + 4yx^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 4x^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{x^2}}{c} \end{aligned} \tag{6}$$

$$\tau'' = \frac{2x}{c\sqrt{x^2}}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{\frac{2x}{c\sqrt{x^2}} - \frac{1}{x}\frac{2\sqrt{x^2}}{c}}{\left(\frac{2\sqrt{x^2}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned}
\tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{x^2}dx}{c} \\
&= \frac{x\sqrt{x^2}}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos(x^2) + c_2 \sin(x^2)
$$

Now the particular solution to this ODE is found

$$
xy'' - y' + 4yx^3 = 8x^3 \sin(x)^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x^2)$$
$$y_2 = \sin(x^2)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x^2) & \sin(x^2) \\ \frac{d}{dx}(\cos(x^2)) & \frac{d}{dx}(\sin(x^2)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x^2) & \sin(x^2) \\ -2x\sin(x^2) & 2x\cos(x^2) \end{vmatrix}$$

Therefore

$$W = (\cos(x^2))(2x\cos(x^2)) - (\sin(x^2))(-2x\sin(x^2))$$

Which simplifies to

$$W = 2\cos(x^2)^2 x + 2\sin(x^2)^2 x$$

Which simplifies to

$$W = 2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8\sin(x^2)x^3\sin(x)^2}{2x^2}\,dx$$

Which simplifies to

$$u_1 = -\int 4 \sin\left(x^2\right) x \sin\left(x\right)^2 dx$$

Hence

$$\begin{aligned} u_1 = \cos\left(x^2\right) - \frac{\cos\left(x^2-2x\right)}{2} \\ + \frac{\sqrt{\pi}\,\sqrt{2}\left(\cos\left(1\right)\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) - \sin\left(1\right)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{2} \\ - \frac{\cos\left(x^2+2x\right)}{2} \\ - \frac{\sqrt{\pi}\,\sqrt{2}\left(\cos\left(1\right)\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right) - \sin\left(1\right)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)\right)}{2} \end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8 \cos\left(x^2\right) x^3 \sin\left(x\right)^2}{2x^2} dx$$

Which simplifies to

$$u_2 = \int 4 \cos\left(x^2\right) x \sin\left(x\right)^2 dx$$

Hence

$$\begin{aligned} u_2 = \sin\left(x^2\right) - \frac{\sin\left(x^2-2x\right)}{2} \\ - \frac{\sqrt{\pi}\,\sqrt{2}\left(\cos\left(1\right)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) + \sin\left(1\right)\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{2} \\ - \frac{\sin\left(x^2+2x\right)}{2} \\ + \frac{\sqrt{\pi}\,\sqrt{2}\left(\cos\left(1\right)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right) + \sin\left(1\right)\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)\right)}{2} \end{aligned}$$

Which simplifies to

$$u_1 = \cos\left(x^2\right) - \cos\left(x^2\right)\cos\left(2x\right) + \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2}$$

$$u_2 = \sin\left(x^2\right) - \sin\left(x^2\right)\cos\left(2x\right) - \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\sin\left(1\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\cos\left(x^2\right) - \cos\left(x^2\right)\cos\left(2x\right) + \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2}\right)\cos\left(x^2\right) + \left(\sin\left(x^2\right) - \sin\left(x^2\right)\cos\left(2x\right) - \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\sin\left(1\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2}\right)\sin\left(x^2\right)$$

Which simplifies to

$$
\begin{aligned}
y_p(x) = &-\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1)+\cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\
&+\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1)+\cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} \\
&-\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1)-\cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\
&+\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1)-\cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2}-\cos(2x)+1
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\cos\left(x^2\right)+c_2\sin\left(x^2\right)\right) \\
&\quad+\left(-\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1)+\cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2}\right. \\
&\quad+\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1)+\cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} \\
&\quad-\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1)-\cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\
&\quad\left.+\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1)-\cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2}-\cos(2x)+1\right) \\
&= c_1\cos\left(x^2\right)+c_2\sin\left(x^2\right) \\
&\quad-\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1)+\cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\
&\quad+\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1)+\cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} \\
&\quad-\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1)-\cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\
&\quad+\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1)-\cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2}-\cos(2x)+1
\end{aligned}
$$

Which simplifies to

$$
y = 1 + c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right) - \frac{\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) \sin\left(x^2+1\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right) \sin\left(x^2+1\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) \cos\left(x^2+1\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right) \cos\left(x^2+1\right)}{2} - \cos\left(2x\right)
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = 1 + c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right) - \frac{\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) \sin\left(x^2+1\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right) \sin\left(x^2+1\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) \cos\left(x^2+1\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right) \cos\left(x^2+1\right)}{2} - \cos\left(2x\right) \tag{1}
$$

<u>Verification of solutions</u>

$$
y = 1 + c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right) - \frac{\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) \sin\left(x^2+1\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right) \sin\left(x^2+1\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) \cos\left(x^2+1\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right) \cos\left(x^2+1\right)}{2} - \cos\left(2x\right)
$$

Verified OK.

2.23.3 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' - xy' + 4yx^4 = 8x^4 \sin(x)^2 \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha) xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta x^\gamma) + c_2 \operatorname{BesselY}(n, \beta x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= 1 \\
\beta &= 1 \\
n &= \frac{1}{2} \\
\gamma &= 2
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x\sqrt{2}\, \sin(x^2)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x\sqrt{2}\, \cos(x^2)}{\sqrt{\pi}\, \sqrt{x^2}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x\sqrt{2}\, \sin(x^2)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x\sqrt{2}\, \cos(x^2)}{\sqrt{\pi}\, \sqrt{x^2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(x^2\right)$$
$$y_2 = \sin\left(x^2\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x^2) & \sin(x^2) \\ \frac{d}{dx}(\cos(x^2)) & \frac{d}{dx}(\sin(x^2)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x^2) & \sin(x^2) \\ -2x\sin(x^2) & 2x\cos(x^2) \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(x^2\right)\right)\left(2x\cos\left(x^2\right)\right) - \left(\sin\left(x^2\right)\right)\left(-2x\sin\left(x^2\right)\right)$$

Which simplifies to

$$W = 2\cos\left(x^2\right)^2 x + 2\sin\left(x^2\right)^2 x$$

Which simplifies to

$$W = 2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8\sin\left(x^2\right) x^4 \sin\left(x\right)^2}{2x^3}\, dx$$

Which simplifies to

$$u_1 = -\int 4\sin\left(x^2\right) x \sin\left(x\right)^2\, dx$$

Hence

$$\begin{aligned}
u_1 = \cos\left(x^2\right) &- \frac{\cos\left(x^2-2x\right)}{2} \\
&+ \frac{\sqrt{\pi}\,\sqrt{2}\left(\cos\left(1\right) \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) - \sin\left(1\right) \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{2} \\
&- \frac{\cos\left(x^2+2x\right)}{2} \\
&- \frac{\sqrt{\pi}\,\sqrt{2}\left(\cos\left(1\right) \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right) - \sin\left(1\right) \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)\right)}{2}
\end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\cos\left(x^2\right) x^4 \sin\left(x\right)^2}{2x^3}\, dx$$

Which simplifies to

$$u_2 = \int 4\cos\left(x^2\right) x \sin\left(x\right)^2\, dx$$

Hence

$$\begin{aligned}
u_2 = \sin\left(x^2\right) &- \frac{\sin\left(x^2-2x\right)}{2} \\
&- \frac{\sqrt{\pi}\,\sqrt{2}\left(\cos\left(1\right) \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) + \sin\left(1\right) \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{2} \\
&- \frac{\sin\left(x^2+2x\right)}{2} \\
&+ \frac{\sqrt{\pi}\,\sqrt{2}\left(\cos\left(1\right) \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right) + \sin\left(1\right) \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)\right)}{2}
\end{aligned}$$

Which simplifies to

$$u_1 = \cos\left(x^2\right) - \cos\left(x^2\right)\cos\left(2x\right) + \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(1\right)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(1\right)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2}$$

$$u_2 = \sin\left(x^2\right) - \sin\left(x^2\right)\cos\left(2x\right) - \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\sin\left(1\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(1\right)\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = {} & \left(\cos\left(x^2\right) - \cos\left(x^2\right)\cos\left(2x\right) + \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(1\right)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(1\right)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2}\right)\cos\left(x^2\right) \\ & + \left(\sin\left(x^2\right) - \sin\left(x^2\right)\cos\left(2x\right) - \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\sin\left(1\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(1\right)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(1\right)\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2}\right)\sin\left(x^2\right) \end{aligned}$$

Which simplifies to

$$y_p(x) = -\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1)+\cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1)+\cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1)-\cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1)-\cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} - \cos(2x) + 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} - \frac{c_2 x\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}\right) \\ &\quad + \left(-\frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1)+\cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\cos(1)+\cos(x^2)\sin(1))\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1)-\cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,(\sin(x^2)\sin(1)-\cos(x^2)\cos(1))\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} - \cos(2x) + 1\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{c_1 x\sqrt{2}\, \sin\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x\sqrt{2}\, \cos\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} \\
& - \frac{\sqrt{2}\, \sqrt{\pi}\, \left(\sin\left(x^2\right)\cos\left(1\right) + \cos\left(x^2\right)\sin\left(1\right)\right) \operatorname{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\
& + \frac{\sqrt{2}\, \sqrt{\pi}\, \left(\sin\left(x^2\right)\cos\left(1\right) + \cos\left(x^2\right)\sin\left(1\right)\right) \operatorname{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} \\
& - \frac{\sqrt{2}\, \sqrt{\pi}\, \left(\sin\left(x^2\right)\sin\left(1\right) - \cos\left(x^2\right)\cos\left(1\right)\right) \operatorname{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\
& + \frac{\sqrt{2}\, \sqrt{\pi}\, \left(\sin\left(x^2\right)\sin\left(1\right) - \cos\left(x^2\right)\cos\left(1\right)\right) \operatorname{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} - \cos\left(2x\right) + 1
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \frac{c_1 x\sqrt{2}\, \sin\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x\sqrt{2}\, \cos\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} \\
& - \frac{\sqrt{2}\, \sqrt{\pi}\, \left(\sin\left(x^2\right)\cos\left(1\right) + \cos\left(x^2\right)\sin\left(1\right)\right) \operatorname{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\
& + \frac{\sqrt{2}\, \sqrt{\pi}\, \left(\sin\left(x^2\right)\cos\left(1\right) + \cos\left(x^2\right)\sin\left(1\right)\right) \operatorname{FresnelC}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} \\
& - \frac{\sqrt{2}\, \sqrt{\pi}\, \left(\sin\left(x^2\right)\sin\left(1\right) - \cos\left(x^2\right)\cos\left(1\right)\right) \operatorname{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\
& + \frac{\sqrt{2}\, \sqrt{\pi}\, \left(\sin\left(x^2\right)\sin\left(1\right) - \cos\left(x^2\right)\cos\left(1\right)\right) \operatorname{FresnelS}\left(\frac{\sqrt{2}\,(1+x)}{\sqrt{\pi}}\right)}{2} - \cos\left(2x\right) + 1
\end{aligned}
$$

Verified OK.

2.23.4 Solving using Kovacic algorithm

Writing the ode as

$$
xy'' - y' + 4yx^3 = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -1 \\ C &= 4x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 84: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-16x^4+3}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= (-4x^2) + \left(\frac{3}{4x^2}\right) \\
&= -4x^2 + \frac{3}{4x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$
\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 2ix \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-16x^4+3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\left(2ix\right) \\
&= -\frac{1}{2x} - 2ix \\
&= -\frac{1}{2x} - 2ix
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{1}{2x} - 2ix\right)dx} \\
&= \frac{e^{-ix^2}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{x}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-i x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{i\,\mathrm{e}^{2 i x^2}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-i x^2}\right) + c_2\left(\mathrm{e}^{-i x^2}\left(-\frac{i\,\mathrm{e}^{2 i x^2}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$xy''-y'+4yx^3=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=c_1\mathrm{e}^{-ix^2}-\frac{ic_2\mathrm{e}^{ix^2}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1,u_2 to be determined, and y_1,y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1&=\mathrm{e}^{-ix^2}\\ y_2&=-\frac{i\mathrm{e}^{ix^2}}{4}\end{aligned}$$

In the Variation of parameters u_1,u_2 are found using

$$u_1=-\int\frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2=\int\frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix}y_1&y_2\\y_1'&y_2'\end{vmatrix}$. Hence

$$W=\begin{vmatrix}\mathrm{e}^{-ix^2}&-\frac{i\mathrm{e}^{ix^2}}{4}\\ \frac{d}{dx}\left(\mathrm{e}^{-ix^2}\right)&\frac{d}{dx}\left(-\frac{i\mathrm{e}^{ix^2}}{4}\right)\end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-i x^2} & -\frac{i \mathrm{e}^{i x^2}}{4} \\ -2 i x \,\mathrm{e}^{-i x^2} & \frac{x \,\mathrm{e}^{i x^2}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-i x^2}\right) \left(\frac{x \,\mathrm{e}^{i x^2}}{2}\right) - \left(-\frac{i \mathrm{e}^{i x^2}}{4}\right) \left(-2 i x \,\mathrm{e}^{-i x^2}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-i x^2} x \,\mathrm{e}^{i x^2}$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2 i \mathrm{e}^{i x^2} x^3 \sin\left(x\right)^2}{x^2} \, dx$$

Which simplifies to

$$u_1 = -\int -2 i \mathrm{e}^{i x^2} x \sin\left(x\right)^2 \, dx$$

Hence

$$\begin{aligned} u_1 = \left(\frac{1}{8} - \frac{i}{8}\right) \Bigg(\sqrt{\pi}\,\mathrm{e}^{-i} \operatorname{erf}\left(\left(\frac{1}{2} - \frac{i}{2}\right) (x-1) \sqrt{2}\right) - \sqrt{\pi}\,\mathrm{e}^{-i} \operatorname{erf}\left(\left(\frac{1}{2} - \frac{i}{2}\right) (1+x) \sqrt{2}\right) \\ + (1+i) \sqrt{2}\,\mathrm{e}^{i x^2} + \left(-\frac{1}{2} - \frac{i}{2}\right) \sqrt{2}\,\mathrm{e}^{i x (x-2)} + \left(-\frac{1}{2} - \frac{i}{2}\right) \sqrt{2}\,\mathrm{e}^{i x (x+2)} \\ + 2 \sqrt{\pi}\,\mathrm{e}^{-i} \operatorname{erf}\left(\left(\frac{1}{2} - \frac{i}{2}\right) \sqrt{2}\right)\Bigg) \sqrt{2} \end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8 \,\mathrm{e}^{-i x^2} x^3 \sin\left(x\right)^2}{x^2} \, dx$$

Which simplifies to

$$u_2 = \int 8 \,\mathrm{e}^{-i x^2} x \sin\left(x\right)^2 \, dx$$

Hence

$$
\begin{aligned}
u_2 = {} & \left(-\frac{1}{2}+\frac{i}{2}\right) \sqrt{\pi}\, \sqrt{2}\, \mathrm{e}^{i} \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right) (x-1) \sqrt{2}\right) \\
& + \left(\frac{1}{2}-\frac{i}{2}\right) \sqrt{\pi}\, \sqrt{2}\, \mathrm{e}^{i} \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right) (1+x) \sqrt{2}\right) + 2i\mathrm{e}^{-i x^{2}} \\
& - i\mathrm{e}^{-i x (x-2)} - i\mathrm{e}^{-i x (x+2)} + (-1+i) \sqrt{\pi}\, \sqrt{2}\, \mathrm{e}^{i} \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right) \sqrt{2}\right)
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = {} & \left(\frac{1}{8}-\frac{i}{8}\right) \left(\sqrt{\pi}\, \mathrm{e}^{-i} \operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right) (x-1) \sqrt{2}\right) - \sqrt{\pi}\, \mathrm{e}^{-i} \operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right) (1+x) \sqrt{2}\right)\right. \\
& + (1+i) \sqrt{2}\, \mathrm{e}^{i x^{2}} + \left(-\frac{1}{2}-\frac{i}{2}\right) \sqrt{2}\, \mathrm{e}^{i x (x-2)} + \left(-\frac{1}{2}-\frac{i}{2}\right) \sqrt{2}\, \mathrm{e}^{i x (x+2)} \\
& \left. + 2\sqrt{\pi}\, \mathrm{e}^{-i} \operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right) \sqrt{2}\right)\right) \sqrt{2}\, \mathrm{e}^{-i x^{2}} \\
& - \frac{i\left(\left(-\frac{1}{2}+\frac{i}{2}\right) \sqrt{\pi}\, \sqrt{2}\, \mathrm{e}^{i} \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right) (x-1) \sqrt{2}\right) + \left(\frac{1}{2}-\frac{i}{2}\right) \sqrt{\pi}\, \sqrt{2}\, \mathrm{e}^{i} \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right) (1+x) \sqrt{2}\right) + 2i\mathrm{e}^{-i x^{2}}\right.}{4}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
y_p(x) = {} & \left(\frac{1}{8}-\frac{i}{8}\right) \sqrt{\pi} \left(\operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right) (x-1) \sqrt{2}\right) - \operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right) (1+x) \sqrt{2}\right)\right. \\
& \left. + 2 \operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right) \sqrt{2}\right)\right) \sqrt{2}\, \mathrm{e}^{-i x^{2}-i} \\
& + \left(\frac{1}{8}+\frac{i}{8}\right) \sqrt{\pi}\, \sqrt{2} \left(\operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right) (x-1) \sqrt{2}\right) - \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right) (1+x) \sqrt{2}\right)\right. \\
& \left. + 2 \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right) \sqrt{2}\right)\right) \mathrm{e}^{i x^{2}+i} - \cos(2x) + 1
\end{aligned}
$$

Therefore the general solution is

$$
y = y_h + y_p
$$

$$
\begin{aligned}
&= \left(c_1 e^{-i x^2} - \frac{i c_2 e^{i x^2}}{4}\right. \\
&\quad + \left(\left(\frac{1}{8}-\frac{i}{8}\right)\sqrt{\pi}\left(\operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)(x-1)\sqrt{2}\right) - \operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)(1+x)\sqrt{2}\right)\right.\right. \\
&\quad \left.\left. + 2\operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{2}\right)\right)\sqrt{2}\, e^{-i x^2 - i} + \left(\frac{1}{8}+\frac{i}{8}\right)\sqrt{\pi}\sqrt{2}\left(\operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)(x-1)\sqrt{2}\right)\right.\right. \\
&\quad \left.\left. - \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)(1+x)\sqrt{2}\right) + 2\operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2}\right)\right) e^{i x^2 + i} - \cos(2x) + 1\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 e^{-i x^2} - \frac{i c_2 e^{i x^2}}{4} + \left(\frac{1}{8}-\frac{i}{8}\right)\sqrt{\pi}\left(\operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)(x-1)\sqrt{2}\right)\right. \\
\left. - \operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)(1+x)\sqrt{2}\right) + 2\operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{2}\right)\right)\sqrt{2}\, e^{-i x^2 - i} \\
+ \left(\frac{1}{8}+\frac{i}{8}\right)\sqrt{\pi}\sqrt{2}\left(\operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)(x-1)\sqrt{2}\right) - \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)(1+x)\sqrt{2}\right)\right. \\
\left. + 2\operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2}\right)\right) e^{i x^2 + i} - \cos(2x) + 1
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 e^{-i x^2} - \frac{i c_2 e^{i x^2}}{4} + \left(\frac{1}{8}-\frac{i}{8}\right)\sqrt{\pi}\left(\operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)(x-1)\sqrt{2}\right)\right. \\
\left. - \operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)(1+x)\sqrt{2}\right) + 2\operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{2}\right)\right)\sqrt{2}\, e^{-i x^2 - i} \\
+ \left(\frac{1}{8}+\frac{i}{8}\right)\sqrt{\pi}\sqrt{2}\left(\operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)(x-1)\sqrt{2}\right) - \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)(1+x)\sqrt{2}\right)\right. \\
\left. + 2\operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2}\right)\right) e^{i x^2 + i} - \cos(2x) + 1
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 124

```
dsolve(x*diff(y(x),x$2)-diff(y(x),x)+4*x^3*y(x)=8*x^3*sin(x)^2,y(x), singsol=all)
```

$$y(x) = \sin(x^2) c_2 + \cos(x^2) c_1 + 1 - \cos(2x) - \frac{\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\sqrt{\pi}\,\sqrt{2}\,\sin(x^2+1)}{2} + \frac{\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\sqrt{\pi}\,\sqrt{2}\,\cos(x^2+1)}{2} + \frac{\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\sqrt{\pi}\,\sqrt{2}\,\sin(x^2+1)}{2} - \frac{\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\sqrt{\pi}\,\sqrt{2}\,\cos(x^2+1)}{2}$$

✓ Solution by Mathematica

Time used: 1.041 (sec). Leaf size: 147

```
DSolve[x*y''[x]-y'[x]+4*x^3*y[x]==8*x^3*Sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-\sqrt{2\pi}\,\text{FresnelC}\left(\sqrt{\frac{2}{\pi}}(x-1)\right)\sin\left(x^2+1\right) + \sqrt{2\pi}\,\text{FresnelC}\left(\sqrt{\frac{2}{\pi}}(x+1)\right)\sin\left(x^2+1\right) + \sqrt{2\pi}\,\text{FresnelS}\left(\sqrt{\frac{2}{\pi}}(x-1)\right)\cos\left(x^2+1\right) - \sqrt{2\pi}\,\text{FresnelS}\left(\sqrt{\frac{2}{\pi}}(x+1)\right)\cos\left(x^2+1\right) + 2c_1\cos\left(x^2\right) + 2c_2\sin\left(x^2\right) - 2\cos(2x) + 2\right)$$

2.24 problem 25

2.24.1 Solving as second order change of variable on x method 2 ode . 879
2.24.2 Solving as second order change of variable on x method 1 ode . 885
2.24.3 Solving as second order bessel ode ode 889
2.24.4 Solving using Kovacic algorithm 892

Internal problem ID [7465]
Internal file name [`OUTPUT/6432_Sunday_June_19_2022_05_02_15_PM_44874972/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{xy'' - y' + 4yx^3 = x^5}$$

2.24.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$xy'' - y' + 4yx^3 = 0$$

In normal form the ode

$$xy'' - y' + 4yx^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 4x^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x) dx\right)} dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{1}{x} dx\right)} dx \\ &= \int e^{\ln(x)}\, dx \\ &= \int x dx \\ &= \frac{x^2}{2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{4x^2}{x^2} \\ &= 4 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + 4y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} + 4\,\mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\lambda_1 = +2i$$
$$\lambda_2 = -2i$$

Which simplifies to

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha \tau}(c_1 \cos(\beta \tau) + c_2 \sin(\beta \tau))$$

Which becomes

$$y(\tau) = e^{0}(c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x^2) + c_2 \sin(x^2)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x^2)$$
$$y_2 = \sin(x^2)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x^2) & \sin(x^2) \\ \frac{d}{dx}(\cos(x^2)) & \frac{d}{dx}(\sin(x^2)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x^2) & \sin(x^2) \\ -2x\sin(x^2) & 2x\cos(x^2) \end{vmatrix}$$

Therefore

$$W = (\cos(x^2))(2x\cos(x^2)) - (\sin(x^2))(-2x\sin(x^2))$$

Which simplifies to

$$W = 2\cos(x^2)^2 x + 2\sin(x^2)^2 x$$

Which simplifies to

$$W = 2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x^2)\, x^5}{2x^2}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(x^2)\, x^3}{2} dx$$

Hence

$$u_1 = -\frac{\sin(x^2)}{4} + \frac{\cos(x^2)\, x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x^2)\, x^5}{2x^2}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(x^2)\, x^3}{2} dx$$

Hence

$$u_2 = \frac{\cos(x^2)}{4} + \frac{\sin(x^2)\, x^2}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\sin(x^2)}{4} + \frac{\cos(x^2)\, x^2}{4}\right) \cos(x^2) + \left(\frac{\cos(x^2)}{4} + \frac{\sin(x^2)\, x^2}{4}\right) \sin(x^2)$$

Which simplifies to

$$y_p(x) = \frac{x^2}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x^2) + c_2 \sin(x^2)) + \left(\frac{x^2}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x^2) + c_2 \sin(x^2) + \frac{x^2}{4} \tag{1}$$

Verification of solutions

$$y = c_1 \cos(x^2) + c_2 \sin(x^2) + \frac{x^2}{4}$$

Verified OK.

2.24.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x, B = -1, C = 4x^3, f(x) = x^5$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$xy'' - y' + 4yx^3 = 0$$

In normal form the ode

$$xy'' - y' + 4yx^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 4x^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{x^2}}{c} \\ \tau'' &= \frac{2x}{c\sqrt{x^2}} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \\
&= \frac{\frac{2x}{c\sqrt{x^2}} - \frac{1}{x}\frac{2\sqrt{x^2}}{c}}{\left(\frac{2\sqrt{x^2}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos\left(c\tau\right) + c_2 \sin\left(c\tau\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{x^2}dx}{c} \\
&= \frac{x\sqrt{x^2}}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right)
$$

Now the particular solution to this ODE is found

$$
xy'' - y' + 4yx^3 = x^5
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(x^2\right)$$
$$y_2 = \sin\left(x^2\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(x^2\right) & \sin\left(x^2\right) \\ \frac{d}{dx}\left(\cos\left(x^2\right)\right) & \frac{d}{dx}\left(\sin\left(x^2\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(x^2\right) & \sin\left(x^2\right) \\ -2x\sin\left(x^2\right) & 2x\cos\left(x^2\right) \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(x^2\right)\right)\left(2x\cos\left(x^2\right)\right) - \left(\sin\left(x^2\right)\right)\left(-2x\sin\left(x^2\right)\right)$$

Which simplifies to

$$W = 2\cos\left(x^2\right)^2 x + 2\sin\left(x^2\right)^2 x$$

Which simplifies to

$$W = 2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin\left(x^2\right) x^5}{2x^2}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(x^2)\, x^3}{2} dx$$

Hence

$$u_1 = -\frac{\sin(x^2)}{4} + \frac{\cos(x^2)\, x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x^2)\, x^5}{2x^2} dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(x^2)\, x^3}{2} dx$$

Hence

$$u_2 = \frac{\cos(x^2)}{4} + \frac{\sin(x^2)\, x^2}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\sin(x^2)}{4} + \frac{\cos(x^2)\, x^2}{4}\right) \cos\left(x^2\right) + \left(\frac{\cos(x^2)}{4} + \frac{\sin(x^2)\, x^2}{4}\right) \sin\left(x^2\right)$$

Which simplifies to

$$y_p(x) = \frac{x^2}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right)\right) + \left(\frac{x^2}{4}\right) \\ &= c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right) + \frac{x^2}{4} \end{aligned}$$

Which simplifies to

$$y = c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right) + \frac{x^2}{4}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x^2) + c_2 \sin(x^2) + \frac{x^2}{4} \tag{1}$$

Verification of solutions

$$y = c_1 \cos(x^2) + c_2 \sin(x^2) + \frac{x^2}{4}$$

Verified OK.

2.24.3 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - xy' + 4yx^4 = x^6 \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2 y'' + xy' + (-n^2 + x^2) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) xy' + (\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta x^\gamma) + c_2 \operatorname{BesselY}(n, \beta x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= 1 \\ \beta &= 1 \\ n &= \frac{1}{2} \\ \gamma &= 2 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x \sqrt{2} \sin(x^2)}{\sqrt{\pi} \sqrt{x^2}} - \frac{c_2 x \sqrt{2} \cos(x^2)}{\sqrt{\pi} \sqrt{x^2}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x \sqrt{2}\, \sin(x^2)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x \sqrt{2}\, \cos(x^2)}{\sqrt{\pi}\, \sqrt{x^2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x^2)$$
$$y_2 = \sin(x^2)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x^2) & \sin(x^2) \\ \frac{d}{dx}(\cos(x^2)) & \frac{d}{dx}(\sin(x^2)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x^2) & \sin(x^2) \\ -2x\sin(x^2) & 2x\cos(x^2) \end{vmatrix}$$

Therefore

$$W = (\cos(x^2))(2x\cos(x^2)) - (\sin(x^2))(-2x\sin(x^2))$$

Which simplifies to

$$W = 2\cos\left(x^2\right)^2 x + 2\sin\left(x^2\right)^2 x$$

Which simplifies to

$$W = 2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin\left(x^2\right) x^6}{2x^3} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin\left(x^2\right) x^3}{2} dx$$

Hence

$$u_1 = -\frac{\sin\left(x^2\right)}{4} + \frac{\cos\left(x^2\right) x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(x^2\right) x^6}{2x^3} dx$$

Which simplifies to

$$u_2 = \int \frac{\cos\left(x^2\right) x^3}{2} dx$$

Hence

$$u_2 = \frac{\cos\left(x^2\right)}{4} + \frac{\sin\left(x^2\right) x^2}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\sin\left(x^2\right)}{4} + \frac{\cos\left(x^2\right) x^2}{4}\right)\cos\left(x^2\right) + \left(\frac{\cos\left(x^2\right)}{4} + \frac{\sin\left(x^2\right) x^2}{4}\right)\sin\left(x^2\right)$$

Which simplifies to

$$y_p(x) = \frac{x^2}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1 x\sqrt{2}\, \sin\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x\sqrt{2}\, \cos\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}}\right) + \left(\frac{x^2}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x\sqrt{2}\, \sin\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x\sqrt{2}\, \cos\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} + \frac{x^2}{4} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x\sqrt{2}\, \sin\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x\sqrt{2}\, \cos\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} + \frac{x^2}{4}
$$

Verified OK.

2.24.4 Solving using Kovacic algorithm

Writing the ode as

$$
x y'' - y' + 4 y x^3 = 0 \tag{1}
$$

$$
A y'' + B y' + C y = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x \\
B &= -1 \\
C &= 4x^3
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 85: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= 2ix
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-16x^4 + 3}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= (-4x^2) + \left(\frac{3}{4x^2}\right) \\
&= -4x^2 + \frac{3}{4x^2}
\end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 2ix \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &&= -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)(2ix) \\
&= -\frac{1}{2x} - 2ix \\
&= -\frac{1}{2x} - 2ix
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4 + 3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\
&= \frac{e^{-ix^2}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-1}{x} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{-ix^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{i\mathrm{e}^{2ix^2}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-ix^2}\right) + c_2 \left(\mathrm{e}^{-ix^2} \left(-\frac{i\mathrm{e}^{2ix^2}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
xy'' - y' + 4yx^3 = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-ix^2} - \frac{ic_2 \mathrm{e}^{ix^2}}{4}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-ix^2}$$

$$y_2 = -\frac{i e^{ix^2}}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-ix^2} & -\frac{i e^{ix^2}}{4} \\ \frac{d}{dx}\left(e^{-ix^2}\right) & \frac{d}{dx}\left(-\frac{i e^{ix^2}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-ix^2} & -\frac{i e^{ix^2}}{4} \\ -2ix\, e^{-ix^2} & \frac{x\, e^{ix^2}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-ix^2}\right)\left(\frac{x\, e^{ix^2}}{2}\right) - \left(-\frac{i e^{ix^2}}{4}\right)\left(-2ix\, e^{-ix^2}\right)$$

Which simplifies to

$$W = e^{-ix^2} x\, e^{ix^2}$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{-\frac{i e^{i x^2} x^5}{4}}{x^2} dx
$$

Which simplifies to

$$
u_1 = -\int -\frac{i e^{i x^2} x^3}{4} dx
$$

Hence

$$
u_1 = -\frac{(-x^2 - i)\, e^{i x^2}}{8} - \frac{i}{8}
$$

And Eq. (3) becomes

$$
u_2 = \int \frac{e^{-i x^2} x^5}{x^2} dx
$$

Which simplifies to

$$
u_2 = \int e^{-i x^2} x^3 dx
$$

Hence

$$
u_2 = -\frac{1}{2} + \frac{i e^{-i x^2} x^2}{2} + \frac{e^{-i x^2}}{2}
$$

Therefore the particular solution, from equation (1) is

$$
y_p(x) = \left(-\frac{(-x^2 - i)\, e^{i x^2}}{8} - \frac{i}{8}\right) e^{-i x^2} - \frac{i\left(-\frac{1}{2} + \frac{i e^{-i x^2} x^2}{2} + \frac{e^{-i x^2}}{2}\right) e^{i x^2}}{4}
$$

Which simplifies to

$$
y_p(x) = \frac{x^2}{4} - \frac{\sin(x^2)}{4}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-i x^2} - \frac{i c_2 e^{i x^2}}{4}\right) + \left(\frac{x^2}{4} - \frac{\sin(x^2)}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-i x^2} - \frac{i c_2 e^{i x^2}}{4} + \frac{x^2}{4} - \frac{\sin(x^2)}{4} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-i x^2} - \frac{i c_2 e^{i x^2}}{4} + \frac{x^2}{4} - \frac{\sin(x^2)}{4}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(x*diff(y(x),x$2)-diff(y(x),x)+4*x^3*y(x)=x^5,y(x), singsol=all)
```

$$y(x) = \sin\left(x^2\right) c_2 + \cos\left(x^2\right) c_1 + \frac{x^2}{4}$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 27

```
DSolve[x*y''[x]-y'[x]+4*x^3*y[x]==x^5,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{4} + c_1 \cos(x^2) + c_2 \sin(x^2)$$

2.25 problem 25

2.25.1 Solving as second order change of variable on x method 2 ode . 903
2.25.2 Solving as second order change of variable on x method 1 ode . 908

Internal problem ID [7466]
Internal file name [`OUTPUT/6433_Sunday_June_19_2022_05_02_17_PM_82117291/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{\cos(x) y'' + y' \sin(x) - 2y \cos(x)^3 = 2 \cos(x)^5}$$

2.25.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$\cos(x) y'' + y' \sin(x) - 2y \cos(x)^3 = 0$$

In normal form the ode

$$\cos(x) y'' + y' \sin(x) - 2y \cos(x)^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$p(x) = \frac{\sin(x)}{\cos(x)}$$
$$q(x) = -2\cos(x)^2$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$
$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int e^{-\left(\int p(x)dx\right)}\,dx \\ &= \int e^{-\left(\int \frac{\sin(x)}{\cos(x)}dx\right)}\,dx \\ &= \int e^{\ln(\cos(x))}\,dx \\ &= \int \cos(x)\,dx \\ &= \sin(x)\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-2\cos(x)^2}{\cos(x)^2} \\ &= -2\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 2y(\tau) &= 0
\end{aligned}
$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Ay''(\tau) + By'(\tau) + Cy(\tau) = 0
$$

Where in the above $A = 1, B = 0, C = -2$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda\tau} - 2\,\mathrm{e}^{\lambda\tau} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$
\lambda^2 - 2 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = 0, C = -2$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-2)} \\
&= \pm\sqrt{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= +\sqrt{2} \\
\lambda_2 &= -\sqrt{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= \sqrt{2} \\
\lambda_2 &= -\sqrt{2}
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\
y(\tau) &= c_1 e^{\left(\sqrt{2}\right)\tau} + c_2 e^{\left(-\sqrt{2}\right)\tau}
\end{aligned}
$$

Or

$$y(\tau) = c_1 e^{\sqrt{2}\,\tau} + c_2 e^{-\sqrt{2}\,\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 e^{\sqrt{2}\,\sin(x)} + c_2 e^{-\sqrt{2}\,\sin(x)}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{\sqrt{2}\,\sin(x)} + c_2 e^{-\sqrt{2}\,\sin(x)}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-\sqrt{2}\,\sin(x)}$$
$$y_2 = e^{\sqrt{2}\,\sin(x)}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-\sqrt{2}\,\sin(x)} & e^{\sqrt{2}\,\sin(x)} \\ \frac{d}{dx}\left(e^{-\sqrt{2}\,\sin(x)}\right) & \frac{d}{dx}\left(e^{\sqrt{2}\,\sin(x)}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-\sqrt{2}\,\sin(x)} & e^{\sqrt{2}\,\sin(x)} \\ -e^{-\sqrt{2}\,\sin(x)}\sqrt{2}\,\cos(x) & \sqrt{2}\,\cos(x)\,e^{\sqrt{2}\,\sin(x)} \end{vmatrix}$$

Therefore

$$W = \left(e^{-\sqrt{2}\,\sin(x)}\right)\left(\sqrt{2}\,\cos(x)\,e^{\sqrt{2}\,\sin(x)}\right) - \left(e^{\sqrt{2}\,\sin(x)}\right)\left(-e^{-\sqrt{2}\,\sin(x)}\sqrt{2}\,\cos(x)\right)$$

Which simplifies to

$$W = 2\sqrt{2}\,\cos(x)$$

Which simplifies to

$$W = 2\sqrt{2}\,\cos(x)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,e^{\sqrt{2}\,\sin(x)}\cos(x)^5}{2\cos(x)^2\sqrt{2}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{\sqrt{2}\,\sin(x)}\cos(x)^3\sqrt{2}}{2}dx$$

Hence

$$u_1 = \frac{e^{\sqrt{2}\,\sin(x)}\sin(x)^2}{2} - \frac{\sqrt{2}\,\sin(x)\,e^{\sqrt{2}\,\sin(x)}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\,e^{-\sqrt{2}\,\sin(x)}\cos(x)^5}{2\cos(x)^2\sqrt{2}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{e^{-\sqrt{2}\,\sin(x)}\cos(x)^3\sqrt{2}}{2}dx$$

Hence

$$u_2 = \frac{e^{-\sqrt{2}\,\sin(x)}\sin(x)^2}{2} + \frac{\sqrt{2}\,\sin(x)\,e^{-\sqrt{2}\,\sin(x)}}{2}$$

Which simplifies to

$$
\begin{aligned}
u_1 &= \frac{e^{\sqrt{2}\,\sin(x)} \sin(x) \left(\sin(x) - \sqrt{2}\right)}{2} \\
u_2 &= \frac{e^{-\sqrt{2}\,\sin(x)} \sin(x) \left(\sin(x) + \sqrt{2}\right)}{2}
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
y_p(x) = \frac{\left(\sin(x) - \sqrt{2}\right) \sin(x)}{2} + \frac{e^{-\sqrt{2}\,\sin(x)} \sin(x) \left(\sin(x) + \sqrt{2}\right) e^{\sqrt{2}\,\sin(x)}}{2}
$$

Which simplifies to

$$
y_p(x) = \sin(x)^2
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{\sqrt{2}\,\sin(x)} + c_2 e^{-\sqrt{2}\,\sin(x)}\right) + \left(\sin(x)^2\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{\sqrt{2}\,\sin(x)} + c_2 e^{-\sqrt{2}\,\sin(x)} + \sin(x)^2 \tag{1}
$$

Verification of solutions

$$
y = c_1 e^{\sqrt{2}\,\sin(x)} + c_2 e^{-\sqrt{2}\,\sin(x)} + \sin(x)^2
$$

Verified OK.

2.25.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = \cos(x)\,, B = \sin(x)\,, C = -2\cos(x)^3\,, f(x) = 2\cos(x)^5$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$\cos(x)\, y'' + y' \sin(x) - 2y \cos(x)^3 = 0$$

In normal form the ode

$$\cos(x)\, y'' + y' \sin(x) - 2y \cos(x)^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= -2\cos(x)^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-2\cos(x)^2}}{c} \\ \tau'' &= \frac{2\sin(x)\cos(x)}{c\sqrt{-2\cos(x)^2}} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{2\sin(x)\cos(x)}{c\sqrt{-2\cos(x)^2}} + \tan(x)\frac{\sqrt{-2\cos(x)^2}}{c}}{\left(\frac{\sqrt{-2\cos(x)^2}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c} \sqrt{q}\, dx \\ &= \frac{\int \sqrt{-2\cos(x)^2} dx}{c} \\ &= -\frac{2\sin(x)\cos(x)}{c\sqrt{-2\cos(x)^2}} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cosh\left(\sqrt{2}\sin(x)\right) + i c_2 \sinh\left(\sqrt{2}\sin(x)\right)$$

Now the particular solution to this ODE is found

$$\cos(x) y'' + y' \sin(x) - 2y\cos(x)^3 = 2\cos(x)^5$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= e^{-\sqrt{2}\sin(x)} \\ y_2 &= e^{\sqrt{2}\sin(x)} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-\sqrt{2}\,\sin(x)} & e^{\sqrt{2}\,\sin(x)} \\ \frac{d}{dx}\left(e^{-\sqrt{2}\,\sin(x)}\right) & \frac{d}{dx}\left(e^{\sqrt{2}\,\sin(x)}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-\sqrt{2}\,\sin(x)} & e^{\sqrt{2}\,\sin(x)} \\ -e^{-\sqrt{2}\,\sin(x)}\sqrt{2}\,\cos(x) & \sqrt{2}\,\cos(x)\,e^{\sqrt{2}\,\sin(x)} \end{vmatrix}$$

Therefore

$$W = \left(e^{-\sqrt{2}\,\sin(x)}\right)\left(\sqrt{2}\,\cos(x)\,e^{\sqrt{2}\,\sin(x)}\right) - \left(e^{\sqrt{2}\,\sin(x)}\right)\left(-e^{-\sqrt{2}\,\sin(x)}\sqrt{2}\,\cos(x)\right)$$

Which simplifies to

$$W = 2\sqrt{2}\,\cos(x)$$

Which simplifies to

$$W = 2\sqrt{2}\,\cos(x)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,e^{\sqrt{2}\,\sin(x)}\cos(x)^5}{2\cos(x)^2\sqrt{2}}dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{\sqrt{2}\,\sin(x)}\cos(x)^3\sqrt{2}}{2}dx$$

Hence

$$u_1 = \frac{e^{\sqrt{2}\,\sin(x)} \sin\left(x\right)^{2}}{2}-\frac{\sqrt{2}\,\sin\left(x\right) e^{\sqrt{2}\,\sin(x)}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\,e^{-\sqrt{2}\,\sin(x)} \cos\left(x\right)^{5}}{2 \cos\left(x\right)^{2} \sqrt{2}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{e^{-\sqrt{2}\,\sin(x)} \cos\left(x\right)^{3} \sqrt{2}}{2} dx$$

Hence

$$u_2 = \frac{e^{-\sqrt{2}\,\sin(x)} \sin\left(x\right)^{2}}{2}+\frac{\sqrt{2}\,\sin\left(x\right) e^{-\sqrt{2}\,\sin(x)}}{2}$$

Which simplifies to

$$\begin{aligned} u_1 &= \frac{e^{\sqrt{2}\,\sin(x)} \sin\left(x\right) \left(\sin\left(x\right)-\sqrt{2}\right)}{2}\\ u_2 &= \frac{e^{-\sqrt{2}\,\sin(x)} \sin\left(x\right) \left(\sin\left(x\right)+\sqrt{2}\right)}{2} \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\sin\left(x\right)-\sqrt{2}\right) \sin\left(x\right)}{2}+\frac{e^{-\sqrt{2}\,\sin(x)} \sin\left(x\right) \left(\sin\left(x\right)+\sqrt{2}\right) e^{\sqrt{2}\,\sin(x)}}{2}$$

Which simplifies to

$$y_p(x) = \sin\left(x\right)^{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cosh\left(\sqrt{2}\,\sin\left(x\right)\right)+i c_2 \sinh\left(\sqrt{2}\,\sin\left(x\right)\right)\right) + \left(\sin\left(x\right)^{2}\right) \\ &= \sin\left(x\right)^{2}+c_1 \cosh\left(\sqrt{2}\,\sin\left(x\right)\right)+i c_2 \sinh\left(\sqrt{2}\,\sin\left(x\right)\right) \end{aligned}$$

Which simplifies to

$$y = \sin(x)^2 + c_1 \cosh\left(\sqrt{2}\, \sin(x)\right) + i c_2 \sinh\left(\sqrt{2}\, \sin(x)\right)$$

Summary

The solution(s) found are the following

$$y = \sin(x)^2 + c_1 \cosh\left(\sqrt{2}\, \sin(x)\right) + i c_2 \sinh\left(\sqrt{2}\, \sin(x)\right) \tag{1}$$

Verification of solutions

$$y = \sin(x)^2 + c_1 \cosh\left(\sqrt{2}\, \sin(x)\right) + i c_2 \sinh\left(\sqrt{2}\, \sin(x)\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(cos(x)*diff(y(x),x$2)+sin(x)*diff(y(x),x)-2*y(x)*cos(x)^3=2*cos(x)^5,y(x), singsol=al
```

$$y(x) = \sinh\left(\sin(x)\sqrt{2}\right) c_2 + \cosh\left(\sin(x)\sqrt{2}\right) c_1 + \frac{1}{2} - \frac{\cos(2x)}{2}$$

✓ Solution by Mathematica

Time used: 17.301 (sec). Leaf size: 167

```
DSolve[Cos[x]*y''[x]+Sin[x]*y'[x]-2*y[x]*Cos[x]^3==2*Cos[x]^5,y[x],x,IncludeSingularSolution
```

$$
\begin{aligned}
y(x) & \\
\rightarrow & \cos\left(\sqrt{-\cos(2x)-1}\tan(x)\right)\int_1^x \cos^2(K[1])\sqrt{-\cos(2K[1])-1}\sin\left(\sqrt{-\cos(2K[1])-1}\tan(K[1])\right) dK \\
& + \sin\left(\sqrt{-\cos(2x)-1}\tan(x)\right)\int_1^x \\
& - \cos^2(K[2])\sqrt{-\cos(2K[2])-1}\cos\left(\sqrt{-\cos(2K[2])-1}\tan(K[2])\right) dK[2] \\
& + c_1\cos\left(\sqrt{-\cos(2x)-1}\tan(x)\right) + c_2\sin\left(\sqrt{-\cos(2x)-1}\tan(x)\right)
\end{aligned}
$$

2.26 problem 26

2.26.1 Solving as second order change of variable on x method 2 ode . 915
2.26.2 Solving as second order change of variable on x method 1 ode . 922

Internal problem ID [7467]
Internal file name [`OUTPUT/6434_Sunday_June_19_2022_05_02_20_PM_32081568/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''+\left(1-\frac{1}{x}\right)y'+4x^2y\,\mathrm{e}^{-2x}=4\left(x^3+x^2\right)\mathrm{e}^{-3x}}$$

2.26.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+\frac{(x-1)\,y'}{x}+4x^2y\,\mathrm{e}^{-2x}=0$$

In normal form the ode

$$y''+\frac{(x-1)\,y'}{x}+4x^2y\,\mathrm{e}^{-2x}=0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= 1 - \frac{1}{x}\\ q(x) &= 4x^2 \mathrm{e}^{-2x}\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx\\ &= \int \mathrm{e}^{-\left(\int \left(1-\frac{1}{x}\right)dx\right)}dx\\ &= \int e^{-x+\ln(x)}\,dx\\ &= \int x\,\mathrm{e}^{-x}dx\\ &= -(1+x)\,\mathrm{e}^{-x}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2}\\ &= \frac{4x^2\mathrm{e}^{-2x}}{x^2\mathrm{e}^{-2x}}\\ &= 4\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + 4y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + 4\, e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\lambda_1 = +2i$$
$$\lambda_2 = -2i$$

Which simplifies to

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^{0}(c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos\left(2(1+x)\,\mathrm{e}^{-x}\right) - c_2 \sin\left(2(1+x)\,\mathrm{e}^{-x}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(2(1+x)\,\mathrm{e}^{-x}\right) - c_2 \sin\left(2(1+x)\,\mathrm{e}^{-x}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= -4\cos\left(\mathrm{e}^{-x}\right)^2 \sin\left(x\,\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right) - 4\sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right)^2 \\
&\quad + 2\sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right) + 2\sin\left(x\,\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right) \\
y_2 &= 4\cos\left(\mathrm{e}^{-x}\right)^2 \cos\left(x\,\mathrm{e}^{-x}\right)^2 - 2\cos\left(\mathrm{e}^{-x}\right)^2 - 2\cos\left(x\,\mathrm{e}^{-x}\right)^2 \\
&\quad + 1 - 4\sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right)\sin\left(x\,\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right)
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -4\cos\left(\mathrm{e}^{-x}\right)^2 \sin\left(x\,\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right) - 4\sin\left(\mathrm{e}^{-x}\right) \cos\left(\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right)^2 + 2\sin\left(\mathrm{e}^{-x}\right) \cos\left(\mathrm{e}^{-x}\right) + 2\,\mathrm{s} \\ \frac{d}{dx}\left(-4\cos\left(\mathrm{e}^{-x}\right)^2 \sin\left(x\,\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right) - 4\sin\left(\mathrm{e}^{-x}\right) \cos\left(\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right)^2 + 2\sin\left(\mathrm{e}^{-x}\right) \cos\left(\mathrm{e}^{-x}\right) + 2\right. \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \\ -8\cos\left(\mathrm{e}^{-x}\right) \sin\left(x\,\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right) \mathrm{e}^{-x} \sin\left(\mathrm{e}^{-x}\right) - 4\cos\left(\mathrm{e}^{-x}\right)^2 \left(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right)^2 + 4\cos\left(\mathrm{e}^{-x}\right. \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(-4\cos\left(\mathrm{e}^{-x}\right)^2 \sin\left(x\,\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right) - 4\sin\left(\mathrm{e}^{-x}\right) \cos\left(\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right)^2 \right. \\ & + 2\sin\left(\mathrm{e}^{-x}\right) \cos\left(\mathrm{e}^{-x}\right) \\ & \left. + 2\sin\left(x\,\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right)\right) \left(8\cos\left(\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right)^2 \mathrm{e}^{-x} \sin\left(\mathrm{e}^{-x}\right) \right. \\ & - 8\cos\left(\mathrm{e}^{-x}\right)^2 \cos\left(x\,\mathrm{e}^{-x}\right) \left(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}\right) \sin\left(x\,\mathrm{e}^{-x}\right) - 4\cos\left(\mathrm{e}^{-x}\right) \mathrm{e}^{-x} \sin\left(\mathrm{e}^{-x}\right) \\ & + 4\cos\left(x\,\mathrm{e}^{-x}\right) \left(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}\right) \sin\left(x\,\mathrm{e}^{-x}\right) + 4\,\mathrm{e}^{-x} \cos\left(\mathrm{e}^{-x}\right)^2 \sin\left(x\,\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right) \\ & - 4\sin\left(\mathrm{e}^{-x}\right)^2 \mathrm{e}^{-x} \sin\left(x\,\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right) \\ & - 4\sin\left(\mathrm{e}^{-x}\right) \cos\left(\mathrm{e}^{-x}\right) \left(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right)^2 \\ & \left. + 4\sin\left(\mathrm{e}^{-x}\right) \cos\left(\mathrm{e}^{-x}\right) \sin\left(x\,\mathrm{e}^{-x}\right)^2 \left(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}\right)\right) \\ & - \left(4\cos\left(\mathrm{e}^{-x}\right)^2 \cos\left(x\,\mathrm{e}^{-x}\right)^2 - 2\cos\left(\mathrm{e}^{-x}\right)^2 - 2\cos\left(x\,\mathrm{e}^{-x}\right)^2 + 1 \right. \\ & \left. -4\sin\left(\mathrm{e}^{-x}\right) \cos\left(\mathrm{e}^{-x}\right) \sin\left(x\,\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right)\right) \left(-8\cos\left(\mathrm{e}^{-x}\right) \sin\left(x\,\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right) \mathrm{e}^{-x} \sin\left(\mathrm{e}^{-x}\right) \right. \\ & - 4\cos\left(\mathrm{e}^{-x}\right)^2 \left(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right)^2 + 4\cos\left(\mathrm{e}^{-x}\right)^2 \sin\left(x\,\mathrm{e}^{-x}\right)^2 \left(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}\right) \\ & + 4\,\mathrm{e}^{-x} \cos\left(\mathrm{e}^{-x}\right)^2 \cos\left(x\,\mathrm{e}^{-x}\right)^2 - 4\sin\left(\mathrm{e}^{-x}\right)^2 \mathrm{e}^{-x} \cos\left(x\,\mathrm{e}^{-x}\right)^2 \\ & + 8\sin\left(\mathrm{e}^{-x}\right) \cos\left(\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right) \left(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}\right) \sin\left(x\,\mathrm{e}^{-x}\right) - 2\,\mathrm{e}^{-x} \cos\left(\mathrm{e}^{-x}\right)^2 \\ & \left. + 2\sin\left(\mathrm{e}^{-x}\right)^2 \mathrm{e}^{-x} + 2\left(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}\right) \cos\left(x\,\mathrm{e}^{-x}\right)^2 - 2\sin\left(x\,\mathrm{e}^{-x}\right)^2 \left(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}\right)\right) \end{aligned}$$

Which simplifies to

$$W = \text{Expression too large to display}$$

Which simplifies to

$$W = -2x\,\mathrm{e}^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\Big(4\cos\left(\mathrm{e}^{-x}\right)^2 \cos\left(x\,\mathrm{e}^{-x}\right)^2 - 2\cos\left(\mathrm{e}^{-x}\right)^2 - 2\cos\left(x\,\mathrm{e}^{-x}\right)^2 + 1 - 4\sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right)\sin\left(x\,\mathrm{e}^{-x}\right)\cos\,(}{-2x\,\mathrm{e}^{-x}}$$

Which simplifies to

$$u_1 = -\int -2x(1+x)\,\mathrm{e}^{-2x}\left(\cos\left(2\,\mathrm{e}^{-x}\right)\cos\left(2x\,\mathrm{e}^{-x}\right) - \sin\left(2\,\mathrm{e}^{-x}\right)\sin\left(2x\,\mathrm{e}^{-x}\right)\right)dx$$

Hence

$$u_1 = -\frac{i(1+x)\,\mathrm{e}^{2i\mathrm{e}^{-x}(-x-1)-x}}{2} - \frac{i(-x-1)\,\mathrm{e}^{2i(1+x)\mathrm{e}^{-x}-x}}{2} + \frac{\cos(2)}{2} + \sin(2) - \frac{\cos\left(2(1+x)\,\mathrm{e}^{-x}\right)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\Big(-4\cos\left(\mathrm{e}^{-x}\right)^2 \sin\left(x\,\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right) - 4\sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right)^2 + 2\sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right) + 2\sin}{-2x\,\mathrm{e}^{-x}}$$

Which simplifies to

$$u_2 = \int 2x(1+x)\,\mathrm{e}^{-2x}\left(\cos\left(2\,\mathrm{e}^{-x}\right)\sin\left(2x\,\mathrm{e}^{-x}\right) + \sin\left(2\,\mathrm{e}^{-x}\right)\cos\left(2x\,\mathrm{e}^{-x}\right)\right)dx$$

Hence

$$u_2 = \frac{(1+x)\,\mathrm{e}^{2i\mathrm{e}^{-x}(-x-1)-x}}{2} + \frac{(1+x)\,\mathrm{e}^{2i(1+x)\mathrm{e}^{-x}-x}}{2} - \cos(2) + \frac{\sin(2)}{2} - \frac{\sin\left(2(1+x)\,\mathrm{e}^{-x}\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = \Bigg(&-\frac{i(1+x)\,\mathrm{e}^{2i\mathrm{e}^{-x}(-x-1)-x}}{2} - \frac{i(-x-1)\,\mathrm{e}^{2i(1+x)\mathrm{e}^{-x}-x}}{2} + \frac{\cos(2)}{2} + \sin(2) \\
&- \frac{\cos(2(1+x)\,\mathrm{e}^{-x})}{2}\Bigg)\Big(-4\cos(\mathrm{e}^{-x})^2 \sin(x\,\mathrm{e}^{-x})\cos(x\,\mathrm{e}^{-x}) \\
&- 4\sin(\mathrm{e}^{-x})\cos(\mathrm{e}^{-x})\cos(x\,\mathrm{e}^{-x})^2 + 2\sin(\mathrm{e}^{-x})\cos(\mathrm{e}^{-x}) \\
&+ 2\sin(x\,\mathrm{e}^{-x})\cos(x\,\mathrm{e}^{-x})\Big) + \Bigg(\frac{(1+x)\,\mathrm{e}^{2i\mathrm{e}^{-x}(-x-1)-x}}{2} + \frac{(1+x)\,\mathrm{e}^{2i(1+x)\mathrm{e}^{-x}-x}}{2} \\
&- \cos(2) + \frac{\sin(2)}{2} - \frac{\sin(2(1+x)\,\mathrm{e}^{-x})}{2}\Bigg)\Big(4\cos(\mathrm{e}^{-x})^2\cos(x\,\mathrm{e}^{-x})^2 \\
&- 2\cos(\mathrm{e}^{-x})^2 - 2\cos(x\,\mathrm{e}^{-x})^2 + 1 - 4\sin(\mathrm{e}^{-x})\cos(\mathrm{e}^{-x})\sin(x\,\mathrm{e}^{-x})\cos(x\,\mathrm{e}^{-x})\Big)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
y_p(x) = &\frac{(-2\cos(2)+\sin(2))\cos(2(1+x)\,\mathrm{e}^{-x})}{2} \\
&+ \frac{(-\cos(2)-2\sin(2))\sin(2(1+x)\,\mathrm{e}^{-x})}{2} + (1+x)\,\mathrm{e}^{-x}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \big(c_1\cos(2(1+x)\,\mathrm{e}^{-x}) - c_2\sin(2(1+x)\,\mathrm{e}^{-x})\big) \\
&\quad + \Bigg(\frac{(-2\cos(2)+\sin(2))\cos(2(1+x)\,\mathrm{e}^{-x})}{2} \\
&\quad + \frac{(-\cos(2)-2\sin(2))\sin(2(1+x)\,\mathrm{e}^{-x})}{2} + (1+x)\,\mathrm{e}^{-x}\Bigg)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &c_1\cos(2(1+x)\,\mathrm{e}^{-x}) - c_2\sin(2(1+x)\,\mathrm{e}^{-x}) \\
&+ \frac{(-2\cos(2)+\sin(2))\cos(2(1+x)\,\mathrm{e}^{-x})}{2} \\
&+ \frac{(-\cos(2)-2\sin(2))\sin(2(1+x)\,\mathrm{e}^{-x})}{2} + (1+x)\,\mathrm{e}^{-x}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y = c_1 \cos\left(2(1+x)\,\mathrm{e}^{-x}\right) - c_2 \sin\left(2(1+x)\,\mathrm{e}^{-x}\right) + \frac{(-2\cos(2)+\sin(2))\cos(2(1+x)\,\mathrm{e}^{-x})}{2} + \frac{(-\cos(2)-2\sin(2))\sin(2(1+x)\,\mathrm{e}^{-x})}{2} + (1+x)\,\mathrm{e}^{-x}$$

Verified OK.

2.26.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = \frac{x-1}{x}, C = 4x^2\mathrm{e}^{-2x}, f(x) = 4x^2(1+x)\,\mathrm{e}^{-3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$y'' + \frac{(x-1)\,y'}{x} + 4x^2y\,\mathrm{e}^{-2x} = 0$$

In normal form the ode

$$y'' + \frac{(x-1)\,y'}{x} + 4x^2y\,\mathrm{e}^{-2x} = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x-1}{x} \\ q(x) &= 4x^2\mathrm{e}^{-2x} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{2\sqrt{x^2 \mathrm{e}^{-2x}}}{c}\\ \tau'' &= \frac{2x\,\mathrm{e}^{-2x} - 2x^2\mathrm{e}^{-2x}}{c\sqrt{x^2\mathrm{e}^{-2x}}}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2}\\ &= \frac{\frac{2x\,\mathrm{e}^{-2x} - 2x^2\mathrm{e}^{-2x}}{c\sqrt{x^2\mathrm{e}^{-2x}}} + \frac{x-1}{x}\frac{2\sqrt{x^2\mathrm{e}^{-2x}}}{c}}{\left(\frac{2\sqrt{x^2\mathrm{e}^{-2x}}}{c}\right)^2}\\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0\\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx\\ &= \frac{\int 2\sqrt{x^2\mathrm{e}^{-2x}}dx}{c}\\ &= -\frac{2(1+x)\sqrt{x^2\mathrm{e}^{-2x}}}{cx}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(2(1+x)\,\mathrm{e}^{-x}\right) - c_2 \sin\left(2(1+x)\,\mathrm{e}^{-x}\right)$$

Now the particular solution to this ODE is found

$$y'' + \frac{(x-1)\,y'}{x} + 4x^2 y\,\mathrm{e}^{-2x} = 4x^2(1+x)\,\mathrm{e}^{-3x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= -4\cos\left(\mathrm{e}^{-x}\right)^2 \sin\left(x\,\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right) - 4\sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right)^2 \\ &\quad + 2\sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right) + 2\sin\left(x\,\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right) \end{aligned}$$

$$\begin{aligned} y_2 &= 4\cos\left(\mathrm{e}^{-x}\right)^2 \cos\left(x\,\mathrm{e}^{-x}\right)^2 - 2\cos\left(\mathrm{e}^{-x}\right)^2 - 2\cos\left(x\,\mathrm{e}^{-x}\right)^2 \\ &\quad + 1 - 4\sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right)\sin\left(x\,\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -4\cos(\mathrm{e}^{-x})^2 \sin(x\,\mathrm{e}^{-x})\cos(x\,\mathrm{e}^{-x}) - 4\sin(\mathrm{e}^{-x})\cos(\mathrm{e}^{-x})\cos(x\,\mathrm{e}^{-x})^2 + 2\sin(\mathrm{e}^{-x})\cos(\mathrm{e}^{-x}) + 2\mathrm{s} \\ \frac{d}{dx}\left(-4\cos(\mathrm{e}^{-x})^2 \sin(x\,\mathrm{e}^{-x})\cos(x\,\mathrm{e}^{-x}) - 4\sin(\mathrm{e}^{-x})\cos(\mathrm{e}^{-x})\cos(x\,\mathrm{e}^{-x})^2 + 2\sin(\mathrm{e}^{-x})\cos(\mathrm{e}^{-x}) + 2\right. \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \\ -8\cos(\mathrm{e}^{-x})\sin(x\,\mathrm{e}^{-x})\cos(x\,\mathrm{e}^{-x})\,\mathrm{e}^{-x}\sin(\mathrm{e}^{-x}) - 4\cos(\mathrm{e}^{-x})^2(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x})\cos(x\,\mathrm{e}^{-x})^2 + 4\cos(\mathrm{e}^{-x} \end{vmatrix}$$

Therefore

$$
\begin{aligned}
W = \Big(&-4\cos\left(e^{-x}\right)^2 \sin\left(x\,e^{-x}\right)\cos\left(x\,e^{-x}\right) - 4\sin\left(e^{-x}\right)\cos\left(e^{-x}\right)\cos\left(x\,e^{-x}\right)^2 \\
&+ 2\sin\left(e^{-x}\right)\cos\left(e^{-x}\right) \\
&+ 2\sin\left(x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)\Big)\Big(8\cos\left(e^{-x}\right)\cos\left(x\,e^{-x}\right)^2 e^{-x}\sin\left(e^{-x}\right) \\
&- 8\cos\left(e^{-x}\right)^2\cos\left(x\,e^{-x}\right)\left(e^{-x} - x\,e^{-x}\right)\sin\left(x\,e^{-x}\right) - 4\cos\left(e^{-x}\right)e^{-x}\sin\left(e^{-x}\right) \\
&+ 4\cos\left(x\,e^{-x}\right)\left(e^{-x} - x\,e^{-x}\right)\sin\left(x\,e^{-x}\right) + 4\,e^{-x}\cos\left(e^{-x}\right)^2\sin\left(x\,e^{-x}\right)\cos\left(x\,e^{-x}\right) \\
&- 4\sin\left(e^{-x}\right)^2 e^{-x}\sin\left(x\,e^{-x}\right)\cos\left(x\,e^{-x}\right) \\
&- 4\sin\left(e^{-x}\right)\cos\left(e^{-x}\right)\left(e^{-x} - x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)^2 \\
&+ 4\sin\left(e^{-x}\right)\cos\left(e^{-x}\right)\sin\left(x\,e^{-x}\right)^2\left(e^{-x} - x\,e^{-x}\right)\Big) \\
&- \Big(4\cos\left(e^{-x}\right)^2\cos\left(x\,e^{-x}\right)^2 - 2\cos\left(e^{-x}\right)^2 - 2\cos\left(x\,e^{-x}\right)^2 + 1 \\
&-4\sin\left(e^{-x}\right)\cos\left(e^{-x}\right)\sin\left(x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)\Big)\Big(-8\cos\left(e^{-x}\right)\sin\left(x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)e^{-x}\sin\left(e^{-x}\right) \\
&- 4\cos\left(e^{-x}\right)^2\left(e^{-x} - x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)^2 + 4\cos\left(e^{-x}\right)^2\sin\left(x\,e^{-x}\right)^2\left(e^{-x} - x\,e^{-x}\right) \\
&+ 4\,e^{-x}\cos\left(e^{-x}\right)^2\cos\left(x\,e^{-x}\right)^2 - 4\sin\left(e^{-x}\right)^2 e^{-x}\cos\left(x\,e^{-x}\right)^2 \\
&+ 8\sin\left(e^{-x}\right)\cos\left(e^{-x}\right)\cos\left(x\,e^{-x}\right)\left(e^{-x} - x\,e^{-x}\right)\sin\left(x\,e^{-x}\right) - 2\,e^{-x}\cos\left(e^{-x}\right)^2 \\
&+ 2\sin\left(e^{-x}\right)^2 e^{-x} + 2\left(e^{-x} - x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)^2 - 2\sin\left(x\,e^{-x}\right)^2\left(e^{-x} - x\,e^{-x}\right)\Big)
\end{aligned}
$$

Which simplifies to

$$
W = \text{Expression too large to display}
$$

Which simplifies to

$$
W = -2x\,e^{-x}
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{4\Big(4\cos\left(e^{-x}\right)^2\cos\left(x\,e^{-x}\right)^2 - 2\cos\left(e^{-x}\right)^2 - 2\cos\left(x\,e^{-x}\right)^2 + 1 - 4\sin\left(e^{-x}\right)\cos\left(e^{-x}\right)\sin\left(x\,e^{-x}\right)\cos\left(
}{-2x\,e^{-x}}
$$

Which simplifies to

$$
u_1 = -\int -2x(1+x)\,e^{-2x}\left(\cos\left(2\,e^{-x}\right)\cos\left(2x\,e^{-x}\right) - \sin\left(2\,e^{-x}\right)\sin\left(2x\,e^{-x}\right)\right)dx
$$

Hence

$$u_1 = -\frac{i(1+x)\,e^{2i e^{-x}(-x-1)-x}}{2} - \frac{i(-x-1)\,e^{2i(1+x)e^{-x}-x}}{2} + \frac{\cos(2)}{2} + \sin(2) - \frac{\cos(2(1+x)\,e^{-x})}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\Big(-4\cos(e^{-x})^2\sin(x\,e^{-x})\cos(x\,e^{-x}) - 4\sin(e^{-x})\cos(e^{-x})\cos(x\,e^{-x})^2 + 2\sin(e^{-x})\cos(e^{-x}) + 2\sin}{-2x\,e^{-x}}$$

Which simplifies to

$$u_2 = \int 2x(1+x)\,e^{-2x}\left(\cos\left(2\,e^{-x}\right)\sin\left(2x\,e^{-x}\right) + \sin\left(2\,e^{-x}\right)\cos\left(2x\,e^{-x}\right)\right)dx$$

Hence

$$u_2 = \frac{(1+x)\,e^{2i e^{-x}(-x-1)-x}}{2} + \frac{(1+x)\,e^{2i(1+x)e^{-x}-x}}{2} - \cos(2) + \frac{\sin(2)}{2} - \frac{\sin(2(1+x)\,e^{-x})}{2}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = \Bigg(&-\frac{i(1+x)\,e^{2i e^{-x}(-x-1)-x}}{2} - \frac{i(-x-1)\,e^{2i(1+x)e^{-x}-x}}{2} + \frac{\cos(2)}{2} + \sin(2) \\ &- \frac{\cos(2(1+x)\,e^{-x})}{2}\Bigg)\left(-4\cos\left(e^{-x}\right)^2\sin\left(x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)\right. \\ &- 4\sin\left(e^{-x}\right)\cos\left(e^{-x}\right)\cos\left(x\,e^{-x}\right)^2 + 2\sin\left(e^{-x}\right)\cos\left(e^{-x}\right) \\ &\left.+ 2\sin\left(x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)\right) + \Bigg(\frac{(1+x)\,e^{2i e^{-x}(-x-1)-x}}{2} + \frac{(1+x)\,e^{2i(1+x)e^{-x}-x}}{2} \\ &- \cos(2) + \frac{\sin(2)}{2} - \frac{\sin(2(1+x)\,e^{-x})}{2}\Bigg)\left(4\cos\left(e^{-x}\right)^2\cos\left(x\,e^{-x}\right)^2\right. \\ &\left.-2\cos\left(e^{-x}\right)^2 - 2\cos\left(x\,e^{-x}\right)^2 + 1 - 4\sin\left(e^{-x}\right)\cos\left(e^{-x}\right)\sin\left(x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)\right) \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{(-2\cos(2) + \sin(2))\cos(2(1+x)\,e^{-x})}{2} + \frac{(-\cos(2) - 2\sin(2))\sin(2(1+x)\,e^{-x})}{2} + (1+x)\,e^{-x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos\left(2(1+x)\,\mathrm{e}^{-x}\right) - c_2 \sin\left(2(1+x)\,\mathrm{e}^{-x}\right)\right) \\
&\quad + \left(\frac{(-2\cos(2)+\sin(2))\cos\left(2(1+x)\,\mathrm{e}^{-x}\right)}{2}\right. \\
&\quad \left. + \frac{(-\cos(2)-2\sin(2))\sin\left(2(1+x)\,\mathrm{e}^{-x}\right)}{2} + (1+x)\,\mathrm{e}^{-x}\right) \\
&= c_1 \cos\left(2(1+x)\,\mathrm{e}^{-x}\right) - c_2 \sin\left(2(1+x)\,\mathrm{e}^{-x}\right) + \frac{(-2\cos(2)+\sin(2))\cos\left(2(1+x)\,\mathrm{e}^{-x}\right)}{2} \\
&\quad + \frac{(-\cos(2)-2\sin(2))\sin\left(2(1+x)\,\mathrm{e}^{-x}\right)}{2} + (1+x)\,\mathrm{e}^{-x}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
y = {} & \frac{(2c_1 - 2\cos(2)+\sin(2))\cos\left(2(1+x)\,\mathrm{e}^{-x}\right)}{2} \\
& + \frac{(-2c_2 - \cos(2) - 2\sin(2))\sin\left(2(1+x)\,\mathrm{e}^{-x}\right)}{2} + (1+x)\,\mathrm{e}^{-x}
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{(2c_1 - 2\cos(2)+\sin(2))\cos\left(2(1+x)\,\mathrm{e}^{-x}\right)}{2} \\
& + \frac{(-2c_2 - \cos(2) - 2\sin(2))\sin\left(2(1+x)\,\mathrm{e}^{-x}\right)}{2} + (1+x)\,\mathrm{e}^{-x}
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y = {} & \frac{(2c_1 - 2\cos(2)+\sin(2))\cos\left(2(1+x)\,\mathrm{e}^{-x}\right)}{2} \\
& + \frac{(-2c_2 - \cos(2) - 2\sin(2))\sin\left(2(1+x)\,\mathrm{e}^{-x}\right)}{2} + (1+x)\,\mathrm{e}^{-x}
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$2)+(1-1/x)*diff(y(x),x)+4*x^2*y(x)*exp(-2*x)=4*(x^2+x^3)*exp(-3*x),y(x),
```

$$y(x) = \sin\left(2(x+1)\,\mathrm{e}^{-x}\right) c_2 + \cos\left(2(x+1)\,\mathrm{e}^{-x}\right) c_1 + \mathrm{e}^{-x}x + \mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.604 (sec). Leaf size: 47

```
DSolve[y''[x]+(1-1/x)*y'[x]+4*x^2*y[x]*Exp[-2*x]==4*(x^2+x^3)*Exp[-3*x],y[x],x,IncludeSingul
```

$$y(x) \to c_1 \cos\left(2e^{-x}(x+1)\right) + e^{-x}\left(x - c_2 e^x \sin\left(2e^{-x}(x+1)\right) + 1\right)$$

2.27 problem 27

2.27.1 Solving as second order change of variable on y method 2 ode . 929
2.27.2 Solving using Kovacic algorithm 936

Internal problem ID [7468]
Internal file name [`OUTPUT/6435_Sunday_June_19_2022_05_02_24_PM_60458086/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - x^2 y' + yx = x^{m+1}}$$

2.27.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -x^2, C = x, f(x) = x^{m+1}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$y'' - x^2 y' + yx = 0$$

In normal form the ode

$$y'' - x^2 y' + yx = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= -x^2\\ q(x) &= x\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - nx + x = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}v''(x) + \left(\frac{2}{x} - x^2\right) v'(x) &= 0\\ v''(x) + \frac{(-x^3+2)\,v'(x)}{x} &= 0\end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(-x^3+2)\,u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u(x^3-2)}{x}
\end{aligned}
$$

Where $f(x) = \frac{x^3-2}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{x^3-2}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{x^3-2}{x}\,dx \\
\ln(u) &= \frac{x^3}{3} - 2\ln(x) + c_1 \\
u &= \mathrm{e}^{\frac{x^3}{3} - 2\ln(x) + c_1} \\
&= c_1 \mathrm{e}^{\frac{x^3}{3} - 2\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_1 \mathrm{e}^{\frac{x^3}{3}}}{x^2}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= \frac{c_1 3^{\frac{2}{3}} (-1)^{\frac{1}{3}} \left(-\frac{3x^2(-1)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{\frac{2}{3}}} + \frac{3\,3^{\frac{1}{3}}(-1)^{\frac{2}{3}}\mathrm{e}^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{2}{3}}} \right)}{9} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(\frac{c_1 3^{\frac{2}{3}} (-1)^{\frac{1}{3}} \left(-\frac{3x^2(-1)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{\frac{2}{3}}} + \frac{3\,3^{\frac{1}{3}}(-1)^{\frac{2}{3}}\mathrm{e}^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{2}{3}}} \right)}{9} + c_2 \right) x \\
&= \frac{\left(-3c_1\mathrm{e}^{\frac{x^3}{3}} + 3c_2 x\right)(-x^3)^{\frac{2}{3}} + x^3 c_1 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3\,(-x^3)^{\frac{2}{3}}}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$y'' - x^2y' + yx = x^{m+1}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}\right)}{3\left(-x^3\right)^{\frac{2}{3}}} - \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3\left(-x^3\right)^{\frac{2}{3}}} - e^{\frac{x^3}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}\right)}{3\left(-x^3\right)^{\frac{2}{3}}} - \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3\left(-x^3\right)^{\frac{2}{3}}} - e^{\frac{x^3}{3}} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}\right)}{3\left(-x^3\right)^{\frac{2}{3}}} - \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3\left(-x^3\right)^{\frac{2}{3}}} - e^{\frac{x^3}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}\right)}{3(-x^3)^{\frac{2}{3}}} - \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3(-x^3)^{\frac{2}{3}}} - e^{\frac{x^3}{3}} \\ 1 & \frac{3^{\frac{2}{3}} x^2 \Gamma\left(\frac{2}{3}\right)}{(-x^3)^{\frac{2}{3}}} + \frac{2\, 3^{\frac{2}{3}} x^5 \Gamma\left(\frac{2}{3}\right)}{3(-x^3)^{\frac{5}{3}}} - \frac{3^{\frac{2}{3}} x^2 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{2}{3}}} - \frac{2\, 3^{\frac{2}{3}} x^5 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3(-x^3)^{\frac{5}{3}}} - \frac{3^{\frac{2}{3}} x^5 e^{\frac{x^3}{3}}}{3(-x^3)^{\frac{2}{3}} \left(-\frac{x^3}{3}\right)^{\frac{1}{3}}} - x^2 e^{\frac{x^3}{3}} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = (x) \Bigg(& \frac{3^{\frac{2}{3}} x^2 \Gamma\left(\frac{2}{3}\right)}{(-x^3)^{\frac{2}{3}}} + \frac{2\, 3^{\frac{2}{3}} x^5 \Gamma\left(\frac{2}{3}\right)}{3\left(-x^3\right)^{\frac{5}{3}}} - \frac{3^{\frac{2}{3}} x^2 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{2}{3}}} - \frac{2\, 3^{\frac{2}{3}} x^5 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3\left(-x^3\right)^{\frac{5}{3}}} \\ & - \frac{3^{\frac{2}{3}} x^5 e^{\frac{x^3}{3}}}{3\left(-x^3\right)^{\frac{2}{3}} \left(-\frac{x^3}{3}\right)^{\frac{1}{3}}} - x^2 e^{\frac{x^3}{3}} \Bigg) - \left(\frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}\right)}{3\left(-x^3\right)^{\frac{2}{3}}} - \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3\left(-x^3\right)^{\frac{2}{3}}} - e^{\frac{x^3}{3}} \right) (1) \end{aligned}$$

Which simplifies to

$$W = \frac{e^{\frac{x^3}{3}} \left(3^{\frac{2}{3}} x^9 - 3(-x^3)^{\frac{5}{3}} \left(-\frac{x^3}{3}\right)^{\frac{1}{3}} x^3 + 3(-x^3)^{\frac{5}{3}} \left(-\frac{x^3}{3}\right)^{\frac{1}{3}} \right)}{3\left(-x^3\right)^{\frac{5}{3}} \left(-\frac{x^3}{3}\right)^{\frac{1}{3}}}$$

Which simplifies to

$$W = e^{\frac{x^3}{3}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}\right)}{3(-x^3)^{\frac{2}{3}}} - \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3(-x^3)^{\frac{2}{3}}} - e^{\frac{x^3}{3}} \right) x^{m+1}}{e^{\frac{x^3}{3}}} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{m+1} \left(-3(-x^3)^{\frac{2}{3}} + x^3 3^{\frac{2}{3}} e^{-\frac{x^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) \right) \right)}{3\left(-x^3\right)^{\frac{2}{3}}} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{\alpha^{m+1} \left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right) \right) \right)}{3\left(-\alpha^3\right)^{\frac{2}{3}}} d\alpha \right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\,x^{m+1}}{e^{\frac{x^3}{3}}}\,dx$$

Which simplifies to

$$u_2 = \int x^{m+2} e^{-\frac{x^3}{3}}\,dx$$

Hence

$$u_2 = \frac{3^{\frac{m}{6}+1} x^m (x^3)^{-\frac{m}{6}} e^{-\frac{x^3}{6}} \operatorname{WhittakerM}\left(\frac{m}{6}, \frac{m}{6}+\frac{1}{2}, \frac{x^3}{3}\right)}{m+3}$$

Which simplifies to

$$u_1 = -\frac{\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{\frac{2}{3}}+\alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha\right)}{3}$$

$$u_2 = \frac{3^{\frac{m}{6}+1} x^m (x^3)^{-\frac{m}{6}} e^{-\frac{x^3}{6}} \operatorname{WhittakerM}\left(\frac{m}{6}, \frac{m}{6}+\frac{1}{2}, \frac{x^3}{3}\right)}{m+3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{\frac{2}{3}}+\alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha\right) x}{3}$$
$$+ \frac{3^{\frac{m}{6}+1} x^m (x^3)^{-\frac{m}{6}} e^{-\frac{x^3}{6}} \operatorname{WhittakerM}\left(\frac{m}{6}, \frac{m}{6}+\frac{1}{2}, \frac{x^3}{3}\right)\left(\frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}\right)}{3(-x^3)^{\frac{2}{3}}} - \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)}{3(-x^3)^{\frac{2}{3}}} - e^{\frac{x^3}{3}}\right)}{m+3}$$

Which simplifies to

$$y_p(x) =$$
$$-\frac{(-x^3)^{\frac{2}{3}} x(m+3)\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{\frac{2}{3}}+\alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha\right) + 9(x^3)^{-\frac{m}{6}}\left(\left((-x^3)^{\frac{2}{3}} e^{\frac{x^3}{6}} x^m 3^{\frac{m}{6}} +\right.\right.}{(-x^3)^{\frac{2}{3}}(9+3m)}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(\left(\frac{c_1 3^{\frac{2}{3}} (-1)^{\frac{1}{3}} \left(-\frac{3x^2(-1)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{\frac{2}{3}}} + \frac{3\,3^{\frac{1}{3}}(-1)^{\frac{2}{3}} e^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{2}{3}}}\right)}{9} + c_2\right) x\right)$$

$$+ \left(-\frac{(-x^3)^{\frac{2}{3}} x(m+3)\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha\right) + 9(x^3)^{-\frac{m}{6}}\left((-x^3)^{\frac{2}{3}} e^{\frac{x^3}{6}} x^m \cdots\right.}{(-x^3)^{\frac{2}{3}}(9+3m)}\right.$$

$$= -\frac{(-x^3)^{\frac{2}{3}} x(m+3)\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha\right) + 9(x^3)^{-\frac{m}{6}}\left((-x^3)^{\frac{2}{3}} e^{\frac{x^3}{6}} x^m 3^{\frac{m}{6}} \cdots\right.}{(-x^3)^{\frac{2}{3}}(9+3m)}$$

$$+ \left(\frac{c_1 3^{\frac{2}{3}} (-1)^{\frac{1}{3}} \left(-\frac{3x^2(-1)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{\frac{2}{3}}} + \frac{3\,3^{\frac{1}{3}}(-1)^{\frac{2}{3}} e^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{2}{3}}}\right)}{9} + c_2\right) x$$

Which simplifies to

$$y = -\frac{\left(9\,\text{WhittakerM}\left(\frac{m}{6}, \frac{m}{6} + \frac{1}{2}, \frac{x^3}{3}\right) 3^{\frac{m}{6}} (x^3)^{-\frac{m}{6}} x^m e^{\frac{x^3}{6}} + \left(\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} \cdots\right.\right.\right.}{\cdots}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(9\,\text{WhittakerM}\left(\frac{m}{6}, \frac{m}{6} + \frac{1}{2}, \frac{x^3}{3}\right) 3^{\frac{m}{6}} (x^3)^{-\frac{m}{6}} x^m e^{\frac{x^3}{6}} + \left(\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} \cdots\right.\right.\right.}{\cdots} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(9\,\mathrm{WhittakerM}\left(\frac{m}{6}, \frac{m}{6}+\frac{1}{2}, \frac{x^3}{3}\right) 3^{\frac{m}{6}} \left(x^3\right)^{-\frac{m}{6}} x^m \mathrm{e}^{\frac{x^3}{6}} + \left(\left(\int_0^x \frac{\alpha^{m+1}\left(-3\left(-\alpha^3\right)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} \mathrm{e}^{-\frac{\alpha^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{\left(-\alpha^3\right)^{\frac{2}{3}}}\right.\right.\right.}{}$$

Verified OK.

2.27.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - x^2y' + yx = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3 - 8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x\left(x^3 - 8\right) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3-8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 86: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} - \frac{2}{x} - \frac{4}{x^4} - \frac{16}{x^7} - \frac{80}{x^{10}} - \frac{448}{x^{13}} - \frac{2688}{x^{16}} - \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x(x^3-8)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^4 - 2x\right) + (0) \\
&= \frac{1}{4}x^4 - 2x
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -2. Now b can be found.

$$
\begin{aligned}
b &= (-2) - (0) \\
&= -2
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x^2}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 2\right) &= -3 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 2\right) &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x(x^3-8)}{4}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	-3	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-}=1$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c \in \Gamma}\left(s(c)[\sqrt{r}]_{c}+\frac{\alpha_{c}^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_{\infty} \\
&= 0+(-)\left(\frac{x^{2}}{2}\right) \\
&= -\frac{x^{2}}{2} \\
&= -\frac{x^{2}}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^{2}-r\right) p=0 \tag{1A}
$$

Let

$$
p(x)=x+a_{0} \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0)+2\left(-\frac{x^{2}}{2}\right)(1)+\left((-x)+\left(-\frac{x^{2}}{2}\right)^{2}-\left(\frac{x(x^{3}-8)}{4}\right)\right) &= 0 \\
x a_{0} &= 0
\end{aligned}
$$

Solving for the coefficients a_{i} in the above using method of undetermined coefficients gives

$$
\{a_{0}=0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x)=x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int -\frac{x^2}{2} dx} \\
&= (x)\, \mathrm{e}^{-\frac{x^3}{6}} \\
&= x\, \mathrm{e}^{-\frac{x^3}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{1} \, dx} \\
&= z_1 e^{\frac{x^3}{6}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^3}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-3(-x^3)^{\frac{2}{3}}\, \mathrm{e}^{\frac{x^3}{3}} + 3^{\frac{2}{3}} x^3 \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3\,(-x^3)^{\frac{2}{3}}\, x} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\frac{-3(-x^3)^{\frac{2}{3}}\,\mathrm{e}^{\frac{x^3}{3}} + 3^{\frac{2}{3}}x^3\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3\left(-x^3\right)^{\frac{2}{3}}x}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - x^2y' + yx = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 x + \frac{c_2\left(-3(-x^3)^{\frac{2}{3}}\,\mathrm{e}^{\frac{x^3}{3}} + 3^{\frac{2}{3}}x^3\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3\left(-x^3\right)^{\frac{2}{3}}}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \frac{-3(-x^3)^{\frac{2}{3}}\,\mathrm{e}^{\frac{x^3}{3}} + 3^{\frac{2}{3}}x^3\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3\left(-x^3\right)^{\frac{2}{3}}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + 3^{\frac{2}{3}} x^3 \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3(-x^3)^{\frac{2}{3}}} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + 3^{\frac{2}{3}} x^3 \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3(-x^3)^{\frac{2}{3}}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + 3^{\frac{2}{3}} x^3 \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3(-x^3)^{\frac{2}{3}}} \\ 1 & \frac{2\left(-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + 3^{\frac{2}{3}} x^3 \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right) x^2}{3(-x^3)^{\frac{5}{3}}} + \frac{\frac{6 e^{\frac{x^3}{3}} x^2}{(-x^3)^{\frac{1}{3}}} - 3(-x^3)^{\frac{2}{3}} x^2 e^{\frac{x^3}{3}} + 3\, 3^{\frac{2}{3}} x^2 \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right) - \frac{3^{\frac{2}{3}} x^5 e^{\frac{x^3}{3}}}{\left(-\frac{x^3}{3}\right)^{\frac{1}{3}}}}{3(-x^3)^{\frac{2}{3}}} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{2\left(-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + 3^{\frac{2}{3}} x^3 \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right) x^2}{3(-x^3)^{\frac{5}{3}}} + \frac{\frac{6 e^{\frac{x^3}{3}} x^2}{(-x^3)^{\frac{1}{3}}} - 3(-x^3)^{\frac{2}{3}} x^2 e^{\frac{x^3}{3}} + 3\, 3^{\frac{2}{3}} x^2 \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right) - \frac{3^{\frac{2}{3}} x^5 e^{\frac{x^3}{3}}}{\left(-\frac{x^3}{3}\right)^{\frac{1}{3}}}}{3(-x^3)^{\frac{2}{3}}}\right) - \left(\frac{-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + 3^{\frac{2}{3}} x^3 \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3(-x^3)^{\frac{2}{3}}}\right)(1)$$

Which simplifies to

$$W = \frac{e^{\frac{x^3}{3}}\left(3^{\frac{2}{3}} x^9 - 3(-x^3)^{\frac{5}{3}}\left(-\frac{x^3}{3}\right)^{\frac{1}{3}} x^3 + 3(-x^3)^{\frac{5}{3}}\left(-\frac{x^3}{3}\right)^{\frac{1}{3}}\right)}{3(-x^3)^{\frac{5}{3}}\left(-\frac{x^3}{3}\right)^{\frac{1}{3}}}$$

Which simplifies to

$$W = e^{\frac{x^3}{3}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\left(-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + 3^{\frac{2}{3}} x^3 \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right) x^{m+1}}{3(-x^3)^{\frac{2}{3}}}}{e^{\frac{x^3}{3}}} dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{m+1}\left(-3(-x^3)^{\frac{2}{3}} + x^3 3^{\frac{2}{3}} e^{-\frac{x^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3\left(-x^3\right)^{\frac{2}{3}}} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{3\left(-\alpha^3\right)^{\frac{2}{3}}} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\, x^{m+1}}{e^{\frac{x^3}{3}}} dx$$

Which simplifies to

$$u_2 = \int x^{m+2} e^{-\frac{x^3}{3}} dx$$

Hence

$$u_2 = \frac{3^{\frac{m}{6}+1} x^m (x^3)^{-\frac{m}{6}} e^{-\frac{x^3}{6}} \operatorname{WhittakerM}\left(\frac{m}{6}, \frac{m}{6} + \frac{1}{2}, \frac{x^3}{3}\right)}{m+3}$$

Which simplifies to

$$u_1 = -\frac{\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha\right)}{3}$$

$$u_2 = \frac{3^{\frac{m}{6}+1} x^m (x^3)^{-\frac{m}{6}} e^{-\frac{x^3}{6}} \operatorname{WhittakerM}\left(\frac{m}{6}, \frac{m}{6} + \frac{1}{2}, \frac{x^3}{3}\right)}{m+3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{\frac{2}{3}}+\alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}}d\alpha\right)x}{3} + \frac{3^{\frac{m}{6}+1}x^m(x^3)^{-\frac{m}{6}} e^{-\frac{x^3}{6}} \operatorname{WhittakerM}\left(\frac{m}{6}, \frac{m}{6}+\frac{1}{2}, \frac{x^3}{3}\right)\left(-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + 3^{\frac{2}{3}}x^3\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)\right)\right)}{3(m+3)(-x^3)^{\frac{2}{3}}}$$

Which simplifies to

$$y_p(x) = -\frac{(-x^3)^{\frac{2}{3}} x(m+3)\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{\frac{2}{3}}+\alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}}d\alpha\right) + 9(x^3)^{-\frac{m}{6}}\left((-x^3)^{\frac{2}{3}} e^{\frac{x^3}{6}} x^m 3^{\frac{m}{6}} +}{(-x^3)^{\frac{2}{3}}(9+3m)}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x + \frac{c_2\left(-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + 3^{\frac{2}{3}}x^3\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)\right)\right)}{3(-x^3)^{\frac{2}{3}}}\right) \\ &+ \left(-\frac{(-x^3)^{\frac{2}{3}} x(m+3)\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{\frac{2}{3}}+\alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}}d\alpha\right) + 9(x^3)^{-\frac{m}{6}}\left((-x^3)^{\frac{2}{3}} e^{\frac{x^3}{6}} x^m}{(-x^3)^{\frac{2}{3}}(9+3m)}\right. \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + \frac{c_2\left(-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + 3^{\frac{2}{3}}x^3\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)\right)\right)}{3(-x^3)^{\frac{2}{3}}} - \frac{(-x^3)^{\frac{2}{3}} x(m+3)\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{\frac{2}{3}}+\alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}}d\alpha\right) + 9(x^3)^{-\frac{m}{6}}\left((-x^3)^{\frac{2}{3}} e^{\frac{x^3}{6}} x^m 3^{\frac{m}{6}} +}{(-x^3)^{\frac{2}{3}}(9+3m)} \tag{1}$$

Verification of solutions

$$y = c_1x + \frac{c_2\left(-3(-x^3)^{\frac{2}{3}}\,\mathrm{e}^{\frac{x^3}{3}} + 3^{\frac{2}{3}}x^3\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3\,(-x^3)^{\frac{2}{3}}}$$
$$- \frac{(-x^3)^{\frac{2}{3}}\,x(m+3)\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}}\mathrm{e}^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}}d\alpha\right) + 9(x^3)^{-\frac{m}{6}}\left((-x^3)^{\frac{2}{3}}\,\mathrm{e}^{\frac{x^3}{6}}x^m 3^{\frac{m}{6}} + \right.}{(-x^3)^{\frac{2}{3}}\,(9+3m)}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 201

```
dsolve(diff(y(x),x$2)-x^2*diff(y(x),x)+x*y(x)=x^(m+1),y(x), singsol=all)
```

$$y(x) = \frac{\left(-3\, 3^{\frac{m}{6}} \mathrm{e}^{\frac{x^3}{6}} \left(x^3\right)^{-\frac{m}{6}} \operatorname{WhittakerM}\left(\frac{m}{6}, \frac{m}{6}+\frac{1}{2}, \frac{x^3}{3}\right) x^m + (m+3) \left(3^{\frac{1}{3}} \mathrm{e}^{\frac{x^3}{3}} c_1 - \frac{\left(\int \frac{\left(-3\left(-x^3\right)^{\frac{2}{3}} + x^3 3^{\frac{2}{3}} \mathrm{e}^{-\frac{x^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma \cdots\right.\right.}{\left(-x^3\right)^{\frac{2}{3}}} \cdots\right.}{3} \cdots\right.\right.}{\left(-x^3\right) \cdots}$$

✓ Solution by Mathematica

Time used: 0.453 (sec). Leaf size: 144

```
DSolve[y''[x]-x^2*y'[x]+x*y[x]==x^(m+1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \int_1^x \frac{e^{-\frac{1}{3}K[1]^3} \Gamma\left(-\frac{1}{3}, -\frac{1}{3}K[1]^3\right) K[1]^{m+1} \sqrt[3]{-K[1]^3}}{3\sqrt[3]{3}} dK[1] - \frac{\sqrt[3]{-x^3} (x^3)^{-m/3} \Gamma\left(-\frac{1}{3}, -\frac{x^3}{3}\right) \left(-3^{m/3} x^m \Gamma\left(\frac{m+3}{3}, \frac{x^3}{3}\right) + c_2 (x^3)^{m/3}\right)}{3\sqrt[3]{3}} + c_1 x$$

2.28 problem 28

2.28.1 Solving using Kovacic algorithm 948

Internal problem ID [7469]
Internal file name [OUTPUT/6436_Sunday_June_19_2022_05_02_30_PM_53292417/index.tex]

Book: Second order enumerated odes
Section: section 2
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - \frac{y'}{\sqrt{x}} + \frac{\left(x + \sqrt{x} - 8\right) y}{4x^2} = 0}$$

2.28.1 Solving using Kovacic algorithm

Writing the ode as

$$4y''x^{\frac{5}{2}} - 4x^2y' + \left(x^{\frac{3}{2}} + x - 8\sqrt{x}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^{\frac{5}{2}} \\ B &= -4x^2 \\ C &= x^{\frac{3}{2}} + x - 8\sqrt{x} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 87: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)(0) \\ &= -\frac{1}{x} \\ &= -\frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4x^2}{4x^{\frac{5}{2}}} \, dx} \\ &= z_1 e^{\sqrt{x}} \\ &= z_1 \left(\mathrm{e}^{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\sqrt{x}}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2}{4x^{\frac{5}{2}}} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2\sqrt{x}}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{\sqrt{x}}}{x}\right) + c_2 \left(\frac{e^{\sqrt{x}}}{x} \left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 e^{\sqrt{x}}}{x} + \frac{c_2 x^2 e^{\sqrt{x}}}{3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 e^{\sqrt{x}}}{x} + \frac{c_2 x^2 e^{\sqrt{x}}}{3}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-1/x^(1/2)*diff(y(x),x)+y(x)/(4*x^2)*(-8+x^(1/2)+x)=0,y(x), singsol=all
```

$$
y(x) = \frac{e^{\sqrt{x}}(c_2 x^3 + c_1)}{x}
$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 30

```
DSolve[y''[x]-1/x^(1/2)*y'[x]+y[x]/(4*x^2)*(-8+x^(1/2)+x)==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{e^{\sqrt{x}}(c_2 x^3 + 3c_1)}{3x}$$

2.29 problem 29

2.29.1 Solving as second order change of variable on y method 1 ode . 955
2.29.2 Solving using Kovacic algorithm 958

Internal problem ID [7470]
Internal file name [`OUTPUT/6437_Sunday_June_19_2022_05_02_32_PM_41088401/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{\cos(x)^2 y'' - 2\cos(x)\sin(x)\, y' + y\cos(x)^2 = 0}$$

2.29.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{2\sin(x)}{\cos(x)}$$
$$q(x) = 1$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= 1 - \frac{\left(-\frac{2\sin(x)}{\cos(x)}\right)'}{2} - \frac{\left(-\frac{2\sin(x)}{\cos(x)}\right)^2}{4} \\
&= 1 - \frac{\left(-\frac{2\sin(x)^2}{\cos(x)^2} - 2\right)}{2} - \frac{\left(\frac{4\sin(x)^2}{\cos(x)^2}\right)}{4} \\
&= 1 - \left(-\frac{\sin(x)^2}{\cos(x)^2} - 1\right) - \frac{\sin(x)^2}{\cos(x)^2} \\
&= 2
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{2\sin(x)}{\cos(x)}}{2}} \\
&= \sec(x)
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x) \sec(x) \tag{4}
$$

Applying this change of variable to the original ode results in

$$
2v(x) + v''(x) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda x} + 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(2)} \\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0\left(c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)\right)$$

Or

$$v(x) = c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \sec(x)$$

Hence (7) becomes

$$y = \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) \sec(x)$$

Summary
The solution(s) found are the following

$$y = \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) \sec(x) \tag{1}$$

Verification of solutions

$$y = \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) \sec(x)$$

Verified OK.

2.29.2 Solving using Kovacic algorithm

Writing the ode as

$$y\cos(x) - 2y' \sin(x) + y'' \cos(x) = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= \cos(x) \\ B &= -2\sin(x) \\ C &= \cos(x) \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 88: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\sqrt{2}\, x\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2\sin(x)}{\cos(x)}\, dx} \\ &= z_1 e^{-\ln(\cos(x))} \\ &= z_1(\sec(x)) \end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(\sqrt{2}\, x\right) \sec(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2\sin(x)}{\cos(x)}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2\ln(\cos(x))}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos\left(\sqrt{2}\, x\right) \sec\left(x\right)\right) + c_2 \left(\cos\left(\sqrt{2}\, x\right) \sec\left(x\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \cos\left(\sqrt{2}\, x\right) \sec\left(x\right) + \frac{c_2 \sin\left(\sqrt{2}\, x\right) \sec\left(x\right) \sqrt{2}}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \cos\left(\sqrt{2}\, x\right) \sec\left(x\right) + \frac{c_2 \sin\left(\sqrt{2}\, x\right) \sec\left(x\right) \sqrt{2}}{2}
$$

Verified OK.
Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 24

```
dsolve(cos(x)^2*diff(y(x),x$2)-2*cos(x)*sin(x)*diff(y(x),x)+y(x)*cos(x)^2=0,y(x), singsol=al
```

$$
y(x) = \sec\left(x\right) \left(c_1 \sin\left(\sqrt{2}\, x\right) + c_2 \cos\left(\sqrt{2}\, x\right)\right)
$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 51

```
DSolve[Cos[x]^2*y''[x]-2*Cos[x]*Sin[x]*y'[x]+y[x]*Cos[x]^2==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{1}{4}e^{-i\sqrt{2}x}\left(4c_1 - i\sqrt{2}c_2e^{2i\sqrt{2}x}\right)\sec(x)$$

2.30 problem 30

2.30.1 Solving as second order change of variable on y method 1 ode . 963
2.30.2 Solving using Kovacic algorithm 970

Internal problem ID [7471]
Internal file name [`OUTPUT/6438_Sunday_June_19_2022_05_02_33_PM_92244890/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' - 4xy' + (4x^2 - 1)\,y = -3\,e^{x^2}\sin(x)$$

2.30.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4xy' + (4x^2 - 1)\,y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -4x$$
$$q(x) = 4x^2 - 1$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= 4x^2 - 1 - \frac{(-4x)'}{2} - \frac{(-4x)^2}{4} \\
&= 4x^2 - 1 - \frac{(-4)}{2} - \frac{(16x^2)}{4} \\
&= 4x^2 - 1 - (-2) - 4x^2 \\
&= 1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-4x}{2}} \\
&= \mathrm{e}^{x^2}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, \mathrm{e}^{x^2} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
-v''(x) - v(x) = 3\sin(x)
$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = f(x)
$$

Where $A = -1, B = 0, C = -1, f(x) = 3\sin(x)$. Let the solution be

$$
v(x) = v_h + v_p
$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$
-v''(x) - v(x) = 0
$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = -1, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$-\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$-\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = -1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(-1)} \pm \frac{1}{(2)(-1)}\sqrt{0^2 - (4)(-1)(-1)} \\ &= \pm - i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= + - i \\ \lambda_2 &= - - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -i \\ \lambda_2 &= i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3 \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{-\sin(x), \cos(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(x), x\sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$v_p = A_1 x \cos(x) + A_2 x \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \sin(x) - 2A_2 \cos(x) = 3 \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = \frac{3x \cos(x)}{2}$$

Therefore the general solution is

$$
\begin{aligned}
v &= v_h + v_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{3x \cos(x)}{2}\right)
\end{aligned}
$$

Now that $v(x)$ is known, then

$$
\begin{aligned}
y &= v(x)\, z(x) \\
&= \left(c_1 \cos(x) + c_2 \sin(x) + \frac{3x \cos(x)}{2}\right)(z(x))
\end{aligned} \tag{7}
$$

But from (5)

$$
z(x) = \mathrm{e}^{x^2}
$$

Hence (7) becomes

$$
y = \left(c_1 \cos(x) + c_2 \sin(x) + \frac{3x \cos(x)}{2}\right) \mathrm{e}^{x^2}
$$

Therefore the homogeneous solution y_h is

$$
y_h = \left(c_1 \cos(x) + c_2 \sin(x) + \frac{3x \cos(x)}{2}\right) \mathrm{e}^{x^2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \cos(x)\, \mathrm{e}^{x^2} \\
y_2 &= -\mathrm{e}^{x^2} \sin(x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x)\,\mathrm{e}^{x^2} & -\mathrm{e}^{x^2}\sin(x) \\ \frac{d}{dx}\left(\cos(x)\,\mathrm{e}^{x^2}\right) & \frac{d}{dx}\left(-\mathrm{e}^{x^2}\sin(x)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x)\,\mathrm{e}^{x^2} & -\mathrm{e}^{x^2}\sin(x) \\ -\mathrm{e}^{x^2}\sin(x) + 2\cos(x)\,x\,\mathrm{e}^{x^2} & -2x\,\mathrm{e}^{x^2}\sin(x) - \cos(x)\,\mathrm{e}^{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\cos(x)\,\mathrm{e}^{x^2}\right)\left(-2x\,\mathrm{e}^{x^2}\sin(x) - \cos(x)\,\mathrm{e}^{x^2}\right) - \left(-\mathrm{e}^{x^2}\sin(x)\right)\left(-\mathrm{e}^{x^2}\sin(x) + 2\cos(x)\,x\,\mathrm{e}^{x^2}\right)$$

Which simplifies to

$$W = -\mathrm{e}^{2x^2}\sin(x)^2 - \mathrm{e}^{2x^2}\cos(x)^2$$

Which simplifies to

$$W = -\mathrm{e}^{2x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{3\,\mathrm{e}^{2x^2}\sin(x)^2}{-\mathrm{e}^{2x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int -3\sin(x)^2\,dx$$

Hence

$$u_1 = -\frac{3\sin(x)\cos(x)}{2} + \frac{3x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-3\cos(x)\,e^{2x^2}\sin(x)}{-e^{2x^2}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{3\sin(2x)}{2}dx$$

Hence

$$u_2 = -\frac{3\cos(2x)}{4}$$

Which simplifies to

$$u_1 = -\frac{3\sin(2x)}{4} + \frac{3x}{2}$$
$$u_2 = -\frac{3\cos(2x)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{3\sin(2x)}{4} + \frac{3x}{2}\right)\cos(x)\,e^{x^2} + \frac{3\cos(2x)\,e^{x^2}\sin(x)}{4}$$

Which simplifies to

$$y_p(x) = -\frac{3\,e^{x^2}(\sin(x) - 2x\cos(x))}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1\cos(x) + c_2\sin(x) + \frac{3x\cos(x)}{2}\right)e^{x^2}\right) + \left(-\frac{3\,e^{x^2}(\sin(x) - 2x\cos(x))}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(c_1 \cos(x) + c_2 \sin(x) + \frac{3x \cos(x)}{2}\right) e^{x^2} - \frac{3\, e^{x^2} (\sin(x) - 2x \cos(x))}{4} \tag{1}$$

Verification of solutions

$$y = \left(c_1 \cos(x) + c_2 \sin(x) + \frac{3x \cos(x)}{2}\right) e^{x^2} - \frac{3\, e^{x^2} (\sin(x) - 2x \cos(x))}{4}$$

Verified OK.

2.30.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4xy' + (4x^2 - 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 89: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\,dx} \\ &= z_1 e^{x^2} \\ &= z_1\left(\mathrm{e}^{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\,\mathrm{e}^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\cos(x)\,\mathrm{e}^{x^2}\right) + c_2\left(\cos(x)\,\mathrm{e}^{x^2}(\tan(x))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-4xy'+\left(4x^2-1\right)y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{x^2}\cos(x)\,c_1 + e^{x^2}\sin(x)\,c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)\,e^{x^2}$$
$$y_2 = e^{x^2}\sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x)\,e^{x^2} & e^{x^2}\sin(x) \\ \frac{d}{dx}\left(\cos(x)\,e^{x^2}\right) & \frac{d}{dx}\left(e^{x^2}\sin(x)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x)\,\mathrm{e}^{x^2} & \mathrm{e}^{x^2}\sin(x) \\ -\mathrm{e}^{x^2}\sin(x) + 2\cos(x)\,x\,\mathrm{e}^{x^2} & 2x\,\mathrm{e}^{x^2}\sin(x) + \cos(x)\,\mathrm{e}^{x^2} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\cos(x)\,\mathrm{e}^{x^2}\right)\left(2x\,\mathrm{e}^{x^2}\sin(x) + \cos(x)\,\mathrm{e}^{x^2}\right) \\ & - \left(\mathrm{e}^{x^2}\sin(x)\right)\left(-\mathrm{e}^{x^2}\sin(x) + 2\cos(x)\,x\,\mathrm{e}^{x^2}\right) \end{aligned}$$

Which simplifies to

$$W = \mathrm{e}^{2x^2}\sin(x)^2 + \mathrm{e}^{2x^2}\cos(x)^2$$

Which simplifies to

$$W = \mathrm{e}^{2x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-3\,\mathrm{e}^{2x^2}\sin(x)^2}{\mathrm{e}^{2x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int -3\sin(x)^2\,dx$$

Hence

$$u_1 = -\frac{3\sin(x)\cos(x)}{2} + \frac{3x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-3\cos(x)\,\mathrm{e}^{2x^2}\sin(x)}{\mathrm{e}^{2x^2}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{3\sin(2x)}{2}\,dx$$

Hence

$$u_2 = \frac{3\cos(2x)}{4}$$

Which simplifies to

$$u_1 = -\frac{3\sin(2x)}{4} + \frac{3x}{2}$$
$$u_2 = \frac{3\cos(2x)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{3\sin(2x)}{4} + \frac{3x}{2}\right)\cos(x)\,\mathrm{e}^{x^2} + \frac{3\cos(2x)\,\mathrm{e}^{x^2}\sin(x)}{4}$$

Which simplifies to

$$y_p(x) = -\frac{3\,\mathrm{e}^{x^2}(\sin(x) - 2x\cos(x))}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{x^2}\cos(x)\,c_1 + \mathrm{e}^{x^2}\sin(x)\,c_2\right) + \left(-\frac{3\,\mathrm{e}^{x^2}(\sin(x) - 2x\cos(x))}{4}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{x^2}(c_1\cos(x) + c_2\sin(x)) - \frac{3\,\mathrm{e}^{x^2}(\sin(x) - 2x\cos(x))}{4}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x^2}(c_1\cos(x) + c_2\sin(x)) - \frac{3\,\mathrm{e}^{x^2}(\sin(x) - 2x\cos(x))}{4} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{x^2}(c_1\cos(x) + c_2\sin(x)) - \frac{3\,\mathrm{e}^{x^2}(\sin(x) - 2x\cos(x))}{4}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)-4*x*diff(y(x),x)+(4*x^2-1)*y(x)=-3*exp(x^2)*sin(x),y(x), singsol=all)
```

$$y(x) = \frac{((2c_2 + 3x)\cos(x) + \sin(x)(2c_1 - 3))\,e^{x^2}}{2}$$

✓ Solution by Mathematica
Time used: 0.094 (sec). Leaf size: 50

```
DSolve[y''[x]-4*x*y'[x]+(4*x^2-1)*y[x]==-3*Exp[x^2]*Sin[x],y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{1}{8}e^{x(x-i)}\left(6x + e^{2ix}(6x + 3i - 4ic_2) - 3i + 8c_1\right)$$

2.31 problem 31

2.31.1 Solving as second order change of variable on y method 1 ode . 977
2.31.2 Solving using Kovacic algorithm 986

Internal problem ID [7472]
Internal file name [`OUTPUT/6439_Sunday_June_19_2022_05_02_35_PM_96893377/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2bxy' + b^2x^2y = x}$$

2.31.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2bxy' + b^2x^2y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -2xb$$
$$q(x) = x^2b^2$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= x^2b^2 - \frac{(-2xb)'}{2} - \frac{(-2xb)^2}{4} \\
&= x^2b^2 - \frac{(-2b)}{2} - \frac{(4x^2b^2)}{4} \\
&= x^2b^2 - (-b) - x^2b^2 \\
&= b
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2}dx\right)} \\
&= e^{-\int \frac{-2xb}{2}} \\
&= \mathrm{e}^{\frac{x^2b}{2}} \qquad (5)
\end{aligned}$$

Hence (3) becomes

$$y = v(x)\,\mathrm{e}^{\frac{x^2b}{2}} \tag{4}$$

Applying this change of variable to the original ode results in

$$\mathrm{e}^{\frac{x^2b}{2}}(bv(x) + v''(x)) = x$$

Which is now solved for $v(x)$ Simplyfing the ode gives

$$bv(x) + v''(x) = x\,\mathrm{e}^{-\frac{x^2b}{2}}$$

This is second order non-homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = f(x)$$

Where $A = 1, B = 0, C = b, f(x) = x\,\mathrm{e}^{-\frac{x^2b}{2}}$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$bv(x) + v''(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = b$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + b\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + b = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = b$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(b)} \\ &= \pm\sqrt{-b}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-b} \\ \lambda_2 &= -\sqrt{-b}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-b} \\ \lambda_2 &= -\sqrt{-b}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ v(x) &= c_1 e^{(\sqrt{-b})x} + c_2 e^{(-\sqrt{-b})x}\end{aligned}$$

Or

$$v(x) = c_1 e^{\sqrt{-b}\,x} + c_2 e^{-\sqrt{-b}\,x}$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 e^{\sqrt{-b}\,x} + c_2 e^{-\sqrt{-b}\,x}$$

The particular solution v_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$v_p(x) = u_1 v_1 + u_2 v_2 \tag{1}$$

Where u_1, u_2 to be determined, and v_1, v_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$v_1 = e^{\sqrt{-b}\,x}$$
$$v_2 = e^{-\sqrt{-b}\,x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{v_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{v_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of v'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\sqrt{-b}\,x} & e^{-\sqrt{-b}\,x} \\ \frac{d}{dx}\left(e^{\sqrt{-b}\,x}\right) & \frac{d}{dx}\left(e^{-\sqrt{-b}\,x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\sqrt{-b}\,x} & e^{-\sqrt{-b}\,x} \\ \sqrt{-b}\, e^{\sqrt{-b}\,x} & -\sqrt{-b}\, e^{-\sqrt{-b}\,x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{\sqrt{-b}\,x}\right)\left(-\sqrt{-b}\,\mathrm{e}^{-\sqrt{-b}\,x}\right) - \left(\mathrm{e}^{-\sqrt{-b}\,x}\right)\left(\sqrt{-b}\,\mathrm{e}^{\sqrt{-b}\,x}\right)$$

Which simplifies to

$$W = -2\,\mathrm{e}^{\sqrt{-b}\,x}\sqrt{-b}\,\mathrm{e}^{-\sqrt{-b}\,x}$$

Which simplifies to

$$W = -2\sqrt{-b}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{-\sqrt{-b}\,x} x\,\mathrm{e}^{-\frac{x^2 b}{2}}}{-2\sqrt{-b}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{x\,\mathrm{e}^{-\frac{x\left(xb+2\sqrt{-b}\right)}{2}}}{2\sqrt{-b}}\,dx$$

Hence

$$u_1 = \frac{-\frac{\mathrm{e}^{-\frac{x^2 b}{2}-\sqrt{-b}\,x}}{b} - \frac{\sqrt{-b}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2}+\frac{\sqrt{-b}\,\sqrt{2}}{2\sqrt{b}}\right)}{2b^{\frac{3}{2}}}}{2\sqrt{-b}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{\sqrt{-b}\,x} x\,\mathrm{e}^{-\frac{x^2 b}{2}}}{-2\sqrt{-b}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{x\,\mathrm{e}^{-\frac{x\left(xb-2\sqrt{-b}\right)}{2}}}{2\sqrt{-b}}\,dx$$

Hence

$$u_2 = -\frac{-\frac{\mathrm{e}^{\sqrt{-b}\,x-\frac{x^2 b}{2}}}{b} + \frac{\sqrt{-b}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2}-\frac{\sqrt{-b}\,\sqrt{2}}{2\sqrt{b}}\right)}{2b^{\frac{3}{2}}}}{2\sqrt{-b}}$$

Therefore the particular solution, from equation (1) is

$$
v_p(x) = \frac{\left(-\frac{e^{-\frac{x^2 b}{2}-\sqrt{-b}\,x}}{b}-\frac{\sqrt{-b}\,\sqrt{\pi}\,e^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2}+\frac{\sqrt{-b}\,\sqrt{2}}{2\sqrt{b}}\right)}{2b^{\frac{3}{2}}}\right)e^{\sqrt{-b}\,x}}{2\sqrt{-b}} - \frac{\left(-\frac{e^{\sqrt{-b}\,x-\frac{x^2 b}{2}}}{b}+\frac{\sqrt{-b}\,\sqrt{\pi}\,e^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2}-\frac{\sqrt{-b}\,\sqrt{2}}{2\sqrt{b}}\right)}{2b^{\frac{3}{2}}}\right)e^{-\sqrt{-b}\,x}}{2\sqrt{-b}}
$$

Which simplifies to

$$
v_p(x) = \frac{\sqrt{\pi}\,\sqrt{2}\left(e^{-\frac{1}{2}-\sqrt{-b}\,x}\,\mathrm{erf}\left(\frac{\sqrt{2}\,(-xb+\sqrt{-b})}{2\sqrt{b}}\right)-e^{-\frac{1}{2}+\sqrt{-b}\,x}\,\mathrm{erf}\left(\frac{\sqrt{2}\,(xb+\sqrt{-b})}{2\sqrt{b}}\right)\right)}{4b^{\frac{3}{2}}}
$$

Therefore the general solution is

$$
\begin{aligned}
v &= v_h + v_p \\
&= \left(c_1 e^{\sqrt{-b}\,x} + c_2 e^{-\sqrt{-b}\,x}\right) \\
&\quad + \left(\frac{\sqrt{\pi}\,\sqrt{2}\left(e^{-\frac{1}{2}-\sqrt{-b}\,x}\,\mathrm{erf}\left(\frac{\sqrt{2}\,(-xb+\sqrt{-b})}{2\sqrt{b}}\right)-e^{-\frac{1}{2}+\sqrt{-b}\,x}\,\mathrm{erf}\left(\frac{\sqrt{2}\,(xb+\sqrt{-b})}{2\sqrt{b}}\right)\right)}{4b^{\frac{3}{2}}}\right)
\end{aligned}
$$

Now that $v(x)$ is known, then

$$
\begin{aligned}
y &= v(x)\,z(x) \\
&= \left(c_1 e^{\sqrt{-b}\,x} + c_2 e^{-\sqrt{-b}\,x} + \frac{\sqrt{\pi}\,\sqrt{2}\left(e^{-\frac{1}{2}-\sqrt{-b}\,x}\,\mathrm{erf}\left(\frac{\sqrt{2}\,(-xb+\sqrt{-b})}{2\sqrt{b}}\right)-e^{-\frac{1}{2}+\sqrt{-b}\,x}\,\mathrm{erf}\left(\frac{\sqrt{2}\,(xb+\sqrt{-b})}{2\sqrt{b}}\right)\right)}{4b^{\frac{3}{2}}}\right)(z(x
\end{aligned} \tag{7}
$$

But from (5)

$$
z(x) = e^{\frac{x^2 b}{2}}
$$

Hence (7) becomes

$$
y = \left(c_1 e^{\sqrt{-b}\,x} + c_2 e^{-\sqrt{-b}\,x} + \frac{\sqrt{\pi}\,\sqrt{2}\left(e^{-\frac{1}{2}-\sqrt{-b}\,x}\,\mathrm{erf}\left(\frac{\sqrt{2}\,(-xb+\sqrt{-b})}{2\sqrt{b}}\right)-e^{-\frac{1}{2}+\sqrt{-b}\,x}\,\mathrm{erf}\left(\frac{\sqrt{2}\,(xb+\sqrt{-b})}{2\sqrt{b}}\right)\right)}{4b^{\frac{3}{2}}}\right)e^{\frac{x^2 b}{2}}
$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 e^{\sqrt{-b}\,x} + c_2 e^{-\sqrt{-b}\,x} + \frac{\sqrt{\pi}\,\sqrt{2}\left(e^{-\frac{1}{2}-\sqrt{-b}\,x}\operatorname{erf}\left(\frac{\sqrt{2}\,(-xb+\sqrt{-b})}{2\sqrt{b}}\right) - e^{-\frac{1}{2}+\sqrt{-b}\,x}\operatorname{erf}\left(\frac{\sqrt{2}\,(xb+\sqrt{-b})}{2\sqrt{b}}\right)\right)}{4b^{\frac{3}{2}}} \right) e^{\frac{x^2 b}{2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{\sqrt{-b}\,x} e^{\frac{x^2 b}{2}}$$
$$y_2 = e^{-\sqrt{-b}\,x} e^{\frac{x^2 b}{2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\sqrt{-b}\,x} e^{\frac{x^2 b}{2}} & e^{-\sqrt{-b}\,x} e^{\frac{x^2 b}{2}} \\ \frac{d}{dx}\left(e^{\sqrt{-b}\,x} e^{\frac{x^2 b}{2}}\right) & \frac{d}{dx}\left(e^{-\sqrt{-b}\,x} e^{\frac{x^2 b}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\sqrt{-b}\,x} e^{\frac{x^2 b}{2}} & e^{-\sqrt{-b}\,x} e^{\frac{x^2 b}{2}} \\ \sqrt{-b}\, e^{\sqrt{-b}\,x} e^{\frac{x^2 b}{2}} + e^{\sqrt{-b}\,x} x b\, e^{\frac{x^2 b}{2}} & -\sqrt{-b}\, e^{-\sqrt{-b}\,x} e^{\frac{x^2 b}{2}} + e^{-\sqrt{-b}\,x} x b\, e^{\frac{x^2 b}{2}} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(e^{\sqrt{-b}\,x} e^{\frac{x^2 b}{2}}\right)\left(-\sqrt{-b}\, e^{-\sqrt{-b}\,x} e^{\frac{x^2 b}{2}} + e^{-\sqrt{-b}\,x} x b\, e^{\frac{x^2 b}{2}}\right) \\ & - \left(e^{-\sqrt{-b}\,x} e^{\frac{x^2 b}{2}}\right)\left(\sqrt{-b}\, e^{\sqrt{-b}\,x} e^{\frac{x^2 b}{2}} + e^{\sqrt{-b}\,x} x b\, e^{\frac{x^2 b}{2}}\right) \end{aligned}$$

Which simplifies to

$$W = -2\sqrt{-b}\, e^{x^2 b} e^{\sqrt{-b}\,x} e^{-\sqrt{-b}\,x}$$

Which simplifies to

$$W = -2\sqrt{-b}\, e^{x^2 b}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-\sqrt{-b}\,x} e^{\frac{x^2 b}{2}} x}{-2\sqrt{-b}\, e^{x^2 b}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{x\, e^{-\frac{x\left(x b + 2\sqrt{-b}\right)}{2}}}{2\sqrt{-b}}\, dx$$

Hence

$$u_1 = \frac{-\frac{e^{-\frac{x^2 b}{2} - \sqrt{-b}\,x}}{b} - \frac{\sqrt{-b}\,\sqrt{\pi}\, e^{-\frac{1}{2}}\sqrt{2}\, \operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2} + \frac{\sqrt{-b}\,\sqrt{2}}{2\sqrt{b}}\right)}{2 b^{\frac{3}{2}}}}{2\sqrt{-b}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{\sqrt{-b}\,x} e^{\frac{x^2 b}{2}} x}{-2\sqrt{-b}\, e^{x^2 b}}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{x\, e^{-\frac{x\left(x b - 2\sqrt{-b}\right)}{2}}}{2\sqrt{-b}}\, dx$$

Hence

$$u_2 = -\frac{-\frac{e^{\sqrt{-b}\,x-\frac{x^2 b}{2}}}{b} + \frac{\sqrt{-b}\,\sqrt{\pi}\,e^{-\frac{1}{2}}\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2} - \frac{\sqrt{-b}\,\sqrt{2}}{2\sqrt{b}}\right)}{2b^{\frac{3}{2}}}}{2\sqrt{-b}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(-\frac{e^{-\frac{x^2 b}{2}-\sqrt{-b}\,x}}{b} - \frac{\sqrt{-b}\,\sqrt{\pi}\,e^{-\frac{1}{2}}\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2} + \frac{\sqrt{-b}\,\sqrt{2}}{2\sqrt{b}}\right)}{2b^{\frac{3}{2}}}\right) e^{\sqrt{-b}\,x} e^{\frac{x^2 b}{2}}}{2\sqrt{-b}} - \frac{\left(-\frac{e^{\sqrt{-b}\,x-\frac{x^2 b}{2}}}{b} + \frac{\sqrt{-b}\,\sqrt{\pi}\,e^{-\frac{1}{2}}\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2} - \frac{\sqrt{-b}\,\sqrt{2}}{2\sqrt{b}}\right)}{2b^{\frac{3}{2}}}\right) e^{-\sqrt{-b}\,x} e^{\frac{x^2 b}{2}}}{2\sqrt{-b}}$$

Which simplifies to

$$y_p(x) = \frac{\sqrt{\pi}\,\sqrt{2}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,(-xb+\sqrt{-b})}{2\sqrt{b}}\right) e^{-\frac{1}{2}+\frac{x^2 b}{2}-\sqrt{-b}\,x} - \operatorname{erf}\left(\frac{\sqrt{2}\,(xb+\sqrt{-b})}{2\sqrt{b}}\right) e^{-\frac{1}{2}+\frac{x^2 b}{2}+\sqrt{-b}\,x}\right)}{4b^{\frac{3}{2}}}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(c_1 e^{\sqrt{-b}\,x} + c_2 e^{-\sqrt{-b}\,x} \right.\right. \\
&\quad \left.\left. + \frac{\sqrt{\pi}\,\sqrt{2}\left(e^{-\frac{1}{2}-\sqrt{-b}\,x}\operatorname{erf}\left(\frac{\sqrt{2}\,(-xb+\sqrt{-b})}{2\sqrt{b}}\right) - e^{-\frac{1}{2}+\sqrt{-b}\,x}\operatorname{erf}\left(\frac{\sqrt{2}\,(xb+\sqrt{-b})}{2\sqrt{b}}\right)\right)}{4b^{\frac{3}{2}}}\right) e^{\frac{x^2 b}{2}}\right) \\
&\quad + \left(\frac{\sqrt{\pi}\,\sqrt{2}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,(-xb+\sqrt{-b})}{2\sqrt{b}}\right) e^{-\frac{1}{2}+\frac{x^2 b}{2}-\sqrt{-b}\,x} - \operatorname{erf}\left(\frac{\sqrt{2}\,(xb+\sqrt{-b})}{2\sqrt{b}}\right) e^{-\frac{1}{2}+\frac{x^2 b}{2}+\sqrt{-b}\,x}\right)}{4b^{\frac{3}{2}}}\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$
y = \left(c_1 e^{\sqrt{-b}\, x} + c_2 e^{-\sqrt{-b}\, x} + \frac{\sqrt{\pi}\, \sqrt{2}\, \left(e^{-\frac{1}{2}-\sqrt{-b}\, x} \operatorname{erf}\left(\frac{\sqrt{2}\, \left(-x b +\sqrt{-b}\right)}{2 \sqrt{b}}\right)-e^{-\frac{1}{2}+\sqrt{-b}\, x} \operatorname{erf}\left(\frac{\sqrt{2}\, \left(x b +\sqrt{-b}\right)}{2 \sqrt{b}}\right)\right)}{4 b^{\frac{3}{2}}}\right) e^{\frac{x^{2} b}{2}} + \frac{\sqrt{\pi}\, \sqrt{2}\, \left(\operatorname{erf}\left(\frac{\sqrt{2}\, \left(-x b +\sqrt{-b}\right)}{2 \sqrt{b}}\right) e^{-\frac{1}{2}+\frac{x^{2} b}{2}-\sqrt{-b}\, x}-\operatorname{erf}\left(\frac{\sqrt{2}\, \left(x b +\sqrt{-b}\right)}{2 \sqrt{b}}\right) e^{-\frac{1}{2}+\frac{x^{2} b}{2}+\sqrt{-b}\, x}\right)}{4 b^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$
y = \left(c_1 e^{\sqrt{-b}\, x} + c_2 e^{-\sqrt{-b}\, x} + \frac{\sqrt{\pi}\, \sqrt{2}\, \left(e^{-\frac{1}{2}-\sqrt{-b}\, x} \operatorname{erf}\left(\frac{\sqrt{2}\, \left(-x b +\sqrt{-b}\right)}{2 \sqrt{b}}\right)-e^{-\frac{1}{2}+\sqrt{-b}\, x} \operatorname{erf}\left(\frac{\sqrt{2}\, \left(x b +\sqrt{-b}\right)}{2 \sqrt{b}}\right)\right)}{4 b^{\frac{3}{2}}}\right) e^{\frac{x^{2} b}{2}} + \frac{\sqrt{\pi}\, \sqrt{2}\, \left(\operatorname{erf}\left(\frac{\sqrt{2}\, \left(-x b +\sqrt{-b}\right)}{2 \sqrt{b}}\right) e^{-\frac{1}{2}+\frac{x^{2} b}{2}-\sqrt{-b}\, x}-\operatorname{erf}\left(\frac{\sqrt{2}\, \left(x b +\sqrt{-b}\right)}{2 \sqrt{b}}\right) e^{-\frac{1}{2}+\frac{x^{2} b}{2}+\sqrt{-b}\, x}\right)}{4 b^{\frac{3}{2}}}
$$

Verified OK.

2.31.2 Solving using Kovacic algorithm

Writing the ode as

$$
y'' - 2bxy' + b^2x^2y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -2xb \\
C &= x^2b^2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-b}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -b \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (-b)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 90: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -b$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{\sqrt{-b}\,x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2xb}{1}\,dx} \\
&= z_1 e^{\frac{x^2 b}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2 b}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{x\left(xb+2\sqrt{-b}\right)}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2xb}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2 b}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2\sqrt{-b}\,x}}{2\sqrt{-b}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{\frac{x(xb+2\sqrt{-b})}{2}} \right) + c_2 \left(e^{\frac{x(xb+2\sqrt{-b})}{2}} \left(-\frac{e^{-2\sqrt{-b}\,x}}{2\sqrt{-b}} \right) \right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2bxy' + b^2x^2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = e^{\frac{x(xb+2\sqrt{-b})}{2}} c_1 - \frac{c_2 e^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= e^{\frac{x(xb+2\sqrt{-b})}{2}} \\
y_2 &= -\frac{e^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\frac{x(xb+2\sqrt{-b})}{2}} & -\frac{e^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}} \\ \frac{d}{dx}\left(e^{\frac{x(xb+2\sqrt{-b})}{2}}\right) & \frac{d}{dx}\left(-\frac{e^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\frac{x(xb+2\sqrt{-b})}{2}} & -\frac{e^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}} \\ \left(xb+\sqrt{-b}\right) e^{\frac{x(xb+2\sqrt{-b})}{2}} & -\frac{(xb-\sqrt{-b}) e^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}} \end{vmatrix}$$

Therefore

$$W = \left(e^{\frac{x(xb+2\sqrt{-b})}{2}}\right)\left(-\frac{\left(xb-\sqrt{-b}\right) e^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}}\right) - \left(-\frac{e^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}}\right)\left(\left(xb+\sqrt{-b}\right) e^{\frac{x(xb+2\sqrt{-b})}{2}}\right)$$

Which simplifies to

$$W = e^{\frac{x(xb+2\sqrt{-b})}{2}} e^{-\frac{x(-xb+2\sqrt{-b})}{2}}$$

Which simplifies to

$$W = e^{x^2 b}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{e^{\frac{x(xb-2\sqrt{-b})}{2}} x}{2\sqrt{-b}}}{e^{x^2 b}} dx$$

Which simplifies to

$$u_1 = -\int -\frac{x\,\mathrm{e}^{-\frac{x\left(xb+2\sqrt{-b}\right)}{2}}}{2\sqrt{-b}}dx$$

Hence

$$u_1 = \frac{-\frac{\mathrm{e}^{-\frac{x^2 b}{2}-\sqrt{-b}\,x}}{b} - \frac{\sqrt{-b}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2}+\frac{\sqrt{-b}\,\sqrt{2}}{2\sqrt{b}}\right)}{2b^{\frac{3}{2}}}}{2\sqrt{-b}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{\frac{x\left(xb+2\sqrt{-b}\right)}{2}}x}{\mathrm{e}^{x^2 b}}\,dx$$

Which simplifies to

$$u_2 = \int x\,\mathrm{e}^{-\frac{x\left(xb-2\sqrt{-b}\right)}{2}}dx$$

Hence

$$u_2 = -\frac{\mathrm{e}^{\sqrt{-b}\,x-\frac{x^2 b}{2}}}{b} + \frac{\sqrt{-b}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2}-\frac{\sqrt{-b}\,\sqrt{2}}{2\sqrt{b}}\right)}{2b^{\frac{3}{2}}}$$

Which simplifies to

$$u_1 = -\frac{\sqrt{-b}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\left(xb+\sqrt{-b}\right)}{2\sqrt{b}}\right)+2\,\mathrm{e}^{-\frac{x\left(xb+2\sqrt{-b}\right)}{2}}\sqrt{b}}{4b^{\frac{3}{2}}\sqrt{-b}}$$

$$u_2 = -\frac{\sqrt{-b}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\left(-xb+\sqrt{-b}\right)}{2\sqrt{b}}\right)+2\,\mathrm{e}^{-\frac{x\left(xb-2\sqrt{-b}\right)}{2}}\sqrt{b}}{2b^{\frac{3}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\sqrt{-b}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\left(xb+\sqrt{-b}\right)}{2\sqrt{b}}\right)+2\,\mathrm{e}^{-\frac{x\left(xb+2\sqrt{-b}\right)}{2}}\sqrt{b}\right)\mathrm{e}^{\frac{x\left(xb+2\sqrt{-b}\right)}{2}}}{4b^{\frac{3}{2}}\sqrt{-b}} + \frac{\left(\sqrt{-b}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\left(-xb+\sqrt{-b}\right)}{2\sqrt{b}}\right)+2\,\mathrm{e}^{-\frac{x\left(xb-2\sqrt{-b}\right)}{2}}\sqrt{b}\right)\mathrm{e}^{\frac{x\left(xb-2\sqrt{-b}\right)}{2}}}{4b^{\frac{3}{2}}\sqrt{-b}}$$

Which simplifies to

$$y_p(x) = \frac{\sqrt{\pi}\,\sqrt{2}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,(-xb+\sqrt{-b})}{2\sqrt{b}}\right) \mathrm{e}^{-\frac{1}{2}+\frac{x^2 b}{2}-\sqrt{-b}\,x} - \operatorname{erf}\left(\frac{\sqrt{2}\,(xb+\sqrt{-b})}{2\sqrt{b}}\right) \mathrm{e}^{-\frac{1}{2}+\frac{x^2 b}{2}+\sqrt{-b}\,x}\right)}{4b^{\frac{3}{2}}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}} c_1 - \frac{c_2 \mathrm{e}^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}}\right) \\ &\quad + \left(\frac{\sqrt{\pi}\,\sqrt{2}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,(-xb+\sqrt{-b})}{2\sqrt{b}}\right) \mathrm{e}^{-\frac{1}{2}+\frac{x^2 b}{2}-\sqrt{-b}\,x} - \operatorname{erf}\left(\frac{\sqrt{2}\,(xb+\sqrt{-b})}{2\sqrt{b}}\right) \mathrm{e}^{-\frac{1}{2}+\frac{x^2 b}{2}+\sqrt{-b}\,x}\right)}{4b^{\frac{3}{2}}}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= \mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}} c_1 - \frac{c_2 \mathrm{e}^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}} \\ &\quad + \frac{\sqrt{\pi}\,\sqrt{2}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,(-xb+\sqrt{-b})}{2\sqrt{b}}\right) \mathrm{e}^{-\frac{1}{2}+\frac{x^2 b}{2}-\sqrt{-b}\,x} - \operatorname{erf}\left(\frac{\sqrt{2}\,(xb+\sqrt{-b})}{2\sqrt{b}}\right) \mathrm{e}^{-\frac{1}{2}+\frac{x^2 b}{2}+\sqrt{-b}\,x}\right)}{4b^{\frac{3}{2}}} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= \mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}} c_1 - \frac{c_2 \mathrm{e}^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}} \\ &\quad + \frac{\sqrt{\pi}\,\sqrt{2}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,(-xb+\sqrt{-b})}{2\sqrt{b}}\right) \mathrm{e}^{-\frac{1}{2}+\frac{x^2 b}{2}-\sqrt{-b}\,x} - \operatorname{erf}\left(\frac{\sqrt{2}\,(xb+\sqrt{-b})}{2\sqrt{b}}\right) \mathrm{e}^{-\frac{1}{2}+\frac{x^2 b}{2}+\sqrt{-b}\,x}\right)}{4b^{\frac{3}{2}}} \end{aligned}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 137

```
dsolve(diff(y(x),x$2)-2*b*x*diff(y(x),x)+b^2*x^2*y(x)=x,y(x), singsol=all)
```

$$y(x) = \frac{4\,\mathrm{e}^{\frac{x\left(bx+2\sqrt{-b}\right)}{2}}c_2 b^{\frac{3}{2}}+4\,\mathrm{e}^{\frac{x\left(bx-2\sqrt{-b}\right)}{2}}c_1 b^{\frac{3}{2}}-\mathrm{erf}\left(\frac{\sqrt{2}\left(bx+\sqrt{-b}\right)}{2\sqrt{b}}\right)\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{b\,x^2}{2}+x\sqrt{-b}-\frac{1}{2}}+\sqrt{2}\,\mathrm{e}^{\frac{b\,x^2}{2}-x\sqrt{-b}-\frac{1}{2}}\sqrt{\pi}\,\mathrm{erf}\left(\right.}{4b^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.427 (sec). Leaf size: 139

```
DSolve[y''[x]-2*b*x*y'[x]+b^2*x^2*y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{\frac{1}{2}\left(\sqrt{b}x-i\right)^2}\left(-\sqrt{2\pi}e^{2i\sqrt{b}x}\mathrm{erf}\left(\frac{\sqrt{b}x+i}{\sqrt{2}}\right)+i\sqrt{2\pi}\mathrm{erfi}\left(\frac{1+i\sqrt{b}x}{\sqrt{2}}\right)+2\sqrt{e}b\left(2\sqrt{b}c_1-ic_2e^{2i\sqrt{b}x}\right)\right)}{4b^{3/2}}$$

2.32 problem 32

2.32.1 Solving as second order change of variable on y method 1 ode . 994
2.32.2 Solving using Kovacic algorithm 1000

Internal problem ID [7473]
Internal file name [`OUTPUT/6440_Sunday_June_19_2022_05_02_38_PM_56345613/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 4xy' + \left(4x^2 - 3\right) y = \mathrm{e}^{x^2}}$$

2.32.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4xy' + \left(4x^2 - 3\right) y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -4x \\ q(x) &= 4x^2 - 3 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= 4x^2 - 3 - \frac{(-4x)'}{2} - \frac{(-4x)^2}{4} \\
&= 4x^2 - 3 - \frac{(-4)}{2} - \frac{(16x^2)}{4} \\
&= 4x^2 - 3 - (-2) - 4x^2 \\
&= -1
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-4x}{2}} \\
&= \mathrm{e}^{x^2}
\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, \mathrm{e}^{x^2} \tag{4}$$

Applying this change of variable to the original ode results in

$$-v(x) + v''(x) = 1$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = 1$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$-v(x) + v''(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ v(x) &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$v(x) = c_1 e^x + c_2 e^{-x}$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 e^x + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$v_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = -1$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= \left(c_1 e^{x} + c_2 e^{-x}\right) + (-1) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 e^{x} + c_2 e^{-x} - 1\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = e^{x^2}$$

Hence (7) becomes

$$y = \left(c_1 e^x + c_2 e^{-x} - 1\right) e^{x^2}$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 e^x + c_2 e^{-x} - 1\right) e^{x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x e^{x^2}$$
$$y_2 = e^{-x} e^{x^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x e^{x^2} & e^{-x} e^{x^2} \\ \frac{d}{dx}\left(e^x e^{x^2}\right) & \frac{d}{dx}\left(e^{-x} e^{x^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x e^{x^2} & e^{-x} e^{x^2} \\ e^x e^{x^2} + 2\,e^x x\,e^{x^2} & -e^{-x} e^{x^2} + 2\,e^{-x} x\,e^{x^2} \end{vmatrix}$$

Therefore

$$W = \left(e^x e^{x^2}\right)\left(-e^{-x} e^{x^2} + 2\,e^{-x} x\,e^{x^2}\right) - \left(e^{-x} e^{x^2}\right)\left(e^x e^{x^2} + 2\,e^x x\,e^{x^2}\right)$$

Which simplifies to

$$W = -2\,e^{2x^2} e^x e^{-x}$$

Which simplifies to

$$W = -2\,e^{2x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-x} e^{2x^2}}{-2\,e^{2x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-x}}{2} dx$$

Hence

$$u_1 = -\frac{e^{-x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^x e^{2x^2}}{-2\,e^{2x^2}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{e^x}{2} dx$$

Hence

$$u_2 = -\frac{e^x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\mathrm{e}^{-x}\mathrm{e}^{x}\mathrm{e}^{x^2}$$

Which simplifies to

$$y_p(x) = -\mathrm{e}^{x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1\mathrm{e}^{x} + c_2\mathrm{e}^{-x} - 1\right)\mathrm{e}^{x^2}\right) + \left(-\mathrm{e}^{x^2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(c_1\mathrm{e}^{x} + c_2\mathrm{e}^{-x} - 1\right)\mathrm{e}^{x^2} - \mathrm{e}^{x^2} \tag{1}$$

Verification of solutions

$$y = \left(c_1\mathrm{e}^{x} + c_2\mathrm{e}^{-x} - 1\right)\mathrm{e}^{x^2} - \mathrm{e}^{x^2}$$

Verified OK.

2.32.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4xy' + \left(4x^2 - 3\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 91: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\,dx} \\
&= z_1 e^{x^2} \\
&= z_1 \left(\mathrm{e}^{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x(x-1)}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{x(x-1)}\right) + c_2\left(e^{x(x-1)}\left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4xy' + \left(4x^2 - 3\right) y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 e^{x(x-1)} + \frac{c_2 e^{x(1+x)}}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= e^{x(x-1)} \\
y_2 &= \frac{e^{x(1+x)}}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{x(x-1)} & \frac{e^{x(1+x)}}{2} \\ \frac{d}{dx}\left(e^{x(x-1)}\right) & \frac{d}{dx}\left(\frac{e^{x(1+x)}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{x(x-1)} & \frac{e^{x(1+x)}}{2} \\ (2x-1)\,e^{x(x-1)} & \frac{(2x+1)e^{x(1+x)}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{x(x-1)}\right)\left(\frac{(2x+1)\,e^{x(1+x)}}{2}\right) - \left(\frac{e^{x(1+x)}}{2}\right)\left((2x-1)\,e^{x(x-1)}\right)$$

Which simplifies to

$$W = e^{x(x-1)}e^{x(1+x)}$$

Which simplifies to

$$W = e^{2x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{x(1+x)}e^{x^2}}{2}}{e^{2x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^x}{2}dx$$

Hence

$$u_1 = -\frac{e^x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{x(x-1)}e^{x^2}}{e^{2x^2}}\,dx$$

Which simplifies to

$$u_2 = \int e^{-x} dx$$

Hence

$$u_2 = -e^{-x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^{x(x-1)} e^x}{2} - \frac{e^{-x} e^{x(1+x)}}{2}$$

Which simplifies to

$$y_p(x) = -e^{x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{x(x-1)} + \frac{c_2 e^{x(1+x)}}{2}\right) + \left(-e^{x^2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x(x-1)} + \frac{c_2 e^{x(1+x)}}{2} - e^{x^2} \tag{1}$$

Verification of solutions

$$y = c_1 e^{x(x-1)} + \frac{c_2 e^{x(1+x)}}{2} - e^{x^2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)-4*x*diff(y(x),x)+(4*x^2-3)*y(x)=exp(x^2),y(x), singsol=all)
```

$$y(x) = e^{x(x+1)}c_2 + e^{x(x-1)}c_1 - e^{x^2}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 34

```
DSolve[y''[x]-4*x*y'[x]+(4*x^2-3)*y[x]==Exp[x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{(x-1)x}\left(-2e^x + c_2e^{2x} + 2c_1\right)$$

2.33 problem 33

2.33.1 Solving as second order change of variable on y method 1 ode . 1007
2.33.2 Solving using Kovacic algorithm 1016

Internal problem ID [7474]
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Book: Second order enumerated odes
Section: section 2
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 2\tan(x)\, y' + 5y = \mathrm{e}^{x^2}\sec(x)}$$

2.33.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2\tan(x)\, y' + 5y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -2\tan(x)$$
$$q(x) = 5$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= 5 - \frac{(-2\tan(x))'}{2} - \frac{(-2\tan(x))^2}{4} \\
&= 5 - \frac{(-2 - 2\tan(x)^2)}{2} - \frac{(4\tan(x)^2)}{4} \\
&= 5 - (-1 - \tan(x)^2) - \tan(x)^2 \\
&= 6
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-2\tan(x)}{2}} \\
&= \sec(x)
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\sec(x) \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(x) + 6v(x) = \mathrm{e}^{x^2}
$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = f(x)
$$

Where $A = 1, B = 0, C = 6, f(x) = \mathrm{e}^{x^2}$. Let the solution be

$$
v(x) = v_h + v_p
$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$
v''(x) + 6v(x) = 0
$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 6$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 6\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(6)} \\ &= \pm i\sqrt{6}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{6} \\ \lambda_2 &= -i\sqrt{6}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{6} \\ \lambda_2 &= -i\sqrt{6}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{6}$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0\left(c_1 \cos\left(\sqrt{6}\,x\right) + c_2 \sin\left(\sqrt{6}\,x\right)\right)$$

Or

$$v(x) = c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right)$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right)$$

The particular solution v_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$v_p(x) = u_1 v_1 + u_2 v_2 \tag{1}$$

Where u_1, u_2 to be determined, and v_1, v_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$v_1 = \cos\left(\sqrt{6}\, x\right)$$
$$v_2 = \sin\left(\sqrt{6}\, x\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{v_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{v_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of v'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\sqrt{6}\, x\right) & \sin\left(\sqrt{6}\, x\right) \\ \frac{d}{dx}\left(\cos\left(\sqrt{6}\, x\right)\right) & \frac{d}{dx}\left(\sin\left(\sqrt{6}\, x\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\sqrt{6}\, x\right) & \sin\left(\sqrt{6}\, x\right) \\ -\sqrt{6}\, \sin\left(\sqrt{6}\, x\right) & \sqrt{6}\, \cos\left(\sqrt{6}\, x\right) \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\sqrt{6}\, x\right)\right)\left(\sqrt{6}\, \cos\left(\sqrt{6}\, x\right)\right) - \left(\sin\left(\sqrt{6}\, x\right)\right)\left(-\sqrt{6}\, \sin\left(\sqrt{6}\, x\right)\right)$$

Which simplifies to

$$W = \cos\left(\sqrt{6}\, x\right)^{2} \sqrt{6} + \sin\left(\sqrt{6}\, x\right)^{2} \sqrt{6}$$

Which simplifies to

$$W = \sqrt{6}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin\left(\sqrt{6}\, x\right) \mathrm{e}^{x^2}}{\sqrt{6}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin\left(\sqrt{6}\, x\right) \mathrm{e}^{x^2} \sqrt{6}}{6} dx$$

Hence

$$u_1 = \frac{\sqrt{6}\, \sqrt{\pi}\, \mathrm{e}^{\frac{3}{2}} \left(2 \operatorname{erf}\left(\frac{\sqrt{6}}{2}\right) + \operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right) - \operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right)\right)}{24}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\sqrt{6}\, x\right) \mathrm{e}^{x^2}}{\sqrt{6}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\cos\left(\sqrt{6}\, x\right) \mathrm{e}^{x^2} \sqrt{6}}{6} dx$$

Hence

$$u_2 = -\frac{i\sqrt{6}\,\sqrt{\pi}\,e^{\frac{3}{2}}\left(\operatorname{erf}\left(ix+\frac{\sqrt{6}}{2}\right)+\operatorname{erf}\left(ix-\frac{\sqrt{6}}{2}\right)\right)}{24}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned}v_p(x) = {} & \frac{\sqrt{6}\,\sqrt{\pi}\,e^{\frac{3}{2}}\left(2\operatorname{erf}\left(\frac{\sqrt{6}}{2}\right)+\operatorname{erf}\left(ix-\frac{\sqrt{6}}{2}\right)-\operatorname{erf}\left(ix+\frac{\sqrt{6}}{2}\right)\right)\cos\left(\sqrt{6}\,x\right)}{24} \\ & - \frac{i\sqrt{6}\,\sqrt{\pi}\,e^{\frac{3}{2}}\left(\operatorname{erf}\left(ix+\frac{\sqrt{6}}{2}\right)+\operatorname{erf}\left(ix-\frac{\sqrt{6}}{2}\right)\right)\sin\left(\sqrt{6}\,x\right)}{24}\end{aligned}$$

Which simplifies to

$$v_p(x) = -\frac{\sqrt{\pi}\,\sqrt{6}\left(\left(i\sin\left(\sqrt{6}\,x\right)-\cos\left(\sqrt{6}\,x\right)\right)\operatorname{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+\left(i\sin\left(\sqrt{6}\,x\right)+\cos\left(\sqrt{6}\,x\right)\right)\operatorname{erf}\left(ix+\frac{\sqrt{6}}{2}\right)-2\,\mathrm{c}}{24}$$

Therefore the general solution is

$$\begin{aligned}v &= v_h + v_p \\ &= \left(c_1\cos\left(\sqrt{6}\,x\right)+c_2\sin\left(\sqrt{6}\,x\right)\right) \\ &\quad + \left(-\frac{\sqrt{\pi}\,\sqrt{6}\left(\left(i\sin\left(\sqrt{6}\,x\right)-\cos\left(\sqrt{6}\,x\right)\right)\operatorname{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+\left(i\sin\left(\sqrt{6}\,x\right)+\cos\left(\sqrt{6}\,x\right)\right)\operatorname{erf}\left(ix+\frac{\sqrt{6}}{2}\right)}{24}\right.\end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\,z(x) \\ &= \left(c_1\cos\left(\sqrt{6}\,x\right)+c_2\sin\left(\sqrt{6}\,x\right)-\frac{\sqrt{\pi}\,\sqrt{6}\left(\left(i\sin\left(\sqrt{6}\,x\right)-\cos\left(\sqrt{6}\,x\right)\right)\operatorname{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+\left(i\sin\left(\sqrt{6}\,x\right)\right.\right.}{24}\right.\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \sec(x)$$

Hence (7) becomes

$$y = \left(c_1 \cos\left(\sqrt{6}\,x\right) + c_2 \sin\left(\sqrt{6}\,x\right) - \frac{\sqrt{\pi}\,\sqrt{6}\,\left(\left(i \sin\left(\sqrt{6}\,x\right) - \cos\left(\sqrt{6}\,x\right)\right) \operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \left(i \sin\left(\sqrt{6}\,x\right)\right.\right.}{24}\right.$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 \cos\left(\sqrt{6}\,x\right) + c_2 \sin\left(\sqrt{6}\,x\right)\right.$$
$$- \frac{\sqrt{\pi}\,\sqrt{6}\,\left(\left(i \sin\left(\sqrt{6}\,x\right) - \cos\left(\sqrt{6}\,x\right)\right) \operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \left(i \sin\left(\sqrt{6}\,x\right) + \cos\left(\sqrt{6}\,x\right)\right) \operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right) - 2\,c\right.}{24}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\sqrt{6}\,x\right) \sec(x)$$

$$y_2 = \sin\left(\sqrt{6}\,x\right) \sec(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\sqrt{6}\,x\right) \sec(x) & \sin\left(\sqrt{6}\,x\right) \sec(x) \\ \frac{d}{dx}\left(\cos\left(\sqrt{6}\,x\right) \sec(x)\right) & \frac{d}{dx}\left(\sin\left(\sqrt{6}\,x\right) \sec(x)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\sqrt{6}\,x\right)\sec\left(x\right) & \sin\left(\sqrt{6}\,x\right)\sec\left(x\right) \\ -\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec\left(x\right)+\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)\tan\left(x\right) & \sqrt{6}\,\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)+\sin\left(\sqrt{6}\,x\right)\sec\left(x\right)\tan \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)\right)\left(\sqrt{6}\,\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)+\sin\left(\sqrt{6}\,x\right)\sec\left(x\right)\tan\left(x\right)\right) \\ -\left(\sin\left(\sqrt{6}\,x\right)\sec\left(x\right)\right)\left(-\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec\left(x\right)+\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)\tan\left(x\right)\right)$$

Which simplifies to

$$W = \sec\left(x\right)^{2}\sqrt{6}\,\cos\left(\sqrt{6}\,x\right)^{2}+\sec\left(x\right)^{2}\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)^{2}$$

Which simplifies to

$$W = \sqrt{6}\,\sec\left(x\right)^{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin\left(\sqrt{6}\,x\right)\sec\left(x\right)^{2}\mathrm{e}^{x^{2}}}{\sqrt{6}\,\sec\left(x\right)^{2}}dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin\left(\sqrt{6}\,x\right)\mathrm{e}^{x^{2}}\sqrt{6}}{6}dx$$

Hence

$$u_1 = \frac{\sqrt{6}\,\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\left(2\,\mathrm{erf}\left(\frac{\sqrt{6}}{2}\right)+\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)-\mathrm{erf}\left(ix+\frac{\sqrt{6}}{2}\right)\right)}{24}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)^{2}\mathrm{e}^{x^{2}}}{\sqrt{6}\,\sec\left(x\right)^{2}}dx$$

Which simplifies to

$$u_2 = \int \frac{\cos\left(\sqrt{6}\,x\right)\mathrm{e}^{x^{2}}\sqrt{6}}{6}dx$$

Hence

$$u_2 = -\frac{i\sqrt{6}\,\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\left(\mathrm{erf}\left(ix + \frac{\sqrt{6}}{2}\right) + \mathrm{erf}\left(ix - \frac{\sqrt{6}}{2}\right)\right)}{24}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\sqrt{6}\,\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\left(2\ \mathrm{erf}\left(\frac{\sqrt{6}}{2}\right) + \mathrm{erf}\left(ix - \frac{\sqrt{6}}{2}\right) - \mathrm{erf}\left(ix + \frac{\sqrt{6}}{2}\right)\right)\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)}{24} - \frac{i\sqrt{6}\,\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\left(\mathrm{erf}\left(ix + \frac{\sqrt{6}}{2}\right) + \mathrm{erf}\left(ix - \frac{\sqrt{6}}{2}\right)\right)\sin\left(\sqrt{6}\,x\right)\sec\left(x\right)}{24}$$

Which simplifies to

$$y_p(x) = -\frac{\sec\left(x\right)\mathrm{e}^{\frac{3}{2}}\left(\left(i\sin\left(\sqrt{6}\,x\right) - \cos\left(\sqrt{6}\,x\right)\right)\mathrm{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \left(i\sin\left(\sqrt{6}\,x\right) + \cos\left(\sqrt{6}\,x\right)\right)\mathrm{erf}\left(ix + \frac{\sqrt{6}}{2}\right) - 2 \cdots}{24}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(\left(c_1\cos\left(\sqrt{6}\,x\right) + c_2\sin\left(\sqrt{6}\,x\right)\right.\right. - \frac{\sqrt{\pi}\,\sqrt{6}\left(\left(i\sin\left(\sqrt{6}\,x\right) - \cos\left(\sqrt{6}\,x\right)\right)\mathrm{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \left(i\sin\left(\sqrt{6}\,x\right) + \cos\left(\sqrt{6}\,x\right)\right)\mathrm{erf}\left(ix + \frac{\sqrt{6}}{2}\right) - 2 \cdots}{24}$$

$$+ \left(-\frac{\sec\left(x\right)\mathrm{e}^{\frac{3}{2}}\left(\left(i\sin\left(\sqrt{6}\,x\right) - \cos\left(\sqrt{6}\,x\right)\right)\mathrm{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \left(i\sin\left(\sqrt{6}\,x\right) + \cos\left(\sqrt{6}\,x\right)\right)\mathrm{erf}\left(ix + \frac{\sqrt{6}}{2} \cdots\right.\right.}{24}\right.$$

Summary

The solution(s) found are the following

$$y = \left(c_1\cos\left(\sqrt{6}\,x\right) + c_2\sin\left(\sqrt{6}\,x\right)\right. \tag{1}$$

$$- \frac{\sqrt{\pi}\,\sqrt{6}\left(\left(i\sin\left(\sqrt{6}\,x\right) - \cos\left(\sqrt{6}\,x\right)\right)\mathrm{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \left(i\sin\left(\sqrt{6}\,x\right) + \cos\left(\sqrt{6}\,x\right)\right)\mathrm{erf}\left(ix + \frac{\sqrt{6}}{2}\right) - 2\,\mathrm{c} \cdots}{24}$$

$$- \frac{\sec\left(x\right)\mathrm{e}^{\frac{3}{2}}\left(\left(i\sin\left(\sqrt{6}\,x\right) - \cos\left(\sqrt{6}\,x\right)\right)\mathrm{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \left(i\sin\left(\sqrt{6}\,x\right) + \cos\left(\sqrt{6}\,x\right)\right)\mathrm{erf}\left(ix + \frac{\sqrt{6}}{2}\right) - 2 \cdots}{24}$$

Verification of solutions

$$y = \left(c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right)\right.$$
$$- \frac{\sqrt{\pi}\,\sqrt{6}\left(\left(i \sin\left(\sqrt{6}\, x\right) - \cos\left(\sqrt{6}\, x\right)\right) \operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \left(i \sin\left(\sqrt{6}\, x\right) + \cos\left(\sqrt{6}\, x\right)\right) \operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right) - 2\,\mathrm{c}}{24}$$
$$- \frac{\sec\left(x\right) \mathrm{e}^{\frac{3}{2}}\left(\left(i \sin\left(\sqrt{6}\, x\right) - \cos\left(\sqrt{6}\, x\right)\right) \operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \left(i \sin\left(\sqrt{6}\, x\right) + \cos\left(\sqrt{6}\, x\right)\right) \operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right) - 2}{24}$$

Verified OK.

2.33.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2\tan\left(x\right) y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2\tan\left(x\right) \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-6}{1} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -6 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -6z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 92: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -6$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\sqrt{6}\, x\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2\tan(x)}{1}\, dx} \\ &= z_1 e^{-\ln(\cos(x))} \\ &= z_1(\sec(x)) \end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(\sqrt{6}\, x\right)\sec(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2\tan(x)}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2\ln(\cos(x))}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\sqrt{6}\, \tan\left(\sqrt{6}\, x\right)}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\cos\left(\sqrt{6}\, x\right)\sec(x)\right) + c_2\left(\cos\left(\sqrt{6}\, x\right)\sec(x)\left(\frac{\sqrt{6}\, \tan\left(\sqrt{6}\, x\right)}{6}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2\tan(x)\, y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \sec(x)\, c_1 \cos\left(\sqrt{6}\, x\right) + \frac{c_2\sqrt{6}\, \sin\left(\sqrt{6}\, x\right) \sec(x)}{6}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\sqrt{6}\, x\right) \sec(x)$$

$$y_2 = \frac{\sqrt{6}\, \sin\left(\sqrt{6}\, x\right) \sec(x)}{6}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\sqrt{6}\,x\right)\sec\left(x\right) & \frac{\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec(x)}{6} \\ \frac{d}{dx}\left(\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)\right) & \frac{d}{dx}\left(\frac{\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec(x)}{6}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\sqrt{6}\,x\right)\sec\left(x\right) & \frac{\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec(x)}{6} \\ -\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec\left(x\right)+\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)\tan\left(x\right) & \cos\left(\sqrt{6}\,x\right)\sec\left(x\right)+\frac{\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec(x)\tan(x)}{6} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)\right)\left(\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)+\frac{\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec\left(x\right)\tan\left(x\right)}{6}\right) - \left(\frac{\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec\left(x\right)}{6}\right)\left(-\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec\left(x\right) + \cos\left(\sqrt{6}\,x\right)\sec\left(x\right)\tan\left(x\right)\right)$$

Which simplifies to

$$W = \cos\left(\sqrt{6}\,x\right)^2\sec\left(x\right)^2+\sec\left(x\right)^2\sin\left(\sqrt{6}\,x\right)^2$$

Which simplifies to

$$W = \sec\left(x\right)^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec(x)^2 e^{x^2}}{6}}{\sec\left(x\right)^2}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin\left(\sqrt{6}\,x\right)e^{x^2}\sqrt{6}}{6}dx$$

Hence

$$u_1 = \frac{\sqrt{6}\,\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\left(2\ \mathrm{erf}\left(\frac{\sqrt{6}}{2}\right)+\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)-\mathrm{erf}\left(ix+\frac{\sqrt{6}}{2}\right)\right)}{24}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)^{2}\mathrm{e}^{x^{2}}}{\sec\left(x\right)^{2}}\,dx$$

Which simplifies to

$$u_2 = \int \cos\left(\sqrt{6}\,x\right)\mathrm{e}^{x^{2}}dx$$

Hence

$$u_2 = -\frac{i\mathrm{e}^{\frac{3}{2}}\sqrt{\pi}\left(\mathrm{erf}\left(ix+\frac{\sqrt{6}}{2}\right)+\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)\right)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\sqrt{6}\,\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\left(2\ \mathrm{erf}\left(\frac{\sqrt{6}}{2}\right)+\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)-\mathrm{erf}\left(ix+\frac{\sqrt{6}}{2}\right)\right)\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)}{24} - \frac{i\sqrt{6}\,\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\left(\mathrm{erf}\left(ix+\frac{\sqrt{6}}{2}\right)+\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)\right)\sin\left(\sqrt{6}\,x\right)\sec\left(x\right)}{24}$$

Which simplifies to

$$y_p(x) = -\frac{\sec\left(x\right)\mathrm{e}^{\frac{3}{2}}\left(\left(i\sin\left(\sqrt{6}\,x\right)-\cos\left(\sqrt{6}\,x\right)\right)\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+\left(i\sin\left(\sqrt{6}\,x\right)+\cos\left(\sqrt{6}\,x\right)\right)\mathrm{erf}\left(ix+\frac{\sqrt{6}}{2}\right)-}{24}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\sec\left(x\right)c_1\cos\left(\sqrt{6}\,x\right)+\frac{c_2\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec\left(x\right)}{6}\right) \\ &\quad + \left(-\frac{\sec\left(x\right)\mathrm{e}^{\frac{3}{2}}\left(\left(i\sin\left(\sqrt{6}\,x\right)-\cos\left(\sqrt{6}\,x\right)\right)\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+\left(i\sin\left(\sqrt{6}\,x\right)+\cos\left(\sqrt{6}\,x\right)\right)\mathrm{erf}\left(ix+\frac{\sqrt{6}}{2}\right.}{24}\right. \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sec(x) c_1 \cos\left(\sqrt{6}\, x\right) + \frac{c_2 \sqrt{6}\, \sin\left(\sqrt{6}\, x\right) \sec(x)}{6} - \frac{\sec(x)\, e^{\frac{3}{2}} \left(\left(i \sin\left(\sqrt{6}\, x\right) - \cos\left(\sqrt{6}\, x\right)\right) \operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \left(i \sin\left(\sqrt{6}\, x\right) + \cos\left(\sqrt{6}\, x\right)\right) \operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right) - 2\right.}{24} \tag{1}$$

Verification of solutions

$$y = \sec(x) c_1 \cos\left(\sqrt{6}\, x\right) + \frac{c_2 \sqrt{6}\, \sin\left(\sqrt{6}\, x\right) \sec(x)}{6} - \frac{\sec(x)\, e^{\frac{3}{2}} \left(\left(i \sin\left(\sqrt{6}\, x\right) - \cos\left(\sqrt{6}\, x\right)\right) \operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \left(i \sin\left(\sqrt{6}\, x\right) + \cos\left(\sqrt{6}\, x\right)\right) \operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right) - 2\right.}{24}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 101

```
dsolve(diff(y(x),x$2)-2*tan(x)*diff(y(x),x)+5*y(x)=exp(x^2)*sec(x),y(x), singsol=all)
```

$$y(x) = -\frac{\left(\sqrt{6}\,e^{\frac{3}{2}}\sqrt{\pi}\left(i\sin\left(\sqrt{6}\,x\right)-\cos\left(\sqrt{6}\,x\right)\right)\operatorname{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+\sqrt{6}\,e^{\frac{3}{2}}\left(i\sin\left(\sqrt{6}\,x\right)+\cos\left(\sqrt{6}\,x\right)\right)\sqrt{\pi}\operatorname{erf}\left(ix\right.}{24}$$

✓ Solution by Mathematica

Time used: 0.249 (sec). Leaf size: 118

```
DSolve[y''[x]-2*Tan[x]*y'[x]+5*y[x]==Exp[x^2]*Sec[x],y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{1}{24}e^{-i\sqrt{6}x}\sec(x)\left(-e^{3/2}\sqrt{6\pi}\operatorname{erf}\left(\sqrt{\frac{3}{2}}-ix\right)-\sqrt{6\pi}e^{\frac{3}{2}+2i\sqrt{6}x}\operatorname{erf}\left(\sqrt{\frac{3}{2}}+ix\right)\right.$$
$$\left.-2i\sqrt{6}c_2e^{2i\sqrt{6}x}+24c_1\right)$$

2.34 problem 34

2.34.1 Solving as second order change of variable on y method 1 ode . 1024
2.34.2 Solving as second order bessel ode ode 1027
2.34.3 Solving using Kovacic algorithm 1028
2.34.4 Maple step by step solution 1031

Internal problem ID [7475]
Internal file name [`OUTPUT/6442_Sunday_June_19_2022_05_02_44_PM_80026835/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - 2xy' + 2(x^2+1)\, y = 0}$$

2.34.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{2x^2+2}{x^2}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{2x^2+2}{x^2} - \frac{\left(-\frac{2}{x}\right)'}{2} - \frac{\left(-\frac{2}{x}\right)^2}{4} \\
&= \frac{2x^2+2}{x^2} - \frac{\left(\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\
&= \frac{2x^2+2}{x^2} - \left(\frac{1}{x^2}\right) - \frac{1}{x^2} \\
&= 2
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{2}{x}}{2}} \\
&= x
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, x \tag{4}
$$

Applying this change of variable to the original ode results in

$$
x^3(2v(x) + v''(x)) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda x} + 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$
\lambda^2 + 2 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(2)} \\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0\left(c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)\right)$$

Or

$$v(x) = c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\,z(x) \\ &= \left(c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)\right)(z(x)) \qquad (7)\end{aligned}$$

But from (5)

$$z(x) = x$$

Hence (7) becomes

$$y = \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) x$$

Summary

The solution(s) found are the following

$$y = \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) x \tag{1}$$

Verification of solutions

$$y = \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) x$$

Verified OK.

2.34.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - 2xy' + \left(2x^2 + 2\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, xy' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{3}{2}$$
$$\beta = \sqrt{2}$$
$$n = -\frac{1}{2}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x^{\frac{3}{2}} \sqrt{2}\, \cos\left(\sqrt{2}\, x\right)}{\sqrt{\pi}\, \sqrt{\sqrt{2}\, x}} + \frac{c_2 x^{\frac{3}{2}} \sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{\sqrt{\pi}\, \sqrt{\sqrt{2}\, x}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{3}{2}} \sqrt{2}\, \cos\left(\sqrt{2}\, x\right)}{\sqrt{\pi}\, \sqrt{\sqrt{2}\, x}} + \frac{c_2 x^{\frac{3}{2}} \sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{\sqrt{\pi}\, \sqrt{\sqrt{2}\, x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{3}{2}} \sqrt{2}\, \cos\left(\sqrt{2}\, x\right)}{\sqrt{\pi}\, \sqrt{\sqrt{2}\, x}} + \frac{c_2 x^{\frac{3}{2}} \sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{\sqrt{\pi}\, \sqrt{\sqrt{2}\, x}}$$

Verified OK.

2.34.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 2xy' + \left(2x^2 + 2\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= 2x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 93: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\sqrt{2}\, x\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(\sqrt{2}\, x\right) x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\cos\left(\sqrt{2}\, x\right) x\right) + c_2\left(\cos\left(\sqrt{2}\, x\right) x\left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos\left(\sqrt{2}\, x\right) x + \frac{c_2 \sin\left(\sqrt{2}\, x\right) \sqrt{2}\, x}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \cos\left(\sqrt{2}\, x\right) x + \frac{c_2 \sin\left(\sqrt{2}\, x\right) \sqrt{2}\, x}{2}
$$

Verified OK.

2.34.4 Maple step by step solution

Let's solve

$x^2 y'' - 2xy' + (2x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(x^2+1)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{2(x^2+1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{2(x^2+1)}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2y''-2xy'+(2x^2+2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x\cdot y'$ to series expansion

    $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

  - Convert $x^2\cdot y''$ to series expansion

    $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\,x^r+a_1r(-1+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(k+r-1)\,(k+r-2)+2a_{k-2})\,x^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{1,2\}$

- Each term must be 0

  $a_1r(-1+r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2)+2a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)(k+r)+2a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{2a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r=1$

  $a_{k+2}=-\frac{2a_k}{(k+2)(k+1)}$

- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{2a_k}{(k+2)(k+1)}, a_1=0\right]$

- Recursion relation for $r=2$

  $a_{k+2}=-\frac{2a_k}{(k+3)(k+2)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{2a_k}{(k+3)(k+2)}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{2a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{2b_k}{(k+3)(k+2)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+2*(1+x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x\left(c_1 \sin\left(\sqrt{2}\, x\right) + c_2 \cos\left(\sqrt{2}\, x\right)\right)$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 48

```
DSolve[x^2*y''[x]-2*x*y'[x]+2*(1+x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-i\sqrt{2}x} x - \frac{i c_2 e^{i\sqrt{2}x} x}{2\sqrt{2}}$$

2.35 problem 35

2.35.1 Solving using Kovacic algorithm 1035
2.35.2 Maple step by step solution 1040

Internal problem ID [7476]
Internal file name [`OUTPUT/6443_Sunday_June_19_2022_05_02_45_PM_90077720/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 4y'x^5 + (x^8 + 6x^4 + 4)\, y = 0}$$

2.35.1 Solving using Kovacic algorithm

Writing the ode as

$$4x^2y'' + 4y'x^5 + (x^8 + 6x^4 + 4)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 4x^5 \\ C &= x^8 + 6x^4 + 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 95: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -1$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -1$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \frac{i\sqrt{3}}{2} - \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} + (-)(0) \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} \\
&= \frac{1 - i\sqrt{3}}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x^2}\right) + \left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x}\right)^2 - \left(-\frac{1}{x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} dx} \\ &= x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{4x^5}{4x^2} \, dx} \\ &= z_1 e^{-\frac{x^4}{8}} \\ &= z_1 \left(e^{-\frac{x^4}{8}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}} e^{-\frac{x^4}{8}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^5}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^4}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i x^{i\sqrt{3}}\sqrt{3}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \mathrm{e}^{-\frac{x^4}{8}}\right) + c_2 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \mathrm{e}^{-\frac{x^4}{8}} \left(-\frac{i x^{i\sqrt{3}}\sqrt{3}}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \mathrm{e}^{-\frac{x^4}{8}} - \frac{i c_2 \sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}} \mathrm{e}^{-\frac{x^4}{8}}}{3} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \mathrm{e}^{-\frac{x^4}{8}} - \frac{i c_2 \sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}} \mathrm{e}^{-\frac{x^4}{8}}}{3}
$$

Verified OK.

2.35.2 Maple step by step solution

Let's solve

$4x^2 y'' + 4y' x^5 + (x^8 + 6x^4 + 4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' x^3 - \frac{(x^8+6x^4+4)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + y' x^3 + \frac{(x^8+6x^4+4) y}{4x^2} = 0$$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $$\left[P_2(x) = x^3, P_3(x) = \frac{x^8+6x^4+4}{4x^2}\right]$$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $$x_0 = 0$$
- Multiply by denominators

  $$4x^2 y'' + 4y' x^5 + (x^8 + 6x^4 + 4) y = 0$$
- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..8$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - Convert $x^5 \cdot y'$ to series expansion

    $$x^5 \cdot y' = \sum_{k=0}^{\infty} a_k (k + r) x^{k+r+4}$$
  - Shift index using $k- >k - 4$

    $$x^5 \cdot y' = \sum_{k=4}^{\infty} a_{k-4} (k - 4 + r) x^{k+r}$$
  - Convert $x^2 \cdot y''$ to series expansion

$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$4a_0(r^2-r+1)\, x^r + 4a_1(r^2+r+1)\, x^{1+r} + 4a_2(r^2+3r+3)\, x^{2+r} + 4a_3(r^2+5r+7)\, x^{3+r} + (4a_4$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r^2 - 4r + 4 = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right\}$

- The coefficients of each power of x must be 0

  $[4a_1(r^2+r+1) = 0, 4a_2(r^2+3r+3) = 0, 4a_3(r^2+5r+7) = 0, 4a_4(r^2+7r+13) + 2a_0(3+2r$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = 0, a_3 = 0, a_4 = -\frac{a_0(3+2r)}{2(r^2+7r+13)}, a_5 = 0, a_6 = 0, a_7 = 0\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4(1+k^2+(2r-1)\, k + r^2 - r)\, a_k + 2a_{k-4}(2k-5+2r) + a_{k-8} = 0$

- Shift index using $k- >k+8$

  $4\left(1+(k+8)^2+(2r-1)\,(k+8)+r^2-r\right) a_{k+8} + 2a_{k+4}(2k+11+2r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+8} = -\frac{4ka_{k+4}+4ra_{k+4}+a_k+22a_{k+4}}{4(k^2+2kr+r^2+15k+15r+57)}$

- Recursion relation for $r = \frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}$

  $a_{k+8} = -\frac{4ka_{k+4}+4\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)a_{k+4}+a_k+22a_{k+4}}{4\left(k^2+2k\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)+\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+15k+\frac{129}{2}-\frac{15\,\mathrm{I}\sqrt{3}}{2}\right)}$

- Solution for $r = \frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}, a_{k+8} = -\frac{4ka_{k+4}+4\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)a_{k+4}+a_k+22a_{k+4}}{4\left(k^2+2k\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)+\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+15k+\frac{129}{2}-\frac{15\,\mathrm{I}\sqrt{3}}{2}\right)}, a_1 = 0, a_2 = 0, a_3 = 0, a_4\right.$

- Recursion relation for $r = \frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}$

  $a_{k+8} = -\frac{4ka_{k+4}+4\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)a_{k+4}+a_k+22a_{k+4}}{4\left(k^2+2k\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)+\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+15k+\frac{129}{2}+\frac{15\,\mathrm{I}\sqrt{3}}{2}\right)}$

- Solution for $r = \frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}}, a_{k+8} = -\frac{4k a_{k+4}+4\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)a_{k+4}+a_k+22a_{k+4}}{4\left(k^2+2k\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)+\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+15k+\frac{129}{2}+\frac{15\,\mathrm{I}\sqrt{3}}{2}\right)}, a_1 = 0, a_2 = 0, a_3 = 0, a_4\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}}\right), a_{k+8} = -\frac{4k a_{k+4}+4\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)a_{k+4}+a_k+22a_{k+4}}{4\left(k^2+2k\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)+\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+15k+\frac{129}{2}-\frac{15\,\mathrm{I}\sqrt{3}}{2}\right)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(4*x^2*diff(y(x),x$2)+4*x^5*diff(y(x),x)+(x^8+6*x^4+4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x}\,\mathrm{e}^{-\frac{x^4}{8}}\left(c_1 x^{\frac{i\sqrt{3}}{2}} + c_2 x^{-\frac{i\sqrt{3}}{2}}\right)$$

✓ Solution by Mathematica

Time used: 0.076 (sec). Leaf size: 62

```
DSolve[4*x^2*y''[x]+4*x^5*y'[x]+(x^8+6*x^4+4)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{3}e^{-\frac{x^4}{8}}x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\left(3c_1 - i\sqrt{3}c_2 x^{i\sqrt{3}}\right)$$

2.36 problem 36

Internal problem ID [7477]
Internal file name [`OUTPUT/6444_Sunday_June_19_2022_05_02_47_PM_12898953/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$x^2y'' + (xy' - y)^2 = 0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integrating f
   --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for dynam
   -> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integratin
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cases
trying symmetries linear in x and y(x)
<- linear symmetries successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 22

```
dsolve(x^2*diff(y(x),x$2)+(x*diff(y(x),x)-y(x))^2=0,y(x), singsol=all)
```

$$y(x) = \left(-\mathrm{e}^{c_1} \operatorname{expIntegral}_1\left(-\ln\left(\frac{1}{x}\right)+c_1\right)+c_2\right)x$$

✓ Solution by Mathematica

Time used: 46.789 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]+(x*y'[x]-y[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(e^{c_1}\operatorname{ExpIntegralEi}(-c_1-\log(x))+c_2)$$
$$y(x) \to c_2 x$$

2.37 problem 37

2.37.1 Solving as second order change of variable on y method 1 ode . 1046
2.37.2 Solving as second order bessel ode ode 1049
2.37.3 Solving using Kovacic algorithm 1050
2.37.4 Maple step by step solution 1053

Internal problem ID [7478]
Internal file name [`OUTPUT/6445_Sunday_June_19_2022_05_02_50_PM_89484997/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + 2y' - yx = 0}$$

2.37.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = -1$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= -1 - \frac{\left(\frac{2}{x}\right)'}{2} - \frac{\left(\frac{2}{x}\right)^2}{4} \\
&= -1 - \frac{\left(-\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\
&= -1 - \left(-\frac{1}{x^2}\right) - \frac{1}{x^2} \\
&= -1
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{2}{x}}{2}} \\
&= \frac{1}{x}
\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{x} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) - v(x) = 0$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ v(x) &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$v(x) = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x)\\ &= \left(c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}\right)(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x}$$

Hence (7) becomes

$$y = \frac{c_1 e^x + c_2 e^{-x}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^x + c_2 e^{-x}}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^x + c_2 e^{-x}}{x}$$

Verified OK.

2.37.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + 2xy' - x^2 y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) xy' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta x^\gamma) + c_2 \operatorname{BesselY}(n, \beta x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= -\frac{1}{2} \\ \beta &= i \\ n &= \frac{1}{2} \\ \gamma &= 1 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{i c_1 \sqrt{2} \sinh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}} - \frac{c_2 \sqrt{2} \cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}}$$

Summary

The solution(s) found are the following

$$y = \frac{i c_1 \sqrt{2}\, \sinh(x)}{\sqrt{x}\, \sqrt{\pi}\, \sqrt{ix}} - \frac{c_2 \sqrt{2}\, \cosh(x)}{\sqrt{x}\, \sqrt{\pi}\, \sqrt{ix}} \tag{1}$$

Verification of solutions

$$y = \frac{i c_1 \sqrt{2}\, \sinh(x)}{\sqrt{x}\, \sqrt{\pi}\, \sqrt{ix}} - \frac{c_2 \sqrt{2}\, \cosh(x)}{\sqrt{x}\, \sqrt{\pi}\, \sqrt{ix}}$$

Verified OK.

2.37.3 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + 2y' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 97: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{e^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^{-x}}{x}\right) + c_2\left(\frac{e^{-x}}{x}\left(\frac{e^{2x}}{2}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 e^{-x}}{x} + \frac{c_2 e^{x}}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{-x}}{x} + \frac{c_2 e^{x}}{2x}$$

Verified OK.

2.37.4 Maple step by step solution

Let's solve

$x y'' + 2y' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} - y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = -1\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

$$xy'' + 2y' - yx = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

$$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  ◦ Shift index using $k- >k-1$

$$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  ◦ Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

  ◦ Shift index using $k- >k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$$

  ◦ Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

  ◦ Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) - a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- Each term must be 0

$$a_1(1+r)\,(2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+r+1)(k+2+r)-a_{k-1}=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+3+r)-a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

  $a_{k+2}=\frac{a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=\frac{a_k}{(k+1)(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

  $a_{k+2}=\frac{a_k}{(k+2)(k+3)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{a_k}{(k+2)(k+3)}, 2a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2}=\frac{a_k}{(k+1)(k+2)}, 0=0, b_{k+2}=\frac{b_k}{(k+2)(k+3)}, 2b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sinh(x) + c_2 \cosh(x)}{x}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 28

```
DSolve[x*y''[x]+2*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-x} + c_2 e^x}{2x}$$

2.38 problem 38

2.38.1 Solving as second order change of variable on y method 1 ode . 1057
2.38.2 Solving as second order bessel ode ode 1060
2.38.3 Solving using Kovacic algorithm 1061
2.38.4 Maple step by step solution 1064

Internal problem ID [7479]
Internal file name [`OUTPUT/6446_Sunday_June_19_2022_05_02_52_PM_23588908/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' + 2y' + yx = 0}$$

2.38.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = 1$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= 1 - \frac{\left(\frac{2}{x}\right)'}{2} - \frac{\left(\frac{2}{x}\right)^2}{4} \\
&= 1 - \frac{\left(-\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\
&= 1 - \left(-\frac{1}{x^2}\right) - \frac{1}{x^2} \\
&= 1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{2}{x}}{2}} \\
&= \frac{1}{x}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = \frac{v(x)}{x} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(x) + v(x) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x) \\ &= (c_1 \cos(x) + c_2 \sin(x))\,(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x}$$

Hence (7) becomes

$$y = \frac{c_1 \cos(x) + c_2 \sin(x)}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos(x) + c_2 \sin(x)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos(x) + c_2 \sin(x)}{x}$$

Verified OK.

2.38.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + 2xy' + x^2 y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + xy' + (-n^2 + x^2) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) xy' + (\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= -\frac{1}{2} \\
\beta &= 1 \\
n &= \frac{1}{2} \\
\gamma &= 1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{2}\,\sin(x)}{x\sqrt{\pi}} - \frac{c_2\sqrt{2}\,\cos(x)}{x\sqrt{\pi}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{2}\,\sin(x)}{x\sqrt{\pi}} - \frac{c_2\sqrt{2}\,\cos(x)}{x\sqrt{\pi}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{2}\,\sin(x)}{x\sqrt{\pi}} - \frac{c_2\sqrt{2}\,\cos(x)}{x\sqrt{\pi}}$$

Verified OK.

2.38.3 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + 2y' + yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 99: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x}\right) + c_2 \left(\frac{\cos(x)}{x}(\tan(x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x}
$$

Verified OK.

2.38.4 Maple step by step solution

Let's solve

$xy'' + 2y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$xy'' + 2y' + yx = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  ◦ Shift index using $k->k-1$

$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  ◦ Convert y' to series expansion

$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

  ◦ Shift index using $k->k+1$

$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$

  ◦ Convert $x \cdot y''$ to series expansion

$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

  ◦ Shift index using $k->k+1$

$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$$a_0 r(1+r)\, x^{-1+r} + a_1 (1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r) + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x) + c_2 \cos(x)}{x}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 37

```
DSolve[x*y''[x]+2*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.39 problem 39

2.39.1 Solving as linear ode 1068
2.39.2 Solving as first order ode lie symmetry lookup ode 1070
2.39.3 Solving as exact ode 1074
2.39.4 Maple step by step solution 1078

Internal problem ID [7480]
Internal file name [`OUTPUT/6447_Sunday_June_19_2022_05_02_53_PM_8051847/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cot(x) = 2\cos(x)}$$

2.39.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \cot(x) \\ q(x) &= 2\cos(x) \end{aligned}$$

Hence the ode is

$$y' + y\cot(x) = 2\cos(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \cot(x)dx} \\ &= \sin(x) \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\,(2\cos(x)) \\
\frac{d}{dx}(\sin(x)\,y) &= (\sin(x))\,(2\cos(x)) \\
d(\sin(x)\,y) &= \sin(2x)\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\sin(x)\,y &= \int \sin(2x)\,dx \\
\sin(x)\,y &= -\frac{\cos(2x)}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$
y = -\frac{\csc(x)\cos(2x)}{2} + c_1\csc(x)
$$

which simplifies to

$$
y = \csc(x)\left(-\cos(x)^2 + c_1 + \frac{1}{2}\right)
$$

Summary
The solution(s) found are the following

$$
y = \csc(x)\left(-\cos(x)^2 + c_1 + \frac{1}{2}\right) \tag{1}
$$

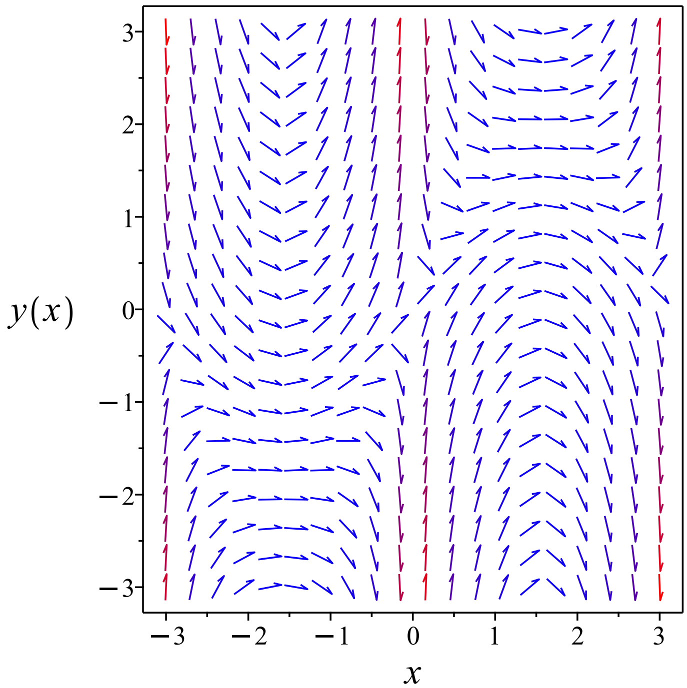


Figure 121: Slope field plot

Verification of solutions

$$y = \csc(x)\left(-\cos(x)^2 + c_1 + \frac{1}{2}\right)$$

Verified OK.

2.39.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y\cot(x) + 2\cos(x)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 101: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sin(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{\sin(x)}} dy
\end{aligned}$$

Which results in

$$S = \sin(x)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -y \cot(x) + 2\cos(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= y \cos(x) \\
S_y &= \sin(x)
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(2x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\cos(2R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\sin(x) = -\frac{\cos(2x)}{2} + c_1$$

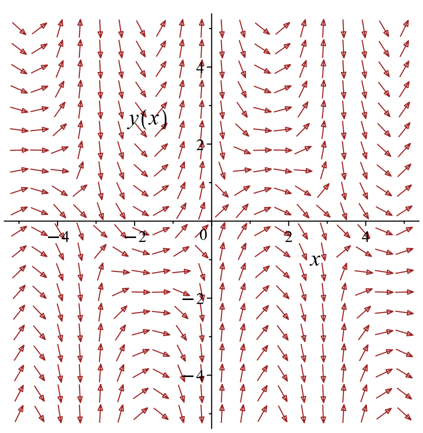
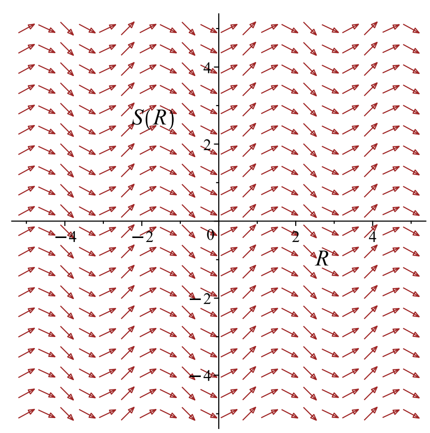
Which simplifies to

$$y\sin(x) = -\frac{\cos(2x)}{2} + c_1$$

Which gives

$$y = -\frac{\cos(2x) - 2c_1}{2\sin(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y\cot(x) + 2\cos(x)$ 	$R = x$ $S = \sin(x)\,y$	$\frac{dS}{dR} = \sin(2R)$ 

Summary

The solution(s) found are the following

$$y = -\frac{\cos(2x) - 2c_1}{2\sin(x)} \tag{1}$$

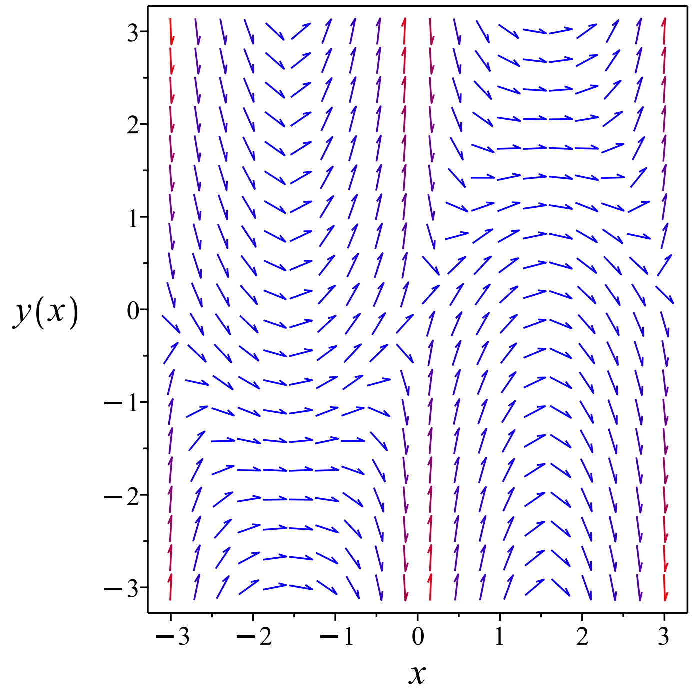


Figure 122: Slope field plot

Verification of solutions

$$y = -\frac{\cos(2x) - 2c_1}{2\sin(x)}$$

Verified OK.

2.39.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-y \cot(x) + 2\cos(x))\,\mathrm{d}x$$
$$(y \cot(x) - 2\cos(x))\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y \cot(x) - 2\cos(x)$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y \cot(x) - 2\cos(x))$$
$$= \cot(x)$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((\cot(x)) - (0)) \\
&= \cot(x)
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \cot(x) \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(\sin(x))} \\
&= \sin(x)
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \sin(x)(y\cot(x) - 2\cos(x)) \\
&= \cos(x)(-2\sin(x) + y)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \sin(x)(1) \\
&= \sin(x)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(\cos(x)(-2\sin(x) + y)) + (\sin(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \cos(x)(-2\sin(x) + y) \mathrm{d}x$$
$$\phi = \sin(x)(-\sin(x) + y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)$. Therefore equation (4) becomes

$$\sin(x) = \sin(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin(x)(-\sin(x) + y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin(x)(-\sin(x) + y)$$

The solution becomes

$$y = \frac{\sin(x)^2 + c_1}{\sin(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin(x)^2 + c_1}{\sin(x)} \tag{1}$$

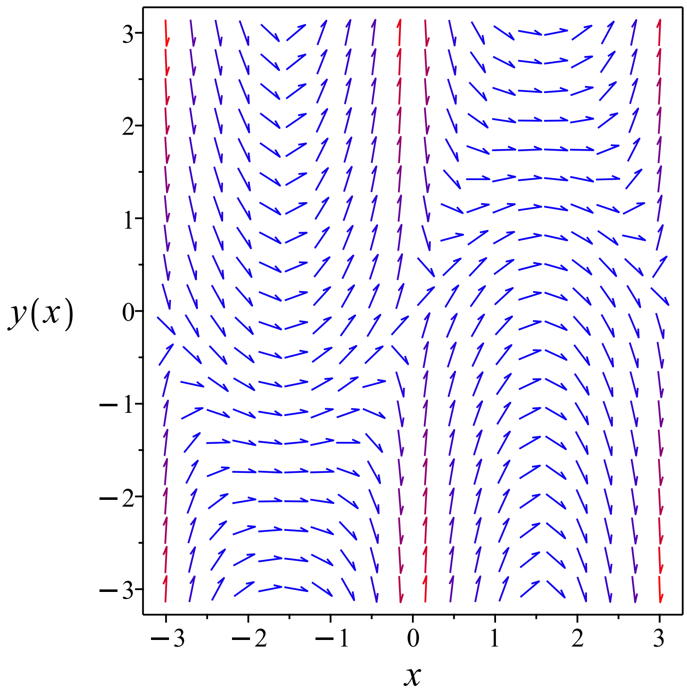


Figure 123: Slope field plot

Verification of solutions

$$y = \frac{\sin(x)^2 + c_1}{\sin(x)}$$

Verified OK.

2.39.4 Maple step by step solution

Let's solve

$y' + y\cot(x) = 2\cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = -y\cot(x) + 2\cos(x)$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + y\cot(x) = 2\cos(x)$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\,(y' + y\cot(x)) = 2\mu(x)\cos(x)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\,(y' + y\cot(x)) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \mu(x)\cot(x)$$

- Solve to find the integrating factor

$$\mu(x) = \sin(x)$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 2\mu(x)\cos(x)\,dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int 2\mu(x)\cos(x)\,dx + c_1$$

- Solve for y

$$y = \frac{\int 2\mu(x)\cos(x)dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \sin(x)$

$$y = \frac{\int 2\sin(x)\cos(x)dx + c_1}{\sin(x)}$$

- Evaluate the integrals on the rhs

$$y = \frac{\sin(x)^2 + c_1}{\sin(x)}$$

- Simplify

$$y = \sin(x) + c_1\csc(x)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)+y(x)*cot(x)=2*cos(x),y(x), singsol=all)
```

$$y(x) = \csc(x)\left(-\cos(x)^2 + c_1 + \frac{1}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 20

```
DSolve[y'[x]+y[x]*Cot[x]==2*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}\csc(x)(\cos(2x) - 2c_1)$$

2.40 problem 40

2.40.1 Solving as exact ode 1081

Internal problem ID [7481]
Internal file name [OUTPUT/6448_Sunday_June_19_2022_05_02_54_PM_2088302/index.tex]

Book: Second order enumerated odes
Section: section 2
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{2xy^2 - y + (y^2 + x + y)\, y' = 0}$$

2.40.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(y^2 + x + y)\,\mathrm{d}y &= (-2x\,y^2 + y)\,\mathrm{d}x \\
(2x\,y^2 - y)\,\mathrm{d}x + (y^2 + x + y)\,\mathrm{d}y &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= 2x\,y^2 - y \\
N(x, y) &= y^2 + x + y
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x\,y^2 - y) \\
&= 4xy - 1
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^2 + x + y) \\
&= 1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{y^2 + x + y}((4xy - 1) - (1)) \\
&= \frac{4xy - 2}{y^2 + x + y}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{2x\,y^2 - y}((1) - (4xy - 1)) \\
&= -\frac{2}{y}
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int -\frac{2}{y} \,\mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2\ln(y)} \\
&= \frac{1}{y^2}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^2}(2x\,y^2 - y) \\
&= \frac{2xy - 1}{y}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}(y^2 + x + y) \\
&= \frac{y^2 + x + y}{y^2}
\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{2xy - 1}{y}\right) + \left(\frac{y^2 + x + y}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= \overline{M} \quad (1)\\
\frac{\partial \phi}{\partial y} &= \overline{N} \quad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2xy-1}{y}\,\mathrm{d}x\\
\phi &= \frac{x(xy-1)}{y} + f(y) \quad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{x^2}{y} - \frac{x(xy-1)}{y^2} + f'(y) \quad (4)\\
&= \frac{x}{y^2} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^2+x+y}{y^2}$. Therefore equation (4) becomes

$$
\frac{y^2+x+y}{y^2} = \frac{x}{y^2} + f'(y) \quad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{y+1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{y+1}{y}\right)\mathrm{d}y\\
f(y) &= \ln(y) + y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(xy-1)}{y} + \ln(y) + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(xy-1)}{y} + \ln(y) + y$$

Summary

The solution(s) found are the following

$$\frac{x(yx-1)}{y} + \ln(y) + y = c_1 \tag{1}$$

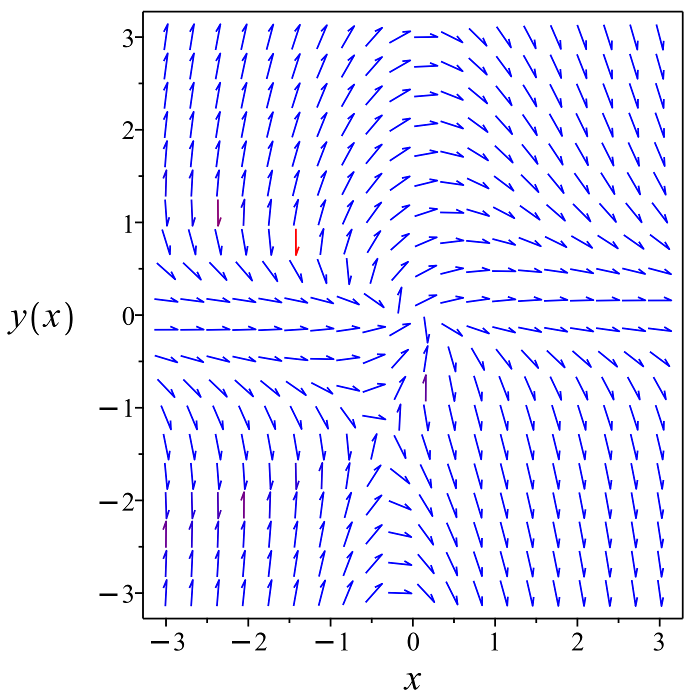


Figure 124: Slope field plot

Verification of solutions

$$\frac{x(yx-1)}{y} + \ln(y) + y = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 28

```
dsolve((2*x*y(x)^2-y(x))+(y(x)^2+x+y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = e^{\mathrm{RootOf}\left(x^2 e^{\_Z} + e^{2\_Z} + c_1 e^{\_Z} + e^{\_Z}\_Z - x\right)}$$

✓ Solution by Mathematica

Time used: 0.191 (sec). Leaf size: 22

```
DSolve[(2*x*y[x]^2-y[x])+(y[x]^2+x+y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[x^2 - \frac{x}{y(x)} + y(x) + \log(y(x)) = c_1, y(x)\right]$$

2.41 problem 41

2.41.1 Solving as riccati ode 1087

Internal problem ID [7482]
Internal file name [`OUTPUT/6449_Sunday_June_19_2022_05_02_59_PM_96072028/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_Riccati, _special]]
```

$$\boxed{y' + y^2 = x}$$

2.41.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -y^2 + x \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y^2 + x$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x$, $f_1(x) = 0$ and $f_2(x) = -1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= x \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) + xu(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \operatorname{AiryAi}(x) + c_2 \operatorname{AiryBi}(x)$$

The above shows that

$$u'(x) = c_1 \operatorname{AiryAi}(1, x) + c_2 \operatorname{AiryBi}(1, x)$$

Using the above in (1) gives the solution

$$y = \frac{c_1 \operatorname{AiryAi}(1, x) + c_2 \operatorname{AiryBi}(1, x)}{c_1 \operatorname{AiryAi}(x) + c_2 \operatorname{AiryBi}(x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 \operatorname{AiryAi}(1, x) + \operatorname{AiryBi}(1, x)}{c_3 \operatorname{AiryAi}(x) + \operatorname{AiryBi}(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{c_3 \operatorname{AiryAi}(1, x) + \operatorname{AiryBi}(1, x)}{c_3 \operatorname{AiryAi}(x) + \operatorname{AiryBi}(x)} \tag{1}$$

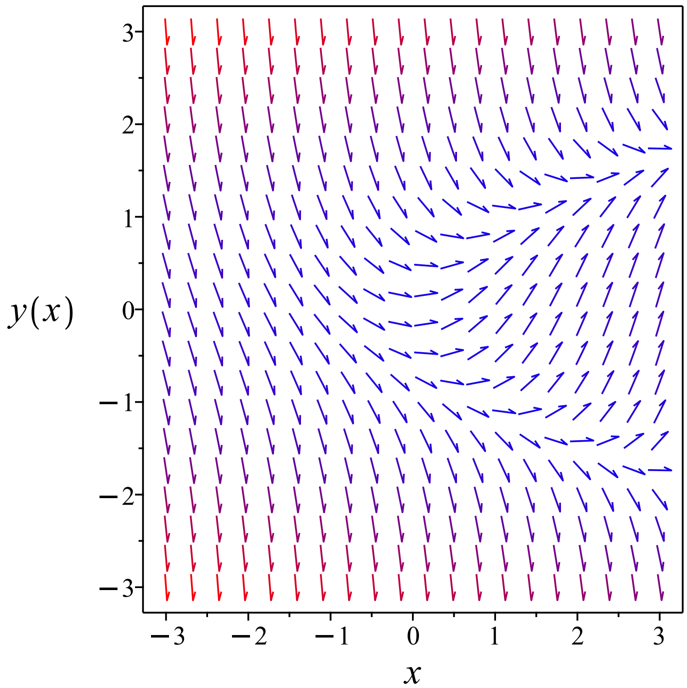


Figure 125: Slope field plot

Verification of solutions

$$y = \frac{c_3 \operatorname{AiryAi}(1, x) + \operatorname{AiryBi}(1, x)}{c_3 \operatorname{AiryAi}(x) + \operatorname{AiryBi}(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)=x-y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \operatorname{AiryAi}(1, x) + \operatorname{AiryBi}(1, x)}{c_1 \operatorname{AiryAi}(x) + \operatorname{AiryBi}(x)}$$

✓ Solution by Mathematica

Time used: 0.118 (sec). Leaf size: 223

```
DSolve[y'[x]==x-y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{-ix^{3/2}\left(2 \operatorname{BesselJ}\left(-\frac{2}{3}, \frac{2}{3}ix^{3/2}\right) + c_1\left(\operatorname{BesselJ}\left(-\frac{4}{3}, \frac{2}{3}ix^{3/2}\right) - \operatorname{BesselJ}\left(\frac{2}{3}, \frac{2}{3}ix^{3/2}\right)\right)\right) - c_1 \operatorname{BesselJ}\left(-\frac{1}{3}, \frac{2}{3}i\right.}{2x\left(\operatorname{BesselJ}\left(\frac{1}{3}, \frac{2}{3}ix^{3/2}\right) + c_1 \operatorname{BesselJ}\left(-\frac{1}{3}, \frac{2}{3}ix^{3/2}\right)\right)}$$

$$y(x) \to \frac{ix^{3/2} \operatorname{BesselJ}\left(-\frac{4}{3}, \frac{2}{3}ix^{3/2}\right) - ix^{3/2} \operatorname{BesselJ}\left(\frac{2}{3}, \frac{2}{3}ix^{3/2}\right) + \operatorname{BesselJ}\left(-\frac{1}{3}, \frac{2}{3}ix^{3/2}\right)}{2x \operatorname{BesselJ}\left(-\frac{1}{3}, \frac{2}{3}ix^{3/2}\right)}$$

2.42 problem 42

Internal problem ID [7483]
Internal file name [`OUTPUT/6450_Sunday_June_19_2022_05_03_01_PM_44498961/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 42.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - y''' - 3y'' + 5y' - 2y = x\,\mathrm{e}^{x} + 3\,\mathrm{e}^{-2x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - y''' - 3y'' + 5y' - 2y = 0$$

The characteristic equation is

$$\lambda^4 - \lambda^3 - 3\lambda^2 + 5\lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = -2$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = e^{-2x}c_1 + c_2e^x + x\,e^x c_3 + x^2e^x c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-2x} \\ y_2 &= e^x \\ y_3 &= x\,e^x \\ y_4 &= x^2e^x \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - y''' - 3y'' + 5y' - 2y = x\,e^x + 3\,e^{-2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,e^x + 3\,e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}, \{x\,e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^x, x^2e^x, e^x, e^{-2x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-2x}\}, \{x\,e^x, e^x\}]$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-2x}\}, \{x\,e^x, x^2e^x\}]$$

Since $x\,e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-2x}\}, \{x^2e^x, x^3e^x\}]$$

Since x^2e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-2x}\}, \{x^3e^x, x^4e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-2x} + A_2 x^3 \mathrm{e}^{x} + A_3 x^4 \mathrm{e}^{x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$72A_3 x\,\mathrm{e}^{x} + 18A_2 \mathrm{e}^{x} + 24A_3 \mathrm{e}^{x} - 27A_1 \mathrm{e}^{-2x} = x\,\mathrm{e}^{x} + 3\,\mathrm{e}^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{9}, A_2 = -\frac{1}{54}, A_3 = \frac{1}{72}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,\mathrm{e}^{-2x}}{9} - \frac{x^3 \mathrm{e}^{x}}{54} + \frac{x^4 \mathrm{e}^{x}}{72}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-2x} c_1 + c_2 \mathrm{e}^{x} + x\,\mathrm{e}^{x} c_3 + x^2 \mathrm{e}^{x} c_4\right) + \left(-\frac{x\,\mathrm{e}^{-2x}}{9} - \frac{x^3 \mathrm{e}^{x}}{54} + \frac{x^4 \mathrm{e}^{x}}{72}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-2x}\left(\left(c_4 x^2 + c_3 x + c_2\right) \mathrm{e}^{3x} + c_1\right) - \frac{x\,\mathrm{e}^{-2x}}{9} - \frac{x^3 \mathrm{e}^{x}}{54} + \frac{x^4 \mathrm{e}^{x}}{72}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2x}\left(\left(c_4 x^2 + c_3 x + c_2\right) \mathrm{e}^{3x} + c_1\right) - \frac{x\,\mathrm{e}^{-2x}}{9} - \frac{x^3 \mathrm{e}^{x}}{54} + \frac{x^4 \mathrm{e}^{x}}{72} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-2x}\left(\left(c_4 x^2 + c_3 x + c_2\right) \mathrm{e}^{3x} + c_1\right) - \frac{x\,\mathrm{e}^{-2x}}{9} - \frac{x^3 \mathrm{e}^{x}}{54} + \frac{x^4 \mathrm{e}^{x}}{72}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 52

```
dsolve(diff(y(x),x$4)-diff(y(x),x$3)-3*diff(y(x),x$2)+5*diff(y(x),x)-2*y(x)=x*exp(x)+3*exp(-
```

$$y(x) = \frac{\left(\left(x^4 - \frac{4x^3}{3} + \left(72c_4 + \frac{4}{3}\right)x^2 + \left(72c_3 - \frac{8}{9}\right)x + 72c_1 + \frac{8}{27}\right)e^{3x} - 8x + 72c_2 - 8\right)e^{-2x}}{72}$$

✓ Solution by Mathematica

Time used: 0.234 (sec). Leaf size: 64

```
DSolve[y''''[x]-y'''[x]-3*y''[x]+5*y'[x]-2*y[x]==x*Exp[x]+3*Exp[-2*x],y[x],x,IncludeSingular
```

$$y(x) \to e^x\left(\frac{x^4}{72} - \frac{x^3}{54} + \left(\frac{1}{54} + c_4\right)x^2 + \left(-\frac{1}{81} + c_3\right)x + \frac{1}{243} + c_2\right) - \frac{1}{9}e^{-2x}(x + 1 - 9c_1)$$

2.43 problem 43

2.43.1 Maple step by step solution 1107

Internal problem ID [7484]
Internal file name [OUTPUT/6451_Sunday_June_19_2022_05_03_04_PM_84564762/index.tex]

Book: Second order enumerated odes
Section: section 2
Problem number: 43.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(x+6)\,y' + 10y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x^2 - 6x\right) y' + 10y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{x+6}{x} \\ q(x) &= \frac{10}{x^2} \end{aligned}$$

Table 104: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+6}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{10}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(-x^2 - 6x\right) y' + 10y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} & x^2\left(\sum_{n=0}^{\infty}(n+r)(n+r-1) a_n x^{n+r-2}\right) \\ & + \left(-x^2 - 6x\right)\left(\sum_{n=0}^{\infty}(n+r) a_n x^{n+r-1}\right) + 10\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-6x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 10 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-6x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 10 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 6x^{n+r} a_n (n+r) + 10 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 6x^r a_0 r + 10 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 6x^r r + 10x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)(r-5)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)(r-5) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 5$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r - 2)(r - 5) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = x^5 \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^2 \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+5}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+2} \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) - a_{n-1}(n + r - 1) - 6a_n(n + r) + 10a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n + r - 1)}{n^2 + 2nr + r^2 - 7n - 7r + 10} \tag{4}$$

Which for the root $r = 5$ becomes

$$a_n = \frac{a_{n-1}(n+4)}{n(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 5$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r}{r^2 - 5r + 4}$$

Which for the root $r = 5$ becomes

$$a_1 = \frac{5}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1+r}{r^3 - 8r^2 + 19r - 12}$$

Which for the root $r = 5$ becomes

$$a_2 = \frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$
a_2	$\frac{1+r}{r^3-8r^2+19r-12}$	$\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{2+r}{r^4 - 10r^3 + 35r^2 - 50r + 24}$$

Which for the root $r = 5$ becomes

$$a_3 = \frac{7}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$
a_2	$\frac{1+r}{r^3-8r^2+19r-12}$	$\frac{3}{4}$
a_3	$\frac{2+r}{r^4-10r^3+35r^2-50r+24}$	$\frac{7}{24}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{3+r}{(-1+r)^2 (r-2) (r-4) (r-3)}$$

Which for the root $r = 5$ becomes

$$a_4 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$
a_2	$\frac{1+r}{r^3-8r^2+19r-12}$	$\frac{3}{4}$
a_3	$\frac{2+r}{r^4-10r^3+35r^2-50r+24}$	$\frac{7}{24}$
a_4	$\frac{3+r}{(-1+r)^2(r-2)(r-4)(r-3)}$	$\frac{1}{12}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{4+r}{r(-1+r)^2 (r-2) (r-4) (r-3)}$$

Which for the root $r = 5$ becomes

$$a_5 = \frac{3}{160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$
a_2	$\frac{1+r}{r^3-8r^2+19r-12}$	$\frac{3}{4}$
a_3	$\frac{2+r}{r^4-10r^3+35r^2-50r+24}$	$\frac{7}{24}$
a_4	$\frac{3+r}{(-1+r)^2(r-2)(r-4)(r-3)}$	$\frac{1}{12}$
a_5	$\frac{4+r}{r(-1+r)^2(r-2)(r-4)(r-3)}$	$\frac{3}{160}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^5\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^5\left(1 + \frac{5x}{4} + \frac{3x^2}{4} + \frac{7x^3}{24} + \frac{x^4}{12} + \frac{3x^5}{160} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= \frac{2+r}{r^4 - 10r^3 + 35r^2 - 50r + 24} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{2+r}{r^4 - 10r^3 + 35r^2 - 50r + 24} &= \lim_{r\to 2} \frac{2+r}{r^4 - 10r^3 + 35r^2 - 50r + 24} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2y'' + (-x^2-6x)\,y' + 10y = 0$ gives

$$
\begin{aligned}
& x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
& + (-x^2-6x)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
& + 10Cy_1(x)\ln(x) + 10\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
& \left(\left(x^2y_1''(x) + (-x^2-6x)\,y_1'(x) + 10y_1(x)\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right. \\
&\left. + \frac{(-x^2-6x)\,y_1(x)}{x}\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
& + (-x^2-6x)\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + 10\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2 y_1''(x) + (-x^2 - 6x)\, y_1'(x) + 10 y_1(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned}
&\left(x^2 \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(-x^2 - 6x)\, y_1(x)}{x}\right) C \\
&+ x^2 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) \\
&+ (-x^2 - 6x) \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) + 10 \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}
&\left(2 \left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x - (7+x) \left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2) (-1+n+r_2)\right) x^2 \\
&+ (-x^2 - 6x) \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) + 10 \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}$$

Since $r_1 = 5$ and $r_2 = 2$ then the above becomes

$$\begin{aligned}
&\left(2 \left(\sum_{n=0}^{\infty} x^{n+4} a_n (n+5)\right) x - (7+x) \left(\sum_{n=0}^{\infty} a_n x^{n+5}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^n b_n (n+2) (1+n)\right) x^2 \\
&+ (-x^2 - 6x) \left(\sum_{n=0}^{\infty} x^{1+n} b_n (n+2)\right) + 10 \left(\sum_{n=0}^{\infty} b_n x^{n+2}\right) = 0
\end{aligned} \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^{n+5} a_n (n+5)\right) + \sum_{n=0}^{\infty} \left(-7C\, x^{n+5} a_n\right) + \sum_{n=0}^{\infty} \left(-C\, x^{n+6} a_n\right) + \left(\sum_{n=0}^{\infty} x^{n+2} b_n \left(n^2+3n+2\right)\right) + \sum_{n=0}^{\infty} \left(-x^{n+3} b_n (n+2)\right) + \sum_{n=0}^{\infty} \left(-6x^{n+2} b_n (n+2)\right) + \left(\sum_{n=0}^{\infty} 10 b_n x^{n+2}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+2} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+5} a_n (n+5) = \sum_{n=3}^{\infty} 2C a_{n-3} (n+2)\, x^{n+2}$$

$$\sum_{n=0}^{\infty} \left(-7C\, x^{n+5} a_n\right) = \sum_{n=3}^{\infty} \left(-7C a_{n-3} x^{n+2}\right)$$

$$\sum_{n=0}^{\infty} \left(-C\, x^{n+6} a_n\right) = \sum_{n=4}^{\infty} \left(-C a_{n-4} x^{n+2}\right)$$

$$\sum_{n=0}^{\infty} \left(-x^{n+3} b_n (n+2)\right) = \sum_{n=1}^{\infty} \left(-b_{n-1} (1+n)\, x^{n+2}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+2$.

$$\left(\sum_{n=3}^{\infty} 2C a_{n-3} (n+2)\, x^{n+2}\right) + \sum_{n=3}^{\infty} \left(-7C a_{n-3} x^{n+2}\right) + \sum_{n=4}^{\infty} \left(-C a_{n-4} x^{n+2}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} b_n \left(n^2+3n+2\right)\right) + \sum_{n=1}^{\infty} \left(-b_{n-1} (1+n)\, x^{n+2}\right) + \sum_{n=0}^{\infty} \left(-6x^{n+2} b_n (n+2)\right) + \left(\sum_{n=0}^{\infty} 10 b_n x^{n+2}\right) = 0 \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-2b_1 - 2b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_1 - 2 = 0$$

Solving the above for b_1 gives

$$b_1 = -1$$

For $n = 2$, Eq (2B) gives

$$-2b_2 - 3b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_2 + 3 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{3}{2}$$

For $n = N$, where $N = 3$ which is the difference between the two roots, we are free to choose $b_3 = 0$. Hence for $n = 3$, Eq (2B) gives

$$3C - 6 = 0$$

Which is solved for C. Solving for C gives

$$C = 2$$

For $n = 4$, Eq (2B) gives

$$(-a_0 + 5a_1) C - 5b_3 + 4b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{21}{2} + 4b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{21}{8}$$

For $n = 5$, Eq (2B) gives

$$(-a_1 + 7a_2) C - 6b_4 + 10b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{95}{4} + 10b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{19}{8}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 2$ and all b_n, then the second solution becomes

$$\begin{aligned}y_2(x) = {} & 2\left(x^5\left(1 + \frac{5x}{4} + \frac{3x^2}{4} + \frac{7x^3}{24} + \frac{x^4}{12} + \frac{3x^5}{160} + O(x^6)\right)\right)\ln(x) \\ & + x^2\left(1 - x + \frac{3x^2}{2} - \frac{21x^4}{8} - \frac{19x^5}{8} + O(x^6)\right)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^5\left(1 + \frac{5x}{4} + \frac{3x^2}{4} + \frac{7x^3}{24} + \frac{x^4}{12} + \frac{3x^5}{160} + O(x^6)\right) \\ &\quad + c_2\left(2\left(x^5\left(1 + \frac{5x}{4} + \frac{3x^2}{4} + \frac{7x^3}{24} + \frac{x^4}{12} + \frac{3x^5}{160} + O(x^6)\right)\right)\ln(x)\right. \\ &\quad \left. + x^2\left(1 - x + \frac{3x^2}{2} - \frac{21x^4}{8} - \frac{19x^5}{8} + O(x^6)\right)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h \\ &= c_1 x^5\left(1 + \frac{5x}{4} + \frac{3x^2}{4} + \frac{7x^3}{24} + \frac{x^4}{12} + \frac{3x^5}{160} + O(x^6)\right) \\ &\quad + c_2\left(2x^5\left(1 + \frac{5x}{4} + \frac{3x^2}{4} + \frac{7x^3}{24} + \frac{x^4}{12} + \frac{3x^5}{160} + O(x^6)\right)\ln(x)\right. \\ &\quad \left. + x^2\left(1 - x + \frac{3x^2}{2} - \frac{21x^4}{8} - \frac{19x^5}{8} + O(x^6)\right)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^5 \left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+O\left(x^6\right)\right) \\
& + c_2 \left(2x^5 \left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+O\left(x^6\right)\right)\ln\left(x\right) \right. \\
& \left. + x^2 \left(1-x+\frac{3x^2}{2}-\frac{21x^4}{8}-\frac{19x^5}{8}+O\left(x^6\right)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x^5 \left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+O\left(x^6\right)\right) \\
& + c_2 \left(2x^5 \left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+O\left(x^6\right)\right)\ln\left(x\right) \right. \\
& \left. + x^2 \left(1-x+\frac{3x^2}{2}-\frac{21x^4}{8}-\frac{19x^5}{8}+O\left(x^6\right)\right)\right)
\end{aligned}
$$

Verified OK.

2.43.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(-x^2-6x\right) y' + 10y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{10y}{x^2} + \frac{(x+6)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+6)y'}{x} + \frac{10y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x+6}{x}, P_3(x) = \frac{10}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -6$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 10$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(x+6)\,y' + 10y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-5+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)\,(k+r-5) - a_{k-1}(k+r-1)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-5+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 5\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)\,(k+r-5) - a_{k-1}(k+r-1) = 0$

- Shift index using $k->k+1$

  $a_{k+1}(k+r-1)\,(k-4+r) - a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+r-1)(k-4+r)}$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$

- Series not valid for $r = 2$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$

- Recursion relation for $r = 5$

  $a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}$

- Solution for $r = 5$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+5}, a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 65

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*(x+6)*diff(y(x),x)+10*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x^2\left(c_1x^3\left(1+\frac{5}{4}x+\frac{3}{4}x^2+\frac{7}{24}x^3+\frac{1}{12}x^4+\frac{3}{160}x^5+\mathrm{O}\left(x^6\right)\right)\right.$$
$$+c_2\left(\ln\left(x\right)\left(24x^3+30x^4+18x^5+\mathrm{O}\left(x^6\right)\right)\right.$$
$$\left.\left.+\left(12-12x+18x^2+26x^3+x^4-9x^5+\mathrm{O}\left(x^6\right)\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 84

```
AsymptoticDSolveValue[x^2*y''[x]-x*(x+6)*y'[x]+10*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{2}x^5(5x+4)\log(x)-\frac{1}{4}x^2\left(3x^4-6x^3-6x^2+4x-4\right)\right)$$
$$+c_2\left(\frac{x^9}{12}+\frac{7x^8}{24}+\frac{3x^7}{4}+\frac{5x^6}{4}+x^5\right)$$

2.44 problem 44

2.44.1 Maple step by step solution 1120

Internal problem ID [7485]
Internal file name [`OUTPUT/6452_Sunday_June_19_2022_05_03_09_PM_22361166/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 44.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[_Bessel]
```

$$\boxed{x^2y'' + xy' + (x^2 - 5)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + (x^2 - 5)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{x^2 - 5}{x^2}$$

Table 106: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2-5}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + x y' + (x^2 - 5) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) + x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + (x^2 - 5) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-5 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-5 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 5 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 5 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - 5 x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 5\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 5 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= \sqrt{5} \\
r_2 &= -\sqrt{5}
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(r^2 - 5)\, x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2\sqrt{5}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\sqrt{5}} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-\sqrt{5}}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n+r)\,(n+r-1) + a_n(n+r) + a_{n-2} - 5a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 5} \tag{4}
$$

Which for the root $r = \sqrt{5}$ becomes

$$
a_n = -\frac{a_{n-2}}{n\,(2\sqrt{5} + n)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \sqrt{5}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 4r - 1}$$

Which for the root $r = \sqrt{5}$ becomes

$$a_2 = -\frac{1}{4 + 4\sqrt{5}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-1}$	$-\frac{1}{4+4\sqrt{5}}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-1}$	$-\frac{1}{4+4\sqrt{5}}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r^2 + 4r - 1)(r^2 + 8r + 11)}$$

Which for the root $r = \sqrt{5}$ becomes

$$a_4 = \frac{1}{32\left(\sqrt{5}+1\right)\left(\sqrt{5}+2\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-1}$	$-\frac{1}{4+4\sqrt{5}}$
a_3	0	0
a_4	$\frac{1}{(r^2+4r-1)(r^2+8r+11)}$	$\frac{1}{32\left(\sqrt{5}+1\right)\left(\sqrt{5}+2\right)}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-1}$	$-\frac{1}{4+4\sqrt{5}}$
a_3	0	0
a_4	$\frac{1}{(r^2+4r-1)(r^2+8r+11)}$	$\frac{1}{32\left(\sqrt{5}+1\right)\left(\sqrt{5}+2\right)}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^{\sqrt{5}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\sqrt{5}}\left(1 - \frac{x^2}{4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}+1\right)\left(\sqrt{5}+2\right)} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-2} - 5b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{n^2 + 2nr + r^2 - 5} \tag{4}$$

Which for the root $r = -\sqrt{5}$ becomes

$$b_n = -\frac{b_{n-2}}{n\left(-2\sqrt{5} + n\right)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\sqrt{5}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{r^2 + 4r - 1}$$

Which for the root $r = -\sqrt{5}$ becomes

$$b_2 = \frac{1}{-4 + 4\sqrt{5}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-1}$	$\frac{1}{-4+4\sqrt{5}}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-1}$	$\frac{1}{-4+4\sqrt{5}}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + 4r - 1)(r^2 + 8r + 11)}$$

Which for the root $r = -\sqrt{5}$ becomes

$$b_4 = \frac{1}{32(\sqrt{5} - 1)(-2 + \sqrt{5})}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-1}$	$\frac{1}{-4+4\sqrt{5}}$
b_3	0	0
b_4	$\frac{1}{(r^2+4r-1)(r^2+8r+11)}$	$\frac{1}{32(\sqrt{5}-1)(-2+\sqrt{5})}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-1}$	$\frac{1}{-4+4\sqrt{5}}$
b_3	0	0
b_4	$\frac{1}{(r^2+4r-1)(r^2+8r+11)}$	$\frac{1}{32\left(\sqrt{5}-1\right)\left(-2+\sqrt{5}\right)}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\sqrt{5}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x^{-\sqrt{5}}\left(1 + \frac{x^2}{-4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}-1\right)\left(-2+\sqrt{5}\right)} + O\left(x^6\right)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\sqrt{5}}\left(1 - \frac{x^2}{4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}+1\right)\left(\sqrt{5}+2\right)} + O\left(x^6\right)\right) \\
&\quad + c_2 x^{-\sqrt{5}}\left(1 + \frac{x^2}{-4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}-1\right)\left(-2+\sqrt{5}\right)} + O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\sqrt{5}}\left(1 - \frac{x^2}{4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}+1\right)\left(\sqrt{5}+2\right)} + O\left(x^6\right)\right) \\
&\quad + c_2 x^{-\sqrt{5}}\left(1 + \frac{x^2}{-4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}-1\right)\left(-2+\sqrt{5}\right)} + O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\sqrt{5}} \left(1 - \frac{x^2}{4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}+1\right)\left(\sqrt{5}+2\right)} + O\left(x^6\right)\right) \\
+ c_2 x^{-\sqrt{5}} \left(1 + \frac{x^2}{-4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}-1\right)\left(-2+\sqrt{5}\right)} + O\left(x^6\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^{\sqrt{5}} \left(1 - \frac{x^2}{4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}+1\right)\left(\sqrt{5}+2\right)} + O\left(x^6\right)\right) \\
+ c_2 x^{-\sqrt{5}} \left(1 + \frac{x^2}{-4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}-1\right)\left(-2+\sqrt{5}\right)} + O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

2.44.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + (x^2-5)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-5)y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(x^2-5)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-5}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -5$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + xy' + \left(x^2 - 5\right) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0\left(r^2 - 5\right) x^r + a_1\left(r^2 + 2r - 4\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k\left(k^2 + 2kr + r^2 - 5\right) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 - 5 = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\sqrt{5}, -\sqrt{5}\right\}$

- Each term must be 0

  $a_1\left(r^2 + 2r - 4\right) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k^2 + 2kr + r^2 - 5) + a_{k-2} = 0$

- Shift index using $k- > k + 2$

  $a_{k+2}\left((k+2)^2 + 2(k+2)\,r + r^2 - 5\right) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k^2+2kr+r^2+4k+4r-1}$

- Recursion relation for $r = \sqrt{5}$

  $a_{k+2} = -\frac{a_k}{k^2+2k\sqrt{5}+4+4k+4\sqrt{5}}$

- Solution for $r = \sqrt{5}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\sqrt{5}}, a_{k+2} = -\frac{a_k}{k^2+2k\sqrt{5}+4+4k+4\sqrt{5}}, a_1 = 0\right]$

- Recursion relation for $r = -\sqrt{5}$

  $a_{k+2} = -\frac{a_k}{k^2-2k\sqrt{5}+4+4k-4\sqrt{5}}$

- Solution for $r = -\sqrt{5}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\sqrt{5}}, a_{k+2} = -\frac{a_k}{k^2-2k\sqrt{5}+4+4k-4\sqrt{5}}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\sqrt{5}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\sqrt{5}}\right), a_{k+2} = -\frac{a_k}{k^2+2k\sqrt{5}+4+4k+4\sqrt{5}}, a_1 = 0, b_{k+2} = -\frac{b_k}{k^2-2k\sqrt{5}+4+4\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 97

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-5)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{-\sqrt{5}} \left(1 + \frac{1}{-4+4\sqrt{5}} x^2 + \frac{1}{32} \frac{1}{(-2+\sqrt{5})(\sqrt{5}-1)} x^4 + \mathrm{O}(x^6)\right) + c_2 x^{\sqrt{5}} \left(1 - \frac{1}{4+4\sqrt{5}} x^2 + \frac{1}{32} \frac{1}{(\sqrt{5}+2)(\sqrt{5}+1)} x^4 + \mathrm{O}(x^6)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 210

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-5)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^4}{\left(-3-\sqrt{5}+\left(1-\sqrt{5}\right)\left(2-\sqrt{5}\right)\right)\left(-1-\sqrt{5}+\left(3-\sqrt{5}\right)\left(4-\sqrt{5}\right)\right)} - \frac{x^2}{-3-\sqrt{5}+\left(1-\sqrt{5}\right)\left(2-\sqrt{5}\right)}+1\right)x^{-\sqrt{5}} + c_1\left(\frac{x^4}{\left(-3+\sqrt{5}+\left(1+\sqrt{5}\right)\left(2+\sqrt{5}\right)\right)\left(-1+\sqrt{5}+\left(3+\sqrt{5}\right)\left(4+\sqrt{5}\right)\right)} - \frac{x^2}{-3+\sqrt{5}+\left(1+\sqrt{5}\right)\left(2+\sqrt{5}\right)}+1\right)x^{\sqrt{5}}$$

2.45 problem 45

2.45.1 Solving as second order bessel ode ode 1125
2.45.2 Maple step by step solution 1126

Internal problem ID [7486]
Internal file name [`OUTPUT/6453_Sunday_June_19_2022_05_03_12_PM_75086495/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 45.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[_Bessel]`

$$\boxed{x^2y'' + xy' + (x^2 - 5)\, y = 0}$$

2.45.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' + (x^2 - 5)\, y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 0\\ \beta &= 1\\ n &= -\sqrt{5}\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \,\text{BesselJ}\left(-\sqrt{5}, x\right) + c_2 \,\text{BesselY}\left(-\sqrt{5}, x\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \,\text{BesselJ}\left(-\sqrt{5}, x\right) + c_2 \,\text{BesselY}\left(-\sqrt{5}, x\right) \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 \,\text{BesselJ}\left(-\sqrt{5}, x\right) + c_2 \,\text{BesselY}\left(-\sqrt{5}, x\right)$$

Verified OK.

2.45.2 Maple step by step solution

Let's solve

$x^2 y'' + x y' + (x^2 - 5)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-5)y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(x^2-5)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-5}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -5$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2y'' + xy' + (x^2 - 5)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(r^2 - 5)\, x^r + a_1(r^2 + 2r - 4)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k^2 + 2kr + r^2 - 5) + a_{k-2})\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 - 5 = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{\sqrt{5}, -\sqrt{5}\right\}$

- Each term must be 0

$a_1(r^2 + 2r - 4) = 0$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k^2+2kr+r^2-5)+a_{k-2}=0$$

- Shift index using $k->k+2$

$$a_{k+2}\left((k+2)^2+2(k+2)\,r+r^2-5\right)+a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a_k}{k^2+2kr+r^2+4k+4r-1}$$

- Recursion relation for $r=\sqrt{5}$

$$a_{k+2}=-\frac{a_k}{k^2+2k\sqrt{5}+4+4k+4\sqrt{5}}$$

- Solution for $r=\sqrt{5}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\sqrt{5}},a_{k+2}=-\frac{a_k}{k^2+2k\sqrt{5}+4+4k+4\sqrt{5}},a_1=0\right]$$

- Recursion relation for $r=-\sqrt{5}$

$$a_{k+2}=-\frac{a_k}{k^2-2k\sqrt{5}+4+4k-4\sqrt{5}}$$

- Solution for $r=-\sqrt{5}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-\sqrt{5}},a_{k+2}=-\frac{a_k}{k^2-2k\sqrt{5}+4+4k-4\sqrt{5}},a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+\sqrt{5}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k-\sqrt{5}}\right),a_{k+2}=-\frac{a_k}{k^2+2k\sqrt{5}+4+4k+4\sqrt{5}},a_1=0,b_{k+2}=-\frac{b_k}{k^2-2k\sqrt{5}+4+4}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-5)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}\left(\sqrt{5}, x\right) + c_2 \operatorname{BesselY}\left(\sqrt{5}, x\right)$$

✓ Solution by Mathematica

Time used: 0.08 (sec). Leaf size: 26

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-5)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 \operatorname{BesselJ}\left(\sqrt{5}, x\right) + c_2 \operatorname{BesselY}\left(\sqrt{5}, x\right)$$

2.46 problem 46

2.46.1 Solving as second order euler ode ode 1131
2.46.2 Solving as linear second order ode solved by an integrating factor ode 1132
2.46.3 Solving as second order change of variable on x method 2 ode . 1133
2.46.4 Solving as second order change of variable on x method 1 ode . 1135
2.46.5 Solving as second order change of variable on y method 1 ode . 1137
2.46.6 Solving as second order change of variable on y method 2 ode . 1139
2.46.7 Solving using Kovacic algorithm 1141
2.46.8 Maple step by step solution 1144

Internal problem ID [7487]
Internal file name [`OUTPUT/6454_Sunday_June_19_2022_05_03_14_PM_32798501/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 46.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 4xy' + 6y = 0}$$

2.46.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 4r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 6 = 0$$

Or

$$r^2 - 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2 x^3 + c_1 x^2$$

Summary

The solution(s) found are the following

$$y = c_2 x^3 + c_1 x^2 \tag{1}$$

Verification of solutions

$$y = c_2 x^3 + c_1 x^2$$

Verified OK.

2.46.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -\frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -\frac{4}{x}\, dx} \\ &= \frac{1}{x^2} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\frac{y}{x^2}\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x^2}\right)' = c_1$$

Integrating again gives

$$\left(\frac{y}{x^2}\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\frac{1}{x^2}}$$

Or

$$y = c_1 x^3 + c_2 x^2$$

Summary

The solution(s) found are the following

$$y = c_1 x^3 + c_2 x^2 \tag{1}$$

Verification of solutions

$$y = c_1 x^3 + c_2 x^2$$

Verified OK.

2.46.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - 4xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{4}{x}dx\right)}dx \\ &= \int e^{4\ln(x)}\,dx \\ &= \int x^4dx \\ &= \frac{x^5}{5} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{6}{x^2}}{x^8} \\ &= \frac{6}{x^{10}} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{x^{10}} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{6}{x^{10}} = \frac{6}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 6 = 0$$

Or

$$25r^2 - 25r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{2}{5}$$
$$r_2 = \frac{3}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{2}{5}} + c_2 \tau^{\frac{3}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}$$

Verified OK.

2.46.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' - 4xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{6}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\\ \tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2}\\ &= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2}\\ &= -\frac{5c\sqrt{6}}{6}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0\\ \frac{d^2}{d\tau^2}y(\tau) - \frac{5c\sqrt{6}\left(\frac{d}{d\tau}y(\tau)\right)}{6} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{5\sqrt{6}\,c\tau}{12}}\left(c_1\cosh\left(\frac{\sqrt{6}\,c\tau}{12}\right) + ic_2\sinh\left(\frac{\sqrt{6}\,c\tau}{12}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{6}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}\,x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{5}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Summary

The solution(s) found are the following

$$
y = x^{\frac{5}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) \tag{1}
$$

Verification of solutions

$$
y = x^{\frac{5}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Verified OK.

2.46.5 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{4}{x} \\
q(x) &= \frac{6}{x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\
&= \frac{6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\
&= \frac{6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{4}{x}}{2}} \\
&= x^2
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, x^2 \tag{4}
$$

Applying this change of variable to the original ode results in

$$
x^4 v''(x) = 0
$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$
v(x) = c_1 x + c_2
$$

Now that $v(x)$ is known, then

$$
\begin{aligned}
y &= v(x)\, z(x) \\
&= (c_1 x + c_2)\, (z(x))
\end{aligned} \tag{7}
$$

But from (5)

$$
z(x) = x^2
$$

Hence (7) becomes

$$y = (c_1 x + c_2) x^2$$

Summary

The solution(s) found are the following

$$y = (c_1 x + c_2) x^2 \tag{1}$$

Verification of solutions

$$y = (c_1 x + c_2) x^2$$

Verified OK.

2.46.6 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' - 4xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x) x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\ \ln(u) &= -2\ln(x) + c_1 \\ u &= \mathrm{e}^{-2\ln(x)+c_1} \\ &= \frac{c_1}{x^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{x} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{x} + c_2\right) x^3 \\
&= (c_2 x - c_1)\, x^2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{c_1}{x} + c_2\right) x^3 \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{x} + c_2\right) x^3
$$

Verified OK.

2.46.7 Solving using Kovacic algorithm

Writing the ode as

$$
x^2 y'' - 4xy' + 6y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= -4x \\
C &= 6
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 109: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\ &= z_1 e^{2\ln(x)} \\ &= z_1\left(x^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^2\right) + c_2\left(x^2(x)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 x^3 + c_1 x^2 \tag{1}$$

Verification of solutions

$$y = c_2 x^3 + c_1 x^2$$

Verified OK.

2.46.8 Maple step by step solution

Let's solve

$x^2 y'' - 4xy' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{x} - \frac{6y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{x} + \frac{6y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' - 4xy' + 6y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

Substitute the change of variables back into the ODE

$$x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)-4\frac{d}{dt}y(t)+6y(t)=0$$

- Simplify

$$\frac{d^2}{dt^2}y(t)-5\frac{d}{dt}y(t)+6y(t)=0$$

- Characteristic polynomial of ODE

$$r^2-5r+6=0$$

- Factor the characteristic polynomial

$$(r-2)(r-3)=0$$

- Roots of the characteristic polynomial

$$r=(2,3)$$

- 1st solution of the ODE

$$y_1(t)=\mathrm{e}^{2t}$$

- 2nd solution of the ODE

$$y_2(t)=\mathrm{e}^{3t}$$

- General solution of the ODE

$$y(t)=c_1y_1(t)+c_2y_2(t)$$

- Substitute in solutions

$$y(t)=c_1\mathrm{e}^{2t}+c_2\mathrm{e}^{3t}$$

- Change variables back using $t=\ln(x)$

$$y=c_2x^3+c_1x^2$$

- Simplify

$$y=x^2(c_2x+c_1)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^2(c_1x + c_2)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 16

```
DSolve[x^2*y''[x]-4*x*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(c_2x + c_1)$$

2.47 problem 47

2.47.1 Maple step by step solution 1147

Internal problem ID [7488]
Internal file name [OUTPUT/6455_Sunday_June_19_2022_05_03_15_PM_25844788/index.tex]

Book: Second order enumerated odes
Section: section 2
Problem number: 47.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y''' - yx = 0}$$

Unable to solve this ODE.

2.47.1 Maple step by step solution

Let's solve

$y''' - yx = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

- Shift index using $k- >k-1$

  $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k}$$

- Convert y''' to series expansion

  $$y''' = \sum_{k=3}^{\infty} a_k k(k-1)(k-2)\, x^{k-3}$$

- Shift index using $k- >k+3$

  $$y''' = \sum_{k=0}^{\infty} a_{k+3}(k+3)(k+2)(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$6a_3 + \left(\sum_{k=1}^{\infty} \left(a_{k+3}(k+3)(k+2)(k+1) - a_{k-1}\right) x^{k}\right) = 0$$

- Each term must be 0

  $$6a_3 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$\left(k^3 + 6k^2 + 11k + 6\right) a_{k+3} - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^3 + 6(k+1)^2 + 11k + 17\right) a_{k+4} - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+4} = \frac{a_k}{k^3 + 9k^2 + 26k + 24}, 6a_3 = 0\right]$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying high order exact linear fully integrable
trying to convert to a linear ODE with constant coefficients
trying differential order: 3; missing the dependent variable
trying Louvillian solutions for 3rd order ODEs, imprimitive case
-> pFq: Equivalence to the 3F2 or one of its 3 confluent cases under a power @ Moebius
<- pFq successful: received ODE is equivalent to the  0F2  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve(diff(y(x),x$3)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \,\mathrm{hypergeom}\left([\,], \left[\frac{1}{2}, \frac{3}{4}\right], \frac{x^4}{64}\right) + c_2 x \,\mathrm{hypergeom}\left([\,], \left[\frac{3}{4}, \frac{5}{4}\right], \frac{x^4}{64}\right) + c_3 x^2 \,\mathrm{hypergeom}\left([\,], \left[\frac{5}{4}, \frac{3}{2}\right], \frac{x^4}{64}\right)$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 76

```
DSolve[y'''[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \,{}_0F_2\left(;\frac{1}{2}, \frac{3}{4}; \frac{x^4}{64}\right) + \frac{1}{8} x \left((2+2i) c_2 \,{}_0F_2\left(;\frac{3}{4}, \frac{5}{4}; \frac{x^4}{64}\right) + i c_3 x \,{}_0F_2\left(;\frac{5}{4}, \frac{3}{2}; \frac{x^4}{64}\right)\right)$$

2.48 problem 48

2.48.1 Existence and uniqueness analysis 1150
2.48.2 Solving as quadrature ode 1151
2.48.3 Maple step by step solution 1152

Internal problem ID [7489]
Internal file name [`OUTPUT/6456_Sunday_June_19_2022_05_03_16_PM_90587072/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 48.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^{\frac{1}{3}} = 0}$$

With initial conditions

$$[y(0) = 0]$$

2.48.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^{\frac{1}{3}} \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{0 \leq y\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^{\frac{1}{3}}\right) \\ &= \frac{1}{3y^{\frac{2}{3}}} \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{0 < y\}$$

But the point $y_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

2.48.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^{\frac{1}{3}}} dy = \int dx$$

$$\frac{3y^{\frac{2}{3}}}{2} = x + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\frac{3y^{\frac{2}{3}}}{2} = x$$

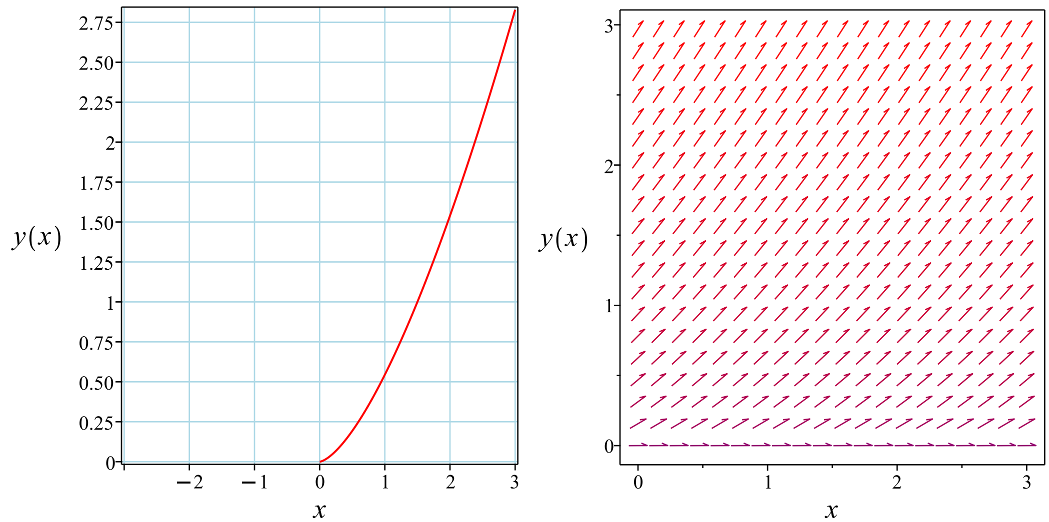
Solving for y from the above gives

$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$$

Summary

The solution(s) found are the following

$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$$

Verified OK.

2.48.3 Maple step by step solution

Let's solve

$\left[y' - y^{\frac{1}{3}} = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{1}{3}}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{1}{3}}} dx = \int 1 dx + c_1$

- Evaluate integral

  $\frac{3y^{\frac{2}{3}}}{2} = x + c_1$

- Solve for y

$y = \frac{(6x+6c_1)^{\frac{3}{2}}}{27}$

- Use initial condition $y(0) = 0$

$0 = \frac{2\sqrt{6}\, c_1^{\frac{3}{2}}}{9}$

- Solve for c_1

$c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$

- Solution to the IVP

$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=y(x)^(1/3),y(0) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 21

```
DSolve[{y'[x]==y[x]^(1/3),{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{3}\sqrt{\frac{2}{3}}x^{3/2}$$

2.49 problem 49

2.49.1 Solution using Matrix exponential method 1154
2.49.2 Solution using explicit Eigenvalue and Eigenvector method . . . 1155
2.49.3 Maple step by step solution 1160

Internal problem ID [7490]
Internal file name [`OUTPUT/6457_Sunday_June_19_2022_05_03_17_PM_34078354/index.tex`]

Book: Second order enumerated odes
Section: section 2
Problem number: 49.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 3x(t) + y(t)$$
$$y'(t) = -x(t) + y(t)$$

2.49.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t}(t+1) & t\,e^{2t} \\ -t\,e^{2t} & e^{2t}(-t+1) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t}(t+1) & t\,e^{2t} \\ -t\,e^{2t} & e^{2t}(-t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(t+1)\,c_1 + t\,e^{2t}c_2 \\ -t\,e^{2t}c_1 + e^{2t}(-t+1)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(tc_1 + c_2t + c_1) \\ -e^{2t}((t-1)\,c_2 + tc_1) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

2.49.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

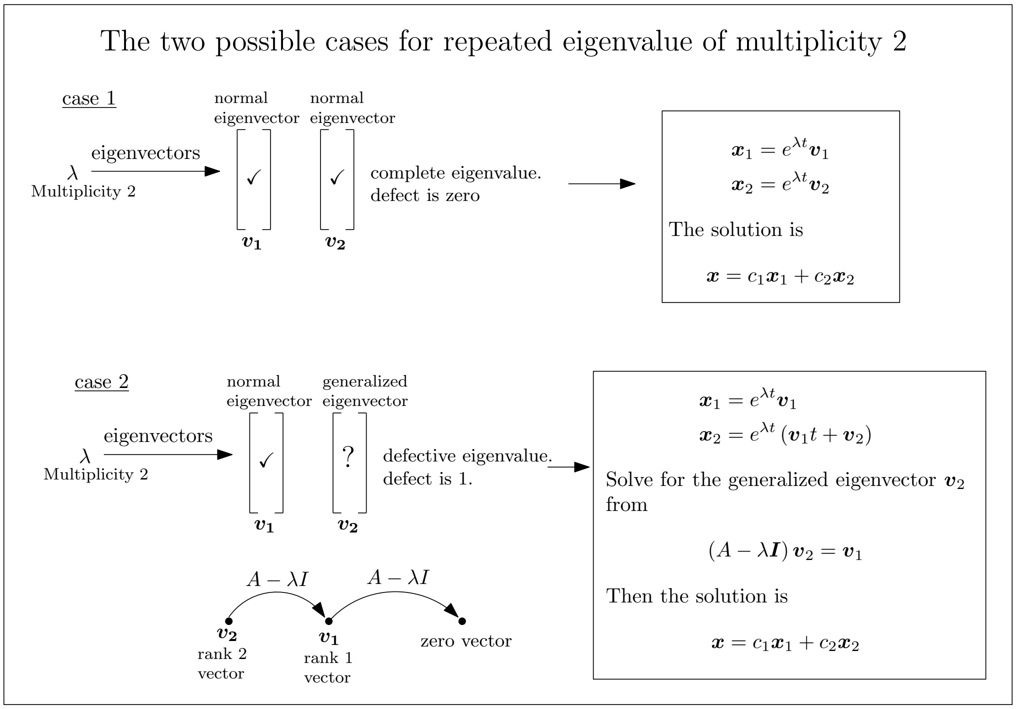


Figure 127: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\ &= \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right) \mathrm{e}^{2t} \\ &= \begin{bmatrix} -\mathrm{e}^{2t}(t+2) \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{2t}(-t-2) \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\mathrm{e}^{2t}((t+2)\, c_2 + c_1) \\ \mathrm{e}^{2t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

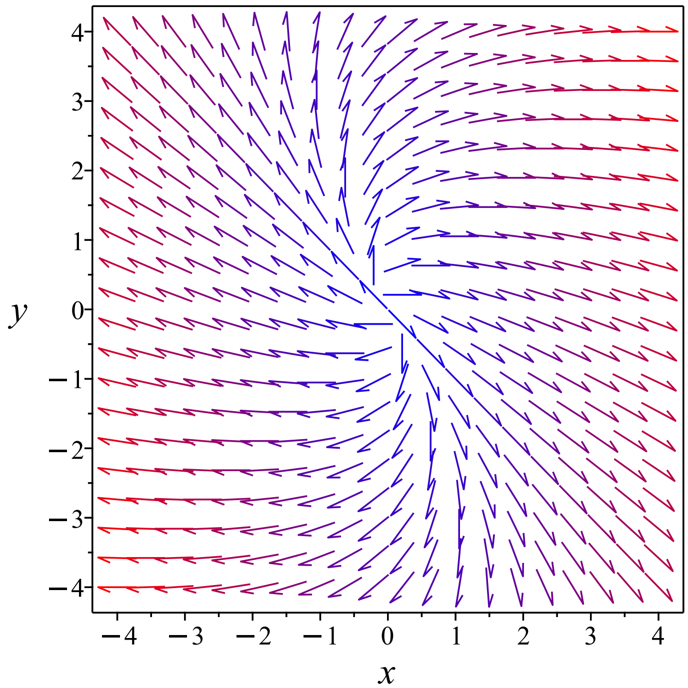


Figure 128: Phase plot

2.49.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t) + y(t), y'(t) = -x(t) + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

  $$\overrightarrow{x}_1(t) = e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, and

  $\overrightarrow{x}_2(t) = e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

  $\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = \left(e^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = e^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$

- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\left[\begin{array}{cc}3 & 1 \\ -1 & 1\end{array}\right]-2\cdot\left[\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c}-1 \\ 1\end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c}-1 \\ 0\end{array}\right]$$

- Second solution from eigenvalue 2

$$\overrightarrow{x}_2(t)=\mathrm{e}^{2t}\cdot\left(t\cdot\left[\begin{array}{c}-1 \\ 1\end{array}\right]+\left[\begin{array}{c}-1 \\ 0\end{array}\right]\right)$$

- General solution to the system of ODEs

$$\overrightarrow{x}=c_1\overrightarrow{x}_1(t)+c_2\overrightarrow{x}_2(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{x}=c_1\mathrm{e}^{2t}\cdot\left[\begin{array}{c}-1 \\ 1\end{array}\right]+c_2\mathrm{e}^{2t}\cdot\left(t\cdot\left[\begin{array}{c}-1 \\ 1\end{array}\right]+\left[\begin{array}{c}-1 \\ 0\end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x(t) \\ y(t)\end{array}\right]=\left[\begin{array}{c}\mathrm{e}^{2t}(-c_2t-c_1-c_2) \\ \mathrm{e}^{2t}(c_2t+c_1)\end{array}\right]$$

- Solution to the system of ODEs

$$\{x(t)=\mathrm{e}^{2t}(-c_2t-c_1-c_2)\,,y(t)=\mathrm{e}^{2t}(c_2t+c_1)\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
dsolve([diff(x(t),t)=3*x(t)+y(t),diff(y(t),t)=-x(t)+y(t)],singsol=all)
```

$$x(t)=\mathrm{e}^{2t}(c_2t+c_1)$$
$$y(t)=-\mathrm{e}^{2t}(c_2t+c_1-c_2)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 42

```
DSolve[{x'[t]==3*x[t]+y[t],y'[t]==-x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True
```

$$x(t) \to e^{2t}(c_1(t+1) + c_2 t)$$
$$y(t) \to e^{2t}(c_2 - (c_1 + c_2)t)$$