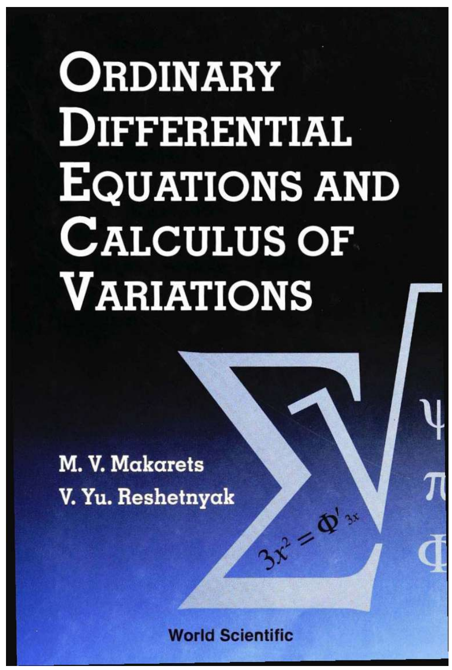
A Solution Manual For

Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995



Nasser M. Abbasi

May 15, 2024

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1.1 problem 1

Internal problem ID [5714]
Internal file name [`OUTPUT/4962_Sunday_June_05_2022_03_15_19_PM_32792222/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x^2}{y} = 0}$$

1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{x^2}{y} \end{aligned}$$

Where $f(x) = x^2$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\, dy = x^2\, dx$$

$$\int \frac{1}{\frac{1}{y}}\, dy = \int x^2\, dx$$

$$\frac{y^2}{2} = \frac{x^3}{3} + c_1$$

Which results in

$$y = \frac{\sqrt{6x^3 + 18c_1}}{3}$$

$$y = -\frac{\sqrt{6x^3 + 18c_1}}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{6x^3 + 18c_1}}{3} \tag{1}$$

$$y = -\frac{\sqrt{6x^3 + 18c_1}}{3} \tag{2}$$

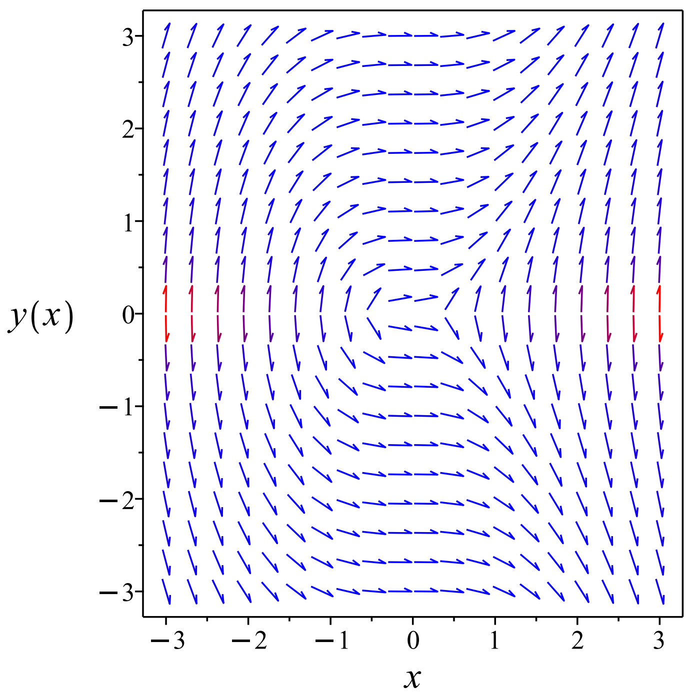


Figure 1: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{6x^3 + 18c_1}}{3}$$

Verified OK.

$$y = -\frac{\sqrt{6x^3 + 18c_1}}{3}$$

Verified OK.

1.1.2 Maple step by step solution

Let's solve

$y' - \frac{x^2}{y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'y = x^2$

- Integrate both sides with respect to x

  $\int y'y dx = \int x^2 dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = \frac{x^3}{3} + c_1$

- Solve for y

  $\left\{y = -\frac{\sqrt{6x^3+18c_1}}{3}, y = \frac{\sqrt{6x^3+18c_1}}{3}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x)=x^2/y(x),y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{6x^3 + 9c_1}}{3}$$
$$y(x) = \frac{\sqrt{6x^3 + 9c_1}}{3}$$

✓ Solution by Mathematica

Time used: 0.084 (sec). Leaf size: 50

```
DSolve[y'[x]==x^2/y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{2}{3}}\sqrt{x^3 + 3c_1}$$
$$y(x) \to \sqrt{\frac{2}{3}}\sqrt{x^3 + 3c_1}$$

1.2 problem 2

Internal problem ID [5715]
Internal file name [`OUTPUT/4963_Sunday_June_05_2022_03_15_20_PM_92366259/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x^2}{y (x^3+1)} = 0}$$

1.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x^2}{y (x^3+1)} \end{aligned}$$

Where $f(x) = \frac{x^2}{x^3+1}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = \frac{x^2}{x^3+1}\,dx$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int \frac{x^2}{x^3+1}\,dx$$

$$\frac{y^2}{2} = \frac{\ln (x^3+1)}{3} + c_1$$

Which results in

$$y = \frac{\sqrt{6 \ln (x^3+1) + 18c_1}}{3}$$
$$y = -\frac{\sqrt{6 \ln (x^3+1) + 18c_1}}{3}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{\sqrt{6 \ln (x^3+1) + 18c_1}}{3} \tag{1}$$
$$y = -\frac{\sqrt{6 \ln (x^3+1) + 18c_1}}{3} \tag{2}$$

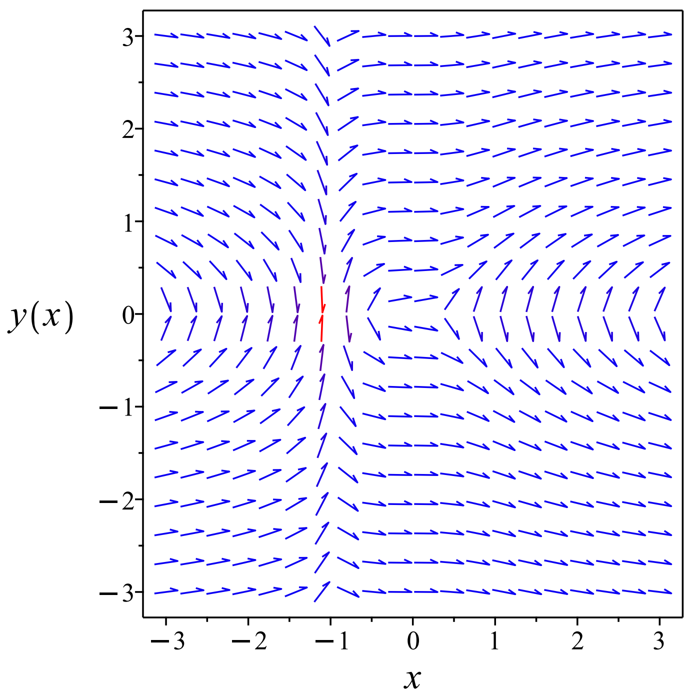


Figure 2: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{6\ln\left(x^3+1\right)+18c_1}}{3}$$

Verified OK.

$$y = -\frac{\sqrt{6\ln\left(x^3+1\right)+18c_1}}{3}$$

Verified OK.

1.2.2 Maple step by step solution

Let's solve

$y' - \frac{x^2}{y(x^3+1)} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'y = \frac{x^2}{x^3+1}$

- Integrate both sides with respect to x

  $\int y'y dx = \int \frac{x^2}{x^3+1}dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = \frac{\ln(x^3+1)}{3} + c_1$

- Solve for y

  $\left\{y = -\frac{\sqrt{6\ln(x^3+1)+18c_1}}{3}, y = \frac{\sqrt{6\ln(x^3+1)+18c_1}}{3}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(x),x)=x^2/(y(x)*(1+x^3)),y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{6\ln(x^3+1)+9c_1}}{3}$$
$$y(x) = \frac{\sqrt{6\ln(x^3+1)+9c_1}}{3}$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 56

```
DSolve[y'[x]==x^2/(y[x]*(1+x^3)),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{2}{3}}\sqrt{\log(x^3+1)+3c_1}$$
$$y(x) \to \sqrt{\frac{2}{3}}\sqrt{\log(x^3+1)+3c_1}$$

1.3 problem 3

Internal problem ID [5716]

Internal file name [`OUTPUT/4964_Sunday_June_05_2022_03_15_22_PM_12531112/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995

Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7

Problem number: 3.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \sin(x)\, y = 0}$$

1.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= y \sin(x) \end{aligned}$$

Where $f(x) = \sin(x)$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\, dy &= \sin(x)\, dx \\ \int \frac{1}{y}\, dy &= \int \sin(x)\, dx \\ \ln(y) &= -\cos(x) + c_1 \\ y &= e^{-\cos(x)+c_1} \\ &= c_1 e^{-\cos(x)} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-\cos(x)} \tag{1}$$

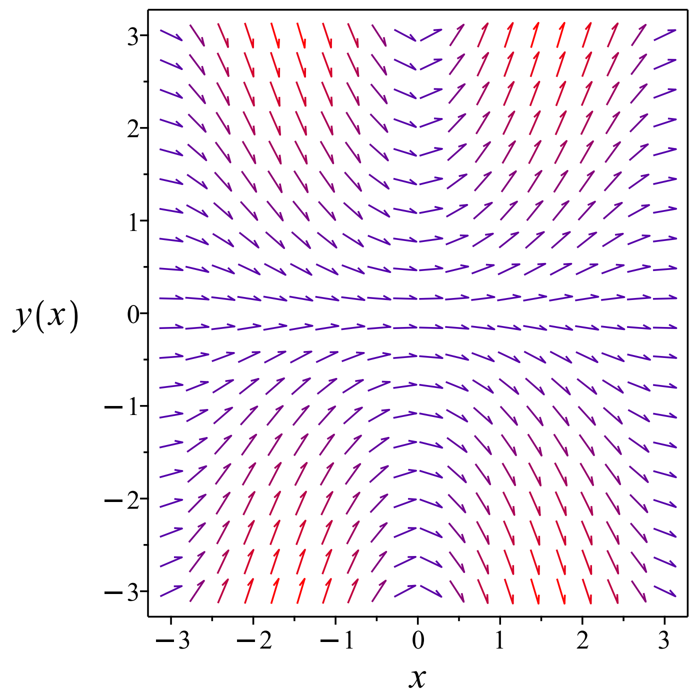


Figure 3: Slope field plot

Verification of solutions

$$y = c_1 e^{-\cos(x)}$$

Verified OK.

1.3.2 Maple step by step solution

Let's solve

$y' - \sin(x)\, y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \sin(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \sin(x) \, dx + c_1$

- Evaluate integral

  $\ln(y) = -\cos(x) + c_1$

- Solve for y

  $y = \mathrm{e}^{-\cos(x)+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)=y(x)*sin(x),y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\cos(x)}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 19

```
DSolve[y'[x]==y[x]*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\cos(x)}$$
$$y(x) \to 0$$

1.4 problem 4

Internal problem ID [5717]
Internal file name [`OUTPUT/4965_Sunday_June_05_2022_03_15_23_PM_88416005/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{xy' - \sqrt{1 - y^2} = 0}$$

1.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{\sqrt{-y^2 + 1}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \sqrt{-y^2 + 1}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\sqrt{-y^2 + 1}}\, dy &= \frac{1}{x}\, dx \\ \int \frac{1}{\sqrt{-y^2 + 1}}\, dy &= \int \frac{1}{x}\, dx \\ \arcsin(y) &= \ln(x) + c_1 \end{aligned}$$

Which results in

$$y = \sin(\ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = \sin(\ln(x) + c_1) \tag{1}$$

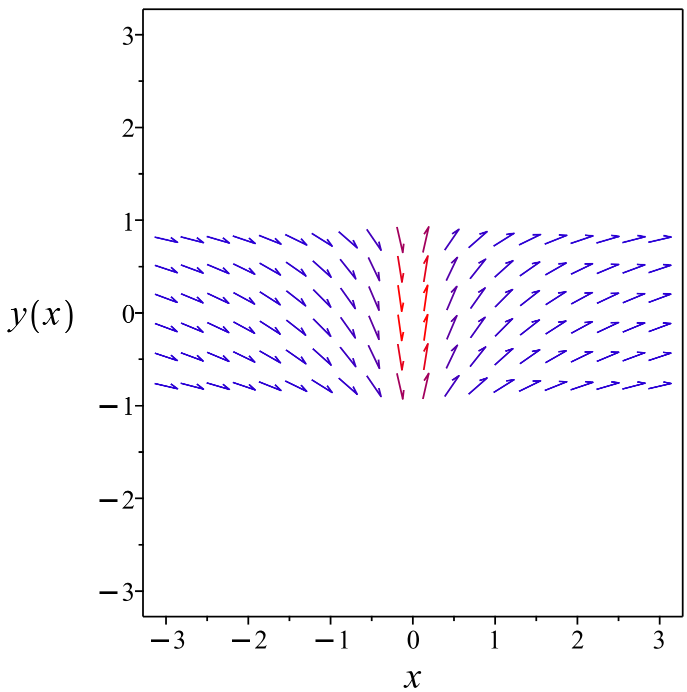


Figure 4: Slope field plot

Verification of solutions

$$y = \sin(\ln(x) + c_1)$$

Verified OK.

1.4.2 Maple step by step solution

Let's solve

$xy' - \sqrt{1-y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y^2}} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y^2}}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $\arcsin(y) = \ln(x) + c_1$

- Solve for y

  $y = \sin(\ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 9

```
dsolve(x*diff(y(x),x)=sqrt(1-y(x)^2),y(x), singsol=all)
```

$$y(x) = \sin(\ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.219 (sec). Leaf size: 29

```
DSolve[x*y'[x]==Sqrt[1-y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \cos(\log(x) + c_1) \\
y(x) &\to -1 \\
y(x) &\to 1 \\
y(x) &\to \text{Interval}[\{-1, 1\}]
\end{aligned}
$$

1.5 problem 5

Internal problem ID [5718]
Internal file name [`OUTPUT/4966_Sunday_June_05_2022_03_15_25_PM_87721000/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x^2}{1+y^2} = 0}$$

1.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x^2}{y^2+1} \end{aligned}$$

Where $f(x) = x^2$ and $g(y) = \frac{1}{y^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y^2+1}}\,dy = x^2\,dx$$

$$\int \frac{1}{\frac{1}{y^2+1}}\,dy = \int x^2\,dx$$

$$\frac{1}{3}y^3 + y = \frac{x^3}{3} + c_1$$

Which results in

$$y = \frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}$$

$$y = -\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}\right)}{2}$$

$$y = -\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}} \tag{1}$$

$$y = -\frac{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}\right)}{2} \tag{2}$$

$$y = -\frac{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}\right)}{2} \tag{3}$$

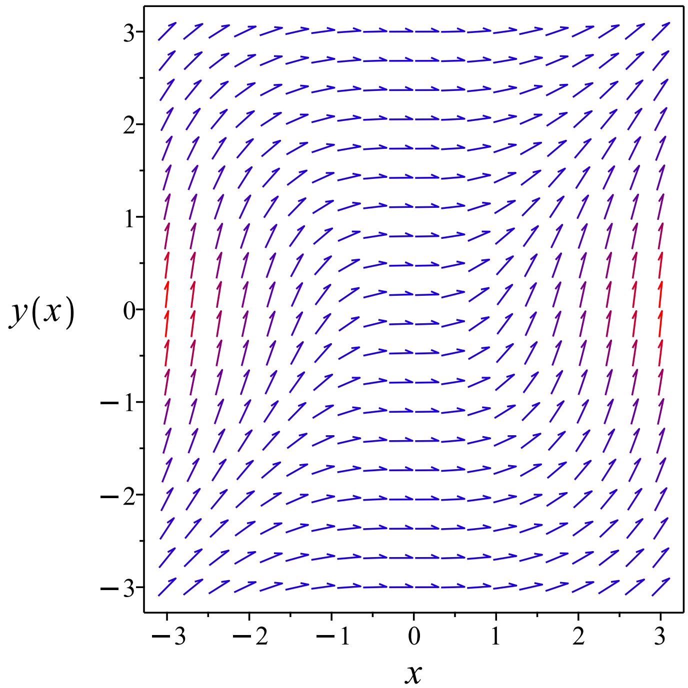


Figure 5: Slope field plot

Verification of solutions

$$y = \frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}$$

Verified OK.

$$\begin{aligned} y = &-\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{4} \\ &+ \frac{1}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}} \\ &+ \frac{i\sqrt{3}\left(\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}\right)}{2} \end{aligned}$$

Verified OK.

$$\begin{aligned} y = &-\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{4} \\ &+ \frac{1}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}} \\ &- \frac{i\sqrt{3}\left(\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}\right)}{2} \end{aligned}$$

Verified OK.

1.5.2 Maple step by step solution

Let's solve

$y' - \frac{x^2}{1+y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $(1+y^2)\, y' = x^2$

- Integrate both sides with respect to x

  $\int (1+y^2)\, y' dx = \int x^2 dx + c_1$

- Evaluate integral

  $\frac{y^3}{3} + y = \frac{x^3}{3} + c_1$

- Solve for y

  $$y = \frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 268

```
dsolve(diff(y(x),x)=x^2/(1+y(x)^2),y(x), singsol=all)
```

$$y(x) = \frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{2}{3}} - 4}{2\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{2}{3}} + 4i\sqrt{3} - 4}{4\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{i\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{2}{3}}\sqrt{3} + 4i\sqrt{3} - \left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{2}{3}} + 4}{4\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 2.179 (sec). Leaf size: 307

```
DSolve[y'[x]==x^2/(1+y[x]^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-2 + \sqrt[3]{2}\left(x^3 + \sqrt{x^6 + 6c_1x^3 + 4 + 9c_1^2} + 3c_1\right)^{2/3}}{2^{2/3}\sqrt[3]{x^3 + \sqrt{x^6 + 6c_1x^3 + 4 + 9c_1^2} + 3c_1}}$$

$$y(x) \to \frac{i\left(\sqrt{3} + i\right)\sqrt[3]{x^3 + \sqrt{x^6 + 6c_1x^3 + 4 + 9c_1^2} + 3c_1}}{2\sqrt[3]{2}} + \frac{1 + i\sqrt{3}}{2^{2/3}\sqrt[3]{x^3 + \sqrt{x^6 + 6c_1x^3 + 4 + 9c_1^2} + 3c_1}}$$

$$y(x) \to \frac{1 - i\sqrt{3}}{2^{2/3}\sqrt[3]{x^3 + \sqrt{x^6 + 6c_1x^3 + 4 + 9c_1^2} + 3c_1}} - \frac{\left(1 + i\sqrt{3}\right)\sqrt[3]{x^3 + \sqrt{x^6 + 6c_1x^3 + 4 + 9c_1^2} + 3c_1}}{2\sqrt[3]{2}}$$

1.6 problem 6

Internal problem ID [5719]
Internal file name [`OUTPUT/4967_Sunday_June_05_2022_03_15_26_PM_94014602/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{xyy' - \sqrt{1+y^2} = 0}$$

1.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\sqrt{y^2+1}}{xy} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \frac{\sqrt{y^2+1}}{y}$. Integrating both sides gives

$$\frac{1}{\frac{\sqrt{y^2+1}}{y}}\, dy = \frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{\sqrt{y^2+1}}{y}}\, dy = \int \frac{1}{x}\, dx$$

$$\sqrt{y^2+1} = \ln(x) + c_1$$

The solution is

$$\sqrt{1+y^2}-\ln(x)-c_1=0$$

Summary

The solution(s) found are the following

$$\sqrt{1+y^2}-\ln(x)-c_1=0 \tag{1}$$

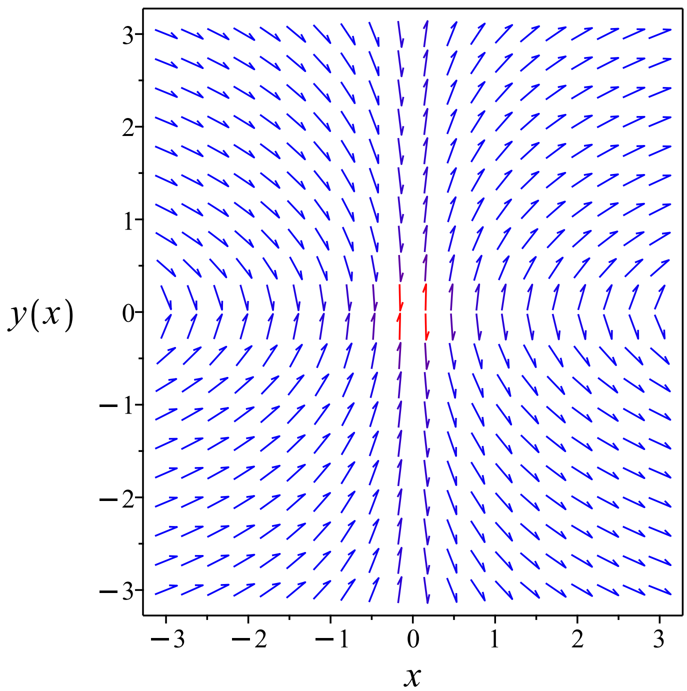


Figure 6: Slope field plot

Verification of solutions

$$\sqrt{1+y^2}-\ln(x)-c_1=0$$

Verified OK.

1.6.2 Maple step by step solution

Let's solve

$xyy' - \sqrt{1+y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{\sqrt{1+y^2}} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{\sqrt{1+y^2}}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $\sqrt{1+y^2} = \ln(x) + c_1$

- Solve for y

  $\left\{y = \sqrt{-1 + c_1^2 + 2c_1 \ln(x) + \ln(x)^2}, y = -\sqrt{-1 + c_1^2 + 2c_1 \ln(x) + \ln(x)^2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(x*y(x)*diff(y(x),x)=sqrt(1+y(x)^2),y(x), singsol=all)
```

$$\ln(x) - \sqrt{1 + y(x)^2} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.229 (sec). Leaf size: 65

```
DSolve[x*y[x]*y'[x]==Sqrt[1+y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -\sqrt{\log^2(x) + 2c_1 \log(x) - 1 + c_1{}^2} \\
y(x) &\to \sqrt{\log^2(x) + 2c_1 \log(x) - 1 + c_1{}^2} \\
y(x) &\to -i \\
y(x) &\to i
\end{aligned}
$$

1.7 problem 7

Internal problem ID [5720]
Internal file name [`OUTPUT/4968_Sunday_June_05_2022_03_15_28_PM_33180416/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\left(x^2-1\right) y'+2xy^2=0}$$

With initial conditions

$$[y(0)=1]$$

1.7.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{2x\,y^2}{x^2-1} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty \leq x < -1, -1 < x < 1, 1 < x \leq \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2x\,y^2}{x^2-1}\right)\\ &= -\frac{4xy}{x^2-1}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty \leq x < -1, -1 < x < 1, 1 < x \leq \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.7.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= -\frac{2x\,y^2}{x^2-1}\end{aligned}$$

Where $f(x) = -\frac{2x}{x^2-1}$ and $g(y) = y^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^2}\,dy &= -\frac{2x}{x^2-1}\,dx\\ \int \frac{1}{y^2}\,dy &= \int -\frac{2x}{x^2-1}\,dx\\ -\frac{1}{y} &= -\ln(x-1) - \ln(1+x) + c_1\end{aligned}$$

Which results in

$$y = \frac{1}{\ln(x-1) + \ln(1+x) - c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{i\pi - c_1}$$

$$c_1 = i\pi - 1$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{-\ln(x-1) - \ln(1+x) - 1 + i\pi}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{-\ln(x-1) - \ln(1+x) - 1 + i\pi} \tag{1}$$

Verification of solutions

$$y = -\frac{1}{-\ln(x-1) - \ln(1+x) - 1 + i\pi}$$

Verified OK.

1.7.3 Maple step by step solution

Let's solve

$[(x^2-1)\,y' + 2xy^2 = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = -\frac{2x}{x^2-1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int -\frac{2x}{x^2-1} dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = -\ln(x-1) - \ln(1+x) + c_1$

- Solve for y

$y = \frac{1}{\ln(x-1)+\ln(1+x)-c_1}$

- Use initial condition $y(0) = 1$

  $1 = \frac{1}{\mathrm{I}\pi - c_1}$

- Solve for c_1

  $c_1 = -1 + \mathrm{I}\pi$

- Substitute $c_1 = -1 + \mathrm{I}\pi$ into general solution and simplify

  $y = \frac{1}{\ln(x-1)+\ln(1+x)+1-\mathrm{I}\pi}$

- Solution to the IVP

  $y = \frac{1}{\ln(x-1)+\ln(1+x)+1-\mathrm{I}\pi}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 20

```
dsolve([(x^2-1)*diff(y(x),x)+2*x*y(x)^2=0,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{1}{-i\pi + \ln(x-1) + \ln(x+1) + 1}$$

✓ Solution by Mathematica

Time used: 0.162 (sec). Leaf size: 26

```
DSolve[{(x^2-1)*y'[x]+2*x*y[x]^2==0,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{i}{i\log(x^2-1) + \pi + i}$$

1.8 problem 8

Internal problem ID [5721]
Internal file name [`OUTPUT/4969_Sunday_June_05_2022_03_15_29_PM_4200083/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 3y^{\frac{2}{3}} = 0}$$

With initial conditions

$$[y(2) = 0]$$

1.8.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= 3y^{\frac{2}{3}} \end{aligned}$$

The y domain of $f(x, y)$ when $x = 2$ is

$$\{0 \leq y\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(3y^{\frac{2}{3}}\right)\\ &= \frac{2}{y^{\frac{1}{3}}}\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 2$ is

$$\{0 < y\}$$

But the point $y_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

1.8.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= 3y^{\frac{2}{3}}\end{aligned}$$

Where $f(x) = 1$ and $g(y) = 3y^{\frac{2}{3}}$. Integrating both sides gives

$$\frac{1}{3y^{\frac{2}{3}}}\,dy = 1\,dx$$

$$\int \frac{1}{3y^{\frac{2}{3}}}\,dy = \int 1\,dx$$

$$y^{\frac{1}{3}} = x + c_1$$

The solution is

$$y^{\frac{1}{3}} - x - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-c_1 - 2 = 0$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$y^{\frac{1}{3}} - x + 2 = 0$$

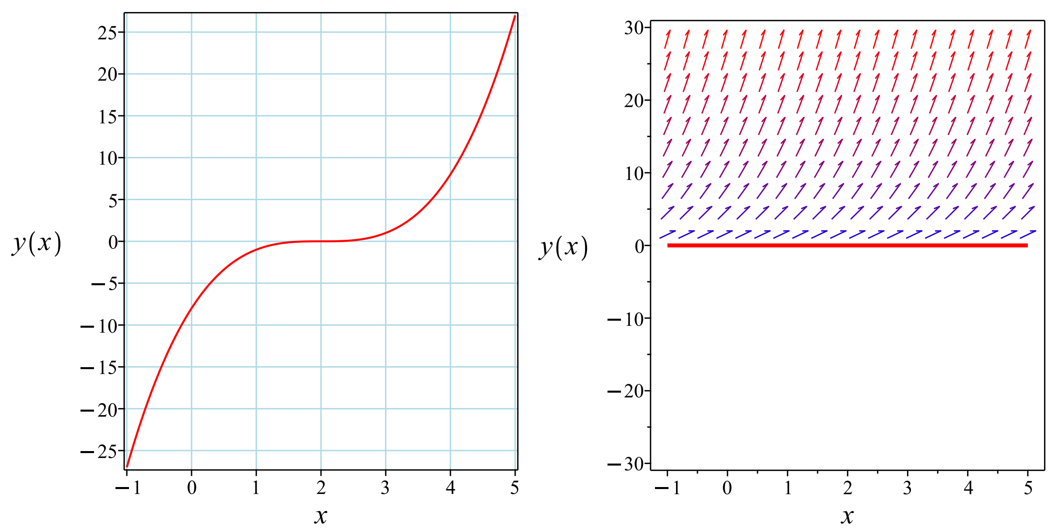
Solving for y from the above gives

$$y = (-2 + x)^3$$

Summary

The solution(s) found are the following

$$y = (-2 + x)^3 \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = (-2 + x)^3$$

Verified OK.

1.8.3 Maple step by step solution

Let's solve

$\left[y' - 3y^{\frac{2}{3}} = 0, y(2) = 0\right]$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

  $\frac{y'}{y^{\frac{2}{3}}} = 3$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{2}{3}}} dx = \int 3dx + c_1$

- Evaluate integral

  $3y^{\frac{1}{3}} = 3x + c_1$

- Solve for y

  $y = x^3 + c_1x^2 + \frac{1}{3}c_1^2x + \frac{1}{27}c_1^3$

- Use initial condition $y(2) = 0$

  $0 = 8 + 4c_1 + \frac{2}{3}c_1^2 + \frac{1}{27}c_1^3$

- Solve for c_1

  $c_1 = (-6, -6, -6)$

- Substitute $c_1 = (-6, -6, -6)$ into general solution and simplify

  $y = (-2 + x)^3$

- Solution to the IVP

  $y = (-2 + x)^3$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=3*y(x)^(2/3),y(2) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 6

```
DSolve[{y'[x]==3*y[x]^(2/3),{y[2]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

1.9 problem 9

Internal problem ID [5722]
Internal file name [`OUTPUT/4970_Sunday_June_05_2022_03_15_33_PM_21795739/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{xy' + y - y^2 = 0}$$

With initial conditions

$$\left[y(1) = \frac{1}{2}\right]$$

1.9.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y(y-1)}{x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = \frac{1}{2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y(y-1)}{x}\right)\\ &= \frac{y-1}{x} + \frac{y}{x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{1}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

1.9.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= \frac{y(y-1)}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y(y-1)$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y\,(y-1)}\,dy &= \frac{1}{x}\,dx\\ \int \frac{1}{y\,(y-1)}\,dy &= \int \frac{1}{x}\,dx\\ \ln(y-1) - \ln(y) &= \ln(x) + c_1\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(y-1)-\ln(y)} = e^{\ln(x)+c_1}$$

Which simplifies to

$$\frac{y-1}{y} = c_2 x$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -\frac{1}{-1 + c_2}$$

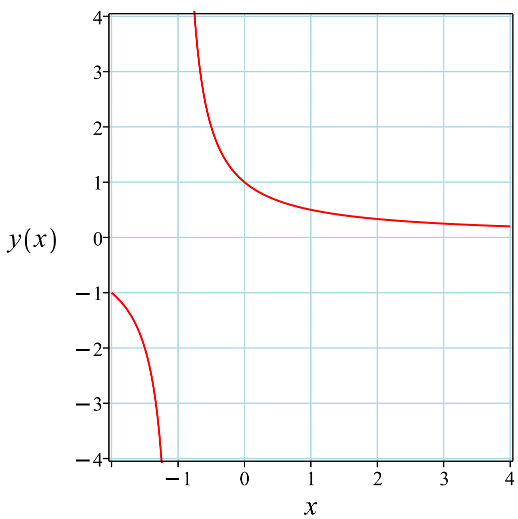
$$c_2 = -1$$

Substituting c_2 found above in the general solution gives

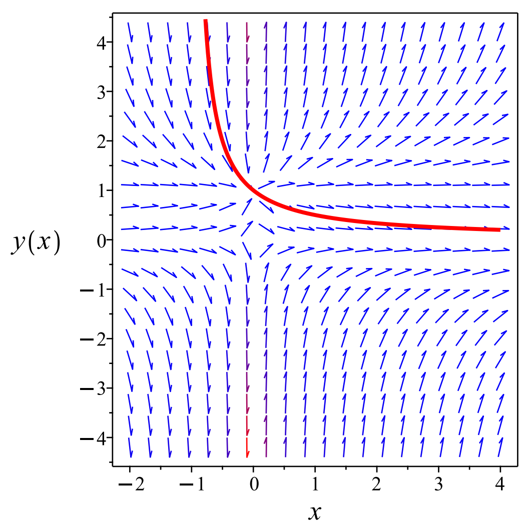
$$y = \frac{1}{1 + x}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{1 + x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{1 + x}$$

Verified OK.

1.9.3 Maple step by step solution

Let's solve

$\left[x y' + y - y^{2} = 0, y(1) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{2}-y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{2}-y} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $\ln(y-1) - \ln(y) = \ln(x) + c_1$

- Solve for y

  $y = -\frac{1}{-1 + x\,\mathrm{e}^{c_1}}$

- Use initial condition $y(1) = \frac{1}{2}$

  $\frac{1}{2} = -\frac{1}{\mathrm{e}^{c_1} - 1}$

- Solve for c_1

  $c_1 = \mathrm{I}\pi$

- Substitute $c_1 = \mathrm{I}\pi$ into general solution and simplify

  $y = \frac{1}{1+x}$

- Solution to the IVP

  $y = \frac{1}{1+x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 9

```
dsolve([x*diff(y(x),x)+y(x)=y(x)^2,y(1) = 1/2],y(x), singsol=all)
```

$$y(x) = \frac{1}{x+1}$$

✓ Solution by Mathematica

Time used: 0.252 (sec). Leaf size: 10

```
DSolve[{x*y'[x]+y[x]==y[x]^2,{y[1]==1/2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{x+1}$$

1.10 problem 10

Internal problem ID [5723]
Internal file name [`OUTPUT/4971_Sunday_June_05_2022_03_15_35_PM_36475819/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2yx^2y' + y^2 = 2}$$

1.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y^2-2}{2y\,x^2} \end{aligned}$$

Where $f(x) = -\frac{1}{2x^2}$ and $g(y) = \frac{y^2-2}{y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{y^2-2}{y}}\,dy &= -\frac{1}{2x^2}\,dx \\ \int \frac{1}{\frac{y^2-2}{y}}\,dy &= \int -\frac{1}{2x^2}\,dx \\ \frac{\ln(y^2-2)}{2} &= \frac{1}{2x} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\sqrt{y^2-2} = e^{\frac{1}{2x}+c_1}$$

Which simplifies to

$$\sqrt{y^2-2} = c_2 e^{\frac{1}{2x}}$$

The solution is

$$\sqrt{y^2-2} = c_2 e^{\frac{1}{2x}+c_1}$$

Summary

The solution(s) found are the following

$$\sqrt{y^2-2} = c_2 e^{\frac{1}{2x}+c_1} \tag{1}$$

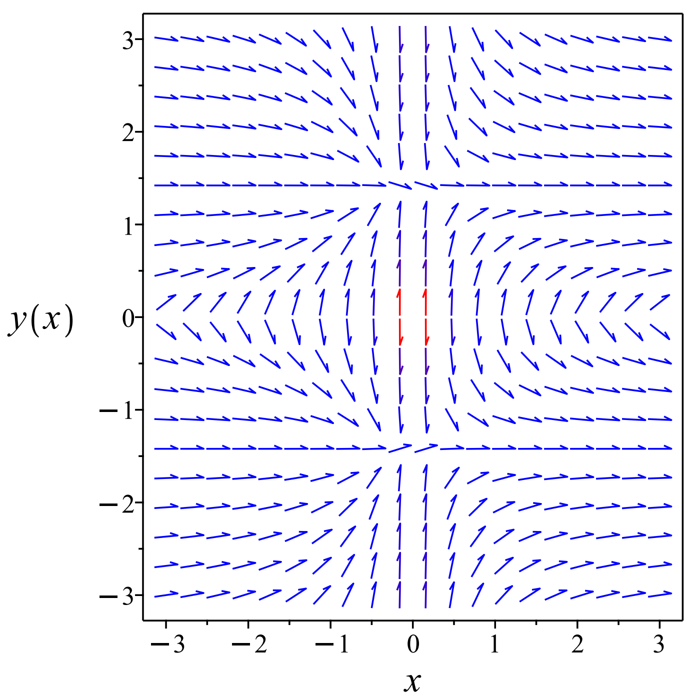


Figure 9: Slope field plot

Verification of solutions

$$\sqrt{y^2-2} = c_2 e^{\frac{1}{2x}+c_1}$$

Verified OK.

1.10.2 Maple step by step solution

Let's solve

$2yx^2y' + y^2 = 2$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{-y^2+2} = \frac{1}{2x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{-y^2+2}dx = \int \frac{1}{2x^2}dx + c_1$

- Evaluate integral

  $-\frac{\ln(y^2-2)}{2} = -\frac{1}{2x} + c_1$

- Solve for y

  $\left\{y = \sqrt{2 + e^{-\frac{2c_1x-1}{x}}}, y = -\sqrt{2 + e^{-\frac{2c_1x-1}{x}}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(2*x^2*y(x)*diff(y(x),x)+y(x)^2=2,y(x), singsol=all)
```

$$y(x) = \sqrt{e^{\frac{1}{x}}c_1 + 2}$$
$$y(x) = -\sqrt{e^{\frac{1}{x}}c_1 + 2}$$

✓ Solution by Mathematica

Time used: 0.289 (sec). Leaf size: 70

```
DSolve[2*x*y[x]*y'[x]+y[x]^2==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{2x + e^{2c_1}}}{\sqrt{x}}$$
$$y(x) \to \frac{\sqrt{2x + e^{2c_1}}}{\sqrt{x}}$$
$$y(x) \to -\sqrt{2}$$
$$y(x) \to \sqrt{2}$$

1.11 problem 11

Internal problem ID [5724]
Internal file name [`OUTPUT/4972_Sunday_June_05_2022_03_15_36_PM_7671639/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - x y^2 - 2xy = 0}$$

1.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= xy(y+2) \end{aligned}$$

Where $f(x) = x$ and $g(y) = y(y+2)$. Integrating both sides gives

$$\frac{1}{y\left(y+2\right)}\, dy = x\, dx$$

$$\int \frac{1}{y\left(y+2\right)}\, dy = \int x\, dx$$

$$\frac{\ln\left(y\right)}{2} - \frac{\ln\left(y+2\right)}{2} = \frac{x^2}{2} + c_1$$

The above can be written as

$$
\begin{aligned}
\left(\frac{1}{2}\right)(\ln(y)-\ln(y+2)) &= \frac{x^2}{2}+2c_1 \\
\ln(y)-\ln(y+2) &= (2)\left(\frac{x^2}{2}+2c_1\right) \\
&= x^2+4c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{\ln(y)-\ln(y+2)} = \mathrm{e}^{x^2+2c_1}
$$

Which simplifies to

$$
\begin{aligned}
\frac{y}{y+2} &= 2c_1\mathrm{e}^{x^2} \\
&= c_2\mathrm{e}^{x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{2c_2\mathrm{e}^{x^2}}{-1+c_2\mathrm{e}^{x^2}} \tag{1}
$$

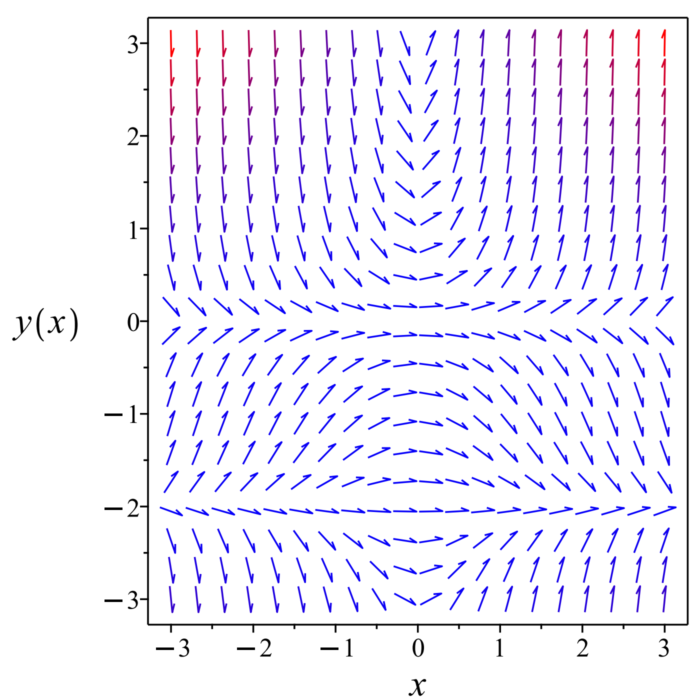


Figure 10: Slope field plot

Verification of solutions

$$y = -\frac{2c_2 e^{x^2}}{-1 + c_2 e^{x^2}}$$

Verified OK.

1.11.2 Maple step by step solution

Let's solve

$y' - xy^2 - 2xy = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(2+y)y} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{(2+y)y} dx = \int x dx + c_1$

- Evaluate integral

  $\frac{\ln(y)}{2} - \frac{\ln(2+y)}{2} = \frac{x^2}{2} + c_1$

- Solve for y

  $y = -\frac{2\,e^{x^2+2c_1}}{-1+e^{x^2+2c_1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)-x*y(x)^2=2*x*y(x),y(x), singsol=all)
```

$$y(x) = \frac{2}{-1 + 2\,\mathrm{e}^{-x^2} c_1}$$

✓ Solution by Mathematica

Time used: 0.276 (sec). Leaf size: 37

```
DSolve[y'[x]-2*x*y[x]^2==2*x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{x^2+c_1}}{-1+e^{x^2+c_1}}$$
$$y(x) \to -1$$
$$y(x) \to 0$$

1.12 problem 12

Internal problem ID [5725]
Internal file name [`OUTPUT/4973_Sunday_June_05_2022_03_15_38_PM_88735436/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{(1 + z')\,\mathrm{e}^{-z} = 1}$$

1.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} z' &= F(t, z) \\ &= f(t)g(z) \\ &= -1 + \mathrm{e}^{z} \end{aligned}$$

Where $f(t) = 1$ and $g(z) = -1 + \mathrm{e}^{z}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-1 + \mathrm{e}^{z}}\, dz &= 1\, dt \\ \int \frac{1}{-1 + \mathrm{e}^{z}}\, dz &= \int 1\, dt \\ \ln\left(-1 + \mathrm{e}^{z}\right) - \ln\left(\mathrm{e}^{z}\right) &= t + c_1 \end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(-1+e^z)-\ln(e^z)} = e^{t+c_1}$$

Which simplifies to

$$-e^{-z} + 1 = c_2 e^t$$

<u>Summary</u>

The solution(s) found are the following

$$z = -\ln\left(1 - c_2 e^t\right) \tag{1}$$

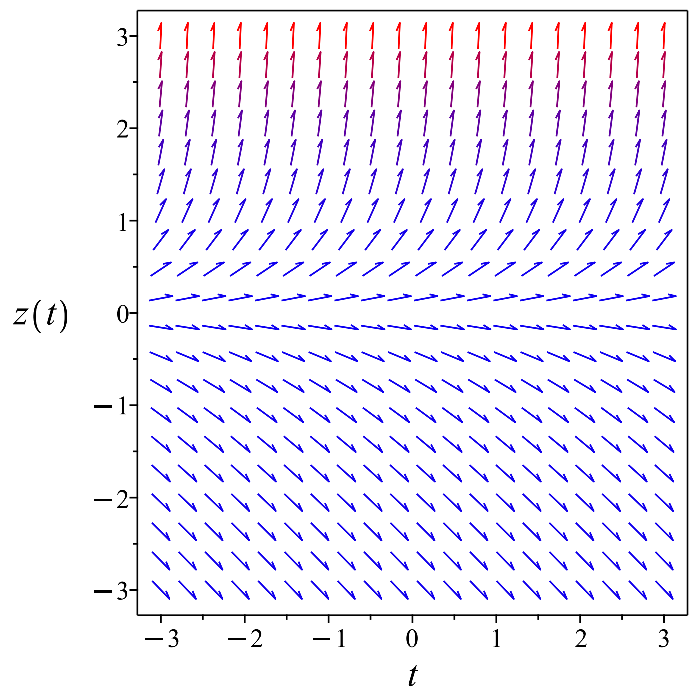


Figure 11: Slope field plot

<u>Verification of solutions</u>

$$z = -\ln\left(1 - c_2 e^t\right)$$

Verified OK.

1.12.2 Maple step by step solution

Let's solve

$(1+z')\,\mathrm{e}^{-z}=1$

- Highest derivative means the order of the ODE is 1

  z'

- Separate variables

  $\frac{z'\mathrm{e}^{-z}}{\mathrm{e}^{-z}-1}=-1$

- Integrate both sides with respect to t

  $\int \frac{z'\mathrm{e}^{-z}}{\mathrm{e}^{-z}-1}dt=\int (-1)\,dt+c_1$

- Evaluate integral

  $-\ln\left(\mathrm{e}^{-z}-1\right)=-t+c_1$

- Solve for z

  $z=-\ln\left(\mathrm{e}^{t-c_1}+1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve((1+diff(z(t),t))*exp(-z(t))=1,z(t), singsol=all)
```

$$z(t)=\ln\left(-\frac{1}{c_1\mathrm{e}^{t}-1}\right)$$

✓ Solution by Mathematica

Time used: 0.722 (sec). Leaf size: 28

```
DSolve[(1+z'[t])*Exp[-z[t]]==1,z[t],t,IncludeSingularSolutions -> True]
```

$$z(t) \to \log\left(\frac{1}{2}\left(1 - \tanh\left(\frac{t + c_1}{2}\right)\right)\right)$$
$$z(t) \to 0$$

1.13 problem 13

Internal problem ID [5726]
Internal file name [`OUTPUT/4974_Sunday_June_05_2022_03_15_39_PM_89504319/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{3x^2 + 4x + 2}{2y - 2} = 0}$$

With initial conditions

$$[y(0) = -1]$$

1.13.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{3x^2 + 4x + 2}{2y - 2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{3x^2 + 4x + 2}{2y - 2}\right) \\
&= -\frac{3x^2 + 4x + 2}{2\left(y - 1\right)^2}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

1.13.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= \frac{\frac{3}{2}x^2 + 2x + 1}{y - 1}
\end{aligned}$$

Where $f(x) = \frac{3}{2}x^2 + 2x + 1$ and $g(y) = \frac{1}{y-1}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{1}{y-1}}\, dy &= \frac{3}{2}x^2 + 2x + 1\, dx \\
\int \frac{1}{\frac{1}{y-1}}\, dy &= \int \frac{3}{2}x^2 + 2x + 1\, dx \\
\frac{1}{2}y^2 - y &= \frac{1}{2}x^3 + x^2 + x + c_1
\end{aligned}$$

Which results in

$$y = 1 + \sqrt{x^3 + 2x^2 + 2c_1 + 2x + 1}$$
$$y = 1 - \sqrt{x^3 + 2x^2 + 2c_1 + 2x + 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = 1 - \sqrt{2c_1 + 1}$$

$$c_1 = \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

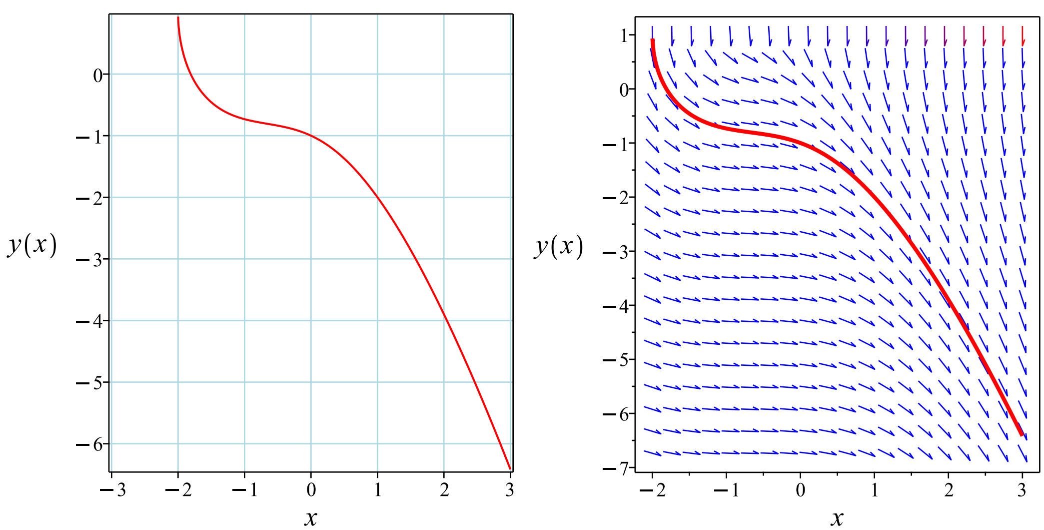
$$-1 = 1 + \sqrt{2c_1 + 1}$$

Warning: Unable to solve for constant of integration.

Summary

The solution(s) found are the following

$$y = 1 - \sqrt{x^3 + 2x^2 + 2$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

Verified OK.

1.13.3 Maple step by step solution

Let's solve

$\left[y' - \frac{3x^2+4x+2}{2y-2} = 0, y(0) = -1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(2y-2) = 3x^2 + 4x + 2$

- Integrate both sides with respect to x

  $\int y'(2y-2)\,dx = \int (3x^2 + 4x + 2)\,dx + c_1$

- Evaluate integral

  $y^2 - 2y = x^3 + 2x^2 + c_1 + 2x$

- Solve for y

  $\left\{y = 1 - \sqrt{x^3 + 2x^2 + c_1 + 2x + 1}, y = 1 + \sqrt{x^3 + 2x^2 + c_1 + 2x + 1}\right\}$

- Use initial condition $y(0) = -1$

  $-1 = 1 - \sqrt{c_1 + 1}$

- Solve for c_1

  $c_1 = 3$

- Substitute $c_1 = 3$ into general solution and simplify

  $y = -\sqrt{(x+2)(x^2+2)} + 1$

- Use initial condition $y(0) = -1$

  $-1 = 1 + \sqrt{c_1 + 1}$

- Solution does not satisfy initial condition

- Solution to the IVP

  $y = -\sqrt{(x+2)(x^2+2)} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 19

```
dsolve([diff(y(x),x)=(3*x^2+4*x+2)/(2*(y(x)-1)),y(0) = -1],y(x), singsol=all)
```

$$y(x) = 1 - \sqrt{(x+2)\left(x^2+2\right)}$$

✓ Solution by Mathematica

Time used: 0.132 (sec). Leaf size: 26

```
DSolve[{y'[x]==(3*x^2+4*x+2)/(2*(y[x]-1)),{y[0]==-1}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to 1 - \sqrt{x^3+2x^2+2x+4}$$

1.14 problem 14

Internal problem ID [5727]
Internal file name [`OUTPUT/4975_Sunday_June_05_2022_03_15_40_PM_2246626/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{-(1+e^{x})\, y y' = -e^{x}}$$

With initial conditions

$$[y(0) = 1]$$

1.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{e^{x}}{(1+e^{x})\, y} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{2i\pi\_Z146 + i\pi < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

1.14.2 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{e^x}{(1+e^x)\,y}
\end{aligned}
$$

Where $f(x) = \frac{e^x}{1+e^x}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{y}}\,dy &= \frac{e^x}{1+e^x}\,dx \\
\int \frac{1}{\frac{1}{y}}\,dy &= \int \frac{e^x}{1+e^x}\,dx \\
\frac{y^2}{2} &= \ln(1+e^x) + c_1
\end{aligned}
$$

Which results in

$$
\begin{aligned}
y &= \sqrt{2\ln(1+e^x) + 2c_1} \\
y &= -\sqrt{2\ln(1+e^x) + 2c_1}
\end{aligned}
$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = -\sqrt{2\ln(2) + 2c_1}
$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = \sqrt{2\ln(2) + 2c_1}
$$

$$
c_1 = -\ln(2) + \frac{1}{2}
$$

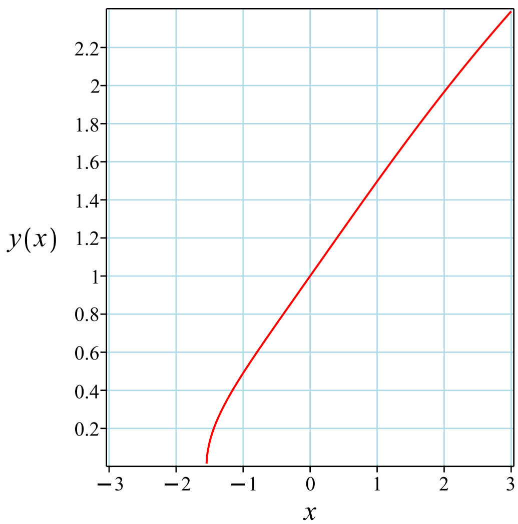
Substituting c_1 found above in the general solution gives

$$
y = \sqrt{2\ln(1+e^x) - 2\ln(2) + 1}
$$

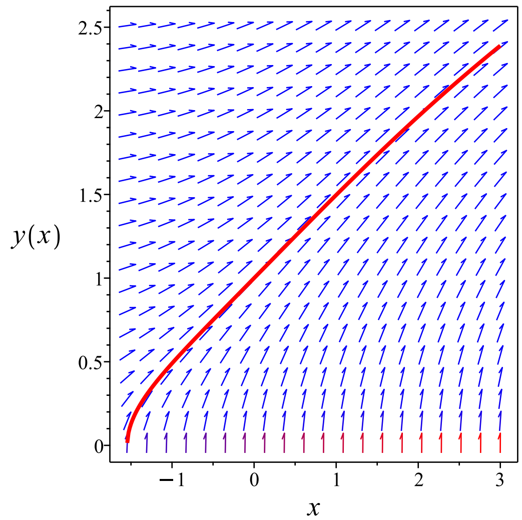
Summary

The solution(s) found are the following

$$y = \sqrt{2 \ln (1 + e^{x}) - 2 \ln (2) + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{2 \ln (1 + e^{x}) - 2 \ln (2) + 1}$$

Verified OK.

1.14.3 Maple step by step solution

Let's solve

$[-(1 + e^{x})\, y y' = -e^{x}, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' y = \frac{e^{x}}{1+e^{x}}$

- Integrate both sides with respect to x

  $\int y' y dx = \int \frac{e^{x}}{1+e^{x}} dx + c_1$

- Evaluate integral

$\frac{y^2}{2} = \ln(1+e^x) + c_1$

- Solve for y

  $\left\{y = \sqrt{2\ln(1+e^x)+2c_1}, y = -\sqrt{2\ln(1+e^x)+2c_1}\right\}$
- Use initial condition $y(0) = 1$

  $1 = \sqrt{2\ln(2)+2c_1}$
- Solve for c_1

  $c_1 = -\ln(2) + \frac{1}{2}$
- Substitute $c_1 = -\ln(2) + \frac{1}{2}$ into general solution and simplify

  $y = \sqrt{2\ln(1+e^x) - 2\ln(2) + 1}$
- Use initial condition $y(0) = 1$

  $1 = -\sqrt{2\ln(2)+2c_1}$
- Solution does not satisfy initial condition
- Solution to the IVP

  $y = \sqrt{2\ln(1+e^x) - 2\ln(2) + 1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.219 (sec). Leaf size: 19

```
dsolve([exp(x)-(1+exp(x))*y(x)*diff(y(x),x)=0,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \sqrt{2\ln(e^x+1) - 2\ln(2) + 1}$$

✓ Solution by Mathematica

Time used: 0.182 (sec). Leaf size: 23

```
DSolve[{Exp[x]-(1+Exp[x])*y[x]*y'[x]==0,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{2\log\left(e^x+1\right)+1-\log(4)}$$

1.15 problem 15

Internal problem ID [5728]
Internal file name [`OUTPUT/4976_Sunday_June_05_2022_03_15_42_PM_72193388/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\frac{y}{x-1}+\frac{xy'}{1+y}=0}$$

1.15.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= -\frac{y(1+y)}{(x-1)\,x}
\end{aligned}$$

Where $f(x)=-\frac{1}{x(x-1)}$ and $g(y)=y(1+y)$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y\,(1+y)}\,dy &= -\frac{1}{x\,(x-1)}\,dx\\
\int\frac{1}{y\,(1+y)}\,dy &= \int-\frac{1}{x\,(x-1)}\,dx\\
-\ln(1+y)+\ln(y) &= -\ln(x-1)+\ln(x)+c_1
\end{aligned}$$

Raising both side to exponential gives

$$e^{-\ln(1+y)+\ln(y)} = e^{-\ln(x-1)+\ln(x)+c_1}$$

Which simplifies to

$$\frac{y}{1+y} = c_2 e^{-\ln(x-1)+\ln(x)}$$

Which simplifies to

$$y = -\frac{c_2 x}{(x-1)\left(-1+\frac{c_2 x}{x-1}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_2 x}{(x-1)\left(-1+\frac{c_2 x}{x-1}\right)} \tag{1}$$

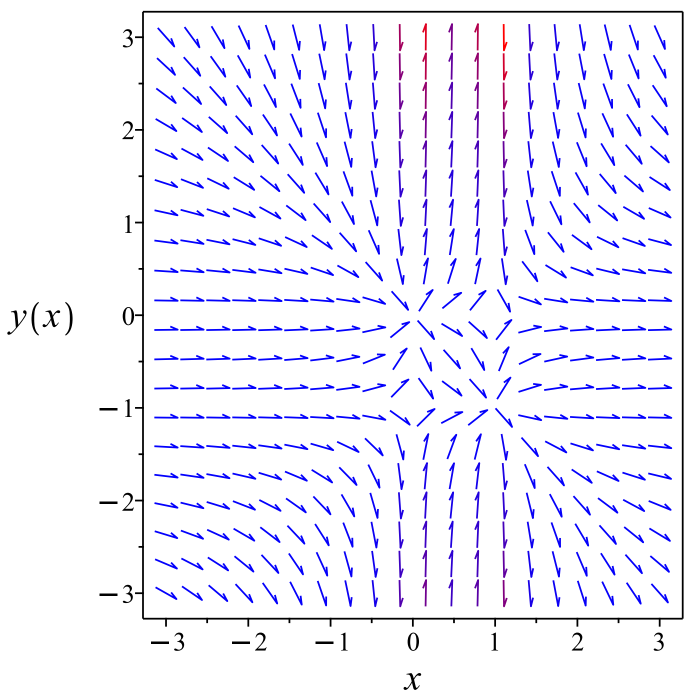


Figure 14: Slope field plot

Verification of solutions

$$y = -\frac{c_2 x}{(x-1)\left(-1+\frac{c_2 x}{x-1}\right)}$$

Verified OK.

1.15.2 Maple step by step solution

Let's solve

$\frac{y}{x-1}+\frac{xy'}{1+y}=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(1+y)}=-\frac{1}{x(x-1)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(1+y)}dx=\int -\frac{1}{x(x-1)}dx+c_1$

- Evaluate integral

  $-\ln(1+y)+\ln(y)=-\ln(x-1)+\ln(x)+c_1$

- Solve for y

  $y=-\frac{x\,\mathrm{e}^{c_1}}{1+x\,\mathrm{e}^{c_1}-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(y(x)/(x-1)+x/(y(x)+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=\frac{x}{-1+c_1(x-1)}$$

✓ Solution by Mathematica

Time used: 0.417 (sec). Leaf size: 33

```
DSolve[y[x]/(x-1)+x/(y[x]+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\rightarrow -\frac{e^{c_1}x}{x+e^{c_1}x-1} \\
y(x) &\rightarrow -1 \\
y(x) &\rightarrow 0
\end{aligned}
$$

1.16 problem 16

Internal problem ID [5729]
Internal file name [`OUTPUT/4977_Sunday_June_05_2022_03_15_44_PM_75195143/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(2y^3+y)\,y' = -2x^3 - x}$$

1.16.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{x(2x^2+1)}{2y^3+y} \end{aligned}$$

Where $f(x) = -x(2x^2+1)$ and $g(y) = \frac{1}{2y^3+y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{2y^3+y}}\,dy = -x\left(2x^2+1\right)\,dx$$

$$\int \frac{1}{\frac{1}{2y^3+y}}\,dy = \int -x\left(2x^2+1\right)\,dx$$

$$\frac{\left(2y^2+1\right)^2}{8} = -\frac{\left(2x^2+1\right)^2}{8} + c_1$$

The solution is

$$\frac{(2y^2+1)^2}{8}+\frac{(2x^2+1)^2}{8}-c_1=0$$

Summary

The solution(s) found are the following

$$\frac{(2y^2+1)^2}{8}+\frac{(2x^2+1)^2}{8}-c_1=0 \tag{1}$$

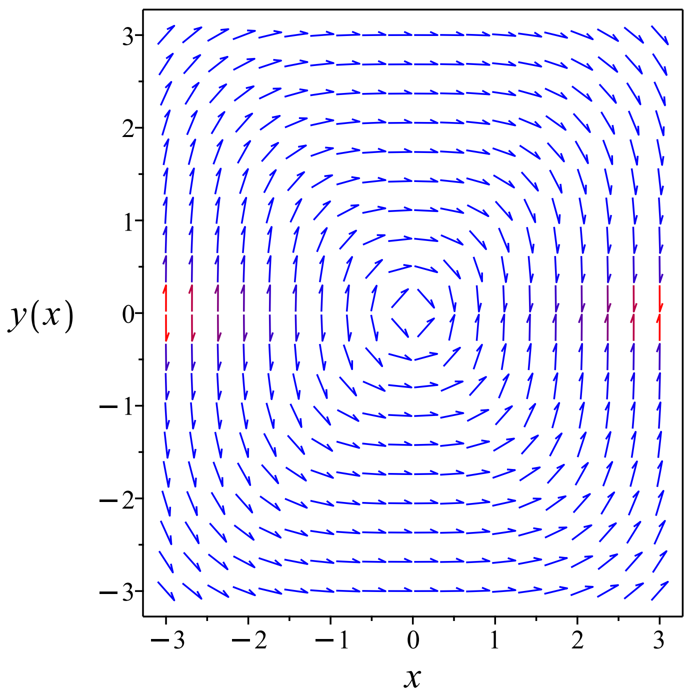


Figure 15: Slope field plot

Verification of solutions

$$\frac{(2y^2+1)^2}{8}+\frac{(2x^2+1)^2}{8}-c_1=0$$

Verified OK.

1.16.2 Maple step by step solution

Let's solve

$(2y^3 + y)\, y' = -2x^3 - x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (2y^3 + y)\, y' dx = \int (-2x^3 - x)\, dx + c_1$

- Evaluate integral

  $\frac{(2y^2+1)^2}{8} = -\frac{(2x^2+1)^2}{8} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 113

```
dsolve((x+2*x^3)+(y(x)+2*y(x)^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-2 - 2\sqrt{-4x^4 - 4x^2 - 8c_1 - 1}}}{2}$$
$$y(x) = \frac{\sqrt{-2 - 2\sqrt{-4x^4 - 4x^2 - 8c_1 - 1}}}{2}$$
$$y(x) = -\frac{\sqrt{-2 + 2\sqrt{-4x^4 - 4x^2 - 8c_1 - 1}}}{2}$$
$$y(x) = \frac{\sqrt{-2 + 2\sqrt{-4x^4 - 4x^2 - 8c_1 - 1}}}{2}$$

✓ Solution by Mathematica

Time used: 2.086 (sec). Leaf size: 151

```
DSolve[(x+2*x^3)+(y[x]+2*y[x]^3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-1-\sqrt{-4x^4-4x^2+1+8c_1}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-1-\sqrt{-4x^4-4x^2+1+8c_1}}}{\sqrt{2}}$$
$$y(x) \to -\frac{\sqrt{-1+\sqrt{-4x^4-4x^2+1+8c_1}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-1+\sqrt{-4x^4-4x^2+1+8c_1}}}{\sqrt{2}}$$

1.17 problem 17

Internal problem ID [5730]
Internal file name [`OUTPUT/4978_Sunday_June_05_2022_03_15_47_PM_86994772/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\frac{y'}{\sqrt{y}} = -\frac{1}{\sqrt{x}}}$$

1.17.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{\sqrt{y}}{\sqrt{x}} \end{aligned}$$

Where $f(x) = -\frac{1}{\sqrt{x}}$ and $g(y) = \sqrt{y}$. Integrating both sides gives

$$\frac{1}{\sqrt{y}}\,dy = -\frac{1}{\sqrt{x}}\,dx$$

$$\int \frac{1}{\sqrt{y}}\,dy = \int -\frac{1}{\sqrt{x}}\,dx$$

$$2\sqrt{y} = -2\sqrt{x} + c_1$$

The solution is

$$2\sqrt{y} + 2\sqrt{x} - c_1 = 0$$

Summary

The solution(s) found are the following

$$2\sqrt{y} + 2\sqrt{x} - c_1 = 0 \tag{1}$$

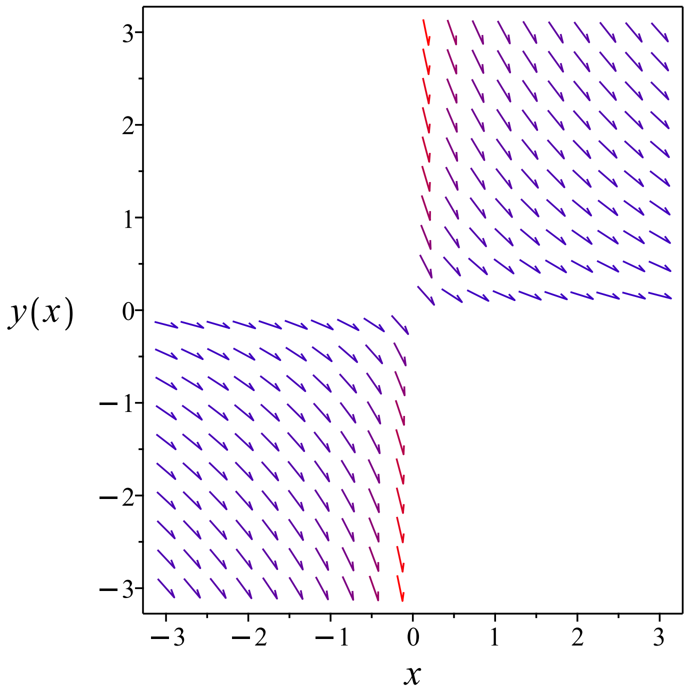


Figure 16: Slope field plot

Verification of solutions

$$2\sqrt{y} + 2\sqrt{x} - c_1 = 0$$

Verified OK.

1.17.2 Maple step by step solution

Let's solve

$\frac{y'}{\sqrt{y}} = -\frac{1}{\sqrt{x}}$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{y}}dx = \int -\frac{1}{\sqrt{x}}dx + c_1$

- Evaluate integral

  $2\sqrt{y} = -2\sqrt{x} + c_1$

- Solve for y

  $y = -\sqrt{x}\, c_1 + \frac{c_1^2}{4} + x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(1/sqrt(x)+diff(y(x),x)/sqrt(y(x))=0,y(x), singsol=all)
```

$$\sqrt{y(x)} + \sqrt{x} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.125 (sec). Leaf size: 21

```
DSolve[1/Sqrt[x]+y'[x]/Sqrt[y[x]]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(-2\sqrt{x}+c_1\right)^2$$

1.18 problem 18

Internal problem ID [5731]
Internal file name [`OUTPUT/4979_Sunday_June_05_2022_03_15_48_PM_32759713/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\frac{y'}{\sqrt{1-y^2}} = -\frac{1}{\sqrt{-x^2+1}}}$$

1.18.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{\sqrt{-y^2+1}}{\sqrt{-x^2+1}} \end{aligned}$$

Where $f(x) = -\frac{1}{\sqrt{-x^2+1}}$ and $g(y) = \sqrt{-y^2+1}$. Integrating both sides gives

$$\frac{1}{\sqrt{-y^2+1}}\,dy = -\frac{1}{\sqrt{-x^2+1}}\,dx$$

$$\int \frac{1}{\sqrt{-y^2+1}}\,dy = \int -\frac{1}{\sqrt{-x^2+1}}\,dx$$

$$\arcsin(y) = \frac{\sqrt{-(x-1)^2-2x+2}}{2} - \arcsin(x) - \frac{\sqrt{-(1+x)^2+2x+2}}{2} + c_1$$

Which results in

$$y = \sin(-\arcsin(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = \sin(-\arcsin(x) + c_1) \tag{1}$$

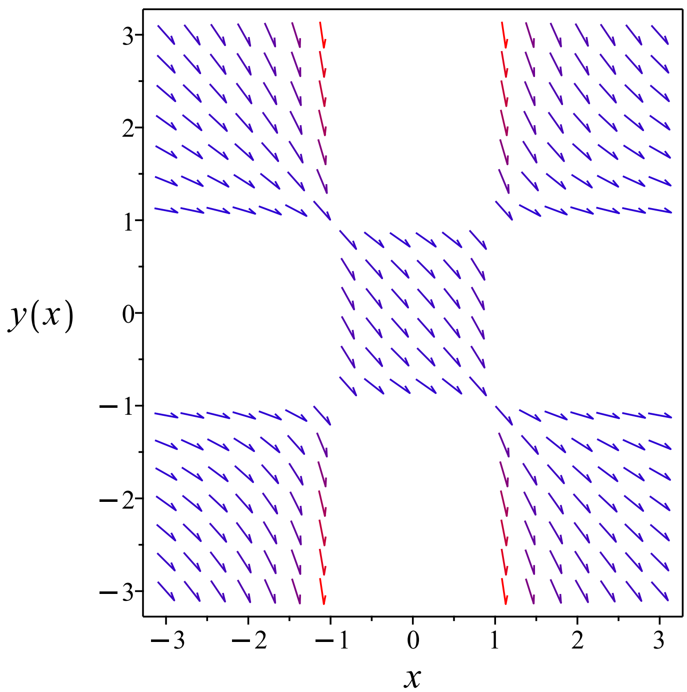


Figure 17: Slope field plot

Verification of solutions

$$y = \sin(-\arcsin(x) + c_1)$$

Verified OK.

1.18.2 Maple step by step solution

Let's solve

$\frac{y'}{\sqrt{1-y^2}} = -\frac{1}{\sqrt{-x^2+1}}$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y^2}} dx = \int -\frac{1}{\sqrt{-x^2+1}} dx + c_1$

- Evaluate integral

  $\arcsin(y) = -\arcsin(x) + c_1$

- Solve for y

  $y = \sin(-\arcsin(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve(1/sqrt(1-x^2)+diff(y(x),x)/sqrt(1-y(x)^2)=0,y(x), singsol=all)
```

$$y(x) = -\sin(\arcsin(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.288 (sec). Leaf size: 37

```
DSolve[1/Sqrt[1-x^2]+y'[x]/Sqrt[1-y[x]^2]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \cos\left(2\arctan\left(\frac{\sqrt{1-x^2}}{x+1}\right)+c_1\right) \\
y(x) &\to \text{Interval}[\{-1,1\}]
\end{aligned}
$$

1.19 problem 19

Internal problem ID [5732]
Internal file name [OUTPUT/4980_Sunday_June_05_2022_03_15_50_PM_91504895/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2x\sqrt{1-y^2}+y'y=0}$$

1.19.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{2x\sqrt{-y^2+1}}{y} \end{aligned}$$

Where $f(x)=-2x$ and $g(y)=\frac{\sqrt{-y^2+1}}{y}$. Integrating both sides gives

$$\frac{1}{\frac{\sqrt{-y^2+1}}{y}}\,dy = -2x\,dx$$

$$\int \frac{1}{\frac{\sqrt{-y^2+1}}{y}}\,dy = \int -2x\,dx$$

$$-\sqrt{-y^2+1} = -x^2+c_1$$

The solution is

$$-\sqrt{1-y^2}+x^2-c_1=0$$

Summary
The solution(s) found are the following

$$-\sqrt{1-y^2}+x^2-c_1=0 \tag{1}$$

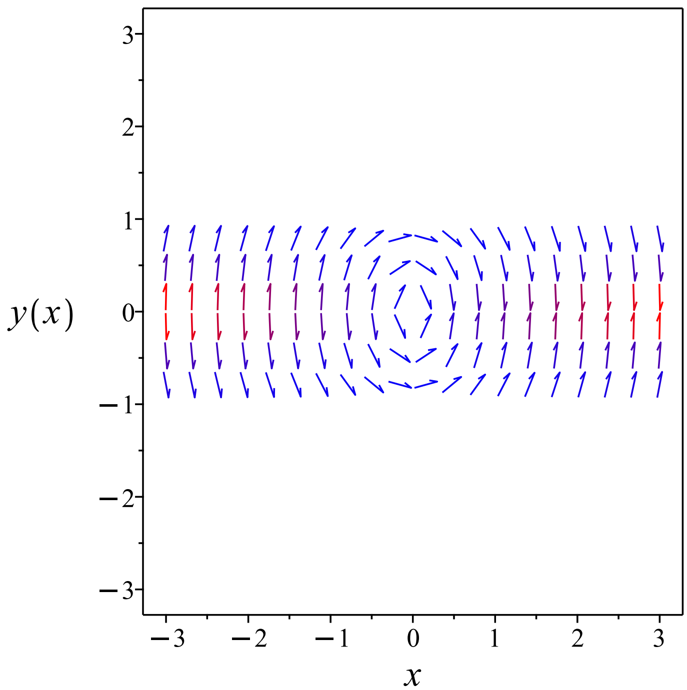


Figure 18: Slope field plot

Verification of solutions

$$-\sqrt{1-y^2}+x^2-c_1=0$$

Verified OK.

1.19.2 Maple step by step solution

Let's solve

$2x\sqrt{1-y^2}+y'y=0$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$\frac{y'y}{\sqrt{1-y^2}}=-2x$

- Integrate both sides with respect to x

$\int \frac{y'y}{\sqrt{1-y^2}}dx=\int -2xdx+c_1$

- Evaluate integral

$-\sqrt{1-y^2}=-x^2+c_1$

- Solve for y

$\left\{y=\sqrt{-x^4+2c_1x^2-c_1^2+1}, y=-\sqrt{-x^4+2c_1x^2-c_1^2+1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(2*x*sqrt(1-y(x)^2)+y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$c_1+x^2+\frac{(y(x)-1)\,(y(x)+1)}{\sqrt{1-y(x)^2}}=0$$

✓ Solution by Mathematica

Time used: 0.288 (sec). Leaf size: 69

```
DSolve[2*x*Sqrt[1-y[x]^2]+y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x^4 + 2c_1x^2 + 1 - c_1{}^2}$$
$$y(x) \to \sqrt{-x^4 + 2c_1x^2 + 1 - c_1{}^2}$$
$$y(x) \to -1$$
$$y(x) \to 1$$

1.20 problem 20

Internal problem ID [5733]
Internal file name [`OUTPUT/4981_Sunday_June_05_2022_03_15_52_PM_62830200/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - (y-1)(1+x) = 0}$$

1.20.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= (y-1)(1+x) \end{aligned}$$

Where $f(x) = 1 + x$ and $g(y) = y - 1$. Integrating both sides gives

$$\frac{1}{y-1}\,dy = 1 + x\,dx$$

$$\int \frac{1}{y-1}\,dy = \int 1 + x\,dx$$

$$\ln(y-1) = \frac{1}{2}x^2 + x + c_1$$

Raising both side to exponential gives

$$y - 1 = e^{\frac{1}{2}x^2 + x + c_1}$$

Which simplifies to

$$y - 1 = c_2 e^{\frac{1}{2}x^2 + x}$$

Summary

The solution(s) found are the following

$$y = c_2 e^{\frac{1}{2}x^2 + x + c_1} + 1 \tag{1}$$

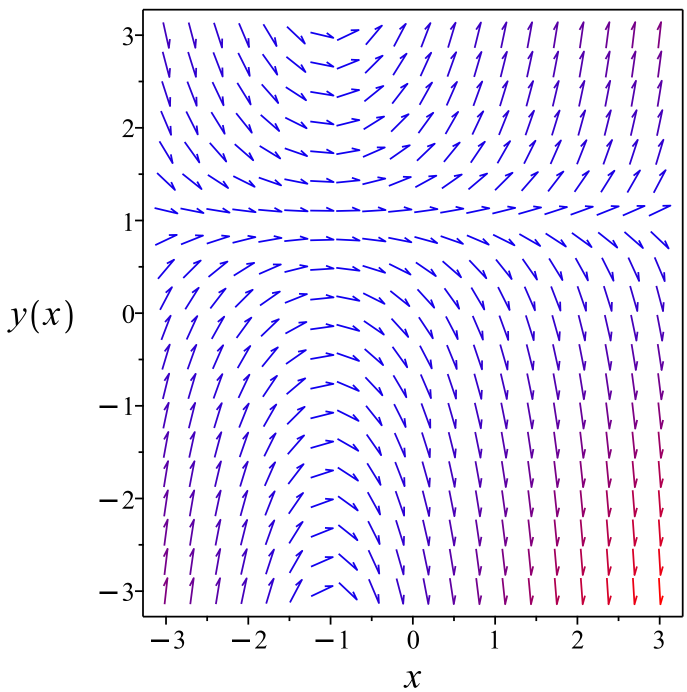


Figure 19: Slope field plot

Verification of solutions

$$y = c_2 e^{\frac{1}{2}x^2 + x + c_1} + 1$$

Verified OK.

1.20.2 Maple step by step solution

Let's solve

$y' - (y-1)(1+x) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y-1} = 1 + x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y-1}dx = \int (1+x)\, dx + c_1$

- Evaluate integral

  $\ln(y-1) = \frac{1}{2}x^2 + x + c_1$

- Solve for y

  $y = \mathrm{e}^{\frac{1}{2}x^2+x+c_1} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)=(y(x)-1)*(x+1),y(x), singsol=all)
```

$$y(x) = 1 + c_1 \mathrm{e}^{\frac{x(x+2)}{2}}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 25

```
DSolve[y'[x]==(y[x]-1)*(x+1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 + c_1 e^{\frac{1}{2}x(x+2)}$$
$$y(x) \to 1$$

1.21 problem 21

Internal problem ID [5734]
Internal file name [`OUTPUT/4982_Sunday_June_05_2022_03_15_53_PM_52181844/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - e^{x-y} = 0}$$

1.21.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= e^x e^{-y} \end{aligned}$$

Where $f(x) = e^x$ and $g(y) = e^{-y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{e^{-y}}\,dy &= e^x\,dx \\ \int \frac{1}{e^{-y}}\,dy &= \int e^x\,dx \\ e^y &= e^x + c_1 \end{aligned}$$

Which results in

$$y = \ln(e^x + c_1)$$

Summary

The solution(s) found are the following

$$y = \ln(e^x + c_1) \tag{1}$$

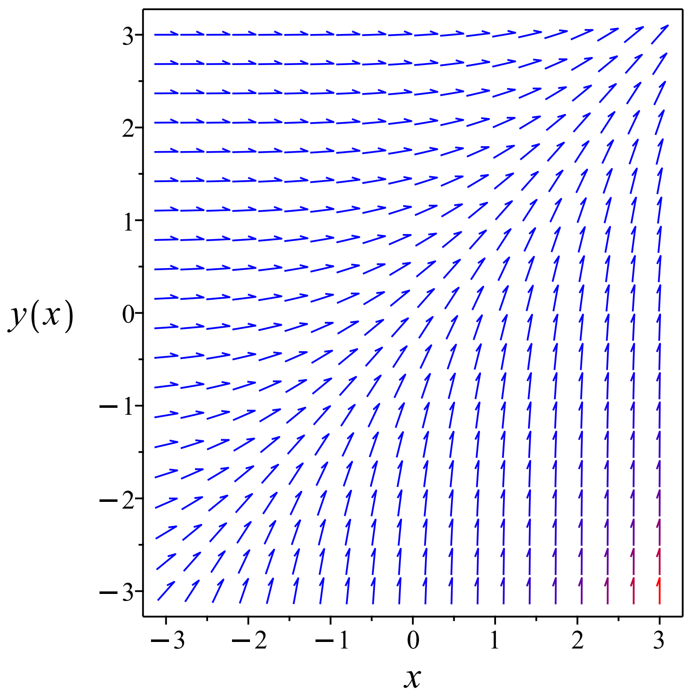


Figure 20: Slope field plot

Verification of solutions

$$y = \ln(e^x + c_1)$$

Verified OK.

1.21.2 Maple step by step solution

Let's solve

$y' - e^{x-y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' e^{y} = e^{x}$

- Integrate both sides with respect to x

  $\int y' e^{y} dx = \int e^{x} dx + c_1$

- Evaluate integral

  $e^{y} = e^{x} + c_1$

- Solve for y

  $y = \ln(e^{x} + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(y(x),x)=exp(x-y(x)),y(x), singsol=all)
```

$$y(x) = \ln(e^{x} + c_1)$$

✓ Solution by Mathematica

Time used: 0.743 (sec). Leaf size: 12

```
DSolve[y'[x]==Exp[x-y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log\left(e^x + c_1\right)$$

1.22 problem 22

Internal problem ID [5735]
Internal file name [`OUTPUT/4983_Sunday_June_05_2022_03_15_54_PM_7171702/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{\sqrt{y}}{\sqrt{x}} = 0}$$

1.22.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\sqrt{y}}{\sqrt{x}} \end{aligned}$$

Where $f(x) = \frac{1}{\sqrt{x}}$ and $g(y) = \sqrt{y}$. Integrating both sides gives

$$\frac{1}{\sqrt{y}}\,dy = \frac{1}{\sqrt{x}}\,dx$$

$$\int \frac{1}{\sqrt{y}}\,dy = \int \frac{1}{\sqrt{x}}\,dx$$

$$2\sqrt{y} = 2\sqrt{x} + c_1$$

The solution is

$$2\sqrt{y} - 2\sqrt{x} - c_1 = 0$$

Summary

The solution(s) found are the following

$$2\sqrt{y} - 2\sqrt{x} - c_1 = 0 \tag{1}$$

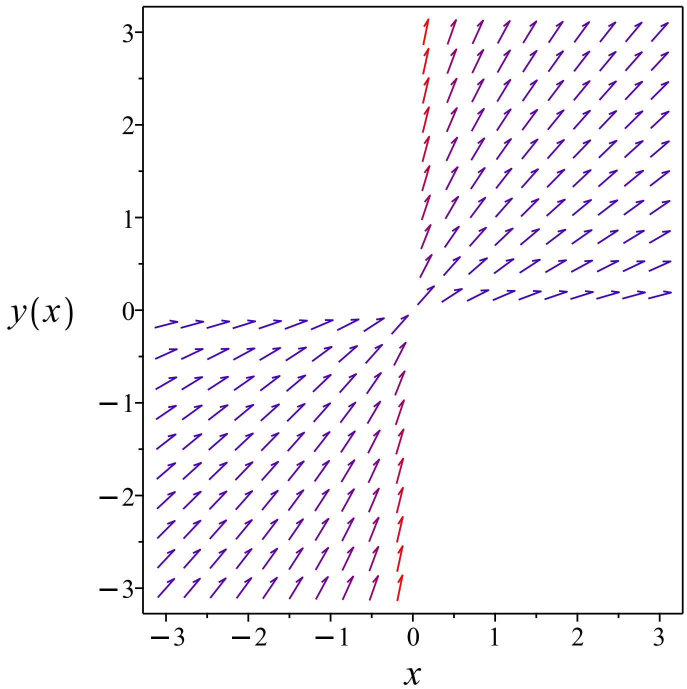


Figure 21: Slope field plot

Verification of solutions

$$2\sqrt{y} - 2\sqrt{x} - c_1 = 0$$

Verified OK.

1.22.2 Maple step by step solution

Let's solve

$y' - \frac{\sqrt{y}}{\sqrt{x}} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{y}} = \frac{1}{\sqrt{x}}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{y}} dx = \int \frac{1}{\sqrt{x}} dx + c_1$

- Evaluate integral

  $2\sqrt{y} = 2\sqrt{x} + c_1$

- Solve for y

  $y = \sqrt{x}\, c_1 + \frac{c_1^2}{4} + x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)=sqrt(y(x))/sqrt(x),y(x), singsol=all)
```

$$\sqrt{y(x)} - \sqrt{x} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.14 (sec). Leaf size: 26

```
DSolve[y'[x]==Sqrt[y[x]]/Sqrt[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(2\sqrt{x} + c_1\right)^2$$
$$y(x) \to 0$$

1.23 problem 23

Internal problem ID [5736]
Internal file name [`OUTPUT/4984_Sunday_June_05_2022_03_15_56_PM_79360025/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{\sqrt{y}}{x} = 0}$$

1.23.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\sqrt{y}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \sqrt{y}$. Integrating both sides gives

$$\frac{1}{\sqrt{y}}\, dy = \frac{1}{x}\, dx$$

$$\int \frac{1}{\sqrt{y}}\, dy = \int \frac{1}{x}\, dx$$

$$2\sqrt{y} = \ln(x) + c_1$$

The solution is

$$2\sqrt{y} - \ln(x) - c_1 = 0$$

Summary

The solution(s) found are the following

$$2\sqrt{y} - \ln(x) - c_1 = 0 \tag{1}$$

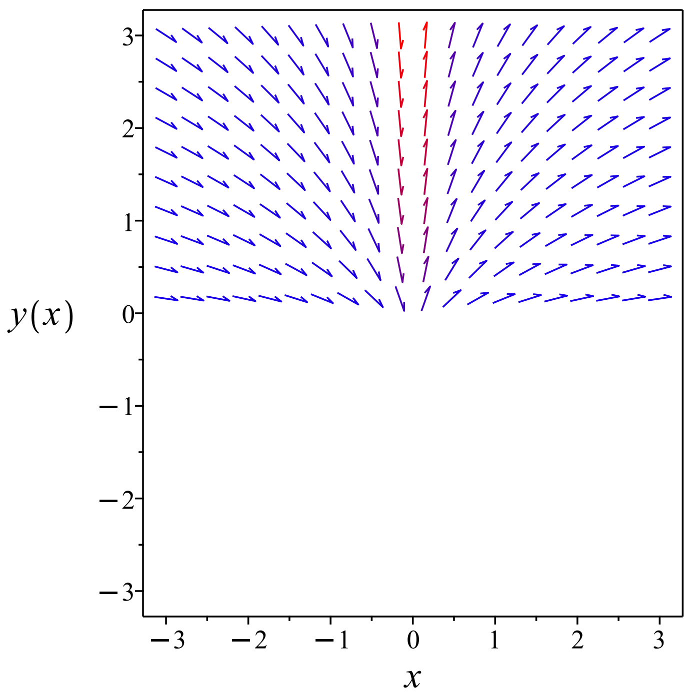


Figure 22: Slope field plot

Verification of solutions

$$2\sqrt{y} - \ln(x) - c_1 = 0$$

Verified OK.

1.23.2 Maple step by step solution

Let's solve

$y' - \frac{\sqrt{y}}{x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{y}} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{y}} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $2\sqrt{y} = \ln(x) + c_1$

- Solve for y

  $y = \frac{\ln(x)^2}{4} + \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)=sqrt(y(x))/x,y(x), singsol=all)
```

$$\sqrt{y(x)} - \frac{\ln(x)}{2} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 21

```
DSolve[y'[x]==Sqrt[y[x]]/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}(\log(x) + c_1)^2$$
$$y(x) \to 0$$

1.24 problem 24

Internal problem ID [5737]
Internal file name [`OUTPUT/4985_Sunday_June_05_2022_03_15_57_PM_66201785/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{z' - 10^{x+z} = 0}$$

1.24.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} z' &= F(x,z) \\ &= f(x)g(z) \\ &= 10^x 10^z \end{aligned}$$

Where $f(x) = 10^x$ and $g(z) = 10^z$. Integrating both sides gives

$$\begin{aligned} \frac{1}{10^z}\,dz &= 10^x\,dx \\ \int \frac{1}{10^z}\,dz &= \int 10^x\,dx \\ -\frac{10^{-z}}{\ln(10)} &= \frac{10^x}{\ln(10)} + c_1 \end{aligned}$$

The solution is

$$-\frac{10^{-z}}{\ln (10)}-\frac{10^{x}}{\ln (10)}-c_1=0$$

Summary

The solution(s) found are the following

$$-\frac{10^{-z}}{\ln (10)}-\frac{10^{x}}{\ln (10)}-c_1=0 \tag{1}$$

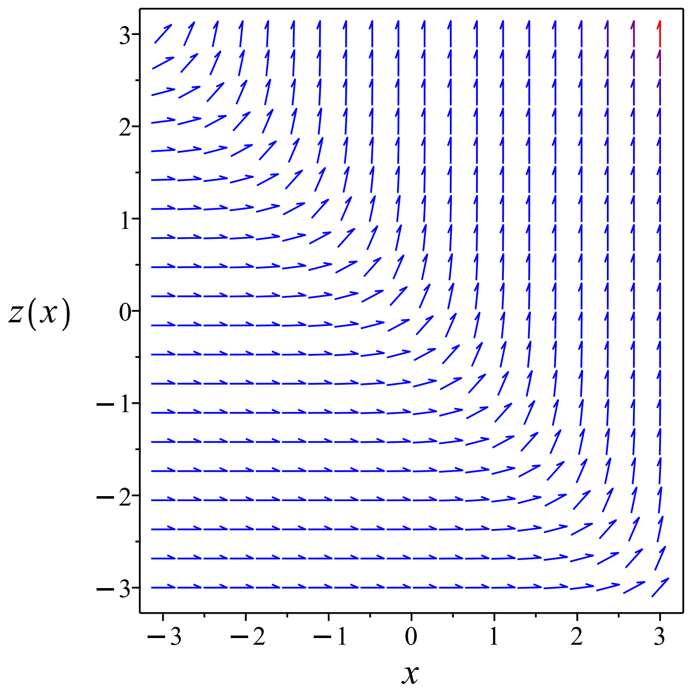


Figure 23: Slope field plot

Verification of solutions

$$-\frac{10^{-z}}{\ln (10)}-\frac{10^{x}}{\ln (10)}-c_1=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve(diff(z(x),x)=10^(x+z(x)),z(x), singsol=all)
```

$$z(x) = \frac{\ln\left(-\frac{1}{c_1 \ln(2)+c_1 \ln(5)+10^x}\right)}{\ln(2)+\ln(5)}$$

✓ Solution by Mathematica

Time used: 0.93 (sec). Leaf size: 24

```
DSolve[z'[x]==10^(x+z[x]),z[x],x,IncludeSingularSolutions -> True]
```

$$z(x) \to -\frac{\log\left(-10^x + c_1(-\log(10))\right)}{\log(10)}$$

1.25 problem 25

Internal problem ID [5738]
Internal file name [`OUTPUT/4986_Sunday_June_05_2022_03_15_59_PM_26046726/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' = -t + 1}$$

1.25.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= f(t)g(x) \\ &= -t + 1 \end{aligned}$$

Where $f(t) = -t + 1$ and $g(x) = 1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{1}\,dx &= -t + 1\,dt \\ \int \frac{1}{1}\,dx &= \int -t + 1\,dt \\ x &= -\frac{1}{2}t^2 + t + c_1 \end{aligned}$$

Which results in

$$x = -\frac{1}{2}t^2 + t + c_1$$

Summary

The solution(s) found are the following

$$x = -\frac{1}{2}t^2 + t + c_1 \tag{1}$$

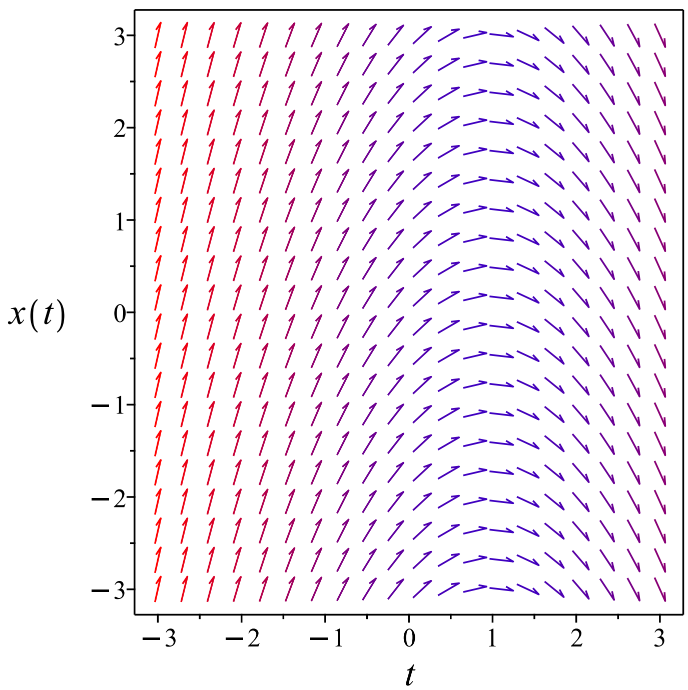


Figure 24: Slope field plot

Verification of solutions

$$x = -\frac{1}{2}t^2 + t + c_1$$

Verified OK.

1.25.2 Maple step by step solution

Let's solve

$x' = -t + 1$

- Highest derivative means the order of the ODE is 1

  x'

- Integrate both sides with respect to t

  $\int x' dt = \int (-t + 1)\, dt + c_1$

- Evaluate integral

  $x = -\frac{1}{2}t^2 + t + c_1$

- Solve for x

  $x = -\frac{1}{2}t^2 + t + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(x(t),t)+t=1,x(t), singsol=all)
```

$$x(t) = -\frac{1}{2}t^2 + t + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 16

```
DSolve[x'[t]+t==1,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{t^2}{2} + t + c_1$$

1.26 problem 26

Internal problem ID [5739]
Internal file name [`OUTPUT/4987_Sunday_June_05_2022_03_16_00_PM_21050741/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - \cos(x - y) = 0}$$

1.26.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \cos(x - y)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \cos(x-y)(b_3 - a_2) - \cos(x-y)^2 a_3 \\ &+ \sin(x-y)(x a_2 + y a_3 + a_1) - \sin(x-y)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} &\sin(x-y) x a_2 - \sin(x-y) x b_2 + \sin(x-y) y a_3 - \sin(x-y) y b_3 - \cos(x-y)^2 a_3 \\ &+ \sin(x-y) a_1 - \sin(x-y) b_1 - \cos(x-y) a_2 + \cos(x-y) b_3 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} &\sin(x-y) x a_2 - \sin(x-y) x b_2 + \sin(x-y) y a_3 - \sin(x-y) y b_3 - \cos(x-y)^2 a_3 \\ &+ \sin(x-y) a_1 - \sin(x-y) b_1 - \cos(x-y) a_2 + \cos(x-y) b_3 + b_2 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &b_2 - \frac{a_3}{2} + \sin(x-y) x a_2 - \sin(x-y) x b_2 + \sin(x-y) y a_3 \\ &- \sin(x-y) y b_3 - \frac{a_3 \cos(-2y+2x)}{2} + \sin(x-y) a_1 \\ &- \sin(x-y) b_1 - \cos(x-y) a_2 + \cos(x-y) b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \cos(x-y), \cos(-2y+2x), \sin(x-y)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \cos(x-y) = v_3, \cos(-2y+2x) = v_4, \sin(x-y) = v_5\}$$

The above PDE (6E) now becomes

$$b_2 - \frac{1}{2} a_3 + v_5 v_1 a_2 - v_5 v_1 b_2 + v_5 v_2 a_3 - v_5 v_2 b_3 - \frac{1}{2} a_3 v_4 + v_5 a_1 - v_5 b_1 - v_3 a_2 + v_3 b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$b_2 - \frac{a_3}{2} + (a_2 - b_2)\, v_1 v_5 + (a_3 - b_3)\, v_2 v_5 + (b_3 - a_2)\, v_3 - \frac{a_3 v_4}{2} + (a_1 - b_1)\, v_5 = 0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -\frac{a_3}{2} &= 0 \\ a_1 - b_1 &= 0 \\ a_2 - b_2 &= 0 \\ a_3 - b_3 &= 0 \\ b_2 - \frac{a_3}{2} &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= 1 - (\cos(x - y))\,(1) \\ &= 1 - \cos(x)\cos(y) - \sin(x)\sin(y) \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{1 - \cos(x)\cos(y) - \sin(x)\sin(y)} dy \end{aligned}$$

Which results in

$$S = \frac{1}{\tan\left(\frac{x}{2} - \frac{y}{2}\right)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \cos(x - y)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{\csc\left(\frac{x}{2} - \frac{y}{2}\right)^2}{2} \\ S_y &= \frac{\csc\left(\frac{x}{2} - \frac{y}{2}\right)^2}{2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\csc\left(\frac{x}{2}-\frac{y}{2}\right)^2 (\cos(x-y)-1)}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\cot\left(\frac{x}{2}-\frac{y}{2}\right) = -x + c_1$$

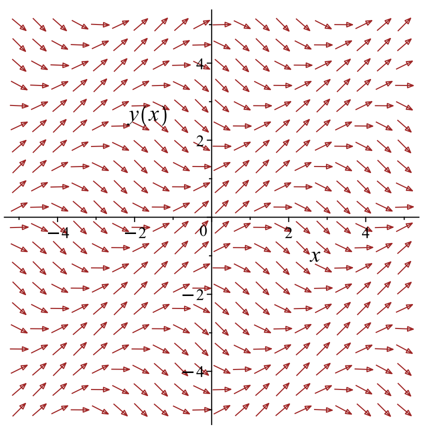
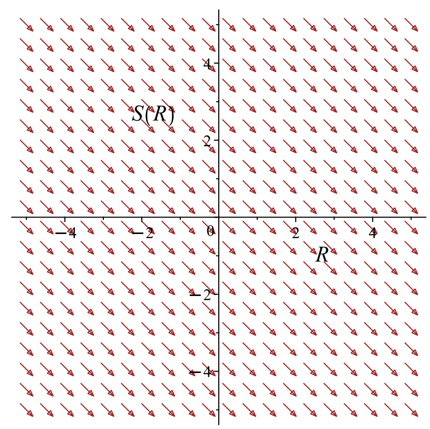
Which simplifies to

$$\cot\left(\frac{x}{2}-\frac{y}{2}\right) = -x + c_1$$

Which gives

$$y = x - 2 \operatorname{arccot}(-x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \cos(x - y)$ 	$R = x$ $S = \cot\left(\frac{x}{2} - \frac{y}{2}\right)$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = x - 2 \operatorname{arccot}(-x + c_1) \tag{1}$$

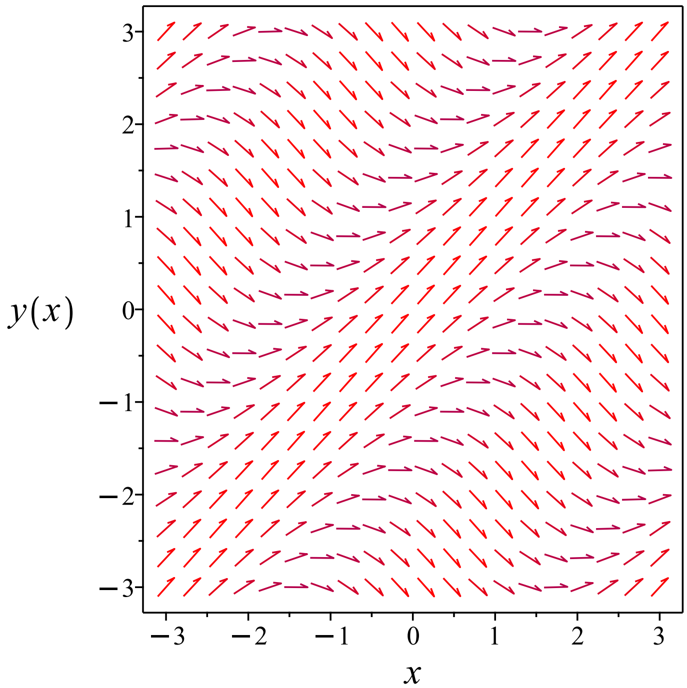


Figure 25: Slope field plot

Verification of solutions

$$y = x - 2 \operatorname{arccot}(-x + c_1)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)=cos(y(x)-x),y(x), singsol=all)
```

$$y(x) = x - 2\operatorname{arccot}(-x + c_1)$$

✓ Solution by Mathematica

Time used: 0.439 (sec). Leaf size: 40

```
DSolve[y'[x]==Cos[y[x]-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + 2\cot^{-1}\left(x - \frac{c_1}{2}\right)$$
$$y(x) \to x + 2\cot^{-1}\left(x - \frac{c_1}{2}\right)$$
$$y(x) \to x$$

1.27 problem 27

Internal problem ID [5740]
Internal file name [`OUTPUT/4988_Sunday_June_05_2022_03_16_06_PM_8099507/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = 2x - 3}$$

1.27.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= 2x - 3 \end{aligned}$$

Hence the ode is

$$y' - y = 2x - 3$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-1)dx} \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(2x-3) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x} y\right) &= \left(\mathrm{e}^{-x}\right)(2x-3) \\
\mathrm{d}\left(\mathrm{e}^{-x} y\right) &= \left((2x-3)\,\mathrm{e}^{-x}\right)\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-x} y &= \int (2x-3)\,\mathrm{e}^{-x}\,\mathrm{d}x \\
\mathrm{e}^{-x} y &= -(2x-1)\,\mathrm{e}^{-x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-x}$ results in

$$
y = -\mathrm{e}^{x}(2x-1)\,\mathrm{e}^{-x} + c_1 \mathrm{e}^{x}
$$

which simplifies to

$$
y = 1 - 2x + c_1 \mathrm{e}^{x}
$$

Summary

The solution(s) found are the following

$$
y = 1 - 2x + c_1 \mathrm{e}^{x} \tag{1}
$$

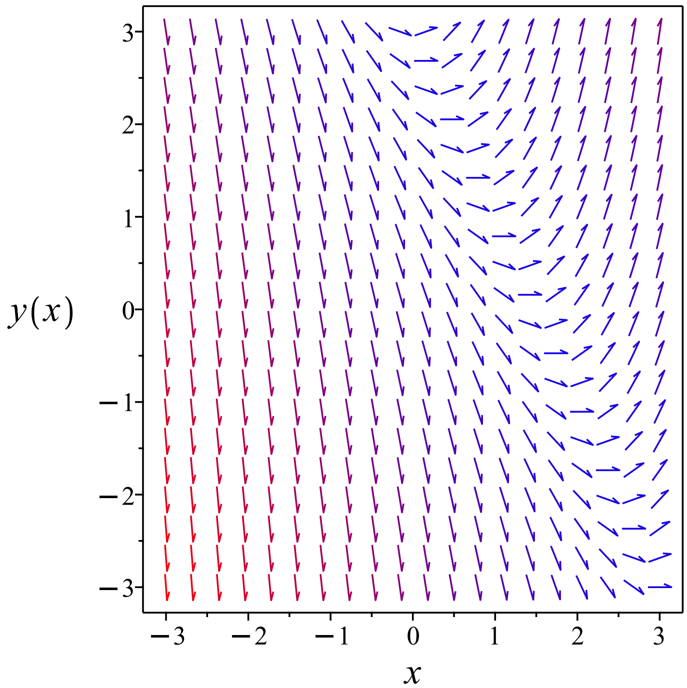


Figure 26: Slope field plot

Verification of solutions

$$y = 1 - 2x + c_1 e^x$$

Verified OK.

1.27.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = y + 2x - 3$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 25: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^x} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = y + 2x - 3$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (2x - 3) e^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (2R - 3) e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -(2R-1)\,\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{-x} = -(2x-1)\,\mathrm{e}^{-x} + c_1$$

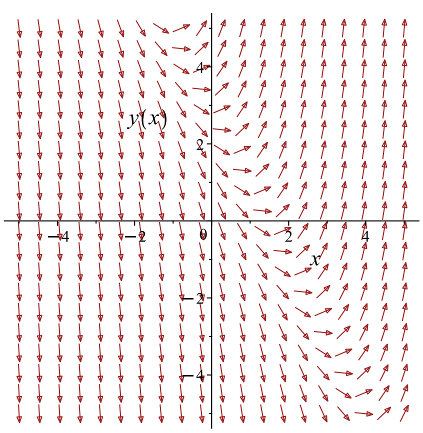
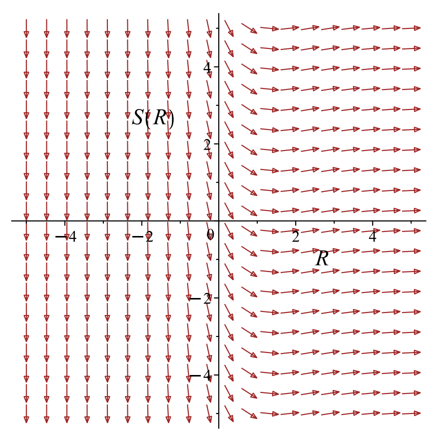
Which simplifies to

$$(2x + y - 1)\,\mathrm{e}^{-x} - c_1 = 0$$

Which gives

$$y = -\left(2x\,\mathrm{e}^{-x} - \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = y + 2x - 3$
</td><td>$R = x$
$S = \mathrm{e}^{-x} y$</td><td>$\frac{dS}{dR} = (2R-3)\,\mathrm{e}^{-R}$
</td></tr>
</table>

<u>Summary</u>
The solution(s) found are the following

$$y = -\left(2x\,\mathrm{e}^{-x} - \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x} \tag{1}$$

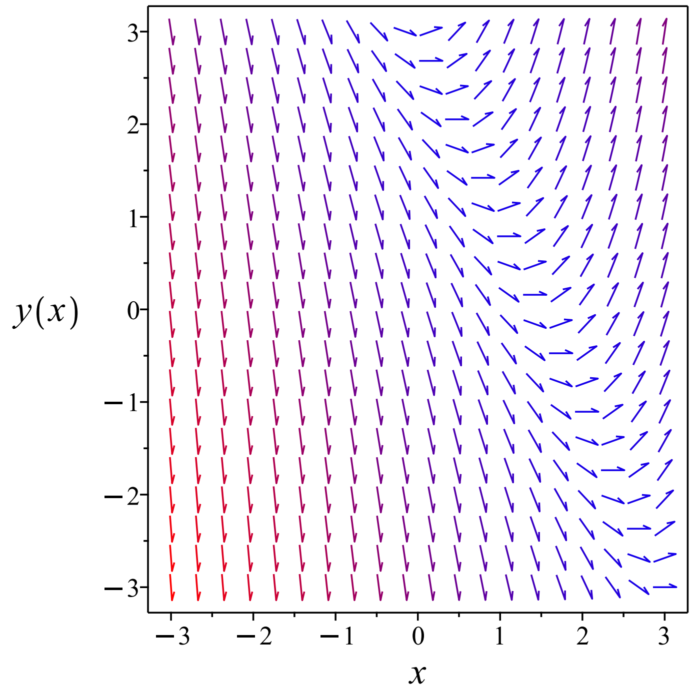


Figure 27: Slope field plot

Verification of solutions

$$y = -\left(2x\,\mathrm{e}^{-x} - \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

Verified OK.

1.27.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (y + 2x - 3)\,\mathrm{d}x\\ (-y - 2x + 3)\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -y - 2x + 3\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y - 2x + 3)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-1) - (0)) \\ &= -1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-x} \\ &= \mathrm{e}^{-x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-x}(-y - 2x + 3) \\ &= (-y - 2x + 3)\,\mathrm{e}^{-x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left((-y - 2x + 3)\,\mathrm{e}^{-x}\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int (-y - 2x + 3)\,\mathrm{e}^{-x} \,\mathrm{d}x$$

$$\phi = (2x + y - 1)\,\mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (2x + y - 1)\,\mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (2x + y - 1)\,\mathrm{e}^{-x}$$

The solution becomes

$$y = -\left(2x\,\mathrm{e}^{-x} - \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = -\left(2x\,\mathrm{e}^{-x} - \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x} \tag{1}$$

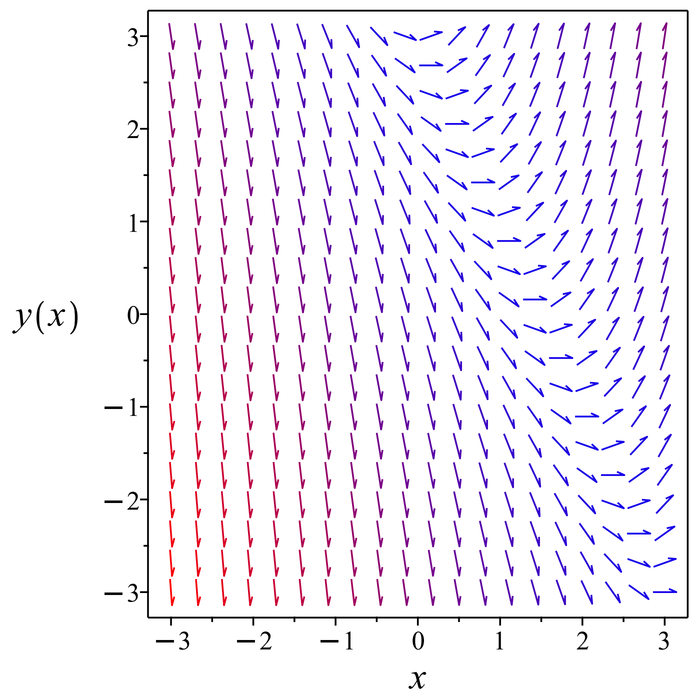


Figure 28: Slope field plot

Verification of solutions

$$y = -\left(2x\,e^{-x} - e^{-x} - c_1\right) e^{x}$$

Verified OK.

1.27.4 Maple step by step solution

Let's solve

$y' - y = 2x - 3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + 2x - 3$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = 2x - 3$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - y) = \mu(x)\,(2x - 3)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\, (y' - y) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, (2x - 3)\, dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, (2x - 3)\, dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)(2x-3)dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-x}$

  $y = \frac{\int (2x-3)\mathrm{e}^{-x} dx + c_1}{\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y = \frac{-(2x-1)\mathrm{e}^{-x} + c_1}{\mathrm{e}^{-x}}$
- Simplify

  $y = 1 - 2x + c_1 \mathrm{e}^{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)-y(x)=2*x-3,y(x), singsol=all)
```

$$y(x) = -2x + 1 + e^{x} c_1$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 16

```
DSolve[y'[x]-y[x]==2*x-3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x + c_1 e^x + 1$$

1.28 problem 28

Internal problem ID [5741]
Internal file name [`OUTPUT/4989_Sunday_June_05_2022_03_16_07_PM_72683433/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeC", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], [_Abel, `2nd type`, `class C`],
  _dAlembert]
```

$$\boxed{(2y + x) y' = 1}$$

With initial conditions

$$[y(0) = -1]$$

1.28.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{1}{2y + x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = -1$ is

$$\{x < 2 \lor 2 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{y < 0 \vee 0 < y\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{1}{2y + x}\right) \\
&= -\frac{2}{(2y + x)^2}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\{x < 2 \vee 2 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{y < 0 \vee 0 < y\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

1.28.2 Solving as homogeneousTypeC ode

Let

$$z = 2y + x \tag{1}$$

Then

$$z'(x) = 2y' + 1$$

Therefore

$$y' = \frac{z'(x)}{2} - \frac{1}{2}$$

Hence the given ode can now be written as

$$\frac{z'(x)}{2} - \frac{1}{2} = \frac{1}{z}$$

This is separable first order ode. Integrating

$$\begin{aligned}
\int dx &= \int \frac{1}{\frac{2}{z} + 1} dz \\
x + c_1 &= z - 2 \ln (2 + z)
\end{aligned}$$

Replacing z back by its value from (1) then the above gives the solution as

$$y = -\operatorname{LambertW}\left(-\frac{e^{-\frac{c_1}{2}-\frac{x}{2}-1}}{2}\right) - \frac{x}{2} - 1$$

$$y = -\operatorname{LambertW}\left(-\frac{e^{-\frac{c_1}{2}-\frac{x}{2}-1}}{2}\right) - \frac{x}{2} - 1$$

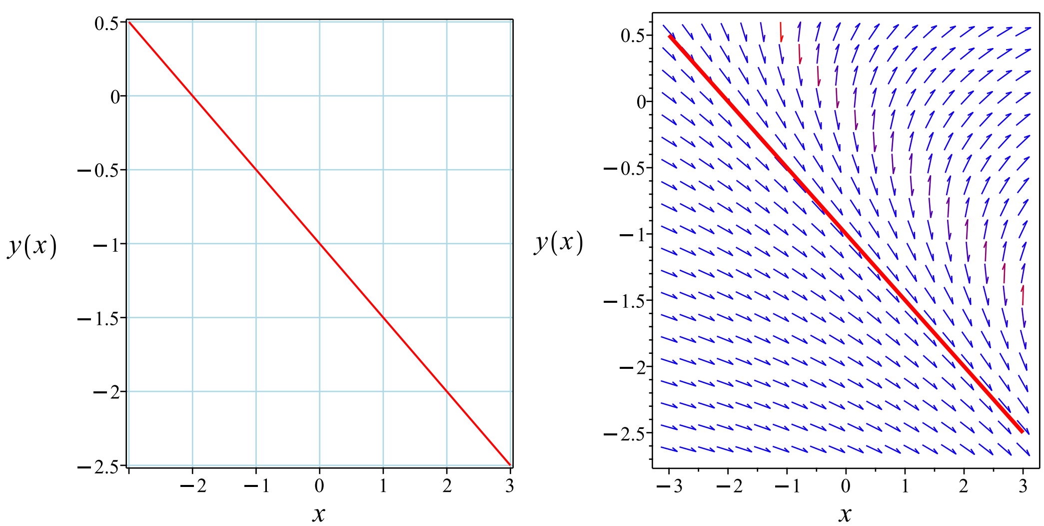
Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\operatorname{LambertW}\left(-\frac{e^{-\frac{c_1}{2}-1}}{2}\right) - 1$$

Unable to solve for constant of integration. Since $\lim_{c_1\to\infty}$ gives $y = -\operatorname{LambertW}\left(-\frac{e^{-\frac{c_1}{2}-\frac{x}{2}-1}}{2}\right) - \frac{x}{2} - 1 = y = -\frac{x}{2} - 1$ and this result satisfies the given initial condition.

Summary

The solution(s) found are the follow



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{x}{2} - 1$$

Verified OK.

1.28.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{1}{2y + x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 28: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 1 \\
\eta(x, y) &= -\frac{1}{2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{-\frac{1}{2}}{1} \\
&= -\frac{1}{2}
\end{aligned}
$$

This is easily solved to give

$$
y = -\frac{x}{2} + c_1
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = y + \frac{x}{2}
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{1}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= x
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{1}{2y + x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= \frac{1}{2} \\ R_y &= 1 \\ S_x &= 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{4y + 2x}{2 + 2y + x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{4R}{2 + 2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2R - 2\ln(1 + R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = 2y + x - 2\ln\left(y + \frac{x}{2} + 1\right) + c_1$$

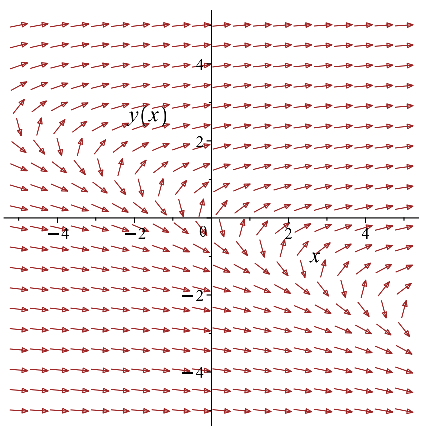
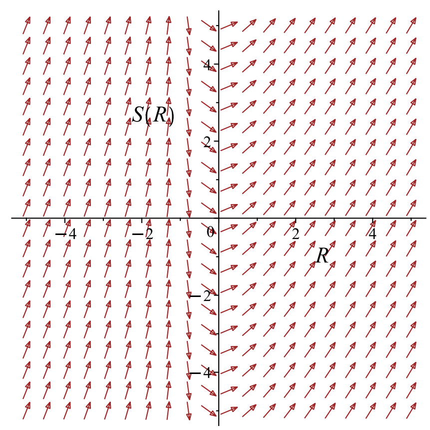
Which simplifies to

$$x = 2y + x - 2\ln\left(y + \frac{x}{2} + 1\right) + c_1$$

Which gives

$$y = -\operatorname{LambertW}\left(-e^{-1-\frac{x}{2}+\frac{c_1}{2}}\right) - 1 - \frac{x}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1}{2y+x}$ 	$R = y + \frac{x}{2}$ $S = x$	$\frac{dS}{dR} = \frac{4R}{2+2R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\operatorname{LambertW}\left(-e^{-1+\frac{c_1}{2}}\right) - 1$$

Unable to solve for constant of integration. Warning: Unable to solve for constant of integration.

<u>Verification of solutions</u> N/A

1.28.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2y+x)\,\mathrm{d}y &= \mathrm{d}x\\ -\,\mathrm{d}x + (2y+x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -1\\ N(x,y) &= 2y+x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2y + x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2y + x}((0) - (1))\\ &= -\frac{1}{2y + x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -1((1) - (0))\\ &= -1\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \,\mathrm{d}y}\\ &= e^{\int -1 \,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-y}\\ &= \mathrm{e}^{-y}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-y}(-1) \\ &= -\mathrm{e}^{-y}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-y}(2y+x) \\ &= (2y+x)\,\mathrm{e}^{-y}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{-y}\right) + \left((2y+x)\,\mathrm{e}^{-y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\mathrm{e}^{-y}\,\mathrm{d}x \\ \phi &= -x\,\mathrm{e}^{-y} + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x\,\mathrm{e}^{-y} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (2y + x)\,\mathrm{e}^{-y}$. Therefore equation (4) becomes

$$(2y + x)\,\mathrm{e}^{-y} = x\,\mathrm{e}^{-y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y\,\mathrm{e}^{-y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(2y\,\mathrm{e}^{-y}\right)\mathrm{d}y$$

$$f(y) = -2(1 + y)\,\mathrm{e}^{-y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x\,\mathrm{e}^{-y} - 2(1 + y)\,\mathrm{e}^{-y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x\,\mathrm{e}^{-y} - 2(1 + y)\,\mathrm{e}^{-y}$$

The solution becomes

$$y = -\frac{x}{2} - \operatorname{LambertW}\left(\frac{c_1 \mathrm{e}^{-\frac{x}{2}-1}}{2}\right) - 1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\operatorname{LambertW}\left(\frac{\mathrm{e}^{-1} c_1}{2}\right) - 1$$

$$c_1 = 0$$

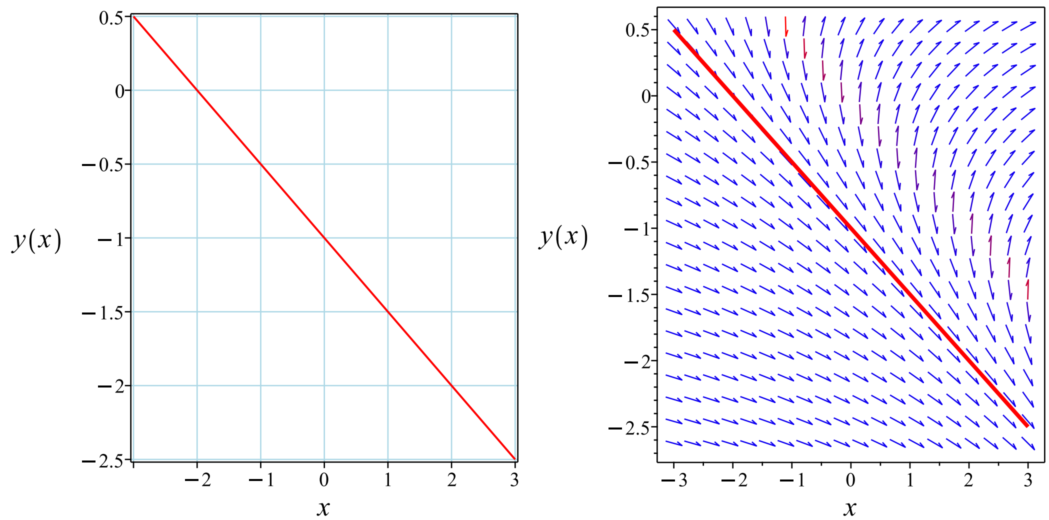
Substituting c_1 found above in the general solution gives

$$y = -\frac{x}{2} - 1$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{2} - 1 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{x}{2} - 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 9

```
dsolve([(x+2*y(x))*diff(y(x),x)=1,y(0) = -1],y(x), singsol=all)
```

$$y(x) = -\frac{x}{2} - 1$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 12

```
DSolve[{(x+2*y[x])*y'[x]==1,{y[0]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{2} - 1$$

1.29 problem 29

Internal problem ID [5742]
Internal file name [`OUTPUT/4990_Sunday_June_05_2022_03_16_09_PM_20786000/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y + y' = 1 + 2x}$$

1.29.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = 1 + 2x$$

Hence the ode is

$$y + y' = 1 + 2x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 1 dx} \\
&= e^x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(1+2x) \\
\frac{d}{dx}(y\,e^x) &= (e^x)(1+2x) \\
d(y\,e^x) &= (e^x(1+2x))\; dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,e^x &= \int e^x(1+2x)\; dx \\
y\,e^x &= (2x-1)\,e^x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$
y = e^x(2x-1)\,e^{-x} + c_1 e^{-x}
$$

which simplifies to

$$
y = 2x - 1 + c_1 e^{-x}
$$

Summary

The solution(s) found are the following

$$
y = 2x - 1 + c_1 e^{-x} \tag{1}
$$

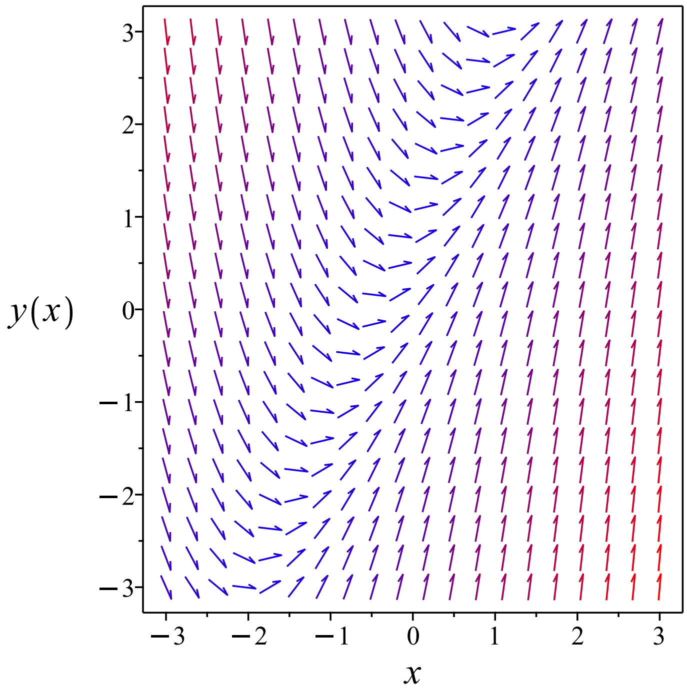


Figure 31: Slope field plot

Verification of solutions

$$y = 2x - 1 + c_1 e^{-x}$$

Verified OK.

1.29.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y + 1 + 2x$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 30: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x}} dy \end{aligned}$$

Which results in

$$S = y\,\mathrm{e}^{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -y + 1 + 2x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y\,\mathrm{e}^{x} \\ S_y &= \mathrm{e}^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{x}(1 + 2x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{R}(1 + 2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = (2R - 1)\,e^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{x} y = (2x - 1)\,e^{x} + c_1$$

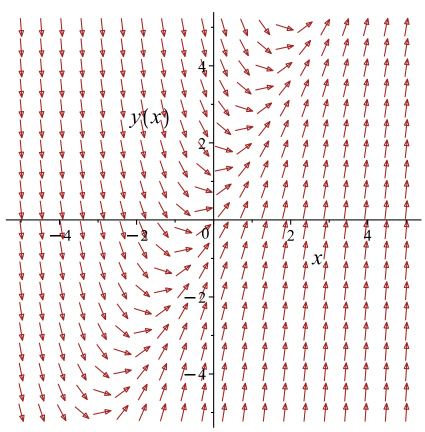
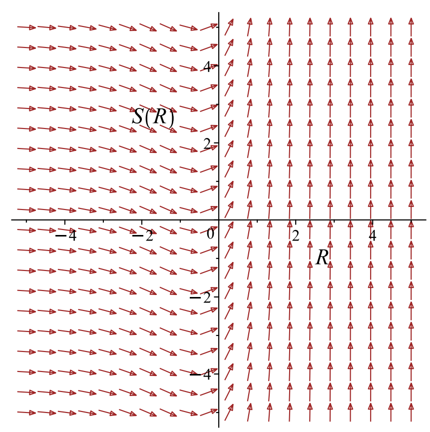
Which simplifies to

$$e^{x} y = (2x - 1)\,e^{x} + c_1$$

Which gives

$$y = (2x\,e^{x} - e^{x} + c_1)\,e^{-x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -y + 1 + 2x$

</td><td>$R = x$
$S = y\,e^{x}$</td><td>$\frac{dS}{dR} = e^{R}(1 + 2R)$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = (2x\,e^{x} - e^{x} + c_1)\,e^{-x} \tag{1}$$

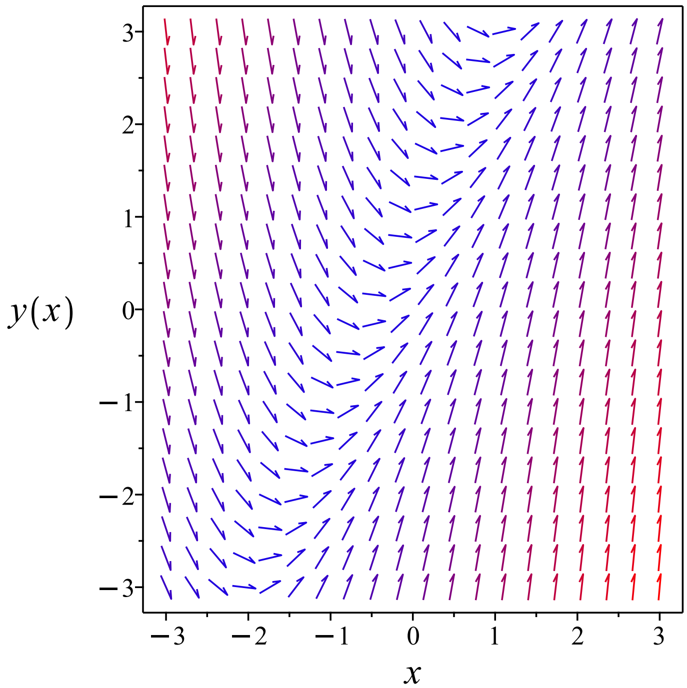


Figure 32: Slope field plot

Verification of solutions

$$y = (2x\,e^{x} - e^{x} + c_1)\,e^{-x}$$

Verified OK.

1.29.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (-y + 1 + 2x)\,\mathrm{d}x\\ (-2x + y - 1)\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -2x + y - 1\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2x + y - 1)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((1) - (0)) \\ &= 1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{x} \\ &= \mathrm{e}^{x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{x}(-2x + y - 1) \\ &= (-2x + y - 1)\,\mathrm{e}^{x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{x}(1) \\ &= \mathrm{e}^{x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ ((-2x + y - 1)\,\mathrm{e}^{x}) + (\mathrm{e}^{x})\,\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int (-2x + y - 1)\,\mathrm{e}^{x} \,\mathrm{d}x$$

$$\phi = (-2x + y + 1)\,\mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (-2x + y + 1)\,\mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (-2x + y + 1)\,\mathrm{e}^{x}$$

The solution becomes

$$y = (2x\,\mathrm{e}^{x} - \mathrm{e}^{x} + c_1)\,\mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = (2x\,\mathrm{e}^{x} - \mathrm{e}^{x} + c_1)\,\mathrm{e}^{-x} \tag{1}$$

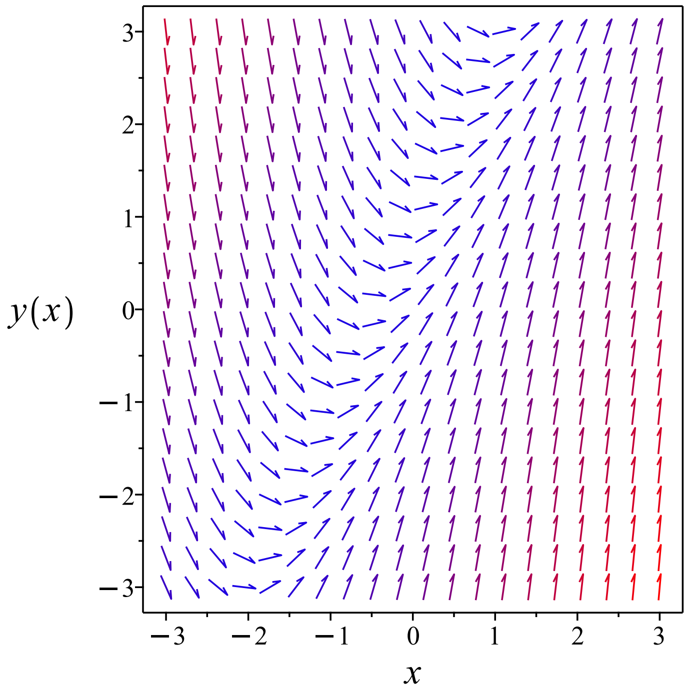


Figure 33: Slope field plot

Verification of solutions

$$y = (2x\,e^{x} - e^{x} + c_1)\,e^{-x}$$

Verified OK.

1.29.4 Maple step by step solution

Let's solve

$y + y' = 1 + 2x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + 1 + 2x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y + y' = 1 + 2x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y + y') = \mu(x)\,(1 + 2x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y+y') = \mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,(1+2x)\,dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,(1+2x)\,dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)(1+2x)dx+c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y = \frac{\int \mathrm{e}^{x}(1+2x)dx+c_1}{\mathrm{e}^{x}}$
- Evaluate the integrals on the rhs

  $y = \frac{(2x-1)\mathrm{e}^{x}+c_1}{\mathrm{e}^{x}}$
- Simplify

  $y = 2x-1+c_1\mathrm{e}^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)+y(x)=2*x+1,y(x), singsol=all)
```

$$y(x) = 2x - 1 + c_1 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 18

```
DSolve[y'[x]+y[x]==2*x+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2x + c_1 e^{-x} - 1$$

1.30 problem 30

Internal problem ID [5743]
Internal file name [`OUTPUT/4991_Sunday_June_05_2022_03_16_10_PM_68440214/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - \cos(x - y - 1) = 0}$$

1.30.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \cos(x - y - 1)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \cos(x - y - 1)(b_3 - a_2) - \cos(x - y - 1)^2 a_3 \\ & + \sin(x - y - 1)(x a_2 + y a_3 + a_1) - \sin(x - y - 1)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} & \sin(x - y - 1) x a_2 - \sin(x - y - 1) x b_2 + \sin(x - y - 1) y a_3 \\ & - \sin(x - y - 1) y b_3 - \cos(x - y - 1)^2 a_3 + \sin(x - y - 1) a_1 \\ & - \sin(x - y - 1) b_1 - \cos(x - y - 1) a_2 + \cos(x - y - 1) b_3 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} & \sin(x - y - 1) x a_2 - \sin(x - y - 1) x b_2 + \sin(x - y - 1) y a_3 \\ & - \sin(x - y - 1) y b_3 - \cos(x - y - 1)^2 a_3 + \sin(x - y - 1) a_1 \\ & - \sin(x - y - 1) b_1 - \cos(x - y - 1) a_2 + \cos(x - y - 1) b_3 + b_2 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} & b_2 - \frac{a_3}{2} + \sin(x - y - 1) x a_2 - \sin(x - y - 1) x b_2 + \sin(x - y - 1) y a_3 \\ & - \sin(x - y - 1) y b_3 - \frac{a_3 \cos(2x - 2y - 2)}{2} + \sin(x - y - 1) a_1 \\ & - \sin(x - y - 1) b_1 - \cos(x - y - 1) a_2 + \cos(x - y - 1) b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \cos(x - y - 1), \cos(2x - 2y - 2), \sin(x - y - 1)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \cos(x - y - 1) = v_3, \cos(2x - 2y - 2) = v_4, \sin(x - y - 1) = v_5\}$$

The above PDE (6E) now becomes

$$b_2 - \frac{1}{2}a_3 + v_5v_1a_2 - v_5v_1b_2 + v_5v_2a_3 - v_5v_2b_3 - \frac{1}{2}a_3v_4 + v_5a_1 - v_5b_1 - v_3a_2 + v_3b_3 = 0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$b_2 - \frac{a_3}{2} + (b_3 - a_2)\, v_3 - \frac{a_3v_4}{2} + (a_1 - b_1)\, v_5 + (a_2 - b_2)\, v_1v_5 + (a_3 - b_3)\, v_2v_5 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -\frac{a_3}{2} &= 0 \\ a_1 - b_1 &= 0 \\ a_2 - b_2 &= 0 \\ a_3 - b_3 &= 0 \\ b_2 - \frac{a_3}{2} &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= 1 - (\cos(x-y-1))\,(1) \\ &= 1 - \cos(x)\cos(1)\cos(y) + \cos(x)\sin(1)\sin(y) - \sin(x)\sin(1)\cos(y) - \sin(x)\cos(1)\sin(y) \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{1 - \cos(x)\cos(1)\cos(y) + \cos(x)\sin(1)\sin(y) - \sin(x)\sin(1)\cos(y) - \sin(x)\cos(1)\sin(y)} dy\end{aligned}$$

Which results in

$$S = \frac{1}{\tan\left(\frac{x}{2} - \frac{y}{2} - \frac{1}{2}\right)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \cos(x-y-1)$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{\csc\left(\frac{x}{2}-\frac{y}{2}-\frac{1}{2}\right)^2}{2} \\
S_y &= \frac{\csc\left(\frac{x}{2}-\frac{y}{2}-\frac{1}{2}\right)^2}{2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{\csc\left(\frac{x}{2}-\frac{y}{2}-\frac{1}{2}\right)^2 (\cos(x-y-1)-1)}{2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -1
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -R + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\cot\left(\frac{x}{2}-\frac{y}{2}-\frac{1}{2}\right) = -x + c_1
$$

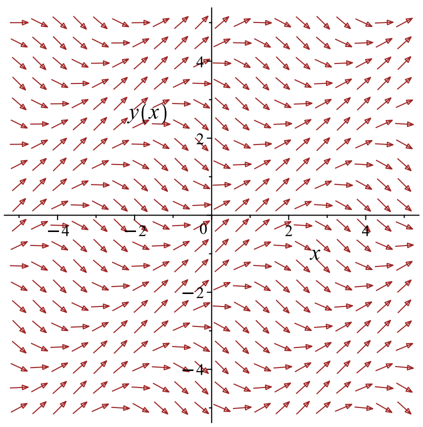
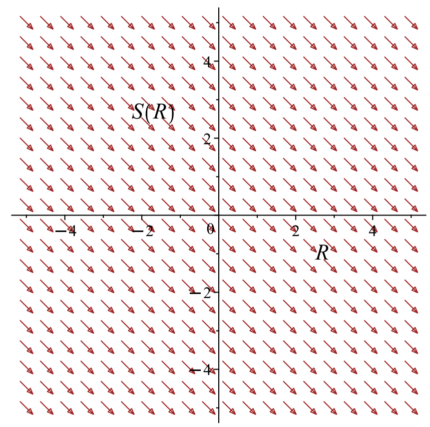
Which simplifies to

$$
\cot\left(\frac{x}{2}-\frac{y}{2}-\frac{1}{2}\right) = -x + c_1
$$

Which gives

$$
y = x - 1 - 2\,\operatorname{arccot}(-x + c_1)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \cos(x - y - 1)$  	$R = x$ $S = \cot\left(\frac{x}{2} - \frac{y}{2} - \frac{1}{2}\right)$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = x - 1 - 2\,\mathrm{arccot}(-x + c_1) \tag{1}$$

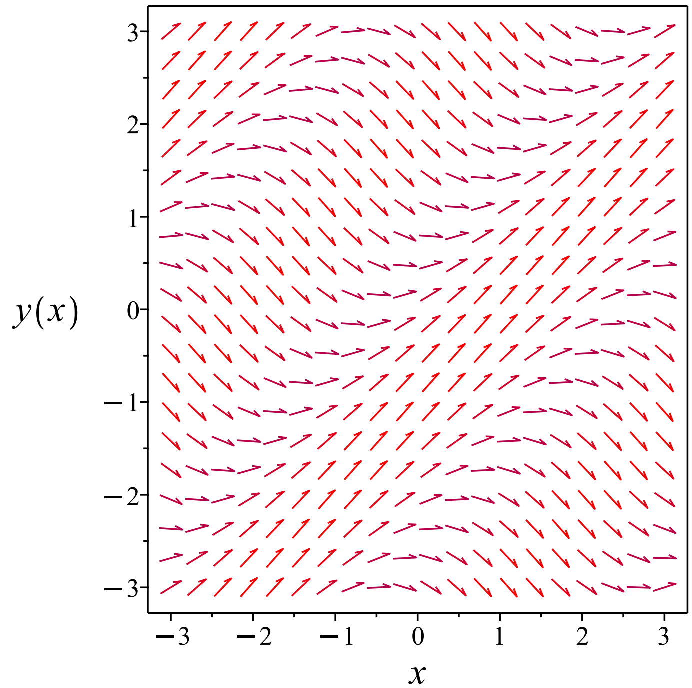


Figure 34: Slope field plot

Verification of solutions

$$y = x - 1 - 2\,\mathrm{arccot}\left(-x + c_1\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)=cos(x-y(x)-1),y(x), singsol=all)
```

$$y(x) = x - 1 - 2 \operatorname{arccot}(-x + c_1)$$

✓ Solution by Mathematica

Time used: 0.551 (sec). Leaf size: 50

```
DSolve[y'[x]==Cos[x-y[x]-1],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - 2\cot^{-1}\left(-x + 1 + \frac{c_1}{2}\right) - 1$$
$$y(x) \to x - 2\cot^{-1}\left(-x + 1 + \frac{c_1}{2}\right) - 1$$
$$y(x) \to x - 1$$

1.31 problem 31

Internal problem ID [5744]
Internal file name [`OUTPUT/4992_Sunday_June_05_2022_03_16_16_PM_77040110/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' + \sin(x+y)^2 = 0}$$

1.31.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\sin(x+y)^2$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \sin(x+y)^2 (b_3 - a_2) - \sin(x+y)^4 a_3 \\ & + 2\sin(x+y)\cos(x+y)(xa_2 + ya_3 + a_1) \\ & + 2\sin(x+y)\cos(x+y)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} & -\sin(x+y)^4 a_3 + 2\sin(x+y)\cos(x+y)\, xa_2 + 2\sin(x+y)\cos(x+y)\, xb_2 \\ & + 2\sin(x+y)\cos(x+y)\, ya_3 + 2\sin(x+y)\cos(x+y)\, yb_3 + \sin(x+y)^2 a_2 \\ & - \sin(x+y)^2 b_3 + 2\sin(x+y)\cos(x+y)\, a_1 + 2\sin(x+y)\cos(x+y)\, b_1 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} & -\sin(x+y)^4 a_3 + 2\sin(x+y)\cos(x+y)\, xa_2 + 2\sin(x+y)\cos(x+y)\, xb_2 \\ & + 2\sin(x+y)\cos(x+y)\, ya_3 + 2\sin(x+y)\cos(x+y)\, yb_3 + \sin(x+y)^2 a_2 \\ & - \sin(x+y)^2 b_3 + 2\sin(x+y)\cos(x+y)\, a_1 + 2\sin(x+y)\cos(x+y)\, b_1 + b_2 \\ & = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} & b_2 - \frac{3a_3}{8} + \frac{a_2}{2} - \frac{b_3}{2} + \frac{a_3\cos(2y+2x)}{2} - \frac{a_3\cos(4y+4x)}{8} + xa_2\sin(2y+2x) \\ & + xb_2\sin(2y+2x) + ya_3\sin(2y+2x) + yb_3\sin(2y+2x) - \frac{a_2\cos(2y+2x)}{2} \\ & + \frac{b_3\cos(2y+2x)}{2} + a_1\sin(2y+2x) + b_1\sin(2y+2x) = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \cos(2y+2x), \cos(4y+4x), \sin(2y+2x)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \cos(2y+2x) = v_3, \cos(4y+4x) = v_4, \sin(2y+2x) = v_5\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
b_2 - \frac{3}{8}a_3 + \frac{1}{2}a_2 - \frac{1}{2}b_3 + \frac{1}{2}a_3 v_3 - \frac{1}{8}a_3 v_4 + v_1 a_2 v_5 + v_1 b_2 v_5 \\
+ v_2 a_3 v_5 + v_2 b_3 v_5 - \frac{1}{2}a_2 v_3 + \frac{1}{2}b_3 v_3 + a_1 v_5 + b_1 v_5 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4, v_5\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
b_2 - \frac{3a_3}{8} + \frac{a_2}{2} - \frac{b_3}{2} + \left(\frac{a_3}{2} - \frac{a_2}{2} + \frac{b_3}{2}\right) v_3 - \frac{a_3 v_4}{8} \\
+ (a_1 + b_1) v_5 + (a_2 + b_2) v_5 v_1 + (a_3 + b_3) v_5 v_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-\frac{a_3}{8} &= 0 \\
a_1 + b_1 &= 0 \\
a_2 + b_2 &= 0 \\
a_3 + b_3 &= 0 \\
\frac{a_3}{2} - \frac{a_2}{2} + \frac{b_3}{2} &= 0 \\
b_2 - \frac{3a_3}{8} + \frac{a_2}{2} - \frac{b_3}{2} &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -b_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = -1$$
$$\eta = 1$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= 1 - \left(-\sin\left(x + y\right)^2\right) \left(-1\right) \\
&= 1 - \sin\left(x\right)^2 \cos\left(y\right)^2 - 2\sin\left(x\right)\cos\left(y\right)\cos\left(x\right)\sin\left(y\right) - \cos\left(x\right)^2 \sin\left(y\right)^2 \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{1 - \sin\left(x\right)^2 \cos\left(y\right)^2 - 2\sin\left(x\right)\cos\left(y\right)\cos\left(x\right)\sin\left(y\right) - \cos\left(x\right)^2 \sin\left(y\right)^2} dy
\end{aligned}$$

Which results in

$$S = \tan\left(x + y\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\sin(x+y)^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \sec(x+y)^2 \\ S_y &= \sec(x+y)^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\tan(x+y) = x + c_1$$

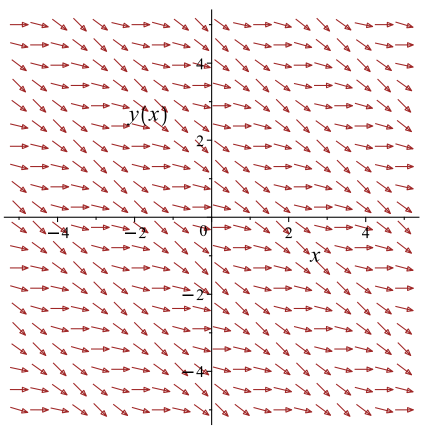
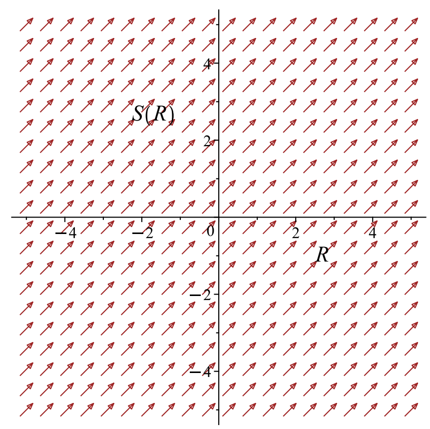
Which simplifies to

$$\tan(x+y) = x + c_1$$

Which gives

$$y = -x + \arctan(x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\sin(x+y)^2$ 	$R = x$ $S = \tan(x+y)$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = -x + \arctan(x + c_1) \tag{1}$$

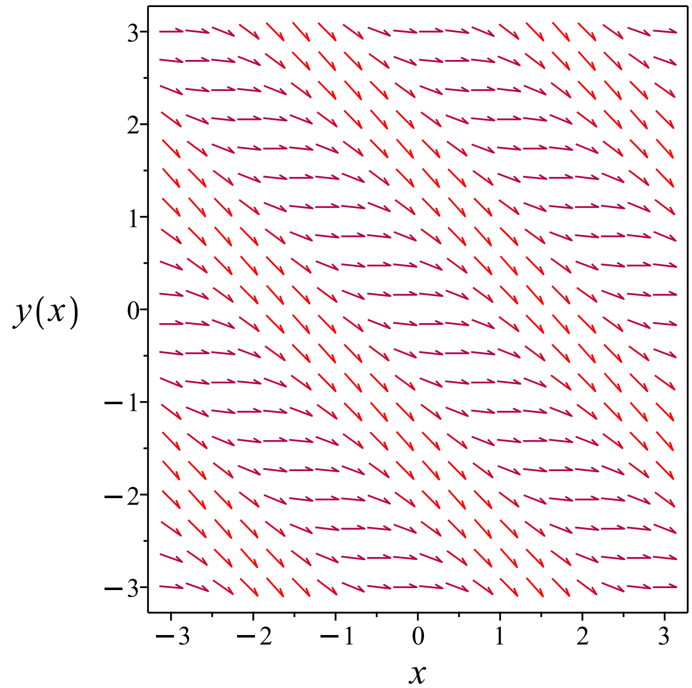


Figure 35: Slope field plot

Verification of solutions

$$y = -x + \arctan(x + c_1)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)+sin(x+y(x))^2=0,y(x), singsol=all)
```

$$y(x) = -x - \arctan(-x + c_1)$$

✓ Solution by Mathematica

Time used: 0.195 (sec). Leaf size: 27

```
DSolve[y'[x]+Sin[x+y[x]]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}[2(\tan(y(x) + x) - \arctan(\tan(y(x) + x))) + 2y(x) = c_1, y(x)]$$

1.32 problem 32

Internal problem ID [5745]
Internal file name [`OUTPUT/4993_Sunday_June_05_2022_03_16_28_PM_87425263/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - 2\sqrt{2x + y + 1} = 0}$$

1.32.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = 2\sqrt{2x + y + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + 2\sqrt{2x+y+1}\,(b_3 - a_2) - 4(2x+y+1)\,a_3 - \frac{2(xa_2 + ya_3 + a_1)}{\sqrt{2x+y+1}} - \frac{xb_2 + yb_3 + b_1}{\sqrt{2x+y+1}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{8a_3\sqrt{2x+y+1}\,x + 4a_3\sqrt{2x+y+1}\,y + 4a_3\sqrt{2x+y+1} - b_2\sqrt{2x+y+1} + 6xa_2 + xb_2 - 4b_3x + 2a_2y}{\sqrt{2x+y+1}}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-8a_3\sqrt{2x+y+1}\,x - 4a_3\sqrt{2x+y+1}\,y - 4a_3\sqrt{2x+y+1} + b_2\sqrt{2x+y+1} \\ &- 6xa_2 - xb_2 + 4b_3x - 2a_2y - 2ya_3 + yb_3 - 2a_1 - 2a_2 - b_1 + 2b_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-2(2x+y+1)\,a_2 + 2(2x+y+1)\,b_3 - 8a_3\sqrt{2x+y+1}\,x - 4a_3\sqrt{2x+y+1}\,y \\ &- 4a_3\sqrt{2x+y+1} + b_2\sqrt{2x+y+1} - 2xa_2 - xb_2 - 2ya_3 - yb_3 - 2a_1 - b_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} &-8a_3\sqrt{2x+y+1}\,x - 4a_3\sqrt{2x+y+1}\,y - 4a_3\sqrt{2x+y+1} + b_2\sqrt{2x+y+1} \\ &- 6xa_2 - xb_2 + 4b_3x - 2a_2y - 2ya_3 + yb_3 - 2a_1 - 2a_2 - b_1 + 2b_3 = 0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{2x+y+1}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{2x+y+1} = v_3\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
-8a_3v_3v_1 - 4a_3v_3v_2 - 6v_1a_2 - 2a_2v_2 - 2v_2a_3 - 4a_3v_3 \\
- v_1b_2 + b_2v_3 + 4b_3v_1 + v_2b_3 - 2a_1 - 2a_2 - b_1 + 2b_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
-8a_3v_3v_1 + (-6a_2 - b_2 + 4b_3)\, v_1 - 4a_3v_3v_2 + (-2a_2 - 2a_3 + b_3)\, v_2 \\
+ (-4a_3 + b_2)\, v_3 - 2a_1 - 2a_2 - b_1 + 2b_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-8a_3 &= 0 \\
-4a_3 &= 0 \\
-4a_3 + b_2 &= 0 \\
-6a_2 - b_2 + 4b_3 &= 0 \\
-2a_2 - 2a_3 + b_3 &= 0 \\
-2a_1 - 2a_2 - b_1 + 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= -2a_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= -2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -2 - \left(2\sqrt{2x+y+1}\right)(1) \\
&= -2 - 2\sqrt{2x+y+1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-2 - 2\sqrt{2x+y+1}} dy
\end{aligned}
$$

Which results in

$$
S = -\sqrt{2x+y+1} - \frac{\ln\left(-1+\sqrt{2x+y+1}\right)}{2} + \frac{\ln\left(\sqrt{2x+y+1}+1\right)}{2} + \frac{\ln(2x+y)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = 2\sqrt{2x+y+1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{-\sqrt{2x+y+1}-1} \\
S_y &= -\frac{1}{2\sqrt{2x+y+1}+2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -1 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -1
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -R + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\sqrt{2x+y+1} - \frac{\ln\left(-1+\sqrt{2x+y+1}\right)}{2} + \frac{\ln\left(\sqrt{2x+y+1}+1\right)}{2} + \frac{\ln(2x+y)}{2} = -x + c_1
$$

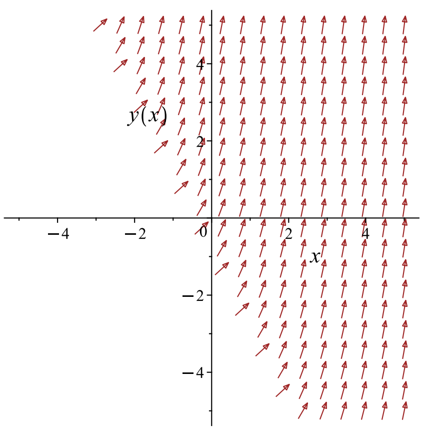
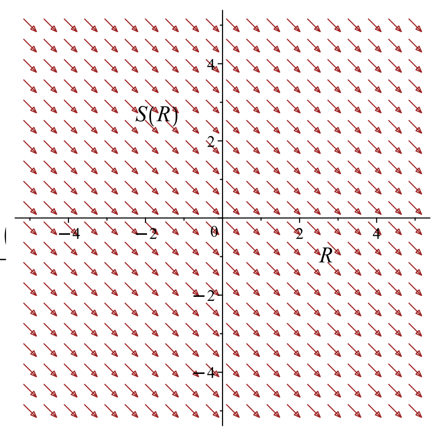
Which simplifies to

$$
-\sqrt{2x+y+1} - \frac{\ln\left(-1+\sqrt{2x+y+1}\right)}{2} + \frac{\ln\left(\sqrt{2x+y+1}+1\right)}{2} + \frac{\ln(2x+y)}{2} = -x + c_1
$$

Which gives

$$
y = \mathrm{e}^{-2\,\mathrm{LambertW}\left(-\mathrm{e}^{-1+c_1-x}\right)-2+2c_1-2x} - 2\,\mathrm{e}^{-\,\mathrm{LambertW}\left(-\mathrm{e}^{-1+c_1-x}\right)-1+c_1-x} - 2x
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2\sqrt{2x+y+1}$ 	$R = x$ $S = -\sqrt{2x+y+1} - \frac{\ln(}{}$ $\frac{+y+1}{2}$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2\,\mathrm{LambertW}\left(-\mathrm{e}^{-1+c_1-x}\right)-2+2c_1-2x} - 2\,\mathrm{e}^{-\mathrm{LambertW}\left(-\mathrm{e}^{-1+c_1-x}\right)-1+c_1-x} - 2x \tag{1}$$

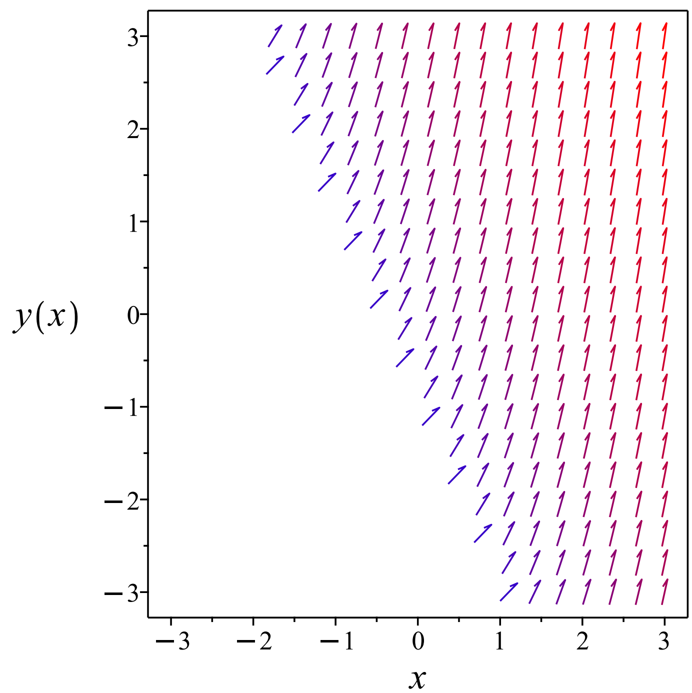


Figure 36: Slope field plot

Verification of solutions

$$y = e^{-2\,\mathrm{LambertW}\left(-e^{-1+c_1-x}\right)-2+2c_1-2x} - 2\,e^{-\mathrm{LambertW}\left(-e^{-1+c_1-x}\right)-1+c_1-x} - 2x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 56

```
dsolve(diff(y(x),x)=2*sqrt(2*x+y(x)+1),y(x), singsol=all)
```

$$x-\sqrt{2x+y(x)+1}-\frac{\ln\left(-1+\sqrt{2x+y(x)+1}\right)}{2}+\frac{\ln\left(\sqrt{2x+y(x)+1}+1\right)}{2}+\frac{\ln(y(x)+2x)}{2}-c_1=0$$

✓ Solution by Mathematica

Time used: 11.43 (sec). Leaf size: 48

```
DSolve[y'[x]==2*Sqrt[2*x+y[x]+1],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to W\left(-e^{-x-\frac{3}{2}+c_1}\right)^2+2W\left(-e^{-x-\frac{3}{2}+c_1}\right)-2x$$
$$y(x)\to -2x$$

1.33 problem 33

Internal problem ID [5746]
Internal file name [`OUTPUT/4994_Sunday_June_05_2022_03_16_35_PM_45372306/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _Riccati]
```

$$\boxed{y' - (y + x + 1)^2 = 0}$$

1.33.1 Solving as homogeneousTypeC ode

Let

$$z = y + x + 1 \tag{1}$$

Then

$$z'(x) = 1 + y'$$

Therefore

$$y' = z'(x) - 1$$

Hence the given ode can now be written as

$$z'(x) - 1 = z^2$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{z^2 + 1} dz$$
$$x + c_1 = \arctan(z)$$

Replacing z back by its value from (1) then the above gives the solution as

$$y = -x - 1 + \tan(x + c_1)$$

$$y = -x - 1 + \tan(x + c_1)$$

Summary
The solution(s) found are the following

$$y = -x - 1 + \tan(x + c_1) \tag{1}$$

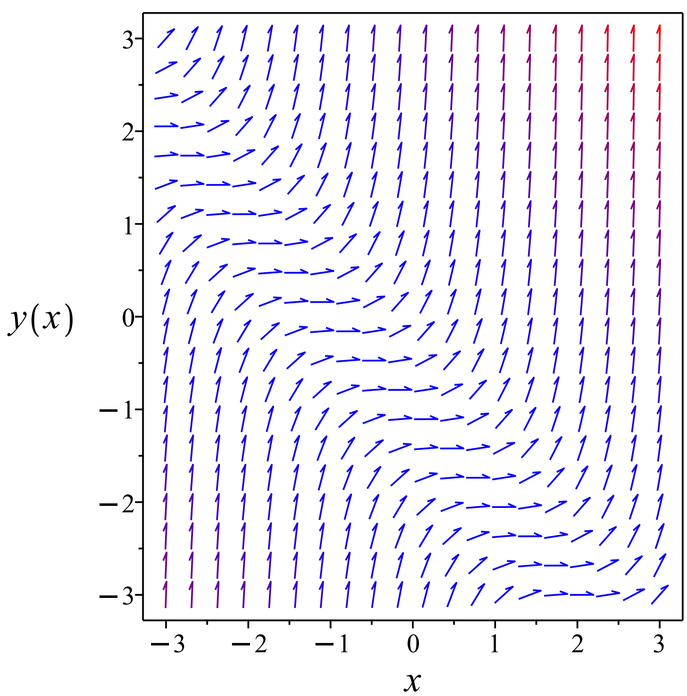


Figure 37: Slope field plot

Verification of solutions

$$y = -x - 1 + \tan(x + c_1)$$

Verified OK.

1.33.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= (x + y + 1)^2 \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 33: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}
\xi(x,y) &= 1 \\
\eta(x,y) &= -1 \qquad \text{(A1)}
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{-1}{1} \\
&= -1
\end{aligned}$$

This is easily solved to give

$$y = -x + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = x + y$$

And S is found from

$$\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{1}
\end{aligned}$$

Integrating gives

$$\begin{aligned}
S &= \int \frac{dx}{T} \\
&= x
\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (x+y+1)^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 1 \\ S_x &= 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{1+(x+y+1)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{1+(R+1)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan(R+1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = \arctan(y+x+1) + c_1$$

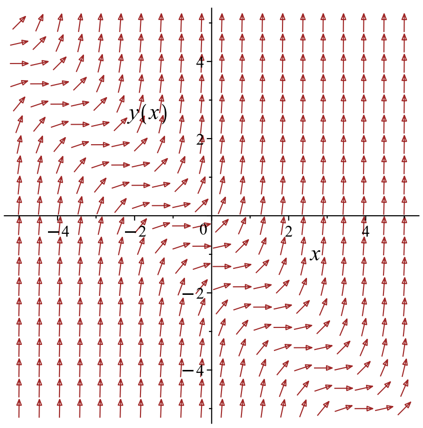
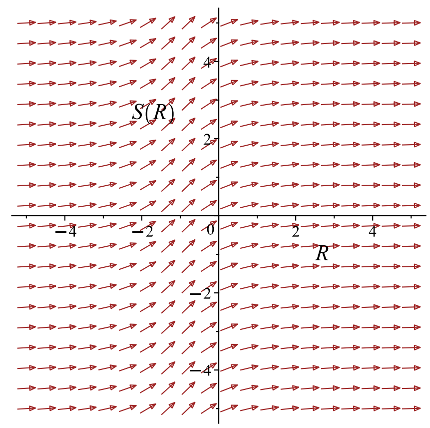
Which simplifies to

$$x = \arctan(y+x+1) + c_1$$

Which gives

$$y = -x - 1 - \tan(-x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (x + y + 1)^2$ 	$R = x + y$ $S = x$	$\frac{dS}{dR} = \frac{1}{1+(R+1)^2}$ 

Summary
The solution(s) found are the following

$$y = -x - 1 - \tan(-x + c_1) \tag{1}$$

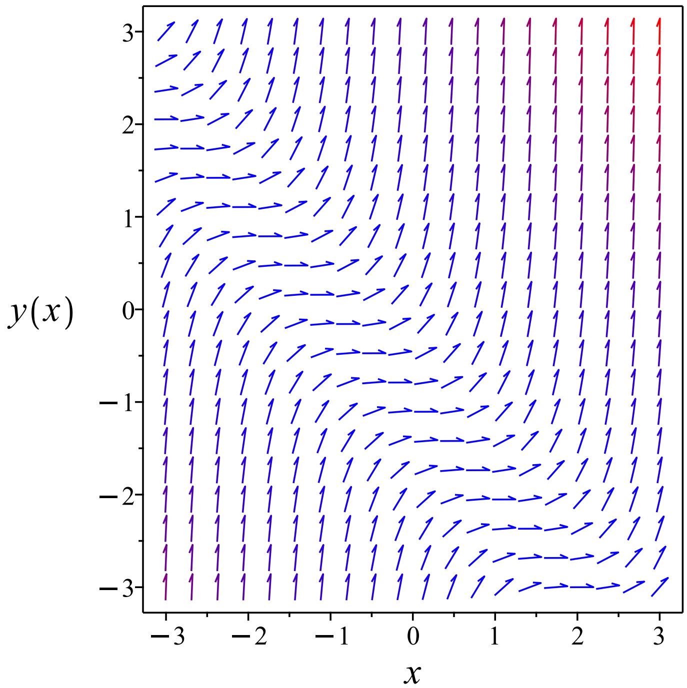


Figure 38: Slope field plot

Verification of solutions

$$y = -x - 1 - \tan(-x + c_1)$$

Verified OK.

1.33.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= (x + y + 1)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 + 2xy + y^2 + 2x + 2y + 1$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = (1+x)^2$, $f_1(x) = 2+2x$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 2+2x \\ f_2^2 f_0 &= (1+x)^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - (2+2x)\, u'(x) + (1+x)^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\frac{x(x+2)}{2}} (\cos(x) c_1 + c_2 \sin(x))$$

The above shows that

$$u'(x) = (((1+x) c_1 + c_2) \cos(x) + \sin(x) (-c_1 + (1+x) c_2))\, \mathrm{e}^{\frac{x(x+2)}{2}}$$

Using the above in (1) gives the solution

$$y = -\frac{((1+x) c_1 + c_2) \cos(x) + \sin(x) (-c_1 + (1+x) c_2)}{\cos(x) c_1 + c_2 \sin(x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(-1 + (-1-x) c_3) \cos(x) - \sin(x) (-c_3 + 1 + x)}{c_3 \cos(x) + \sin(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{(-1 + (-1 - x) c_3) \cos(x) - \sin(x) (-c_3 + 1 + x)}{c_3 \cos(x) + \sin(x)} \tag{1}$$

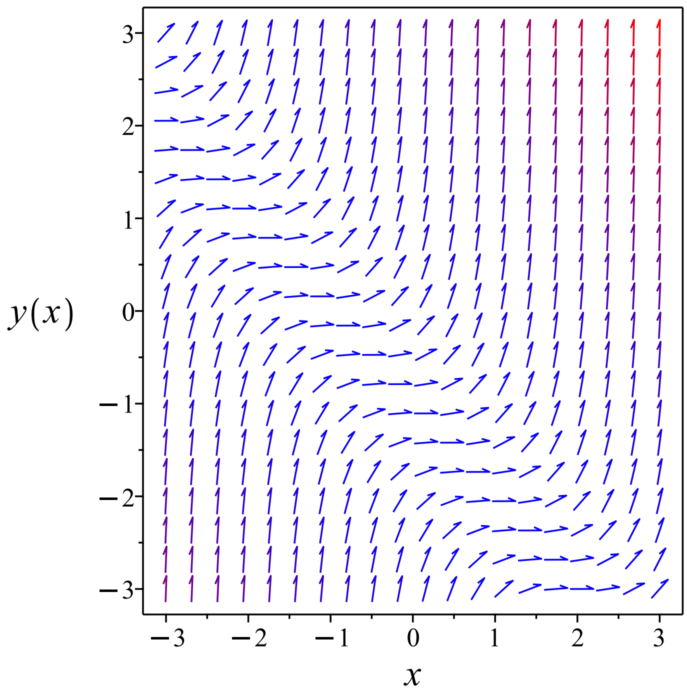


Figure 39: Slope field plot

Verification of solutions

$$y = \frac{(-1 + (-1 - x) c_3) \cos(x) - \sin(x) (-c_3 + 1 + x)}{c_3 \cos(x) + \sin(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)=(x+y(x)+1)^2,y(x), singsol=all)
```

$$y(x) = -x - 1 - \tan\left(-x + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.498 (sec). Leaf size: 15

```
DSolve[y'[x]==(x+y[x]+1)^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x + \tan(x + c_1) - 1$$

1.34 problem 34

Internal problem ID [5747]
Internal file name [`OUTPUT/4995_Sunday_June_05_2022_03_16_37_PM_34066439/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y^2 + x y^2 + \left(x^2 - y x^2\right) y' = 0}$$

1.34.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2(1+x)}{x^2\,(y-1)} \end{aligned}$$

Where $f(x) = \frac{1+x}{x^2}$ and $g(y) = \frac{y^2}{y-1}$. Integrating both sides gives

$$\frac{1}{\frac{y^2}{y-1}}\, dy = \frac{1+x}{x^2}\, dx$$

$$\int \frac{1}{\frac{y^2}{y-1}}\, dy = \int \frac{1+x}{x^2}\, dx$$

$$\ln\left(y\right) + \frac{1}{y} = \ln\left(x\right) - \frac{1}{x} + c_1$$

Which results in

$$y = \mathrm{e}^{\frac{\ln(x)x+\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{\ln(x)x+c_1x-1}{x}}\right)x+c_1x-1}{x}}$$

Which simplifies to

$$y = x\,\mathrm{e}^{\mathrm{LambertW}\left(-\frac{\mathrm{e}^{\frac{1}{x}}\mathrm{e}^{-c_1}}{x}\right)}\mathrm{e}^{c_1}\mathrm{e}^{-\frac{1}{x}}$$

<u>Summary</u>

The solution(s) found are the following

$$y = x\,\mathrm{e}^{\mathrm{LambertW}\left(-\frac{\mathrm{e}^{\frac{1}{x}}\mathrm{e}^{-c_1}}{x}\right)}\mathrm{e}^{c_1}\mathrm{e}^{-\frac{1}{x}} \tag{1}$$

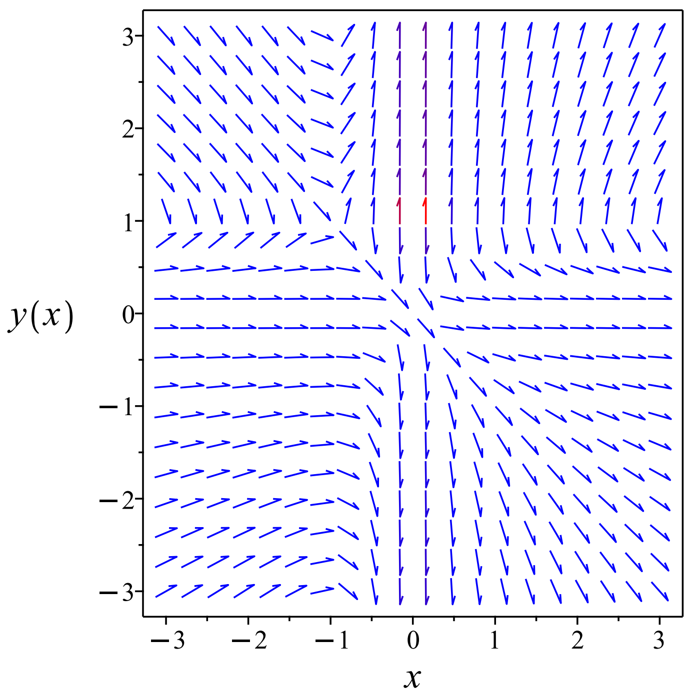


Figure 40: Slope field plot

<u>Verification of solutions</u>

$$y = x\,\mathrm{e}^{\mathrm{LambertW}\left(-\frac{\mathrm{e}^{\frac{1}{x}}\mathrm{e}^{-c_1}}{x}\right)}\mathrm{e}^{c_1}\mathrm{e}^{-\frac{1}{x}}$$

Verified OK.

1.34.2 Maple step by step solution

Let's solve

$y^2+xy^2+(x^2-yx^2)\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(y-1)}{y^2}=\frac{1+x}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'(y-1)}{y^2}dx=\int \frac{1+x}{x^2}dx+c_1$

- Evaluate integral

  $\ln(y)+\frac{1}{y}=\ln(x)-\frac{1}{x}+c_1$

- Solve for y

  $y=\mathrm{e}^{\frac{\ln(x)x+\mathit{LambertW}\left(-\mathrm{e}^{-\frac{\ln(x)x+c_1x-1}{x}}\right)x+c_1x-1}{x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve((y(x)^2+x*y(x)^2)+(x^2-x^2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=x\,\mathrm{e}^{\frac{\mathrm{LambertW}\left(-\frac{\mathrm{e}^{\frac{-c_1x+1}{x}}}{x}\right)x+c_1x-1}{x}}$$

✓ Solution by Mathematica

Time used: 5.623 (sec). Leaf size: 30

```
DSolve[(y[x]^2+x*y[x]^2)+(x^2-x^2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{W\left(-\frac{e^{\frac{1}{x}-c_1}}{x}\right)}$$
$$y(x) \to 0$$

1.35 problem 35

Internal problem ID [5748]
Internal file name [`OUTPUT/4996_Sunday_June_05_2022_03_16_38_PM_17586316/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.1 Separable equations problems. page 7
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(1+y^2)\left(e^{2x}-y'e^{y}\right)-(1+y)\,y'=0}$$

1.35.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{e^{2x}(y^2+1)}{y^2e^{y}+e^{y}+y+1}\end{aligned}$$

Where $f(x)=e^{2x}$ and $g(y)=\frac{y^2+1}{y^2e^{y}+e^{y}+y+1}$. Integrating both sides gives

$$\frac{1}{\frac{y^2+1}{y^2e^{y}+e^{y}+y+1}}\,dy = e^{2x}\,dx$$

$$\int \frac{1}{\frac{y^2+1}{y^2e^{y}+e^{y}+y+1}}\,dy = \int e^{2x}\,dx$$

$$\arctan(y)+\frac{\ln(y^2+1)}{2}+e^{y}=\frac{e^{2x}}{2}+c_1$$

The solution is

$$\arctan(y) + \frac{\ln(1+y^2)}{2} + e^{y} - \frac{e^{2x}}{2} - c_1 = 0$$

Summary

The solution(s) found are the following

$$\arctan(y) + \frac{\ln(1+y^2)}{2} + e^{y} - \frac{e^{2x}}{2} - c_1 = 0 \tag{1}$$

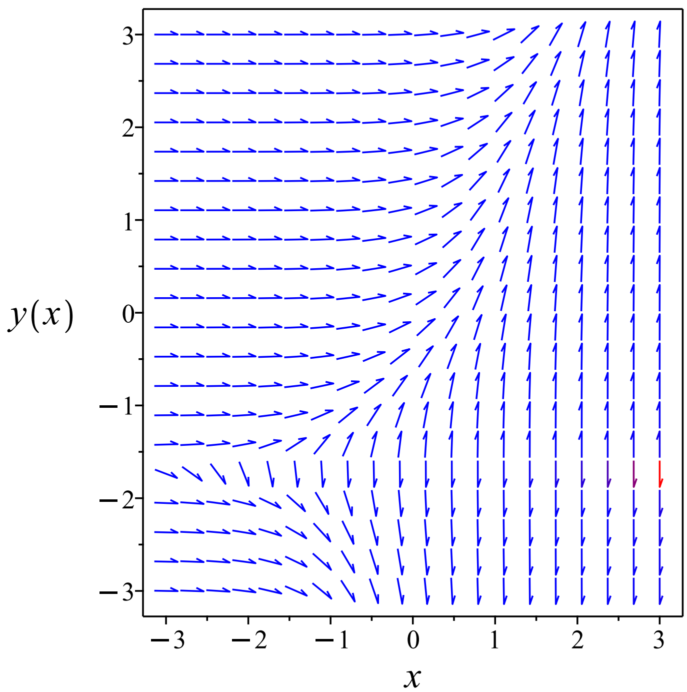


Figure 41: Slope field plot

Verification of solutions

$$\arctan(y) + \frac{\ln(1+y^2)}{2} + e^{y} - \frac{e^{2x}}{2} - c_1 = 0$$

Verified OK.

1.35.2 Maple step by step solution

Let's solve

$\left(1+y^{2}\right)\left(\mathrm{e}^{2x}-y'\mathrm{e}^{y}\right)-\left(1+y\right)y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'\left(-\left(1+y^{2}\right)\mathrm{e}^{y}-1-y\right)}{1+y^{2}}=-\mathrm{e}^{2x}$

- Integrate both sides with respect to x

  $\int \frac{y'\left(-\left(1+y^{2}\right)\mathrm{e}^{y}-1-y\right)}{1+y^{2}}dx=\int -\mathrm{e}^{2x}dx+c_{1}$

- Evaluate integral

  $-\arctan\left(y\right)-\frac{\ln\left(1+y^{2}\right)}{2}-\mathrm{e}^{y}=-\frac{\mathrm{e}^{2x}}{2}+c_{1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 30

```
dsolve((1+y(x)^2)*(exp(2*x)-exp(y(x))*diff(y(x),x))-(1+y(x))*diff(y(x),x)=0,y(x), singsol=al
```

$$\frac{\mathrm{e}^{2x}}{2}-\arctan\left(y(x)\right)-\frac{\ln\left(1+y(x)^{2}\right)}{2}-\mathrm{e}^{y(x)}+c_{1}=0$$

✓ Solution by Mathematica

Time used: 0.696 (sec). Leaf size: 70

```
DSolve[(1+y[x]^2)*(Exp[2*x]-Exp[y[x]]*y'[x])-(1+y[x])*y'[x]==0,y[x],x,IncludeSingularSolutio
```

$$y(x) \to \text{InverseFunction}\left[e^{\#1} + \left(\frac{1}{2} - \frac{i}{2}\right)\log(-\#1 + i) + \left(\frac{1}{2} + \frac{i}{2}\right)\log(\#1 + i)\&\right]\left[\frac{e^{2x}}{2} + c_1\right]$$

$$y(x) \to -i$$

$$y(x) \to i$$

2 Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12

2.1 problem 1

Internal problem ID [5749]
Internal file name [`OUTPUT/4997_Sunday_June_05_2022_03_16_40_PM_73523658/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-y+(x+y)\,y'=-x}$$

2.1.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-x+y}{x+y} \end{aligned} \tag{1}$$

An ode of the form $y'=\frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M=-x+y$ and $N=x+y$ are both homogeneous and of the same order $n=1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u=\frac{y}{x}$, or $y=ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = \frac{u-1}{u+1}$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\frac{u(x)-1}{u(x)+1} - u(x)}{x}$$

Or

$$u'(x) - \frac{\frac{u(x)-1}{u(x)+1} - u(x)}{x} = 0$$

Or

$$u'(x)\, xu(x) + u'(x)\, x + u(x)^2 + 1 = 0$$

Or

$$(u(x) + 1)\, xu'(x) + u(x)^2 + 1 = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2+1}{(u+1)\, x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u+1}}\, du = -\frac{1}{x}\, dx$$
$$\int \frac{1}{\frac{u^2+1}{u+1}}\, du = \int -\frac{1}{x}\, dx$$
$$\frac{\ln(u^2+1)}{2} + \arctan(u) = -\ln(x) + c_2$$

The solution is

$$\frac{\ln\left(u(x)^2 + 1\right)}{2} + \arctan(u(x)) + \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\frac{\ln\left(\frac{y^2}{x^2} + 1\right)}{2} + \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2}+\arctan\left(\frac{y}{x}\right)+\ln\left(x\right)-c_2=0 \tag{1}$$

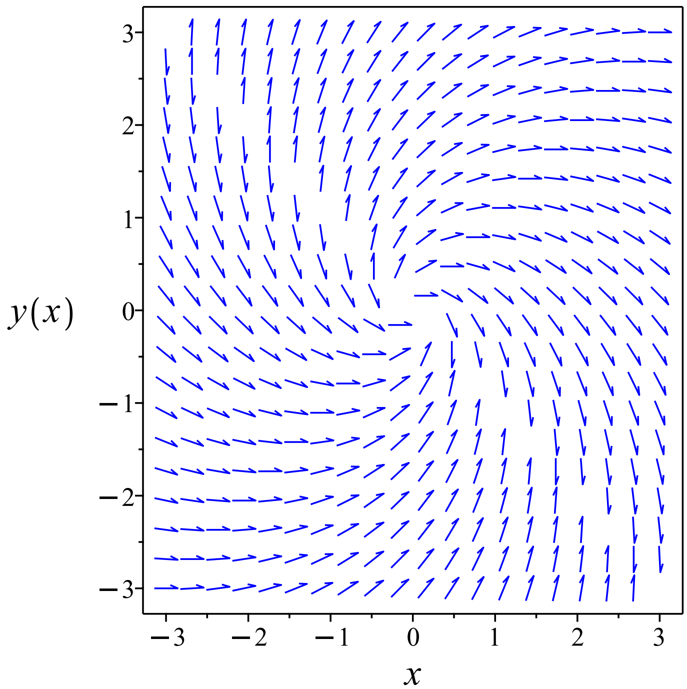


Figure 42: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2}+\arctan\left(\frac{y}{x}\right)+\ln\left(x\right)-c_2=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve((x-y(x))+(x+y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \tan\left(\text{RootOf}\left(2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x\right) + 2c_1\right)\right) x$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 34

```
DSolve[(x-y[x])+(x+y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\arctan\left(\frac{y(x)}{x}\right) + \frac{1}{2}\log\left(\frac{y(x)^2}{x^2} + 1\right) = -\log(x) + c_1, y(x)\right]$$

2.2 problem 2

Internal problem ID [5750]
Internal file name [`OUTPUT/4998_Sunday_June_05_2022_03_16_41_PM_64507826/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y - 2xy + x^2y' = 0}$$

2.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y(2x-1)}{x^2} \end{aligned}$$

Where $f(x) = \frac{2x-1}{x^2}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{2x-1}{x^2}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{2x-1}{x^2}\,dx \\
\ln(y) &= 2\ln(x) + \frac{1}{x} + c_1 \\
y &= \mathrm{e}^{2\ln(x)+\frac{1}{x}+c_1} \\
&= c_1 \mathrm{e}^{2\ln(x)+\frac{1}{x}}
\end{aligned}$$

Which simplifies to

$$y = c_1 x^2 \mathrm{e}^{\frac{1}{x}}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 \mathrm{e}^{\frac{1}{x}} \tag{1}$$

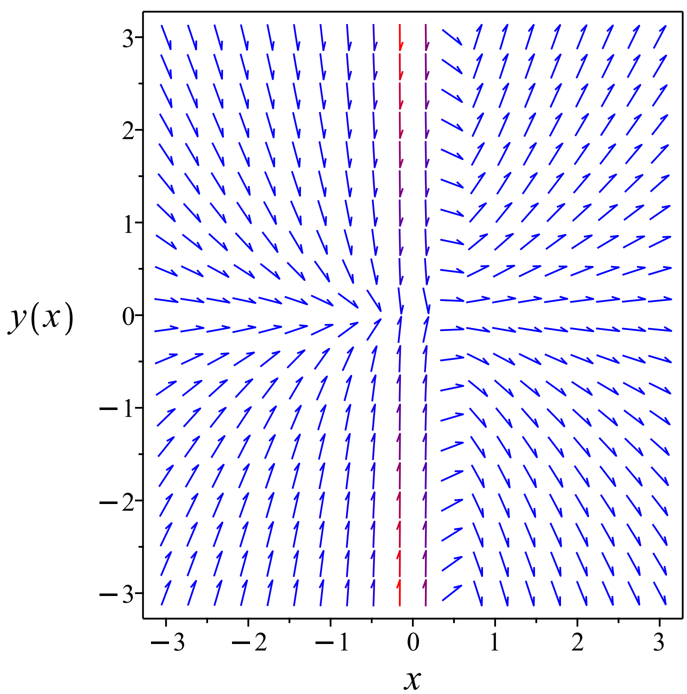


Figure 43: Slope field plot

Verification of solutions

$$y = c_1 x^2 e^{\frac{1}{x}}$$

Verified OK.

2.2.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2x-1}{x^2} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{y(2x-1)}{x^2} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{2x-1}{x^2}dx} \\ &= e^{-2\ln(x)-\frac{1}{x}} \end{aligned}$$

Which simplifies to

$$\mu = \frac{e^{-\frac{1}{x}}}{x^2}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(\frac{e^{-\frac{1}{x}}y}{x^2}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{e^{-\frac{1}{x}}y}{x^2} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{e^{-\frac{1}{x}}}{x^2}$ results in

$$y = c_1 x^2 e^{\frac{1}{x}}$$

Summary
The solution(s) found are the following

$$y = c_1 x^2 e^{\frac{1}{x}} \tag{1}$$

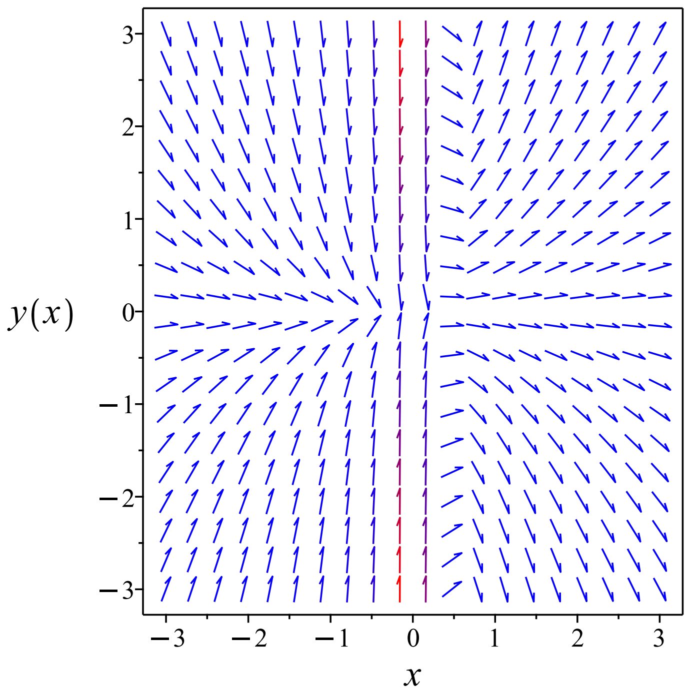


Figure 44: Slope field plot

Verification of solutions

$$y = c_1 x^2 e^{\frac{1}{x}}$$

Verified OK.

2.2.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x - 2x^2u(x) + x^2(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u(x-1)}{x^2}
\end{aligned}$$

Where $f(x) = \frac{x-1}{x^2}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= \frac{x-1}{x^2}\,dx\\ \int \frac{1}{u}\,du &= \int \frac{x-1}{x^2}\,dx\\ \ln(u) &= \ln(x) + \frac{1}{x} + c_2\\ u &= \mathrm{e}^{\ln(x)+\frac{1}{x}+c_2}\\ &= c_2\mathrm{e}^{\ln(x)+\frac{1}{x}}\end{aligned}$$

Which simplifies to

$$u(x) = c_2 x\,\mathrm{e}^{\frac{1}{x}}$$

Therefore the solution y is

$$\begin{aligned}y &= ux\\ &= x^2 c_2 \mathrm{e}^{\frac{1}{x}}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = x^2 c_2 \mathrm{e}^{\frac{1}{x}} \tag{1}$$

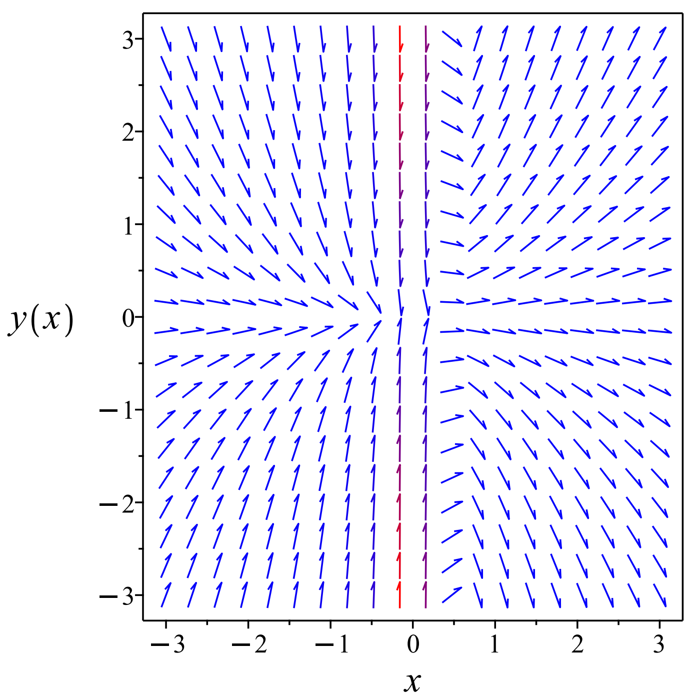


Figure 45: Slope field plot

Verification of solutions

$$y = x^2 c_2 e^{\frac{1}{x}}$$

Verified OK.

2.2.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(2x-1)}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 37: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= e^{2\ln(x)+\frac{1}{x}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{2\ln(x)+\frac{1}{x}}} dy \end{aligned}$$

Which results in

$$S = \frac{e^{-\frac{1}{x}} y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(2x-1)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y(1-2x)\, e^{-\frac{1}{x}}}{x^4} \\ S_y &= \frac{e^{-\frac{1}{x}}}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{e^{-\frac{1}{x}} y}{x^2} = c_1$$

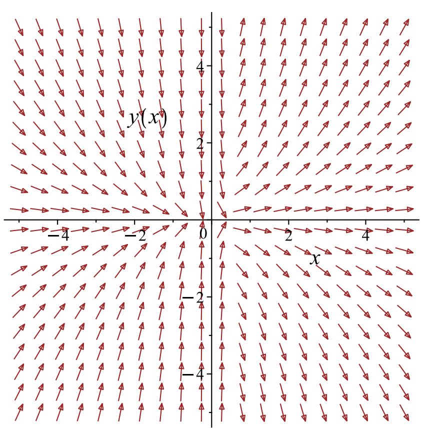
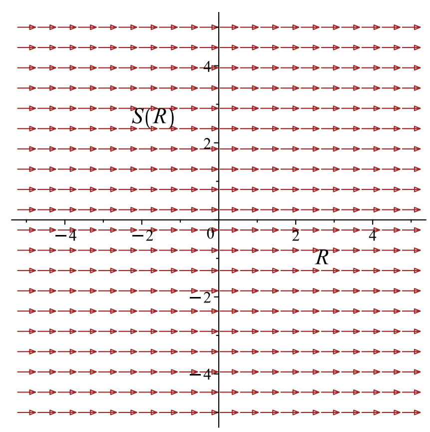
Which simplifies to

$$\frac{e^{-\frac{1}{x}} y}{x^2} = c_1$$

Which gives

$$y = c_1 x^2 e^{\frac{1}{x}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(2x-1)}{x^2}$ 	$R = x$ $S = \frac{e^{-\frac{1}{x}} y}{x^2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 x^2 e^{\frac{1}{x}} \tag{1}$$

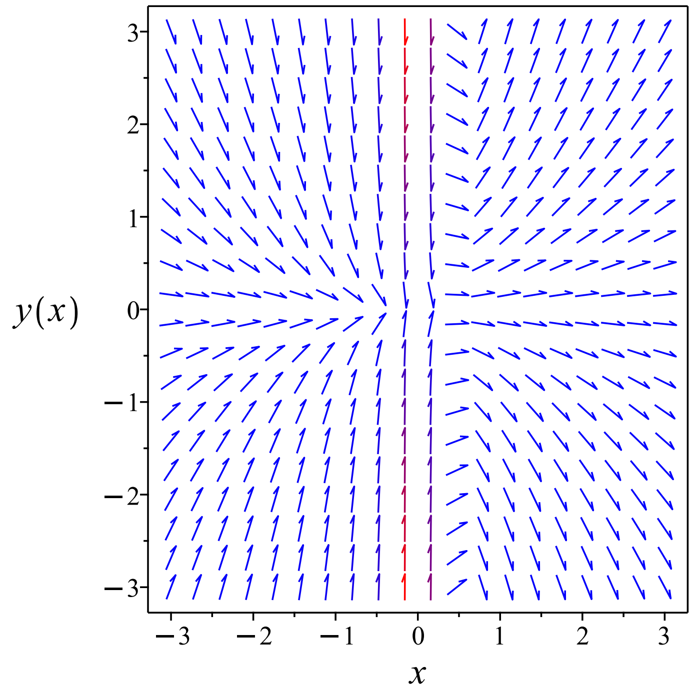


Figure 46: Slope field plot

Verification of solutions

$$y = c_1 x^2 \mathrm{e}^{\frac{1}{x}}$$

Verified OK.

2.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y}\right) \mathrm{d}y = \left(\frac{2x-1}{x^2}\right) \mathrm{d}x$$
$$\left(-\frac{2x-1}{x^2}\right) \mathrm{d}x + \left(\frac{1}{y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{2x-1}{x^2}$$
$$N(x, y) = \frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2x-1}{x^2}\right) \\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{2x-1}{x^2} \, \mathrm{d}x$$
$$\phi = -2\ln(x) - \frac{1}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2\ln(x) - \frac{1}{x} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2\ln(x) - \frac{1}{x} + \ln(y)$$

The solution becomes

$$y = e^{\frac{2\ln(x)x + c_1 x + 1}{x}}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{2\ln(x)x + c_1 x + 1}{x}} \tag{1}$$

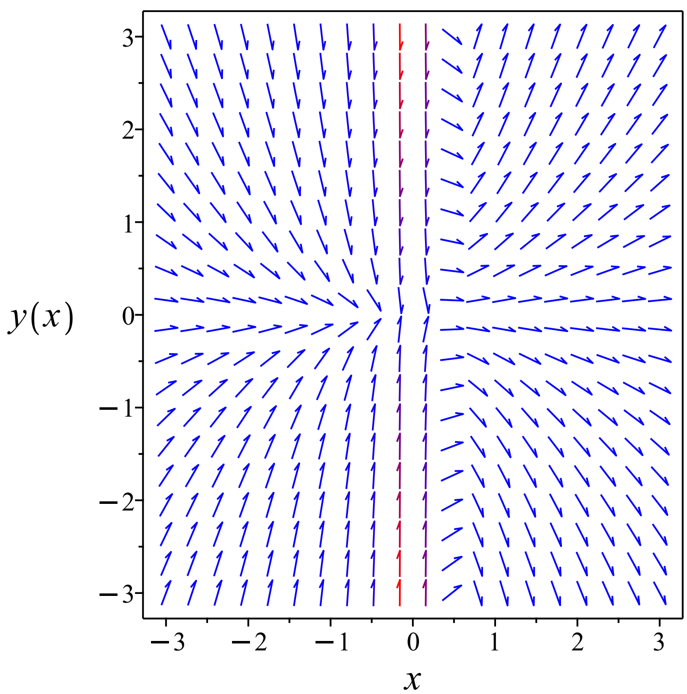


Figure 47: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{2\ln(x)x+c_1x+1}{x}}$$

Verified OK.

2.2.6 Maple step by step solution

Let's solve

$y - 2xy + x^2y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{2x-1}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{2x-1}{x^2}dx + c_1$

- Evaluate integral

  $\ln(y) = 2\ln(x) + \frac{1}{x} + c_1$

- Solve for y

  $y = \mathrm{e}^{\frac{2\ln(x)x+c_1x+1}{x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 13

```
dsolve((y(x)-2*x*y(x))+x^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1\mathrm{e}^{\frac{1}{x}}x^2$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 21

```
DSolve[(y[x]-2*x*y[x])+x^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\frac{1}{x}} x^2$$
$$y(x) \to 0$$

2.3 problem 3

Internal problem ID [5751]
Internal file name [`OUTPUT/4999_Sunday_June_05_2022_03_16_43_PM_84291016/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{2xy' - y(2x^2 - y^2) = 0}$$

2.3.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(-2x^2 + y^2)}{2x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 40: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= y^3 \mathrm{e}^{-x^2} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^3 e^{-x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{e^{x^2}}{2y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(-2x^2 + y^2)}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x e^{x^2}}{y^2} \\ S_y &= \frac{e^{x^2}}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{e^{x^2}}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{e^{R^2}}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\text{expIntegral}_1(-R^2)}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

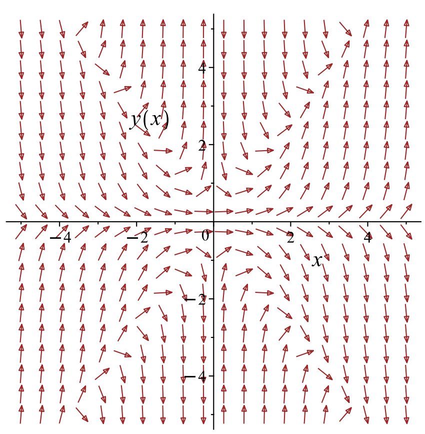
$$-\frac{e^{x^2}}{2y^2} = \frac{\text{expIntegral}_1(-x^2)}{4} + c_1$$

Which simplifies to

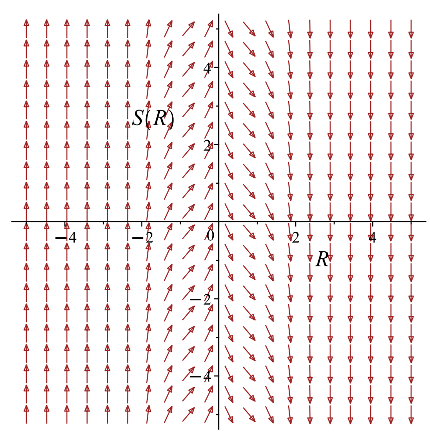
$$-\frac{e^{x^2}}{2y^2} = \frac{\text{expIntegral}_1(-x^2)}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{y(-2x^2+y^2)}{2x}$


</td><td>$R = x$

$S = -\frac{e^{x^2}}{2y^2}$</td><td>$\frac{dS}{dR} = -\frac{e^{R^2}}{2R}$


</td></tr>
</table>

Summary

The solution(s) found are the following

$$-\frac{e^{x^2}}{2y^2} = \frac{\text{expIntegral}_1(-x^2)}{4} + c_1 \tag{1}$$

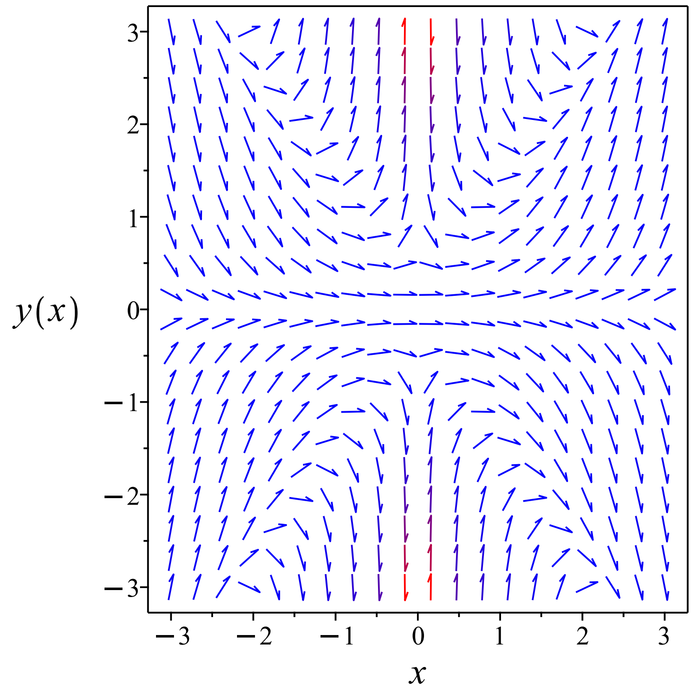


Figure 48: Slope field plot

Verification of solutions

$$-\frac{e^{x^2}}{2y^2} = \frac{\text{expIntegral}_1(-x^2)}{4} + c_1$$

Verified OK.

2.3.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(-2x^2+y^2)}{2x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = xy - \frac{1}{2x}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= x \\
f_1(x) &= -\frac{1}{2x} \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y'\frac{1}{y^3} = \frac{x}{y^2} - \frac{1}{2x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{2}{y^3}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= w(x)\,x - \frac{1}{2x} \\
w' &= -2xw + \frac{1}{x}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 2x \\
q(x) &= \frac{1}{x}
\end{aligned}
$$

Hence the ode is

$$w'(x) + 2w(x)\,x = \frac{1}{x}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int 2x dx} \\
&= \mathrm{e}^{x^2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(\frac{1}{x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2} w\right) &= \left(\mathrm{e}^{x^2}\right)\left(\frac{1}{x}\right) \\
\mathrm{d}\left(\mathrm{e}^{x^2} w\right) &= \left(\frac{\mathrm{e}^{x^2}}{x}\right) \mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\mathrm{e}^{x^2} w &= \int \frac{\mathrm{e}^{x^2}}{x} \,\mathrm{d}x \\
\mathrm{e}^{x^2} w &= -\frac{\mathrm{expIntegral}_1\left(-x^2\right)}{2} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x^2}$ results in

$$w(x) = -\frac{\mathrm{e}^{-x^2}\,\mathrm{expIntegral}_1\left(-x^2\right)}{2} + c_1 \mathrm{e}^{-x^2}$$

which simplifies to

$$w(x) = \mathrm{e}^{-x^2}\left(-\frac{\mathrm{expIntegral}_1\left(-x^2\right)}{2} + c_1\right)$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = \mathrm{e}^{-x^2}\left(-\frac{\mathrm{expIntegral}_1\left(-x^2\right)}{2} + c_1\right)$$

Solving for y gives

$$\begin{aligned}
y(x) &= \frac{\sqrt{2}}{\sqrt{\mathrm{e}^{-x^2}\left(-\,\mathrm{expIntegral}_1\left(-x^2\right) + 2c_1\right)}} \\
y(x) &= -\frac{\sqrt{2}}{\sqrt{\mathrm{e}^{-x^2}\left(-\,\mathrm{expIntegral}_1\left(-x^2\right) + 2c_1\right)}}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{2}}{\sqrt{e^{-x^2}\left(-\operatorname{expIntegral}_1\left(-x^2\right)+2c_1\right)}} \tag{1}$$

$$y = -\frac{\sqrt{2}}{\sqrt{e^{-x^2}\left(-\operatorname{expIntegral}_1\left(-x^2\right)+2c_1\right)}} \tag{2}$$

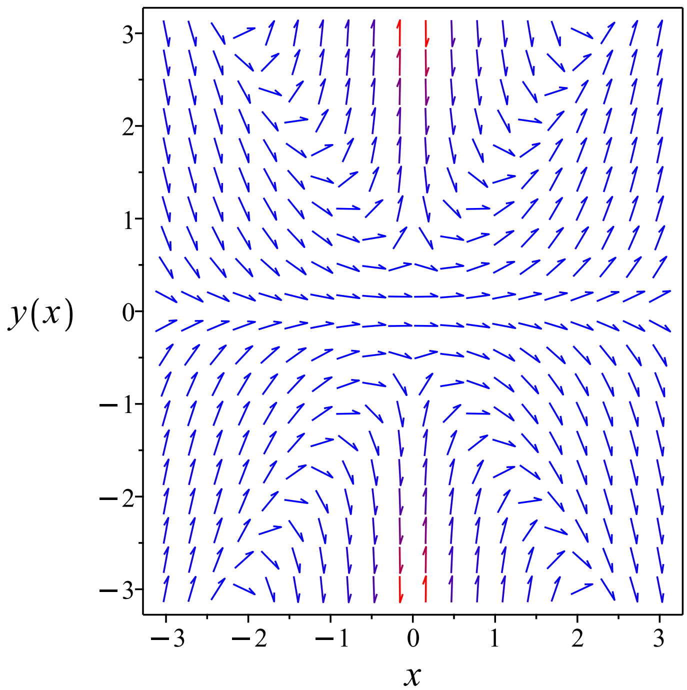


Figure 49: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{2}}{\sqrt{e^{-x^2}\left(-\operatorname{expIntegral}_1\left(-x^2\right)+2c_1\right)}}$$

Verified OK.

$$y = -\frac{\sqrt{2}}{\sqrt{e^{-x^2}\left(-\operatorname{expIntegral}_1\left(-x^2\right)+2c_1\right)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 83

```
dsolve(2*x*diff(y(x),x)=y(x)*(2*x^2-y(x)^2),y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{2}\,\sqrt{\left(2c_1 - \text{expIntegral}_1\left(-x^2\right)\right) e^{x^2}}}{-2c_1 + \text{expIntegral}_1\left(-x^2\right)}$$
$$y(x) = \frac{\sqrt{2}\,\sqrt{\left(2c_1 - \text{expIntegral}_1\left(-x^2\right)\right) e^{x^2}}}{2c_1 - \text{expIntegral}_1\left(-x^2\right)}$$

✓ Solution by Mathematica

Time used: 0.269 (sec). Leaf size: 65

```
DSolve[2*x*y'[x]==y[x]*(2*x^2-y[x]^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{\frac{x^2}{2}}}{\sqrt{\frac{\text{ExpIntegralEi}(x^2)}{2} + c_1}}$$
$$y(x) \to \frac{e^{\frac{x^2}{2}}}{\sqrt{\frac{\text{ExpIntegralEi}(x^2)}{2} + c_1}}$$
$$y(x) \to 0$$

2.4 problem 4

Internal problem ID [5752]
Internal file name [`OUTPUT/5000_Sunday_June_05_2022_03_16_46_PM_53928815/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y^2 + x^2 y' - x y y' = 0}$$

2.4.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y^2}{x(-x+y)} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = -y^2$ and $N = x(x - y)$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx} x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = \frac{u^2}{u-1}$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\frac{u(x)^2}{u(x)-1} - u(x)}{x}$$

Or

$$u'(x) - \frac{\frac{u(x)^2}{u(x)-1} - u(x)}{x} = 0$$

Or

$$u'(x)\, xu(x) - u'(x)\, x - u(x) = 0$$

Or

$$x(u(x) - 1)\, u'(x) - u(x) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u}{x\,(u-1)} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{u}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u}{u-1}}\, du = \frac{1}{x}\, dx$$
$$\int \frac{1}{\frac{u}{u-1}}\, du = \int \frac{1}{x}\, dx$$
$$u - \ln(u) = \ln(x) + c_2$$

The solution is

$$u(x) - \ln(u(x)) - \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\frac{y}{x} - \ln\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y}{x} - \ln\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}$$

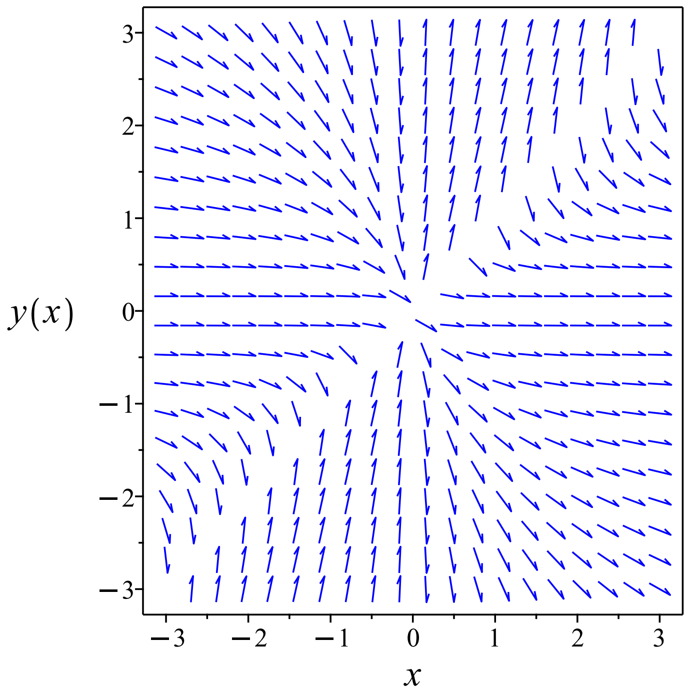


Figure 50: Slope field plot

Verification of solutions

$$\frac{y}{x} - \ln\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 17

```
dsolve(y(x)^2+x^2*diff(y(x),x)=x*y(x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = -x \operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

✓ Solution by Mathematica

Time used: 2.289 (sec). Leaf size: 25

```
DSolve[y[x]^2+x^2*y'[x]==x*y[x]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -xW\left(-\frac{e^{-c_1}}{x}\right)$$
$$y(x) \to 0$$

2.5 problem 5

Internal problem ID [5753]
Internal file name [OUTPUT/5001_Sunday_June_05_2022_03_16_48_PM_72453845/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{(x^2 + y^2)\,y' - 2xy = 0}$$

2.5.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{2xy}{x^2 + y^2} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = 2xy$ and $N = x^2 + y^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= \frac{2u}{u^2+1} \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\frac{2u(x)}{u(x)^2+1} - u(x)}{x}
\end{aligned}$$

Or

$$u'(x) - \frac{\frac{2u(x)}{u(x)^2+1} - u(x)}{x} = 0$$

Or

$$u'(x)\,u(x)^2\,x + u(x)^3 + u'(x)\,x - u(x) = 0$$

Or

$$x\left(u(x)^2+1\right)u'(x) + u(x)^3 - u(x) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u^3-u}{x\,(u^2+1)}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^3-u}{u^2+1}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{u^3-u}{u^2+1}}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{\frac{u^3-u}{u^2+1}}\,du &= \int -\frac{1}{x}\,dx \\
\ln(u+1) + \ln(u-1) - \ln(u) &= -\ln(x) + c_2
\end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(u+1)+\ln(u-1)-\ln(u)} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{u^2-1}{u} = \frac{c_3}{x}$$

The solution is

$$\frac{u(x)^2 - 1}{u(x)} = \frac{c_3}{x}$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\frac{x\left(\frac{y^2}{x^2} - 1\right)}{y} = \frac{c_3}{x}$$

Which simplifies to

$$-\frac{(x-y)(x+y)}{y} = c_3$$

Summary

The solution(s) found are the following

$$-\frac{(x-y)(x+y)}{y} = c_3 \tag{1}$$

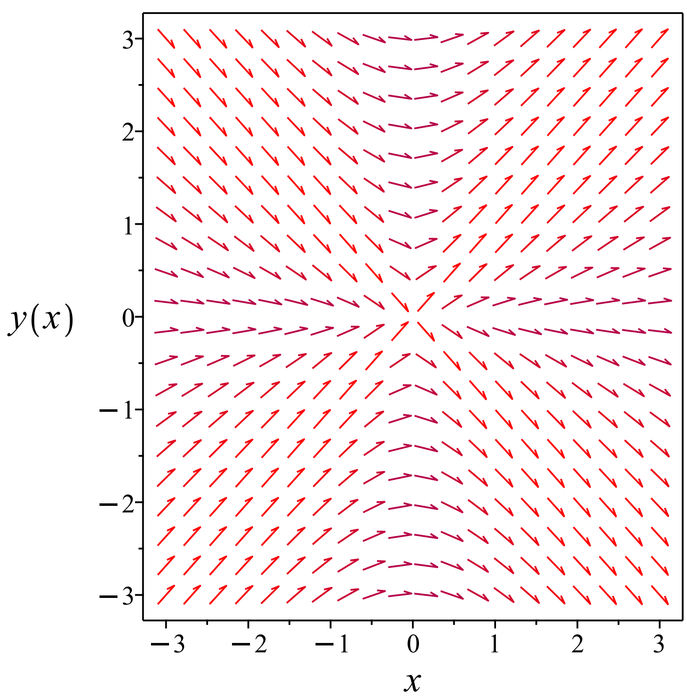


Figure 51: Slope field plot

Verification of solutions

$$-\frac{(x-y)(x+y)}{y}=c_3$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 47

```
dsolve((x^2+y(x)^2)*diff(y(x),x)=2*x*y(x),y(x), singsol=all)
```

$$y(x)=\frac{1-\sqrt{4c_1^2x^2+1}}{2c_1}$$
$$y(x)=\frac{1+\sqrt{4c_1^2x^2+1}}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.931 (sec). Leaf size: 70

```
DSolve[(x^2+y[x]^2)*y'[x]==2*x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{1}{2}\left(-\sqrt{4x^2+e^{2c_1}}-e^{c_1}\right)$$
$$y(x)\to\frac{1}{2}\left(\sqrt{4x^2+e^{2c_1}}-e^{c_1}\right)$$
$$y(x)\to 0$$

2.6 problem 6

Internal problem ID [5754]
Internal file name [`OUTPUT/5002_Sunday_June_05_2022_03_16_54_PM_25028875/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{-y + xy' - \tan\left(\frac{y}{x}\right)x = 0}$$

2.6.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y + \tan\left(\frac{y}{x}\right)x}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = y + \tan\left(\frac{y}{x}\right)x$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{du}{dx}x + u = \tan(u) + u$$
$$\frac{du}{dx} = \frac{\tan(u(x))}{x}$$

Or

$$u'(x) - \frac{\tan(u(x))}{x} = 0$$

Or

$$u'(x)\,x - \tan(u(x)) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\tan(u)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \tan(u)$. Integrating both sides gives

$$\frac{1}{\tan(u)}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{\tan(u)}\,du = \int \frac{1}{x}\,dx$$
$$\ln(\sin(u)) = \ln(x) + c_2$$

Raising both side to exponential gives

$$\sin(u) = e^{\ln(x)+c_2}$$

Which simplifies to

$$\sin(u) = c_3 x$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$y = x \arcsin(c_3 x\, e^{c_2})$$

<u>Summary</u>
The solution(s) found are the following

$$y = x \arcsin(c_3 x\, e^{c_2}) \tag{1}$$

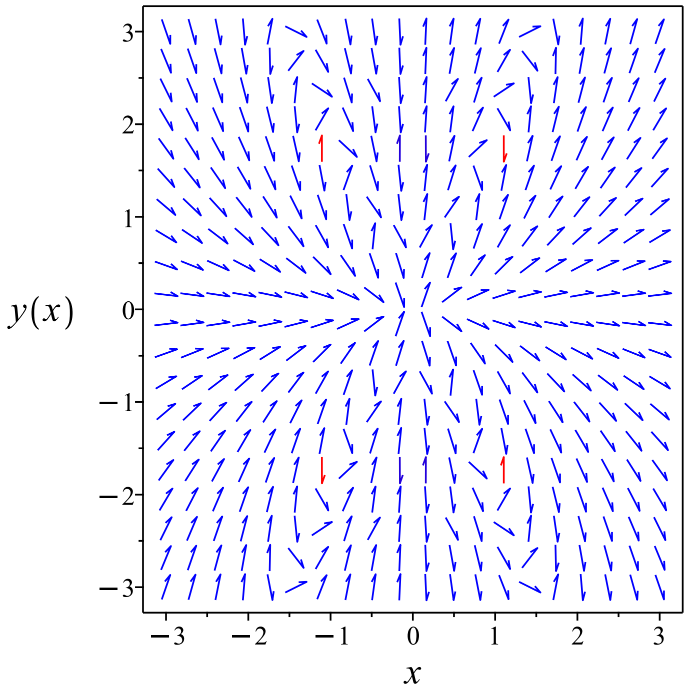


Figure 52: Slope field plot

Verification of solutions

$$y = x \arcsin\left(c_3 x\, e^{c_2}\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 10

```
dsolve(x*diff(y(x),x)-y(x)=x*tan(y(x)/x),y(x), singsol=all)
```

$$y(x) = \arcsin(c_1 x)\, x$$

✓ Solution by Mathematica

Time used: 6.102 (sec). Leaf size: 19

```
DSolve[x*y'[x]-y[x]==x*Tan[y[x]/x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \arcsin\left(e^{c_1} x\right)$$
$$y(x) \to 0$$

2.7 problem 7

Internal problem ID [5755]
Internal file name [`OUTPUT/5003_Sunday_June_05_2022_03_16_56_PM_90566151/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{xy' - y + x\,\mathrm{e}^{\frac{y}{x}} = 0}$$

2.7.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x\,\mathrm{e}^{\frac{y}{x}} - y}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = y - x\,\mathrm{e}^{\frac{y}{x}}$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{du}{dx}x + u = -e^{u} + u$$
$$\frac{du}{dx} = -\frac{e^{u(x)}}{x}$$

Or

$$u'(x) + \frac{e^{u(x)}}{x} = 0$$

Or

$$u'(x)\, x + e^{u(x)} = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{e^{u}}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = e^{u}$. Integrating both sides gives

$$\frac{1}{e^{u}}\, du = -\frac{1}{x}\, dx$$
$$\int \frac{1}{e^{u}}\, du = \int -\frac{1}{x}\, dx$$
$$-e^{-u} = -\ln(x) + c_2$$

The solution is

$$-e^{-u(x)} + \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$-e^{-\frac{y}{x}} + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$-e^{-\frac{y}{x}} + \ln(x) - c_2 = 0 \tag{1}$$

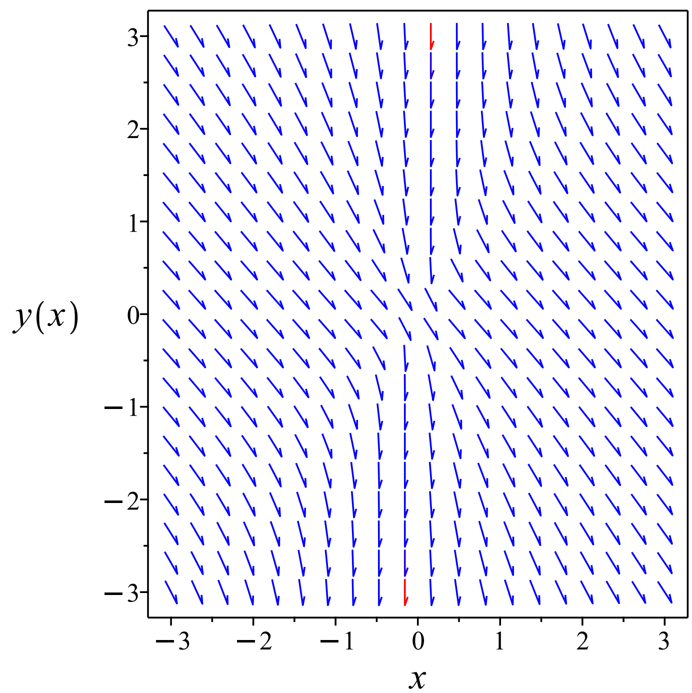


Figure 53: Slope field plot

Verification of solutions

$$-e^{-\frac{y}{x}} + \ln(x) - c_2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x)=y(x)-x*exp(y(x)/x),y(x), singsol=all)
```

$$y(x) = -\ln\left(\ln\left(x\right) + c_1\right) x$$

✓ Solution by Mathematica

Time used: 0.348 (sec). Leaf size: 16

```
DSolve[x*y'[x]==y[x]-x*Exp[y[x]/x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x \log(\log(x) - c_1)$$

2.8 problem 8

Internal problem ID [5756]
Internal file name [`OUTPUT/5004_Sunday_June_05_2022_03_16_58_PM_49807557/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{-y + xy' - (x+y)\ln\left(\frac{x+y}{x}\right) = 0}$$

2.8.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\ln\left(\frac{x+y}{x}\right)x + \ln\left(\frac{x+y}{x}\right)y + y}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = \ln\left(\frac{x+y}{x}\right)x + \ln\left(\frac{x+y}{x}\right)y + y$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode.

Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}\frac{du}{dx}x + u &= \ln(u+1) + \ln(u+1)u + u \\ \frac{du}{dx} &= \frac{\ln(u(x)+1) + \ln(u(x)+1)u(x)}{x}\end{aligned}$$

Or

$$u'(x) - \frac{\ln(u(x)+1) + \ln(u(x)+1)u(x)}{x} = 0$$

Or

$$u'(x)x - \ln(u(x)+1)u(x) - \ln(u(x)+1) = 0$$

Or

$$(-u(x)-1)\ln(u(x)+1) + u'(x)x = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\ln(u+1)(u+1)}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \ln(u+1)(u+1)$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\ln(u+1)(u+1)}\,du &= \frac{1}{x}\,dx \\ \int \frac{1}{\ln(u+1)(u+1)}\,du &= \int \frac{1}{x}\,dx \\ \ln(\ln(u+1)) &= \ln(x) + c_2\end{aligned}$$

Raising both side to exponential gives

$$\ln(u+1) = e^{\ln(x)+c_2}$$

Which simplifies to

$$\ln(u+1) = c_3 x$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$y = x\left(\mathrm{e}^{c_3 x\,\mathrm{e}^{c_2}} - 1\right)$$

Summary

The solution(s) found are the following

$$y = x\left(\mathrm{e}^{c_3 x\,\mathrm{e}^{c_2}} - 1\right) \tag{1}$$

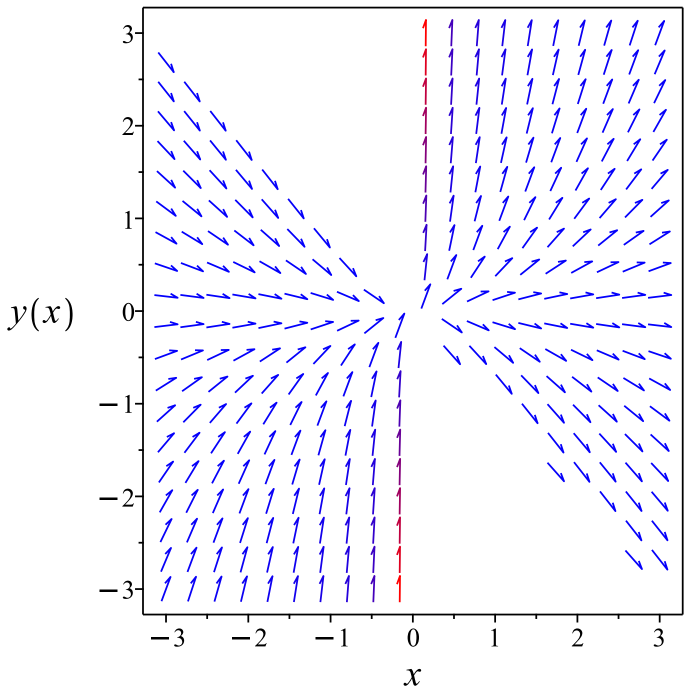


Figure 54: Slope field plot

Verification of solutions

$$y = x\left(\mathrm{e}^{c_3 x\,\mathrm{e}^{c_2}} - 1\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x)-y(x)=(x+y(x))*ln((x+y(x))/x),y(x), singsol=all)
```

$$y(x) = x(-1 + e^{c_1 x})$$

✓ Solution by Mathematica

Time used: 0.406 (sec). Leaf size: 24

```
DSolve[x*y'[x]-y[x]==(x+y[x])*Log[ (x+y[x])/x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\left(-1 + e^{e^{-c_1 x}}\right)$$
$$y(x) \to 0$$

2.9 problem 9

Internal problem ID [5757]
Internal file name [`OUTPUT/5005_Sunday_June_05_2022_03_17_00_PM_32810271/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{xy' - y\cos\left(\frac{y}{x}\right) = 0}$$

2.9.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y\cos\left(\frac{y}{x}\right)}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = y\cos\left(\frac{y}{x}\right)$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{du}{dx}x + u = u\cos(u)$$
$$\frac{du}{dx} = \frac{u(x)\cos(u(x)) - u(x)}{x}$$

Or

$$u'(x) - \frac{u(x)\cos(u(x)) - u(x)}{x} = 0$$

Or

$$u'(x)\,x - u(x)\cos(u(x)) + u(x) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(-1+\cos(u))}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u(-1+\cos(u))$. Integrating both sides gives

$$\frac{1}{u(-1+\cos(u))}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{u(-1+\cos(u))}\,du = \int \frac{1}{x}\,dx$$
$$\int^{u} \frac{1}{\_a(-1+\cos(\_a))}d\_a = \ln(x) + c_2$$

Which results in

$$\int^{u} \frac{1}{\_a(-1+\cos(\_a))}d\_a = \ln(x) + c_2$$

The solution is

$$\int^{u(x)} \frac{1}{\_a(-1+\cos(\_a))}d\_a - \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\int^{\frac{y}{x}} \frac{1}{\_a(-1+\cos(\_a))}d\_a - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\int^{\frac{y}{x}} \frac{1}{\_a\left(-1+\cos\left(\_a\right)\right)}d\_a - \ln\left(x\right) - c_2 = 0 \tag{1}$$

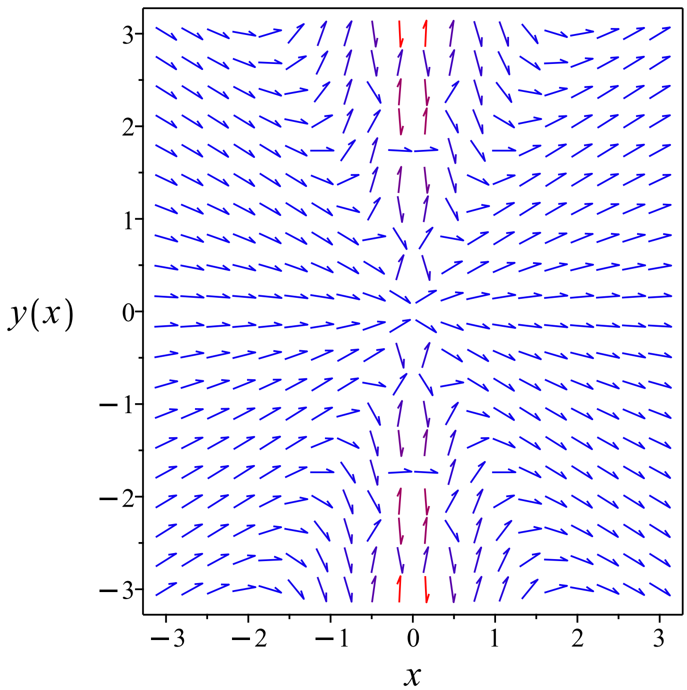


Figure 55: Slope field plot

Verification of solutions

$$\int^{\frac{y}{x}} \frac{1}{\_a\left(-1+\cos\left(\_a\right)\right)}d\_a - \ln\left(x\right) - c_2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x*diff(y(x),x)=y(x)*cos(y(x)/x),y(x), singsol=all)
```

$$y(x) = \text{RootOf}\left(\ln(x) + c_1 - \left(\int^{\_Z} \frac{1}{\_a(-1+\cos(\_a))} d\_a\right)\right) x$$

✓ Solution by Mathematica

Time used: 2.086 (sec). Leaf size: 33

```
DSolve[x*y'[x]==y[x]*Cos[y[x]/x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\int_1^{\frac{y(x)}{x}} \frac{1}{(\cos(K[1])-1)K[1]} dK[1] = \log(x) + c_1, y(x)\right]$$

2.10 problem 10

Internal problem ID [5758]
Internal file name [OUTPUT/5006_Sunday_June_05_2022_03_17_02_PM_33152817/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y + \sqrt{xy} - xy' = 0}$$

2.10.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y + \sqrt{xy}}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = y + \sqrt{xy}$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{du}{dx}x + u = u + \sqrt{u}$$
$$\frac{du}{dx} = \frac{\sqrt{u(x)}}{x}$$

Or

$$u'(x) - \frac{\sqrt{u(x)}}{x} = 0$$

Or

$$u'(x)\,x - \sqrt{u(x)} = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\sqrt{u}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \sqrt{u}$. Integrating both sides gives

$$\frac{1}{\sqrt{u}}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{\sqrt{u}}\,du = \int \frac{1}{x}\,dx$$
$$2\sqrt{u} = \ln(x) + c_2$$

The solution is

$$2\sqrt{u(x)} - \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$2\sqrt{\frac{y}{x}} - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$2\sqrt{\frac{y}{x}} - \ln(x) - c_2 = 0 \tag{1}$$

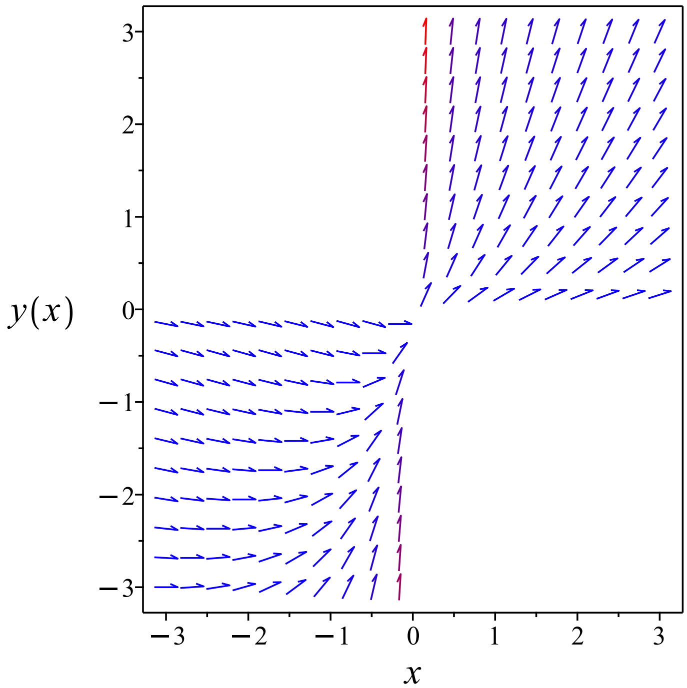


Figure 56: Slope field plot

Verification of solutions

$$2\sqrt{\frac{y}{x}} - \ln(x) - c_2 = 0$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve((y(x)+sqrt(x*y(x)))-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$-\frac{y(x)}{\sqrt{xy(x)}}+\frac{\ln(x)}{2}-c_1=0$$

✓ Solution by Mathematica

Time used: 0.183 (sec). Leaf size: 17

```
DSolve[(y[x]+Sqrt[x*y[x]])-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{1}{4}x(\log(x)+c_1)^2$$

2.11 problem 11

Internal problem ID [5759]
Internal file name [`OUTPUT/5007_Sunday_June_05_2022_03_17_05_PM_93867860/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy' - \sqrt{x^2 - y^2} - y = 0}$$

2.11.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{\sqrt{x^2 - y^2} + y}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = \sqrt{x^2 - y^2} + y$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx} x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x} x + u &= \sqrt{-u^2 + 1} + u \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\sqrt{-u(x)^2 + 1}}{x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{\sqrt{-u(x)^2 + 1}}{x} = 0
$$

Or

$$
u'(x)\, x - \sqrt{-u(x)^2 + 1} = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= \frac{\sqrt{-u^2 + 1}}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \sqrt{-u^2 + 1}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\sqrt{-u^2 + 1}}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{\sqrt{-u^2 + 1}}\, du &= \int \frac{1}{x}\, dx \\
\arcsin(u) &= \ln(x) + c_2
\end{aligned}
$$

The solution is

$$
\arcsin(u(x)) - \ln(x) - c_2 = 0
$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$
\arcsin\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0
$$

Summary
The solution(s) found are the following

$$
\arcsin\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}
$$

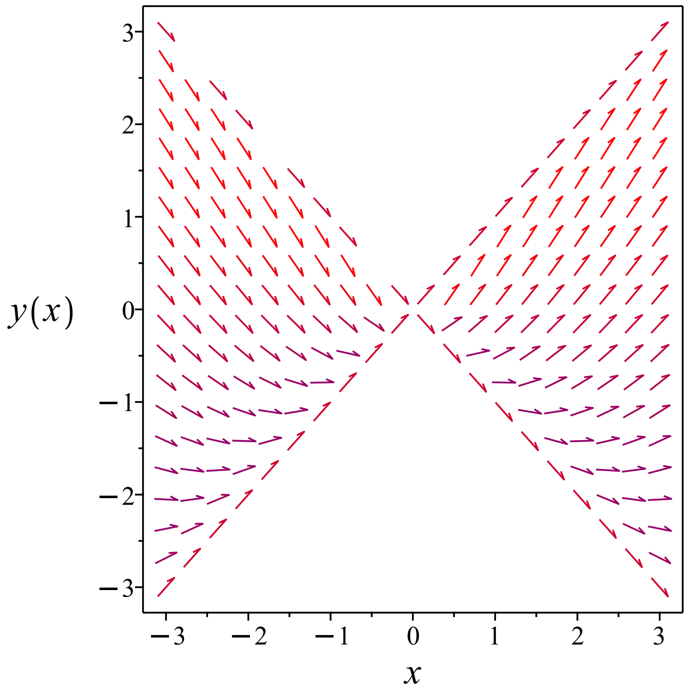


Figure 57: Slope field plot

Verification of solutions

$$\arcsin\left(\frac{y}{x}\right) - \ln\left(x\right) - c_2 = 0$$

Verified OK. {0 < x}

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x*diff(y(x),x)-sqrt(x^2-y(x)^2)-y(x)=0,y(x), singsol=all)
```

$$-\arctan\left(\frac{y(x)}{\sqrt{x^2-y(x)^2}}\right)+\ln(x)-c_1=0$$

✓ Solution by Mathematica

Time used: 0.243 (sec). Leaf size: 18

```
DSolve[x*y'[x]-Sqrt[x^2-y[x]^2]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -x\cosh(i\log(x)+c_1)$$

2.12 problem 12

Internal problem ID [5760]
Internal file name [`OUTPUT/5008_Sunday_June_05_2022_03_17_08_PM_48810216/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y-(x-y)\,y'=-x}$$

2.12.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x+y}{-x+y} \end{aligned} \tag{1}$$

An ode of the form $y'=\frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M=x+y$ and $N=x-y$ are both homogeneous and of the same order $n=1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u=\frac{y}{x}$, or $y=ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x+u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = \frac{-u-1}{u-1}$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\frac{-u(x)-1}{u(x)-1} - u(x)}{x}$$

Or

$$u'(x) - \frac{\frac{-u(x)-1}{u(x)-1} - u(x)}{x} = 0$$

Or

$$u'(x)\, xu(x) - u'(x)\, x + u(x)^2 + 1 = 0$$

Or

$$x(u(x) - 1)\, u'(x) + u(x)^2 + 1 = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u^2+1}{x(u-1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}}\, du = -\frac{1}{x}\, dx$$
$$\int \frac{1}{\frac{u^2+1}{u-1}}\, du = \int -\frac{1}{x}\, dx$$
$$\frac{\ln(u^2+1)}{2} - \arctan(u) = -\ln(x) + c_2$$

The solution is

$$\frac{\ln\left(u(x)^2+1\right)}{2} - \arctan(u(x)) + \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Summary
The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2}-\arctan\left(\frac{y}{x}\right)+\ln\left(x\right)-c_2=0 \tag{1}$$

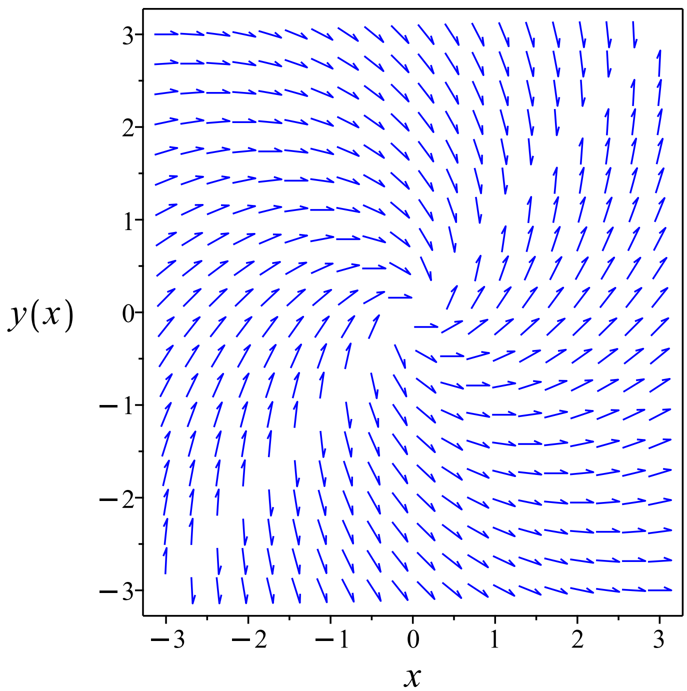


Figure 58: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2}-\arctan\left(\frac{y}{x}\right)+\ln\left(x\right)-c_2=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve((x+y(x))-(x-y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \tan\left(\text{RootOf}\left(-2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x\right) + 2c_1\right)\right)x$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 36

```
DSolve[(x+y[x])-(x-y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{2}\log\left(\frac{y(x)^2}{x^2}+1\right) - \arctan\left(\frac{y(x)}{x}\right) = -\log(x) + c_1, y(x)\right]$$

2.13 problem 13

Internal problem ID [5761]
Internal file name [`OUTPUT/5009_Sunday_June_05_2022_03_17_11_PM_61223389/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{2xy - y^2 + (y^2 + 2xy - x^2)\, y' = -x^2}$$

With initial conditions

$$[y(1) = -1]$$

2.13.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-x^2 - 2xy + y^2}{-x^2 + 2xy + y^2} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = x^2 + 2xy - y^2$ and $N = x^2 - 2xy - y^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= \frac{u^2 - 2u - 1}{u^2 + 2u - 1} \\ \frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\frac{u(x)^2 - 2u(x) - 1}{u(x)^2 + 2u(x) - 1} - u(x)}{x}\end{aligned}$$

Or

$$u'(x) - \frac{\frac{u(x)^2 - 2u(x) - 1}{u(x)^2 + 2u(x) - 1} - u(x)}{x} = 0$$

Or

$$u'(x)\, u(x)^2\, x + 2u'(x)\, u(x)\, x + u(x)^3 - u'(x)\, x + u(x)^2 + u(x) + 1 = 0$$

Or

$$x\left(u(x)^2 + 2u(x) - 1\right) u'(x) + \left(u(x) + 1\right)\left(u(x)^2 + 1\right) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{(u + 1)(u^2 + 1)}{x(u^2 + 2u - 1)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{(u+1)(u^2+1)}{u^2+2u-1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{(u+1)(u^2+1)}{u^2+2u-1}}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{\frac{(u+1)(u^2+1)}{u^2+2u-1}}\, du &= \int -\frac{1}{x}\, dx \\ -\ln(u + 1) + \ln\left(u^2 + 1\right) &= -\ln(x) + c_2\end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{-\ln(u+1) + \ln(u^2+1)} = \mathrm{e}^{-\ln(x) + c_2}$$

Which simplifies to

$$\frac{u^2+1}{u+1} = \frac{c_3}{x}$$

The solution is

$$\frac{u(x)^2+1}{u(x)+1} = \frac{c_3}{x}$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\frac{\frac{y^2}{x^2}+1}{\frac{y}{x}+1} = \frac{c_3}{x}$$

Which simplifies to

$$\frac{x^2+y^2}{x+y} = c_3$$

Writing the solution as

$$c_1(x^2+y^2) = x+y$$

Where $c_1 = \frac{1}{c_3}$ and solving for c_1 after applying initial conditions gives $c_1 = 0$. Hence the above solution becomes

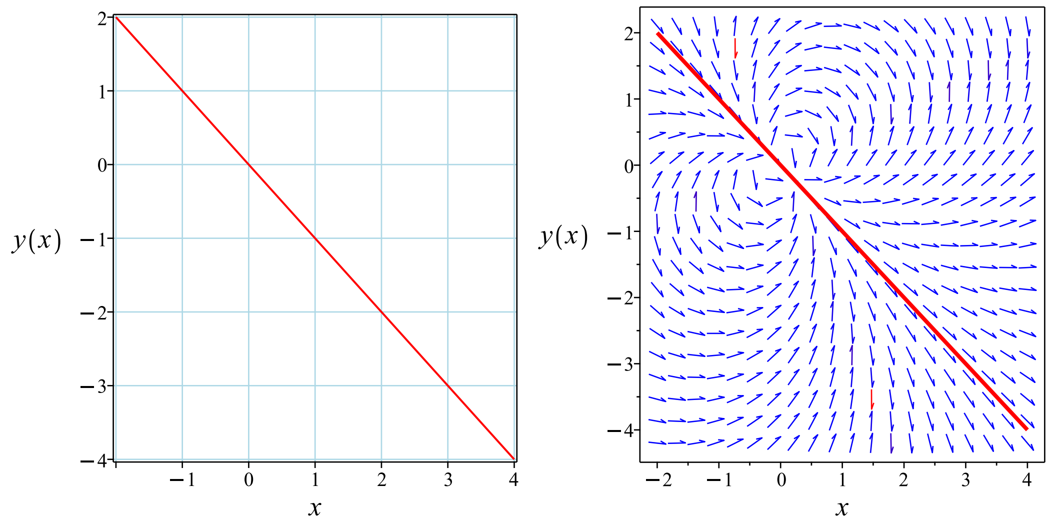
$$0 = x+y$$

Solving for y from the above gives

$$y = -x$$

Summary
The solution(s) found are the following

$$y = -x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 7

```
dsolve([(x^2+2*x*y(x)-y(x)^2)+(y(x)^2+2*x*y(x)-x^2)*diff(y(x),x)=0,y(1) = -1],y(x), singsol=
```

$$y(x) = -x$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{(x^2+2*x*y[x]-y[x]^2)+(y[x]^2+2*x*y[x]-x^2)*y'[x]==0,{y[1]==-1}},y[x],x,IncludeSingu
```

{}

2.14 problem 14

Internal problem ID [5762]
Internal file name [OUTPUT/5010_Sunday_June_05_2022_03_17_14_PM_18960310/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-y + xy' - y'y = 0}$$

2.14.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y}{-x+y} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = y$ and $N = x - y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = -\frac{u}{u-1}$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{-\frac{u(x)}{u(x)-1} - u(x)}{x}$$

Or

$$u'(x) - \frac{-\frac{u(x)}{u(x)-1} - u(x)}{x} = 0$$

Or

$$u'(x)\, xu(x) - u'(x)\, x + u(x)^2 = 0$$

Or

$$x(u(x) - 1)\, u'(x) + u(x)^2 = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2}{x\,(u-1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{u-1}}\,du = -\frac{1}{x}\,dx$$
$$\int \frac{1}{\frac{u^2}{u-1}}\,du = \int -\frac{1}{x}\,dx$$
$$\ln(u) + \frac{1}{u} = -\ln(x) + c_2$$

The solution is

$$\ln(u(x)) + \frac{1}{u(x)} + \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\ln\left(\frac{y}{x}\right) + \frac{x}{y} + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\ln\left(\frac{y}{x}\right)+\frac{x}{y}+\ln\left(x\right)-c_2=0 \tag{1}$$

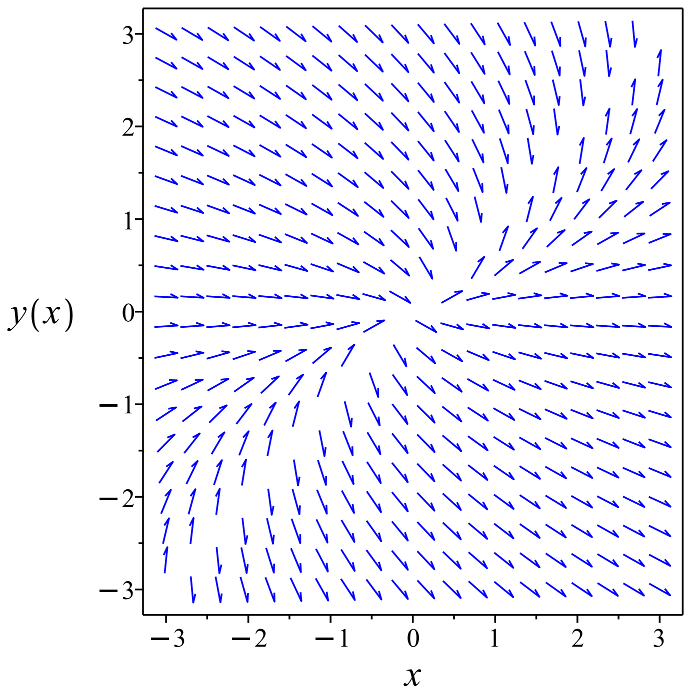


Figure 60: Slope field plot

Verification of solutions

$$\ln\left(\frac{y}{x}\right)+\frac{x}{y}+\ln\left(x\right)-c_2=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x)-y(x)=y(x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = -\frac{x}{\text{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)}$$

✓ Solution by Mathematica

Time used: 3.949 (sec). Leaf size: 25

```
DSolve[x*y'[x]-y[x]==y[x]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{W\left(-e^{-c_1}x\right)}$$
$$y(x) \to 0$$

2.15 problem 15

Internal problem ID [5763]
Internal file name [`OUTPUT/5011_Sunday_June_05_2022_03_17_15_PM_86048617/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y^2 + (x^2 - xy)\, y' = 0}$$

2.15.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2}{x(-x+y)} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = -y^2$ and $N = x(x-y)$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is

homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{du}{dx}x + u = \frac{u^2}{u-1}$$
$$\frac{du}{dx} = \frac{\frac{u(x)^2}{u(x)-1} - u(x)}{x}$$

Or

$$u'(x) - \frac{\frac{u(x)^2}{u(x)-1} - u(x)}{x} = 0$$

Or

$$u'(x)\, xu(x) - u'(x)\, x - u(x) = 0$$

Or

$$x(u(x) - 1)\, u'(x) - u(x) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u}{x\,(u-1)} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{u}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u}{u-1}}\, du = \frac{1}{x}\, dx$$
$$\int \frac{1}{\frac{u}{u-1}}\, du = \int \frac{1}{x}\, dx$$
$$u - \ln(u) = \ln(x) + c_2$$

The solution is

$$u(x) - \ln(u(x)) - \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\frac{y}{x} - \ln\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y}{x} - \ln\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}$$

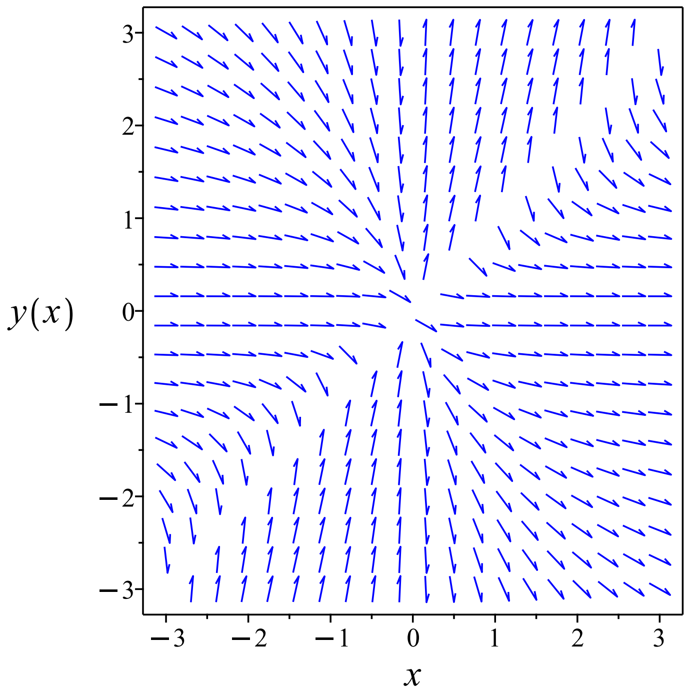


Figure 61: Slope field plot

Verification of solutions

$$\frac{y}{x} - \ln\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 17

```
dsolve(y(x)^2+(x^2-x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -x \operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

✓ Solution by Mathematica

Time used: 2.172 (sec). Leaf size: 25

```
DSolve[y[x]^2+(x^2-x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -xW\left(-\frac{e^{-c_1}}{x}\right)$$

$$y(x) \to 0$$

2.16 problem 16

Internal problem ID [5764]
Internal file name [`OUTPUT/5012_Sunday_June_05_2022_03_17_17_PM_77468873/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{xy + y^2 - x^2y' = -x^2}$$

2.16.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + xy + y^2}{x^2} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = x^2+xy+y^2$ and $N = x^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{du}{dx}x + u &= u^2 + u + 1 \\
\frac{du}{dx} &= \frac{u(x)^2 + 1}{x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{u(x)^2 + 1}{x} = 0
$$

Or

$$
u'(x)\,x - u(x)^2 - 1 = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= \frac{u^2 + 1}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2 + 1$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u^2 + 1}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{u^2 + 1}\,du &= \int \frac{1}{x}\,dx \\
\arctan(u) &= \ln(x) + c_2
\end{aligned}
$$

The solution is

$$
\arctan(u(x)) - \ln(x) - c_2 = 0
$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$
\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0
$$

Summary

The solution(s) found are the following

$$
\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}
$$

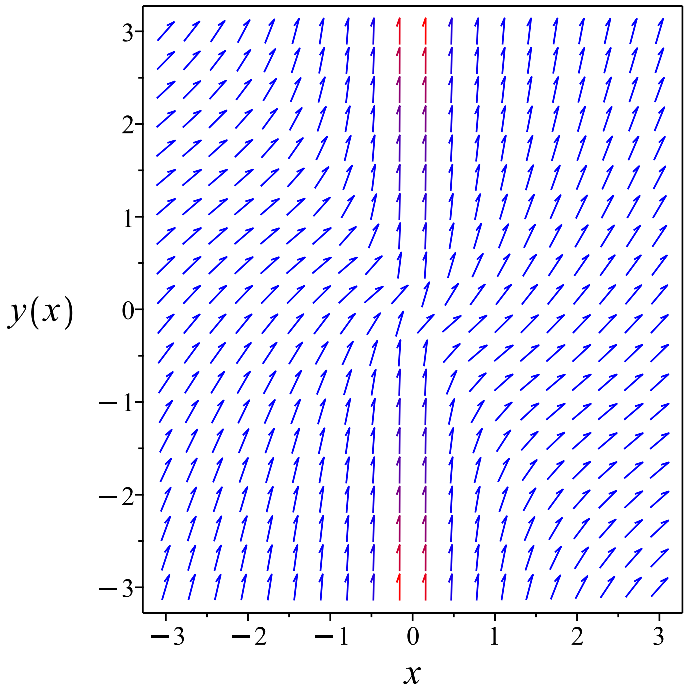


Figure 62: Slope field plot

Verification of solutions

$$\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve((x^2+x*y(x)+y(x)^2)=x^2*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \tan(\ln(x) + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.188 (sec). Leaf size: 13

```
DSolve[(x^2+x*y[x]+y[x]^2)==x^2*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \tan(\log(x) + c_1)$$

2.17 problem 17

Internal problem ID [5765]
Internal file name [`OUTPUT/5013_Sunday_June_05_2022_03_17_19_PM_71187612/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[[_homogeneous, `class A`], _rational, _dAlembert]`

$$\boxed{\frac{1}{x^2-xy+y^2}-\frac{y'}{2y^2-xy}=0}$$

2.17.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(-x+2y)}{x^2-xy+y^2} \end{aligned} \tag{1}$$

An ode of the form $y'=\frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M=-y(x-2y)$ and $N=x^2-xy+y^2$ are both homogeneous and of the same order $n=2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u=\frac{y}{x}$, or $y=ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}u}{\mathrm{d}x}x+u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{du}{dx}x + u = \frac{2u^2 - u}{u^2 - u + 1}$$
$$\frac{du}{dx} = \frac{\frac{2u(x)^2 - u(x)}{u(x)^2 - u(x) + 1} - u(x)}{x}$$

Or

$$u'(x) - \frac{\frac{2u(x)^2 - u(x)}{u(x)^2 - u(x) + 1} - u(x)}{x} = 0$$

Or

$$u'(x)\, u(x)^2\, x - u'(x)\, u(x)\, x + u(x)^3 + u'(x)\, x - 3u(x)^2 + 2u(x) = 0$$

Or

$$x\left(u(x)^2 - u(x) + 1\right) u'(x) + u(x)^3 - 3u(x)^2 + 2u(x) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u(u^2 - 3u + 2)}{x\,(u^2 - u + 1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u^2-3u+2)}{u^2-u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u^2-3u+2)}{u^2-u+1}}\, du = -\frac{1}{x}\, dx$$
$$\int \frac{1}{\frac{u(u^2-3u+2)}{u^2-u+1}}\, du = \int -\frac{1}{x}\, dx$$
$$-\ln(u-1) + \frac{\ln(u)}{2} + \frac{3\ln(u-2)}{2} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{-\ln(u-1) + \frac{\ln(u)}{2} + \frac{3\ln(u-2)}{2}} = e^{-\ln(x) + c_2}$$

Which simplifies to

$$\frac{\sqrt{u}\,(u-2)^{\frac{3}{2}}}{u-1} = \frac{c_3}{x}$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$y = \frac{c_3^2 \left(\operatorname{RootOf}\left(x^2\_Z^8 + 2x^2\_Z^6 - \_Z^4 c_3^2 - 2\_Z^2 c_3^2 - c_3^2\right)^2 + 1\right)^2}{x \operatorname{RootOf}\left(x^2\_Z^8 + 2x^2\_Z^6 - \_Z^4 c_3^2 - 2\_Z^2 c_3^2 - c_3^2\right)^6}$$

Summary

The solution(s) found are the following

$$y = \frac{c_3^2 \left(\operatorname{RootOf}\left(x^2\_Z^8 + 2x^2\_Z^6 - \_Z^4 c_3^2 - 2\_Z^2 c_3^2 - c_3^2\right)^2 + 1\right)^2}{x \operatorname{RootOf}\left(x^2\_Z^8 + 2x^2\_Z^6 - \_Z^4 c_3^2 - 2\_Z^2 c_3^2 - c_3^2\right)^6} \tag{1}$$

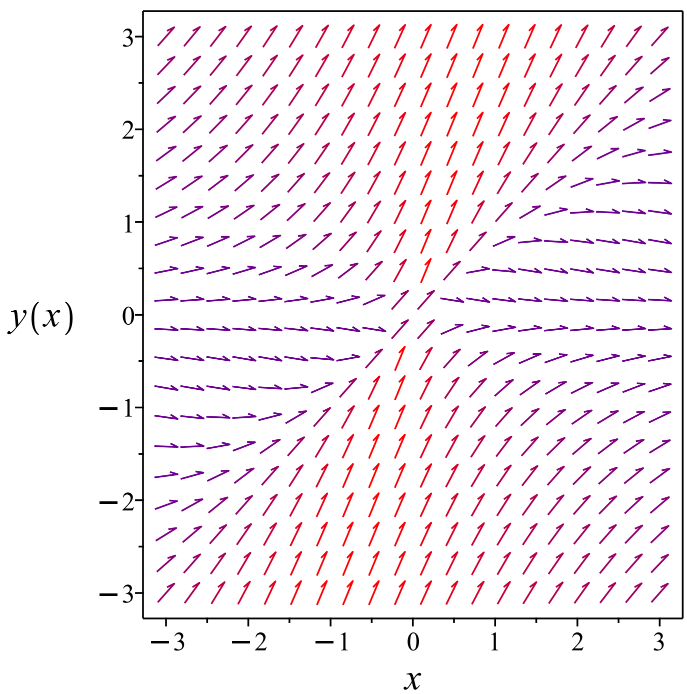


Figure 63: Slope field plot

Verification of solutions

$$y = \frac{c_3^2 \left(\operatorname{RootOf}\left(x^2\_Z^8 + 2x^2\_Z^6 - \_Z^4 c_3^2 - 2\_Z^2 c_3^2 - c_3^2\right)^2 + 1\right)^2}{x \operatorname{RootOf}\left(x^2\_Z^8 + 2x^2\_Z^6 - \_Z^4 c_3^2 - 2\_Z^2 c_3^2 - c_3^2\right)^6}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 5.532 (sec). Leaf size: 40

```
dsolve(1/(x^2-x*y(x)+y(x)^2)=1/(2*y(x)^2-x*y(x))*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \left(\operatorname{RootOf}\left(\_Z^8 c_1 x^2 + 2\_Z^6 c_1 x^2 - \_Z^4 - 2\_Z^2 - 1\right)^2 + 2\right) x$$

✓ Solution by Mathematica

Time used: 60.201 (sec). Leaf size: 1805

```
DSolve[1/(x^2-x*y[x]+y[x]^2)==1/(2*y[x]^2-x*y[x])*y'[x],y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \rightarrow \frac{1}{6}\left(-\sqrt{3}\sqrt{\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+e^{6c_1}}+\frac{e^{4c_1}}{\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+}}}\right.$$
$$-\sqrt{3}\sqrt{-\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+e^{6c_1}}-\frac{e^{4c_1}}{\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+e^{6}}}}$$
$$\left.+9x\right)$$

$$y(x) \rightarrow \frac{1}{6}\left(-\sqrt{3}\sqrt{\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+e^{6c_1}}+\frac{e^{4c_1}}{\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+}}}\right.$$
$$+\sqrt{3}\sqrt{-\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+e^{6c_1}}-\frac{e^{4c_1}}{\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+e^{6}}}}$$
$$\left.+9x\right)$$

$$y(x)$$

$$\frac{1}{}\left(\quad\frac{e^{4c_1}}{}\right.$$

2.18 problem 18

Internal problem ID [5766]
Internal file name [OUTPUT/5014_Sunday_June_05_2022_03_17_23_PM_2691186/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y' - \frac{2xy}{3x^2 - y^2} = 0}$$

2.18.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{2xy}{-3x^2 + y^2} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = 2xy$ and $N = 3x^2 - y^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= -\frac{2u}{u^2-3} \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{-\frac{2u(x)}{u(x)^2-3} - u(x)}{x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{-\frac{2u(x)}{u(x)^2-3} - u(x)}{x} = 0
$$

Or

$$
u'(x)\,u(x)^2\,x + u(x)^3 - 3u'(x)\,x - u(x) = 0
$$

Or

$$
x\left(u(x)^2 - 3\right)u'(x) + u(x)^3 - u(x) = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u^3-u}{x\left(u^2-3\right)}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^3-u}{u^2-3}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^3-u}{u^2-3}}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{\frac{u^3-u}{u^2-3}}\,du &= \int -\frac{1}{x}\,dx \\
-\ln(u+1) - \ln(u-1) + 3\ln(u) &= -\ln(x) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{-\ln(u+1)-\ln(u-1)+3\ln(u)} = \mathrm{e}^{-\ln(x)+c_2}
$$

Which simplifies to

$$
\frac{u^3}{u^2-1} = \frac{c_3}{x}
$$

The solution is

$$\frac{u(x)^3}{u(x)^2 - 1} = \frac{c_3}{x}$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\frac{y^3}{x^3 \left(\frac{y^2}{x^2} - 1\right)} = \frac{c_3}{x}$$

Which simplifies to

$$-\frac{y^3}{(x-y)(x+y)} = c_3$$

Summary
The solution(s) found are the following

$$-\frac{y^3}{(x-y)(x+y)} = c_3 \tag{1}$$

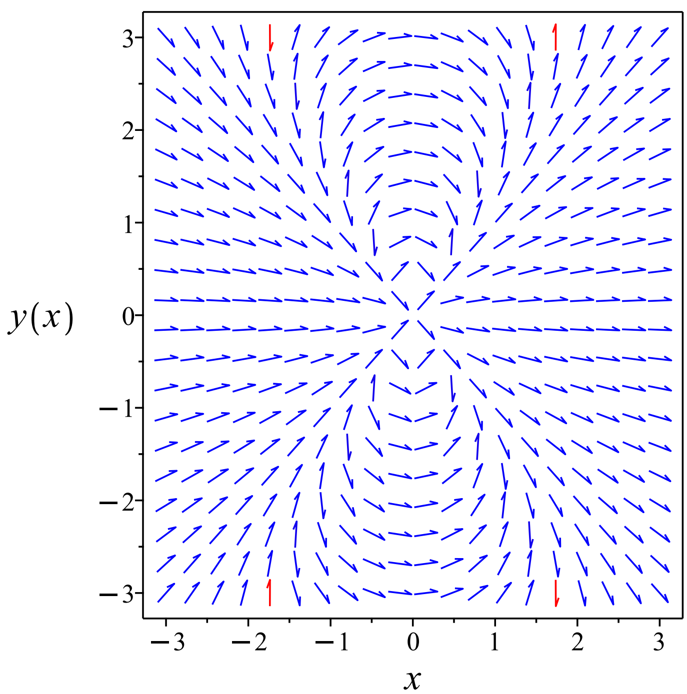


Figure 64: Slope field plot

Verification of solutions

$$-\frac{y^3}{(x-y)(x+y)} = c_3$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 317

```
dsolve(diff(y(x),x)=2*x*y(x)/(3*x^2-y(x)^2),y(x), singsol=all)
```

$$y(x) = \frac{1+\frac{\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2+8\right)^{\frac{1}{3}}}{2}+\frac{2}{\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2+8\right)^{\frac{1}{3}}}}{3c_1}$$

$$y(x) = -\frac{(1+i\sqrt{3})\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2+8\right)^{\frac{2}{3}}-4i\sqrt{3}-4\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2\right.}{12\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2+8\right)^{\frac{1}{3}}c_1}$$

$$y(x) = \frac{(i\sqrt{3}-1)\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2+8\right)^{\frac{2}{3}}-4i\sqrt{3}+4\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2+8\right.}{12\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2+8\right)^{\frac{1}{3}}c_1}$$

✓ Solution by Mathematica

Time used: 60.196 (sec). Leaf size: 458

```
DSolve[y'[x]==2*x*y[x]/(3*x^2-y[x]^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}\left(\frac{\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}}{\sqrt[3]{2}} + \frac{\sqrt[3]{2}e^{2c_1}}{\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}} - e^{c_1}\right)$$

$$y(x) \to \frac{i(\sqrt{3}+i)\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}}{6\sqrt[3]{2}} - \frac{i(\sqrt{3}-i)e^{2c_1}}{3\ 2^{2/3}\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}} - \frac{e^{c_1}}{3}$$

$$y(x) \to -\frac{i(\sqrt{3}-i)\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}}{6\sqrt[3]{2}} + \frac{i(\sqrt{3}+i)e^{2c_1}}{3\ 2^{2/3}\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}} - \frac{e^{c_1}}{3}$$

2.19 problem 19

Internal problem ID [5767]
Internal file name [`OUTPUT/5015_Sunday_June_05_2022_03_17_25_PM_7693094/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y' - \frac{x}{y} - \frac{y}{x} = 0}$$

With initial conditions

$$[y(-1) = 0]$$

2.19.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{x^2 + y^2}{xy} \end{aligned}$$

$f(x,y)$ is not defined at $y = 0$ therefore existence and uniqueness theorem do not apply.

2.19.2 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^2+y^2}{xy} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = x^2 + y^2$ and $N = xy$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dx}x + u &= \frac{1}{u} + u \\ \frac{du}{dx} &= \frac{1}{u(x)\,x} \end{aligned}$$

Or

$$u'(x) - \frac{1}{u(x)\,x} = 0$$

Or

$$u'(x)\,u(x)\,x - 1 = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{1}{ux} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{1}{u}}\, du = \frac{1}{x}\, dx$$
$$\int \frac{1}{\frac{1}{u}}\, du = \int \frac{1}{x}\, dx$$
$$\frac{u^2}{2} = \ln(x) + c_2$$

The solution is

$$\frac{u(x)^2}{2} - \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\frac{y^2}{2x^2} - \ln(x) - c_2 = 0$$

Substituting initial conditions and solving for c_2 gives $c_2 = -i\pi$. Hence the solution becomes Solving for y from the above gives

$$y = \sqrt{-2i\pi + 2\ln(x)}\, x$$
$$y = -\sqrt{-2i\pi + 2\ln(x)}\, x$$

Summary
The solution(s) found are the following

$$y = \sqrt{-2i\pi + 2\ln(x)}\, x \tag{1}$$
$$y = -\sqrt{-2i\pi + 2\ln(x)}\, x \tag{2}$$

Verification of solutions

$$y = \sqrt{-2i\pi + 2\ln(x)}\, x$$

Verified OK.

$$y = -\sqrt{-2i\pi + 2\ln(x)}\, x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 34

```
dsolve([diff(y(x),x)=x/y(x)+y(x)/x,y(-1) = 0],y(x), singsol=all)
```

$$y(x) = \sqrt{2\ln(x) - 2i\pi}\, x$$
$$y(x) = -\sqrt{2\ln(x) - 2i\pi}\, x$$

✓ Solution by Mathematica

Time used: 0.19 (sec). Leaf size: 48

```
DSolve[{y'[x]==x/y[x]+y[x]/x,{y[-1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2}x\sqrt{\log(x) - i\pi}$$
$$y(x) \to \sqrt{2}x\sqrt{\log(x) - i\pi}$$

2.20 problem 20

Internal problem ID [5768]
Internal file name [OUTPUT/5016_Sunday_June_05_2022_03_17_28_PM_96250479/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy' - y - \sqrt{y^2 - x^2} = 0}$$

2.20.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y + \sqrt{-x^2 + y^2}}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = y + \sqrt{-x^2 + y^2}$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = u + \sqrt{u^2 - 1}$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\sqrt{u(x)^2 - 1}}{x}$$

Or

$$u'(x) - \frac{\sqrt{u(x)^2 - 1}}{x} = 0$$

Or

$$u'(x)\,x - \sqrt{u(x)^2 - 1} = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\sqrt{u^2-1}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \sqrt{u^2 - 1}$. Integrating both sides gives

$$\frac{1}{\sqrt{u^2-1}}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{\sqrt{u^2-1}}\,du = \int \frac{1}{x}\,dx$$
$$\ln\left(u + \sqrt{u^2-1}\right) = \ln(x) + c_2$$

Raising both side to exponential gives

$$u + \sqrt{u^2-1} = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$u + \sqrt{u^2-1} = c_3 x$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$y = \frac{\left(\mathrm{e}^{2c_2}c_3^2x^2 + 1\right)\mathrm{e}^{-c_2}}{2c_3}$$

Summary
The solution(s) found are the following

$$y = \frac{(e^{2c_2} c_3^2 x^2 + 1) e^{-c_2}}{2c_3} \tag{1}$$

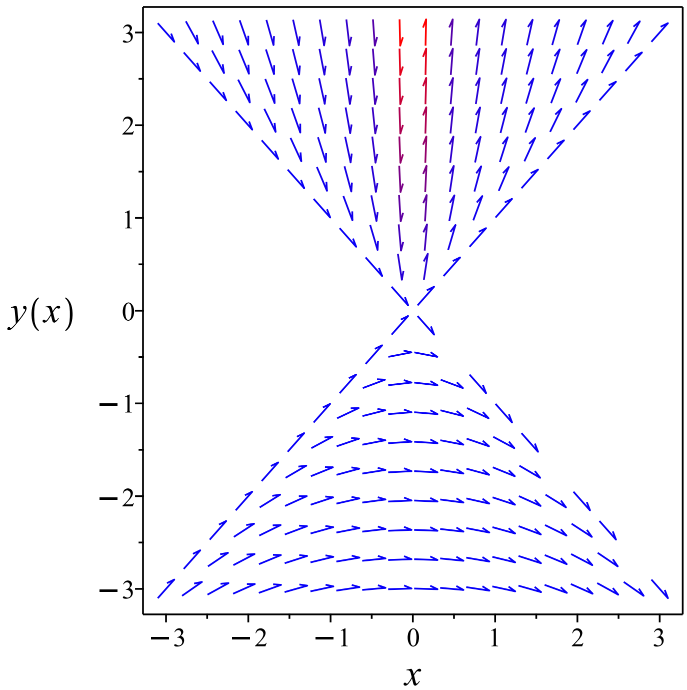


Figure 65: Slope field plot

Verification of solutions

$$y = \frac{(e^{2c_2} c_3^2 x^2 + 1) e^{-c_2}}{2c_3}$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve(x*diff(y(x),x)=y(x)+sqrt(y(x)^2-x^2),y(x), singsol=all)
```

$$\frac{-c_1 x^2 + y(x) + \sqrt{y(x)^2 - x^2}}{x^2} = 0$$

✓ Solution by Mathematica

Time used: 0.366 (sec). Leaf size: 14

```
DSolve[x*y'[x]==y[x]+Sqrt[y[x]^2-x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\cosh(\log(x) + c_1)$$

2.21 problem 21

Internal problem ID [5769]
Internal file name [OUTPUT/5017_Sunday_June_05_2022_03_17_30_PM_64848144/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y + \left(2\sqrt{xy} - x\right) y' = 0}$$

2.21.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y}{2\sqrt{xy} - x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = -y$ and $N = 2\sqrt{xy} - x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx} x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = \frac{u}{-2\sqrt{u}+1}$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\frac{u(x)}{-2\sqrt{u(x)}+1} - u(x)}{x}$$

Or

$$u'(x) - \frac{\frac{u(x)}{-2\sqrt{u(x)}+1} - u(x)}{x} = 0$$

Or

$$2u'(x)\,x\sqrt{u(x)} - u'(x)\,x + 2u(x)^{\frac{3}{2}} = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u^{\frac{3}{2}}}{x\left(2\sqrt{u}-1\right)} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u^{\frac{3}{2}}}{2\sqrt{u}-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^{\frac{3}{2}}}{2\sqrt{u}-1}}\,du = -\frac{2}{x}\,dx$$
$$\int \frac{1}{\frac{u^{\frac{3}{2}}}{2\sqrt{u}-1}}\,du = \int -\frac{2}{x}\,dx$$
$$\frac{2}{\sqrt{u}} + 2\ln(u) = -2\ln(x) + c_2$$

The solution is

$$\frac{2}{\sqrt{u(x)}} + 2\ln(u(x)) + 2\ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\frac{2}{\sqrt{\frac{y}{x}}} + 2\ln\left(\frac{y}{x}\right) + 2\ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{2}{\sqrt{\frac{y}{x}}}+2\ln\left(\frac{y}{x}\right)+2\ln\left(x\right)-c_2=0 \tag{1}$$

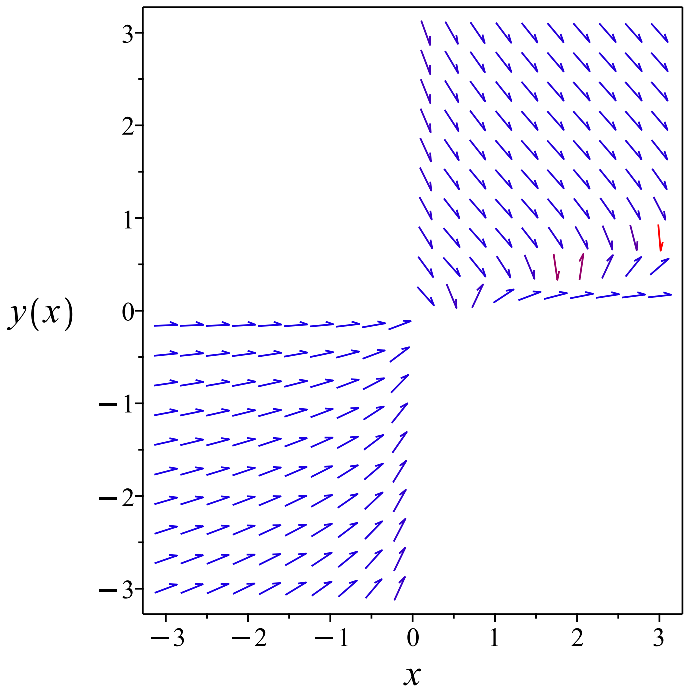


Figure 66: Slope field plot

Verification of solutions

$$\frac{2}{\sqrt{\frac{y}{x}}}+2\ln\left(\frac{y}{x}\right)+2\ln\left(x\right)-c_2=0$$

Verified OK. {0 < x}

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(y(x)+(2*sqrt(x*y(x))-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$\ln\left(y(x)\right)+\frac{x}{\sqrt{xy\left(x\right)}}-c_{1}=0$$

✓ Solution by Mathematica

Time used: 0.23 (sec). Leaf size: 33

```
DSolve[y[x]+(2*Sqrt[x*y[x]]-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{2}{\sqrt{\frac{y(x)}{x}}}+2\log\left(\frac{y(x)}{x}\right)=-2\log(x)+c_{1},y(x)\right]$$

2.22 problem 22

Internal problem ID [5770]
Internal file name [OUTPUT/5018_Sunday_June_05_2022_03_17_32_PM_73721832/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{xy' - \ln\left(\frac{y}{x}\right) y = 0}$$

2.22.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\ln\left(\frac{y}{x}\right) y}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = \ln\left(\frac{y}{x}\right) y$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{du}{dx}x + u &= \ln(u)\, u \\
\frac{du}{dx} &= \frac{\ln(u(x))\, u(x) - u(x)}{x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{\ln(u(x))\, u(x) - u(x)}{x} = 0
$$

Or

$$
u'(x)\, x - \ln(u(x))\, u(x) + u(x) = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u(\ln(u) - 1)}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u(\ln(u) - 1)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u(\ln(u) - 1)}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{u(\ln(u) - 1)}\, du &= \int \frac{1}{x}\, dx \\
\ln(\ln(u) - 1) &= \ln(x) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\ln(u) - 1 = e^{\ln(x) + c_2}
$$

Which simplifies to

$$
\ln(u) - 1 = c_3 x
$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$
y = x\, e^{c_3 x\, e^{c_2} + 1}
$$

Summary

The solution(s) found are the following

$$
y = x\, e^{c_3 x\, e^{c_2} + 1} \tag{1}
$$

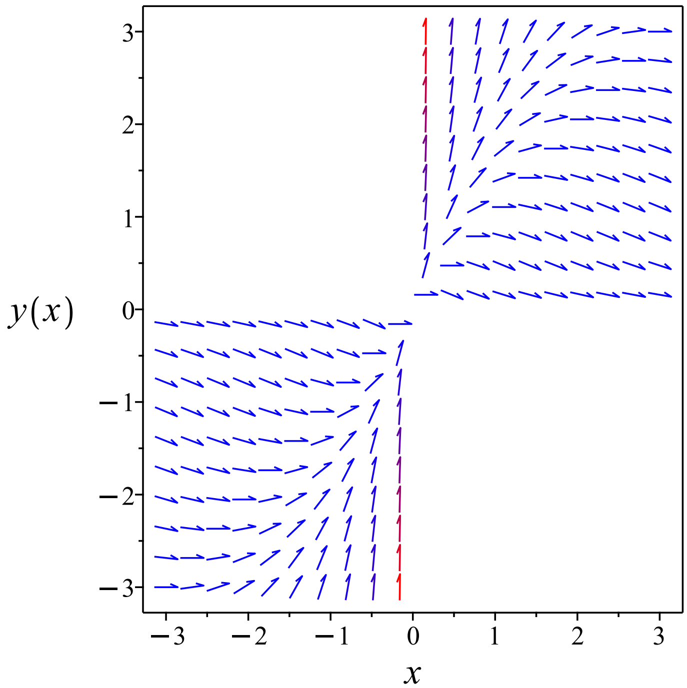


Figure 67: Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}^{c_3 x\,\mathrm{e}^{c_2}+1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x)=y(x)*ln(y(x)/x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{c_1 x+1} x$$

✓ Solution by Mathematica

Time used: 0.199 (sec). Leaf size: 24

```
DSolve[x*y'[x]==y[x]*Log[y[x]/x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x e^{1+e^{c_1 x}}$$
$$y(x) \to e x$$

2.23 problem 23

Internal problem ID [5771]
Internal file name [`OUTPUT/5019_Sunday_June_05_2022_03_17_34_PM_26513288/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 23.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "quadrature", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y'(y+y')-x(x+y)=0}$$

With initial conditions

$$[y(0)=0]$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y'=x \tag{1}$$
$$y'=-x-y \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

2.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= x \end{aligned}$$

Hence the ode is

$$y' = x$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Integrating both sides gives

$$\begin{aligned} y &= \int x \, \mathrm{d}x \\ &= \frac{x^2}{2} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = \frac{x^2}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{2} \tag{1}$$

Verification of solutions

$$y = \frac{x^2}{2}$$

Verified OK.
Solving equation (2)

2.23.2 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -x \end{aligned}$$

Hence the ode is

$$y + y' = -x$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)(-x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x}) = (\mathrm{e}^{x})(-x)$$
$$\mathrm{d}(y\,\mathrm{e}^{x}) = (-x\,\mathrm{e}^{x})\ \mathrm{d}x$$

Integrating gives

$$y\,\mathrm{e}^{x} = \int -x\,\mathrm{e}^{x}\,\mathrm{d}x$$
$$y\,\mathrm{e}^{x} = -(x-1)\,\mathrm{e}^{x} + c_2$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$y = -\mathrm{e}^{-x}(x-1)\,\mathrm{e}^{x} + c_2\mathrm{e}^{-x}$$

which simplifies to

$$y = 1 - x + c_2\mathrm{e}^{-x}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_2 + 1$$

$$c_2 = -1$$

Substituting c_2 found above in the general solution gives

$$y = 1 - \mathrm{e}^{-x} - x$$

Summary
The solution(s) found are the following

$$y = 1 - \mathrm{e}^{-x} - x \tag{1}$$

Verification of solutions

$$y = 1 - \mathrm{e}^{-x} - x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 9

```
dsolve([diff(y(x),x)*(diff(y(x),x)+y(x))=x*(x+y(x)),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 28

```
DSolve[{y'[x]*(y'[x]+y[x])==x*(x+y[x]),{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2}$$
$$y(x) \to -x - e^{-x} + 1$$

2.24 problem 24

Internal problem ID [5772]
Internal file name [OUTPUT/5020_Sunday_June_05_2022_03_17_39_PM_50656827/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 24.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{(xy' + y)^2 - y'y^2 = 0}$$

2.24.1 Solving as homogeneous ode

Solving for y' gives

$$y' = \frac{\left(-2x + y + \sqrt{y^2 - 4xy}\right) y}{2x^2} \tag{1}$$
$$y' = \frac{\left(-2x + y - \sqrt{y^2 - 4xy}\right) y}{2x^2} \tag{2}$$

Now ODE (1) is solved In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{\left(-2x + y + \sqrt{-4xy + y^2}\right) y}{2x^2} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = \left(-2x + y + \sqrt{-4xy + y^2}\right) y$ and $N = 2x^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= \frac{u\left(\sqrt{(u-4)\,u} + u - 2\right)}{2}\\ \frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\frac{u(x)\left(\sqrt{(u(x)-4)u(x)}+u(x)-2\right)}{2} - u(x)}{x}\end{aligned}$$

Or

$$u'(x) - \frac{\frac{u(x)\left(\sqrt{(u(x)-4)u(x)}+u(x)-2\right)}{2} - u(x)}{x} = 0$$

Or

$$2u'(x)\,x - u(x)^2 - u(x)\sqrt{(u(x)-4)\,u(x)} + 4u(x) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{u\left(\sqrt{(u-4)\,u} + u - 4\right)}{2x}\end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(u) = u\left(\sqrt{(u-4)\,u} + u - 4\right)$. Integrating both sides gives

$$\frac{1}{u\left(\sqrt{(u-4)\,u} + u - 4\right)}\,du = \frac{1}{2x}\,dx$$

$$\int \frac{1}{u\left(\sqrt{(u-4)\,u} + u - 4\right)}\,du = \int \frac{1}{2x}\,dx$$

$$-\frac{\sqrt{u^2-4u}}{16} + \frac{\ln\left(-2+u+\sqrt{u^2-4u}\right)}{8} + \frac{\sqrt{(u-4)^2 - 16 + 4u}}{16} + \frac{\ln\left(-2+u+\sqrt{(u-4)^2-16+4u}\right)}{8} - \frac{\ln(u)}{4} = \frac{\ln(x)}{2} + c_2$$

The solution is

$$-\frac{\sqrt{u\left(x\right)^{2}-4u\left(x\right)}}{16}+\frac{\ln\left(-2+u(x)+\sqrt{u\left(x\right)^{2}-4u\left(x\right)}\right)}{8}$$
$$+\frac{\sqrt{\left(u\left(x\right)-4\right)^{2}-16+4u\left(x\right)}}{16}$$
$$+\frac{\ln\left(-2+u(x)+\sqrt{\left(u\left(x\right)-4\right)^{2}-16+4u\left(x\right)}\right)}{8}-\frac{\ln\left(u(x)\right)}{4}-\frac{\ln\left(x\right)}{2}-c_{2}=0$$

Now u in the above solution is replaced back by y using $u=\frac{y}{x}$ which results in the solution

$$-\frac{\sqrt{\frac{y^{2}}{x^{2}}-\frac{4y}{x}}}{16}+\frac{\ln\left(-2+\frac{y}{x}+\sqrt{\frac{y^{2}}{x^{2}}-\frac{4y}{x}}\right)}{8}+\frac{\sqrt{\left(\frac{y}{x}-4\right)^{2}-16+\frac{4y}{x}}}{16}+\frac{\ln\left(-2+\frac{y}{x}+\sqrt{\left(\frac{y}{x}-4\right)^{2}-16+\frac{4y}{x}}\right)}{8}-$$

Now ODE (2) is solved In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= \frac{\left(-2x+y-\sqrt{-4xy+y^{2}}\right)y}{2x^{2}} \end{aligned} \tag{1}$$

An ode of the form $y'=\frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^{n}x,t^{n}y)=t^{n}f(x,y)$$

In this case, it can be seen that both $M=-\left(2x-y+\sqrt{-4xy+y^{2}}\right)y$ and $N=2x^{2}$ are both homogeneous and of the same order $n=2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u=\frac{y}{x}$, or $y=ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}u}{\mathrm{d}x}x+u$$

Applying the transformation $y=ux$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}x}x+u &= -\frac{u\left(\sqrt{\left(u-4\right)u}-u+2\right)}{2}\\ \frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{-\frac{u(x)\left(\sqrt{\left(u(x)-4\right)u(x)}-u(x)+2\right)}{2}-u(x)}{x} \end{aligned}$$

Or

$$u'(x) - \frac{-\frac{u(x)\left(\sqrt{(u(x)-4)u(x)}-u(x)+2\right)}{2} - u(x)}{x} = 0$$

Or

$$2u'(x)\,x - u(x)^2 + u(x)\sqrt{(u(x)-4)\,u(x)} + 4u(x) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u\left(-\sqrt{(u-4)\,u}+u-4\right)}{2x}
\end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(u) = u\left(-\sqrt{(u-4)\,u}+u-4\right)$. Integrating both sides gives

$$\frac{1}{u\left(-\sqrt{(u-4)\,u}+u-4\right)}\,du = \frac{1}{2x}\,dx$$

$$\int \frac{1}{u\left(-\sqrt{(u-4)\,u}+u-4\right)}\,du = \int \frac{1}{2x}\,dx$$

$$\frac{\sqrt{u^2-4u}}{16} - \frac{\ln\left(-2+u+\sqrt{u^2-4u}\right)}{8} - \frac{\sqrt{(u-4)^2-16+4u}}{16} - \frac{\ln\left(-2+u+\sqrt{(u-4)^2-16+4u}\right)}{8} - \frac{\ln(u)}{4} = \frac{\ln(x)}{2} + c_4$$

The solution is

$$\frac{\sqrt{u(x)^2-4u(x)}}{16} - \frac{\ln\left(-2+u(x)+\sqrt{u(x)^2-4u(x)}\right)}{8} - \frac{\sqrt{(u(x)-4)^2-16+4u(x)}}{16} - \frac{\ln\left(-2+u(x)+\sqrt{(u(x)-4)^2-16+4u(x)}\right)}{8} - \frac{\ln(u(x))}{4} - \frac{\ln(x)}{2} - c_4 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\frac{\sqrt{\frac{y^2}{x^2}-\frac{4y}{x}}}{16}-\frac{\ln\left(-2+\frac{y}{x}+\sqrt{\frac{y^2}{x^2}-\frac{4y}{x}}\right)}{8}-\frac{\sqrt{\left(\frac{y}{x}-4\right)^2-16+\frac{4y}{x}}}{16}-\frac{\ln\left(-2+\frac{y}{x}+\sqrt{\left(\frac{y}{x}-4\right)^2-16+\frac{4y}{x}}\right)}{8}-\text{l}$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{\sqrt{\frac{y^2-4xy}{x^2}}\,x+y-2x}{x}\right)}{4}-\frac{\ln\left(\frac{y}{x}\right)}{4}-\frac{\ln(x)}{2}-c_2=0 \tag{1}$$

$$-\frac{\ln\left(\frac{\sqrt{\frac{y^2-4xy}{x^2}}\,x+y-2x}{x}\right)}{4}-\frac{\ln\left(\frac{y}{x}\right)}{4}-\frac{\ln(x)}{2}-c_4=0 \tag{2}$$

Verification of solutions

$$\frac{\ln\left(\frac{\sqrt{\frac{y^2-4xy}{x^2}}\,x+y-2x}{x}\right)}{4}-\frac{\ln\left(\frac{y}{x}\right)}{4}-\frac{\ln(x)}{2}-c_2=0$$

Verified OK. `{0 < x}`

$$-\frac{\ln\left(\frac{\sqrt{\frac{y^2-4xy}{x^2}}\,x+y-2x}{x}\right)}{4}-\frac{\ln\left(\frac{y}{x}\right)}{4}-\frac{\ln(x)}{2}-c_4=0$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   <- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 124

```
dsolve((x*diff(y(x),x)+y(x))^2=y(x)^2*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = 4x$$
$$y(x) = 0$$
$$y(x) = -\frac{2c_1^2(-\sqrt{2}\,c_1 + x)}{-2c_1^2 + x^2}$$
$$y(x) = -\frac{2c_1^2(\sqrt{2}\,c_1 + x)}{-2c_1^2 + x^2}$$
$$y(x) = \frac{c_1^3\sqrt{2} - 2c_1^2 x}{-2c_1^2 + 4x^2}$$
$$y(x) = \frac{c_1^2(\sqrt{2}\,c_1 + 2x)}{2c_1^2 - 4x^2}$$

✓ Solution by Mathematica

Time used: 0.501 (sec). Leaf size: 62

```
DSolve[(x*y'[x]+y[x])^2==y[x]^2*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{4e^{-2c_1}}{2 + e^{2c_1}x}$$
$$y(x) \to -\frac{e^{-2c_1}}{2 + 4e^{2c_1}x}$$
$$y(x) \to 0$$
$$y(x) \to 4x$$

2.25 problem 25

Internal problem ID [5773]
Internal file name [OUTPUT/5021_Sunday_June_05_2022_03_17_45_PM_40910443/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 25.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x^2 y'^2 - 3xyy' + 2y^2 = 0}$$

2.25.1 Solving as homogeneous ode

Solving for y' gives

$$y' = \frac{y}{x} \tag{1}$$
$$y' = \frac{2y}{x} \tag{2}$$

Now ODE (1) is solved In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = y$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}\frac{du}{dx}x + u &= u\\ \frac{du}{dx} &= 0\end{aligned}$$

Or

$$u'(x) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. Integrating both sides gives

$$\begin{aligned}u(x) &= \int 0 \, dx\\ &= c_2\end{aligned}$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$y = c_2 x$$

Now ODE (2) is solved In canonical form, the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= \frac{2y}{x}\end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = 2y$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{du}{dx}x + u = 2u$$
$$\frac{du}{dx} = \frac{u(x)}{x}$$

Or

$$u'(x) - \frac{u(x)}{x} = 0$$

Or

$$u'(x)\,x - u(x) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= \frac{1}{x}\,dx \\ \int \frac{1}{u}\,du &= \int \frac{1}{x}\,dx \\ \ln(u) &= \ln(x) + c_4 \\ u &= \mathrm{e}^{\ln(x)+c_4} \\ &= c_4 x \end{aligned}$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$y = c_4 x^2$$

Summary

The solution(s) found are the following

$$y = c_2 x \tag{1}$$
$$y = c_4 x^2 \tag{2}$$

Verification of solutions

$$y = c_2 x$$

Verified OK.

$$y = c_4 x^2$$

Verified OK.

2.25.2 Maple step by step solution

Let's solve

$x^2y'^2 - 3xyy' + 2y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = x\,\mathrm{e}^{c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x)^2-3*x*y(x)*diff(y(x),x)+2*y(x)^2=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2$$
$$y(x) = c_1 x$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 24

```
DSolve[x^2*(y'[x])^2-3*x*y[x]*y'[x]+2*y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x$$
$$y(x) \to c_1 x^2$$
$$y(x) \to 0$$

2.26 problem 26

Internal problem ID [5774]
Internal file name [`OUTPUT/5022_Sunday_June_05_2022_03_17_48_PM_88856494/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{-y + xy' - \sqrt{x^2 + y^2} = 0}$$

2.26.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y + \sqrt{x^2 + y^2}}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = y + \sqrt{x^2 + y^2}$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= u + \sqrt{u^2+1} \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\sqrt{u(x)^2+1}}{x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{\sqrt{u(x)^2+1}}{x} = 0
$$

Or

$$
u'(x)\,x - \sqrt{u(x)^2+1} = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{\sqrt{u^2+1}}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \sqrt{u^2+1}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\sqrt{u^2+1}}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{\sqrt{u^2+1}}\,du &= \int \frac{1}{x}\,dx \\
\operatorname{arcsinh}(u) &= \ln(x) + c_2
\end{aligned}
$$

The solution is

$$
\operatorname{arcsinh}(u(x)) - \ln(x) - c_2 = 0
$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$
\operatorname{arcsinh}\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0
$$

Summary

The solution(s) found are the following

$$
\operatorname{arcsinh}\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}
$$

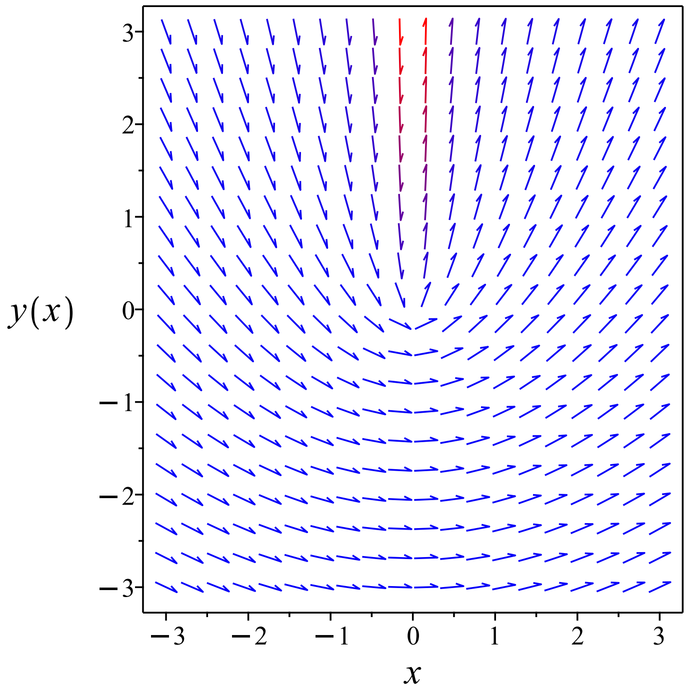


Figure 68: Slope field plot

Verification of solutions

$$\operatorname{arcsinh}\left(\frac{y}{x}\right)-\ln\left(x\right)-c_2=0$$

Verified OK. {0 < x}

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(x*diff(y(x),x)-y(x)=sqrt(x^2+y(x)^2),y(x), singsol=all)
```

$$\frac{-c_1 x^2 + y(x) + \sqrt{x^2 + y(x)^2}}{x^2} = 0$$

✓ Solution by Mathematica

Time used: 0.331 (sec). Leaf size: 27

```
DSolve[x*y'[x]-y[x]==Sqrt[x^2+y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-c_1} \left(-1 + e^{2c_1} x^2\right)$$

2.27 problem 27

Internal problem ID [5775]
Internal file name [`OUTPUT/5023_Sunday_June_05_2022_03_17_51_PM_23504009/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 27.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{yy'^2 + 2xy' - y = 0}$$

2.27.1 Solving as homogeneous ode

Solving for y' gives

$$y' = \frac{-x + \sqrt{x^2 + y^2}}{y} \tag{1}$$

$$y' = -\frac{x + \sqrt{x^2 + y^2}}{y} \tag{2}$$

Now ODE (1) is solved In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{-x + \sqrt{x^2 + y^2}}{y} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = -x + \sqrt{x^2 + y^2}$ and $N = y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = \frac{\sqrt{u^2+1} - 1}{u}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\frac{\sqrt{u(x)^2+1}-1}{u(x)} - u(x)}{x}$$

Or

$$u'(x) - \frac{\frac{\sqrt{u(x)^2+1}-1}{u(x)} - u(x)}{x} = 0$$

Or

$$u'(x)\, u(x)\, x + u(x)^2 - \sqrt{u(x)^2 + 1} + 1 = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2 - \sqrt{u^2+1} + 1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2 - \sqrt{u^2+1}+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-\sqrt{u^2+1}+1}{u}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u^2-\sqrt{u^2+1}+1}{u}}\, du = \int -\frac{1}{x}\, dx$$

$$-\operatorname{arctanh}\left(\frac{1}{\sqrt{u^2+1}}\right) + \ln(u) = -\ln(x) + c_2$$

The solution is

$$-\operatorname{arctanh}\left(\frac{1}{\sqrt{u(x)^2+1}}\right) + \ln(u(x)) + \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$-\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{y^2}{x^2}+1}}\right)+\ln\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Now ODE (2) is solved In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x+\sqrt{x^2+y^2}}{y} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = -x - \sqrt{x^2+y^2}$ and $N = y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dx}x + u &= \frac{-\sqrt{u^2+1}-1}{u} \\ \frac{du}{dx} &= \frac{\frac{-\sqrt{u(x)^2+1}-1}{u(x)} - u(x)}{x} \end{aligned}$$

Or

$$u'(x) - \frac{\frac{-\sqrt{u(x)^2+1}-1}{u(x)} - u(x)}{x} = 0$$

Or

$$u'(x)\,u(x)\,x + u(x)^2 + \sqrt{u(x)^2+1} + 1 = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2+\sqrt{u^2+1}+1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+\sqrt{u^2+1}+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+\sqrt{u^2+1}+1}{u}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u^2+\sqrt{u^2+1}+1}{u}}\, du = \int -\frac{1}{x}\, dx$$

$$\operatorname{arctanh}\left(\frac{1}{\sqrt{u^2+1}}\right) + \ln(u) = -\ln(x) + c_4$$

The solution is

$$\operatorname{arctanh}\left(\frac{1}{\sqrt{u(x)^2+1}}\right) + \ln(u(x)) + \ln(x) - c_4 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{y^2}{x^2}+1}}\right) + \ln\left(\frac{y}{x}\right) + \ln(x) - c_4 = 0$$

Summary

The solution(s) found are the following

$$-\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{x^2+y^2}{x^2}}}\right) + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

$$\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{x^2+y^2}{x^2}}}\right) + \ln\left(\frac{y}{x}\right) + \ln(x) - c_4 = 0 \tag{2}$$

Verification of solutions

$$-\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{x^2+y^2}{x^2}}}\right) + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Verified OK. `{0 < x}`

$$\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{x^2+y^2}{x^2}}}\right) + \ln\left(\frac{y}{x}\right) + \ln(x) - c_4 = 0$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   <- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 71

```
dsolve(y(x)*diff(y(x),x)^2+2*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = -ix$$
$$y(x) = ix$$
$$y(x) = 0$$
$$y(x) = \sqrt{c_1 (c_1 - 2x)}$$
$$y(x) = \sqrt{c_1 (c_1 + 2x)}$$
$$y(x) = -\sqrt{c_1 (c_1 - 2x)}$$
$$y(x) = -\sqrt{c_1 (c_1 + 2x)}$$

✓ Solution by Mathematica

Time used: 0.451 (sec). Leaf size: 126

```
DSolve[y[x]*(y'[x])^2+2*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -e^{\frac{c_1}{2}}\sqrt{-2x+e^{c_1}} \\
y(x) &\to e^{\frac{c_1}{2}}\sqrt{-2x+e^{c_1}} \\
y(x) &\to -e^{\frac{c_1}{2}}\sqrt{2x+e^{c_1}} \\
y(x) &\to e^{\frac{c_1}{2}}\sqrt{2x+e^{c_1}} \\
y(x) &\to 0 \\
y(x) &\to -ix \\
y(x) &\to ix
\end{aligned}
$$

2.28 problem 28

Internal problem ID [5776]
Internal file name [`OUTPUT/5024_Sunday_June_05_2022_03_17_55_PM_58980371/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{2y + x}{x} = 0}$$

2.28.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{2y + x}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = -2y - x$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is

homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{du}{dx}x + u = -2u - 1$$
$$\frac{du}{dx} = \frac{-3u(x) - 1}{x}$$

Or

$$u'(x) - \frac{-3u(x) - 1}{x} = 0$$

Or

$$u'(x)\,x + 3u(x) + 1 = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-3u - 1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -3u - 1$. Integrating both sides gives

$$\frac{1}{-3u - 1}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{-3u - 1}\,du = \int \frac{1}{x}\,dx$$
$$-\frac{\ln(-3u - 1)}{3} = \ln(x) + c_2$$

Raising both side to exponential gives

$$\frac{1}{(-3u - 1)^{\frac{1}{3}}} = e^{\ln(x) + c_2}$$

Which simplifies to

$$\frac{1}{(-3u - 1)^{\frac{1}{3}}} = c_3 x$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$y = -\frac{(c_3^3 x^3 e^{3c_2} + 1) e^{-3c_2}}{3x^2 c_3^3}$$

Summary

The solution(s) found are the following

$$y = -\frac{(c_3^3 x^3 e^{3c_2} + 1) e^{-3c_2}}{3x^2 c_3^3} \tag{1}$$

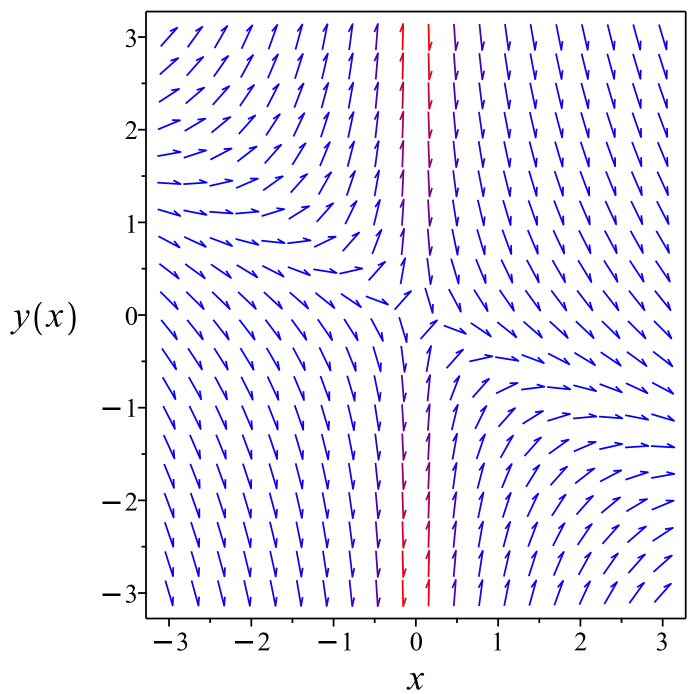


Figure 69: Slope field plot

Verification of solutions

$$y = -\frac{(c_3^3 x^3 e^{3c_2} + 1) e^{-3c_2}}{3x^2 c_3^3}$$

Verified OK.

2.28.2 Maple step by step solution

Let's solve

$y' + \frac{2y+x}{x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -1 - \frac{2y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = -1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = -\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int -\mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int -\mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

  $y = \frac{\int -x^2 dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{x^3}{3} + c_1}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)+(x+2*y(x))/x=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{3} + \frac{c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 17

```
DSolve[y'[x]+(x+2*y[x])/x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{3} + \frac{c_1}{x^2}$$

2.29 problem 29

Internal problem ID [5777]
Internal file name [`OUTPUT/5025_Sunday_June_05_2022_03_17_57_PM_13065767/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{y}{x+y} = 0}$$

2.29.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y}{x+y} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = y$ and $N = x + y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is

homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}\frac{du}{dx}x + u &= \frac{u}{u+1} \\ \frac{du}{dx} &= \frac{\frac{u(x)}{u(x)+1} - u(x)}{x}\end{aligned}$$

Or

$$u'(x) - \frac{\frac{u(x)}{u(x)+1} - u(x)}{x} = 0$$

Or

$$u'(x)\,xu(x) + u'(x)\,x + u(x)^2 = 0$$

Or

$$(u(x) + 1)\,xu'(x) + u(x)^2 = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u^2}{(u+1)\,x}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2}{u+1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u^2}{u+1}}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{\frac{u^2}{u+1}}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u) - \frac{1}{u} &= -\ln(x) + c_2\end{aligned}$$

The solution is

$$\ln(u(x)) - \frac{1}{u(x)} + \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\ln\left(\frac{y}{x}\right) - \frac{x}{y} + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\ln\left(\frac{y}{x}\right) - \frac{x}{y} + \ln(x) - c_2 = 0 \tag{1}$$

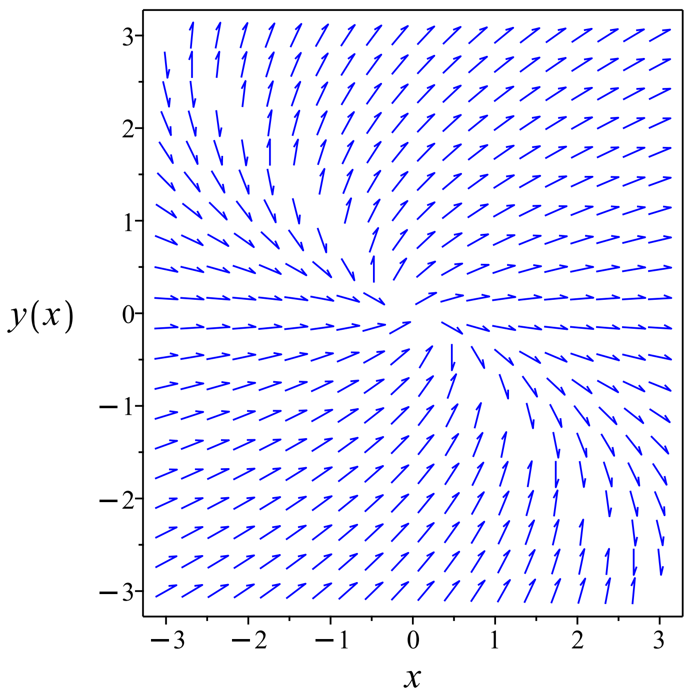


Figure 70: Slope field plot

Verification of solutions

$$\ln\left(\frac{y}{x}\right) - \frac{x}{y} + \ln(x) - c_2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)=y(x)/(x+y(x)),y(x), singsol=all)
```

$$y(x) = \frac{x}{\text{LambertW}\left(x\,e^{c_1}\right)}$$

✓ Solution by Mathematica

Time used: 3.517 (sec). Leaf size: 23

```
DSolve[y'[x]==y[x]/(x+y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{W\left(e^{-c_1}x\right)}$$
$$y(x) \to 0$$

2.30 problem 30

Internal problem ID [5778]
Internal file name [`OUTPUT/5026_Sunday_June_05_2022_03_17_58_PM_95821168/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

Unable to solve or complete the solution.

$$\boxed{xy' - \frac{y}{2} = x}$$

With initial conditions

$$[y(0) = 0]$$

2.30.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{2x}$$
$$q(x) = 1$$

Hence the ode is

$$y' - \frac{y}{2x} = 1$$

The domain of $p(x) = -\frac{1}{2x}$ is

$$\{x < 0 \lor 0 < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

2.30.2 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{2x + y}{2x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = 2x + y$ and $N = 2x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dx}x + u &= 1 + \frac{u}{2} \\ \frac{du}{dx} &= \frac{1 - \frac{u(x)}{2}}{x} \end{aligned}$$

Or

$$u'(x) - \frac{1 - \frac{u(x)}{2}}{x} = 0$$

Or

$$2u'(x)\, x + u(x) - 2 = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-\frac{u}{2}+1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -\frac{u}{2} + 1$. Integrating both sides gives

$$\frac{1}{-\frac{u}{2}+1}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{-\frac{u}{2}+1}\,du = \int \frac{1}{x}\,dx$$

$$-2\ln(u-2) = \ln(x) + c_2$$

Raising both side to exponential gives

$$\frac{1}{(u-2)^2} = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{1}{(u-2)^2} = c_3 x$$

Which simplifies to

$$\frac{1}{(u(x)-2)^2} = c_3 x\,\mathrm{e}^{c_2}$$

The solution is

$$\frac{1}{(u(x)-2)^2} = c_3 x\,\mathrm{e}^{c_2}$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\frac{1}{\left(\frac{y}{x}-2\right)^2} = c_3 x\,\mathrm{e}^{c_2}$$

Which simplifies to

$$\frac{x}{(2x-y)^2} = c_3\mathrm{e}^{c_2}$$

<u>Verification of solutions</u> N/A

2.30.3 Maple step by step solution

Let's solve

$\left[x y' -\frac{y}{2}=x, y \left(0\right)=0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=1+\frac{y}{2 x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'-\frac{y}{2 x}=1$

- The ODE is linear; multiply by an integrating factor $\mu \left(x \right)$

  $\mu \left(x \right) \left(y'-\frac{y}{2 x}\right)=\mu \left(x \right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{d x}\left(\mu \left(x \right) y\right)$

  $\mu \left(x \right) \left(y'-\frac{y}{2 x}\right)=\mu' \left(x \right) y+\mu \left(x \right) y'$

- Isolate $\mu' \left(x \right)$

  $\mu' \left(x \right)=-\frac{\mu \left(x \right)}{2 x}$

- Solve to find the integrating factor

  $\mu \left(x \right)=\frac{1}{\sqrt{x}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{d x}\left(\mu \left(x \right) y\right)\right)d x =\int \mu \left(x \right)d x +c_{1}$

- Evaluate the integral on the lhs

  $\mu \left(x \right) y=\int \mu \left(x \right)d x +c_{1}$

- Solve for y

  $y=\frac{\int \mu \left(x \right)d x +c_{1}}{\mu \left(x \right)}$

- Substitute $\mu \left(x \right)=\frac{1}{\sqrt{x}}$

  $y=\sqrt{x}\, \left(\int \frac{1}{\sqrt{x}}d x +c_{1}\right)$

- Evaluate the integrals on the rhs

  $y=\sqrt{x}\, \left(2 \sqrt{x}+c_{1}\right)$

- Use initial condition $y \left(0\right)=0$

$0 = 0$

- Solve for c_1

  $c_1 = c_1$

- Substitute $c_1 = c_1$ into general solution and simplify

  $y = \sqrt{x}\,(2\sqrt{x} + c_1)$

- Solution to the IVP

  $y = \sqrt{x}\,(2\sqrt{x} + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 13

```
dsolve([x*diff(y(x),x)=x+1/2*y(x),y(0) = 0],y(x), singsol=all)
```

$$y(x) = 2x + \sqrt{x}\,c_1$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 17

```
DSolve[{x*y'[x]==x+1/2*y[x],{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2x + c_1\sqrt{x}$$

2.31 problem Example 3

Internal problem ID [5779]
Internal file name [`OUTPUT/5027_Sunday_June_05_2022_03_18_00_PM_4330689/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: Example 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{x + y - 2}{y - x - 4} = 0}$$

2.31.1 Solving as polynomial ode

This is ODE of type polynomial. Where the RHS of the ode is ratio of equations of two lines. Writing the ODE in the form

$$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_3}$$

Where $a_1 = -1, b_1 = -1, c_1 = 2, a_2 = 1, b_2 = -1, c_2 = 4$. There are now two possible solution methods. The first case is when the two lines $a_1x + b_1y + c_1$,$a_2x + b_2y + c_3$ are not parallel and the second case is if they are parallel. If they are not parallel, then the transformation $X = x - x_0$, $Y = y - y_0$ converts the ODE to a homogeneous ODE. The values x_0, y_0 have to be determined. If they are parallel then a transformation $U(x) = a_1x + b_1y$ converts the given ODE in y to a separable ODE in $U(x)$. The first case is when $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ and the second case when $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. From the above we see that

$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$. Hence this is case one where lines are not parallel. Using the transformation

$$
\begin{aligned}
X &= x - x_0 \\
Y &= y - y_0
\end{aligned}
$$

Where the constants x_0, y_0 are obtained by solving the following two linear algebraic equations

$$
\begin{aligned}
a_1 x_0 + b_1 y_0 + c_1 &= 0 \\
a_2 x_0 + b_2 y_0 + c_2 &= 0
\end{aligned}
$$

Substituting the values for $a_1, b_1, c_1, a_2, b_2, c_2$ gives

$$
\begin{aligned}
-x_0 - y_0 + 2 &= 0 \\
x_0 - y_0 + 4 &= 0
\end{aligned}
$$

Solving for x_0, y_0 from the above gives

$$
\begin{aligned}
x_0 &= -1 \\
y_0 &= 3
\end{aligned}
$$

Therefore the transformation becomes

$$
\begin{aligned}
X &= x + 1 \\
Y &= y - 3
\end{aligned}
$$

Using this transformation in $y' - \frac{x+y-2}{y-x-4} = 0$ result in

$$
\frac{dY}{dX} = \frac{-X - Y}{-Y + X}
$$

This is now a homogeneous ODE which will now be solved for $Y(X)$. In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{X + Y}{Y - X} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -X - Y$ and $N = -Y + X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{u+1}{u-1}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)+1}{u(X)-1} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{u(X)+1}{u(X)-1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) - \left(\frac{d}{dX}u(X)\right)X + u(X)^2 - 2u(X) - 1 = 0$$

Or

$$X(u(X)-1)\left(\frac{d}{dX}u(X)\right) + u(X)^2 - 2u(X) - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X,u)\\ &= f(X)g(u)\\ &= -\frac{u^2-2u-1}{X(u-1)}\end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-2u-1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-2u-1}{u-1}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2-2u-1}{u-1}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(u^2-2u-1)}{2} = -\ln(X) + c_3$$

Raising both side to exponential gives

$$\sqrt{u^2-2u-1} = \mathrm{e}^{-\ln(X)+c_3}$$

Which simplifies to

$$\sqrt{u^2-2u-1}=\frac{c_4}{X}$$

Which simplifies to

$$\sqrt{u(X)^2-2u(X)-1}=\frac{c_4 e^{c_3}}{X}$$

The solution is

$$\sqrt{u(X)^2-2u(X)-1}=\frac{c_4 e^{c_3}}{X}$$

Now u in the above solution is replaced back by Y using $u=\frac{Y}{X}$ which results in the solution

$$\sqrt{\frac{Y(X)^2}{X^2}-\frac{2Y(X)}{X}-1}=\frac{c_4 e^{c_3}}{X}$$

The solution is implicit $\sqrt{\frac{Y(X)^2-2Y(X)X-X^2}{X^2}}=\frac{c_4 e^{c_3}}{X}$. Replacing $Y=y-y_0, X=x-x_0$ gives

$$\sqrt{\frac{-(1+x)^2-2(y-3)(1+x)+(y-3)^2}{(1+x)^2}}=\frac{c_4 e^{c_3}}{1+x}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{-(1+x)^2-2(y-3)(1+x)+(y-3)^2}{(1+x)^2}}=\frac{c_4 e^{c_3}}{1+x} \tag{1}$$

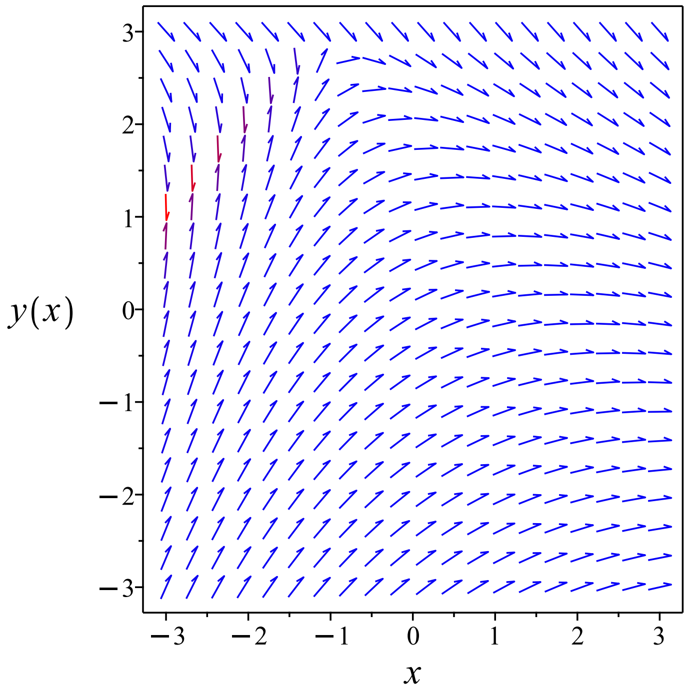


Figure 71: Slope field plot

Verification of solutions

$$\sqrt{\frac{-\left(1+x\right)^{2}-2\left(y-3\right)\left(1+x\right)+\left(y-3\right)^{2}}{\left(1+x\right)^{2}}}=\frac{c_{4} \mathrm{e}^{c_{3}}}{1+x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.265 (sec). Leaf size: 30

```
dsolve(diff(y(x),x)=(x+y(x)-2)/(y(x)-x-4),y(x), singsol=all)
```

$$y(x) = \frac{-\sqrt{2\,(x+1)^2\,c_1^2+1}+(x+4)\,c_1}{c_1}$$

✓ Solution by Mathematica

Time used: 0.807 (sec). Leaf size: 59

```
DSolve[y'[x]==(x+y[x]-2)/(y[x]-x-4),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -i\sqrt{-2x^2-4x-16-c_1}+x+4$$
$$y(x) \to i\sqrt{-2x^2-4x-16-c_1}+x+4$$

2.32 problem Example 4

Internal problem ID [5780]
Internal file name [`OUTPUT/5028_Sunday_June_05_2022_03_18_03_PM_69075019/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: Example 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-4y+(x+y-2)\,y'=-2x-6}$$

2.32.1 Solving as polynomial ode

This is ODE of type polynomial. Where the RHS of the ode is ratio of equations of two lines. Writing the ODE in the form

$$y'=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_3}$$

Where $a_1=-2, b_1=4, c_1=-6, a_2=1, b_2=1, c_2=-2$. There are now two possible solution methods. The first case is when the two lines $a_1x+b_1y+c_1$,$a_2x+b_2y+c_3$ are not parallel and the second case is if they are parallel. If they are not parallel, then the transformation $X=x-x_0$, $Y=y-y_0$ converts the ODE to a homogeneous ODE. The values x_0, y_0 have to be determined. If they are parallel then a transformation $U(x)=a_1x+b_1y$ converts the given ODE in y to a separable ODE in $U(x)$. The first case is when $\frac{a_1}{b_1}\neq\frac{a_2}{b_2}$ and the second case when $\frac{a_1}{b_1}=\frac{a_2}{b_2}$. From the above we see that $\frac{a_1}{b_1}\neq\frac{a_2}{b_2}$. Hence this is case one where lines are not parallel. Using the transformation

$$X=x-x_0$$
$$Y=y-y_0$$

Where the constants x_0, y_0 are obtained by solving the following two linear algebraic equations

$$
\begin{aligned}
a_1x_0 + b_1y_0 + c_1 &= 0 \\
a_2x_0 + b_2y_0 + c_2 &= 0
\end{aligned}
$$

Substituting the values for $a_1, b_1, c_1, a_2, b_2, c_2$ gives

$$
\begin{aligned}
-2x_0 + 4y_0 - 6 &= 0 \\
x_0 + y_0 - 2 &= 0
\end{aligned}
$$

Solving for x_0, y_0 from the above gives

$$
\begin{aligned}
x_0 &= \frac{1}{3} \\
y_0 &= \frac{5}{3}
\end{aligned}
$$

Therefore the transformation becomes

$$
\begin{aligned}
X &= x - \frac{1}{3} \\
Y &= y - \frac{5}{3}
\end{aligned}
$$

Using this transformation in $-4y + (x + y - 2)\, y' = -2x - 6$ result in

$$
\frac{dY}{dX} = \frac{-2X + 4Y}{X + Y}
$$

This is now a homogeneous ODE which will now be solved for $Y(X)$. In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{-2X + 4Y}{X + Y} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -2X + 4Y$ and $N = X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{4u-2}{u+1}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{4u(X)-2}{u(X)+1} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{4u(X)-2}{u(X)+1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) + \left(\frac{d}{dX}u(X)\right)X + u(X)^2 - 3u(X) + 2 = 0$$

Or

$$(u(X)+1)\,X\left(\frac{d}{dX}u(X)\right) + u(X)^2 - 3u(X) + 2 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X,u)\\ &= f(X)g(u)\\ &= -\frac{u^2-3u+2}{(u+1)\,X}\end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-3u+2}{u+1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u^2-3u+2}{u+1}}\,du &= -\frac{1}{X}\,dX\\ \int \frac{1}{\frac{u^2-3u+2}{u+1}}\,du &= \int -\frac{1}{X}\,dX\\ -2\ln(u-1) + 3\ln(u-2) &= -\ln(X) + c_3\end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{-2\ln(u-1)+3\ln(u-2)} = \mathrm{e}^{-\ln(X)+c_3}$$

Which simplifies to

$$\frac{(u-2)^3}{(u-1)^2} = \frac{c_4}{X}$$

The solution is

$$\frac{(u(X)-2)^3}{(u(X)-1)^2} = \frac{c_4}{X}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\left(\frac{Y(X)}{X}-2\right)^3}{\left(\frac{Y(X)}{X}-1\right)^2} = \frac{c_4}{X}$$

Which simplifies to

$$-\frac{(-Y(X)+2X)^3}{(-Y(X)+X)^2} = c_4$$

The solution is implicit $-\frac{(-Y(X)+2X)^3}{(-Y(X)+X)^2} = c_4$. Replacing $Y = y - y_0, X = x - x_0$ gives

$$-\frac{(-y+1+2x)^3}{\left(\frac{4}{3}+x-y\right)^2} = c_4$$

Summary
The solution(s) found are the following

$$-\frac{(-y+1+2x)^3}{\left(\frac{4}{3}+x-y\right)^2} = c_4 \tag{1}$$

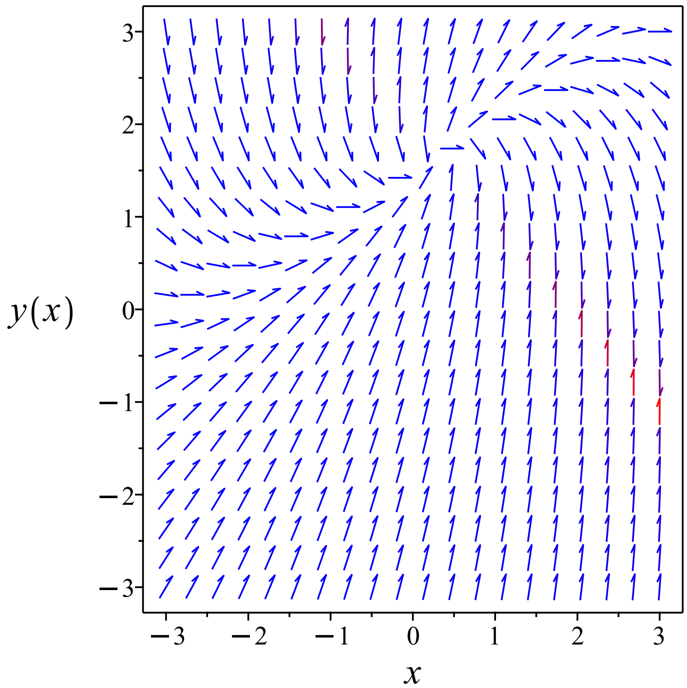


Figure 72: Slope field plot

Verification of solutions

$$-\frac{(-y+1+2x)^3}{\left(\frac{4}{3}+x-y\right)^2} = c_4$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 198

```
dsolve((2*x-4*y(x)+6)+(x+y(x)-2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{2\left(\left(\frac{i\sqrt{3}}{72}-\frac{1}{72}\right)\left(36\sqrt{3}\left(x-\frac{1}{3}\right)c_1^2\sqrt{\frac{243\left(x-\frac{1}{3}\right)^2 c_1-12x+4}{c_1}}+8+972\left(x-\frac{1}{3}\right)^2 c_1^2+\left(-216x+72\right)c_1\right)^{\frac{2}{3}}+\left(\cdots\right.}{\left(36\sqrt{3}\left(x-\frac{1}{3}\right)c_1^2\sqrt{\frac{243(x\cdots}{}}\right.}$$

✓ Solution by Mathematica

Time used: 60.144 (sec). Leaf size: 2563

```
DSolve[(2*x-4*y[x]+6)+(x+y[x]-2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

2.33 problem 31

Internal problem ID [5781]
Internal file name [`OUTPUT/5029_Sunday_June_05_2022_03_18_06_PM_70744627/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{2y - x + 5}{2x - y - 4} = 0}$$

2.33.1 Solving as polynomial ode

This is ODE of type polynomial. Where the RHS of the ode is ratio of equations of two lines. Writing the ODE in the form

$$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_3}$$

Where $a_1 = -1, b_1 = 2, c_1 = 5, a_2 = 2, b_2 = -1, c_2 = -4$. There are now two possible solution methods. The first case is when the two lines $a_1x + b_1y + c_1$,$a_2x + b_2y + c_3$ are not parallel and the second case is if they are parallel. If they are not parallel, then the transformation $X = x - x_0$, $Y = y - y_0$ converts the ODE to a homogeneous ODE. The values x_0, y_0 have to be determined. If they are parallel then a transformation $U(x) = a_1x + b_1y$ converts the given ODE in y to a separable ODE in $U(x)$. The first case is when $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ and the second case when $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. From the above we see that

$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$. Hence this is case one where lines are not parallel. Using the transformation

$$
\begin{aligned}
X &= x - x_0 \\
Y &= y - y_0
\end{aligned}
$$

Where the constants x_0, y_0 are obtained by solving the following two linear algebraic equations

$$
\begin{aligned}
a_1 x_0 + b_1 y_0 + c_1 &= 0 \\
a_2 x_0 + b_2 y_0 + c_2 &= 0
\end{aligned}
$$

Substituting the values for $a_1, b_1, c_1, a_2, b_2, c_2$ gives

$$
\begin{aligned}
-x_0 + 2y_0 + 5 &= 0 \\
2x_0 - y_0 - 4 &= 0
\end{aligned}
$$

Solving for x_0, y_0 from the above gives

$$
\begin{aligned}
x_0 &= 1 \\
y_0 &= -2
\end{aligned}
$$

Therefore the transformation becomes

$$
\begin{aligned}
X &= x - 1 \\
Y &= y + 2
\end{aligned}
$$

Using this transformation in $y' - \frac{2y-x+5}{2x-y-4} = 0$ result in

$$
\frac{dY}{dX} = \frac{2Y - X}{2X - Y}
$$

This is now a homogeneous ODE which will now be solved for $Y(X)$. In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{2Y - X}{-2X + Y}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = 2Y - X$ and $N = 2X - Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-2u+1}{u-2}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-2u(X)+1}{u(X)-2} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-2u(X)+1}{u(X)-2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) - 2\left(\frac{d}{dX}u(X)\right)X + u(X)^2 - 1 = 0$$

Or

$$X(u(X)-2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X,u)\\ &= f(X)g(u)\\ &= -\frac{u^2-1}{X(u-2)}\end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-1}{u-2}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u^2-1}{u-2}}\,du &= -\frac{1}{X}\,dX\\ \int \frac{1}{\frac{u^2-1}{u-2}}\,du &= \int -\frac{1}{X}\,dX\\ -\frac{\ln(u-1)}{2} + \frac{3\ln(u+1)}{2} &= -\ln(X) + c_3\end{aligned}$$

The above can be written as

$$
\begin{aligned}
\frac{-\ln (u-1)+3\ln (u+1)}{2} &= -\ln (X)+c_3 \\
-\ln (u-1)+3\ln (u+1) &= (2)\left(-\ln (X)+c_3\right) \\
&= -2\ln (X)+2c_3
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{-\ln(u-1)+3\ln(u+1)} = e^{-2\ln(X)+2c_3}
$$

Which simplifies to

$$
\begin{aligned}
\frac{(u+1)^3}{u-1} &= \frac{2c_3}{X^2} \\
&= \frac{c_4}{X^2}
\end{aligned}
$$

Which simplifies to

$$
\frac{(u(X)+1)^3}{u(X)-1} = \frac{c_4 e^{2c_3}}{X^2}
$$

The solution is

$$
\frac{(u(X)+1)^3}{u(X)-1} = \frac{c_4 e^{2c_3}}{X^2}
$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$
\frac{\left(\frac{Y(X)}{X}+1\right)^3}{\frac{Y(X)}{X}-1} = \frac{c_4 e^{2c_3}}{X^2}
$$

Which simplifies to

$$
-\frac{(Y(X)+X)^3}{-Y(X)+X} = c_4 e^{2c_3}
$$

The solution is implicit $-\frac{(Y(X)+X)^3}{-Y(X)+X} = c_4 e^{2c_3}$. Replacing $Y = y - y_0, X = x - x_0$ gives

$$
-\frac{(y+x+1)^3}{-y-3+x} = c_4 e^{2c_3}
$$

Summary

The solution(s) found are the following

$$
-\frac{(y+x+1)^3}{-y-3+x} = c_4 e^{2c_3} \tag{1}
$$

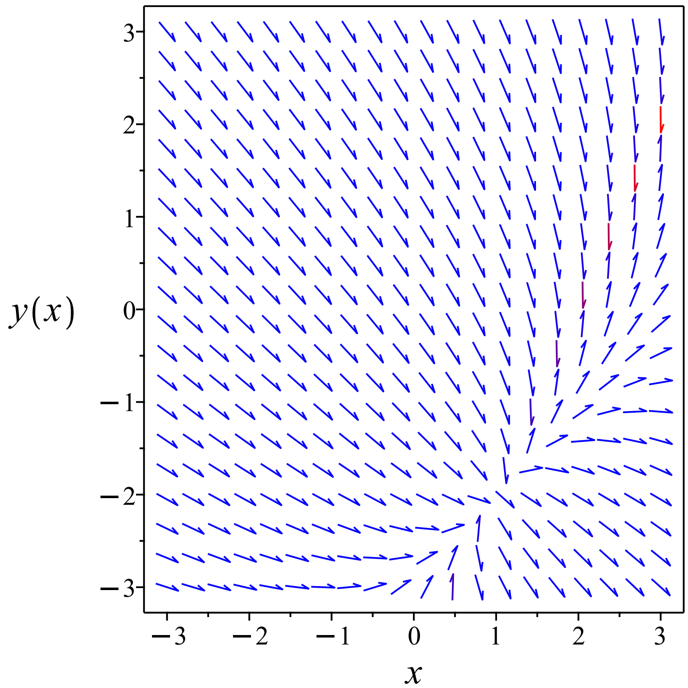


Figure 73: Slope field plot

Verification of solutions

$$-\frac{(y+x+1)^3}{-y-3+x} = c_4 e^{2c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.391 (sec). Leaf size: 117

```
dsolve(diff(y(x),x)=(2*y(x)-x+5)/(2*x-y(x)-4),y(x), singsol=all)
```

$$y(x) = -\frac{\left(i\sqrt{3}-1\right)\left(27c_1(x-1)+3\sqrt{3}\,\sqrt{27\left(x-1\right)^2 c_1^2-1}\right)^{\frac{2}{3}}-3i\sqrt{3}-3+6\left(3\sqrt{3}\,\sqrt{27\left(x-1\right)^2 c_1^2-1+2}\right.}{6\left(27c_1\left(x-1\right)+3\sqrt{3}\,\sqrt{27\left(x-1\right)^2 c_1^2-1}\right)^{\frac{1}{3}} c_1}$$

✓ Solution by Mathematica

Time used: 60.196 (sec). Leaf size: 1601

```
DSolve[y'[x]==(2*y[x]-x+5)/(2*x-y[x]-4),y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

2.34 problem 32

Internal problem ID [5782]
Internal file name [OUTPUT/5030_Sunday_June_05_2022_03_18_09_PM_56810907/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' + \frac{4x + 3y + 15}{2x + y + 7} = 0}$$

2.34.1 Solving as polynomial ode

This is ODE of type polynomial. Where the RHS of the ode is ratio of equations of two lines. Writing the ODE in the form

$$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_3}$$

Where $a_1 = -4, b_1 = -3, c_1 = -15, a_2 = 2, b_2 = 1, c_2 = 7$. There are now two possible solution methods. The first case is when the two lines $a_1x + b_1y + c_1$,$a_2x + b_2y + c_3$ are not parallel and the second case is if they are parallel. If they are not parallel, then the transformation $X = x - x_0$, $Y = y - y_0$ converts the ODE to a homogeneous ODE. The values x_0, y_0 have to be determined. If they are parallel then a transformation $U(x) = a_1x + b_1y$ converts the given ODE in y to a separable ODE in $U(x)$. The first case is when $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ and the second case when $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. From the above we see that

$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$. Hence this is case one where lines are not parallel. Using the transformation

$$
\begin{aligned}
X &= x - x_0 \\
Y &= y - y_0
\end{aligned}
$$

Where the constants x_0, y_0 are obtained by solving the following two linear algebraic equations

$$
\begin{aligned}
a_1 x_0 + b_1 y_0 + c_1 &= 0 \\
a_2 x_0 + b_2 y_0 + c_2 &= 0
\end{aligned}
$$

Substituting the values for $a_1, b_1, c_1, a_2, b_2, c_2$ gives

$$
\begin{aligned}
-4x_0 - 3y_0 - 15 &= 0 \\
2x_0 + y_0 + 7 &= 0
\end{aligned}
$$

Solving for x_0, y_0 from the above gives

$$
\begin{aligned}
x_0 &= -3 \\
y_0 &= -1
\end{aligned}
$$

Therefore the transformation becomes

$$
\begin{aligned}
X &= x + 3 \\
Y &= y + 1
\end{aligned}
$$

Using this transformation in $y' + \frac{4x+3y+15}{2x+y+7} = 0$ result in

$$
\frac{dY}{dX} = \frac{-4X - 3Y}{2X + Y}
$$

This is now a homogeneous ODE which will now be solved for $Y(X)$. In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{4X + 3Y}{2X + Y} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -4X - 3Y$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-3u - 4}{u + 2}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-3u(X)-4}{u(X)+2} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-3u(X)-4}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) + 2\left(\frac{d}{dX}u(X)\right)X + u(X)^2 + 5u(X) + 4 = 0$$

Or

$$X(u(X) + 2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 5u(X) + 4 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X, u)\\ &= f(X)g(u)\\ &= -\frac{u^2 + 5u + 4}{X\,(u + 2)}\end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+5u+4}{u+2}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u^2+5u+4}{u+2}}\,du &= -\frac{1}{X}\,dX\\ \int \frac{1}{\frac{u^2+5u+4}{u+2}}\,du &= \int -\frac{1}{X}\,dX\\ \frac{\ln(u+1)}{3} + \frac{2\ln(u+4)}{3} &= -\ln(X) + c_3\end{aligned}$$

The above can be written as

$$
\begin{aligned}
\frac{\ln(u+1)+2\ln(u+4)}{3} &= -\ln(X)+c_3 \\
\ln(u+1)+2\ln(u+4) &= (3)\left(-\ln(X)+c_3\right) \\
&= -3\ln(X)+3c_3
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{\ln(u+1)+2\ln(u+4)} = e^{-3\ln(X)+3c_3}
$$

Which simplifies to

$$
\begin{aligned}
(u+1)(u+4)^2 &= \frac{3c_3}{X^3} \\
&= \frac{c_4}{X^3}
\end{aligned}
$$

Which simplifies to

$$
(u(X)+1)(u(X)+4)^2 = \frac{c_4 e^{3c_3}}{X^3}
$$

The solution is

$$
(u(X)+1)(u(X)+4)^2 = \frac{c_4 e^{3c_3}}{X^3}
$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$
\left(\frac{Y(X)}{X}+1\right)\left(\frac{Y(X)}{X}+4\right)^2 = \frac{c_4 e^{3c_3}}{X^3}
$$

Which simplifies to

$$
(Y(X)+X)(Y(X)+4X)^2 = c_4 e^{3c_3}
$$

The solution is implicit $(Y(X)+X)(Y(X)+4X)^2 = c_4 e^{3c_3}$. Replacing $Y = y-y_0, X = x-x_0$ gives

$$
(4+y+x)(4x+13+y)^2 = c_4 e^{3c_3}
$$

<u>Summary</u>

The solution(s) found are the following

$$
(4+y+x)(4x+13+y)^2 = c_4 e^{3c_3} \tag{1}
$$

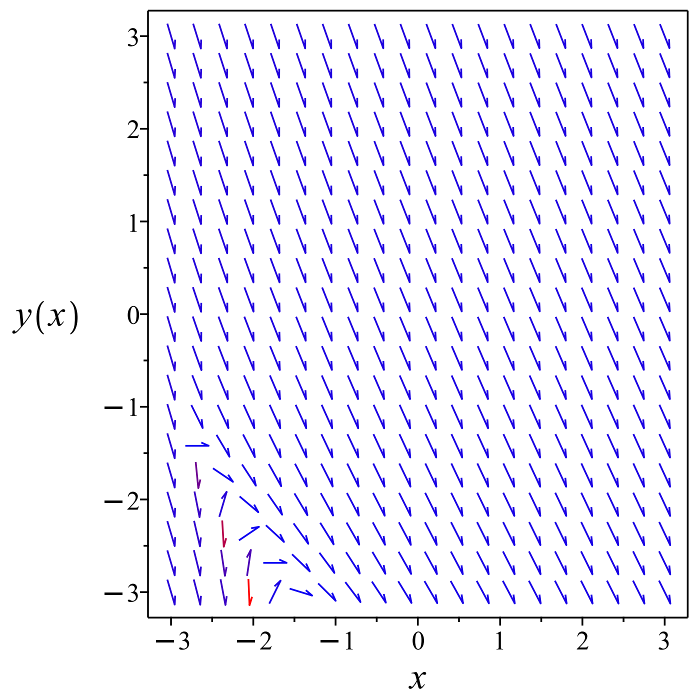


Figure 74: Slope field plot

Verification of solutions

$$(4+y+x)(4x+13+y)^2 = c_4 e^{3c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.234 (sec). Leaf size: 227

```
dsolve(diff(y(x),x)=-(4*x+3*y(x)+15)/(2*x+y(x)+7),y(x), singsol=all)
```

$$y(x) = \frac{-24(x+3)^2 c_1 \left(x+\frac{10}{3}\right)\left(4\sqrt{-4\left(-\frac{1}{4}+(x+3)^3 c_1\right)(x+3)^6 c_1^2}+4(x^3+9x^2+27x+27)c_1\right)^{\frac{2}{3}}+i\left(-1\right.}{}$$

✓ Solution by Mathematica

Time used: 60.066 (sec). Leaf size: 763

```
DSolve[y'[x]==-(4*x+3*y[x]+15)/(2*x+y[x]+7),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{}{\text{Root}\left[\#1^6\left(16x^6+288x^5+2160x^4+8640x^3+19440x^2+23328x+11664+16e^{12c_1}\right)+\#1^4\left(-24x^4-\right.\right.} - 2x - 7$$

$$y(x) \to \frac{}{\text{Root}\left[\#1^6\left(16x^6+288x^5+2160x^4+8640x^3+19440x^2+23328x+11664+16e^{12c_1}\right)+\#1^4\left(-24x^4-\right.\right.} - 2x - 7$$

$$y(x) \to \frac{}{\text{Root}\left[\#1^6\left(16x^6+288x^5+2160x^4+8640x^3+19440x^2+23328x+11664+16e^{12c_1}\right)+\#1^4\left(-24x^4-\right.\right.} - 2x - 7$$

$$y(x) \to \frac{}{\text{Root}\left[\#1^6\left(16x^6+288x^5+2160x^4+8640x^3+19440x^2+23328x+11664+16e^{12c_1}\right)+\#1^4\left(-24x^4-\right.\right.} - 2x - 7$$

$$y(x) \to \frac{}{\text{Root}\left[\#1^6\left(16x^6+288x^5+2160x^4+8640x^3+19440x^2+23328x+11664+16e^{12c_1}\right)+\#1^4\left(-24x^4-\right.\right.} - 2x - 7$$

$$y(x) \to \frac{}{\text{Root}\left[\#1^6\left(16x^6+288x^5+2160x^4+8640x^3+19440x^2+23328x+11664+16e^{12c_1}\right)+\#1^4\left(-24x^4-\right.\right.} - 2x - 7$$

2.35 problem 33

Internal problem ID [5783]
Internal file name [OUTPUT/5031_Sunday_June_05_2022_03_18_12_PM_39213340/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{x + 3y - 5}{x - y - 1} = 0}$$

2.35.1 Solving as polynomial ode

This is ODE of type polynomial. Where the RHS of the ode is ratio of equations of two lines. Writing the ODE in the form

$$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_3}$$

Where $a_1 = 1, b_1 = 3, c_1 = -5, a_2 = 1, b_2 = -1, c_2 = -1$. There are now two possible solution methods. The first case is when the two lines $a_1x + b_1y + c_1$,$a_2x + b_2y + c_3$ are not parallel and the second case is if they are parallel. If they are not parallel, then the transformation $X = x - x_0$, $Y = y - y_0$ converts the ODE to a homogeneous ODE. The values x_0, y_0 have to be determined. If they are parallel then a transformation $U(x) = a_1x + b_1y$ converts the given ODE in y to a separable ODE in $U(x)$. The first case is when $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ and the second case when $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. From the above we see that

$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$. Hence this is case one where lines are not parallel. Using the transformation

$$X = x - x_0$$
$$Y = y - y_0$$

Where the constants x_0, y_0 are obtained by solving the following two linear algebraic equations

$$a_1 x_0 + b_1 y_0 + c_1 = 0$$
$$a_2 x_0 + b_2 y_0 + c_2 = 0$$

Substituting the values for $a_1, b_1, c_1, a_2, b_2, c_2$ gives

$$x_0 + 3y_0 - 5 = 0$$
$$x_0 - y_0 - 1 = 0$$

Solving for x_0, y_0 from the above gives

$$x_0 = 2$$
$$y_0 = 1$$

Therefore the transformation becomes

$$X = x - 2$$
$$Y = y - 1$$

Using this transformation in $y' - \frac{x+3y-5}{x-y-1} = 0$ result in

$$\frac{dY}{dX} = \frac{X + 3Y}{X - Y}$$

This is now a homogeneous ODE which will now be solved for $Y(X)$. In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{X + 3Y}{-X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = X + 3Y$ and $N = X - Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-3u - 1}{u - 1}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-3u(X)-1}{u(X)-1} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-3u(X)-1}{u(X)-1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) - \left(\frac{d}{dX}u(X)\right)X + u(X)^2 + 2u(X) + 1 = 0$$

Or

$$X(u(X) - 1)\left(\frac{d}{dX}u(X)\right) + (u(X) + 1)^2 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X, u)\\ &= f(X)g(u)\\ &= -\frac{(u+1)^2}{X(u-1)}\end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{(u+1)^2}{u-1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{(u+1)^2}{u-1}}\,du &= -\frac{1}{X}\,dX\\ \int \frac{1}{\frac{(u+1)^2}{u-1}}\,du &= \int -\frac{1}{X}\,dX\\ \ln(u+1) + \frac{2}{u+1} &= -\ln(X) + c_3\end{aligned}$$

The solution is

$$\ln\left(u(X)+1\right)+\frac{2}{u\left(X\right)+1}+\ln\left(X\right)-c_3=0$$

Now u in the above solution is replaced back by Y using $u=\frac{Y}{X}$ which results in the solution

$$\ln\left(\frac{Y(X)}{X}+1\right)+\frac{2}{\frac{Y(X)}{X}+1}+\ln\left(X\right)-c_3=0$$

The solution is implicit $\ln\left(\frac{Y(X)+X}{X}\right)+\frac{2X}{Y(X)+X}+\ln\left(X\right)-c_3=0$. Replacing $Y=y-y_0, X=x-x_0$ gives

$$\ln\left(\frac{x+y-3}{-2+x}\right)+\frac{2x-4}{x+y-3}+\ln\left(-2+x\right)-c_3=0$$

Summary

The solution(s) found are the following

$$\ln\left(\frac{x+y-3}{-2+x}\right)+\frac{2x-4}{x+y-3}+\ln\left(-2+x\right)-c_3=0 \tag{1}$$

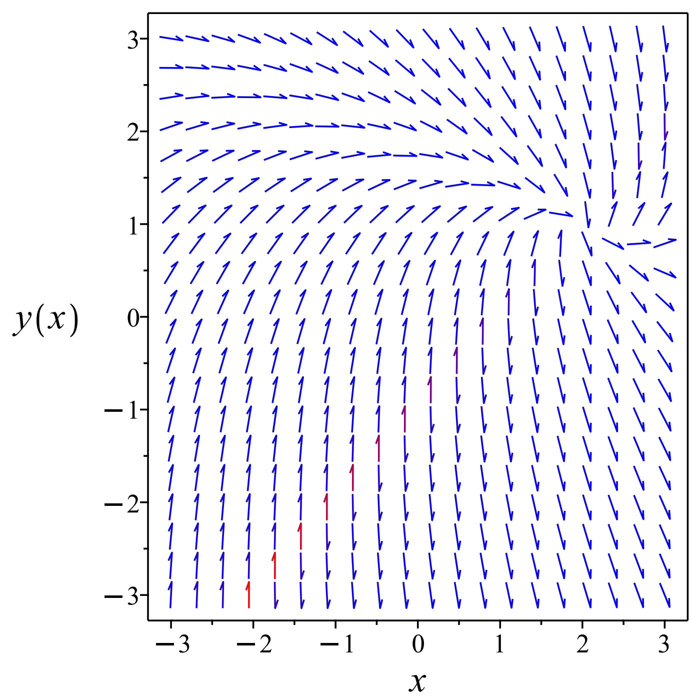


Figure 75: Slope field plot

Verification of solutions

$$\ln\left(\frac{x+y-3}{-2+x}\right)+\frac{2x-4}{x+y-3}+\ln\left(-2+x\right)-c_3=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.188 (sec). Leaf size: 32

```
dsolve(diff(y(x),x)=(x+3*y(x)-5)/(x-y(x)-1),y(x), singsol=all)
```

$$y(x)=\frac{(-x+3)\operatorname{LambertW}\left(2c_1(-2+x)\right)-2x+4}{\operatorname{LambertW}\left(2c_1\left(-2+x\right)\right)}$$

✓ Solution by Mathematica

Time used: 1.041 (sec). Leaf size: 148

```
DSolve[y'[x]==(x+3*y[x]-5)/(x-y[x]-1),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{2^{2/3}\left(x\log\left(-\frac{y(x)+x-3}{-y(x)+x-1}\right)-(x-3)\log\left(\frac{x-2}{-y(x)+x-1}\right)-3\log\left(-\frac{y(x)+x-3}{-y(x)+x-1}\right)-y(x)\left(\log\left(\frac{x-2}{-y(x)+x-1}\right.\right.\right.}{9(y(x)+x-3)}\right.$$

2.36 problem 34

Internal problem ID [5784]
Internal file name [`OUTPUT/5032_Sunday_June_05_2022_03_18_15_PM_44718510/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational]
```

$$\boxed{y' - \frac{2(2+y)^2}{(y+x+1)^2} = 0}$$

2.36.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{2(2+Y(X)+y_0)^2}{(Y(X)+y_0+X+x_0+1)^2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = -2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{2Y(X)^2}{X^2 + 2Y(X)X + Y(X)^2}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{2Y^2}{X^2 + 2YX + Y^2} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = 2Y^2$ and $N = X^2 + 2YX + Y^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{2u^2}{(u+1)^2} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{2u(X)^2}{(u(X)+1)^2} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{\frac{2u(X)^2}{(u(X)+1)^2} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX}u(X)\right)u(X)^2 X + 2\left(\frac{d}{dX}u(X)\right)u(X)\,X + u(X)^3 + \left(\frac{d}{dX}u(X)\right)X + u(X) = 0
$$

Or

$$
X(u(X)+1)^2\left(\frac{d}{dX}u(X)\right) + u(X)^3 + u(X) = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X)g(u) \\
&= -\frac{u(u^2+1)}{X(u+1)^2}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u(u^2+1)}{(u+1)^2}$. Integrating both sides gives

$$\frac{1}{\frac{u(u^2+1)}{(u+1)^2}} du = -\frac{1}{X} dX$$

$$\int \frac{1}{\frac{u(u^2+1)}{(u+1)^2}} du = \int -\frac{1}{X} dX$$

$$2 \arctan(u) + \ln(u) = -\ln(X) + c_2$$

The solution is

$$2 \arctan(u(X)) + \ln(u(X)) + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$2 \arctan\left(\frac{Y(X)}{X}\right) + \ln\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$2 \arctan\left(\frac{Y(X)}{X}\right) + \ln\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 2$$
$$X = 1 + x$$

Then the solution in y becomes

$$2 \arctan\left(\frac{2+y}{x-1}\right) + \ln\left(\frac{2+y}{x-1}\right) + \ln(x-1) - c_2 = 0$$

<u>Summary</u>
The solution(s) found are the following

$$2 \arctan\left(\frac{2+y}{x-1}\right) + \ln\left(\frac{2+y}{x-1}\right) + \ln(x-1) - c_2 = 0 \tag{1}$$

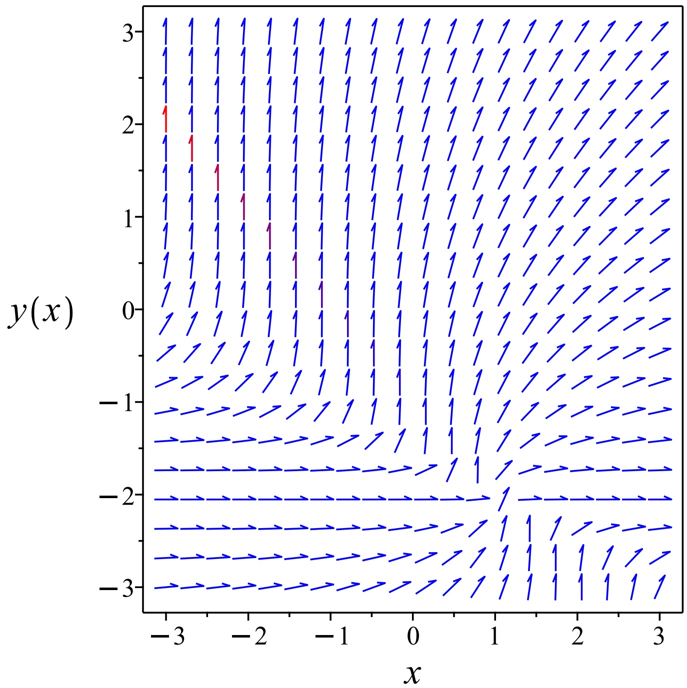


Figure 76: Slope field plot

Verification of solutions

$$2\arctan\left(\frac{2+y}{x-1}\right)+\ln\left(\frac{2+y}{x-1}\right)+\ln(x-1)-c_2=0$$

Verified OK.

2.36.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{2(y+2)^2}{(x+y+1)^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{2(y+2)^2 (b_3 - a_2)}{(x+y+1)^2} - \frac{4(y+2)^4 a_3}{(x+y+1)^4} + \frac{4(y+2)^2 (x a_2 + y a_3 + a_1)}{(x+y+1)^3} - \left(\frac{4y+8}{(x+y+1)^2} - \frac{4(y+2)^2}{(x+y+1)^3}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^4 b_2 + 2x^2y^2a_2 + 2x^2y^2b_2 - 2x^2y^2b_3 + 4x\,y^3a_3 + 4x\,y^3b_2 - 2y^4a_2 + y^4b_2 + 2y^4b_3 - 4x^3b_2 + 8x^2ya_2 - 4x^2y}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &x^4b_2 + 2x^2y^2a_2 + 2x^2y^2b_2 - 2x^2y^2b_3 + 4x\,y^3a_3 + 4x\,y^3b_2 - 2y^4a_2 + y^4b_2 \\ &+ 2y^4b_3 - 4x^3b_2 + 8x^2ya_2 - 4x^2yb_1 + 4x^2yb_2 + 4x\,y^2a_1 + 16x\,y^2a_3 \\ &- 4x\,y^2b_1 + 16x\,y^2b_2 + 12x\,y^2b_3 + 4y^3a_1 - 12y^3a_2 - 12y^3a_3 + 4y^3b_2 \\ &+ 16y^3b_3 + 8x^2a_2 - 8x^2b_1 + 6x^2b_2 + 8x^2b_3 + 16xya_1 + 16xya_3 \\ &- 8xyb_1 + 24xyb_2 + 32xyb_3 + 20y^2a_1 - 26y^2a_2 - 64y^2a_3 + 4y^2b_1 \\ &+ 6y^2b_2 + 38y^2b_3 + 16xa_1 + 12xb_2 + 16xb_3 + 32ya_1 - 24ya_2 - 112ya_3 \\ &+ 12yb_1 + 4yb_2 + 32yb_3 + 16a_1 - 8a_2 - 64a_3 + 8b_1 + b_2 + 8b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&2a_2v_1^2v_2^2 - 2a_2v_2^4 + 4a_3v_1v_2^3 + b_2v_1^4 + 2b_2v_1^2v_2^2 + 4b_2v_1v_2^3 + b_2v_2^4 - 2b_3v_1^2v_2^2 \\
&+ 2b_3v_2^4 + 4a_1v_1v_2^2 + 4a_1v_2^3 + 8a_2v_1^2v_2 - 12a_2v_2^3 + 16a_3v_1v_2^2 - 12a_3v_2^3 \\
&- 4b_1v_1^2v_2 - 4b_1v_1v_2^2 - 4b_2v_1^3 + 4b_2v_1^2v_2 + 16b_2v_1v_2^2 + 4b_2v_2^3 + 12b_3v_1v_2^2 \\
&+ 16b_3v_2^3 + 16a_1v_1v_2 + 20a_1v_2^2 + 8a_2v_1^2 - 26a_2v_2^2 + 16a_3v_1v_2 - 64a_3v_2^2 \\
&- 8b_1v_1^2 - 8b_1v_1v_2 + 4b_1v_2^2 + 6b_2v_1^2 + 24b_2v_1v_2 + 6b_2v_2^2 + 8b_3v_1^2 + 32b_3v_1v_2 \\
&+ 38b_3v_2^2 + 16a_1v_1 + 32a_1v_2 - 24a_2v_2 - 112a_3v_2 + 12b_1v_2 + 12b_2v_1 \\
&+ 4b_2v_2 + 16b_3v_1 + 32b_3v_2 + 16a_1 - 8a_2 - 64a_3 + 8b_1 + b_2 + 8b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&b_2v_1^4 - 4b_2v_1^3 + (2a_2 + 2b_2 - 2b_3)\,v_1^2v_2^2 + (8a_2 - 4b_1 + 4b_2)\,v_1^2v_2 \\
&+ (8a_2 - 8b_1 + 6b_2 + 8b_3)\,v_1^2 + (4a_3 + 4b_2)\,v_1v_2^3 \\
&+ (4a_1 + 16a_3 - 4b_1 + 16b_2 + 12b_3)\,v_1v_2^2 \\
&+ (16a_1 + 16a_3 - 8b_1 + 24b_2 + 32b_3)\,v_1v_2 + (16a_1 + 12b_2 + 16b_3)\,v_1 \\
&+ (-2a_2 + b_2 + 2b_3)\,v_2^4 + (4a_1 - 12a_2 - 12a_3 + 4b_2 + 16b_3)\,v_2^3 \\
&+ (20a_1 - 26a_2 - 64a_3 + 4b_1 + 6b_2 + 38b_3)\,v_2^2 \\
&+ (32a_1 - 24a_2 - 112a_3 + 12b_1 + 4b_2 + 32b_3)\,v_2 \\
&+ 16a_1 - 8a_2 - 64a_3 + 8b_1 + b_2 + 8b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-4b_2 &= 0 \\
4a_3 + 4b_2 &= 0 \\
16a_1 + 12b_2 + 16b_3 &= 0 \\
-2a_2 + b_2 + 2b_3 &= 0 \\
2a_2 + 2b_2 - 2b_3 &= 0 \\
8a_2 - 4b_1 + 4b_2 &= 0 \\
8a_2 - 8b_1 + 6b_2 + 8b_3 &= 0 \\
4a_1 - 12a_2 - 12a_3 + 4b_2 + 16b_3 &= 0 \\
4a_1 + 16a_3 - 4b_1 + 16b_2 + 12b_3 &= 0 \\
16a_1 + 16a_3 - 8b_1 + 24b_2 + 32b_3 &= 0 \\
16a_1 - 8a_2 - 64a_3 + 8b_1 + b_2 + 8b_3 &= 0 \\
20a_1 - 26a_2 - 64a_3 + 4b_1 + 6b_2 + 38b_3 &= 0 \\
32a_1 - 24a_2 - 112a_3 + 12b_1 + 4b_2 + 32b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -b_3 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 2b_3 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x - 1 \\
\eta &= y + 2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y + 2 - \left(\frac{2(y+2)^2}{(x+y+1)^2}\right)(x-1) \\
&= \frac{y\,x^2 + y^3 + 2x^2 - 2xy + 6y^2 - 4x + 13y + 10}{x^2 + 2xy + y^2 + 2x + 2y + 1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y\,x^2 + y^3 + 2x^2 - 2xy + 6y^2 - 4x + 13y + 10}{x^2 + 2xy + y^2 + 2x + 2y + 1}} dy
\end{aligned}$$

Which results in

$$S = \ln(y+2) + 2\arctan\left(\frac{2y+4}{2x-2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2(y+2)^2}{(x+y+1)^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-2y-4}{x^2+y^2-2x+4y+5} \\
S_y &= \frac{(x+y+1)^2}{(y+2)\left(x^2+y^2-2x+4y+5\right)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

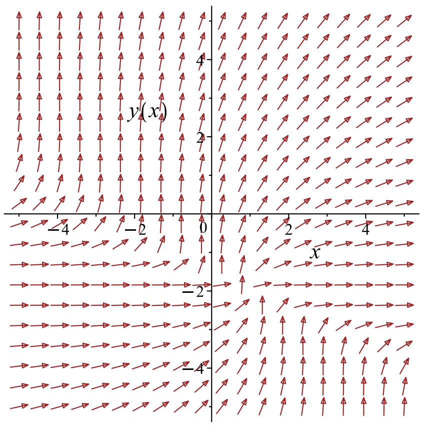
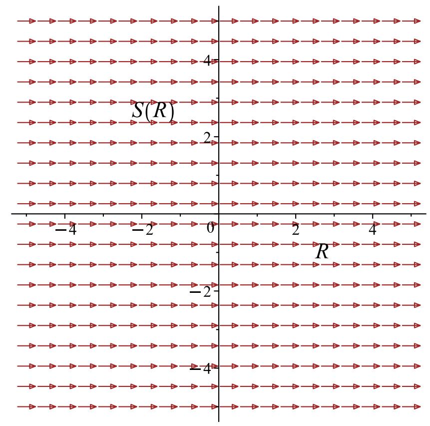
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln(2+y) + 2\arctan\left(\frac{2+y}{x-1}\right) = c_1
$$

Which simplifies to

$$
\ln(2+y) + 2\arctan\left(\frac{2+y}{x-1}\right) = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2(y+2)^2}{(x+y+1)^2}$ 	$R = x$ $S = \ln(y+2) + 2\,\mathrm{arctan}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\ln(2+y) + 2\arctan\left(\frac{2+y}{x-1}\right) = c_1 \tag{1}$$

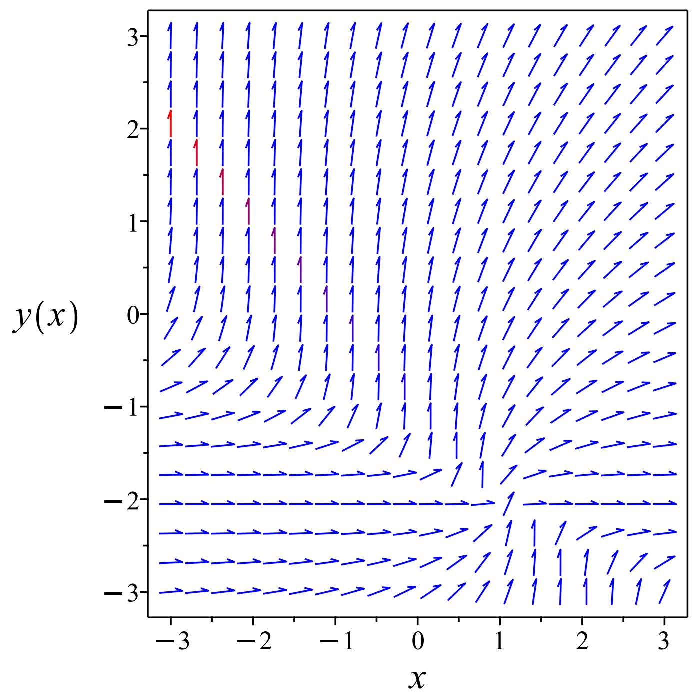


Figure 77: Slope field plot

Verification of solutions

$$\ln(2+y)+2\arctan\left(\frac{2+y}{x-1}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 25

```
dsolve(diff(y(x),x)=2*((y(x)+2)/(x+y(x)+1))^2,y(x), singsol=all)
```

$$y(x) = -2 - \tan\left(\operatorname{RootOf}\left(-2\_Z + \ln\left(\tan\left(\_Z\right)\right) + \ln\left(x-1\right) + c_1\right)\right)(x-1)$$

✓ Solution by Mathematica

Time used: 0.138 (sec). Leaf size: 27

```
DSolve[y'[x]==2*((y[x]+2)/(x+y[x]+1))^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[2\arctan\left(\frac{1-x}{y(x)+2}\right) + \log(y(x)+2) = c_1, y(x)\right]$$

2.37 problem 35

Internal problem ID [5785]
Internal file name [OUTPUT/5033_Sunday_June_05_2022_03_18_17_PM_72205197/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y-(4x+2y-3)\,y'=-1-2x}$$

2.37.1 Solving as polynomial ode

This is ODE of type polynomial. Where the RHS of the ode is ratio of equations of two lines. Writing the ODE in the form

$$y'=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_3}$$

Where $a_1=2, b_1=1, c_1=1, a_2=4, b_2=2, c_2=-3$. There are now two possible solution methods. The first case is when the two lines $a_1x+b_1y+c_1$,$a_2x+b_2y+c_3$ are not parallel and the second case is if they are parallel. If they are not parallel, then the transformation $X=x-x_0$, $Y=y-y_0$ converts the ODE to a homogeneous ODE. The values x_0, y_0 have to be determined. If they are parallel then a transformation $U(x)=a_1x+b_1y$ converts the given ODE in y to a separable ODE in $U(x)$. The first case is when $\frac{a_1}{b_1}\neq\frac{a_2}{b_2}$ and the second case when $\frac{a_1}{b_1}=\frac{a_2}{b_2}$. From the above we see that $\frac{a_1}{b_1}=\frac{2}{1}=2$ and $\frac{a_2}{b_2}=\frac{4}{2}=2$. Hence this is case two, where the lines are parallel. Let $U(x)=2x+y$. Solving for y gives

$$y=-2x+U(x)$$

Taking derivative w.r.t x gives

$$y' = -2 + U'(x)$$

Substituting the above into the ODE results in the ODE

$$-2x + U(x) - (2U(x) - 3)(-2 + U'(x)) = -1 - 2x$$

Or

$$(-2U(x) + 3) U'(x) - 2x + 5U(x) - 6 = -1 - 2x$$

Or

$$U'(x) = \frac{5U(x) - 5}{2U(x) - 3}$$

Which is now solved as separable in $U(x)$. In canonical form the ODE is

$$\begin{aligned} U' &= F(x, U) \\ &= f(x)g(U) \\ &= \frac{5U - 5}{2U - 3} \end{aligned}$$

Where $f(x) = 1$ and $g(U) = \frac{5U-5}{2U-3}$. Integrating both sides gives

$$\frac{1}{\frac{5U-5}{2U-3}}\, dU = 1\, dx$$

$$\int \frac{1}{\frac{5U-5}{2U-3}}\, dU = \int 1\, dx$$

$$\frac{2U}{5} - \frac{\ln(U - 1)}{5} = c_2 + x$$

The solution is

$$\frac{2U(x)}{5} - \frac{\ln(U(x) - 1)}{5} - c_2 - x = 0$$

The solution $\frac{2U(x)}{5} - \frac{\ln(U(x)-1)}{5} - c_2 - x = 0$ is converted to y using $U(x) = 2x + y$. Which gives

$$-\frac{x}{5} + \frac{2y}{5} - \frac{\ln(2x + y - 1)}{5} - c_2 = 0$$

Summary

The solution(s) found are the following

$$-\frac{x}{5} + \frac{2y}{5} - \frac{\ln(2x + y - 1)}{5} - c_2 = 0 \tag{1}$$

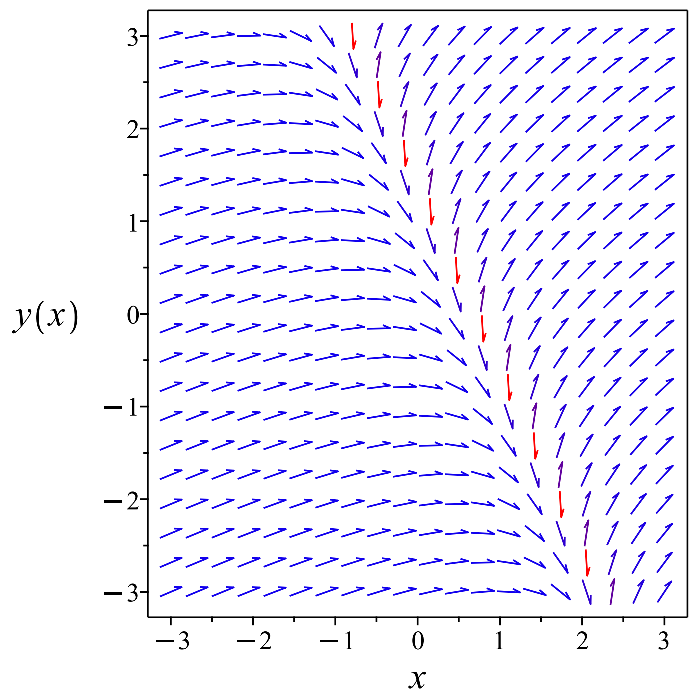


Figure 78: Slope field plot

Verification of solutions

$$-\frac{x}{5}+\frac{2y}{5}-\frac{\ln\left(2x+y-1\right)}{5}-c_2=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 23

```
dsolve((2*x+y(x)+1)-(4*x+2*y(x)-3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\text{LambertW}\left(-2\,\mathrm{e}^{-5x+2+5c_1}\right)}{2} - 2x + 1$$

✓ Solution by Mathematica

Time used: 11.239 (sec). Leaf size: 35

```
DSolve[(2*x+y[x]+1)-(4*x+2*y[x]-3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}W\left(-e^{-5x-1+c_1}\right) - 2x + 1$$
$$y(x) \to 1 - 2x$$

2.38 problem 36

Internal problem ID [5786]
Internal file name [`OUTPUT/5034_Sunday_June_05_2022_03_18_19_PM_14609277/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{-y+(y-x+2)\,y'=1-x}$$

2.38.1 Solving as polynomial ode

This is ODE of type polynomial. Where the RHS of the ode is ratio of equations of two lines. Writing the ODE in the form

$$y'=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_3}$$

Where $a_1=1, b_1=-1, c_1=-1, a_2=1, b_2=-1, c_2=-2$. There are now two possible solution methods. The first case is when the two lines $a_1x+b_1y+c_1$,$a_2x+b_2y+c_3$ are not parallel and the second case is if they are parallel. If they are not parallel, then the transformation $X=x-x_0$, $Y=y-y_0$ converts the ODE to a homogeneous ODE. The values x_0, y_0 have to be determined. If they are parallel then a transformation $U(x)=a_1x+b_1y$ converts the given ODE in y to a separable ODE in $U(x)$. The first case is when $\frac{a_1}{b_1}\neq\frac{a_2}{b_2}$ and the second case when $\frac{a_1}{b_1}=\frac{a_2}{b_2}$. From the above we see that $\frac{a_1}{b_1}=\frac{1}{-1}=-1$ and $\frac{a_2}{b_2}=\frac{1}{-1}=-1$. Hence this is case two, where the lines are parallel.

Let $U(x) = x - y$. Solving for y gives

$$y = x - U(x)$$

Taking derivative w.r.t x gives

$$y' = 1 - U'(x)$$

Substituting the above into the ODE results in the ODE

$$-x + U(x) + (-U(x) + 2)(1 - U'(x)) = 1 - x$$

Or

$$(U(x) - 2) U'(x) - x + 2 = 1 - x$$

Or

$$U'(x) = -\frac{1}{U(x) - 2}$$

Which is now solved as separable in $U(x)$. In canonical form the ODE is

$$\begin{aligned} U' &= F(x, U) \\ &= f(x)g(U) \\ &= -\frac{1}{U - 2} \end{aligned}$$

Where $f(x) = 1$ and $g(U) = -\frac{1}{U-2}$. Integrating both sides gives

$$\frac{1}{-\frac{1}{U-2}} \, dU = 1 \, dx$$

$$\int \frac{1}{-\frac{1}{U-2}} \, dU = \int 1 \, dx$$

$$-\frac{1}{2}U^2 + 2U = c_2 + x$$

The solution is

$$-\frac{U(x)^2}{2} + 2U(x) - c_2 - x = 0$$

The solution $-\frac{U(x)^2}{2} + 2U(x) - c_2 - x = 0$ is converted to y using $U(x) = x - y$. Which gives

$$-\frac{(x - y)^2}{2} + x - 2y - c_2 = 0$$

Summary
The solution(s) found are the following

$$-\frac{(x-y)^{2}}{2}+x-2y-c_{2}=0 \tag{1}$$

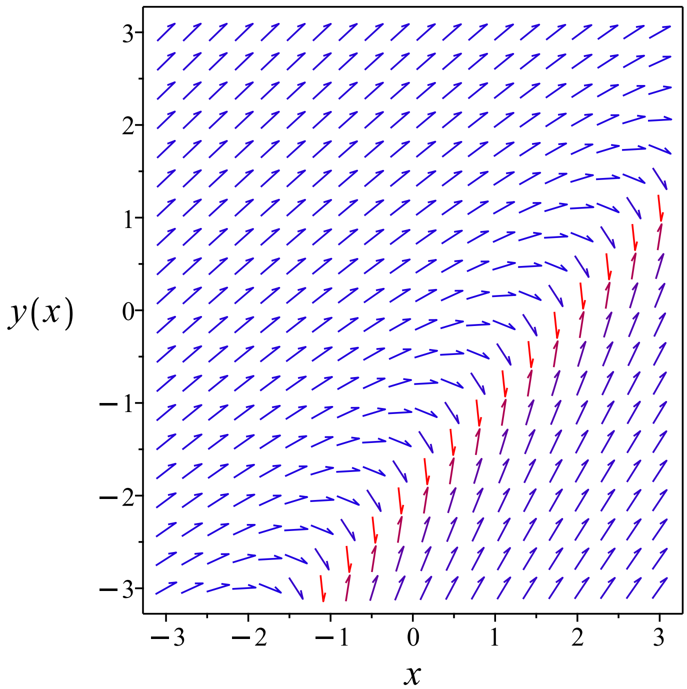


Figure 79: Slope field plot

Verification of solutions

$$-\frac{(x-y)^{2}}{2}+x-2y-c_{2}=0$$

Verified OK.

2.38.2 Maple step by step solution

Let's solve

$-y+(y-x+2)\,y'=1-x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y) = 0$

- Compute derivative of lhs

  $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$

- Evaluate derivatives

  $-1 = -1$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (x-y-1)\,dx + f_1(y)$

- Evaluate integral

  $F(x,y) = \frac{x^2}{2} - xy - x + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $y - x + 2 = -x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = y + 2$

- Solve for $f_1(y)$

  $f_1(y) = \frac{1}{2}y^2 + 2y$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = \frac{1}{2}x^2 - xy - x + \frac{1}{2}y^2 + 2y$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{1}{2}x^2 - xy - x + \frac{1}{2}y^2 + 2y = c_1$

- Solve for y

  $\left\{y = x - 2 - \sqrt{2c_1 - 2x + 4}, y = x - 2 + \sqrt{2c_1 - 2x + 4}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
dsolve((x-y(x)-1)+(y(x)-x+2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = x - 2 - \sqrt{2c_1 - 2x + 4}$$
$$y(x) = x - 2 + \sqrt{2c_1 - 2x + 4}$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 49

```
DSolve[(x-y[x]-1)+(y[x]-x+2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - i\sqrt{2x - 4 - c_1} - 2$$
$$y(x) \to x + i\sqrt{2x - 4 - c_1} - 2$$

2.39 problem 37

Internal problem ID [5787]
Internal file name [`OUTPUT/5035_Sunday_June_05_2022_03_18_21_PM_30532492/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(x+4y)\,y'-3y=2x-5}$$

2.39.1 Solving as polynomial ode

This is ODE of type polynomial. Where the RHS of the ode is ratio of equations of two lines. Writing the ODE in the form

$$y'=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_3}$$

Where $a_1=2, b_1=3, c_1=-5, a_2=1, b_2=4, c_2=0$. There are now two possible solution methods. The first case is when the two lines $a_1x+b_1y+c_1$,$a_2x+b_2y+c_3$ are not parallel and the second case is if they are parallel. If they are not parallel, then the transformation $X=x-x_0$, $Y=y-y_0$ converts the ODE to a homogeneous ODE. The values x_0, y_0 have to be determined. If they are parallel then a transformation $U(x)=a_1x+b_1y$ converts the given ODE in y to a separable ODE in $U(x)$. The first case is when $\frac{a_1}{b_1}\neq\frac{a_2}{b_2}$ and the second case when $\frac{a_1}{b_1}=\frac{a_2}{b_2}$. From the above we see that $\frac{a_1}{b_1}\neq\frac{a_2}{b_2}$. Hence this is case one where lines are not parallel. Using the transformation

$$X=x-x_0$$
$$Y=y-y_0$$

Where the constants x_0, y_0 are obtained by solving the following two linear algebraic equations

$$\begin{aligned}a_1x_0 + b_1y_0 + c_1 &= 0\\ a_2x_0 + b_2y_0 + c_2 &= 0\end{aligned}$$

Substituting the values for $a_1, b_1, c_1, a_2, b_2, c_2$ gives

$$\begin{aligned}2x_0 + 3y_0 - 5 &= 0\\ x_0 + 4y_0 &= 0\end{aligned}$$

Solving for x_0, y_0 from the above gives

$$\begin{aligned}x_0 &= 4\\ y_0 &= -1\end{aligned}$$

Therefore the transformation becomes

$$\begin{aligned}X &= x - 4\\ Y &= y + 1\end{aligned}$$

Using this transformation in $(x + 4y)\,y' - 3y = 2x - 5$ result in

$$\frac{dY}{dX} = \frac{2X + 3Y}{X + 4Y}$$

This is now a homogeneous ODE which will now be solved for $Y(X)$. In canonical form, the ODE is

$$\begin{aligned}Y' &= F(X, Y)\\ &= \frac{2X + 3Y}{X + 4Y}\end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^nX, t^nY) = t^n f(X, Y)$$

In this case, it can be seen that both $M = 2X + 3Y$ and $N = X + 4Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{3u+2}{4u+1}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{3u(X)+2}{4u(X)+1} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{3u(X)+2}{4u(X)+1} - u(X)}{X} = 0$$

Or

$$4\left(\frac{d}{dX}u(X)\right)Xu(X) + \left(\frac{d}{dX}u(X)\right)X + 4u(X)^2 - 2u(X) - 2 = 0$$

Or

$$-2 + X(4u(X)+1)\left(\frac{d}{dX}u(X)\right) + 4u(X)^2 - 2u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X,u)\\ &= f(X)g(u)\\ &= -\frac{2(2u^2-u-1)}{X\,(4u+1)}\end{aligned}$$

Where $f(X) = -\frac{2}{X}$ and $g(u) = \frac{2u^2-u-1}{4u+1}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2-u-1}{4u+1}}\,du = -\frac{2}{X}\,dX$$

$$\int \frac{1}{\frac{2u^2-u-1}{4u+1}}\,du = \int -\frac{2}{X}\,dX$$

$$\frac{5\ln(u-1)}{3} + \frac{\ln(2u+1)}{3} = -2\ln(X) + c_3$$

The above can be written as

$$\begin{aligned}\frac{5\ln(u-1) + \ln(2u+1)}{3} &= -2\ln(X) + c_3\\ 5\ln(u-1) + \ln(2u+1) &= (3)\,(-2\ln(X) + c_3)\\ &= -6\ln(X) + 3c_3\end{aligned}$$

Raising both side to exponential gives

$$e^{5\ln(u-1)+\ln(2u+1)} = e^{-6\ln(X)+3c_3}$$

Which simplifies to

$$\begin{aligned}(u-1)^5(2u+1) &= \frac{3c_3}{X^6}\\ &= \frac{c_4}{X^6}\end{aligned}$$

Which simplifies to

$$u(X) = \text{RootOf}\left(2\_Z^6 - 9\_Z^5 + 15\_Z^4 - 10\_Z^3 - \frac{c_4 e^{3c_3}}{X^6} + 3\_Z - 1\right)$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = X\,\text{RootOf}\left(2\_Z^6X^6 - 9\_Z^5X^6 + 15\_Z^4X^6 - 10\_Z^3X^6 + 3\_ZX^6 - X^6 - c_4e^{3c_3}\right)$$

The solution is

$$Y(X) = X\,\text{RootOf}\left(2\_Z^6X^6 - 9\_Z^5X^6 + 15\_Z^4X^6 - 10\_Z^3X^6 + 3\_ZX^6 - X^6 - c_4e^{3c_3}\right)$$

Replacing $Y = y - y_0, X = x - x_0$ gives

$$1+y = (-4+x)\,\text{RootOf}\left(2\_Z^6(-4+x)^6 - 9\_Z^5(-4+x)^6 + 15\_Z^4(-4+x)^6 - 10\_Z^3(-4+x)^6 + 3\_Z$$

Or

$$y = (-4+x)\,\text{RootOf}\left(2\_Z^6(-4+x)^6 - 9\_Z^5(-4+x)^6 + 15\_Z^4(-4+x)^6 - 10\_Z^3(-4+x)^6 + 3\_Z(-$$

Summary

The solution(s) found are the following

$$\begin{aligned}y = (-4+x)\,\text{RootOf}\big(&(2x^6 - 48x^5 + 480x^4 - 2560x^3 + 7680x^2 - 12288x + 8192)\_Z^6\\ &+ (-9x^6 + 216x^5 - 2160x^4 + 11520x^3 - 34560x^2 + 55296x - 36864)\_Z^5\\ &+ (15x^6 - 360x^5 + 3600x^4 - 19200x^3 + 57600x^2 - 92160x + 61440)\_Z^4\\ &+ (-10x^6 + 240x^5 - 2400x^4 + 12800x^3 - 38400x^2 + 61440x - 40960)\_Z^3\\ &+ (3x^6 - 72x^5 + 720x^4 - 3840x^3 + 11520x^2 - 18432x + 12288)\_Z - x^6 + 24x^5\\ &- 240x^4 + 1280x^3 - c_4e^{3c_3} - 3840x^2 + 6144x - 4096\big) - 1\end{aligned} \tag{1}$$

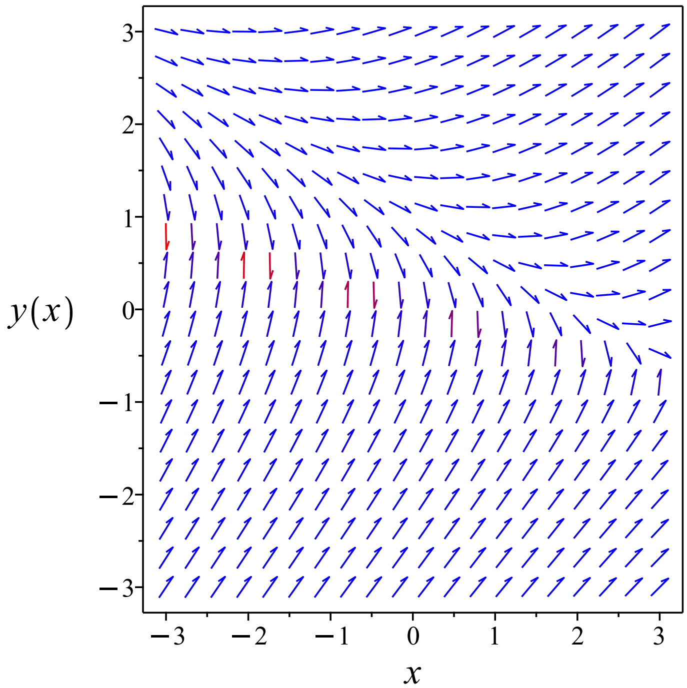


Figure 80: Slope field plot

Verification of solutions

$$y = (-4+x)\,\mathrm{RootOf}\Big((2x^6-48x^5+480x^4-2560x^3+7680x^2-12288x+8192)\_Z^6 + (-9x^6+216x^5-2160x^4+11520x^3-34560x^2+55296x-36864)\_Z^5 + (15x^6-360x^5+3600x^4-19200x^3+57600x^2-92160x+61440)\_Z^4 + (-10x^6+240x^5-2400x^4+12800x^3-38400x^2+61440x-40960)\_Z^3 + (3x^6-72x^5+720x^4-3840x^3+11520x^2-18432x+12288)\_Z - x^6 + 24x^5 - 240x^4 + 1280x^3 - c_4 e^{3c_3} - 3840x^2 + 6144x - 4096\Big) - 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.437 (sec). Leaf size: 186

```
dsolve((x+4*y(x))*diff(y(x),x)=2*x+3*y(x)-5,y(x), singsol=all)
```

$$y(x) = \frac{(x-5)\operatorname{RootOf}\left(\_Z^{36}+\left(3c_1x^6-72c_1x^5+720c_1x^4-3840c_1x^3+11520c_1x^2-18432c_1x+12288c_1\right)\_Z\right.}{\operatorname{RootOf}\left(\_Z^{36}+\left(3c_1x^6-72c_1x^5+720c_1x^4-3840c_1x^3+11520c_1x^2-18432c_1x+12288c_1\right)\_Z\right.}$$

✓ Solution by Mathematica

Time used: 60.076 (sec). Leaf size: 805

```
DSolve[(x+4*y[x])*y'[x]==2*x+3*y[x]-5,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{4} + \frac{}{4\text{Root}\left[\#1^6\left(-3125x^6 + 75000x^5 - 750000x^4 + 4000000x^3 - 12000000x^2 + 19200000x - 12800000\right.\right.}$$

$$y(x) \to -\frac{x}{4} + \frac{}{4\text{Root}\left[\#1^6\left(-3125x^6 + 75000x^5 - 750000x^4 + 4000000x^3 - 12000000x^2 + 19200000x - 12800000\right.\right.}$$

$$y(x) \to -\frac{x}{4} + \frac{}{4\text{Root}\left[\#1^6\left(-3125x^6 + 75000x^5 - 750000x^4 + 4000000x^3 - 12000000x^2 + 19200000x - 12800000\right.\right.}$$

$$y(x) \to -\frac{x}{4} + \frac{}{4\text{Root}\left[\#1^6\left(-3125x^6 + 75000x^5 - 750000x^4 + 4000000x^3 - 12000000x^2 + 19200000x - 12800000\right.\right.}$$

$$y(x) \to -\frac{x}{4} + \frac{}{4\text{Root}\left[\#1^6\left(-3125x^6 + 75000x^5 - 750000x^4 + 4000000x^3 - 12000000x^2 + 19200000x - 12800000\right.\right.}$$

$$y(x) \to -\frac{x}{4} + \frac{}{4\text{Root}\left[\#1^6\left(-3125x^6 + 75000x^5 - 750000x^4 + 4000000x^3 - 12000000x^2 + 19200000x - 12800000\right.\right.}$$

2.40 problem 38

Internal problem ID [5788]
Internal file name [`OUTPUT/5036_Sunday_June_05_2022_03_18_24_PM_72469879/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y-(2x+y-4)\,y'=-2}$$

2.40.1 Solving as homogeneousTypeMapleC ode

Let $Y=y+y_0$ and $X=x+x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X)=\frac{2+Y(X)+y_0}{2X+2x_0+Y\left(X\right)+y_0-4}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0=3$$
$$y_0=-2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X)=\frac{Y(X)}{2X+Y\left(X\right)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{Y}{2X + Y}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = Y$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= \frac{u}{u + 2} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)}{u(X)+2} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) - \frac{\frac{u(X)}{u(X)+2} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX} u(X)\right) X u(X) + 2\left(\frac{d}{dX} u(X)\right) X + u(X)^2 + u(X) = 0
$$

Or

$$
X(u(X) + 2) \left(\frac{d}{dX} u(X)\right) + u(X)^2 + u(X) = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= -\frac{u(u + 1)}{X\,(u + 2)}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u(u+1)}{u+2}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u(u+1)}{u+2}} du &= -\frac{1}{X} dX \\
\int \frac{1}{\frac{u(u+1)}{u+2}} du &= \int -\frac{1}{X} dX \\
-\ln(u+1) + 2\ln(u) &= -\ln(X) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{-\ln(u+1)+2\ln(u)} = e^{-\ln(X)+c_2}
$$

Which simplifies to

$$
\frac{u^2}{u+1} = \frac{c_3}{X}
$$

The solution is

$$
\frac{u(X)^2}{u(X)+1} = \frac{c_3}{X}
$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$
\frac{Y(X)^2}{\left(\frac{Y(X)}{X} + 1\right) X^2} = \frac{c_3}{X}
$$

Which simplifies to

$$
\frac{Y(X)^2}{Y(X)+X} = c_3
$$

Using the solution for $Y(X)$

$$
\frac{Y(X)^2}{Y(X)+X} = c_3
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$Y = y - 2$$
$$X = x + 3$$

Then the solution in y becomes

$$\frac{(2 + y)^2}{-1 + y + x} = c_3$$

<u>Summary</u>
The solution(s) found are the following

$$\frac{(2 + y)^2}{-1 + y + x} = c_3 \tag{1}$$

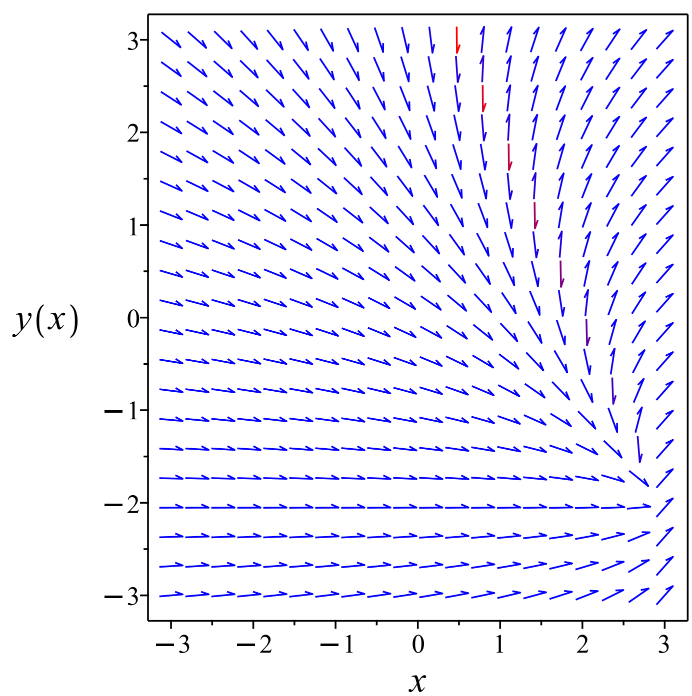


Figure 81: Slope field plot

<u>Verification of solutions</u>

$$\frac{(2 + y)^2}{-1 + y + x} = c_3$$

Verified OK.

2.40.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y+2}{2x+y-4} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{(y+2)(b_3-a_2)}{2x+y-4} - \frac{(y+2)^2 a_3}{(2x+y-4)^2} + \frac{2(y+2)(x a_2 + y a_3 + a_1)}{(2x+y-4)^2} \\
&- \left(\frac{1}{2x+y-4} - \frac{y+2}{(2x+y-4)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{2x^2 b_2 + 4xy b_2 - y^2 a_2 + y^2 a_3 + y^2 b_2 + y^2 b_3 - 2x b_1 - 10x b_2 + 4x b_3 + 2y a_1 + 2y a_2 - 8y b_2 + 4y b_3 + 4a_1 + 8}{(2x+y-4)^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&2x^2 b_2 + 4xy b_2 - y^2 a_2 + y^2 a_3 + y^2 b_2 + y^2 b_3 - 2x b_1 - 10x b_2 + 4x b_3 \\
&+ 2y a_1 + 2y a_2 - 8y b_2 + 4y b_3 + 4a_1 + 8a_2 - 4a_3 + 6b_1 + 16b_2 - 8b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-a_2v_2^2 + a_3v_2^2 + 2b_2v_1^2 + 4b_2v_1v_2 + b_2v_2^2 + b_3v_2^2 + 2a_1v_2 + 2a_2v_2 - 2b_1v_1 \\ &- 10b_2v_1 - 8b_2v_2 + 4b_3v_1 + 4b_3v_2 + 4a_1 + 8a_2 - 4a_3 + 6b_1 + 16b_2 - 8b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &2b_2v_1^2 + 4b_2v_1v_2 + (-2b_1 - 10b_2 + 4b_3)\,v_1 + (-a_2 + a_3 + b_2 + b_3)\,v_2^2 \\ &+ (2a_1 + 2a_2 - 8b_2 + 4b_3)\,v_2 + 4a_1 + 8a_2 - 4a_3 + 6b_1 + 16b_2 - 8b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2b_2 &= 0 \\ 4b_2 &= 0 \\ -2b_1 - 10b_2 + 4b_3 &= 0 \\ 2a_1 + 2a_2 - 8b_2 + 4b_3 &= 0 \\ -a_2 + a_3 + b_2 + b_3 &= 0 \\ 4a_1 + 8a_2 - 4a_3 + 6b_1 + 16b_2 - 8b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -a_3 - 3b_3 \\ a_2 &= a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 2b_3 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x - 3$$
$$\eta = y + 2$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y + 2 - \left(\frac{y+2}{2x+y-4}\right)(x-3) \\
&= \frac{xy + y^2 + 2x + y - 2}{2x + y - 4} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{xy+y^2+2x+y-2}{2x+y-4}} dy
\end{aligned}$$

Which results in

$$S = -\ln(y + x - 1) + 2\ln(y + 2)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y + 2}{2x + y - 4}$$

Evaluating all the partial derivatives gives

$$R_x = 1$$
$$R_y = 0$$
$$S_x = -\frac{1}{y + x - 1}$$
$$S_y = \frac{2x + y - 4}{(y + 2)(y + x - 1)}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

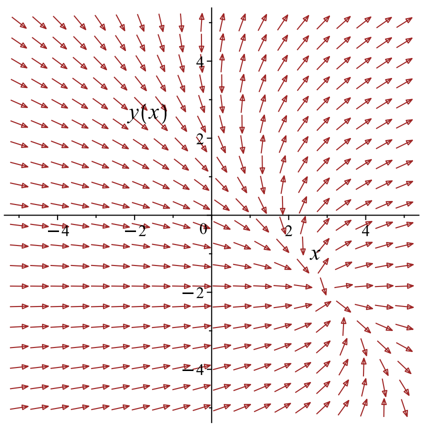
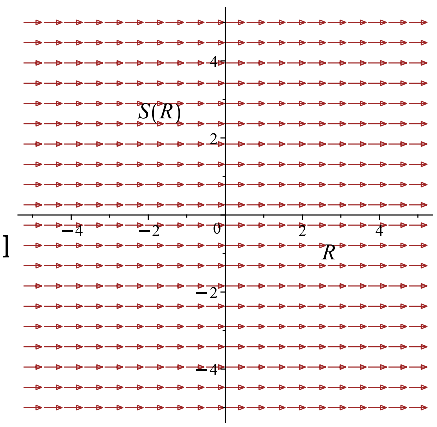
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(-1 + y + x) + 2\ln(2 + y) = c_1$$

Which simplifies to

$$-\ln(-1 + y + x) + 2\ln(2 + y) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+2}{2x+y-4}$ 	$R = x$ $S = -\ln\left(y + x - 1\right) + 2\mathrm{l}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$-\ln\left(-1 + y + x\right) + 2\ln\left(2 + y\right) = c_1 \tag{1}$$

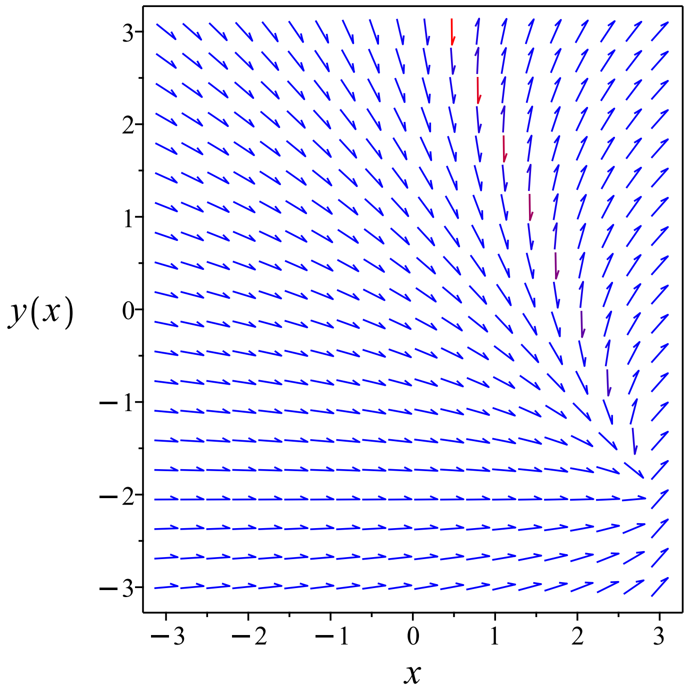


Figure 82: Slope field plot

Verification of solutions

$$-\ln(-1+y+x)+2\ln(2+y)=c_1$$

Verified OK.

2.40.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2x - y + 4)\,\mathrm{d}y &= (-y - 2)\,\mathrm{d}x \\ (y + 2)\,\mathrm{d}x + (-2x - y + 4)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y + 2 \\ N(x, y) &= -2x - y + 4\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y + 2) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2x - y + 4) \\ &= -2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{-2x - y + 4}((1) - (-2)) \\
&= -\frac{3}{2x + y - 4}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y + 2}((-2) - (1)) \\
&= -\frac{3}{y + 2}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{3}{y+2} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(y+2)} \\
&= \frac{1}{(y + 2)^3}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{(y + 2)^3}(y + 2) \\
&= \frac{1}{(y + 2)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(y+2)^3}(-2x-y+4) \\
&= \frac{-2x-y+4}{(y+2)^3}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{1}{(y+2)^2}\right) + \left(\frac{-2x-y+4}{(y+2)^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{1}{(y+2)^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x}{(y+2)^2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{2x}{(y+2)^3} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-2x-y+4}{(y+2)^3}$. Therefore equation (4) becomes

$$
\frac{-2x-y+4}{(y+2)^3} = -\frac{2x}{(y+2)^3} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{-4+y}{(y+2)^3}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{4-y}{(y+2)^3}\right)\mathrm{d}y$$

$$f(y) = \frac{1}{y+2} - \frac{3}{(y+2)^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x}{(y+2)^2} + \frac{1}{y+2} - \frac{3}{(y+2)^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x}{(y+2)^2} + \frac{1}{y+2} - \frac{3}{(y+2)^2}$$

Summary

The solution(s) found are the following

$$\frac{x}{(2+y)^2} + \frac{1}{2+y} - \frac{3}{(2+y)^2} = c_1 \tag{1}$$

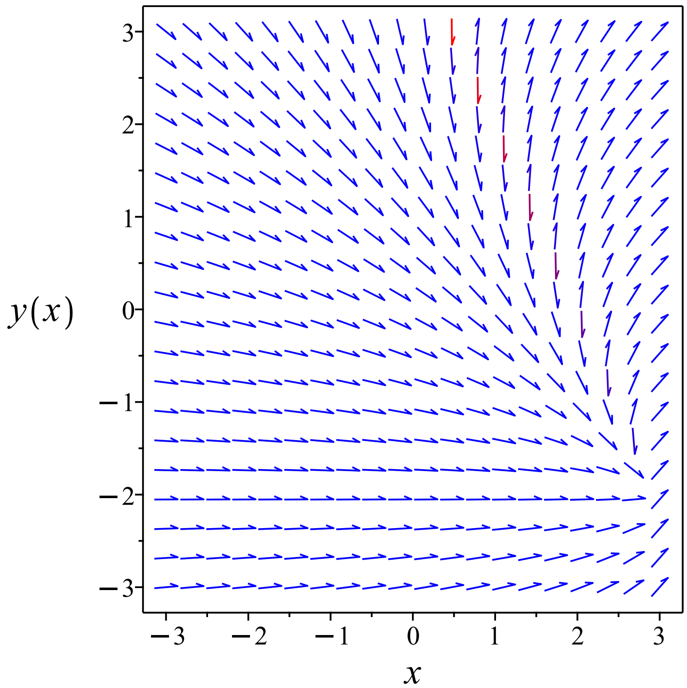


Figure 83: Slope field plot

Verification of solutions

$$\frac{x}{(2+y)^2} + \frac{1}{2+y} - \frac{3}{(2+y)^2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
dsolve(y(x)+2=(2*x+y(x)-4)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{-4c_1 + 1 + \sqrt{1 + 4(x-3)c_1}}{2c_1}$$
$$y(x) = \frac{-4c_1 + 1 - \sqrt{1 + 4(x-3)c_1}}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.237 (sec). Leaf size: 82

```
DSolve[y[x]+2==(2*x+y[x]-4)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{1 + 4c_1(x-3)} - 1 + 4c_1}{2c_1}$$
$$y(x) \to \frac{\sqrt{1 + 4c_1(x-3)} + 1 - 4c_1}{2c_1}$$
$$y(x) \to -2$$
$$y(x) \to \text{Indeterminate}$$
$$y(x) \to 1 - x$$

2.41 problem 39

Internal problem ID [5789]
Internal file name [`OUTPUT/5037_Sunday_June_05_2022_03_18_25_PM_75963944/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _dAlembert]
```

$$\boxed{(1+y')\ln\left(\frac{x+y}{x+3}\right)-\frac{x+y}{x+3}=0}$$

2.41.1 Solving as homogeneousTypeMapleC ode

Let $Y=y+y_0$ and $X=x+x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X)=-\frac{\ln\left(\frac{X+x_0+Y(X)+y_0}{X+x_0+3}\right)(X+x_0)-Y(X)-y_0+3\ln\left(\frac{X+x_0+Y(X)+y_0}{X+x_0+3}\right)-X-x_0}{\ln\left(\frac{X+x_0+Y(X)+y_0}{X+x_0+3}\right)(X+x_0+3)}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0=-3$$
$$y_0=3$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{\ln\left(\frac{X+Y(X)}{X}\right)X - Y(X) - X}{\ln\left(\frac{X+Y(X)}{X}\right)X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{-\ln\left(\frac{X+Y}{X}\right)X + Y + X}{\ln\left(\frac{X+Y}{X}\right)X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -\ln\left(\frac{X+Y}{X}\right)X + Y + X$ and $N = \ln\left(\frac{X+Y}{X}\right)X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-\ln(u+1) + 1 + u}{\ln(u+1)} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-\ln(u(X)+1)+1+u(X)}{\ln(u(X)+1)} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-\ln(u(X)+1)+1+u(X)}{\ln(u(X)+1)} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)\ln(u(X)+1)X + u(X)\ln(u(X)+1) + \ln(u(X)+1) - u(X) - 1 = 0$$

Or

$$\left(X\left(\frac{d}{dX}u(X)\right) + u(X) + 1\right)\ln(u(X)+1) - u(X) - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X,u) \\
&= f(X)g(u) \\
&= -\frac{(u+1)\left(\ln\left(u+1\right)-1\right)}{\ln\left(u+1\right)X}
\end{aligned}
$$

Where $f(X)=-\frac{1}{X}$ and $g(u)=\frac{(u+1)(\ln(u+1)-1)}{\ln(u+1)}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{(u+1)(\ln(u+1)-1)}{\ln(u+1)}}\,du &= -\frac{1}{X}\,dX \\
\int \frac{1}{\frac{(u+1)(\ln(u+1)-1)}{\ln(u+1)}}\,du &= \int -\frac{1}{X}\,dX \\
\ln\left(u+1\right)+\ln\left(\ln\left(u+1\right)-1\right) &= -\ln\left(X\right)+c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{\ln(u+1)+\ln(\ln(u+1)-1)} = \mathrm{e}^{-\ln(X)+c_2}
$$

Which simplifies to

$$
(u+1)\left(\ln\left(u+1\right)-1\right) = \frac{c_3}{X}
$$

Now u in the above solution is replaced back by Y using $u=\frac{Y}{X}$ which results in the solution

$$
Y(X) = X\left(\mathrm{e}^{\mathrm{LambertW}\left(\frac{c_3\mathrm{e}^{-1}}{X}\right)+1}-1\right)
$$

Using the solution for $Y(X)$

$$
Y(X) = X\left(\mathrm{e}^{\mathrm{LambertW}\left(\frac{c_3\mathrm{e}^{-1}}{X}\right)+1}-1\right)
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y+y_0 \\
X &= x+x_0
\end{aligned}
$$

Or

$$Y = 3 + y$$
$$X = x - 3$$

Then the solution in y becomes

$$y - 3 = (x + 3)\left(e^{\operatorname{LambertW}\left(\frac{c_3 e^{-1}}{x+3}\right)+1} - 1\right)$$

Summary

The solution(s) found are the following

$$y - 3 = (x + 3)\left(e^{\operatorname{LambertW}\left(\frac{c_3 e^{-1}}{x+3}\right)+1} - 1\right) \tag{1}$$

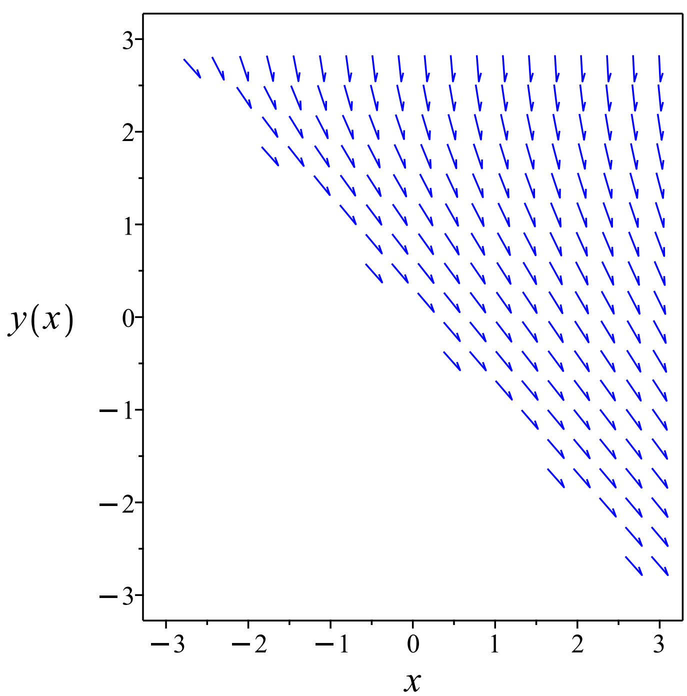


Figure 84: Slope field plot

Verification of solutions

$$y - 3 = (x + 3)\left(e^{\operatorname{LambertW}\left(\frac{c_3 e^{-1}}{x+3}\right)+1} - 1\right)$$

Verified OK.

2.41.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\ln\left(\frac{x+y}{x+3}\right) x - y + 3\ln\left(\frac{x+y}{x+3}\right) - x}{\ln\left(\frac{x+y}{x+3}\right)(x+3)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{\left(\ln\left(\frac{x+y}{x+3}\right) x - y + 3\ln\left(\frac{x+y}{x+3}\right) - x\right)(b_3 - a_2)}{\ln\left(\frac{x+y}{x+3}\right)(x+3)} \\
&- \frac{\left(\ln\left(\frac{x+y}{x+3}\right) x - y + 3\ln\left(\frac{x+y}{x+3}\right) - x\right)^2 a_3}{\ln\left(\frac{x+y}{x+3}\right)^2 (x+3)^2} \\
&- \left(-\frac{\frac{\left(\frac{1}{x+3} - \frac{x+y}{(x+3)^2}\right)(x+3)x}{x+y} + \ln\left(\frac{x+y}{x+3}\right) + \frac{3\left(\frac{1}{x+3} - \frac{x+y}{(x+3)^2}\right)(x+3)}{x+y} - 1}{\ln\left(\frac{x+y}{x+3}\right)(x+3)}\right. \\
&+ \frac{\left(\ln\left(\frac{x+y}{x+3}\right) x - y + 3\ln\left(\frac{x+y}{x+3}\right) - x\right)\left(\frac{1}{x+3} - \frac{x+y}{(x+3)^2}\right)}{\ln\left(\frac{x+y}{x+3}\right)^2 (x+y)} \\
&\left.+ \frac{\ln\left(\frac{x+y}{x+3}\right) x - y + 3\ln\left(\frac{x+y}{x+3}\right) - x}{\ln\left(\frac{x+y}{x+3}\right)(x+3)^2}\right)(x a_2 + y a_3 + a_1) - \left(-\frac{\frac{x}{x+y} - 1 + \frac{3}{x+y}}{\ln\left(\frac{x+y}{x+3}\right)(x+3)}\right. \\
&\left.+ \frac{\ln\left(\frac{x+y}{x+3}\right) x - y + 3\ln\left(\frac{x+y}{x+3}\right) - x}{\ln\left(\frac{x+y}{x+3}\right)^2 (x+3)(x+y)}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$\frac{3a_1 + 3b_1 + 2\ln\left(\frac{x+y}{x+3}\right) xya_3 - 2y^2a_3 - ya_1 - x^2a_3 + x^2b_2 + xb_1 + 9\ln\left(\frac{x+y}{x+3}\right)^2 a_2 - 9\ln\left(\frac{x+y}{x+3}\right)^2 a_3 + 9b_2\ln\left(\right.}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&3a_1 + 3b_1 + 2\ln\left(\frac{x+y}{x+3}\right) xya_3 - 2y^2a_3 - ya_1 - x^2a_3 + x^2b_2 \\
&+ xb_1 + 9\ln\left(\frac{x+y}{x+3}\right)^2 a_2 - 9\ln\left(\frac{x+y}{x+3}\right)^2 a_3 + 9b_2\ln\left(\frac{x+y}{x+3}\right)^2 \\
&- 9\ln\left(\frac{x+y}{x+3}\right)^2 b_3 - 3\ln\left(\frac{x+y}{x+3}\right) a_1 - 3\ln\left(\frac{x+y}{x+3}\right) b_1 \\
&+ 3xa_2 + 3ya_3 + 3xb_2 + 3yb_3 - xya_2 - 2xya_3 + xyb_3 \\
&+ \ln\left(\frac{x+y}{x+3}\right)^2 x^2a_2 - \ln\left(\frac{x+y}{x+3}\right)^2 x^2a_3 + \ln\left(\frac{x+y}{x+3}\right)^2 x^2b_2 \\
&- \ln\left(\frac{x+y}{x+3}\right)^2 x^2b_3 + 6\ln\left(\frac{x+y}{x+3}\right)^2 xa_2 - 6\ln\left(\frac{x+y}{x+3}\right)^2 xa_3 \\
&+ 6\ln\left(\frac{x+y}{x+3}\right)^2 xb_2 - 6\ln\left(\frac{x+y}{x+3}\right)^2 xb_3 - \ln\left(\frac{x+y}{x+3}\right) x^2a_2 \\
&+ 2\ln\left(\frac{x+y}{x+3}\right) x^2a_3 - \ln\left(\frac{x+y}{x+3}\right) x^2b_2 + \ln\left(\frac{x+y}{x+3}\right) x^2b_3 \\
&+ \ln\left(\frac{x+y}{x+3}\right) y^2a_3 - 6\ln\left(\frac{x+y}{x+3}\right) xa_2 + 6\ln\left(\frac{x+y}{x+3}\right) xa_3 \\
&- \ln\left(\frac{x+y}{x+3}\right) xb_1 - 3\ln\left(\frac{x+y}{x+3}\right) xb_2 + 3\ln\left(\frac{x+y}{x+3}\right) xb_3 \\
&+ \ln\left(\frac{x+y}{x+3}\right) ya_1 - 3\ln\left(\frac{x+y}{x+3}\right) ya_2 + 3\ln\left(\frac{x+y}{x+3}\right) ya_3 = 0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&(x+3)\Bigg(-x^2ya_2-3x^2ya_3+9\ln\left(\frac{x+y}{x+3}\right)xya_3\\
&+\ln\left(\frac{x+y}{x+3}\right)^2x^2ya_2-\ln\left(\frac{x+y}{x+3}\right)^2x^2ya_3\\
&+\ln\left(\frac{x+y}{x+3}\right)^2x^2yb_2-\ln\left(\frac{x+y}{x+3}\right)^2x^2yb_3\\
&+6\ln\left(\frac{x+y}{x+3}\right)^2xya_2-6\ln\left(\frac{x+y}{x+3}\right)^2xya_3\\
&+6\ln\left(\frac{x+y}{x+3}\right)^2xyb_2-6\ln\left(\frac{x+y}{x+3}\right)^2xyb_3\\
&-\ln\left(\frac{x+y}{x+3}\right)x^2ya_2+4\ln\left(\frac{x+y}{x+3}\right)x^2ya_3\\
&-\ln\left(\frac{x+y}{x+3}\right)x^2yb_2+\ln\left(\frac{x+y}{x+3}\right)x^2yb_3\\
&+3\ln\left(\frac{x+y}{x+3}\right)x\,y^2a_3+\ln\left(\frac{x+y}{x+3}\right)xya_1\\
&-9\ln\left(\frac{x+y}{x+3}\right)xya_2-\ln\left(\frac{x+y}{x+3}\right)xyb_1\\
&-3\ln\left(\frac{x+y}{x+3}\right)xyb_2+3\ln\left(\frac{x+y}{x+3}\right)xyb_3\\
&+3y^2a_3+3ya_1+3x^2b_2+3xb_1+x^2yb_2+x^2yb_3\\
&-x\,y^2a_2-4x\,y^2a_3+x\,y^2b_3-xya_1+xyb_1\\
&+3xyb_2+\ln\left(\frac{x+y}{x+3}\right)^2x^3a_2-\ln\left(\frac{x+y}{x+3}\right)^2x^3a_3\\
&+\ln\left(\frac{x+y}{x+3}\right)^2x^3b_2-\ln\left(\frac{x+y}{x+3}\right)^2x^3b_3\\
&-\ln\left(\frac{x+y}{x+3}\right)x^3a_2+2\ln\left(\frac{x+y}{x+3}\right)x^3a_3\\
&-\ln\left(\frac{x+y}{x+3}\right)x^3b_2+\ln\left(\frac{x+y}{x+3}\right)x^3b_3\\
&+\ln\left(\frac{x+y}{x+3}\right)y^3a_3+9\ln\left(\frac{x+y}{x+3}\right)^2ya_2\\
&-9\ln\left(\frac{x+y}{x+3}\right)^2ya_3+9\ln\left(\frac{x+y}{x+3}\right)^2yb_2\\
&-9\ln\left(\frac{x+y}{x+3}\right)^2yb_3-\ln\left(\frac{x+y}{x+3}\right)x^2b_1\\
&+\ln\left(\frac{x+y}{x+3}\right)y^2a_1-3\ln\left(\frac{x+y}{x+3}\right)y^2a_2\\
&-3\ln\left(\frac{x+y}{x+3}\right)xa_1-3\ln\left(\frac{x+y}{x+3}\right)yb_1-x^3a_3+x^3b_2\\
&-2y^3a_3+3x^2a_2+x^2b_1-y^2a_1+3y^2b_3+3xa_1+3yb_1
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \ln\left(\frac{x+y}{x+3}\right)\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \ln\left(\frac{x+y}{x+3}\right) = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&(v_1+3)\left(v_3^2v_1^3a_2 + v_3^2v_1^2v_2a_2 - v_3^2v_1^3a_3 - v_3^2v_1^2v_2a_3 + v_3^2v_1^3b_2\right.\\
&+ v_3^2v_1^2v_2b_2 - v_3^2v_1^3b_3 - v_3^2v_1^2v_2b_3 - v_3v_1^3a_2 - v_3v_1^2v_2a_2 + 6v_3^2v_1^2a_2\\
&+ 6v_3^2v_1v_2a_2 + 2v_3v_1^3a_3 + 4v_3v_1^2v_2a_3 - 6v_3^2v_1^2a_3 + 3v_3v_1v_2^2a_3\\
&- 6v_3^2v_1v_2a_3 + v_3v_2^3a_3 - v_3v_1^3b_2 - v_3v_1^2v_2b_2 + 6v_3^2v_1^2b_2 + 6v_3^2v_1v_2b_2\\
&+ v_3v_1^3b_3 + v_3v_1^2v_2b_3 - 6v_3^2v_1^2b_3 - 6v_3^2v_1v_2b_3 + v_3v_1v_2a_1 + v_3v_2^2a_1\\
&- v_1^2v_2a_2 - 6v_3v_1^2a_2 - v_1v_2^2a_2 - 9v_3v_1v_2a_2 + 9v_3^2v_1a_2 - 3v_3v_2^2a_2\\
&+ 9v_3^2v_2a_2 - v_1^3a_3 - 3v_1^2v_2a_3 + 6v_3v_1^2a_3 - 4v_1v_2^2a_3 + 9v_3v_1v_2a_3\\
&- 9v_3^2v_1a_3 - 2v_2^3a_3 + 3v_3v_2^2a_3 - 9v_3^2v_2a_3 - v_3v_1^2b_1 - v_3v_1v_2b_1\\
&+ v_1^3b_2 + v_1^2v_2b_2 - 3v_3v_1^2b_2 - 3v_3v_1v_2b_2 + 9v_3^2v_1b_2 + 9v_3^2v_2b_2\\
&+ v_1^2v_2b_3 + 3v_3v_1^2b_3 + v_1v_2^2b_3 + 3v_3v_1v_2b_3 - 9v_3^2v_1b_3 - 9v_3^2v_2b_3\\
&- v_1v_2a_1 - 3v_3v_1a_1 - v_2^2a_1 - 3v_3v_2a_1 + 3v_1^2a_2 + 3v_1v_2a_2\\
&+ 3v_1v_2a_3 + 3v_2^2a_3 + v_1^2b_1 + v_1v_2b_1 - 3v_3v_1b_1 - 3v_3v_2b_1 + 3v_1^2b_2\\
&\left.+ 3v_1v_2b_2 + 3v_1v_2b_3 + 3v_2^2b_3 + 3v_1a_1 + 3v_2a_1 + 3v_1b_1 + 3v_2b_1\right) = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(a_2 - a_3 + b_2 - b_3)\, v_3^2 v_1^4 + (a_2 - a_3 + b_2 - b_3)\, v_2 v_3^2 v_1^3 + (-a_2 + 4a_3 - b_2 + b_3)\, v_2 v_3 v_1^3 \\
&+ 3a_3 v_2^2 v_3 v_1^2 + (9a_2 - 9a_3 + 9b_2 - 9b_3)\, v_2 v_3^2 v_1^2 + (a_1 - 12a_2 + 21a_3 - b_1 - 6b_2 + 6b_3)\, v_2 v_3 v_1^2 \\
&+ a_3 v_2^3 v_3 v_1 + (a_1 - 3a_2 + 12a_3)\, v_2^2 v_3 v_1 + (27a_2 - 27a_3 + 27b_2 - 27b_3)\, v_2 v_3^2 v_1 \\
&+ (-27a_2 + 27a_3 - 6b_1 - 9b_2 + 9b_3)\, v_2 v_3 v_1 + (3a_1 - 9a_2 + 9a_3)\, v_2^2 v_3 \\
&+ (27a_2 - 27a_3 + 27b_2 - 27b_3)\, v_2 v_3^2 + (-9a_1 - 9b_1)\, v_2 v_3 - 6v_2^3 a_3 + (-a_2 - 4a_3 + b_3)\, v_2^2 v_1^2 \\
&+ (-a_1 - 6a_3 + b_1 + 6b_2 + 6b_3)\, v_2 v_1^2 + (27a_2 - 27a_3 + 27b_2 - 27b_3)\, v_3^2 v_1^2 \\
&+ (-3a_1 - 18a_2 + 18a_3 - 6b_1 - 9b_2 + 9b_3)\, v_3 v_1^2 + (3a_1 + 9a_2 + 6b_1 + 9b_2)\, v_1^2 \\
&- 2a_3 v_2^3 v_1 + (-a_1 - 3a_2 - 9a_3 + 6b_3)\, v_2^2 v_1 + (9a_2 + 9a_3 + 6b_1 + 9b_2 + 9b_3)\, v_2 v_1 \\
&+ (27a_2 - 27a_3 + 27b_2 - 27b_3)\, v_3^2 v_1 + (-9a_1 - 9b_1)\, v_3 v_1 + (9a_1 + 9b_1)\, v_1 \\
&+ (-3a_1 + 9a_3 + 9b_3)\, v_2^2 + (9a_1 + 9b_1)\, v_2 + 3v_3 v_2^3 a_3 + (-a_2 + 2a_3 - b_2 + b_3)\, v_3 v_1^4 \\
&+ (-a_3 + b_2)\, v_1^4 + (-a_2 - 3a_3 + b_2 + b_3)\, v_2 v_1^3 + (9a_2 - 9a_3 + 9b_2 - 9b_3)\, v_3^2 v_1^3 \\
&+ (-9a_2 + 12a_3 - b_1 - 6b_2 + 6b_3)\, v_3 v_1^3 + (3a_2 - 3a_3 + b_1 + 6b_2)\, v_1^3 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_3 &= 0\\
-6a_3 &= 0\\
-2a_3 &= 0\\
3a_3 &= 0\\
-9a_1 - 9b_1 &= 0\\
9a_1 + 9b_1 &= 0\\
-a_3 + b_2 &= 0\\
-3a_1 + 9a_3 + 9b_3 &= 0\\
a_1 - 3a_2 + 12a_3 &= 0\\
3a_1 - 9a_2 + 9a_3 &= 0\\
-a_2 - 4a_3 + b_3 &= 0\\
-a_1 - 3a_2 - 9a_3 + 6b_3 &= 0\\
3a_1 + 9a_2 + 6b_1 + 9b_2 &= 0\\
-a_2 - 3a_3 + b_2 + b_3 &= 0\\
-a_2 + 2a_3 - b_2 + b_3 &= 0\\
-a_2 + 4a_3 - b_2 + b_3 &= 0\\
a_2 - a_3 + b_2 - b_3 &= 0\\
3a_2 - 3a_3 + b_1 + 6b_2 &= 0\\
9a_2 - 9a_3 + 9b_2 - 9b_3 &= 0\\
27a_2 - 27a_3 + 27b_2 - 27b_3 &= 0\\
-a_1 - 6a_3 + b_1 + 6b_2 + 6b_3 &= 0\\
-27a_2 + 27a_3 - 6b_1 - 9b_2 + 9b_3 &= 0\\
-9a_2 + 12a_3 - b_1 - 6b_2 + 6b_3 &= 0\\
9a_2 + 9a_3 + 6b_1 + 9b_2 + 9b_3 &= 0\\
-3a_1 - 18a_2 + 18a_3 - 6b_1 - 9b_2 + 9b_3 &= 0\\
a_1 - 12a_2 + 21a_3 - b_1 - 6b_2 + 6b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 3b_3 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= -3b_3 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x+3 \\
\eta &= -3+y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -3+y-\left(-\frac{\ln\left(\frac{x+y}{x+3}\right)x - y + 3\ln\left(\frac{x+y}{x+3}\right) - x}{\ln\left(\frac{x+y}{x+3}\right)(x+3)}\right)(x+3) \\
&= \frac{\ln\left(\frac{x+y}{x+3}\right)x + \ln\left(\frac{x+y}{x+3}\right)y - x - y}{\ln\left(\frac{x+y}{x+3}\right)} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{\ln\left(\frac{x+y}{x+3}\right)x+\ln\left(\frac{x+y}{x+3}\right)y-x-y}{\ln\left(\frac{x+y}{x+3}\right)}} dy
\end{aligned}
$$

Which results in

$$
S = \ln\left(\ln\left(\frac{x+y}{x+3}\right)x+\ln\left(\frac{x+y}{x+3}\right)y-x-y\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{\ln\left(\frac{x+y}{x+3}\right)x - y + 3\ln\left(\frac{x+y}{x+3}\right) - x}{\ln\left(\frac{x+y}{x+3}\right)(x+3)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{x+y} + \frac{-3+y}{(x+3)(x+y)(\ln(x+3)-\ln(x+y)+1)} \\
S_y &= \frac{1}{x+y} + \frac{1}{(x+y)(-\ln(x+3)+\ln(x+y)-1)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{\ln\left(\frac{x+y}{x+3}\right) - \ln(x+y) + \ln(x+3)}{(\ln(x+3)-\ln(x+y)+1)\ln\left(\frac{x+y}{x+3}\right)(x+3)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x+y) + \ln(-\ln(x+3) + \ln(x+y) - 1) = c_1$$

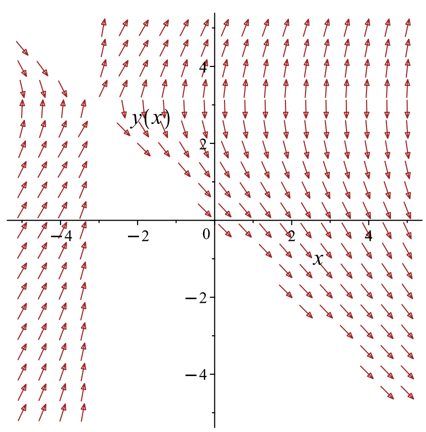
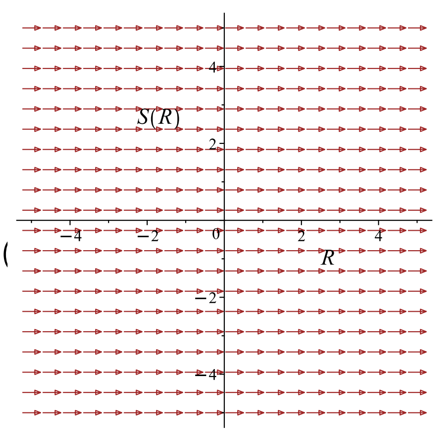
Which simplifies to

$$\ln(x+y) + \ln(-\ln(x+3) + \ln(x+y) - 1) = c_1$$

Which gives

$$y = e^{\operatorname{LambertW}\left(\frac{e^{-1+c_1}}{x+3}\right)+1} x + 3\, e^{\operatorname{LambertW}\left(\frac{e^{-1+c_1}}{x+3}\right)+1} - x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\ln\left(\frac{x+y}{x+3}\right)x - y + 3\ln\left(\frac{x+y}{x+3}\right) - x}{\ln\left(\frac{x+y}{x+3}\right)(x+3)}$ 	$R = x$ $S = \ln(x+y) + \ln(-\ln($	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = e^{\operatorname{LambertW}\left(\frac{e^{-1+c_1}}{x+3}\right)+1} x + 3\, e^{\operatorname{LambertW}\left(\frac{e^{-1+c_1}}{x+3}\right)+1} - x \tag{1}$$

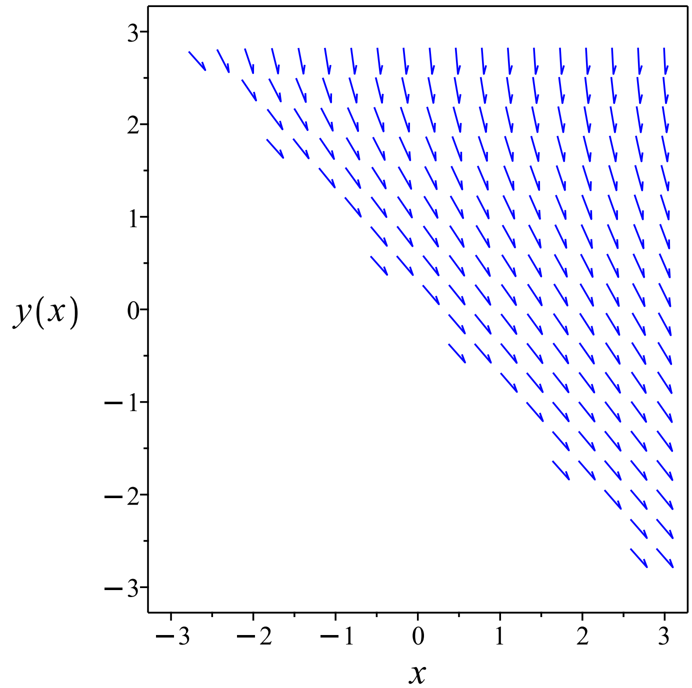


Figure 85: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\mathrm{LambertW}\left(\frac{\mathrm{e}^{-1+c_1}}{x+3}\right)+1} x + 3\,\mathrm{e}^{\mathrm{LambertW}\left(\frac{\mathrm{e}^{-1+c_1}}{x+3}\right)+1} - x$$

Verified OK.

2.41.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\left(\ln\left(\frac{x+y}{x+3}\right)\right) \mathrm{d}y = \left(-\ln\left(\frac{x+y}{x+3}\right) + \frac{x+y}{x+3}\right) \mathrm{d}x
$$

$$
\left(\ln\left(\frac{x+y}{x+3}\right) - \frac{x+y}{x+3}\right) \mathrm{d}x + \left(\ln\left(\frac{x+y}{x+3}\right)\right) \mathrm{d}y = 0 \tag{2A}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= \ln\left(\frac{x+y}{x+3}\right) - \frac{x+y}{x+3} \\
N(x, y) &= \ln\left(\frac{x+y}{x+3}\right)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\ln\left(\frac{x+y}{x+3}\right) - \frac{x+y}{x+3}\right) \\
&= \frac{1}{x+y} - \frac{1}{x+3}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\ln\left(\frac{x+y}{x+3}\right)\right) \\
&= \frac{3-y}{(x+3)(x+y)}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int M \,\mathrm{d}x
$$

$$
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int \ln\left(\frac{x+y}{x+3}\right) - \frac{x+y}{x+3}\,\mathrm{d}x
$$

$$
\phi = (3-y)\ln\left(\frac{-3+y}{x+3}\right) + \ln\left(\frac{x+y}{x+3}\right)(x+y) + (3-y)\ln(x+3) - x + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= -\ln\left(\frac{-3+y}{x+3}\right) + \frac{3-y}{-3+y} + 1 + \ln\left(\frac{x+y}{x+3}\right) - \ln(x+3) + f'(y) \qquad (4) \\
&= -\ln\left(\frac{-3+y}{x+3}\right) + \ln\left(\frac{x+y}{x+3}\right) - \ln(x+3) + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \ln\left(\frac{x+y}{x+3}\right)$. Therefore equation (4) becomes

$$
\ln\left(\frac{x+y}{x+3}\right) = -\ln\left(\frac{-3+y}{x+3}\right) + \ln\left(\frac{x+y}{x+3}\right) - \ln(x+3) + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \ln\left(\frac{-3+y}{x+3}\right) + \ln(x+3)
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\ln\left(\frac{-3+y}{x+3}\right)+\ln\left(x+3\right)\right)\mathrm{d}y$$

$$f(y) = (x+3)\left(\left(\frac{y}{x+3}-\frac{3}{x+3}\right)\ln\left(\frac{y}{x+3}-\frac{3}{x+3}\right)-\frac{y}{x+3}+\frac{3}{x+3}\right)+y\ln\left(x+3\right)+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = (3-y)\ln\left(\frac{-3+y}{x+3}\right)+\ln\left(\frac{x+y}{x+3}\right)(x+y)+(3-y)\ln\left(x+3\right)-x + (x+3)\left(\left(\frac{y}{x+3}-\frac{3}{x+3}\right)\ln\left(\frac{y}{x+3}-\frac{3}{x+3}\right)-\frac{y}{x+3}+\frac{3}{x+3}\right)+y\ln\left(x+3\right)+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (3-y)\ln\left(\frac{-3+y}{x+3}\right)+\ln\left(\frac{x+y}{x+3}\right)(x+y)+(3-y)\ln\left(x+3\right)-x + (x+3)\left(\left(\frac{y}{x+3}-\frac{3}{x+3}\right)\ln\left(\frac{y}{x+3}-\frac{3}{x+3}\right)-\frac{y}{x+3}+\frac{3}{x+3}\right)+y\ln\left(x+3\right)$$

The solution becomes

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-\frac{(3-c_1+3\ln(x+3))\mathrm{e}^{-1}}{x+3}\right)+1}x+3\,\mathrm{e}^{\mathrm{LambertW}\left(-\frac{(3-c_1+3\ln(x+3))\mathrm{e}^{-1}}{x+3}\right)+1}-x$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-\frac{(3-c_1+3\ln(x+3))\mathrm{e}^{-1}}{x+3}\right)+1}x+3\,\mathrm{e}^{\mathrm{LambertW}\left(-\frac{(3-c_1+3\ln(x+3))\mathrm{e}^{-1}}{x+3}\right)+1}-x \tag{1}$$

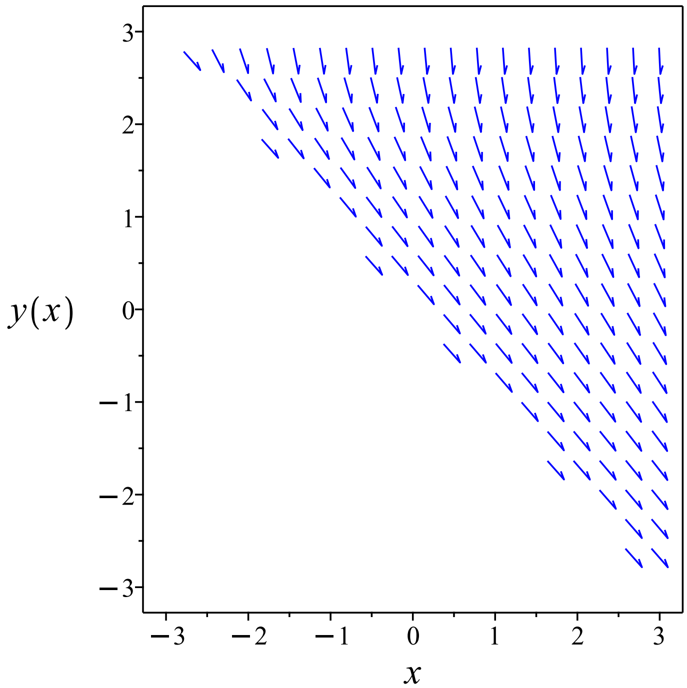


Figure 86: Slope field plot

Verification of solutions

$$y = e^{\text{LambertW}\left(-\frac{(3-c_1+3\ln(x+3))e^{-1}}{x+3}\right)+1} x + 3\,e^{\text{LambertW}\left(-\frac{(3-c_1+3\ln(x+3))e^{-1}}{x+3}\right)+1} - x$$

Verified OK.

2.41.4 Maple step by step solution

Let's solve

$(1 + y')\ln\left(\frac{x+y}{x+3}\right) - \frac{x+y}{x+3} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ∘ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - ∘ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $\frac{1}{x+y}-\frac{1}{x+3}=\frac{\left(\frac{1}{x+3}-\frac{x+y}{(x+3)^2}\right)(x+3)}{x+y}$

- Simplify

  $\frac{1}{x+y}-\frac{1}{x+3}=\frac{3-y}{(x+3)(x+y)}$

- Condition met, ODE is exact

* Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

* Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int \left(\ln\left(\frac{x+y}{x+3}\right)-\frac{x+y}{x+3}\right)dx+f_1(y)$

* Evaluate integral

  $F(x,y)=-x-(-3+y)\ln(x+3)-(-3+y)\left(\ln\left(\frac{-3+y}{x+3}\right)-\frac{\ln\left(1+\frac{-3+y}{x+3}\right)(x+3)\left(1+\frac{-3+y}{x+3}\right)}{-3+y}\right)+f_1(y)$

* Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

* Compute derivative

  $\ln\left(\frac{x+y}{x+3}\right)=-\ln(x+3)-\ln\left(\frac{-3+y}{x+3}\right)+\frac{\ln\left(1+\frac{-3+y}{x+3}\right)(x+3)\left(1+\frac{-3+y}{x+3}\right)}{-3+y}-(-3+y)\left(\frac{\ln\left(1+\frac{-3+y}{x+3}\right)(x+3)\left(1+\frac{-3+}{x+}\right.}{(-3+y)^2}\right.$

* Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=\ln\left(\frac{x+y}{x+3}\right)+\ln(x+3)+\ln\left(\frac{-3+y}{x+3}\right)-\frac{\ln\left(1+\frac{-3+y}{x+3}\right)(x+3)\left(1+\frac{-3+y}{x+3}\right)}{-3+y}+(-3+y)\left(\frac{\ln\left(1+\frac{-3+y}{x+3}\right)(x+}{(-3+}\right.$

* Solve for $f_1(y)$

  $f_1(y)=-(x+3)\,dilog\left(\frac{y}{x+3}+1-\frac{3}{x+3}\right)+(-x-3)(x+3)\left(\frac{\left(\frac{y}{x+3}+1-\frac{3}{x+3}\right)\ln\left(\frac{y}{x+3}+1-\frac{3}{x+3}\right)-\frac{y}{x+3}-1+\frac{3}{x+}}{x+3}\right.$

* Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=-x-(-3+y)\ln(x+3)-(-3+y)\left(\ln\left(\frac{-3+y}{x+3}\right)-\frac{\ln\left(1+\frac{-3+y}{x+3}\right)(x+3)\left(1+\frac{-3+y}{x+3}\right)}{-3+y}\right)-(x+3$

* Substitute $F(x,y)$ into the solution of the ODE

  $-x-(-3+y)\ln(x+3)-(-3+y)\left(\ln\left(\frac{-3+y}{x+3}\right)-\frac{\ln\left(1+\frac{-3+y}{x+3}\right)(x+3)\left(1+\frac{-3+y}{x+3}\right)}{-3+y}\right)-(x+3)\,dilog\left(\frac{y}{x+3}\right.$

* Solve for y

$$y = \mathrm{e}^{\mathit{LambertW}\left(-\frac{3-c_1+3\ln(x+3)}{\mathrm{e}(x+3)}\right)+1}x + 3\,\mathrm{e}^{\mathit{LambertW}\left(-\frac{3-c_1+3\ln(x+3)}{\mathrm{e}(x+3)}\right)+1} - x$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.25 (sec). Leaf size: 40

```
dsolve((diff(y(x),x)+1)*ln((y(x)+x)/(x+3))=(y(x)+x)/(x+3),y(x), singsol=all)
```

$$y(x) = \frac{-x\,\mathrm{LambertW}\left(\frac{\mathrm{e}^{-1}}{(x+3)c_1}\right)c_1 + 1}{\mathrm{LambertW}\left(\frac{\mathrm{e}^{-1}}{(x+3)c_1}\right)c_1}$$

✓ Solution by Mathematica

Time used: 0.226 (sec). Leaf size: 30

```
DSolve[(y'[x]+1)*Log[(y[x]+x)/(x+3)]==(y[x]+x)/(x+3),y[x],x,IncludeSingularSolutions -> True
```

$$\mathrm{Solve}\left[-y(x) + (y(x)+x)\log\left(\frac{y(x)+x}{x+3}\right) - x = c_1, y(x)\right]$$

2.42 problem 40

Internal problem ID [5790]
Internal file name [OUTPUT/5038_Sunday_June_05_2022_03_18_32_PM_90959887/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{x - 2y + 5}{y - 2x - 4} = 0}$$

2.42.1 Solving as polynomial ode

This is ODE of type polynomial. Where the RHS of the ode is ratio of equations of two lines. Writing the ODE in the form

$$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_3}$$

Where $a_1 = -1, b_1 = 2, c_1 = -5, a_2 = 2, b_2 = -1, c_2 = 4$. There are now two possible solution methods. The first case is when the two lines $a_1x + b_1y + c_1$,$a_2x + b_2y + c_3$ are not parallel and the second case is if they are parallel. If they are not parallel, then the transformation $X = x - x_0$, $Y = y - y_0$ converts the ODE to a homogeneous ODE. The values x_0, y_0 have to be determined. If they are parallel then a transformation $U(x) = a_1x + b_1y$ converts the given ODE in y to a separable ODE in $U(x)$. The first case is when $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ and the second case when $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. From the above we see that

$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$. Hence this is case one where lines are not parallel. Using the transformation

$$
\begin{aligned}
X &= x - x_0 \\
Y &= y - y_0
\end{aligned}
$$

Where the constants x_0, y_0 are obtained by solving the following two linear algebraic equations

$$
\begin{aligned}
a_1 x_0 + b_1 y_0 + c_1 &= 0 \\
a_2 x_0 + b_2 y_0 + c_2 &= 0
\end{aligned}
$$

Substituting the values for $a_1, b_1, c_1, a_2, b_2, c_2$ gives

$$
\begin{aligned}
-x_0 + 2y_0 - 5 &= 0 \\
2x_0 - y_0 + 4 &= 0
\end{aligned}
$$

Solving for x_0, y_0 from the above gives

$$
\begin{aligned}
x_0 &= -1 \\
y_0 &= 2
\end{aligned}
$$

Therefore the transformation becomes

$$
\begin{aligned}
X &= x + 1 \\
Y &= y - 2
\end{aligned}
$$

Using this transformation in $y' - \frac{x-2y+5}{y-2x-4} = 0$ result in

$$
\frac{dY}{dX} = \frac{-X + 2Y}{-Y + 2X}
$$

This is now a homogeneous ODE which will now be solved for $Y(X)$. In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{-X + 2Y}{Y - 2X} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -X + 2Y$ and $N = -Y + 2X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}X}X + u = \frac{-2u+1}{u-2}$$

$$\frac{\mathrm{d}u}{\mathrm{d}X} = \frac{\frac{-2u(X)+1}{u(X)-2} - u(X)}{X}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-2u(X)+1}{u(X)-2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) - 2\left(\frac{d}{dX}u(X)\right)X + u(X)^2 - 1 = 0$$

Or

$$X(u(X)-2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{u^2-1}{X(u-2)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-1}{u-2}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{u-2}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2-1}{u-2}}\,du = \int -\frac{1}{X}\,dX$$

$$-\frac{\ln(u-1)}{2} + \frac{3\ln(u+1)}{2} = -\ln(X) + c_3$$

The above can be written as

$$
\begin{aligned}
\frac{-\ln(u-1)+3\ln(u+1)}{2} &= -\ln(X)+c_3 \\
-\ln(u-1)+3\ln(u+1) &= (2)(-\ln(X)+c_3) \\
&= -2\ln(X)+2c_3
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{-\ln(u-1)+3\ln(u+1)} = e^{-2\ln(X)+2c_3}
$$

Which simplifies to

$$
\begin{aligned}
\frac{(u+1)^3}{u-1} &= \frac{2c_3}{X^2} \\
&= \frac{c_4}{X^2}
\end{aligned}
$$

Which simplifies to

$$
\frac{(u(X)+1)^3}{u(X)-1} = \frac{c_4 e^{2c_3}}{X^2}
$$

The solution is

$$
\frac{(u(X)+1)^3}{u(X)-1} = \frac{c_4 e^{2c_3}}{X^2}
$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$
\frac{\left(\frac{Y(X)}{X}+1\right)^3}{\frac{Y(X)}{X}-1} = \frac{c_4 e^{2c_3}}{X^2}
$$

Which simplifies to

$$
-\frac{(Y(X)+X)^3}{-Y(X)+X} = c_4 e^{2c_3}
$$

The solution is implicit $-\frac{(Y(X)+X)^3}{-Y(X)+X} = c_4 e^{2c_3}$. Replacing $Y = y - y_0, X = x - x_0$ gives

$$
-\frac{(-1+y+x)^3}{3+x-y} = c_4 e^{2c_3}
$$

Summary

The solution(s) found are the following

$$
-\frac{(-1+y+x)^3}{3+x-y} = c_4 e^{2c_3} \tag{1}
$$

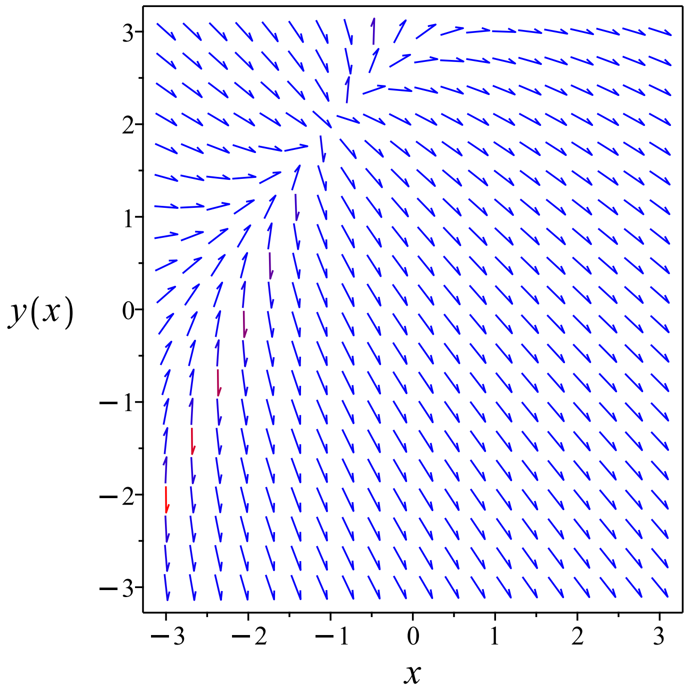


Figure 87: Slope field plot

<u>Verification of solutions</u>

$$-\frac{(-1+y+x)^3}{3+x-y} = c_4 e^{2c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 117

```
dsolve(diff(y(x),x)=(x-2*y(x)+5)/(y(x)-2*x-4),y(x), singsol=all)
```

$$y(x) = \frac{\frac{1}{2} + \frac{\left(1-i\sqrt{3}\right)\left(27(x+1)c_1+3\sqrt{3}\sqrt{27(x+1)^2c_1^2-1}\right)^{\frac{2}{3}}}{6} + \frac{i\sqrt{3}}{2} - \left(3\sqrt{3}\sqrt{27\left(x+1\right)^2c_1^2-1}+27c_1x+27c_1\right)^{\frac{1}{3}}\left(x-1\right)c}{\left(27\left(x+1\right)c_1+3\sqrt{3}\sqrt{27\left(x+1\right)^2c_1^2-1}\right)^{\frac{1}{3}}c_1}$$

✓ Solution by Mathematica

Time used: 60.297 (sec). Leaf size: 1601

```
DSolve[y'[x]==(x-2*y[x]+5)/(y[x]-2*x-4),y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

2.43 problem 41

Internal problem ID [5791]
Internal file name [OUTPUT/5039_Sunday_June_05_2022_03_18_35_PM_8446703/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{3x - y + 1}{2x + y + 4} = 0}$$

2.43.1 Solving as polynomial ode

This is ODE of type polynomial. Where the RHS of the ode is ratio of equations of two lines. Writing the ODE in the form

$$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_3}$$

Where $a_1 = 3, b_1 = -1, c_1 = 1, a_2 = 2, b_2 = 1, c_2 = 4$. There are now two possible solution methods. The first case is when the two lines $a_1x + b_1y + c_1$,$a_2x + b_2y + c_3$ are not parallel and the second case is if they are parallel. If they are not parallel, then the transformation $X = x - x_0, Y = y - y_0$ converts the ODE to a homogeneous ODE. The values x_0, y_0 have to be determined. If they are parallel then a transformation $U(x) = a_1x + b_1y$ converts the given ODE in y to a separable ODE in $U(x)$. The first case is when $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ and the second case when $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. From the above we see that

$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$. Hence this is case one where lines are not parallel. Using the transformation

$$X = x - x_0$$
$$Y = y - y_0$$

Where the constants x_0, y_0 are obtained by solving the following two linear algebraic equations

$$a_1 x_0 + b_1 y_0 + c_1 = 0$$
$$a_2 x_0 + b_2 y_0 + c_2 = 0$$

Substituting the values for $a_1, b_1, c_1, a_2, b_2, c_2$ gives

$$3x_0 - y_0 + 1 = 0$$
$$2x_0 + y_0 + 4 = 0$$

Solving for x_0, y_0 from the above gives

$$x_0 = -1$$
$$y_0 = -2$$

Therefore the transformation becomes

$$X = x + 1$$
$$Y = y + 2$$

Using this transformation in $y' - \frac{3x-y+1}{2x+y+4} = 0$ result in

$$\frac{dY}{dX} = \frac{3X - Y}{2X + Y}$$

This is now a homogeneous ODE which will now be solved for $Y(X)$. In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{-3X + Y}{2X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = 3X - Y$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-u+3}{u+2} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-u(X)+3}{u(X)+2} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-u(X)+3}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) + 2\left(\frac{d}{dX}u(X)\right)X + u(X)^2 + 3u(X) - 3 = 0$$

Or

$$X(u(X)+2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 3u(X) - 3 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{u^2+3u-3}{X(u+2)}\end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+3u-3}{u+2}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u^2+3u-3}{u+2}}\,du &= -\frac{1}{X}\,dX \\ \int \frac{1}{\frac{u^2+3u-3}{u+2}}\,du &= \int -\frac{1}{X}\,dX\end{aligned}$$

$$\frac{\ln(u^2+3u-3)}{2} - \frac{\sqrt{21}\,\operatorname{arctanh}\left(\frac{(2u+3)\sqrt{21}}{21}\right)}{21} = -\ln(X) + c_3$$

The solution is

$$\frac{\ln\left(u(X)^2+3u(X)-3\right)}{2}-\frac{\sqrt{21}\,\operatorname{arctanh}\left(\frac{(2u(X)+3)\sqrt{21}}{21}\right)}{21}+\ln(X)-c_3=0$$

Now u in the above solution is replaced back by Y using $u=\frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+\frac{3Y(X)}{X}-3\right)}{2}-\frac{\sqrt{21}\,\operatorname{arctanh}\left(\frac{\left(\frac{2Y(X)}{X}+3\right)\sqrt{21}}{21}\right)}{21}+\ln(X)-c_3=0$$

The solution is implicit $\frac{\ln\left(\frac{Y(X)^2}{X^2}+\frac{3Y(X)}{X}-3\right)}{2}-\frac{\sqrt{21}\,\operatorname{arctanh}\left(\frac{(2Y(X)+3X)\sqrt{21}}{21X}\right)}{21}+\ln(X)-c_3=0$. Replacing $Y=y-y_0, X=x-x_0$ gives

$$\frac{\ln\left(\frac{(2+y)^2}{(1+x)^2}+\frac{6+3y}{1+x}-3\right)}{2}-\frac{\sqrt{21}\,\operatorname{arctanh}\left(\frac{(2y+7+3x)\sqrt{21}}{21+21x}\right)}{21}+\ln(1+x)-c_3=0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{(2+y)^2}{(1+x)^2}+\frac{6+3y}{1+x}-3\right)}{2}-\frac{\sqrt{21}\,\operatorname{arctanh}\left(\frac{(2y+7+3x)\sqrt{21}}{21+21x}\right)}{21}+\ln(1+x)-c_3=0 \quad (1)$$

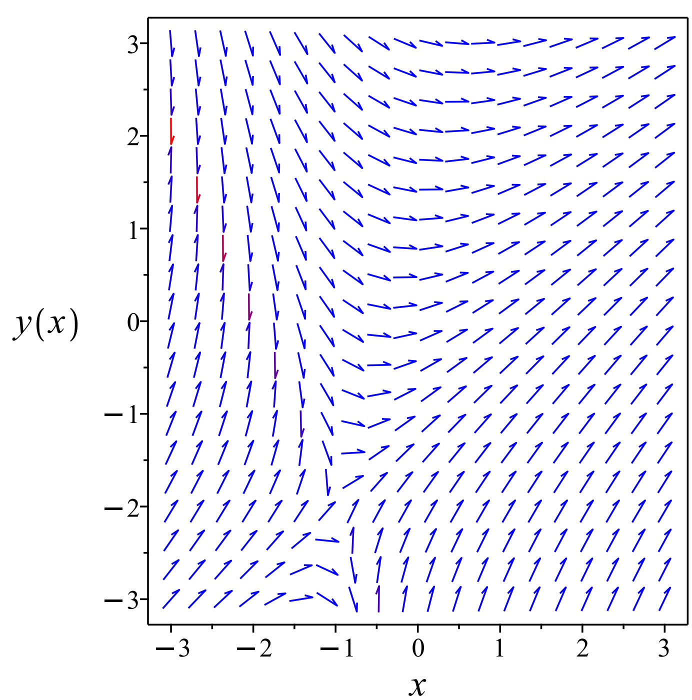


Figure 88: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{(2+y)^2}{(1+x)^2}+\frac{6+3y}{1+x}-3\right)}{2}-\frac{\sqrt{21}\,\operatorname{arctanh}\left(\frac{(2y+7+3x)\sqrt{21}}{21+21x}\right)}{21}+\ln(1+x)-c_3=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.797 (sec). Leaf size: 67

```
dsolve(diff(y(x),x)=(3*x-y(x)+1)/(2*x+y(x)+4),y(x), singsol=all)
```

$$-\frac{\ln\left(\frac{y(x)^2+(3x+7)y(x)-3x^2+7}{(x+1)^2}\right)}{2}+\frac{\sqrt{21}\,\operatorname{arctanh}\left(\frac{(2y(x)+7+3x)\sqrt{21}}{21x+21}\right)}{21}-\ln(x+1)-c_1=0$$

✓ Solution by Mathematica

Time used: 0.136 (sec). Leaf size: 79

```
DSolve[y'[x]==(3*x-y[x]+1)/(2*x+y[x]+4),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[2\sqrt{21}\operatorname{arctanh}\left(\frac{-\frac{10(x+1)}{y(x)+2(x+2)}-1}{\sqrt{21}}\right)\right.$$
$$\left.+21\left(\log\left(-\frac{-3x^2+y(x)^2+(3x+7)y(x)+7}{5(x+1)^2}\right)+2\log(x+1)-10c_1\right)=0,y(x)\right]$$

2.44 problem Example 5

Internal problem ID [5792]
Internal file name [`OUTPUT/5040_Sunday_June_05_2022_03_18_40_PM_42512006/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: Example 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$
\boxed{2xy' + \left(y^4x^2 + 1\right) y = 0}
$$

2.44.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{(y^4x^2 + 1)\, y}{2x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 47: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= y^5 x^2 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^5 x^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{4x^2 y^4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{(y^4 x^2 + 1) y}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{2x^3 y^4} \\ S_y &= \frac{1}{y^5 x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}$$

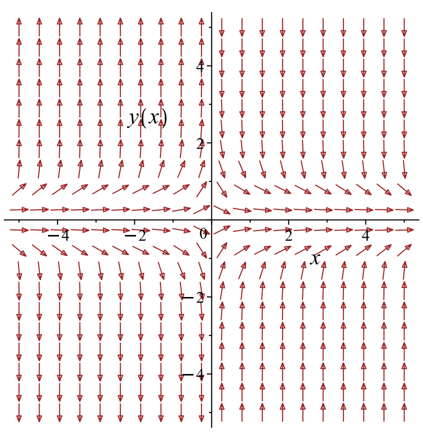
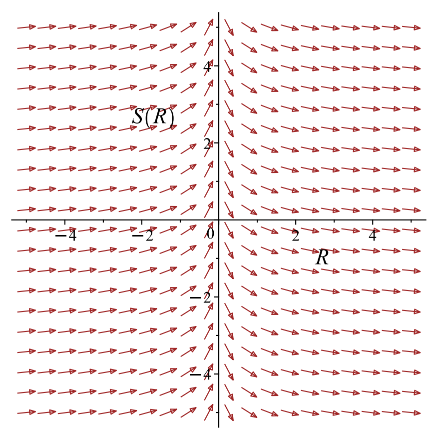
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{4x^2y^4} = -\frac{\ln(x)}{2} + c_1$$

Which simplifies to

$$-\frac{1}{4x^2y^4} = -\frac{\ln(x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{(y^4x^2+1)y}{2x}$ 	$R = x$ $S = -\frac{1}{4x^2y^4}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$-\frac{1}{4x^2y^4} = -\frac{\ln(x)}{2} + c_1 \tag{1}$$

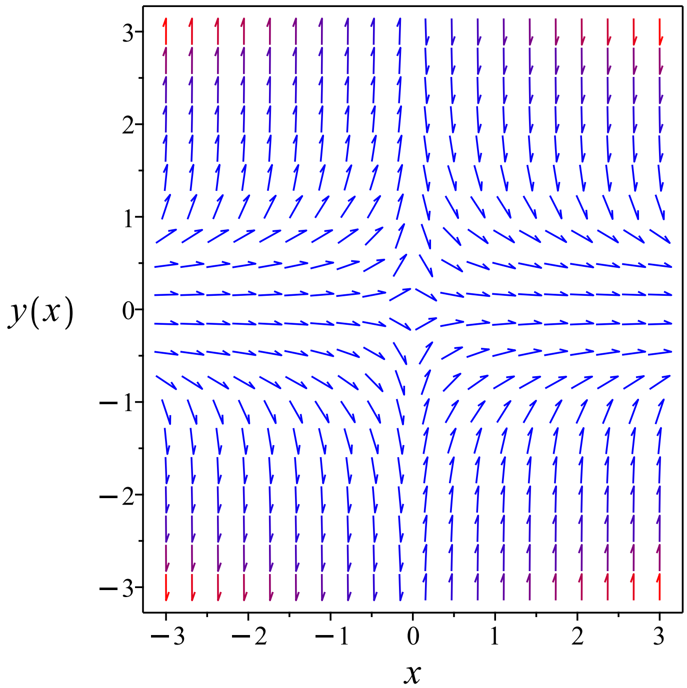


Figure 89: Slope field plot

Verification of solutions

$$-\frac{1}{4x^2y^4} = -\frac{\ln(x)}{2} + c_1$$

Verified OK.

2.44.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{(y^4x^2+1)\,y}{2x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{2x}y - \frac{x}{2}y^5 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{2x} \\
f_1(x) &= -\frac{x}{2} \\
n &= 5
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^5$ gives

$$
y' \frac{1}{y^5} = -\frac{1}{2x\,y^4} - \frac{x}{2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^4}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{4}{y^5} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{4} &= -\frac{w(x)}{2x} - \frac{x}{2} \\
w' &= \frac{2w}{x} + 2x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= 2x
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = 2x$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{2}{x}dx} \\
&= \frac{1}{x^2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(2x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)(2x) \\
\mathrm{d}\left(\frac{w}{x^2}\right) &= \left(\frac{2}{x}\right)\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{w}{x^2} &= \int \frac{2}{x}\,\mathrm{d}x \\
\frac{w}{x^2} &= 2\ln(x) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = 2\ln(x)\,x^2 + c_1 x^2$$

which simplifies to

$$w(x) = x^2(2\ln(x) + c_1)$$

Replacing w in the above by $\frac{1}{y^4}$ using equation (5) gives the final solution.

$$\frac{1}{y^4} = x^2(2\ln(x) + c_1)$$

Solving for y gives

$$y(x) = \frac{1}{\sqrt{\sqrt{2\ln\left(x\right)+c_1}\, x}}$$
$$y(x) = -\frac{1}{\sqrt{\sqrt{2\ln\left(x\right)+c_1}\, x}}$$
$$y(x) = -\frac{1}{\sqrt{-\sqrt{2\ln\left(x\right)+c_1}\, x}}$$
$$y(x) = \frac{1}{\sqrt{-\sqrt{2\ln\left(x\right)+c_1}\, x}}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{\sqrt{2\ln\left(x\right)+c_1}\, x}} \tag{1}$$
$$y = -\frac{1}{\sqrt{\sqrt{2\ln\left(x\right)+c_1}\, x}} \tag{2}$$
$$y = -\frac{1}{\sqrt{-\sqrt{2\ln\left(x\right)+c_1}\, x}} \tag{3}$$
$$y = \frac{1}{\sqrt{-\sqrt{2\ln\left(x\right)+c_1}\, x}} \tag{4}$$

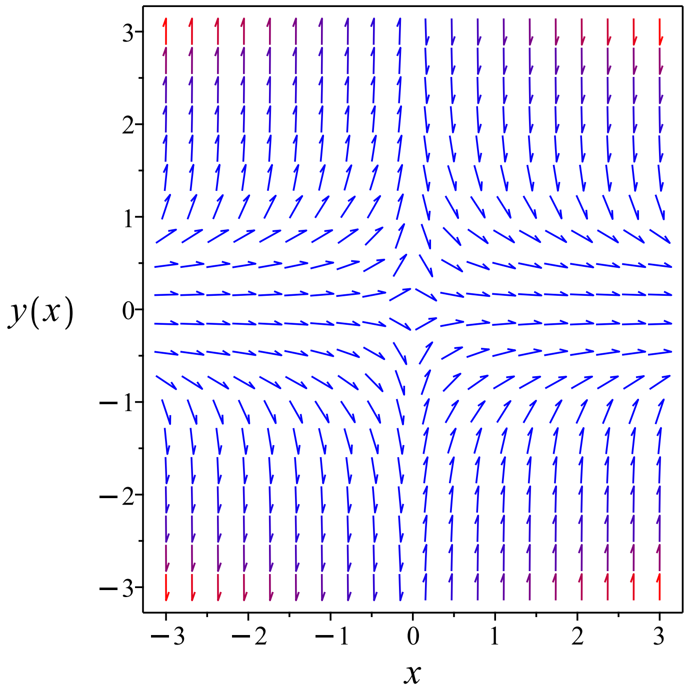


Figure 90: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{\sqrt{2\ln(x) + c_1}\, x}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{\sqrt{2\ln(x) + c_1}\, x}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{-\sqrt{2\ln(x) + c_1}\, x}}$$

Verified OK.

$$y = \frac{1}{\sqrt{-\sqrt{2\ln(x) + c_1}\, x}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 67

```
dsolve(2*x*diff(y(x),x)+(x^2*y(x)^4+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{\sqrt{2\ln(x) + c_1}\, x}}$$
$$y(x) = \frac{1}{\sqrt{-\sqrt{2\ln(x) + c_1}\, x}}$$
$$y(x) = -\frac{1}{\sqrt{\sqrt{2\ln(x) + c_1}\, x}}$$
$$y(x) = -\frac{1}{\sqrt{-\sqrt{2\ln(x) + c_1}\, x}}$$

✓ Solution by Mathematica

Time used: 1.552 (sec). Leaf size: 92

```
DSolve[2*x*y'[x]+(x^2*y[x]^4+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt[4]{x^2(2\log(x)+c_1)}}$$
$$y(x) \to -\frac{i}{\sqrt[4]{x^2(2\log(x)+c_1)}}$$
$$y(x) \to \frac{i}{\sqrt[4]{x^2(2\log(x)+c_1)}}$$
$$y(x) \to \frac{1}{\sqrt[4]{x^2(2\log(x)+c_1)}}$$
$$y(x) \to 0$$

2.45 problem Example 6

Internal problem ID [5793]
Internal file name [`OUTPUT/5041_Sunday_June_05_2022_03_18_44_PM_19033746/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: Example 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[[_homogeneous, `class G`], _rational]`

$$\boxed{2xy'(x - y^2) + y^3 = 0}$$

2.45.1 Solving as isobaric ode

Solving for y' gives

$$y' = \frac{y^3}{2x(-x + y^2)} \tag{1}$$

Each of the above ode's is now solved

Solving ode 1

An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = \frac{y^3}{2x(-x + y^2)} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = \frac{1}{2}$$

Since the ode is isobaric of order $m = \frac{1}{2}$, then the substitution

$$\begin{aligned} y &= x u^m \\ &= u\sqrt{x} \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$\frac{2xu'(x) + u(x)}{2\sqrt{x}} = \frac{u(x)^3}{\sqrt{x}\,\left(2u(x)^2 - 2\right)}$$

Or

$$u'(x) = \frac{u(x)}{2xu(x)^2 - 2x}$$

Which is now solved as separable in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u}{2x\left(u^2 - 1\right)} \end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(u) = \frac{u}{u^2-1}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u}{u^2-1}}\,du &= \frac{1}{2x}\,dx \\ \int \frac{1}{\frac{u}{u^2-1}}\,du &= \int \frac{1}{2x}\,dx \\ \frac{u^2}{2} - \ln(u) &= \frac{\ln(x)}{2} + c_1 \end{aligned}$$

The solution is

$$\frac{u(x)^2}{2} - \ln(u(x)) - \frac{\ln(x)}{2} - c_1 = 0$$

Now $u(x)$ in the above solution is replaced back by y using $u = \frac{y}{\sqrt{x}}$ which results in the solution

$$\frac{y^2}{2x} - \ln\left(\frac{y}{\sqrt{x}}\right) - \frac{\ln(x)}{2} - c_1 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2x} - \ln\left(\frac{y}{\sqrt{x}}\right) - \frac{\ln(x)}{2} - c_1 = 0 \tag{1}$$

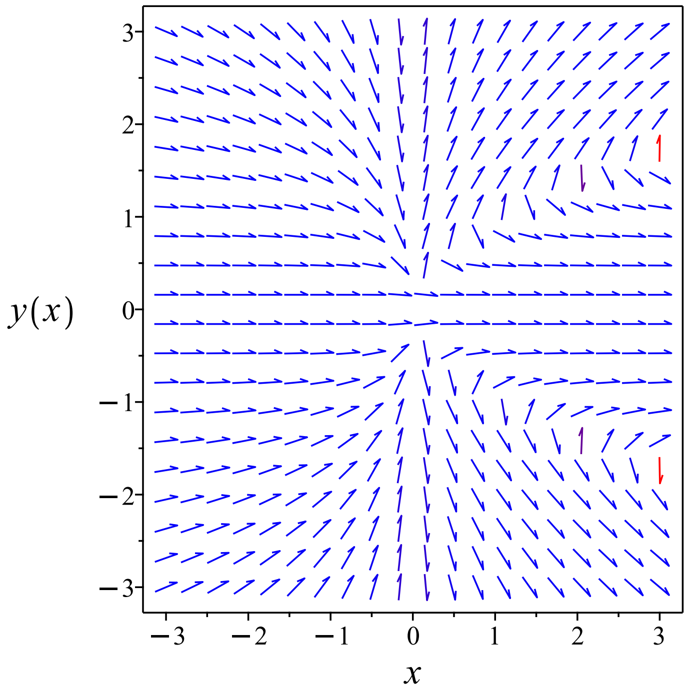


Figure 91: Slope field plot

Verification of solutions

$$\frac{y^2}{2x} - \ln\left(\frac{y}{\sqrt{x}}\right) - \frac{\ln(x)}{2} - c_1 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 28

```
dsolve(2*x*diff(y(x),x)*(x-y(x)^2)+y(x)^3=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{\frac{c_1}{2}}}{\sqrt{-\frac{e^{c_1}}{x \operatorname{LambertW}\left(-\frac{e^{c_1}}{x}\right)}}}$$

✓ Solution by Mathematica

Time used: 2.287 (sec). Leaf size: 60

```
DSolve[2*x*y'[x]*(x-y[x]^2)+y[x]^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -i\sqrt{x}\sqrt{W\left(-\frac{e^{c_1}}{x}\right)}$$
$$y(x) \to i\sqrt{x}\sqrt{W\left(-\frac{e^{c_1}}{x}\right)}$$
$$y(x) \to 0$$

2.46 problem 42

Internal problem ID [5794]
Internal file name [`OUTPUT/5042_Sunday_June_05_2022_03_18_46_PM_36692102/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 42.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Riccati]
```

$$\boxed{x^3(y'-x)-y^2=0}$$

2.46.1 Solving as isobaric ode

Solving for y' gives

$$y' = \frac{x^4+y^2}{x^3} \tag{1}$$

Each of the above ode's is now solved

<u>Solving ode 1</u>

An ode $y' = f(x,y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x,y) \tag{1}$$

Where here

$$f(x,y) = \frac{x^4+y^2}{x^3} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 2$$

Since the ode is isobaric of order $m = 2$, then the substitution

$$\begin{aligned} y &= x u^m \\ &= u\, x^2 \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$x(u'(x)\, x + 2u(x)) = x\left(1 + u(x)^2\right)$$

Or

$$u'(x) = \frac{\left(u(x) - 1\right)^2}{x}$$

Which is now solved as separable in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= \frac{u^2 - 2u + 1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2 - 2u + 1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u^2 - 2u + 1}\, du &= \frac{1}{x}\, dx \\ \int \frac{1}{u^2 - 2u + 1}\, du &= \int \frac{1}{x}\, dx \\ -\frac{1}{u - 1} &= \ln(x) + c_1 \end{aligned}$$

The solution is

$$-\frac{1}{u(x) - 1} - \ln(x) - c_1 = 0$$

Now $u(x)$ in the above solution is replaced back by y using $u = \frac{y}{x^2}$ which results in the solution

$$-\frac{1}{\frac{y}{x^2} - 1} - \ln(x) - c_1 = 0$$

Summary

The solution(s) found are the following

$$-\frac{1}{\frac{y}{x^2} - 1} - \ln(x) - c_1 = 0 \tag{1}$$

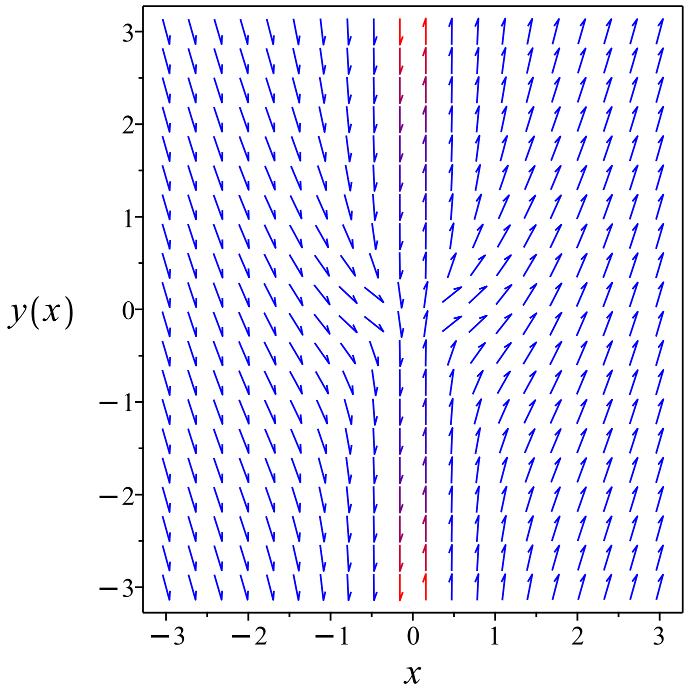


Figure 92: Slope field plot

Verification of solutions

$$-\frac{1}{\frac{y}{x^2}-1}-\ln(x)-c_1=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve(x^3*(diff(y(x),x)-x)=y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{x^2(\ln(x) - c_1 - 1)}{\ln(x) - c_1}$$

✓ Solution by Mathematica

Time used: 0.157 (sec). Leaf size: 29

```
DSolve[x^3*(y'[x]-x)==y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2(\log(x) - 1 + c_1)}{\log(x) + c_1}$$
$$y(x) \to x^2$$

2.47 problem 43

Internal problem ID [5795]
Internal file name [`OUTPUT/5043_Sunday_June_05_2022_03_18_48_PM_32788025/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 43.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{2x^2y' - y^3 - xy = 0}$$

2.47.1 Solving as isobaric ode

Solving for y' gives

$$y' = \frac{y(y^2 + x)}{2x^2} \tag{1}$$

Each of the above ode's is now solved

Solving ode 1

An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = \frac{y(y^2 + x)}{2x^2} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = \frac{1}{2}$$

Since the ode is isobaric of order $m = \frac{1}{2}$, then the substitution

$$\begin{aligned} y &= x u^m \\ &= u\sqrt{x} \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$\frac{2xu'(x) + u(x)}{2\sqrt{x}} = \frac{u(x)\left(u(x)^2 + 1\right)}{2\sqrt{x}}$$

Or

$$u'(x) = \frac{u(x)^3}{2x}$$

Which is now solved as separable in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^3}{2x} \end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(u) = u^3$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u^3}\,du &= \frac{1}{2x}\,dx \\ \int \frac{1}{u^3}\,du &= \int \frac{1}{2x}\,dx \\ -\frac{1}{2u^2} &= \frac{\ln(x)}{2} + c_1 \end{aligned}$$

The solution is

$$-\frac{1}{2u(x)^2} - \frac{\ln(x)}{2} - c_1 = 0$$

Now $u(x)$ in the above solution is replaced back by y using $u = \frac{y}{\sqrt{x}}$ which results in the solution

$$-\frac{x}{2y^2} - \frac{\ln(x)}{2} - c_1 = 0$$

Summary
The solution(s) found are the following

$$-\frac{x}{2y^2} - \frac{\ln(x)}{2} - c_1 = 0 \tag{1}$$

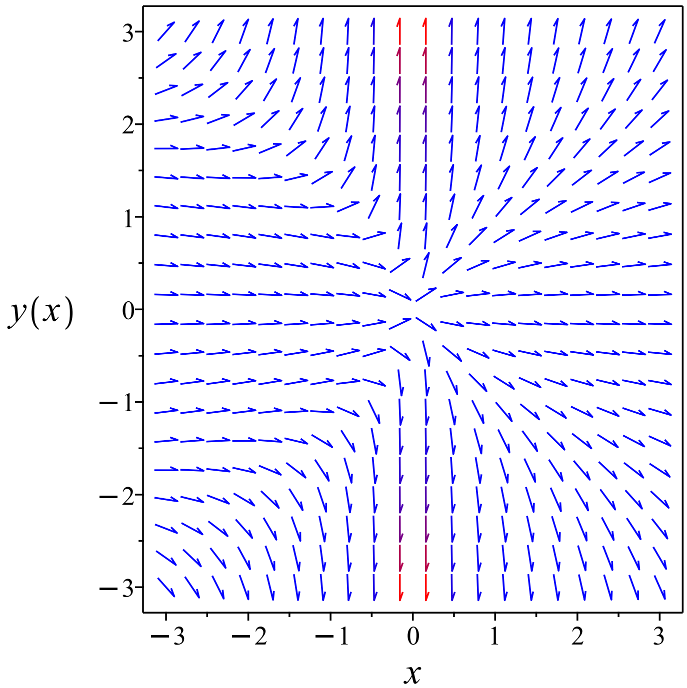


Figure 93: Slope field plot

Verification of solutions

$$-\frac{x}{2y^2} - \frac{\ln(x)}{2} - c_1 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
dsolve(2*x^2*diff(y(x),x)=y(x)^3+x*y(x),y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{(-\ln(x) + c_1)\, x}}{\ln(x) - c_1}$$
$$y(x) = \frac{\sqrt{(-\ln(x) + c_1)\, x}}{-\ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.158 (sec). Leaf size: 49

```
DSolve[2*x^2*y'[x]==y[x]^3+x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{x}}{\sqrt{-\log(x) + c_1}}$$
$$y(x) \to \frac{\sqrt{x}}{\sqrt{-\log(x) + c_1}}$$
$$y(x) \to 0$$

2.48 problem 44

Internal problem ID [5796]
Internal file name [OUTPUT/5044_Sunday_June_05_2022_03_18_51_PM_9301031/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
  class B`]]
```

$$\boxed{y + x(1 + 2xy)\, y' = 0}$$

2.48.1 Solving as isobaric ode

Solving for y' gives

$$y' = -\frac{y}{x\,(1 + 2xy)} \tag{1}$$

Each of the above ode's is now solved

Solving ode 1

An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = -\frac{y}{x\,(1 + 2xy)} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = -1$$

Since the ode is isobaric of order $m = -1$, then the substitution

$$\begin{aligned} y &= x u^m \\ &= \frac{u}{x} \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$\frac{u'(x)\, x - u(x)}{x^2} = -\frac{u(x)}{x^2 \left(1 + 2u\left(x\right)\right)}$$

Or

$$u'(x) = \frac{2u(x)^2}{x \left(1 + 2u\left(x\right)\right)}$$

Which is now solved as separable in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= \frac{2u^2}{x\left(1 + 2u\right)} \end{aligned}$$

Where $f(x) = \frac{2}{x}$ and $g(u) = \frac{u^2}{1+2u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2}{1+2u}}\, du &= \frac{2}{x}\, dx \\ \int \frac{1}{\frac{u^2}{1+2u}}\, du &= \int \frac{2}{x}\, dx \\ 2\ln\left(u\right) - \frac{1}{u} &= 2\ln\left(x\right) + c_1 \end{aligned}$$

The solution is

$$2\ln\left(u(x)\right) - \frac{1}{u\left(x\right)} - 2\ln\left(x\right) - c_1 = 0$$

Now $u(x)$ in the above solution is replaced back by y using $u = \frac{y}{\frac{1}{x}}$ which results in the solution

$$2\ln\left(xy\right) - \frac{1}{xy} - 2\ln\left(x\right) - c_1 = 0$$

Summary

The solution(s) found are the following

$$2\ln\left(xy\right) - \frac{1}{xy} - 2\ln\left(x\right) - c_1 = 0 \tag{1}$$

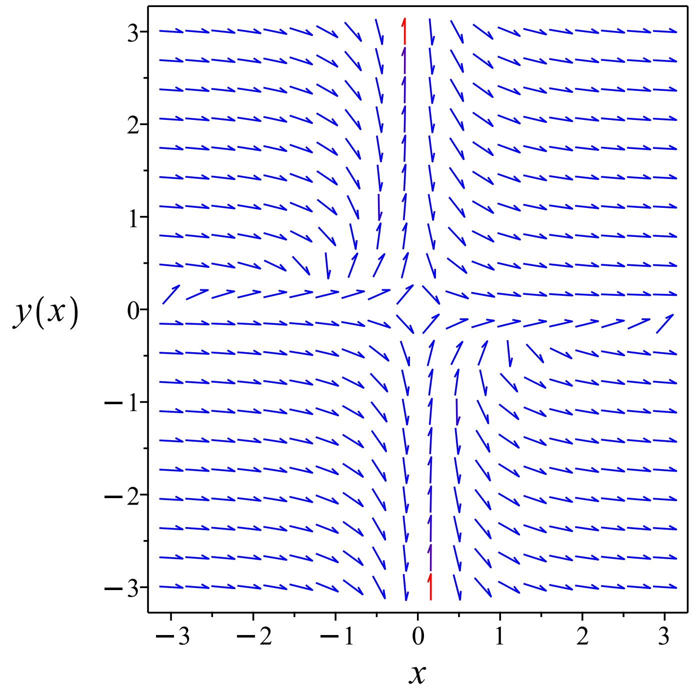


Figure 94: Slope field plot

Verification of solutions

$$2\ln(xy) - \frac{1}{xy} - 2\ln(x) - c_1 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve(y(x)+x*(2*x*y(x)+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{2\,\mathrm{LambertW}\left(\frac{c_1}{2x}\right)x}$$

✓ Solution by Mathematica

Time used: 60.506 (sec). Leaf size: 36

```
DSolve[y[x]+x*(2*x*y[x]+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2xW\left(\frac{e^{\frac{1}{2}\left(-2-9\sqrt[3]{-2}c_1\right)}}{x}\right)}$$

2.49 problem 45

Internal problem ID [5797]
Internal file name [`OUTPUT/5045_Sunday_June_05_2022_03_18_53_PM_28399984/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 45.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Chini]
```

$$\boxed{2y' - 4\sqrt{y} = -x}$$

2.49.1 Solving as isobaric ode

Solving for y' gives

$$y' = -\frac{x}{2} + 2\sqrt{y} \tag{1}$$

Each of the above ode's is now solved

Solving ode 1

An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = -\frac{x}{2} + 2\sqrt{y} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 2$$

Since the ode is isobaric of order $m = 2$, then the substitution

$$\begin{aligned} y &= x u^m \\ &= u\, x^2 \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$x(u'(x)\, x + 2u(x)) = -\frac{x}{2} + 2\sqrt{x^2 u(x)}$$

Or

$$u'(x) = \frac{-4xu(x) + 4\sqrt{x^2 u(x)} - x}{2x^2}$$

Simplifying the above ode, assuming $x > 0$ gives

$$u'(x) = \frac{-4u(x) + 4\sqrt{u(x)} - 1}{2x}$$

Which is now solved as separable in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-2u + 2\sqrt{u} - \frac{1}{2}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -2u + 2\sqrt{u} - \frac{1}{2}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-2u + 2\sqrt{u} - \frac{1}{2}}\, du &= \frac{1}{x}\, dx \\ \int \frac{1}{-2u + 2\sqrt{u} - \frac{1}{2}}\, du &= \int \frac{1}{x}\, dx \end{aligned}$$

$$\frac{1}{4\sqrt{u} - 2} - \frac{\ln(2\sqrt{u} - 1)}{2} + \frac{1}{4\sqrt{u} + 2} + \frac{\ln(2\sqrt{u} + 1)}{2} - \frac{\ln(4u - 1)}{2} + \frac{1}{4u - 1} = \ln(x) + c_1$$

The solution is

$$\begin{aligned} &\frac{1}{4\sqrt{u(x)} - 2} - \frac{\ln\left(2\sqrt{u(x)} - 1\right)}{2} + \frac{1}{4\sqrt{u(x)} + 2} \\ &+ \frac{\ln\left(2\sqrt{u(x)} + 1\right)}{2} - \frac{\ln(4u(x) - 1)}{2} + \frac{1}{4u(x) - 1} - \ln(x) - c_1 = 0 \end{aligned}$$

Now $u(x)$ in the above solution is replaced back by y using $u = \frac{y}{x^2}$ which results in the solution

$$\frac{1}{4\sqrt{\frac{y}{x^2}}-2}-\frac{\ln\left(2\sqrt{\frac{y}{x^2}}-1\right)}{2}+\frac{1}{4\sqrt{\frac{y}{x^2}}+2}+\frac{\ln\left(2\sqrt{\frac{y}{x^2}}+1\right)}{2}-\frac{\ln\left(\frac{4y}{x^2}-1\right)}{2}+\frac{1}{\frac{4y}{x^2}-1}-\ln(x)-c_1 = 0$$

Summary

The solution(s) found are the following

$$\frac{1}{4\sqrt{\frac{y}{x^2}}-2}-\frac{\ln\left(2\sqrt{\frac{y}{x^2}}-1\right)}{2}+\frac{1}{4\sqrt{\frac{y}{x^2}}+2}+\frac{\ln\left(2\sqrt{\frac{y}{x^2}}+1\right)}{2} \\ -\frac{\ln\left(\frac{4y}{x^2}-1\right)}{2}+\frac{1}{\frac{4y}{x^2}-1}-\ln(x)-c_1 = 0 \tag{1}$$

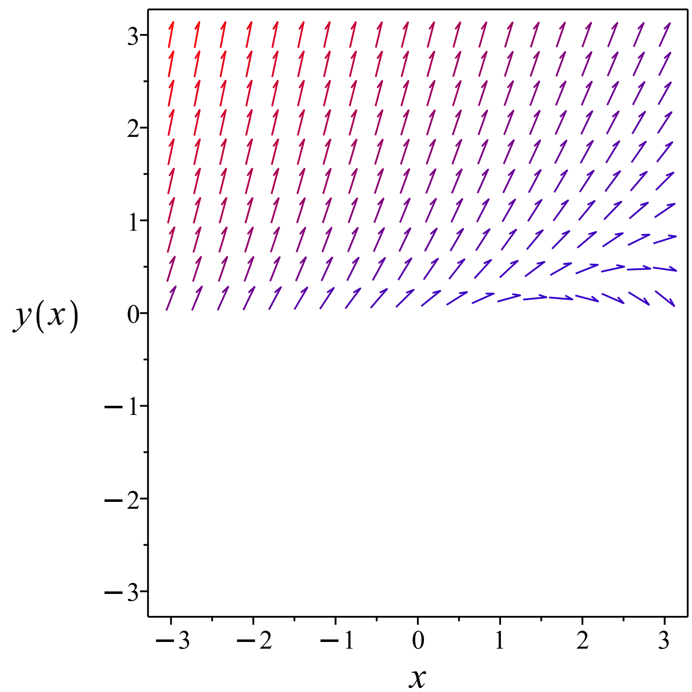


Figure 95: Slope field plot

Verification of solutions

$$\frac{1}{4\sqrt{\frac{y}{x^2}}-2}-\frac{\ln\left(2\sqrt{\frac{y}{x^2}}-1\right)}{2}+\frac{1}{4\sqrt{\frac{y}{x^2}}+2}+\frac{\ln\left(2\sqrt{\frac{y}{x^2}}+1\right)}{2} \\ -\frac{\ln\left(\frac{4y}{x^2}-1\right)}{2}+\frac{1}{\frac{4y}{x^2}-1}-\ln(x)-c_1 = 0$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 100

```
dsolve(2*diff(y(x),x)+x=4*sqrt(y(x)),y(x), singsol=all)
```

$$\frac{(-x^2+4y(x))\ln\left(\frac{x^2-4y(x)}{x^2}\right)+2i(x^2-4y(x))\arctan\left(2\sqrt{-\frac{y(x)}{x^2}}\right)-4i\sqrt{-\frac{y(x)}{x^2}}\,x^2+4(-c_1+2\ln(x))\,y(x)}{x^2-4y(x)}$$

$= 0$

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 49

```
DSolve[2*y'[x]+x==4*Sqrt[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[4\left(\frac{4}{4\sqrt{\frac{y(x)}{x^2}}+2}+2\log\left(4\sqrt{\frac{y(x)}{x^2}}+2\right)\right)=-8\log(x)+c_1,y(x)\right]$$

2.50 problem 46

Internal problem ID [5798]
Internal file name [`OUTPUT/5046_Sunday_June_05_2022_03_18_55_PM_27794649/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 46.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Riccati, _special]]
```

$$\boxed{y' - y^2 = -\frac{2}{x^2}}$$

2.50.1 Solving as isobaric ode

Solving for y' gives

$$y' = \frac{y^2x^2 - 2}{x^2} \tag{1}$$

Each of the above ode's is now solved

Solving ode 1

An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = \frac{y^2x^2 - 2}{x^2} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = -1$$

Since the ode is isobaric of order $m = -1$, then the substitution

$$\begin{aligned} y &= x u^m \\ &= \frac{u}{x} \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$\frac{u'(x)\, x - u(x)}{x^2} = \frac{u(x)^2 - 2}{x^2}$$

Or

$$u'(x) = \frac{u(x)^2 + u(x) - 2}{x}$$

Which is now solved as separable in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= \frac{u^2 + u - 2}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2 + u - 2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u^2 + u - 2}\, du &= \frac{1}{x}\, dx \\ \int \frac{1}{u^2 + u - 2}\, du &= \int \frac{1}{x}\, dx \\ \frac{\ln(u-1)}{3} - \frac{\ln(u+2)}{3} &= \ln(x) + c_1 \end{aligned}$$

The above can be written as

$$\begin{aligned} \left(\frac{1}{3}\right) (\ln(u-1) - \ln(u+2)) &= \ln(x) + 2c_1 \\ \ln(u-1) - \ln(u+2) &= (3)(\ln(x) + 2c_1) \\ &= 3\ln(x) + 6c_1 \end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(u-1) - \ln(u+2)} = e^{3\ln(x) + 3c_1}$$

Which simplifies to

$$\begin{aligned}\frac{u-1}{u+2} &= 3c_1x^3 \\ &= c_2x^3\end{aligned}$$

Now $u(x)$ in the above solution is replaced back by y using $u = \frac{y}{\frac{1}{x}}$ which results in the solution

$$y = -\frac{2c_2x^3+1}{x\left(c_2x^3-1\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{2c_2x^3+1}{x\left(c_2x^3-1\right)} \tag{1}$$

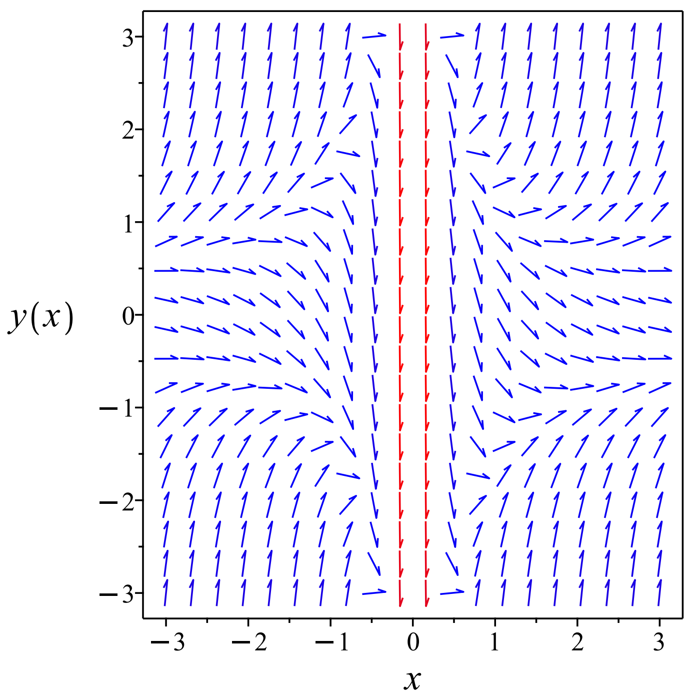


Figure 96: Slope field plot

Verification of solutions

$$y = -\frac{2c_2x^3+1}{x\left(c_2x^3-1\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)=y(x)^2-2/x^2,y(x), singsol=all)
```

$$y(x) = \frac{2x^3 + c_1}{x(-x^3 + c_1)}$$

✓ Solution by Mathematica

Time used: 0.14 (sec). Leaf size: 32

```
DSolve[y'[x]==y[x]^2-2/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-2x^3 + c_1}{x(x^3 + c_1)}$$
$$y(x) \to \frac{1}{x}$$

2.51 problem 47

Internal problem ID [5799]
Internal file name [`OUTPUT/5047_Sunday_June_05_2022_03_18_58_PM_62390225/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 47.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[[_homogeneous, `class G`]]`

$$\boxed{2xy' + y - y^2\sqrt{x - y^2x^2} = 0}$$

2.51.1 Solving as isobaric ode

Solving for y' gives

$$y' = \frac{y\left(y\sqrt{x - y^2x^2} - 1\right)}{2x} \tag{1}$$

Each of the above ode's is now solved

Solving ode 1

An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = \frac{y\left(y\sqrt{x - y^2x^2} - 1\right)}{2x} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = -\frac{1}{2}$$

Since the ode is isobaric of order $m = -\frac{1}{2}$, then the substitution

$$\begin{aligned} y &= x u^m \\ &= \frac{u}{\sqrt{x}} \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$\frac{2u'(x)\,x - u(x)}{2x^{\frac{3}{2}}} = \frac{u(x)\left(u(x)\sqrt{x - x u(x)^2} - \sqrt{x}\right)}{2x^2}$$

Or

$$u'(x) = \frac{u(x)^2\sqrt{x - x u(x)^2}}{2x^{\frac{3}{2}}}$$

Simplifying the above ode, assuming $x > 0$ gives

$$u'(x) = \frac{\sqrt{1 - u(x)^2}\,u(x)^2}{2x}$$

Which is now solved as separable in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\sqrt{-u^2+1}\,u^2}{2x} \end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(u) = \sqrt{-u^2+1}\,u^2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\sqrt{-u^2+1}\,u^2}\,du &= \frac{1}{2x}\,dx \\ \int \frac{1}{\sqrt{-u^2+1}\,u^2}\,du &= \int \frac{1}{2x}\,dx \\ -\frac{\sqrt{-u^2+1}}{u} &= \frac{\ln(x)}{2} + c_1 \end{aligned}$$

The solution is

$$-\frac{\sqrt{1 - u(x)^2}}{u(x)} - \frac{\ln(x)}{2} - c_1 = 0$$

Now $u(x)$ in the above solution is replaced back by y using $u = \frac{y}{\frac{1}{\sqrt{x}}}$ which results in the solution

$$-\frac{\sqrt{-x y^2+1}}{y\sqrt{x}}-\frac{\ln\left(x\right)}{2}-c_1=0$$

Summary

The solution(s) found are the following

$$-\frac{\sqrt{-x y^2+1}}{y\sqrt{x}}-\frac{\ln\left(x\right)}{2}-c_1=0 \tag{1}$$

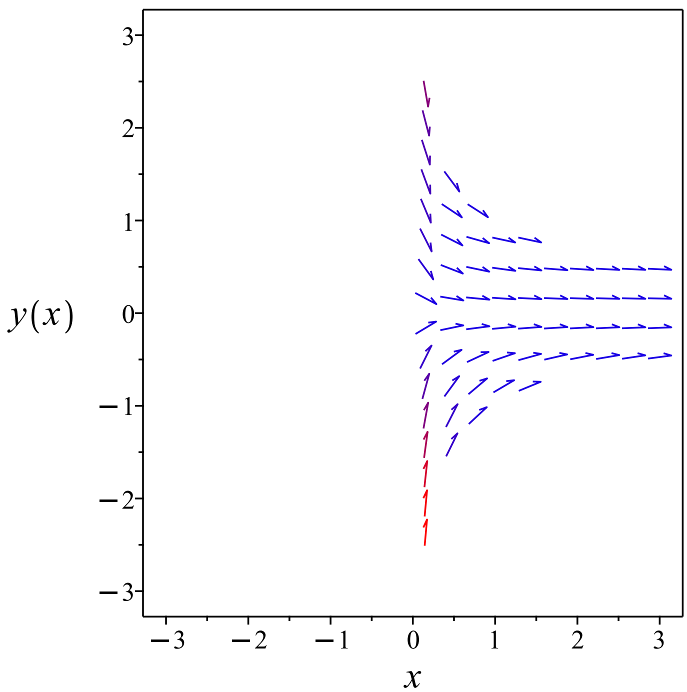


Figure 97: Slope field plot

Verification of solutions

$$-\frac{\sqrt{-x y^2+1}}{y\sqrt{x}}-\frac{\ln\left(x\right)}{2}-c_1=0$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(2*x*diff(y(x),x)+y(x)=y(x)^2*sqrt(x-x^2*y(x)^2),y(x), singsol=all)
```

$$-\frac{-1+xy(x)^2}{y(x)\sqrt{-x\left(-1+xy(x)^2\right)}}+\frac{\ln(x)}{2}-c_1=0$$

✓ Solution by Mathematica

Time used: 1.852 (sec). Leaf size: 62

```
DSolve[2*x*y'[x]+y[x]==y[x]^2*Sqrt[x-x^2*y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to-\frac{2}{\sqrt{x\left(\log^2(x)-2c_1\log(x)+4+c_1{}^2\right)}}$$
$$y(x)\to\frac{2}{\sqrt{x\left(\log^2(x)-2c_1\log(x)+4+c_1{}^2\right)}}$$
$$y(x)\to0$$

2.52 problem 48

Internal problem ID [5800]
Internal file name [OUTPUT/5048_Sunday_June_05_2022_03_19_01_PM_62397227/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 48.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{\frac{2xyy'}{3} - \sqrt{x^6 - y^4} - y^2 = 0}$$

2.52.1 Solving as isobaric ode

Solving for y' gives

$$y' = \frac{\frac{3\sqrt{x^6-y^4}}{2} + \frac{3y^2}{2}}{xy} \tag{1}$$

Each of the above ode's is now solved

Solving ode 1

An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = \frac{\frac{3\sqrt{x^6-y^4}}{2} + \frac{3y^2}{2}}{xy} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = \frac{3}{2}$$

Since the ode is isobaric of order $m = \frac{3}{2}$, then the substitution

$$\begin{aligned} y &= x u^m \\ &= u\, x^{\frac{3}{2}} \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$\frac{\sqrt{x}\,(2xu'(x) + 3u(x))}{2} = \frac{\frac{3\sqrt{x^6\left(1-u(x)^4\right)}}{2} + \frac{3x^3 u(x)^2}{2}}{x^{\frac{5}{2}} u\left(x\right)}$$

Or

$$u'(x) = \frac{3\sqrt{x^6\left(1 - u\left(x\right)^4\right)}}{2x^4 u\left(x\right)}$$

Simplifying the above ode, assuming $x > 0$ gives

$$u'(x) = \frac{3\sqrt{1 - u\left(x\right)^4}}{2xu\left(x\right)}$$

Which is now solved as separable in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{3\sqrt{-u^4+1}}{2xu} \end{aligned}$$

Where $f(x) = \frac{3}{2x}$ and $g(u) = \frac{\sqrt{-u^4+1}}{u}$. Integrating both sides gives

$$\frac{1}{\frac{\sqrt{-u^4+1}}{u}}\,du = \frac{3}{2x}\,dx$$

$$\int \frac{1}{\frac{\sqrt{-u^4+1}}{u}}\,du = \int \frac{3}{2x}\,dx$$

$$\frac{\arcsin\left(u^2\right)}{2} = \frac{3\ln\left(x\right)}{2} + c_1$$

The solution is

$$\frac{\arcsin\left(u(x)^2\right)}{2} - \frac{3\ln(x)}{2} - c_1 = 0$$

Now $u(x)$ in the above solution is replaced back by y using $u = \frac{y}{x^{\frac{3}{2}}}$ which results in the solution

$$\frac{\arcsin\left(\frac{y^2}{x^3}\right)}{2} - \frac{3\ln(x)}{2} - c_1 = 0$$

Summary

The solution(s) found are the following

$$\frac{\arcsin\left(\frac{y^2}{x^3}\right)}{2} - \frac{3\ln(x)}{2} - c_1 = 0 \tag{1}$$

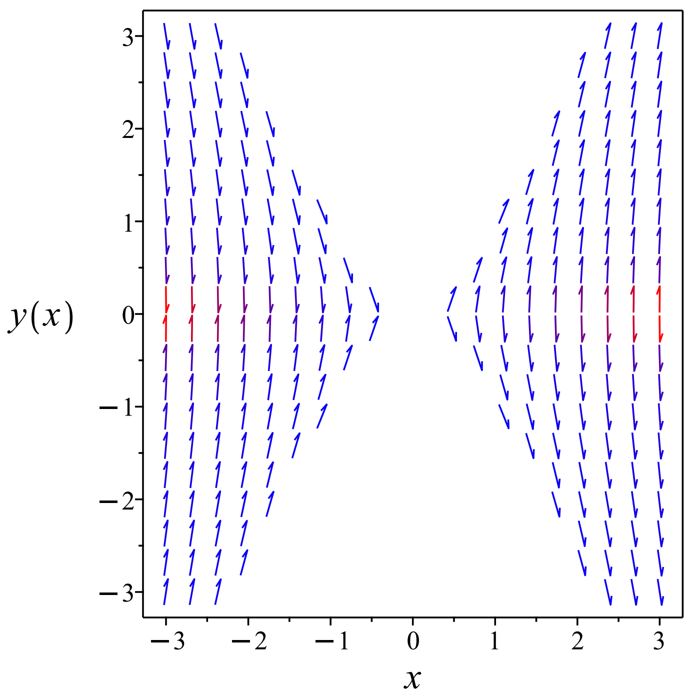


Figure 98: Slope field plot

Verification of solutions

$$\frac{\arcsin\left(\frac{y^2}{x^3}\right)}{2} - \frac{3\ln(x)}{2} - c_1 = 0$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
trying an integrating factor from the invariance group
<- integrating factor successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 102

```
dsolve(2/3*x*y(x)*diff(y(x),x)=sqrt(x^6-y(x)^4)+y(x)^2,y(x), singsol=all)
```

$$-\left(\int_{\_b}^{x}\frac{\sqrt{\_a^{6}-y\left(x\right)^{4}}+y\left(x\right)^{2}}{\sqrt{\_a^{6}-y\left(x\right)^{4}}\,\_a}d\_a\right)+\frac{2\left(\int^{y(x)}\frac{\_f\left(3\sqrt{x^{6}-\_f^{4}}\left(\int_{\_b}^{x}\frac{\_a^{5}}{\left(\_a^{6}-\_f^{4}\right)^{\frac{3}{2}}}d\_a\right)+1\right)}{\sqrt{x^{6}-\_f^{4}}}d\_f\right)}{3}+c_{1}=0$$

✓ Solution by Mathematica

Time used: 6.948 (sec). Leaf size: 128

```
DSolve[2/3*x*y[x]*y'[x]==Sqrt[x^6-y[x]^4]+y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^{3/2}}{\sqrt[4]{\sec^2\left(-\frac{\log(x^6)}{2}+3c_1\right)}}$$

$$y(x) \to -\frac{ix^{3/2}}{\sqrt[4]{\sec^2\left(-\frac{\log(x^6)}{2}+3c_1\right)}}$$

$$y(x) \to \frac{ix^{3/2}}{\sqrt[4]{\sec^2\left(-\frac{\log(x^6)}{2}+3c_1\right)}}$$

$$y(x) \to \frac{x^{3/2}}{\sqrt[4]{\sec^2\left(-\frac{\log(x^6)}{2}+3c_1\right)}}$$

2.53 problem 49

Internal problem ID [5801]
Internal file name [`OUTPUT/5049_Sunday_June_05_2022_03_19_11_PM_86253345/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 49.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{2y+\left(y x^{2}+1\right) x y'=0}$$

2.53.1 Solving as isobaric ode

Solving for y' gives

$$y' = -\frac{2y}{\left(y x^{2}+1\right) x} \tag{1}$$

Each of the above ode's is now solved

Solving ode 1

An ode $y'=f(x,y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x,y) \tag{1}$$

Where here

$$f(x,y) = -\frac{2y}{\left(y x^{2}+1\right) x} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = -2$$

Since the ode is isobaric of order $m = -2$, then the substitution

$$\begin{aligned}
y &= x u^m \\
&= \frac{u}{x^2}
\end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$\frac{u'(x)\,x - 2u(x)}{x^3} = -\frac{2u(x)}{x^3\,(u\,(x)+1)}$$

Or

$$u'(x) = \frac{2u(x)^2}{x\,(u\,(x)+1)}$$

Which is now solved as separable in $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{2u^2}{x\,(u+1)}
\end{aligned}$$

Where $f(x) = \frac{2}{x}$ and $g(u) = \frac{u^2}{u+1}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{u^2}{u+1}}\,du &= \frac{2}{x}\,dx \\
\int \frac{1}{\frac{u^2}{u+1}}\,du &= \int \frac{2}{x}\,dx \\
\ln\,(u) - \frac{1}{u} &= 2\ln\,(x) + c_1
\end{aligned}$$

The solution is

$$\ln\,(u(x)) - \frac{1}{u\,(x)} - 2\ln\,(x) - c_1 = 0$$

Now $u(x)$ in the above solution is replaced back by y using $u = \frac{y}{\frac{1}{x^2}}$ which results in the solution

$$\ln\left(y x^2\right) - \frac{1}{y x^2} - 2\ln\,(x) - c_1 = 0$$

Summary

The solution(s) found are the following

$$\ln\left(y x^2\right) - \frac{1}{y x^2} - 2\ln\,(x) - c_1 = 0 \tag{1}$$

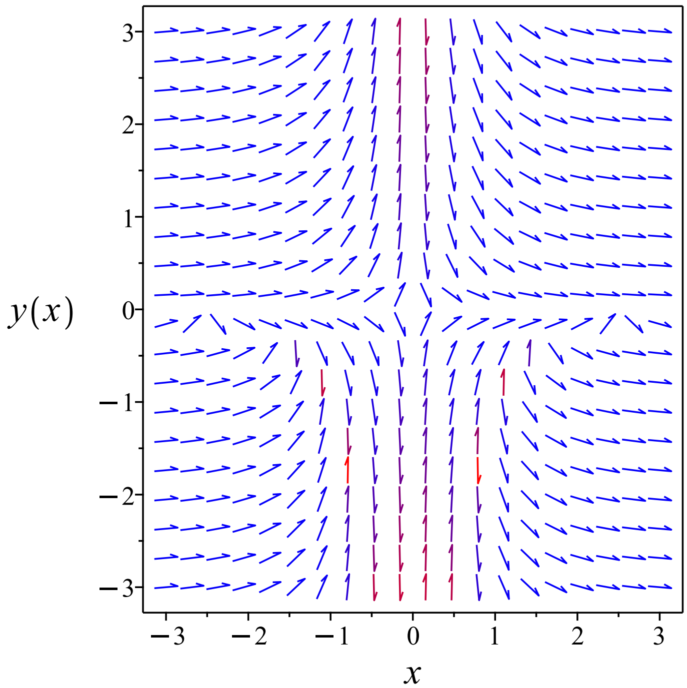


Figure 99: Slope field plot

Verification of solutions

$$\ln \left(y x^{2}\right)-\frac{1}{y x^{2}}-2 \ln \left(x \right)-c_{1} = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 16

```
dsolve(2*y(x)+(x^2*y(x)+1)*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{\mathrm{LambertW}\left(\frac{c_1}{x^2}\right) x^2}$$

✓ Solution by Mathematica

Time used: 60.405 (sec). Leaf size: 33

```
DSolve[2*y[x]+(x^2*y[x]+1)*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{x^2 W\left(\frac{e^{\frac{1}{2}\left(-2-9\sqrt[3]{-2}c_1\right)}}{x^2}\right)}$$

2.54 problem 50

Internal problem ID [5802]
Internal file name [`OUTPUT/5050_Sunday_June_05_2022_03_19_13_PM_71776004/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 50.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y(xy+1)+(1-xy)\,xy'=0}$$

2.54.1 Solving as isobaric ode

Solving for y' gives

$$y'=\frac{y(xy+1)}{(xy-1)\,x} \tag{1}$$

Each of the above ode's is now solved

Solving ode 1

An ode $y'=f(x,y)$ is isobaric if

$$f(tx,t^m y)=t^{m-1}f(x,y) \tag{1}$$

Where here

$$f(x,y)=\frac{y(xy+1)}{(xy-1)\,x} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m=-1$$

Since the ode is isobaric of order $m = -1$, then the substitution

$$\begin{aligned} y &= x u^m \\ &= \frac{u}{x} \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$\frac{u'(x)\,x - u(x)}{x^2} = \frac{u(x)\,(u(x)+1)}{x^2\,(u\,(x)-1)}$$

Or

$$u'(x) = \frac{2u(x)^2}{x\,(u\,(x)-1)}$$

Which is now solved as separable in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{2u^2}{x\,(u-1)} \end{aligned}$$

Where $f(x) = \frac{2}{x}$ and $g(u) = \frac{u^2}{u-1}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2}{u-1}}\,du &= \frac{2}{x}\,dx \\ \int \frac{1}{\frac{u^2}{u-1}}\,du &= \int \frac{2}{x}\,dx \\ \ln\,(u) + \frac{1}{u} &= 2\ln\,(x) + c_1 \end{aligned}$$

The solution is

$$\ln\,(u(x)) + \frac{1}{u\,(x)} - 2\ln\,(x) - c_1 = 0$$

Now $u(x)$ in the above solution is replaced back by y using $u = \frac{y}{\frac{1}{x}}$ which results in the solution

$$\ln\,(xy) + \frac{1}{xy} - 2\ln\,(x) - c_1 = 0$$

<u>Summary</u>

The solution(s) found are the following

$$\ln\,(xy) + \frac{1}{xy} - 2\ln\,(x) - c_1 = 0 \tag{1}$$

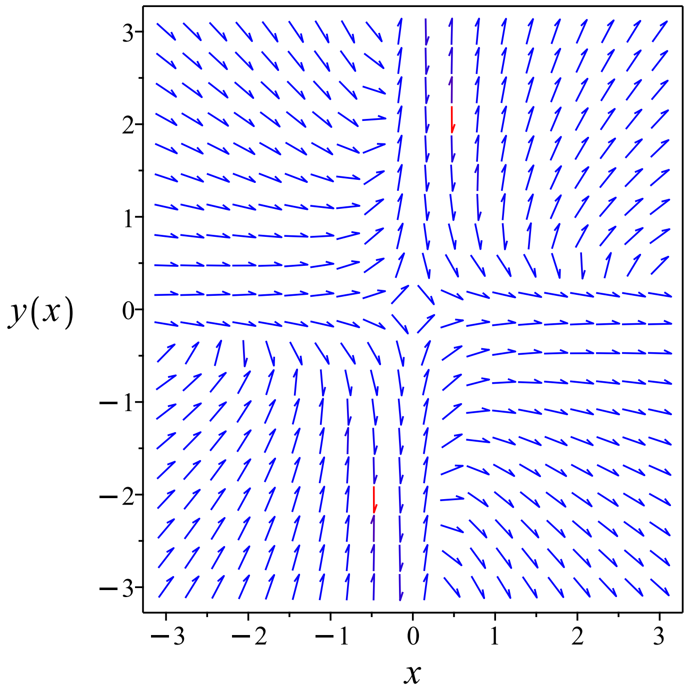


Figure 100: Slope field plot

Verification of solutions

$$\ln(xy) + \frac{1}{xy} - 2\ln(x) - c_1 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve(y(x)*(1+x*y(x))+(1-x*y(x))*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{1}{\operatorname{LambertW}\left(-\frac{c_1}{x^2}\right) x}$$

✓ Solution by Mathematica

Time used: 6.096 (sec). Leaf size: 35

```
DSolve[y[x]*(1+x*y[x])+(1-x*y[x])*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{xW\left(\frac{e^{-1+\frac{9c_1}{2^{2/3}}}}{x^2}\right)}$$

$$y(x) \to 0$$

2.55 problem 51

Internal problem ID [5803]
Internal file name [`OUTPUT/5051_Sunday_June_05_2022_03_19_15_PM_17080372/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 51.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y\left(y^2x^2+1\right)+\left(y^2x^2-1\right)xy'=0}$$

2.55.1 Solving as isobaric ode

Solving for y' gives

$$y' = -\frac{y(y^2x^2+1)}{(y^2x^2-1)\,x} \tag{1}$$

Each of the above ode's is now solved

Solving ode 1

An ode $y' = f(x,y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x,y) \tag{1}$$

Where here

$$f(x,y) = -\frac{y(y^2x^2+1)}{(y^2x^2-1)\,x} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = -1$$

Since the ode is isobaric of order $m = -1$, then the substitution

$$\begin{aligned} y &= x u^m \\ &= \frac{u}{x} \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$\frac{u'(x)\,x - u(x)}{x^2} = -\frac{u(x)\left(u(x)^2 + 1\right)}{x^2\left(u\left(x\right)^2 - 1\right)}$$

Or

$$u'(x) = -\frac{2u(x)}{x\left(u\left(x\right)^2 - 1\right)}$$

Which is now solved as separable in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x\left(u^2 - 1\right)} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u}{u^2-1}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u}{u^2-1}}\,du &= -\frac{2}{x}\,dx \\ \int \frac{1}{\frac{u}{u^2-1}}\,du &= \int -\frac{2}{x}\,dx \\ \frac{u^2}{2} - \ln\left(u\right) &= -2\ln\left(x\right) + c_1 \end{aligned}$$

The solution is

$$\frac{u(x)^2}{2} - \ln\left(u(x)\right) + 2\ln\left(x\right) - c_1 = 0$$

Now $u(x)$ in the above solution is replaced back by y using $u = \frac{y}{\frac{1}{x}}$ which results in the solution

$$\frac{y^2x^2}{2} - \ln\left(xy\right) + 2\ln\left(x\right) - c_1 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2x^2}{2} - \ln\left(xy\right) + 2\ln\left(x\right) - c_1 = 0 \tag{1}$$

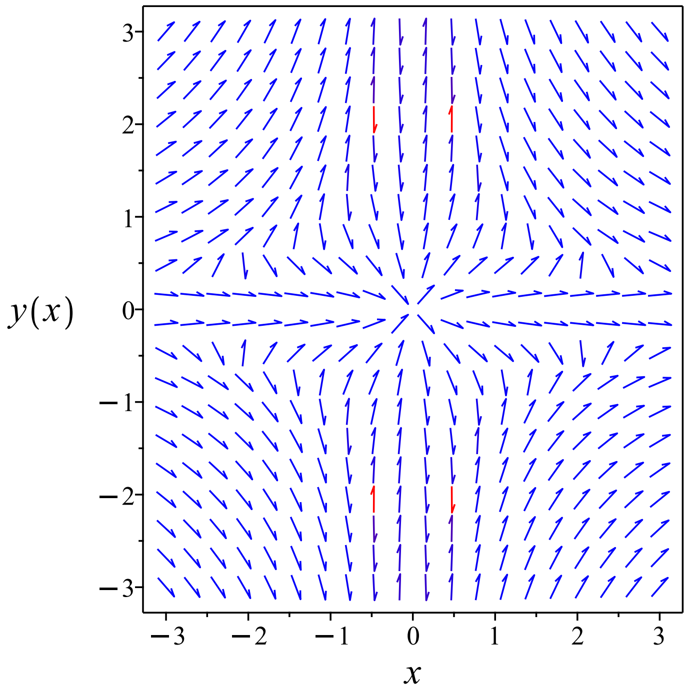


Figure 101: Slope field plot

Verification of solutions

$$\frac{y^2x^2}{2} - \ln(xy) + 2\ln(x) - c_1 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 33

```
dsolve(y(x)*(x^2*y(x)^2+1)+(x^2*y(x)^2-1)*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{-2c_1}x}{\sqrt{-\frac{x^4 e^{-4c_1}}{\text{LambertW}\left(-x^4 e^{-4c_1}\right)}}}$$

✓ Solution by Mathematica

Time used: 31.376 (sec). Leaf size: 60

```
DSolve[y[x]*(x^2*y[x]^2+1)+(x^2*y[x]^2-1)*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\frac{i\sqrt{W\left(-e^{-2c_1}x^4\right)}}{x}$$
$$y(x) \to \frac{i\sqrt{W\left(-e^{-2c_1}x^4\right)}}{x}$$
$$y(x) \to 0$$

2.56 problem 52

Internal problem ID [5804]
Internal file name [`OUTPUT/5052_Sunday_June_05_2022_03_19_17_PM_88188863/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 52.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{(x^2 - y^4)\, y' - xy = 0}$$

2.56.1 Solving as isobaric ode

Solving for y' gives

$$y' = -\frac{xy}{y^4 - x^2} \tag{1}$$

Each of the above ode's is now solved

Solving ode 1

An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = -\frac{xy}{y^4 - x^2} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = \frac{1}{2}$$

Since the ode is isobaric of order $m = \frac{1}{2}$, then the substitution

$$\begin{aligned} y &= x u^m \\ &= u\sqrt{x} \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$\frac{2xu'(x) + u(x)}{2\sqrt{x}} = \frac{u(x)}{\left(-u\left(x\right)^{4}+1\right)\sqrt{x}}$$

Or

$$u'(x) = -\frac{u(x)\left(u(x)^{4}+1\right)}{2u\left(x\right)^{4} x - 2x}$$

Which is now solved as separable in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(u^4+1)}{2\left(u^4-1\right)x} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{u(u^4+1)}{u^4-1}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u(u^4+1)}{u^4-1}}\,du &= -\frac{1}{2x}\,dx \\ \int \frac{1}{\frac{u(u^4+1)}{u^4-1}}\,du &= \int -\frac{1}{2x}\,dx \\ \frac{\ln\left(u^4+1\right)}{2} - \ln\left(u\right) &= -\frac{\ln\left(x\right)}{2} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{\frac{\ln\left(u^4+1\right)}{2} - \ln(u)} = \mathrm{e}^{-\frac{\ln(x)}{2} + c_1}$$

Which simplifies to

$$\frac{\sqrt{u^4+1}}{u} = \frac{c_2}{\sqrt{x}}$$

The solution is

$$\frac{\sqrt{u\left(x\right)^{4}+1}}{u\left(x\right)} = \frac{c_2}{\sqrt{x}}$$

Now $u(x)$ in the above solution is replaced back by y using $u = \frac{y}{\sqrt{x}}$ which results in the solution

$$\frac{\sqrt{\frac{y^4}{x^2}+1}\,\sqrt{x}}{y} = \frac{c_2}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$\frac{\sqrt{\frac{y^4}{x^2}+1}\,\sqrt{x}}{y} = \frac{c_2}{\sqrt{x}} \tag{1}$$

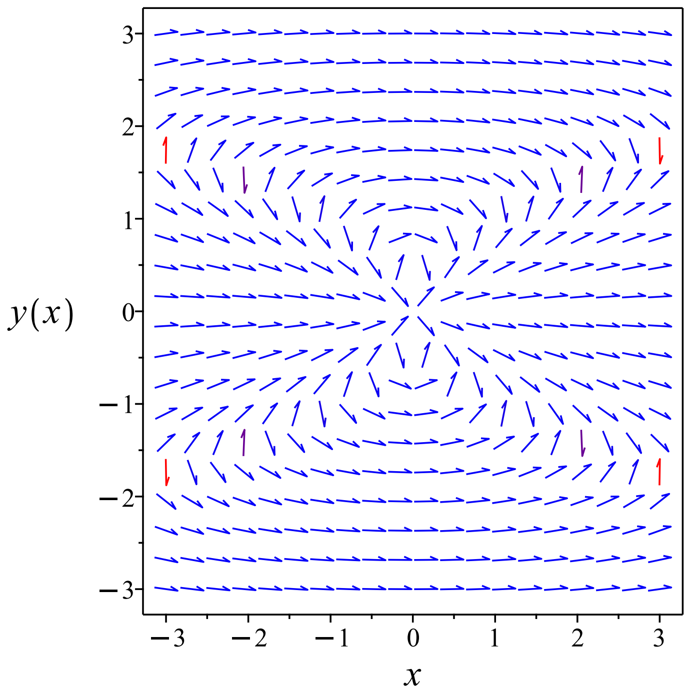


Figure 102: Slope field plot

Verification of solutions

$$\frac{\sqrt{\frac{y^4}{x^2}+1}\,\sqrt{x}}{y} = \frac{c_2}{\sqrt{x}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 97

```
dsolve((x^2-y(x)^4)*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-2\sqrt{c_1^2 - 4x^2} + 2c_1}}{2}$$
$$y(x) = \frac{\sqrt{-2\sqrt{c_1^2 - 4x^2} + 2c_1}}{2}$$
$$y(x) = -\frac{\sqrt{2\sqrt{c_1^2 - 4x^2} + 2c_1}}{2}$$
$$y(x) = \frac{\sqrt{2\sqrt{c_1^2 - 4x^2} + 2c_1}}{2}$$

✓ Solution by Mathematica

Time used: 5.14 (sec). Leaf size: 122

```
DSolve[(x^2-y[x]^4)*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-\sqrt{-x^2+c_1{}^2}-c_1}$$
$$y(x) \to \sqrt{-\sqrt{-x^2+c_1{}^2}-c_1}$$
$$y(x) \to -\sqrt{\sqrt{-x^2+c_1{}^2}-c_1}$$
$$y(x) \to \sqrt{\sqrt{-x^2+c_1{}^2}-c_1}$$
$$y(x) \to 0$$

2.57 problem 53

Internal problem ID [5805]
Internal file name [OUTPUT/5053_Sunday_June_05_2022_03_19_20_PM_20550564/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.2 Homogeneous equations problems. page 12
Problem number: 53.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{y\left(1+\sqrt{y^4x^2-1}\right)+2xy'=0}$$

2.57.1 Solving as isobaric ode

Solving for y' gives

$$y' = -\frac{y\left(1+\sqrt{y^4x^2-1}\right)}{2x} \tag{1}$$

Each of the above ode's is now solved

Solving ode 1

An ode $y' = f(x,y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x,y) \tag{1}$$

Where here

$$f(x,y) = -\frac{y\left(1+\sqrt{y^4x^2-1}\right)}{2x} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = -\frac{1}{2}$$

Since the ode is isobaric of order $m = -\frac{1}{2}$, then the substitution

$$\begin{aligned} y &= x u^m \\ &= \frac{u}{\sqrt{x}} \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$\frac{2u'(x)\, x - u(x)}{2x^{\frac{3}{2}}} = -\frac{u(x) \left(1 + \sqrt{u\left(x\right)^{4} - 1}\right)}{2x^{\frac{3}{2}}}$$

Or

$$u'(x) = -\frac{u(x) \sqrt{u\left(x\right)^{4} - 1}}{2x}$$

Which is now solved as separable in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{u\sqrt{u^4 - 1}}{2x} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = u\sqrt{u^4 - 1}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u\sqrt{u^4 - 1}}\, du &= -\frac{1}{2x}\, dx \\ \int \frac{1}{u\sqrt{u^4 - 1}}\, du &= \int -\frac{1}{2x}\, dx \\ -\frac{\arctan\left(\frac{1}{\sqrt{u^4 - 1}}\right)}{2} &= -\frac{\ln\left(x\right)}{2} + c_1 \end{aligned}$$

The solution is

$$-\frac{\arctan\left(\frac{1}{\sqrt{u(x)^4 - 1}}\right)}{2} + \frac{\ln\left(x\right)}{2} - c_1 = 0$$

Now $u(x)$ in the above solution is replaced back by y using $u = \frac{y}{\frac{1}{\sqrt{x}}}$ which results in the solution

$$-\frac{\arctan\left(\frac{1}{\sqrt{y^4 x^2 - 1}}\right)}{2} + \frac{\ln\left(x\right)}{2} - c_1 = 0$$

Summary
The solution(s) found are the following

$$
-\frac{\arctan\left(\frac{1}{\sqrt{y^{4} x^{2}-1}}\right)}{2}+\frac{\ln\left(x\right)}{2}-c_{1}=0 \tag{1}
$$

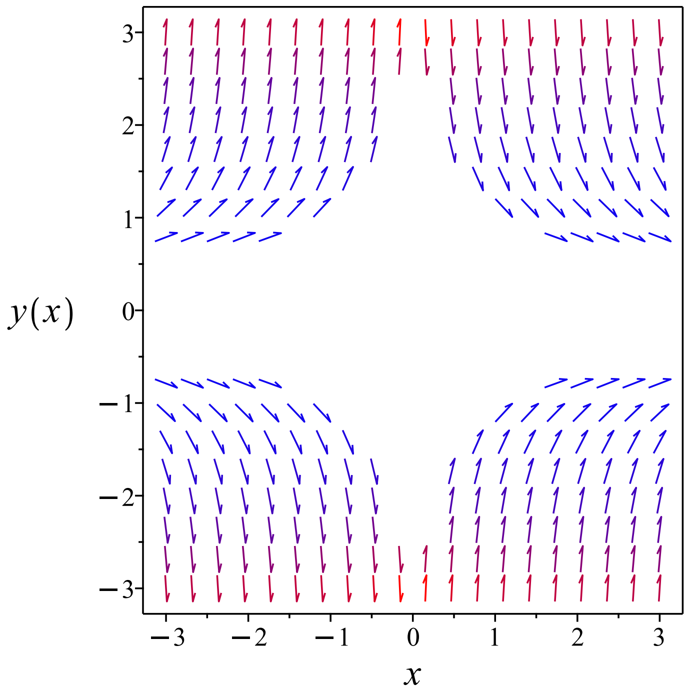


Figure 103: Slope field plot

Verification of solutions

$$
-\frac{\arctan\left(\frac{1}{\sqrt{y^{4} x^{2}-1}}\right)}{2}+\frac{\ln\left(x\right)}{2}-c_{1}=0
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve(y(x)*(1+sqrt(x^2*y(x)^4-1))+2*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\operatorname{RootOf}\left(-\ln(x) + c_1 - 2\left(\int^{\_Z} \frac{1}{\_a\sqrt{\_a^4-1}} d\_a\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.188 (sec). Leaf size: 40

```
DSolve[y[x]*(1+Sqrt[x^2*y[x]^4-1])+2*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\operatorname{Solve}\left[\arctan\left(\sqrt{x^2y(x)^4-1}\right) + \frac{1}{2}\log\left(x^2y(x)^4\right) - 2\log(y(x)) = c_1, y(x)\right]$$

3 Chapter 1. First order differential equations. Section 1.3. Exact equations problems. page 24

3.1 problem 1

Internal problem ID [5806]
Internal file name [`OUTPUT/5054_Sunday_June_05_2022_03_19_23_PM_65831039/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.3. Exact equations problems. page 24
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{x(2-9x y^{2})+y(4y^{2}-6x^{3}) y' = 0}$$

3.1.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(y\left(-6x^3 + 4y^2\right)\right) \mathrm{d}y &= \left(-x\left(-9y^2 x + 2\right)\right) \mathrm{d}x \\
\left(x\left(-9y^2 x + 2\right)\right) \mathrm{d}x + \left(y\left(-6x^3 + 4y^2\right)\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= x\left(-9y^2 x + 2\right) \\
N(x, y) &= y\left(-6x^3 + 4y^2\right)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x\left(-9y^2 x + 2\right)\right) \\
&= -18y\, x^2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y\left(-6x^3 + 4y^2\right)\right) \\
&= -18y\, x^2
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int x(-9y^2x + 2) \, \mathrm{d}x$$
$$\phi = -3y^2x^3 + x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -6y\,x^3 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y(-6x^3 + 4y^2)$. Therefore equation (4) becomes

$$y(-6x^3 + 4y^2) = -6y\,x^3 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y^3$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (4y^3) \, \mathrm{d}y$$
$$f(y) = y^4 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -3y^2x^3 + y^4 + x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -3y^2x^3 + y^4 + x^2$$

Summary

The solution(s) found are the following

$$-3x^3y^2 + y^4 + x^2 = c_1 \tag{1}$$

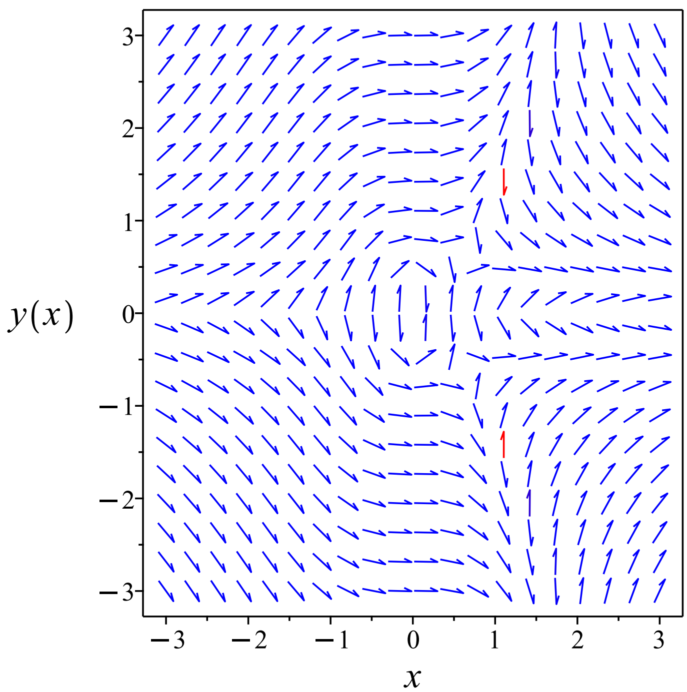


Figure 104: Slope field plot

Verification of solutions

$$-3x^3y^2 + y^4 + x^2 = c_1$$

Verified OK.

3.1.2 Maple step by step solution

Let's solve

$x(2-9xy^2)+y(4y^2-6x^3)\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $-18y\,x^2=-18y\,x^2$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int x(-9y^2x+2)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=-3y^2x^3+x^2+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $y(-6x^3+4y^2)=-6y\,x^3+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=6y\,x^3+y(-6x^3+4y^2)$

- Solve for $f_1(y)$

  $f_1(y)=y^4$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = -3y^2x^3 + y^4 + x^2$$

- Substitute $F(x,y)$ into the solution of the ODE

$$-3y^2x^3 + y^4 + x^2 = c_1$$

- Solve for y

$$\left\{y = -\frac{\sqrt{6x^3-2\sqrt{9x^6-4x^2+4c_1}}}{2}, y = \frac{\sqrt{6x^3-2\sqrt{9x^6-4x^2+4c_1}}}{2}, y = -\frac{\sqrt{6x^3+2\sqrt{9x^6-4x^2+4c_1}}}{2}, y = \frac{\sqrt{6x^3+2\sqrt{9x^6-4}}}{2}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 125

```
dsolve(x*(2-9*x*y(x)^2)+y(x)*(4*y(x)^2-6*x^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{6x^3 - 2\sqrt{9x^6 - 4x^2 - 4c_1}}}{2}$$

$$y(x) = \frac{\sqrt{6x^3 - 2\sqrt{9x^6 - 4x^2 - 4c_1}}}{2}$$

$$y(x) = -\frac{\sqrt{6x^3 + 2\sqrt{9x^6 - 4x^2 - 4c_1}}}{2}$$

$$y(x) = \frac{\sqrt{6x^3 + 2\sqrt{9x^6 - 4x^2 - 4c_1}}}{2}$$

✓ Solution by Mathematica

Time used: 5.767 (sec). Leaf size: 163

```
DSolve[x*(2-9*x*y[x]^2)+y[x]*(4*y[x]^2-6*x^3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to -\frac{\sqrt{3x^3 - \sqrt{9x^6 - 4x^2 + 4c_1}}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{3x^3 - \sqrt{9x^6 - 4x^2 + 4c_1}}}{\sqrt{2}}$$

$$y(x) \to -\frac{\sqrt{3x^3 + \sqrt{9x^6 - 4x^2 + 4c_1}}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{3x^3 + \sqrt{9x^6 - 4x^2 + 4c_1}}}{\sqrt{2}}$$

3.2 problem 2

Internal problem ID [5807]
Internal file name [`OUTPUT/5055_Sunday_June_05_2022_03_19_25_PM_95258704/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.3. Exact equations problems. page 24
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_1st_order, `_with_symmetry_[F(x),G(y)]`]]
```

$$\boxed{\frac{y}{x}+\left(y^{3}+\ln \left(x \right)\right) y^{\prime}=0}$$

3.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(y^3 + \ln(x)\right)\mathrm{d}y = \left(-\frac{y}{x}\right)\mathrm{d}x$$
$$\left(\frac{y}{x}\right)\mathrm{d}x + \left(y^3 + \ln(x)\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{y}{x}$$
$$N(x, y) = y^3 + \ln(x)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$
$$= \frac{1}{x}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y^3+\ln\left(x\right)\right) \\ &= \frac{1}{x}\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{y}{x}\,\mathrm{d}x\end{aligned}$$
$$\phi = y\ln\left(x\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \ln\left(x\right) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=y^3+\ln\left(x\right)$. Therefore equation (4) becomes

$$y^3+\ln\left(x\right) = \ln\left(x\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^3$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(y^3\right)\mathrm{d}y \\ f(y) &= \frac{y^4}{4} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y \ln(x) + \frac{y^4}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y \ln(x) + \frac{y^4}{4}$$

Summary

The solution(s) found are the following

$$\ln(x)\, y + \frac{y^4}{4} = c_1 \tag{1}$$

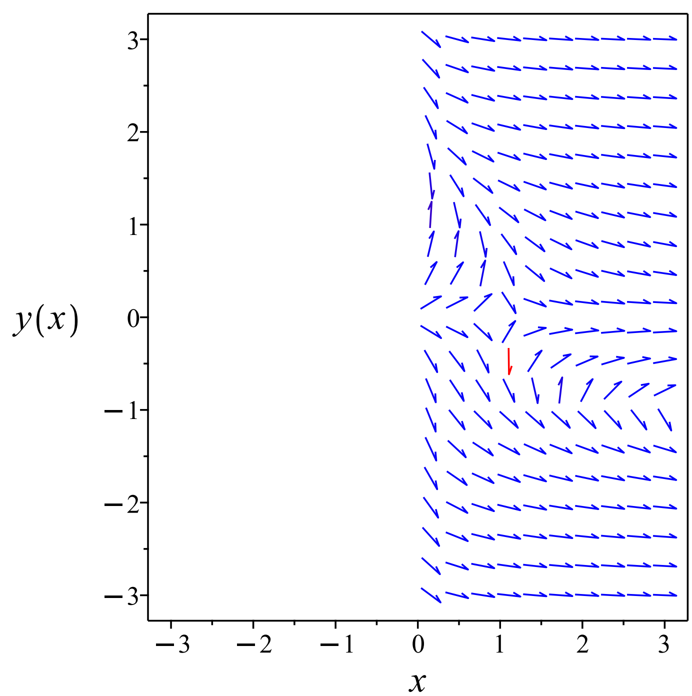


Figure 105: Slope field plot

Verification of solutions

$$\ln(x)\, y + \frac{y^4}{4} = c_1$$

Verified OK.

3.2.2 Maple step by step solution

Let's solve

$\frac{y}{x}+\left(y^{3}+\ln \left(x \right)\right) y^{\prime}=0$

- Highest derivative means the order of the ODE is 1

  $y^{\prime}$

- ☐ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^{2} function

    $F^{\prime}(x , y)=0$

  - Compute derivative of lhs

    $F^{\prime}(x , y)+\left(\frac{\partial}{\partial y}F(x , y)\right) y^{\prime}=0$

  - Evaluate derivatives

    $\frac{1}{x}=\frac{1}{x}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x , y)=c_{1}, M(x , y)=F^{\prime}(x , y), N(x , y)=\frac{\partial}{\partial y}F(x , y)\right]$

- Solve for $F(x , y)$ by integrating $M(x , y)$ with respect to x

  $F(x , y)=\int \frac{y}{x}d x +f_{1}(y)$

- Evaluate integral

  $F(x , y)=y \ln \left(x \right)+f_{1}(y)$

- Take derivative of $F(x , y)$ with respect to y

  $N(x , y)=\frac{\partial}{\partial y}F(x , y)$

- Compute derivative

  $y^{3}+\ln \left(x \right)=\ln \left(x \right)+\frac{d}{d y}f_{1}(y)$

- Isolate for $\frac{d}{d y}f_{1}(y)$

  $\frac{d}{d y}f_{1}(y)=y^{3}$

- Solve for $f_{1}(y)$

  $f_{1}(y)=\frac{y^{4}}{4}$

- Substitute $f_{1}(y)$ into equation for $F(x , y)$

$F(x,y) = y\ln(x) + \frac{y^4}{4}$

- Substitute $F(x,y)$ into the solution of the ODE

  $y\ln(x) + \frac{y^4}{4} = c_1$
- Solve for y

  $y = \mathit{RootOf}\left(\_Z^4 + 4\_Z\ln(x) - 4c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(y(x)/x+(y(x)^3+ln(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\ln(x)\, y(x) + \frac{y(x)^4}{4} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 60.188 (sec). Leaf size: 1025

```
DSolve[y[x]/x+(y[x]^3+Log[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{\sqrt{\frac{\sqrt[3]{3}\left(9\log^2(x)+\sqrt{81\log^4(x)+192c_1{}^3}\right)^{2/3}-4\ 3^{2/3}c_1}{\sqrt[3]{9\log^2(x)+\sqrt{81\log^4(x)+192c_1{}^3}}}}}{\sqrt{6}}$$

$$-\frac{1}{2}\sqrt{\frac{8c_1}{\sqrt[3]{3}\sqrt[3]{9\log^2(x)+\sqrt{81\log^4(x)+192c_1{}^3}}}-\frac{2\sqrt[3]{9\log^2(x)+\sqrt{81\log^4(x)+192c_1{}^3}}}{3^{2/3}}-\frac{\quad}{\sqrt{\frac{\sqrt[3]{3}\left(9\log^2(x)\cdots\right.}{\sqrt[3]{9\log^2(\cdots}}}}}$$

$$y(x)\to \frac{1}{2}\left(\frac{\sqrt{\frac{2\sqrt[3]{3}\left(9\log^2(x)+\sqrt{81\log^4(x)+192c_1{}^3}\right)^{2/3}-8\ 3^{2/3}c_1}{\sqrt[3]{9\log^2(x)+\sqrt{81\log^4(x)+192c_1{}^3}}}}}{\sqrt{3}}\right.$$

$$+\sqrt{\frac{8c_1}{\sqrt[3]{3}\sqrt[3]{9\log^2(x)+\sqrt{81\log^4(x)+192c_1{}^3}}}-\frac{2\sqrt[3]{9\log^2(x)+\sqrt{81\log^4(x)+192c_1{}^3}}}{3^{2/3}}-\frac{\quad}{\sqrt{\frac{\sqrt[3]{3}\left(9\log^2(x)+\cdots\right.}{\sqrt[3]{9\log^2(x)\cdots}}}}}$$

$$y(x)\to -\frac{\sqrt{\frac{\sqrt[3]{3}\left(9\log^2(x)+\sqrt{81\log^4(x)+192c_1{}^3}\right)^{2/3}-4\ 3^{2/3}c_1}{\sqrt[3]{9\log^2(x)+\sqrt{81\log^4(x)+192c_1{}^3}}}}}{\sqrt{6}}$$

$$-\frac{1}{2}\sqrt{\frac{8c_1}{\sqrt[3]{3}\sqrt[3]{9\log^2(x)+\sqrt{81\log^4(x)+192c_1{}^3}}}-\frac{2\sqrt[3]{9\log^2(x)+\sqrt{81\log^4(x)+192c_1{}^3}}}{3^{2/3}}+\frac{\quad}{\sqrt{\frac{\sqrt[3]{3}\left(9\log^2(x)\cdots\right.}{\sqrt[3]{9\log^2(\cdots}}}}}$$

$y(x)$

3.3 problem 3

Internal problem ID [5808]
Internal file name [`OUTPUT/5056_Sunday_June_05_2022_03_19_27_PM_67885355/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.3. Exact equations problems. page 24
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y'(2y-2) = -2x-3}$$

3.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-x-\frac{3}{2}}{y-1} \end{aligned}$$

Where $f(x) = -x - \frac{3}{2}$ and $g(y) = \frac{1}{y-1}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y-1}}\,dy = -x - \frac{3}{2}\,dx$$

$$\int \frac{1}{\frac{1}{y-1}}\,dy = \int -x - \frac{3}{2}\,dx$$

$$\frac{1}{2}y^2 - y = -\frac{1}{2}x^2 - \frac{3}{2}x + c_1$$

Which results in

$$y = 1 + \sqrt{-x^2 + 2c_1 - 3x + 1}$$

$$y = 1 - \sqrt{-x^2 + 2c_1 - 3x + 1}$$

Summary

The solution(s) found are the following

$$y = 1 + \sqrt{-x^2 + 2c_1 - 3x + 1} \tag{1}$$

$$y = 1 - \sqrt{-x^2 + 2c_1 - 3x + 1} \tag{2}$$

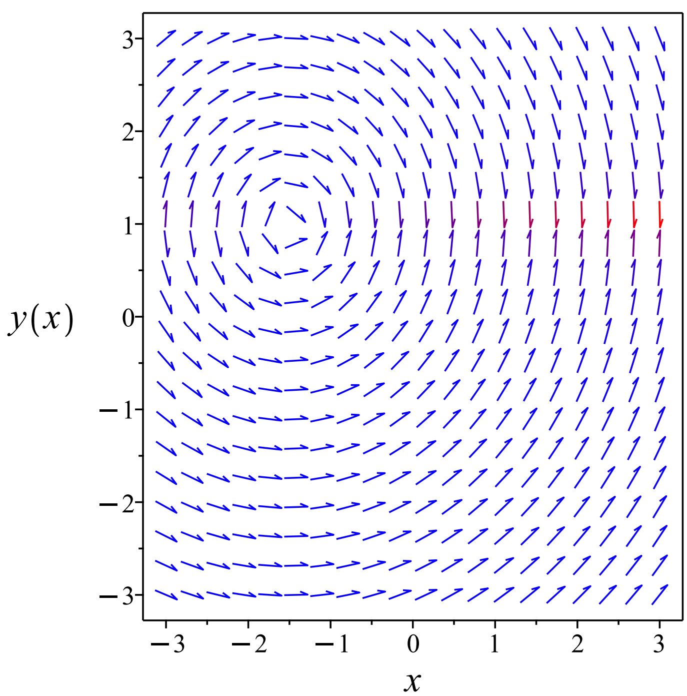


Figure 106: Slope field plot

Verification of solutions

$$y = 1 + \sqrt{-x^2 + 2c_1 - 3x + 1}$$

Verified OK.

$$y = 1 - \sqrt{-x^2 + 2c_1 - 3x + 1}$$

Verified OK.

3.3.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-2x - 3}{2y - 2} \tag{1}$$

Which becomes

$$(2y - 2)\,dy = (-2x - 3)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-2x - 3)\,dx = d\left(-x^2 - 3x\right)$$

Hence (2) becomes

$$(2y - 2)\,dy = d\left(-x^2 - 3x\right)$$

Integrating both sides gives gives these solutions

$$y = 1 + \sqrt{-x^2 + c_1 - 3x + 1} + c_1$$
$$y = 1 - \sqrt{-x^2 + c_1 - 3x + 1} + c_1$$

Summary
The solution(s) found are the following

$$y = 1 + \sqrt{-x^2 + c_1 - 3x + 1} + c_1 \tag{1}$$
$$y = 1 - \sqrt{-x^2 + c_1 - 3x + 1} + c_1 \tag{2}$$

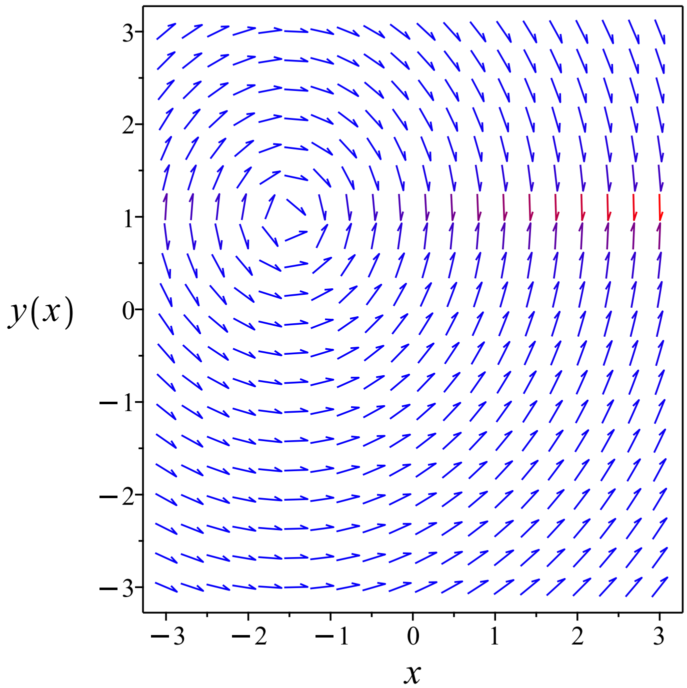


Figure 107: Slope field plot

Verification of solutions

$$y = 1 + \sqrt{-x^2 + c_1 - 3x + 1} + c_1$$

Verified OK.

$$y = 1 - \sqrt{-x^2 + c_1 - 3x + 1} + c_1$$

Verified OK.

3.3.3 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{2X + 2x_0 + 3}{2\left(Y\left(X\right) + y_0 - 1\right)}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -\frac{3}{2}$$
$$y_0 = 1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{X}{Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= -\frac{X}{Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -X$ and $N = Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= -\frac{1}{u} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{-\frac{1}{u(X)} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{-\frac{1}{u(X)} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) u(X)\, X + u(X)^2 + 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{u^2+1}{uX} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u}} du = -\frac{1}{X} dX$$
$$\int \frac{1}{\frac{u^2+1}{u}} du = \int -\frac{1}{X} dX$$
$$\frac{\ln (u^2+1)}{2} = -\ln (X) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+1} = \mathrm{e}^{-\ln(X)+c_2}$$

Which simplifies to

$$\sqrt{u^2+1} = \frac{c_3}{X}$$

Which simplifies to

$$\sqrt{u(X)^2+1} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

The solution is

$$\sqrt{u(X)^2+1} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\sqrt{\frac{Y(X)^2}{X^2}+1} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

Using the solution for $Y(X)$

$$\sqrt{\frac{Y(X)^2+X^2}{X^2}} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$
\begin{aligned}
Y &= 1 + y \\
X &= x - \frac{3}{2}
\end{aligned}
$$

Then the solution in y becomes

$$
\sqrt{\frac{(y-1)^2 + \left(x+\frac{3}{2}\right)^2}{\left(x+\frac{3}{2}\right)^2}} = \frac{c_3 e^{c_2}}{x+\frac{3}{2}}
$$

Summary

The solution(s) found are the following

$$
\sqrt{\frac{(y-1)^2 + \left(x+\frac{3}{2}\right)^2}{\left(x+\frac{3}{2}\right)^2}} = \frac{c_3 e^{c_2}}{x+\frac{3}{2}} \tag{1}
$$

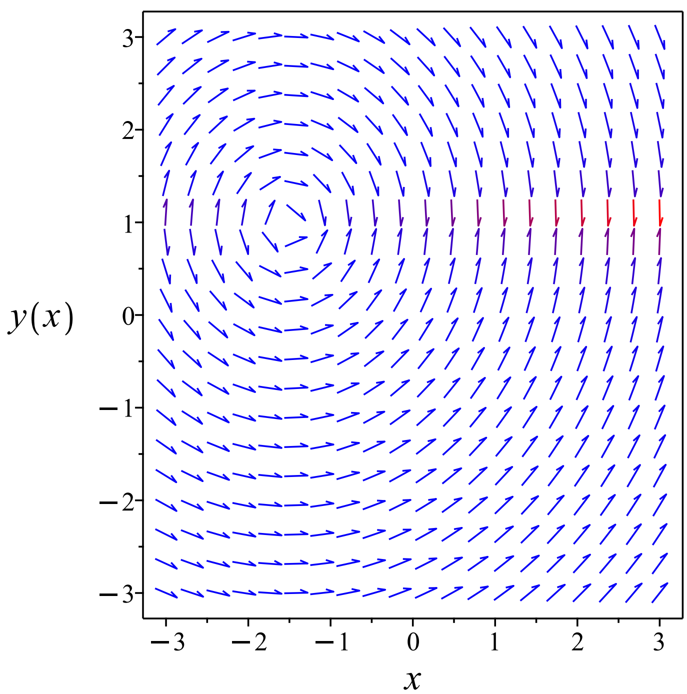


Figure 108: Slope field plot

Verification of solutions

$$\sqrt{\frac{(y-1)^2+\left(x+\frac{3}{2}\right)^2}{\left(x+\frac{3}{2}\right)^2}} = \frac{c_3 e^{c_2}}{x+\frac{3}{2}}$$

Verified OK.

3.3.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2x+3}{2(y-1)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 51: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{-x - \frac{3}{2}} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{-x-\frac{3}{2}}} dx \end{aligned}$$

Which results in

$$S = -\frac{1}{2}x^2 - \frac{3}{2}x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2x + 3}{2(y - 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -x - \frac{3}{2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y - 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R - 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{2}R^2 - R + c_1 \tag{4}$$

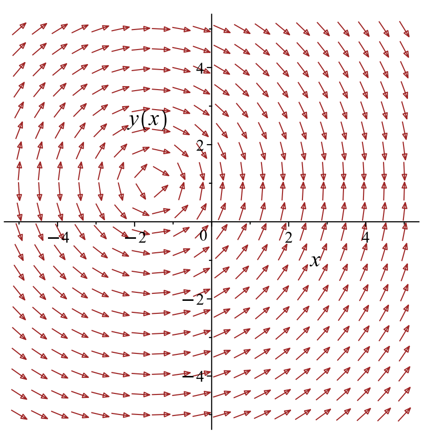
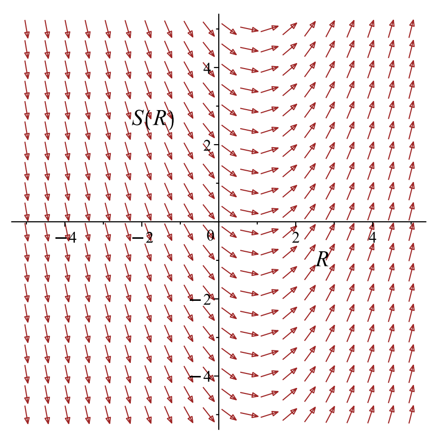
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{2}x^2 - \frac{3}{2}x = \frac{y^2}{2} - y + c_1$$

Which simplifies to

$$-\frac{1}{2}x^2 - \frac{3}{2}x = \frac{y^2}{2} - y + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+3}{2(y-1)}$ 	$R = y$ $S = -\frac{1}{2}x^2 - \frac{3}{2}x$	$\frac{dS}{dR} = R - 1$ 

Summary

The solution(s) found are the following

$$-\frac{1}{2}x^2 - \frac{3}{2}x = \frac{y^2}{2} - y + c_1 \tag{1}$$

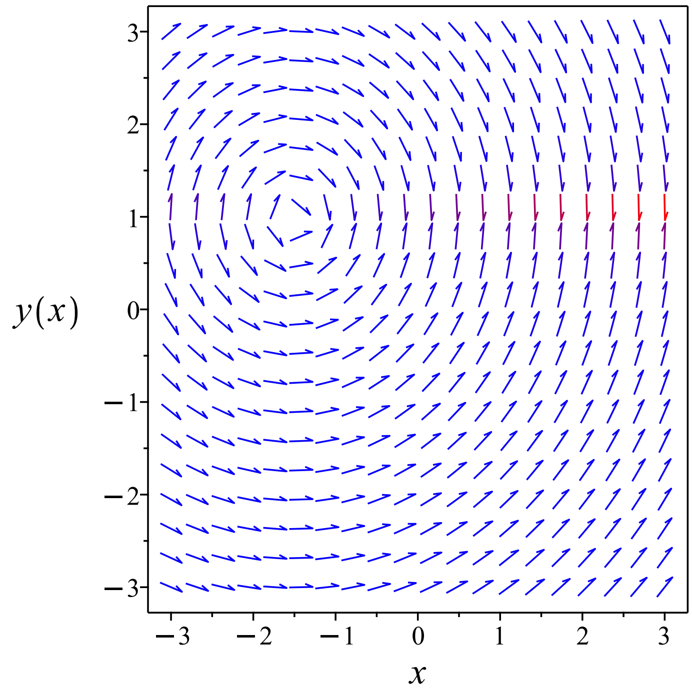


Figure 109: Slope field plot

Verification of solutions

$$-\frac{1}{2}x^2 - \frac{3}{2}x = \frac{y^2}{2} - y + c_1$$

Verified OK.

3.3.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (-2y + 2)\,\mathrm{d}y &= (2x + 3)\,\mathrm{d}x \\ (-2x - 3)\,\mathrm{d}x + (-2y + 2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -2x - 3 \\ N(x, y) &= -2y + 2 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2x - 3) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2y + 2) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -2x - 3 \, \mathrm{d}x$$
$$\phi = -x^2 - 3x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -2y + 2$. Therefore equation (4) becomes

$$-2y + 2 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -2y + 2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-2y + 2) \, \mathrm{d}y$$
$$f(y) = -y^2 + 2y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 - y^2 - 3x + 2y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 - y^2 - 3x + 2y$$

Summary
The solution(s) found are the following

$$-y^2 - x^2 + 2y - 3x = c_1 \tag{1}$$

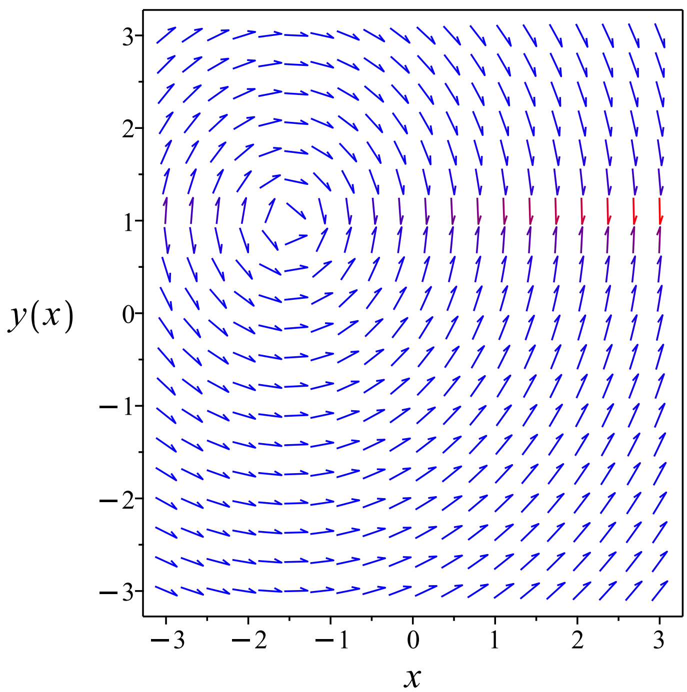


Figure 110: Slope field plot

Verification of solutions

$$-y^2 - x^2 + 2y - 3x = c_1$$

Verified OK.

3.3.6 Maple step by step solution

Let's solve

$y'(2y-2) = -2x-3$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'(2y-2)\,dx = \int (-2x-3)\,dx + c_1$

- Evaluate integral

  $y^2 - 2y = -x^2 + c_1 - 3x$

- Solve for y

  $\left\{y = 1-\sqrt{-x^2+c_1-3x+1}, y = 1+\sqrt{-x^2+c_1-3x+1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve((2*x+3)+(2*y(x)-2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 1-\sqrt{-x^2-c_1-3x+1}$$
$$y(x) = 1+\sqrt{-x^2-c_1-3x+1}$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 51

```
DSolve[(2*x+3)+(2*y[x]-2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 - \sqrt{-x^2 - 3x + 1 + 2c_1}$$
$$y(x) \to 1 + \sqrt{-x^2 - 3x + 1 + 2c_1}$$

3.4 problem 4

Internal problem ID [5809]
Internal file name [`OUTPUT/5057_Sunday_June_05_2022_03_19_29_PM_99560819/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 1. First order differential equations. Section 1.3. Exact equations problems. page 24
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{4y + (2x - 2y)\, y' = -2x}$$

3.4.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$4u(x)\, x + (2x - 2u(x)\, x)\, (u'(x)\, x + u(x)) = -2x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{u^2 - 3u - 1}{(u - 1)\, x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-3u-1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-3u-1}{u-1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2-3u-1}{u-1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln\left(u^2-3u-1\right)}{2} - \frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{(2u-3)\sqrt{13}}{13}\right)}{13} = -\ln\left(x\right) + c_2$$

The solution is

$$\frac{\ln\left(u(x)^2-3u(x)-1\right)}{2} - \frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{(2u(x)-3)\sqrt{13}}{13}\right)}{13} + \ln\left(x\right) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{y^2}{x^2}-\frac{3y}{x}-1\right)}{2} - \frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{\left(\frac{2y}{x}-3\right)\sqrt{13}}{13}\right)}{13} + \ln\left(x\right) - c_2 = 0$$

$$\frac{\ln\left(\frac{y^2}{x^2}-\frac{3y}{x}-1\right)}{2} - \frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{(2y-3x)\sqrt{13}}{13x}\right)}{13} + \ln\left(x\right) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{x^2}-\frac{3y}{x}-1\right)}{2} - \frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{(2y-3x)\sqrt{13}}{13x}\right)}{13} + \ln\left(x\right) - c_2 = 0 \tag{1}$$

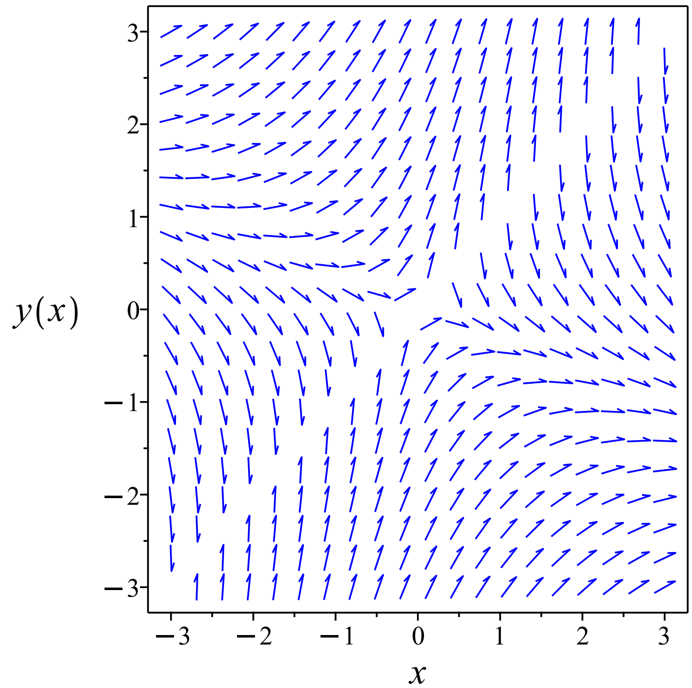


Figure 111: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}-\frac{3y}{x}-1\right)}{2}-\frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{(2y-3x)\sqrt{13}}{13x}\right)}{13}+\ln(x)-c_2=0$$

Verified OK.

3.4.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{2y+x}{-x+y}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(2y+x)(b_3-a_2)}{-x+y} - \frac{(2y+x)^2 a_3}{(-x+y)^2} \\ & - \left(\frac{1}{-x+y} + \frac{2y+x}{(-x+y)^2}\right)(xa_2 + ya_3 + a_1) \\ & - \left(\frac{2}{-x+y} - \frac{2y+x}{(-x+y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^2a_2 - x^2a_3 + 4x^2b_2 - x^2b_3 - 2xya_2 - 4xya_3 - 2xyb_2 + 2xyb_3 - 2y^2a_2 - 7y^2a_3 + y^2b_2 + 2y^2b_3 + 3xb_1 - 3}{(x-y)^2}$$

$= 0$

Setting the numerator to zero gives

$$\begin{aligned} & x^2a_2 - x^2a_3 + 4x^2b_2 - x^2b_3 - 2xya_2 - 4xya_3 - 2xyb_2 \\ & + 2xyb_3 - 2y^2a_2 - 7y^2a_3 + y^2b_2 + 2y^2b_3 + 3xb_1 - 3ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & a_2v_1^2 - 2a_2v_1v_2 - 2a_2v_2^2 - a_3v_1^2 - 4a_3v_1v_2 - 7a_3v_2^2 + 4b_2v_1^2 \\ & - 2b_2v_1v_2 + b_2v_2^2 - b_3v_1^2 + 2b_3v_1v_2 + 2b_3v_2^2 - 3a_1v_2 + 3b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(a_2 - a_3 + 4b_2 - b_3) v_1^2 + (-2a_2 - 4a_3 - 2b_2 + 2b_3) v_1 v_2 \\ + 3b_1 v_1 + (-2a_2 - 7a_3 + b_2 + 2b_3) v_2^2 - 3a_1 v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-3a_1 &= 0 \\ 3b_1 &= 0 \\ -2a_2 - 7a_3 + b_2 + 2b_3 &= 0 \\ -2a_2 - 4a_3 - 2b_2 + 2b_3 &= 0 \\ a_2 - a_3 + 4b_2 - b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0 \\ a_2 &= -3a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 0 \\ b_2 &= a_3 \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x \\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(\frac{2y + x}{-x + y}\right)(x) \\ &= \frac{x^2 + 3xy - y^2}{x - y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2+3xy-y^2}{x-y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln\left(-x^2 - 3xy + y^2\right)}{2} - \frac{\sqrt{13}\, \operatorname{arctanh}\left(\frac{(-3x+2y)\sqrt{13}}{13x}\right)}{13}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2y + x}{-x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2y + x}{x^2 + 3xy - y^2} \\ S_y &= \frac{x - y}{x^2 + 3xy - y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

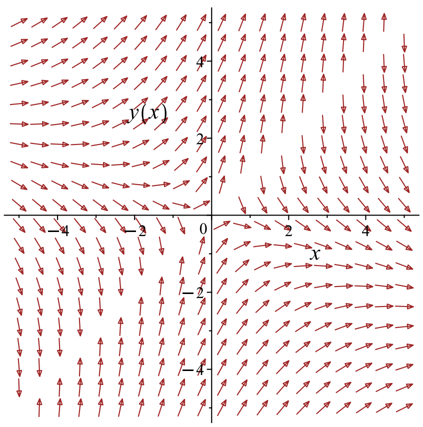
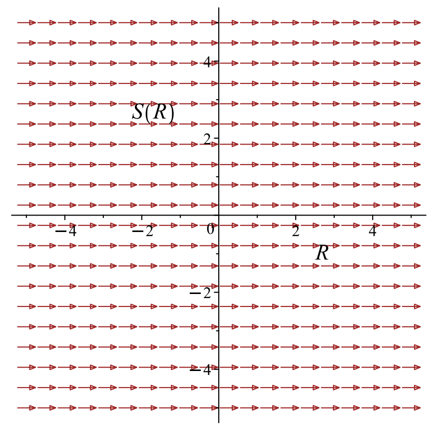
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2 - 3xy - x^2)}{2} + \frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{(3x-2y)\sqrt{13}}{13x}\right)}{13} = c_1$$

Which simplifies to

$$\frac{\ln(y^2 - 3xy - x^2)}{2} + \frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{(3x-2y)\sqrt{13}}{13x}\right)}{13} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y+x}{-x+y}$ 	$R = x$ $S = \frac{\ln(-x^2 - 3xy + y^2)}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y^2 - 3xy - x^2)}{2} + \frac{\sqrt{13}\,\mathrm{arctanh}\left(\frac{(3x-2y)\sqrt{13}}{13x}\right)}{13} = c_1 \tag{1}$$

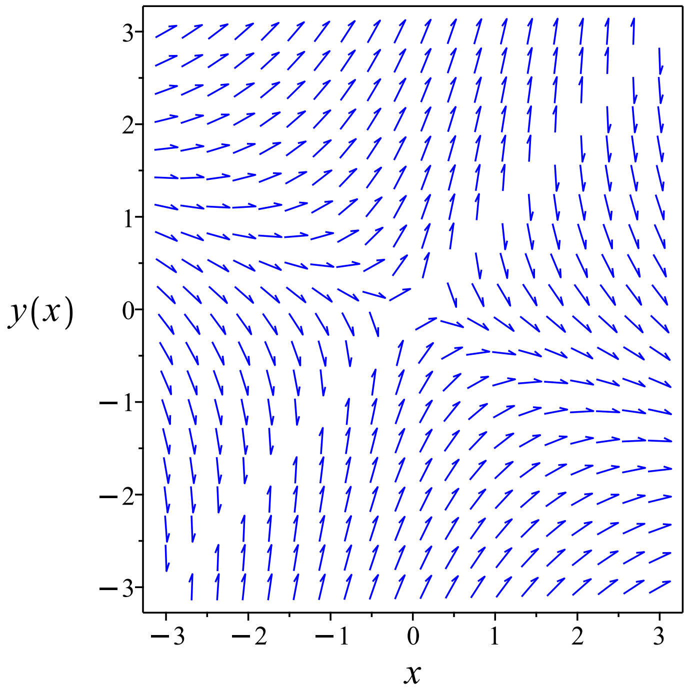


Figure 112: Slope field plot

Verification of solutions

$$\frac{\ln\left(y^2-3xy-x^2\right)}{2}+\frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{(3x-2y)\sqrt{13}}{13x}\right)}{13}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 55

```
dsolve((2*x+4*y(x))+(2*x-2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$-\frac{\ln\left(\frac{-x^2-3xy(x)+y(x)^2}{x^2}\right)}{2}+\frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{(2y(x)-3x)\sqrt{13}}{13x}\right)}{13}-\ln(x)-c_1=0$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 51

```
DSolve[(2*x+3)+(2*y[x]-2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to 1-\sqrt{-x^2-3x+1+2c_1}$$
$$y(x)\to 1+\sqrt{-x^2-3x+1+2c_1}$$

4 Chapter 2. Linear homogeneous equations. Section 2.2 problems. page 95

4.1 problem 49

Internal problem ID [5810]
Internal file name [`OUTPUT/5058_Sunday_June_05_2022_03_19_33_PM_2999641/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.2 problems. page 95
Problem number: 49.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' - y = 0}$$

4.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(-1)}\\ &= -1 \pm \sqrt{2}\end{aligned}$$

Hence

$$\lambda_1 = -1 + \sqrt{2}$$
$$\lambda_2 = -1 - \sqrt{2}$$

Which simplifies to

$$\lambda_1 = \sqrt{2} - 1$$
$$\lambda_2 = -1 - \sqrt{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\sqrt{2}-1\right)x} + c_2 e^{\left(-1-\sqrt{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\left(\sqrt{2}-1\right)x} + c_2 \mathrm{e}^{\left(-1-\sqrt{2}\right)x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\left(\sqrt{2}-1\right)x} + c_2 \mathrm{e}^{\left(-1-\sqrt{2}\right)x} \tag{1}$$

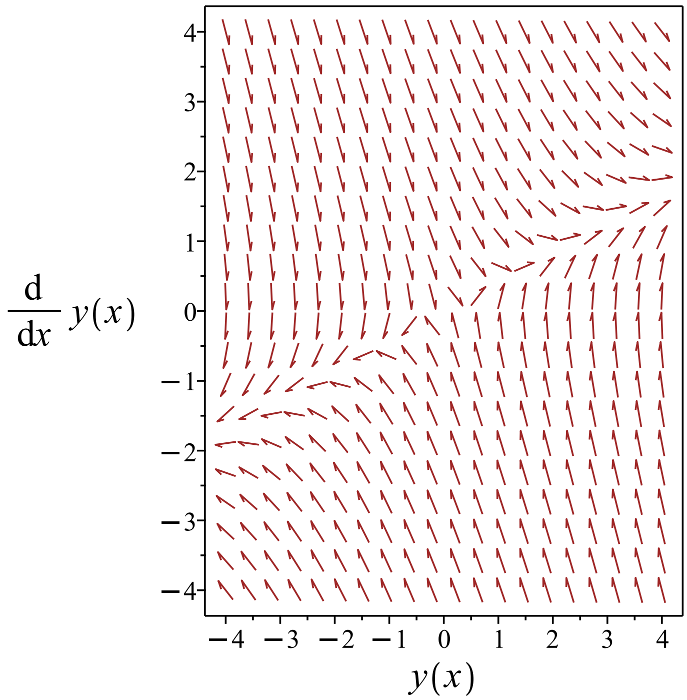


Figure 113: Slope field plot

Verification of solutions

$$y = c_1 e^{(\sqrt{2}-1)x} + c_2 e^{(-1-\sqrt{2})x}$$

Verified OK.

4.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 54: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x\sqrt{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\left(1+\sqrt{2}\right)x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\sqrt{2}\,\mathrm{e}^{2x\sqrt{2}}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\left(1+\sqrt{2}\right)x}\right) + c_2 \left(\mathrm{e}^{-\left(1+\sqrt{2}\right)x} \left(\frac{\sqrt{2}\,\mathrm{e}^{2x\sqrt{2}}}{4}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\left(1+\sqrt{2}\right)x} + \frac{c_2 \sqrt{2}\,\mathrm{e}^{\left(\sqrt{2}-1\right)x}}{4} \tag{1}$$

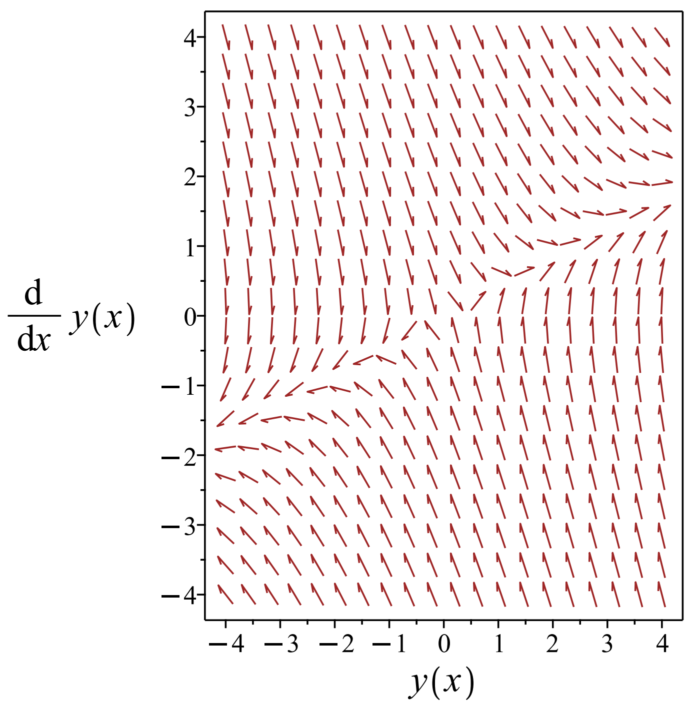


Figure 114: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-\left(1+\sqrt{2}\right)x} + \frac{c_2 \sqrt{2}\,\mathrm{e}^{\left(\sqrt{2}-1\right)x}}{4}$$

Verified OK.

4.1.3 Maple step by step solution

Let's solve

$y'' + 2y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r - 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2) \pm \left(\sqrt{8}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-1 - \sqrt{2}, \sqrt{2} - 1\right)$

- 1st solution of the ODE

  $y_1(x) = e^{\left(-1-\sqrt{2}\right)x}$

- 2nd solution of the ODE

  $y_2(x) = e^{\left(\sqrt{2}-1\right)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\left(-1-\sqrt{2}\right)x} + c_2 e^{\left(\sqrt{2}-1\right)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\left(\sqrt{2}-1\right)x} + c_2 e^{-\left(1+\sqrt{2}\right)x}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 34

```
DSolve[y''[x]+2*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\left(\left(1+\sqrt{2}\right)x\right)}\left(c_2 e^{2\sqrt{2}x} + c_1\right)$$

4.2 problem 50

Internal problem ID [5811]
Internal file name [`OUTPUT/5059_Sunday_June_05_2022_03_19_34_PM_88198110/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.2 problems. page 95
Problem number: 50.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + \frac{y'}{x} - \frac{y}{x^2} = 0}$$

The ode can be written as

$$x^2y'' + xy' - y = 0$$

Which shows it is a Euler ODE.

4.2.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 1 = 0$$

Or

$$r^2 - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2 x$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + c_2 x \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + c_2 x$$

Verified OK.

4.2.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\ &= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)} dx \\ &= \int e^{-\ln(x)}\, dx \\ &= \int \frac{1}{x} dx \\ &= \ln(x) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= -1 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - y(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} - \mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1 \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +1 \\ \lambda_2 &= -1 \end{aligned}$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}$$

Or

$$y(\tau) = c_1 \mathrm{e}^{\tau} + c_2 \mathrm{e}^{-\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^2 + c_2}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2 + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2 + c_2}{x}$$

Verified OK.

4.2.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' + x y' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-\frac{1}{x^2}}}{c} \\ \tau'' &= \frac{1}{c\sqrt{-\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{c\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx\\ &= \frac{\int \sqrt{-\frac{1}{x^2}}dx}{c}\\ &= \frac{\sqrt{-\frac{1}{x^2}}\,x\ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{(ic_2 + c_1)\,x^2 - ic_2 + c_1}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{(ic_2 + c_1)\,x^2 - ic_2 + c_1}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{(ic_2 + c_1)\,x^2 - ic_2 + c_1}{2x}$$

Verified OK.

4.2.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= \frac{1}{x}\\ q(x) &= -\frac{1}{x^2}\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \frac{3v'(x)}{x} = 0$$

$$v''(x) + \frac{3v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= \mathrm{e}^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{c_1}{2x^2} + c_2\right) x \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{2x^2} + c_2\right) x
$$

Verified OK.

4.2.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (x^2y'' + xy' - y)\,dx &= 0 \\
x^2y' - xy &= c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right) \\ d\left(\frac{y}{x}\right) &= \left(\frac{c_1}{x^3}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{c_1}{x^3} dx \\ \frac{y}{x} &= -\frac{c_1}{2x^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x} + c_2 x$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{2x} + c_2 x \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{2x} + c_2 x$$

Verified OK.

4.2.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -1 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2\right)(0) + (x)(1) + (-1)(x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$x^3v'' + (3x^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$x^2(u'(x)\, x + 3u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= \mathrm{e}^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{x^3} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x^3}\, \mathrm{d}x \\ &= -\frac{c_1}{2x^2} + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (x)\left(-\frac{c_1}{2x^2} + c_2\right) \\ &= \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x \tag{1}$$

Verification of solutions

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x$$

Verified OK.

4.2.7 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2 y'' + x y' - y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2 y'' + x y' - y)\, dx = 0$$
$$x^2 y' - x y = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x) y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x} dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right)\\ \mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{c_1}{x^3}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{c_1}{x^3}\,\mathrm{d}x\\ \frac{y}{x} &= -\frac{c_1}{2x^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x} + c_2 x$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{2x} + c_2 x \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{2x} + c_2 x$$

Verified OK.

4.2.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + x y' - y = 0 \tag{1}$$

$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= x^2\\ B &= x\\ C &= -1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 56: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{x}{x^2} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} dx$$
$$= y_1 \left(\frac{x^2}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^2}{2}\right)\right)$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 x}{2}$$

Verified OK.

4.2.9 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$p(x) = x^2$$
$$q(x) = x$$
$$r(x) = -1$$
$$s(x) = 0$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 1
\end{aligned}
$$

Therefore (1) becomes

$$
2 - (1) + (-1) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx
$$

Substituting the above values for p, q, r, s gives

$$
x^2 y' - xy = c_1
$$

We now have a first order ode to solve which is

$$
x^2 y' - xy = c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{y}{x} = \frac{c_1}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right)$$
$$d\left(\frac{y}{x}\right) = \left(\frac{c_1}{x^3}\right) dx$$

Integrating gives

$$\frac{y}{x} = \int \frac{c_1}{x^3}\, dx$$
$$\frac{y}{x} = -\frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x} + c_2 x$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{2x} + c_2 x \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{2x} + c_2 x$$

Verified OK.

4.2.10 Maple step by step solution

Let's solve

$x^2 y'' + x y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' + x y' - y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2 \left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{d}{dt}y(t) - y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-t}$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{t}$

- General solution of the ODE

$y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{-t} + c_2 e^{t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x} + c_2 x$

- Simplify

  $y = \frac{c_1}{x} + c_2 x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)+1/x*diff(y(x),x)-1/x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 x^2 + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 16

```
DSolve[y''[x]+1/x*y'[x]-1/x^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x} + c_2 x$$

4.3 problem 51

Internal problem ID [5812]
Internal file name [`OUTPUT/5060_Sunday_June_05_2022_03_19_35_PM_86961686/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.2 problems. page 95
Problem number: 51.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{(x^2+1)\,y''+xy'+y=0}$$

4.3.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$(x^2+1)\,y''+xy'+y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=\frac{x}{x^2+1}$$
$$q(x)=\frac{1}{x^2+1}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{x}{x^2+1} dx\right)} dx \\
&= \int e^{-\frac{\ln\left(x^2+1\right)}{2}} dx \\
&= \int \frac{1}{\sqrt{x^2+1}} dx \\
&= \operatorname{arcsinh}(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2+1}}{\frac{1}{x^2+1}} \\
&= 1
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) + y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$y(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(\operatorname{arcsinh}(x)) + c_2 \sin(\operatorname{arcsinh}(x))$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(\operatorname{arcsinh}(x)) + c_2 \sin(\operatorname{arcsinh}(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(\operatorname{arcsinh}(x)) + c_2 \sin(\operatorname{arcsinh}(x))$$

Verified OK.

4.3.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$\left(x^2+1\right) y'' + xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x}{x^2+1} \\ q(x) &= \frac{1}{x^2+1} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{1}{c\sqrt{x^2+1}} \\ \tau'' &= -\frac{x}{c\left(x^2+1\right)^{\frac{3}{2}}}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{x}{c(x^2+1)^{\frac{3}{2}}}+\frac{x}{x^2+1}\frac{1}{c\sqrt{x^2+1}}}{\left(\frac{1}{c\sqrt{x^2+1}}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)''+p_1 y(\tau)'+q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau)+c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1\cos(c\tau)+c_2\sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \frac{1}{\sqrt{x^2+1}}dx}{c} \\ &= \frac{\operatorname{arcsinh}(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1\cos(\operatorname{arcsinh}(x))+c_2\sin(\operatorname{arcsinh}(x))$$

Summary

The solution(s) found are the following

$$y = c_1\cos(\operatorname{arcsinh}(x))+c_2\sin(\operatorname{arcsinh}(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(\operatorname{arcsinh}(x)) + c_2 \sin(\operatorname{arcsinh}(x))$$

Verified OK.

4.3.3 Solving using Kovacic algorithm

Writing the ode as

$$\begin{align*} (x^2+1)\,y'' + xy' + y &= 0 \tag{1} \\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= x^2+1 \\ B &= x \tag{3} \\ C &= 1 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5x^2-2}{4\,(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{align*} s &= -5x^2-2 \\ t &= 4(x^2+1)^2 \end{align*}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-5x^2 - 2}{4\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 58: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(-i+x)^2} - \frac{3}{16(i+x)^2} + \frac{7i}{16(-i+x)} - \frac{7i}{16(i+x)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(-i+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(i+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-5x^2 - 2}{4(x^2+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
i	2	$\{1, 2, 3\}$
$-i$	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(i))} + \frac{1}{(x-(-i))}\right) \\ &= \frac{1}{2i+2x} + \frac{1}{-2i+2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2i+2x} + \frac{1}{-2i+2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2i + 2x} + \frac{1}{-2i + 2x}\right) w + \frac{5x^2 + 4}{4\left(i + x\right)^2 \left(-x + i\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{x + 2\sqrt{-x^2 - 1}}{2x^2 + 2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{x+2\sqrt{-x^2-1}}{2x^2+2} dx} \\ &= \left(x^2 + 1\right)^{\frac{1}{4}} \mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2+1} \, dx} \\ &= z_1 e^{-\frac{\ln\left(x^2+1\right)}{4}} \\ &= z_1 \left(\frac{1}{\left(x^2 + 1\right)^{\frac{1}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln\left(x^2+1\right)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{2\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)}}{\sqrt{x^2+1}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)}\right) + c_2\left(\mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)}\left(\int \frac{\mathrm{e}^{2\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)}}{\sqrt{x^2+1}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)} + c_2\mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)}\left(\int \frac{\mathrm{e}^{2\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)}}{\sqrt{x^2+1}}dx\right) \tag{1}
$$

Verification of solutions

$$
y = c_1\mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)} + c_2\mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)}\left(\int \frac{\mathrm{e}^{2\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)}}{\sqrt{x^2+1}}dx\right)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x^2+1)*diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(\operatorname{arcsinh}(x)) + c_2 \cos(\operatorname{arcsinh}(x))$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 43

```
DSolve[(x^2+1)*y''[x]+x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos\left(\log\left(\sqrt{x^2+1}-x\right)\right) - c_2 \sin\left(\log\left(\sqrt{x^2+1}-x\right)\right)$$

4.4 problem 52

Internal problem ID [5813]
Internal file name [`OUTPUT/5061_Sunday_June_05_2022_03_19_37_PM_32704329/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.2 problems. page 95
Problem number: 52.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' - y' \cot(x) + y \cos(x) = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0,
   Change of variables used:
      [x = arcsin(t)]
   Linear ODE actually solved:
      t*(-t^2+1)^(1/2)*u(t)-diff(u(t),t)+(-t^3+t)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 2.0 (sec). Leaf size: 49

```
dsolve(diff(y(x),x$2)-cot(x)*diff(y(x),x)+cos(x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = (1+\cos(x))\,\mathrm{HeunC}\left(0,1,-1,-2,\frac{3}{2},\frac{\cos(x)}{2}+\frac{1}{2}\right)\left(c_1 + c_2\left(\int^{\cos(x)} \frac{1}{(\_a+1)^2\,\mathrm{HeunC}\left(0,1,-1,-2,\frac{3}{2},\frac{\_a}{2}+\frac{1}{2}\right)^2} d\_a\right)\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]-Cot[x]*y'[x]+Cos[x]*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

4.5 problem 53

Internal problem ID [5814]
Internal file name [`OUTPUT/5062_Sunday_June_05_2022_03_19_45_PM_33985016/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.2 problems. page 95
Problem number: 53.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{y'' + \frac{y'}{x} + y x^2 = 0}$$

4.5.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + x y' + y x^4 = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + x y' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta x^\gamma) + c_2 \operatorname{BesselY}(n, \beta x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 0$$
$$\beta = \frac{1}{2}$$
$$n = 0$$
$$\gamma = 2$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \,\text{BesselJ}\left(0, \frac{x^2}{2}\right) + c_2 \,\text{BesselY}\left(0, \frac{x^2}{2}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \,\text{BesselJ}\left(0, \frac{x^2}{2}\right) + c_2 \,\text{BesselY}\left(0, \frac{x^2}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \,\text{BesselJ}\left(0, \frac{x^2}{2}\right) + c_2 \,\text{BesselY}\left(0, \frac{x^2}{2}\right)$$

Verified OK.

4.5.2 Maple step by step solution

Let's solve

$y x^3 + y'' x + y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - y x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + y x^2 = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = x^2\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$yx^3 + y''x + y' = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^3 \cdot y$ to series expansion

  $$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$

- ○ Shift index using $k- >k - 3$

  $$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$

- ○ Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1}$$

- ○ Shift index using $k- >k + 1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\, x^{k+r}$$

- ○ Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r-1}$$

- ○ Shift index using $k- >k + 1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\,(k + r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + a_1 (1+r)^2 x^r + a_2 (2+r)^2 x^{1+r} + a_3 (3+r)^2 x^{2+r} + \left(\sum_{k=3}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_{k-3}\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- The coefficients of each power of x must be 0

  $\left[a_1(1+r)^2 = 0, a_2(2+r)^2 = 0, a_3(3+r)^2 = 0\right]$
- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + a_{k-3} = 0$
- Shift index using $k- >k+3$

  $a_{k+4}(k+4)^2 + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{a_k}{(k+4)^2}$
- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{a_k}{(k+4)^2}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_k}{(k+4)^2}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+1/x*diff(y(x),x)+x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}\left(0, \frac{x^2}{2}\right) + c_2 \operatorname{BesselY}\left(0, \frac{x^2}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.088 (sec). Leaf size: 31

```
DSolve[y''[x]+1/x*y'[x]+x^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}\left(0, \frac{x^2}{2}\right) + 2c_2 \operatorname{BesselY}\left(0, \frac{x^2}{2}\right)$$

4.6 problem 54

Internal problem ID [5815]
Internal file name [OUTPUT/5063_Sunday_June_05_2022_03_19_48_PM_22354661/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.2 problems. page 95
Problem number: 54.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(-x^2+1)\,y''+2x(-x^2+1)\,y'-2y=0}$$

4.6.1 Solving using Kovacic algorithm

Writing the ode as

$$(-x^4+x^2)\,y''+(-2x^3+2x)\,y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+x^2 \\ B &= -2x^3+2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{x^2 (x^2 - 1)} \tag{6}$$

Comparing the above to (5) shows that

$$s = -2$$
$$t = x^2 (x^2 - 1)$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{2}{x^2 (x^2 - 1)}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 60: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2(x^2 - 1)$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 1. There is a pole at $x = -1$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 1$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x-1} + \frac{1}{1+x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{x^2 (x^2 - 1)}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	1	0	0	1
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= 1 - (0) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{x - 1} - \frac{1}{x} + (-)(0) \\ &= \frac{1}{x - 1} - \frac{1}{x} \\ &= \frac{1}{x^2 - x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x-1} - \frac{1}{x}\right)(1) + \left(\left(-\frac{1}{(x-1)^2} + \frac{1}{x^2}\right) + \left(\frac{1}{x-1} - \frac{1}{x}\right)^2 - \left(-\frac{2}{x^2\,(x^2-1)}\right)\right) = 0$$

$$\frac{-2a_0 + 2}{x^3 - x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (1+x)\, e^{\int \left(\frac{1}{x-1} - \frac{1}{x}\right) dx} \\ &= (1+x)\, e^{\ln(x-1) - \ln(x)} \\ &= \frac{x^2 - 1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x^3 + 2x}{-x^4 + x^2}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 - 1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+2x}{-x^4+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{1}{4x+4} - \frac{\ln(1+x)}{4} - \frac{1}{4x-4} + \frac{\ln(x-1)}{4} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2-1}{x^2} \right) + c_2 \left(\frac{x^2-1}{x^2} \left(-\frac{1}{4x+4} - \frac{\ln(1+x)}{4} - \frac{1}{4x-4} + \frac{\ln(x-1)}{4} \right) \right)
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_1(x^2-1)}{x^2} + \frac{c_2(-\ln(1+x)\, x^2 + \ln(x-1)\, x^2 + \ln(1+x) - \ln(x-1) - 2x)}{4x^2} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1(x^2-1)}{x^2} + \frac{c_2(-\ln(1+x)\, x^2 + \ln(x-1)\, x^2 + \ln(1+x) - \ln(x-1) - 2x)}{4x^2}$$

Verified OK.

4.6.2 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$\left(-x^4+x^2\right) y''+\left(-2x^3+2x\right) y'-2y=0 \tag{1}$$

Becomes

$$y''+p(x)\, y'+q(x)\, y=r(x) \tag{2}$$

Where

$$\begin{aligned}
p(x)&=\frac{2}{x}\\
q(x)&=\frac{2}{x^2\left(x^2-1\right)}\\
r(x)&=0
\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}
\xi^{''}-(\xi\, p)'+\xi q&=0\\
\xi^{''}-\left(\frac{2\xi(x)}{x}\right)'+\left(\frac{2\xi(x)}{x^2\left(x^2-1\right)}\right)&=0\\
\xi''(x)-\frac{2\xi'(x)}{x}+\left(\frac{2}{x^2}+\frac{2}{x^2\left(x^2-1\right)}\right)\xi(x)&=0
\end{aligned}$$

Which is solved for $\xi(x)$. Given an ode of the form

$$A\xi''(x)+B\xi'(x)+C\xi=F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$\xi=Bv$$

This results in

$$\begin{aligned}
\xi'&=B'v+v'B\\
\xi''&=B''v+B'v'+v''B+v'B'\\
&=v''B+2v'+B'+B''v
\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}
A(v''B+2v'B'+B''v)+B(B'v+v'B)+CBv&=0\\
ABv''+\left(2AB'+B^2\right)v'+\left(AB''+BB'+CB\right)v&=0 \qquad (1)
\end{aligned}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $\xi = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^3 - x \\ B &= -2x^2 + 2 \\ C &= 2x \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^3 - x\right)(-4) + \left(-2x^2 + 2\right)(-4x) + (2x)\left(-2x^2 + 2\right) \\ &= -4x^3 + 4x - 2\left(-2x^2 + 2\right) x \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2x^5 + 4x^3 - 2xv'' + \left(-4x^4 + 4\right) v' = 0$$

Now by applying $v' = u$ the above becomes

$$\left(-2x^5 + 4x^3 - 2x\right) u'(x) + \left(-4x^4 + 4\right) u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2(x^2 + 1) u}{x (x^2 - 1)} \end{aligned}$$

Where $f(x) = -\frac{2(x^2+1)}{x(x^2-1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{2(x^2+1)}{x\,(x^2-1)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2(x^2+1)}{x\,(x^2-1)}\,dx \\
\ln(u) &= -2\ln(1+x) - 2\ln(x-1) + 2\ln(x) + c_1 \\
u &= \mathrm{e}^{-2\ln(1+x)-2\ln(x-1)+2\ln(x)+c_1} \\
&= c_1\mathrm{e}^{-2\ln(1+x)-2\ln(x-1)+2\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_1x^2}{(1+x)^2\,(x-1)^2}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1x^2}{(1+x)^2\,(x-1)^2}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1x^2}{(1+x)^2\,(x-1)^2}\,\mathrm{d}x \\
&= c_1\left(-\frac{1}{4\,(1+x)} - \frac{\ln(1+x)}{4} - \frac{1}{4\,(x-1)} + \frac{\ln(x-1)}{4}\right) + c_2
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
\xi(x) &= Bv \\
&= (-2x^2+2)\left(c_1\left(-\frac{1}{4\,(1+x)} - \frac{\ln(1+x)}{4} - \frac{1}{4\,(x-1)} + \frac{\ln(x-1)}{4}\right) + c_2\right) \\
&= \frac{(-x^2+1)\,c_1\ln(x-1)}{2} + \frac{(x^2-1)\,c_1\ln(1+x)}{2} - 2c_2x^2 + c_1x + 2c_2
\end{aligned}
$$

The original ode (2) now reduces to first order ode

$$
\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}
\end{aligned}
$$

$$
y' + y\left(\frac{2}{x} - \frac{-xc_3\ln(x-1) + \frac{(-x^2+1)c_3}{2x-2} + xc_3\ln(1+x) + \frac{(x^2-1)c_3}{2+2x} - 4c_2x + c_3}{\frac{(-x^2+1)c_3\ln(x-1)}{2} + \frac{(x^2-1)c_3\ln(1+x)}{2} - 2c_2x^2 + c_3x + 2c_2}\right) = 0
$$

Which is now a first order ode. This is now solved for y. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{2y(\ln(1+x)\,c_3 - \ln(x-1)\,c_3 - 2c_3x - 4c_2)}{(\ln(1+x)\,c_3x^2 - \ln(x-1)\,c_3x^2 - 4c_2x^2 - \ln(1+x)\,c_3 + \ln(x-1)\,c_3 + 2c_3x + 4c_2)\,x}
\end{aligned}
$$

Where $f(x) = \frac{2\ln(1+x)c_3 - 2\ln(x-1)c_3 - 4c_3x - 8c_2}{(\ln(1+x)c_3x^2 - \ln(x-1)c_3x^2 - 4c_2x^2 - \ln(1+x)c_3 + \ln(x-1)c_3 + 2c_3x + 4c_2)x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{2\ln(1+x)\,c_3 - 2\ln(x-1)\,c_3 - 4c_3x - 8c_2}{(\ln(1+x)\,c_3x^2 - \ln(x-1)\,c_3x^2 - 4c_2x^2 - \ln(1+x)\,c_3 + \ln(x-1)\,c_3 + 2c_3x + 4c_2)\,x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{2\ln(1+x)\,c_3 - 2\ln(x-1)\,c_3 - 4c_3x - 8c_2}{(\ln(1+x)\,c_3x^2 - \ln(x-1)\,c_3x^2 - 4c_2x^2 - \ln(1+x)\,c_3 + \ln(x-1)\,c_3 + 2c_3x + 4c_2)\,x}\,dx \\
\ln(y) &= -2\ln(x) + \ln\left((x-1)^2\,c_3\ln(x-1) - \ln(1+x)\,(x-1)^2\,c_3 + 4(x-1)^2\,c_2 + 2(x-1)\,c_3\ln(x - \right. \\
y &= \mathrm{e}^{-2\ln(x)+\ln\left((x-1)^2c_3\ln(x-1)-\ln(1+x)(x-1)^2c_3+4(x-1)^2c_2+2(x-1)c_3\ln(x-1)-2\ln(1+x)(x-1)c_3+8c_2(x-1)-2c_3(x-1)-2c\right.} \\
&= c_3\mathrm{e}^{-2\ln(x)+\ln\left((x-1)^2c_3\ln(x-1)-\ln(1+x)(x-1)^2c_3+4(x-1)^2c_2+2(x-1)c_3\ln(x-1)-2\ln(1+x)(x-1)c_3+8c_2(x-1)-2c_3(x-1)-\right.}
\end{aligned}
$$

Which simplifies to

$$
y = c_3\left(-\ln(1+x)\,c_3 + \ln(x-1)\,c_3 + 4c_2 + \frac{\ln(1+x)\,c_3}{x^2} - \frac{\ln(x-1)\,c_3}{x^2} - \frac{2c_3}{x} - \frac{4c_2}{x^2}\right)
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = c_3\left(4 - \frac{4}{x^2}\right)c_2 + c_3\left(-\ln(1+x)\,c_3 + \ln(x-1)\,c_3 + \frac{\ln(1+x)\,c_3}{x^2} - \frac{\ln(x-1)\,c_3}{x^2} - \frac{2c_3}{x}\right)
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_3\left(4 - \frac{4}{x^2}\right)c_2 \\
& + c_3\left(-\ln(1+x)\,c_3 + \ln(x-1)\,c_3 + \frac{\ln(1+x)\,c_3}{x^2} - \frac{\ln(x-1)\,c_3}{x^2} - \frac{2c_3}{x}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = c_3\left(4 - \frac{4}{x^2}\right)c_2 + c_3\left(-\ln(1+x)\,c_3 + \ln(x-1)\,c_3 + \frac{\ln(1+x)\,c_3}{x^2} - \frac{\ln(x-1)\,c_3}{x^2} - \frac{2c_3}{x}\right)
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(x^2*(1-x^2)*diff(y(x),x$2)+2*x*(1-x^2)*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2(x^2-1)\ln(x-1)+(-x^2+1)\,c_2\ln(x+1)+2c_1x^2-2c_2x-2c_1}{2x^2}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 56

```
DSolve[x^2*(1-x^2)*y''[x]+2*x*(1-x^2)*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{-4c_1x^2-c_2(x^2-1)\log(1-x)+c_2(x^2-1)\log(x+1)+2c_2x+4c_1}{4x^2}$$

4.7 problem 55

Internal problem ID [5816]
Internal file name [`OUTPUT/5064_Sunday_June_05_2022_03_19_52_PM_38625738/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.2 problems. page 95
Problem number: 55.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(-x^2+1)\,y''-xy'+y=0}$$

4.7.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$\left(-x^{2}+1\right) y''-x y'+y=0 \tag{1}$$

Becomes

$$y''+p(x) y'+q(x) y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x}{x^{2}-1} \\ q(x) &= \frac{1}{-x^{2}+1} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^{2}}{d\tau^{2}}y(\tau)+p_{1}\left(\frac{d}{d\tau}y(\tau)\right)+q_{1}y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_{1}(\tau) = \frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'\left(x\right)^{2}} \tag{4}$$

$$q_{1}(\tau) = \frac{q(x)}{\tau'\left(x\right)^{2}} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x)+p(x)\,\tau'(x)=0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int {\mathrm e}^{-\left(\int p(x)d x\right)}d x \\ &= \int {\mathrm e}^{-\left(\int \frac{x}{x^{2}-1}d x\right)}d x \\ &= \int e^{-\frac{\ln\left(x-1\right)}{2}-\frac{\ln\left(1+x\right)}{2}} \,dx \\ &= \int \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}d x \\ &= \frac{\sqrt{\left(x-1\right)\left(1+x\right)}\,\ln\left(x+\sqrt{x^{2}-1}\right)}{\sqrt{x-1}\,\sqrt{1+x}} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{-x^2+1}}{\frac{1}{(x-1)(1+x)}} \\ &= -1 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - y(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda \tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda \tau} - \mathrm{e}^{\lambda \tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda \tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1 \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +1 \\ \lambda_2 &= -1 \end{aligned}$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}$$

Or

$$y(\tau) = c_1 e^{\tau} + c_2 e^{-\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(x + \sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x + \sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(x + \sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x + \sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} \tag{1}$$

Verification of solutions

$$y = c_1 \left(x + \sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x + \sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}}$$

Verified OK.

4.7.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$\left(-x^2+1\right) y'' - x y' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{x}{x^2-1}$$
$$q(x) = -\frac{1}{x^2-1}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-\frac{1}{x^2-1}}}{c} \\ \tau'' &= \frac{x}{c\sqrt{-\frac{1}{x^2-1}}\,(x^2-1)^2} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{x}{c\sqrt{-\frac{1}{x^2-1}}\,(x^2-1)^2} + \frac{x}{x^2-1}\frac{\sqrt{-\frac{1}{x^2-1}}}{c}}{\left(\frac{\sqrt{-\frac{1}{x^2-1}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{-\frac{1}{x^2-1}}dx}{c} \\
&= \frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
\begin{aligned}
y = {} & c_1 \cos\left(\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \\
& + c_2 \sin\left(\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 \cos\left(\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \\
& + c_2 \sin\left(\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 \cos\left(\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \\
& + c_2 \sin\left(\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)
\end{aligned}
$$

Verified OK.

4.7.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$(-x^2+1)\,y''-xy'+y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=\frac{x}{x^2-1}\\ q(x)&=-\frac{1}{x^2-1}\end{aligned}$$

Applying change of variables on the depndent variable $y=v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right)v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right)v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{n}{x^2-1}-\frac{1}{x^2-1}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}v''(x)+\left(\frac{2}{x}+\frac{x}{x^2-1}\right)v'(x)&=0\\ v''(x)+\frac{(3x^2-2)\,v'(x)}{x^3-x}&=0\end{aligned} \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x)+\frac{(3x^2-2)\,u(x)}{x^3-x}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{u(3x^2-2)}{x\,(x^2-1)}
\end{aligned}$$

Where $f(x)=-\frac{3x^2-2}{x(x^2-1)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{3x^2-2}{x\,(x^2-1)}\,dx\\
\int \frac{1}{u}\,du &= \int -\frac{3x^2-2}{x\,(x^2-1)}\,dx\\
\ln(u) &= -\frac{\ln(1+x)}{2}-\frac{\ln(x-1)}{2}-2\ln(x)+c_1\\
u &= \mathrm{e}^{-\frac{\ln(1+x)}{2}-\frac{\ln(x-1)}{2}-2\ln(x)+c_1}\\
&= c_1\mathrm{e}^{-\frac{\ln(1+x)}{2}-\frac{\ln(x-1)}{2}-2\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x)=\frac{c_1}{\sqrt{1+x}\,\sqrt{x-1}\,x^2}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x)\\
v(x) &= \int u(x)\,dx+c_2\\
&= \frac{\sqrt{x-1}\,\sqrt{1+x}\,c_1}{x}+c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n\\
&= \left(\frac{\sqrt{x-1}\,\sqrt{1+x}\,c_1}{x}+c_2\right)x\\
&= c_1\sqrt{x-1}\,\sqrt{1+x}+c_2x
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(\frac{\sqrt{x-1}\,\sqrt{1+x}\,c_1}{x} + c_2\right) x \tag{1}$$

Verification of solutions

$$y = \left(\frac{\sqrt{x-1}\,\sqrt{1+x}\,c_1}{x} + c_2\right) x$$

Verified OK.

4.7.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(\left(-x^2+1\right) y'' - xy' + y\right) dx = 0$$
$$xy - \left(x^2-1\right) y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{x}{x^2-1}$$
$$q(x) = -\frac{c_1}{x^2-1}$$

Hence the ode is

$$y' - \frac{xy}{x^2-1} = -\frac{c_1}{x^2-1}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{x}{x^2-1}dx}$$
$$= e^{-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2}}$$

Which simplifies to

$$\mu = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(-\frac{c_1}{x^2-1}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{x-1}\,\sqrt{1+x}}\right) = \left(\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}\right)\left(-\frac{c_1}{x^2-1}\right)$$
$$\mathrm{d}\left(\frac{y}{\sqrt{x-1}\,\sqrt{1+x}}\right) = \left(-\frac{c_1}{(x^2-1)\sqrt{x-1}\,\sqrt{1+x}}\right)\mathrm{d}x$$

Integrating gives

$$\frac{y}{\sqrt{x-1}\,\sqrt{1+x}} = \int -\frac{c_1}{(x^2-1)\sqrt{x-1}\,\sqrt{1+x}}\,\mathrm{d}x$$
$$\frac{y}{\sqrt{x-1}\,\sqrt{1+x}} = \frac{\sqrt{x-1}\,\sqrt{1+x}\,x c_1}{x^2-1} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$ results in

$$y = \frac{(x-1)(1+x)\,x c_1}{x^2-1} + c_2\sqrt{x-1}\,\sqrt{1+x}$$

which simplifies to

$$y = c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x}$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x} \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x}$$

Verified OK.

4.7.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + (2AB' + B^2)\,v' + (AB'' + BB' + CB)\,v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)\,v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)\,u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= -x^2 + 1 \\ B &= -x \\ C &= 1 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (-x^2 + 1)\,(0) + (-x)\,(-1) + (1)\,(-x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$x^3 - xv'' + (3x^2 - 2)\,v' = 0$$

Now by applying $v' = u$ the above becomes

$$(x^3 - x)\,u'(x) + (3x^2 - 2)\,u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(3x^2-2)}{x\,(x^2-1)} \end{aligned}$$

Where $f(x) = -\frac{3x^2-2}{x(x^2-1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{3x^2-2}{x\,(x^2-1)}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{3x^2-2}{x\,(x^2-1)}\,dx \\ \ln(u) &= -\frac{\ln(1+x)}{2} - \frac{\ln(x-1)}{2} - 2\ln(x) + c_1 \\ u &= \mathrm{e}^{-\frac{\ln(1+x)}{2}-\frac{\ln(x-1)}{2}-2\ln(x)+c_1} \\ &= c_1 \mathrm{e}^{-\frac{\ln(1+x)}{2}-\frac{\ln(x-1)}{2}-2\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1}{\sqrt{1+x}\,\sqrt{x-1}\,x^2}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{\sqrt{1+x}\,\sqrt{x-1}\,x^2} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{\sqrt{1+x}\,\sqrt{x-1}\,x^2}\,\mathrm{d}x \\ &= \frac{\sqrt{x-1}\,\sqrt{1+x}\,c_1}{x} + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (-x)\left(\frac{\sqrt{x-1}\,\sqrt{1+x}\,c_1}{x} + c_2\right) \\ &= -c_1\sqrt{x-1}\,\sqrt{1+x} - c_2 x \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -c_1\sqrt{x-1}\,\sqrt{1+x} - c_2 x \tag{1}$$

Verification of solutions

$$y = -c_1\sqrt{x-1}\,\sqrt{1+x} - c_2 x$$

Verified OK.

4.7.6 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(-x^2+1)\,y'' - xy' + y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left((-x^2+1)\,y'' - xy' + y\right) dx = 0$$
$$xy - (x^2-1)\,y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{x}{x^2-1}$$
$$q(x) = -\frac{c_1}{x^2-1}$$

Hence the ode is

$$y' - \frac{xy}{x^2-1} = -\frac{c_1}{x^2-1}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{x}{x^2-1}dx}$$
$$= e^{-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2}}$$

Which simplifies to

$$\mu = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(-\frac{c_1}{x^2-1}\right)\\
\frac{d}{dx}\left(\frac{y}{\sqrt{x-1}\,\sqrt{1+x}}\right) &= \left(\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}\right)\left(-\frac{c_1}{x^2-1}\right)\\
d\left(\frac{y}{\sqrt{x-1}\,\sqrt{1+x}}\right) &= \left(-\frac{c_1}{(x^2-1)\sqrt{x-1}\,\sqrt{1+x}}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{\sqrt{x-1}\,\sqrt{1+x}} &= \int -\frac{c_1}{(x^2-1)\sqrt{x-1}\,\sqrt{1+x}}\,dx\\
\frac{y}{\sqrt{x-1}\,\sqrt{1+x}} &= \frac{\sqrt{x-1}\,\sqrt{1+x}\,x c_1}{x^2-1} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$ results in

$$y = \frac{(x-1)(1+x)\,x c_1}{x^2-1} + c_2\sqrt{x-1}\,\sqrt{1+x}$$

which simplifies to

$$y = c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x}$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x} \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x}$$

Verified OK.

4.7.7 Solving using Kovacic algorithm

Writing the ode as

$$(-x^2+1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x^2-6}{4\,(x^2-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2-6 \\ t &= 4(x^2-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^2-6}{4\,(x^2-1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y=z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 61: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16 (x - 1)^2} + \frac{9}{16 (x - 1)} - \frac{9}{16 (1 + x)} - \frac{3}{16 (1 + x)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^2 - 6}{4\left(x^2 - 1\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^2 - 6}{4\left(x^2 - 1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+}\right) \\ &= \frac{3}{2} - \left(\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{3}{4\left(x-1\right)} + \frac{3}{4\left(1+x\right)} + (0) \\ &= \frac{3}{4\left(x-1\right)} + \frac{3}{4\left(1+x\right)} \\ &= \frac{3x}{2x^2-2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4\left(x-1\right)} + \frac{3}{4\left(1+x\right)}\right)(0) + \left(\left(-\frac{3}{4\left(x-1\right)^2} - \frac{3}{4\left(1+x\right)^2}\right) + \left(\frac{3}{4\left(x-1\right)} + \frac{3}{4\left(1+x\right)}\right)^2 - \left(\frac{3}{4(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{4(x-1)} + \frac{3}{4(1+x)}\right) dx} \\
&= (x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{-x^2+1} \, dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{4} - \frac{\ln(1+x)}{4}} \\
&= z_1 \left(\frac{1}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x-1}\,\sqrt{1+x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{-x^2+1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{x}{\sqrt{x-1}\,\sqrt{1+x}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{x-1}\,\sqrt{1+x}\right) + c_2 \left(\sqrt{x-1}\,\sqrt{1+x}\left(-\frac{x}{\sqrt{x-1}\,\sqrt{1+x}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x-1}\,\sqrt{1+x} - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x-1}\,\sqrt{1+x} - c_2 x$$

Verified OK.

4.7.8 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= -x^2 + 1 \\ q(x) &= -x \\ r(x) &= 1 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= -2 \\ q'(x) &= -1 \end{aligned}$$

Therefore (1) becomes

$$-2 - (-1) + (1) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$\left(-x^2+1\right) y'+x y=c_1$$

We now have a first order ode to solve which is

$$\left(-x^2+1\right) y'+x y=c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x) y=q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{x}{x^2-1} \\
q(x) &= -\frac{c_1}{x^2-1}
\end{aligned}$$

Hence the ode is

$$y'-\frac{x y}{x^2-1}=-\frac{c_1}{x^2-1}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{x}{x^2-1} d x} \\
&= \mathrm{e}^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}}
\end{aligned}$$

Which simplifies to

$$\mu=\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d} x}(\mu y) &= (\mu)\left(-\frac{c_1}{x^2-1}\right) \\
\frac{\mathrm{d}}{\mathrm{d} x}\left(\frac{y}{\sqrt{x-1}\,\sqrt{1+x}}\right) &= \left(\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}\right)\left(-\frac{c_1}{x^2-1}\right) \\
\mathrm{d}\left(\frac{y}{\sqrt{x-1}\,\sqrt{1+x}}\right) &= \left(-\frac{c_1}{\left(x^2-1\right) \sqrt{x-1}\,\sqrt{1+x}}\right) \mathrm{d} x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{\sqrt{x-1}\,\sqrt{1+x}} &= \int -\frac{c_1}{\left(x^2-1\right) \sqrt{x-1}\,\sqrt{1+x}} \mathrm{d} x \\
\frac{y}{\sqrt{x-1}\,\sqrt{1+x}} &= \frac{\sqrt{x-1}\,\sqrt{1+x}\, x c_1}{x^2-1}+c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$ results in

$$y = \frac{(x-1)\,(1+x)\,x c_1}{x^2-1} + c_2\sqrt{x-1}\,\sqrt{1+x}$$

which simplifies to

$$y = c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x}$$

Summary
The solution(s) found are the following

$$y = c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x} \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x}$$

Verified OK.

4.7.9 Maple step by step solution

Let's solve

$(-x^2+1)\,y'' - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{xy'}{x^2-1} + \frac{y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{xy'}{x^2-1} - \frac{y}{x^2-1} = 0$

- Multiply by denominators of ODE

  $(-x^2+1)\,y'' - xy' + y = 0$

- Make a change of variables

  $\theta = \arccos(x)$

- Calculate y' with change of variables

  $y' = \left(\frac{d}{d\theta}y(\theta)\right)\theta'(x)$

- Compute 1st derivative y'

$y' = -\frac{\frac{d}{d\theta}y(\theta)}{\sqrt{-x^2+1}}$

- Calculate y'' with change of variables

  $y'' = \left(\frac{d^2}{d\theta^2}y(\theta)\right)\theta'(x)^2 + \theta''(x)\left(\frac{d}{d\theta}y(\theta)\right)$

- Compute 2nd derivative y''

  $y'' = \frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1} - \frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}$

- Apply the change of variables to the ODE

  $(-x^2+1)\left(\frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1} - \frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}\right) + \frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{\sqrt{-x^2+1}} + y = 0$

- Multiply through

  $-\frac{\left(\frac{d^2}{d\theta^2}y(\theta)\right)x^2}{-x^2+1} + \frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1} + \frac{x^3\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}} - \frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}} + \frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{\sqrt{-x^2+1}} + y = 0$

- Simplify ODE

  $y + \frac{d^2}{d\theta^2}y(\theta) = 0$

- ODE is that of a harmonic oscillator with given general solution

  $y(\theta) = c_1 \sin(\theta) + c_2 \cos(\theta)$

- Revert back to x

  $y = c_1 \sin(\arccos(x)) + c_2 \cos(\arccos(x))$

- Use trig identity to simplify $\sin(\arccos(x))$

  $\sin(\arccos(x)) = \sqrt{-x^2+1}$

- Simplify solution to the ODE

  $y = c_1\sqrt{-x^2+1} + c_2 x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve((1-x^2)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 \sqrt{x-1}\,\sqrt{x+1}$$

✓ Solution by Mathematica

Time used: 0.193 (sec). Leaf size: 97

```
DSolve[(1-x^2)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cosh\left(\frac{2\sqrt{1-x^2}\arctan\left(\frac{\sqrt{1-x^2}}{x+1}\right)}{\sqrt{x^2-1}}\right) - ic_2 \sinh\left(\frac{2\sqrt{1-x^2}\arctan\left(\frac{\sqrt{1-x^2}}{x+1}\right)}{\sqrt{x^2-1}}\right)$$

4.8 problem 56

Internal problem ID [5817]
Internal file name [OUTPUT/5065_Sunday_June_05_2022_03_19_53_PM_91565691/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.2 problems. page 95
Problem number: 56.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_3rd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y''' - 2y''x + 4x^2y' + 8yx^3 = 0}$$

Unable to solve this ODE.

4.8.1 Maple step by step solution

Let's solve

$y''' - 2y''x + 4x^2y' + 8yx^3 = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^3 \cdot y$ to series expansion

$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+3}$

- Shift index using $k- >k-3$

  $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^k$

- Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

- Shift index using $k- >k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=1}^{\infty} a_k k(k-1)\, x^{k-1}$

- Shift index using $k- >k+1$

  $x \cdot y'' = \sum_{k=0}^{\infty} a_{k+1}(k+1)\, k\, x^k$

- Convert y''' to series expansion

  $y''' = \sum_{k=3}^{\infty} a_k k(k-1)\,(k-2)\, x^{k-3}$

- Shift index using $k- >k+3$

  $y''' = \sum_{k=0}^{\infty} a_{k+3}(k+3)\,(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $6a_3 + (24a_4 - 4a_2)\, x + (60a_5 - 12a_3 + 4a_1)\, x^2 + \left(\sum_{k=3}^{\infty} (a_{k+3}(k+3)\,(k+2)\,(k+1) - 2a_{k+1}(k+1$

- The coefficients of each power of x must be 0

  $[6a_3 = 0, 24a_4 - 4a_2 = 0, 60a_5 - 12a_3 + 4a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_3 = 0, a_4 = \frac{a_2}{6}, a_5 = -\frac{a_1}{15}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $k^3 a_{k+3} + (-2a_{k+1} + 6a_{k+3})\, k^2 + (4a_{k-1} - 2a_{k+1} + 11a_{k+3})\, k + 8a_{k-3} - 4a_{k-1} + 6a_{k+3} = 0$

- Shift index using $k- >k+3$

$$(k+3)^3 a_{k+6} + (-2a_{k+4} + 6a_{k+6})(k+3)^2 + (4a_{k+2} - 2a_{k+4} + 11a_{k+6})(k+3) + 8a_k - 4a_{k+2} +$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = \frac{2(k^2 a_{k+4} - 2k a_{k+2} + 7k a_{k+4} - 4a_k - 4a_{k+2} + 12a_{k+4})}{k^3 + 15k^2 + 74k + 120}, a_3 = 0, a_4 = \frac{a_2}{6}, a_5 = -\frac{a_1}{15}\right]$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying high order exact linear fully integrable
trying to convert to a linear ODE with constant coefficients
trying differential order: 3; missing the dependent variable
trying Louvillian solutions for 3rd order ODEs, imprimitive case
-> pFq: Equivalence to the 3F2 or one of its 3 confluent cases under a power @ Moebius
-> pFq: Equivalence to the 3F2 or one of its 3 confluent cases under a power @ Moebius
trying a solution in terms of MeijerG functions
-> pFq: Equivalence to the 3F2 or one of its 3 confluent cases under a power @ Moebius
-> pFq: Equivalence to the 3F2 or one of its 3 confluent cases under a power @ Moebius
trying a solution in terms of MeijerG functions
   checking if the LODE is of Euler type
<- no solution through differential factorization was found
trying reduction of order using simple exponentials
--- Trying Lie symmetry methods, high order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$3)-2*x*diff(y(x),x$2)+4*x^2*diff(y(x),x)+8*x^3*y(x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'''[x]-2*x*y''[x]+4*x^2*y'[x]+8*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

4.9 problem 57

Internal problem ID [5818]
Internal file name [`OUTPUT/5066_Sunday_June_05_2022_03_19_55_PM_1567324/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.2 problems. page 95
Problem number: 57.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + x(1-x)\,y' + \mathrm{e}^{x} y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+x*(1-x)*diff(y(x),x)+exp(x)*y(x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+x*(1-x)*y'[x]+Exp[x]*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

4.10 problem 58

Internal problem ID [5819]
Internal file name [`OUTPUT/5067_Sunday_June_05_2022_03_19_57_PM_12886167/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.2 problems. page 95
Problem number: 58.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + 2xy' + 4y = 0}$$

4.10.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 2xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 2r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 2r + 4 = 0$$

Or

$$r^2 + r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -\frac{1}{2} - \frac{i\sqrt{15}}{2} \\ r_2 &= -\frac{1}{2} + \frac{i\sqrt{15}}{2} \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = -\frac{1}{2}$ and $\beta = -\frac{\sqrt{15}}{2}$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha + i\beta} + c_2 x^{\alpha - i\beta} \\ &= x^{\alpha}\left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^{\alpha}\left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^{\alpha}\left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = -\frac{1}{2}, \beta = -\frac{\sqrt{15}}{2}$, the above becomes

$$y = x^{-\frac{1}{2}}\left(c_1 e^{-\frac{i\sqrt{15}\,\ln(x)}{2}} + c_2 e^{\frac{i\sqrt{15}\,\ln(x)}{2}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = \frac{1}{\sqrt{x}}\left(c_1 \cos\left(\frac{\sqrt{15}\,\ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\,\ln(x)}{2}\right)\right)$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 \cos\left(\frac{\sqrt{15}\,\ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\,\ln(x)}{2}\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos\left(\frac{\sqrt{15}\,\ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\,\ln(x)}{2}\right)}{\sqrt{x}}$$

Verified OK.

4.10.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' + 2xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int \frac{2}{x}dx\right)} dx \\
&= \int e^{-2\ln(x)}\, dx \\
&= \int \frac{1}{x^2} dx \\
&= -\frac{1}{x}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{4}{x^2}}{\frac{1}{x^4}} \\
&= 4x^2
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + 4x^2 y(\tau) &= 0
\end{aligned}$$

But in terms of τ

$$4x^2 = \frac{4}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2} y(\tau) + \frac{4y(\tau)}{\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2} y(\tau)\right) \tau^2 + 4y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2 (r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 4\tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r + 4\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 + 4 = 0$$

Or

$$r^2 - r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= \frac{1}{2} - \frac{i\sqrt{15}}{2} \\ r_2 &= \frac{1}{2} + \frac{i\sqrt{15}}{2} \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{\sqrt{15}}{2}$. Hence the solution becomes

$$\begin{aligned} y(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\ &= c_1 \tau^{\alpha + i\beta} + c_2 \tau^{\alpha - i\beta} \\ &= \tau^{\alpha} \left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta} \right) \\ &= \tau^{\alpha} \left(c_1 e^{\ln\left(\tau^{i\beta}\right)} + c_2 e^{\ln\left(\tau^{-i\beta}\right)} \right) \\ &= \tau^{\alpha} \left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)} \right) \end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{\sqrt{15}}{2}$, the above becomes

$$y(\tau) = \tau^{\frac{1}{2}} \left(c_1 e^{-\frac{i\sqrt{15}\,\ln(\tau)}{2}} + c_2 e^{\frac{i\sqrt{15}\,\ln(\tau)}{2}} \right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y(\tau) = \sqrt{\tau} \left(c_1 \cos\left(\frac{\sqrt{15}\,\ln(\tau)}{2} \right) + c_2 \sin\left(\frac{\sqrt{15}\,\ln(\tau)}{2} \right) \right)$$

The above solution is now transformed back to y using (6) which results in

$$y = \sqrt{-\frac{1}{x}}\left(c_1 \cos\left(\frac{\sqrt{15}\,\ln\left(-\frac{1}{x}\right)}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\,\ln\left(-\frac{1}{x}\right)}{2}\right)\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \sqrt{-\frac{1}{x}}\left(c_1 \cos\left(\frac{\sqrt{15}\,\ln\left(-\frac{1}{x}\right)}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\,\ln\left(-\frac{1}{x}\right)}{2}\right)\right) \tag{1}$$

<u>Verification of solutions</u>

$$y = \sqrt{-\frac{1}{x}}\left(c_1 \cos\left(\frac{\sqrt{15}\,\ln\left(-\frac{1}{x}\right)}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\,\ln\left(-\frac{1}{x}\right)}{2}\right)\right)$$

Verified OK.

4.10.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' + 2xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{2}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= \frac{c}{2}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{c\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-\frac{c\tau}{4}}\left(c_1 \cos\left(\frac{c\sqrt{15}\,\tau}{4}\right) + c_2 \sin\left(\frac{c\sqrt{15}\,\tau}{4}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$y = \frac{c_1 \cos\left(\frac{\sqrt{15}\,\ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\,\ln(x)}{2}\right)}{\sqrt{x}}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 \cos\left(\frac{\sqrt{15}\,\ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\,\ln(x)}{2}\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos\left(\frac{\sqrt{15}\,\ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\,\ln(x)}{2}\right)}{\sqrt{x}}$$

Verified OK.

4.10.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' + 2xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{2n}{x^2}+\frac{4}{x^2}=0 \tag{5}$$

Solving (5) for n gives

$$n=-\frac{1}{2}+\frac{i\sqrt{15}}{2} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x)+\left(\frac{-1+i\sqrt{15}}{x}+\frac{2}{x}\right)v'(x)&=0\\ v''(x)+\frac{\left(i\sqrt{15}+1\right)v'(x)}{x}&=0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x)+\frac{\left(i\sqrt{15}+1\right)u(x)}{x}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u'&=F(x,u)\\ &=f(x)g(u)\\ &=\frac{\left(-1-i\sqrt{15}\right)u}{x} \end{aligned}$$

Where $f(x)=\frac{-1-i\sqrt{15}}{x}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du&=\frac{-1-i\sqrt{15}}{x}\,dx\\ \int\frac{1}{u}\,du&=\int\frac{-1-i\sqrt{15}}{x}\,dx\\ \ln(u)&=\left(-1-i\sqrt{15}\right)\ln(x)+c_1\\ u&=\mathrm{e}^{\left(-1-i\sqrt{15}\right)\ln(x)+c_1}\\ &=c_1\mathrm{e}^{\left(-1-i\sqrt{15}\right)\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-i\sqrt{15}}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= \frac{i\sqrt{15}\, c_1 x^{-i\sqrt{15}}}{15} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(\frac{i\sqrt{15}\, c_1 x^{-i\sqrt{15}}}{15} + c_2\right) x^{-\frac{1}{2}+\frac{i\sqrt{15}}{2}} \\ &= \frac{x^{-\frac{1}{2}-\frac{i\sqrt{15}}{2}} \left(i\sqrt{15}\, c_1 + 15 c_2 x^{i\sqrt{15}}\right)}{15} \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \left(\frac{i\sqrt{15}\, c_1 x^{-i\sqrt{15}}}{15} + c_2\right) x^{-\frac{1}{2}+\frac{i\sqrt{15}}{2}} \tag{1}$$

<u>Verification of solutions</u>

$$y = \left(\frac{i\sqrt{15}\, c_1 x^{-i\sqrt{15}}}{15} + c_2\right) x^{-\frac{1}{2}+\frac{i\sqrt{15}}{2}}$$

Verified OK.

4.10.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 2xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{4}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 64: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{4}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -4$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{15}}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{15}}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{4}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -4$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{15}}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{15}}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{4}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + \frac{i\sqrt{15}}{2}$	$\frac{1}{2} - \frac{i\sqrt{15}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + \frac{i\sqrt{15}}{2}$	$\frac{1}{2} - \frac{i\sqrt{15}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2} - \frac{i\sqrt{15}}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \frac{i\sqrt{15}}{2} - \left(\frac{1}{2} - \frac{i\sqrt{15}}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{15}}{2}}{x} + (-)\,(0) \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{15}}{2}}{x} \\
&= \frac{1 - i\sqrt{15}}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{\frac{1}{2} - \frac{i\sqrt{15}}{2}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{i\sqrt{15}}{2}}{x^2}\right) + \left(\frac{\frac{1}{2} - \frac{i\sqrt{15}}{2}}{x}\right)^2 - \left(-\frac{4}{x^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \frac{\frac{1}{2} - \frac{i\sqrt{15}}{2}}{x} dx} \\
&= x^{\frac{1}{2} - \frac{i\sqrt{15}}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{-\frac{1}{2}-\frac{i\sqrt{15}}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{i x^{i\sqrt{15}}\sqrt{15}}{15}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{-\frac{1}{2}-\frac{i\sqrt{15}}{2}}\right) + c_2\left(x^{-\frac{1}{2}-\frac{i\sqrt{15}}{2}}\left(-\frac{i x^{i\sqrt{15}}\sqrt{15}}{15}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{-\frac{1}{2}-\frac{i\sqrt{15}}{2}} - \frac{i c_2 \sqrt{15}\, x^{-\frac{1}{2}+\frac{i\sqrt{15}}{2}}}{15} \tag{1}
$$

Verification of solutions

$$y = c_1 x^{-\frac{1}{2}-\frac{i\sqrt{15}}{2}} - \frac{i c_2 \sqrt{15}\, x^{-\frac{1}{2}+\frac{i\sqrt{15}}{2}}}{15}$$

Verified OK.

4.10.6 Maple step by step solution

Let's solve

$x^2 y'' + 2xy' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - \frac{4y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + \frac{4y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' + 2xy' + 4y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)+2\frac{d}{dt}y(t)+4y(t)=0$$

- Simplify

$$\frac{d^2}{dt^2}y(t)+\frac{d}{dt}y(t)+4y(t)=0$$

- Characteristic polynomial of ODE

$$r^2+r+4=0$$

- Use quadratic formula to solve for r

$$r=\frac{(-1)\pm(\sqrt{-15})}{2}$$

- Roots of the characteristic polynomial

$$r=\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2},-\frac{1}{2}+\frac{\mathrm{I}\sqrt{15}}{2}\right)$$

- 1st solution of the ODE

$$y_1(t)=\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{15}\,t}{2}\right)$$

- 2nd solution of the ODE

$$y_2(t)=\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{15}\,t}{2}\right)$$

- General solution of the ODE

$$y(t)=c_1y_1(t)+c_2y_2(t)$$

- Substitute in solutions

$$y(t)=c_1\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{15}\,t}{2}\right)+c_2\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{15}\,t}{2}\right)$$

- Change variables back using $t=\ln(x)$

$$y=\frac{c_1\cos\left(\frac{\sqrt{15}\,\ln(x)}{2}\right)}{\sqrt{x}}+\frac{c_2\sin\left(\frac{\sqrt{15}\,\ln(x)}{2}\right)}{\sqrt{x}}$$

- Simplify

$$y=\frac{c_1\cos\left(\frac{\sqrt{15}\,\ln(x)}{2}\right)}{\sqrt{x}}+\frac{c_2\sin\left(\frac{\sqrt{15}\,\ln(x)}{2}\right)}{\sqrt{x}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(x^2*diff(y(x),x$2)+2*x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin\left(\frac{\sqrt{15}\,\ln(x)}{2}\right) + c_2 \cos\left(\frac{\sqrt{15}\,\ln(x)}{2}\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 42

```
DSolve[x^2*y''[x]+2*x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 \cos\left(\frac{1}{2}\sqrt{15}\log(x)\right) + c_1 \sin\left(\frac{1}{2}\sqrt{15}\log(x)\right)}{\sqrt{x}}$$

4.11 problem 59

Internal problem ID [5820]
Internal file name [`OUTPUT/5068_Sunday_June_05_2022_03_19_59_PM_84530589/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.2 problems. page 95
Problem number: 59.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_high_order, _with_linear_symmetries]]
```

$$\boxed{x^4y'''' - x^2y'' + y = 0}$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\ y'''' &= \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4} \end{aligned}$$

Substituting these back into

$$x^4y'''' - x^2y'' + y = 0$$

gives

$$-x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^4\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4} + x^\lambda = 0$$

Which simplifies to

$$-\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^\lambda + x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$-\lambda(\lambda-1)+\lambda(\lambda-1)(\lambda-2)(\lambda-3)+1=0$$

Simplifying gives the characteristic equation as

$$\lambda^4-6\lambda^3+10\lambda^2-5\lambda+1=0$$

Solving the above gives the following roots

$$\lambda_1=\frac{3}{2}+\frac{\sqrt{6}\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}}{12}+\frac{\sqrt{6}\sqrt{-\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+24\cdots}}{\cdots}$$

$$\lambda_2=\frac{3}{2}+\frac{\sqrt{6}\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}}{12}-\frac{\sqrt{6}\sqrt{-\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+24\cdots}}{\cdots}$$

$$\lambda_3=\frac{3}{2}-\frac{\sqrt{6}\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}}{12}+\frac{i\sqrt{6}\sqrt{\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+24\cdots}}{\cdots}$$

$$\lambda_4=\frac{3}{2}-\frac{\sqrt{6}\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}}{12}-\frac{i\sqrt{6}\sqrt{\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+24\cdots}}{\cdots}$$

This table summarises the result

root
$\frac{3}{2}+\frac{\sqrt{6}\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}}{12}-\frac{\sqrt{6}\sqrt{-\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+24\sqrt{6}\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+2\cdots}}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}\sqrt{\cdots}}$
$\frac{3}{2}-\frac{\sqrt{6}\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}}{12}\pm\frac{\sqrt{6}\sqrt{\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+24\sqrt{6}\left(908+12\sqrt{993}\right)^{\frac{1}{3}}-28\cdots}}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}\sqrt{\cdots}}$
$\frac{3}{2}+\frac{\sqrt{6}\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}}{12}+\frac{\sqrt{6}\sqrt{-\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+24\sqrt{6}\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+2\cdots}}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}\sqrt{\cdots}}$

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \cdots$ basis solutions. And so on. Using the above show that the solution is

Expression too large to display

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = x^{\frac{3}{2}+\frac{\sqrt{6}\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}}{12}-\frac{\sqrt{6}\sqrt{\frac{-\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+24\sqrt{6}\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+\cdots}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}\sqrt{\cdots}}}}{\cdots}}$$

$$y_2 = x^{\frac{3}{2}-\frac{\sqrt{6}\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}}{12}}\cos\left(\frac{\sqrt{6}\sqrt{\frac{\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+24\sqrt{6}\left(908\cdots\right.}{\left(908\cdots\right.}}}{\cdots}\cdots\right.$$

$$y_3 = x^{\frac{3}{2}-\frac{\sqrt{6}\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}}{12}}\sin\left(\frac{\sqrt{6}\sqrt{\frac{\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+24\sqrt{6}\left(908\cdots\right.}{\left(908\cdots\right.}}}{\cdots}\cdots\right.$$

$$y_4 = x^{\frac{3}{2}+\frac{\sqrt{6}\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}}{12}+\frac{\sqrt{6}\sqrt{\frac{-\sqrt{\frac{\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+14\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+88}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}}}\left(908+12\sqrt{993}\right)^{\frac{2}{3}}+24\sqrt{6}\left(908+12\sqrt{993}\right)^{\frac{1}{3}}+\cdots}{\left(908+12\sqrt{993}\right)^{\frac{1}{3}}\sqrt{\cdots}}}}{\cdots}}$$

Summary

The solution(s) found are the following

$$\text{Expression too large to display} \tag{1}$$

Verification of solutions

$$\text{Expression too large to display}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(x^4*diff(y(x),x$4)-x^2*diff(y(x),x$2)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \sum_{\_a=1}^{4} x^{\mathrm{RootOf}\left(\_Z^4-6\_Z^3+10\_Z^2-5\_Z+1,\mathrm{index}=\_a\right)} \_C\_a$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 130

```
DSolve[x^4*y''''[x]-x^2*y''[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} y(x) \rightarrow\ & c_4 x^{\mathrm{Root}\left[\#1^4-6\#1^3+10\#1^2-5\#1+1\&,4\right]} + c_3 x^{\mathrm{Root}\left[\#1^4-6\#1^3+10\#1^2-5\#1+1\&,3\right]} \\ & + c_1 x^{\mathrm{Root}\left[\#1^4-6\#1^3+10\#1^2-5\#1+1\&,1\right]} + c_2 x^{\mathrm{Root}\left[\#1^4-6\#1^3+10\#1^2-5\#1+1\&,2\right]} \end{aligned}$$

4.12 problem 60

Internal problem ID [5821]
Internal file name [`OUTPUT/5069_Sunday_June_05_2022_03_20_07_PM_8085769/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.2 problems. page 95
Problem number: 60.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{\left(x^{2}+1\right) y''+x y'+y=0}$$

4.12.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$\left(x^{2}+1\right) y''+x y'+y=0 \tag{1}$$

Becomes

$$y''+p(x)\, y'+q(x)\, y=0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{x}{x^{2}+1}\\
q(x) &= \frac{1}{x^{2}+1}
\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{x}{x^2+1}dx\right)}dx \\
&= \int e^{-\frac{\ln\left(x^2+1\right)}{2}}\,dx \\
&= \int \frac{1}{\sqrt{x^2+1}}dx \\
&= \operatorname{arcsinh}(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2+1}}{\frac{1}{x^2+1}} \\
&= 1
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} + \mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$y(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(\operatorname{arcsinh}(x)) + c_2 \sin(\operatorname{arcsinh}(x))$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(\operatorname{arcsinh}(x)) + c_2 \sin(\operatorname{arcsinh}(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(\operatorname{arcsinh}(x)) + c_2 \sin(\operatorname{arcsinh}(x))$$

Verified OK.

4.12.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$\left(x^2+1\right) y'' + xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x}{x^2+1} \\ q(x) &= \frac{1}{x^2+1} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{1}{c\sqrt{x^2+1}} \\ \tau'' &= -\frac{x}{c\left(x^2+1\right)^{\frac{3}{2}}} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \\ &= \frac{-\frac{x}{c(x^2+1)^{\frac{3}{2}}} + \frac{x}{x^2+1}\frac{1}{c\sqrt{x^2+1}}}{\left(\frac{1}{c\sqrt{x^2+1}}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos\left(c\tau\right) + c_2 \sin\left(c\tau\right)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \frac{1}{\sqrt{x^2+1}}dx}{c} \\ &= \frac{\operatorname{arcsinh}\left(x\right)}{c} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(\operatorname{arcsinh}\left(x\right)\right) + c_2 \sin\left(\operatorname{arcsinh}\left(x\right)\right)$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 \cos\left(\operatorname{arcsinh}\left(x\right)\right) + c_2 \sin\left(\operatorname{arcsinh}\left(x\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(\operatorname{arcsinh}(x)) + c_2 \sin(\operatorname{arcsinh}(x))$$

Verified OK.

4.12.3 Solving using Kovacic algorithm

Writing the ode as

$$\left(x^2+1\right) y'' + xy' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5x^2-2}{4\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5x^2-2 \\ t &= 4\left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-5x^2 - 2}{4\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 66: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4\left(x^2+1\right)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16 (-i + x)^2} - \frac{3}{16 (i + x)^2} + \frac{7i}{16 (-i + x)} - \frac{7i}{16 (i + x)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(-i+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(i+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-5x^2 - 2}{4 (x^2 + 1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
i	2	$\{1, 2, 3\}$
$-i$	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(i))} + \frac{1}{(x-(-i))}\right) \\ &= \frac{1}{2i+2x} + \frac{1}{-2i+2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2i+2x} + \frac{1}{-2i+2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2i + 2x} + \frac{1}{-2i + 2x}\right) w + \frac{5x^2 + 4}{4\left(i + x\right)^2 \left(-x + i\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{x + 2\sqrt{-x^2 - 1}}{2x^2 + 2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{x + 2\sqrt{-x^2 - 1}}{2x^2 + 2} dx} \\
&= \left(x^2 + 1\right)^{\frac{1}{4}} \mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2 - 1}}\right)}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2 + 1} \, dx} \\
&= z_1 e^{-\frac{\ln\left(x^2 + 1\right)}{4}} \\
&= z_1 \left(\frac{1}{\left(x^2 + 1\right)^{\frac{1}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2 - 1}}\right)}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{x}{x^2+1} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{-\frac{\ln(x^2+1)}{2}}}{(y_1)^2} dx$$
$$= y_1 \left(\int \frac{\mathrm{e}^{2 \arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)}}{\sqrt{x^2+1}} dx \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)} \right) + c_2 \left(\mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)} \left(\int \frac{\mathrm{e}^{2 \arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)}}{\sqrt{x^2+1}} dx \right) \right)$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)} + c_2 \mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)} \left(\int \frac{\mathrm{e}^{2 \arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)}}{\sqrt{x^2+1}} dx \right) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)} + c_2 \mathrm{e}^{-\arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)} \left(\int \frac{\mathrm{e}^{2 \arctan\left(\frac{x}{\sqrt{-x^2-1}}\right)}}{\sqrt{x^2+1}} dx \right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve((1+x^2)*diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(\operatorname{arcsinh}(x)) + c_2 \cos(\operatorname{arcsinh}(x))$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 43

```
DSolve[(1+x^2)*y''[x]+x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos\left(\log\left(\sqrt{x^2+1}-x\right)\right) - c_2 \sin\left(\log\left(\sqrt{x^2+1}-x\right)\right)$$

5 Chapter 2. Linear homogeneous equations. Section 2.3.4 problems. page 104

5.1 problem 1

Internal problem ID [5822]
Internal file name [`OUTPUT/5070_Sunday_June_05_2022_03_20_09_PM_19112425/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.3.4 problems. page 104
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + xy' + y = 2x\,\mathrm{e}^{x} - 1}$$

5.1.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + xy' + y)\,dx = \int (2x\,\mathrm{e}^{x} - 1)\,dx$$
$$y' + xy = -x + 2x\,\mathrm{e}^{x} - 2\,\mathrm{e}^{x} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = x$$
$$q(x) = (2x-2)\,e^{x} - x + c_1$$

Hence the ode is

$$y' + xy = (2x-2)\,e^{x} - x + c_1$$

The integrating factor μ is

$$\mu = e^{\int x dx}$$
$$= e^{\frac{x^2}{2}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left((2x-2)\,e^{x} - x + c_1\right)$$
$$\frac{d}{dx}\left(e^{\frac{x^2}{2}} y\right) = \left(e^{\frac{x^2}{2}}\right)\left((2x-2)\,e^{x} - x + c_1\right)$$
$$d\left(e^{\frac{x^2}{2}} y\right) = \left(\left((2x-2)\,e^{x} - x + c_1\right)e^{\frac{x^2}{2}}\right)\,dx$$

Integrating gives

$$e^{\frac{x^2}{2}} y = \int \left((2x-2)\,e^{x} - x + c_1\right)e^{\frac{x^2}{2}}\,dx$$

$$e^{\frac{x^2}{2}} y = -\frac{i c_1 \sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} - e^{\frac{x^2}{2}} + 2i\sqrt{\pi}\,e^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2} + \frac{i\sqrt{2}}{2}\right) + 2\,e^{\frac{1}{2}x^2+x} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x^2}{2}}$ results in

$$y = e^{-\frac{x^2}{2}}\left(-\frac{i c_1 \sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} - e^{\frac{x^2}{2}} + 2i\sqrt{\pi}\,e^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2} + \frac{i\sqrt{2}}{2}\right) + 2\,e^{\frac{1}{2}x^2+x}\right) + e^{-\frac{x^2}{2}} c_2$$

which simplifies to

$$y = 2i\sqrt{2}\,\sqrt{\pi}\,e^{-\frac{1}{2}-\frac{x^2}{2}}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right) + \frac{\left(-i c_1\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) + 2c_2\right)e^{-\frac{x^2}{2}}}{2} + 2\,e^{x} - 1$$

Summary

The solution(s) found are the following

$$y = 2i\sqrt{2}\,\sqrt{\pi}\,e^{-\frac{1}{2}-\frac{x^2}{2}}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right) + \frac{\left(-i c_1\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) + 2c_2\right)e^{-\frac{x^2}{2}}}{2} + 2\,e^{x} - 1 \tag{1}$$

Verification of solutions

$$y = 2i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}-\frac{x^2}{2}}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right)$$
$$+ \frac{\left(-ic_1\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+2c_2\right)\mathrm{e}^{-\frac{x^2}{2}}}{2} + 2\,\mathrm{e}^{x} - 1$$

Verified OK.

5.1.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + xy' + y = 2x\,\mathrm{e}^{x} - 1$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + xy' + y)\,dx = \int (2x\,\mathrm{e}^{x} - 1)\,dx$$
$$y' + xy = -x + 2x\,\mathrm{e}^{x} - 2\,\mathrm{e}^{x} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = x$$
$$q(x) = (2x-2)\,\mathrm{e}^{x} - x + c_1$$

Hence the ode is

$$y' + xy = (2x-2)\,\mathrm{e}^{x} - x + c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int x dx}$$
$$= \mathrm{e}^{\frac{x^2}{2}}$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left((2x-2)\,\mathrm{e}^{x}-x+c_1\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{x^2}{2}} y\right) &= \left(\mathrm{e}^{\frac{x^2}{2}}\right)\left((2x-2)\,\mathrm{e}^{x}-x+c_1\right) \\
\mathrm{d}\left(\mathrm{e}^{\frac{x^2}{2}} y\right) &= \left(\left((2x-2)\,\mathrm{e}^{x}-x+c_1\right)\mathrm{e}^{\frac{x^2}{2}}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\mathrm{e}^{\frac{x^2}{2}} y = \int \left((2x-2)\,\mathrm{e}^{x}-x+c_1\right)\mathrm{e}^{\frac{x^2}{2}}\,\mathrm{d}x
$$

$$
\mathrm{e}^{\frac{x^2}{2}} y = -\frac{i c_1 \sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} - \mathrm{e}^{\frac{x^2}{2}} + 2i\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}+\frac{i\sqrt{2}}{2}\right) + 2\,\mathrm{e}^{\frac{1}{2}x^2+x} + c_2
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{x^2}{2}}$ results in

$$
y = \mathrm{e}^{-\frac{x^2}{2}}\left(-\frac{i c_1 \sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} - \mathrm{e}^{\frac{x^2}{2}} + 2i\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}+\frac{i\sqrt{2}}{2}\right) + 2\,\mathrm{e}^{\frac{1}{2}x^2+x}\right) + \mathrm{e}^{-\frac{x^2}{2}} c_2
$$

which simplifies to

$$
y = 2i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}-\frac{x^2}{2}}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right) + \frac{\left(-i c_1 \sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+2c_2\right)\mathrm{e}^{-\frac{x^2}{2}}}{2} + 2\,\mathrm{e}^{x} - 1
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & 2i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}-\frac{x^2}{2}}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right) \\
& + \frac{\left(-i c_1 \sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+2c_2\right)\mathrm{e}^{-\frac{x^2}{2}}}{2} + 2\,\mathrm{e}^{x} - 1
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & 2i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}-\frac{x^2}{2}}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right) \\
& + \frac{\left(-i c_1 \sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+2c_2\right)\mathrm{e}^{-\frac{x^2}{2}}}{2} + 2\,\mathrm{e}^{x} - 1
\end{aligned}
$$

Verified OK.

5.1.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + xy' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 67: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞.

Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{x}{2} - \frac{1}{2x} - \frac{1}{4x^3} - \frac{1}{4x^5} - \frac{5}{16x^7} - \frac{7}{16x^9} - \frac{21}{32x^{11}} - \frac{33}{32x^{13}} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{1}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{1}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= -1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_{\infty} \\
&= 0 + (-)\left(\frac{x}{2}\right) \\
&= -\frac{x}{2} \\
&= -\frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{x}{2}\right)(0) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{1}{2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\, x}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\, x}{2}\right)}{2} \right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + xy' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-\frac{x^2}{2}} - \frac{i c_2 \mathrm{e}^{-\frac{x^2}{2}} \sqrt{\pi}\,\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\, x}{2}\right)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}}$$
$$y_2 = -\frac{i\mathrm{e}^{-\frac{x^2}{2}}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-\frac{x^2}{2}} & -\frac{i\mathrm{e}^{-\frac{x^2}{2}}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \\ \frac{d}{dx}\left(\mathrm{e}^{-\frac{x^2}{2}}\right) & \frac{d}{dx}\left(-\frac{i\mathrm{e}^{-\frac{x^2}{2}}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-\frac{x^2}{2}} & -\frac{i\mathrm{e}^{-\frac{x^2}{2}}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \\ -x\,\mathrm{e}^{-\frac{x^2}{2}} & \frac{ix\,\mathrm{e}^{-\frac{x^2}{2}}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} + \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{e}^{\frac{x^2}{2}} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-\frac{x^2}{2}}\right)\left(\frac{ix\,\mathrm{e}^{-\frac{x^2}{2}}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} + \mathrm{e}^{-\frac{x^2}{2}}\mathrm{e}^{\frac{x^2}{2}}\right) - \left(-\frac{i\mathrm{e}^{-\frac{x^2}{2}}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)\left(-x\,\mathrm{e}^{-\frac{x^2}{2}}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-x^2}\mathrm{e}^{\frac{x^2}{2}}$$

Which simplifies to

$$W = \mathrm{e}^{-\frac{x^2}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{i\mathrm{e}^{-\frac{x^2}{2}}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)(2x\,\mathrm{e}^{x}-1)}{2}}{\mathrm{e}^{-\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int -i\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)\sqrt{\pi}\left(x\,\mathrm{e}^{x} - \frac{1}{2}\right)dx$$

Hence

$$u_1 = -1 + 2\,\mathrm{e}^{\frac{1}{2}x^2+x} + 2i\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right) - \mathrm{e}^{\frac{x^2}{2}} - \frac{i(2(1-x)\,\mathrm{e}^{x}+x)\,\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} - 2i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\operatorname{erf}\left(\frac{i\sqrt{2}}{2}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-\frac{x^2}{2}}(2x\,\mathrm{e}^{x}-1)}{\mathrm{e}^{-\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int (2x\,\mathrm{e}^x - 1)\,dx$$

Hence

$$u_2 = -x + 2x\,\mathrm{e}^x - 2\,\mathrm{e}^x$$

Which simplifies to

$$\begin{aligned} u_1 &= -1 + 2i\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right) - \mathrm{e}^{\frac{x^2}{2}} + 2\,\mathrm{e}^{\frac{x(x+2)}{2}} \\ &\quad + \frac{i(2(x-1)\,\mathrm{e}^x - x)\,\sqrt{2}\,\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} - 2i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\,\mathrm{erf}\left(\frac{i\sqrt{2}}{2}\right) \\ u_2 &= (2x-2)\,\mathrm{e}^x - x \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) &= \left(-1 + 2i\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right) - \mathrm{e}^{\frac{x^2}{2}} + 2\,\mathrm{e}^{\frac{x(x+2)}{2}}\right. \\ &\quad \left. + \frac{i(2(x-1)\,\mathrm{e}^x - x)\,\sqrt{2}\,\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} - 2i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\,\mathrm{erf}\left(\frac{i\sqrt{2}}{2}\right)\right)\mathrm{e}^{-\frac{x^2}{2}} \\ &\quad - \frac{i((2x-2)\,\mathrm{e}^x - x)\,\mathrm{e}^{-\frac{x^2}{2}}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \end{aligned}$$

Which simplifies to

$$y_p(x) = 2\sqrt{2}\left(i\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right) + \mathrm{erfi}\left(\frac{\sqrt{2}}{2}\right)\right)\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}-\frac{x^2}{2}} + 2\,\mathrm{e}^x - \mathrm{e}^{-\frac{x^2}{2}} - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-\frac{x^2}{2}} - \frac{ic_2\mathrm{e}^{-\frac{x^2}{2}}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right) \\ &\quad + \left(2\sqrt{2}\left(i\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right) + \mathrm{erfi}\left(\frac{\sqrt{2}}{2}\right)\right)\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}-\frac{x^2}{2}} + 2\,\mathrm{e}^x - \mathrm{e}^{-\frac{x^2}{2}} - 1\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-\frac{x^2}{2}} \left(c_1 - \frac{i \,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) \sqrt{2}\,\sqrt{\pi}\,c_2}{2}\right.$$
$$\left. + 2\sqrt{2}\left(i \,\mathrm{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right) + \mathrm{erfi}\left(\frac{\sqrt{2}}{2}\right)\right) \sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}-\frac{x^2}{2}} + 2\,\mathrm{e}^{x} - \mathrm{e}^{-\frac{x^2}{2}} - 1\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{x^2}{2}} \left(c_1 - \frac{i \,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) \sqrt{2}\,\sqrt{\pi}\,c_2}{2}\right.$$
$$\left. + 2\sqrt{2}\left(i \,\mathrm{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right) + \mathrm{erfi}\left(\frac{\sqrt{2}}{2}\right)\right) \sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}-\frac{x^2}{2}} + 2\,\mathrm{e}^{x} - \mathrm{e}^{-\frac{x^2}{2}} - 1\right) \tag{1}$$

<u>Verification of solutions</u>

$$y = \mathrm{e}^{-\frac{x^2}{2}} \left(c_1 - \frac{i \,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) \sqrt{2}\,\sqrt{\pi}\,c_2}{2}\right.$$
$$\left. + 2\sqrt{2}\left(i \,\mathrm{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right) + \mathrm{erfi}\left(\frac{\sqrt{2}}{2}\right)\right) \sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}-\frac{x^2}{2}} + 2\,\mathrm{e}^{x} - \mathrm{e}^{-\frac{x^2}{2}} - 1\right)$$

Verified OK.

5.1.4 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x \\ r(x) &= 1 \\ s(x) &= 2x\,\mathrm{e}^{x} - 1 \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 1
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (1) + (1) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\ dx
$$

Substituting the above values for p, q, r, s gives

$$
y' + xy = \int 2x\, \mathrm{e}^{x} - 1\, dx
$$

We now have a first order ode to solve which is

$$
y' + xy = -x + 2x\, \mathrm{e}^{x} - 2\, \mathrm{e}^{x} + c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= x \\
q(x) &= (2x - 2)\, \mathrm{e}^{x} - x + c_1
\end{aligned}
$$

Hence the ode is

$$
y' + xy = (2x - 2)\, \mathrm{e}^{x} - x + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int x dx} \\
&= \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left((2x-2)\,e^{x}-x+c_1\right)\\
\frac{d}{dx}\left(e^{\frac{x^2}{2}}y\right) &= \left(e^{\frac{x^2}{2}}\right)\left((2x-2)\,e^{x}-x+c_1\right)\\
d\left(e^{\frac{x^2}{2}}y\right) &= \left(\left((2x-2)\,e^{x}-x+c_1\right)e^{\frac{x^2}{2}}\right)\,dx
\end{aligned}
$$

Integrating gives

$$
e^{\frac{x^2}{2}}y = \int \left((2x-2)\,e^{x}-x+c_1\right)e^{\frac{x^2}{2}}\,dx
$$

$$
e^{\frac{x^2}{2}}y = -\frac{ic_1\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} - e^{\frac{x^2}{2}} + 2i\sqrt{\pi}\,e^{-\frac{1}{2}}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}+\frac{i\sqrt{2}}{2}\right) + 2\,e^{\frac{1}{2}x^2+x} + c_2
$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x^2}{2}}$ results in

$$
y = e^{-\frac{x^2}{2}}\left(-\frac{ic_1\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} - e^{\frac{x^2}{2}} + 2i\sqrt{\pi}\,e^{-\frac{1}{2}}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}+\frac{i\sqrt{2}}{2}\right) + 2\,e^{\frac{1}{2}x^2+x}\right) + e^{-\frac{x^2}{2}}c_2
$$

which simplifies to

$$
y = 2i\sqrt{2}\,\sqrt{\pi}\,e^{-\frac{1}{2}-\frac{x^2}{2}}\operatorname{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right) + \frac{\left(-ic_1\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+2c_2\right)e^{-\frac{x^2}{2}}}{2} + 2\,e^{x} - 1
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y = {} & 2i\sqrt{2}\,\sqrt{\pi}\,e^{-\frac{1}{2}-\frac{x^2}{2}}\operatorname{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right)\\
& + \frac{\left(-ic_1\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+2c_2\right)e^{-\frac{x^2}{2}}}{2} + 2\,e^{x} - 1
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y = {} & 2i\sqrt{2}\,\sqrt{\pi}\,e^{-\frac{1}{2}-\frac{x^2}{2}}\operatorname{erf}\left(\frac{i\sqrt{2}\,(1+x)}{2}\right)\\
& + \frac{\left(-ic_1\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+2c_2\right)e^{-\frac{x^2}{2}}}{2} + 2\,e^{x} - 1
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 56

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+y(x)=2*x*exp(x)-1,y(x), singsol=all)
```

$$y(x) = 2i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{x^2}{2}-\frac{1}{2}}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+1)}{2}\right)+\left(c_1\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+c_2\right)\mathrm{e}^{-\frac{x^2}{2}}+2\,\mathrm{e}^{x}-1$$

✓ Solution by Mathematica

Time used: 0.148 (sec). Leaf size: 53

```
DSolve[y''[x]+x*y'[x]+y[x]==2*x*Exp[x]-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{x^2}{2}}\left(\int_1^x e^{\frac{K[1]^2}{2}}\left(c_1+2e^{K[1]}(K[1]-1)-K[1]\right)dK[1]+c_2\right)$$

5.2 problem 2

Internal problem ID [5823]
Internal file name [`OUTPUT/5071_Sunday_June_05_2022_03_20_11_PM_23099807/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.3.4 problems. page 104
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y''x + xy' - y = x^2 + 2x}$$

5.2.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x, B = x, C = -1, f(x) = x^2 + 2x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$y''x + xy' - y = 0$$

In normal form the ode

$$y''x + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -\frac{1}{x} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x} - \frac{1}{x} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} + 1\right) v'(x) &= 0 \\ v''(x) + \frac{(x+2)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(x+2)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{(x+2)\, u}{x} \end{aligned}$$

Where $f(x) = -\frac{x+2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{x+2}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{x+2}{x}\, dx \\ \ln(u) &= -x - 2\ln(x) + c_1 \\ u &= e^{-x-2\ln(x)+c_1} \\ &= c_1 e^{-x-2\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 e^{-x}}{x^2}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= c_1 \left(-\frac{e^{-x}}{x} + \operatorname{expIntegral}_1(x) \right) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(c_1 \left(-\frac{e^{-x}}{x} + \operatorname{expIntegral}_1(x) \right) + c_2 \right) x \\ &= -c_1 e^{-x} + x(c_1 \operatorname{expIntegral}_1(x) + c_2) \end{aligned}$$

Now the particular solution to this ODE is found

$$y''x + xy' - y = x^2 + 2x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \text{expIntegral}_1(x)\, x - e^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \text{expIntegral}_1(x)\, x - e^{-x} \\ \frac{d}{dx}(x) & \frac{d}{dx}(\text{expIntegral}_1(x)\, x - e^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \text{expIntegral}_1(x)\, x - e^{-x} \\ 1 & \text{expIntegral}_1(x) \end{vmatrix}$$

Therefore

$$W = (x)\,(\text{expIntegral}_1(x)) - \left(\text{expIntegral}_1(x)\, x - e^{-x}\right)(1)$$

Which simplifies to

$$W = -\left(e^{x} e^{-x} x - x - 1\right) e^{-x}$$

Which simplifies to

$$W = e^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\text{expIntegral}_1\left(x\right) x - e^{-x}\right)\left(x^2 + 2x\right)}{x\, e^{-x}}\, dx$$

Which simplifies to

$$u_1 = -\int (x + 2)\left(\text{expIntegral}_1\left(x\right) x\, e^{x} - 1\right) dx$$

Hence

$$u_1 = -\left(\int_0^x (\alpha + 2)\left(\text{expIntegral}_1\left(\alpha\right) \alpha\, e^{\alpha} - 1\right) d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x^2 + 2x)}{x\, e^{-x}}\, dx$$

Which simplifies to

$$u_2 = \int x(x + 2)\, e^{x} dx$$

Hence

$$u_2 = x^2 e^{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\left(\int_0^x (\alpha+2)\left(\text{expIntegral}_1\left(\alpha\right) \alpha\, e^{\alpha} - 1\right) d\alpha\right) x + x^2 e^{x}\left(\text{expIntegral}_1\left(x\right) x - e^{-x}\right)$$

Which simplifies to

$$y_p(x) = x\left(\text{expIntegral}_1\left(x\right) x^2 e^{x} - \left(\int_0^x (\alpha + 2)\left(\text{expIntegral}_1\left(\alpha\right) \alpha\, e^{\alpha} - 1\right) d\alpha\right) - x\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(c_1\left(-\frac{e^{-x}}{x} + \text{expIntegral}_1(x)\right) + c_2\right) x\right) \\
&\quad + \left(x\left(\text{expIntegral}_1(x)\, x^2 e^x - \left(\int_0^x (\alpha+2)\left(\text{expIntegral}_1(\alpha)\, \alpha\, e^\alpha - 1\right) d\alpha\right) - x\right)\right) \\
&= x\left(\text{expIntegral}_1(x)\, x^2 e^x - \left(\int_0^x (\alpha+2)\left(\text{expIntegral}_1(\alpha)\, \alpha\, e^\alpha - 1\right) d\alpha\right) - x\right) \\
&\quad + \left(c_1\left(-\frac{e^{-x}}{x} + \text{expIntegral}_1(x)\right) + c_2\right) x
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
y = &-\left(\int_0^x (\alpha+2)\left(\text{expIntegral}_1(\alpha)\, \alpha\, e^\alpha - 1\right) d\alpha\right) x \\
&- c_1 e^{-x} + x\left(\left(x^2 e^x + c_1\right) \text{expIntegral}_1(x) - x + c_2\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &-\left(\int_0^x (\alpha+2)\left(\text{expIntegral}_1(\alpha)\, \alpha\, e^\alpha - 1\right) d\alpha\right) x \\
&- c_1 e^{-x} + x\left(\left(x^2 e^x + c_1\right) \text{expIntegral}_1(x) - x + c_2\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = &-\left(\int_0^x (\alpha+2)\left(\text{expIntegral}_1(\alpha)\, \alpha\, e^\alpha - 1\right) d\alpha\right) x \\
&- c_1 e^{-x} + x\left(\left(x^2 e^x + c_1\right) \text{expIntegral}_1(x) - x + c_2\right)
\end{aligned}
$$

Verified OK.

5.2.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x \\ B &= x \\ C &= -1 \\ F &= x^2 + 2x \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (x)(0) + (x)(1) + (-1)(x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$x^2v'' + (x^2 + 2x)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$(u'(x)\, x + (x + 2)\, u(x))\, x = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{(x + 2)\, u}{x} \end{aligned}$$

Where $f(x) = -\frac{x+2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{x + 2}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{x + 2}{x}\, dx \\ \ln(u) &= -x - 2\ln(x) + c_1 \\ u &= e^{-x-2\ln(x)+c_1} \\ &= c_1 e^{-x-2\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 e^{-x}}{x^2}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1 e^{-x}}{x^2} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1 e^{-x}}{x^2}\, dx \\ &= c_1 \left(-\frac{e^{-x}}{x} + \text{expIntegral}_1(x) \right) + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (x)\left(c_1\left(-\frac{e^{-x}}{x} + \text{expIntegral}_1(x)\right) + c_2\right) \\
&= -c_1 e^{-x} + x(c_1 \text{expIntegral}_1(x) + c_2)
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \text{expIntegral}_1(x)\, x - e^{-x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x & \text{expIntegral}_1(x)\, x - e^{-x} \\ \frac{d}{dx}(x) & \frac{d}{dx}(\text{expIntegral}_1(x)\, x - e^{-x}) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} x & \text{expIntegral}_1(x)\, x - e^{-x} \\ 1 & \text{expIntegral}_1(x) \end{vmatrix}
$$

Therefore

$$W = (x)\left(\text{expIntegral}_1\left(x\right)\right) - \left(\text{expIntegral}_1\left(x\right) x - e^{-x}\right)(1)$$

Which simplifies to

$$W = -\left(e^{x} e^{-x} x - x - 1\right) e^{-x}$$

Which simplifies to

$$W = e^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\text{expIntegral}_1\left(x\right) x - e^{-x}\right)\left(x^2 + 2x\right)}{x\, e^{-x}}\, dx$$

Which simplifies to

$$u_1 = -\int (x+2)\left(\text{expIntegral}_1\left(x\right) x\, e^{x} - 1\right) dx$$

Hence

$$u_1 = -\left(\int_0^x (\alpha+2)\left(\text{expIntegral}_1\left(\alpha\right) \alpha\, e^{\alpha} - 1\right) d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x^2 + 2x)}{x\, e^{-x}}\, dx$$

Which simplifies to

$$u_2 = \int x(x+2)\, e^{x} dx$$

Hence

$$u_2 = x^2 e^{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\left(\int_0^x (\alpha+2)\left(\text{expIntegral}_1\left(\alpha\right) \alpha\, e^{\alpha} - 1\right) d\alpha\right) x + x^2 e^{x}\left(\text{expIntegral}_1\left(x\right) x - e^{-x}\right)$$

Which simplifies to

$$y_p(x) = x\left(\text{expIntegral}_1(x)\,x^2\mathrm{e}^x - \left(\int_0^x (\alpha+2)\left(\text{expIntegral}_1(\alpha)\,\alpha\,\mathrm{e}^\alpha - 1\right)d\alpha\right) - x\right)$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-c_1\mathrm{e}^{-x} + x(c_1\,\text{expIntegral}_1(x) + c_2)\right) + \left(x\left(\text{expIntegral}_1(x)\,x^2\mathrm{e}^x - \left(\int_0^x (\alpha+2)\left(\text{expIntegral}_1\right.\right.\right.\right. \\ &= -\left(\int_0^x (\alpha+2)\left(\text{expIntegral}_1(\alpha)\,\alpha\,\mathrm{e}^\alpha - 1\right)d\alpha\right)x - c_1\mathrm{e}^{-x} + x\left(\left(x^2\mathrm{e}^x + c_1\right)\text{expIntegral}_1(x) - x + c\right. \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &-\left(\int_0^x (\alpha+2)\left(\text{expIntegral}_1(\alpha)\,\alpha\,\mathrm{e}^\alpha - 1\right)d\alpha\right)x \\ &- c_1\mathrm{e}^{-x} + x\left(\left(x^2\mathrm{e}^x + c_1\right)\text{expIntegral}_1(x) - x + c_2\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = &-\left(\int_0^x (\alpha+2)\left(\text{expIntegral}_1(\alpha)\,\alpha\,\mathrm{e}^\alpha - 1\right)d\alpha\right)x \\ &- c_1\mathrm{e}^{-x} + x\left(\left(x^2\mathrm{e}^x + c_1\right)\text{expIntegral}_1(x) - x + c_2\right) \end{aligned}$$

Verified OK.

5.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y''x + xy' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x+4}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x + 4 \\ t &= 4x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x+4}{4x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 68: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 1 - 1 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{5}{x^4} + \frac{14}{x^5} - \frac{42}{x^6} + \frac{132}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x+4}{4x} \\
&= Q + \frac{R}{4x} \\
&= \left(\frac{1}{4}\right) + \left(\frac{1}{x}\right) \\
&= \frac{1}{4} + \frac{1}{x}
\end{aligned}
$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$
\begin{aligned}
b &= (1) - (0) \\
&= 1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{1}{\frac{1}{2}}-0\right) &&= 1\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{1}{\frac{1}{2}}-0\right) &&= -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x+4}{4x}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right)\\
&= 1 - (1)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{x} \\
&= \frac{1}{2} + \frac{1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{1}{x}\right)(0) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{2} + \frac{1}{x}\right)^2 - \left(\frac{x+4}{4x}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{1}{x}\right) dx} \\
&= x \, \mathrm{e}^{\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x} \, dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\text{expIntegral}_1(x)\, x - e^{-x}}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2 \left(x \left(\frac{\text{expIntegral}_1(x)\, x - e^{-x}}{x}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y''x + xy' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x + c_2 \left(\text{expIntegral}_1(x)\, x - e^{-x}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= \text{expIntegral}_1(x)\, x - \mathrm{e}^{-x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \text{expIntegral}_1(x)\, x - \mathrm{e}^{-x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\text{expIntegral}_1(x)\, x - \mathrm{e}^{-x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \text{expIntegral}_1(x)\, x - \mathrm{e}^{-x} \\ 1 & \text{expIntegral}_1(x) \end{vmatrix}$$

Therefore

$$W = (x)\left(\text{expIntegral}_1(x)\right) - \left(\text{expIntegral}_1(x)\, x - \mathrm{e}^{-x}\right)(1)$$

Which simplifies to

$$W = \mathrm{e}^{-x}$$

Which simplifies to

$$W = \mathrm{e}^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(\text{expIntegral}_1(x)\, x - e^{-x})\,(x^2+2x)}{x\, e^{-x}}\, dx$$

Which simplifies to

$$u_1 = -\int (x+2)\,(\text{expIntegral}_1(x)\, x\, e^{x} - 1)\, dx$$

Hence

$$u_1 = -\left(\int_0^x (\alpha+2)\,(\text{expIntegral}_1(\alpha)\, \alpha\, e^{\alpha} - 1)\, d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x^2+2x)}{x\, e^{-x}}\, dx$$

Which simplifies to

$$u_2 = \int x(x+2)\, e^{x} dx$$

Hence

$$u_2 = x^2 e^{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\left(\int_0^x (\alpha+2)\,(\text{expIntegral}_1(\alpha)\, \alpha\, e^{\alpha} - 1)\, d\alpha\right) x + x^2 e^{x}\left(\text{expIntegral}_1(x)\, x - e^{-x}\right)$$

Which simplifies to

$$y_p(x) = x\left(\text{expIntegral}_1(x)\, x^2 e^{x} - \left(\int_0^x (\alpha+2)\,(\text{expIntegral}_1(\alpha)\, \alpha\, e^{\alpha} - 1)\, d\alpha\right) - x\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x + c_2\left(\text{expIntegral}_1(x)\, x - e^{-x}\right)\right) \\ &\quad + \left(x\left(\text{expIntegral}_1(x)\, x^2 e^{x} - \left(\int_0^x (\alpha+2)\,(\text{expIntegral}_1(\alpha)\, \alpha\, e^{\alpha} - 1)\, d\alpha\right) - x\right)\right) \end{aligned}$$

Which simplifies to

$$
y = -c_2 e^{-x} + x(\operatorname{expIntegral}_1(x)\, c_2 + c_1) + x\left(\operatorname{expIntegral}_1(x)\, x^2 e^x - \left(\int_0^x (\alpha+2)\left(\operatorname{expIntegral}_1(\alpha)\, \alpha\, e^{\alpha} - 1\right) d\alpha\right) - x\right)
$$

Summary

The solution(s) found are the following

$$
y = -c_2 e^{-x} + x(\operatorname{expIntegral}_1(x)\, c_2 + c_1) + x\left(\operatorname{expIntegral}_1(x)\, x^2 e^x - \left(\int_0^x (\alpha+2)\left(\operatorname{expIntegral}_1(\alpha)\, \alpha\, e^{\alpha} - 1\right) d\alpha\right) - x\right) \tag{1}
$$

Verification of solutions

$$
y = -c_2 e^{-x} + x(\operatorname{expIntegral}_1(x)\, c_2 + c_1) + x\left(\operatorname{expIntegral}_1(x)\, x^2 e^x - \left(\int_0^x (\alpha+2)\left(\operatorname{expIntegral}_1(\alpha)\, \alpha\, e^{\alpha} - 1\right) d\alpha\right) - x\right)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(x*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=x^2+2*x,y(x), singsol=all)
```

$$
y(x) = -c_2 e^{-x} + x(c_2 \operatorname{expIntegral}_1(x) + x + c_1)
$$

✓ Solution by Mathematica

Time used: 0.274 (sec). Leaf size: 31

```
DSolve[x*y''[x]+x*y'[x]-y[x]==x^2+2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -c_2 x \,\text{ExpIntegralEi}(-x) + x^2 + c_1 x - c_2 e^{-x}$$

5.3 problem 3

Internal problem ID [5824]
Internal file name [`OUTPUT/5072_Sunday_June_05_2022_03_20_15_PM_23298935/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.3.4 problems. page 104
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' + xy' - y = x^2 + 2x}$$

5.3.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = x^2 + 2x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 1 = 0$$

Or

$$r^2 - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2x$$

Next, we find the particular solution to the ODE

$$x^2y'' + xy' - y = x^2 + 2x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & x \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x \\ -\frac{1}{x^2} & 1 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)(1) - (x)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{2}{x}$$

Which simplifies to

$$W = \frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x(x^2 + 2x)}{2x} \, dx$$

Which simplifies to

$$u_1 = -\int \left(\frac{1}{2}x^2 + x\right) dx$$

Hence

$$u_1 = -\frac{1}{6}x^3 - \frac{1}{2}x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x^2+2x}{x}}{2x} \, dx$$

Which simplifies to

$$u_2 = \int \frac{x+2}{2x} dx$$

Hence

$$u_2 = \frac{x}{2} + \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{1}{6}x^3 - \frac{1}{2}x^2}{x} + \left(\frac{x}{2} + \ln(x)\right) x$$

Which simplifies to

$$y_p(x) = \frac{x(2x - 3 + 6\ln(x))}{6}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{x(2x - 3 + 6\ln(x))}{6} + \frac{c_1}{x} + c_2 x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x(2x - 3 + 6\ln(x))}{6} + \frac{c_1}{x} + c_2 x \tag{1}
$$

Verification of solutions

$$
y = \frac{x(2x - 3 + 6\ln(x))}{6} + \frac{c_1}{x} + c_2 x
$$

Verified OK.

5.3.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' + xy' - y = 0
$$

In normal form the ode

$$
x^2y'' + xy' - y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -\frac{1}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\
&= \int e^{-\ln(x)}\,dx \\
&= \int \frac{1}{x}dx \\
&= \ln(x) \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= -1 \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} - e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\ y(\tau) &= c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{\tau} + c_2 e^{-\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^2 + c_2}{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x^2 + c_2}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^2+2x}{x}}{-2x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{-x-2}{2x} dx$$

Hence

$$u_1 = \frac{x}{2} + \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x^2+2x)}{-2x}\, dx$$

Which simplifies to

$$u_2 = \int \left(-\frac{1}{2}x^2 - x\right) dx$$

Hence

$$u_2 = -\frac{1}{6}x^3 - \frac{1}{2}x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{1}{6}x^3 - \frac{1}{2}x^2}{x} + \left(\frac{x}{2} + \ln(x)\right)x$$

Which simplifies to

$$y_p(x) = \frac{x(2x - 3 + 6\ln(x))}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1x^2 + c_2}{x}\right) + \left(\frac{x(2x - 3 + 6\ln(x))}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1x^2 + c_2}{x} + \frac{x(2x - 3 + 6\ln(x))}{6} \tag{1}$$

Verification of solutions

$$y = \frac{c_1x^2 + c_2}{x} + \frac{x(2x - 3 + 6\ln(x))}{6}$$

Verified OK.

5.3.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = x^2 + 2x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - y = 0$$

In normal form the ode

$$x^2y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-\frac{1}{x^2}}}{c} \\ \tau'' &= \frac{1}{c\sqrt{-\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{c\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q}\, dx \\
&= \frac{\int \sqrt{-\frac{1}{x^2}} dx}{c} \\
&= \frac{\sqrt{-\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{(ic_2 + c_1) x^2 - ic_2 + c_1}{2x}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + xy' - y = x^2 + 2x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \frac{1}{x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^2+2x}{x}}{-2x} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{-x-2}{2x} dx$$

Hence

$$u_1 = \frac{x}{2} + \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x^2 + 2x)}{-2x}\, dx$$

Which simplifies to

$$u_2 = \int \left(-\frac{1}{2}x^2 - x\right) dx$$

Hence

$$u_2 = -\frac{1}{6}x^3 - \frac{1}{2}x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{1}{6}x^3 - \frac{1}{2}x^2}{x} + \left(\frac{x}{2} + \ln(x)\right) x$$

Which simplifies to

$$y_p(x) = \frac{x(2x - 3 + 6\ln(x))}{6}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}\right) + \left(\frac{x(2x - 3 + 6\ln(x))}{6}\right) \\
&= \frac{x(2x - 3 + 6\ln(x))}{6} + \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}
\end{aligned}$$

Which simplifies to

$$y = \frac{6\ln(x)\, x^2 + 2x^3 + (3ic_2 + 3c_1 - 3)\, x^2 - 3ic_2 + 3c_1}{6x}$$

Summary
The solution(s) found are the following

$$y = \frac{6 \ln(x) x^2 + 2x^3 + (3ic_2 + 3c_1 - 3) x^2 - 3ic_2 + 3c_1}{6x} \tag{1}$$

Verification of solutions

$$y = \frac{6 \ln(x) x^2 + 2x^3 + (3ic_2 + 3c_1 - 3) x^2 - 3ic_2 + 3c_1}{6x}$$

Verified OK.

5.3.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = x^2 + 2x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - y = 0$$

In normal form the ode

$$x^2y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \frac{3v'(x)}{x} = 0$$
$$v''(x) + \frac{3v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= \mathrm{e}^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{c_1}{2x^2} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(-\frac{c_1}{2x^2} + c_2\right) x \\ &= \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' + xy' - y = x^2 + 2x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= \frac{1}{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^2+2x}{x}}{-2x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{-x-2}{2x}dx$$

Hence

$$u_1 = \frac{x}{2} + \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x^2+2x)}{-2x}\,dx$$

Which simplifies to

$$u_2 = \int \left(-\frac{1}{2}x^2 - x\right) dx$$

Hence

$$u_2 = -\frac{1}{6}x^3 - \frac{1}{2}x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{1}{6}x^3 - \frac{1}{2}x^2}{x} + \left(\frac{x}{2} + \ln(x)\right) x$$

Which simplifies to

$$y_p(x) = \frac{x(2x - 3 + 6\ln(x))}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x\right) + \left(\frac{x(2x - 3 + 6\ln(x))}{6}\right) \\ &= \frac{x(2x - 3 + 6\ln(x))}{6} + \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

Which simplifies to

$$y = \frac{x(2x - 3 + 6\ln(x))}{6} + \left(-\frac{c_1}{2x^2} + c_2\right) x$$

Summary
The solution(s) found are the following

$$y = \frac{x(2x - 3 + 6\ln(x))}{6} + \left(-\frac{c_1}{2x^2} + c_2\right) x \tag{1}$$

Verification of solutions

$$y = \frac{x(2x - 3 + 6\ln(x))}{6} + \left(-\frac{c_1}{2x^2} + c_2\right) x$$

Verified OK.

5.3.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x^2y'' + xy' - y\right) dx = \int \left(x^2 + 2x\right) dx$$

$$x^2y' - xy = \frac{1}{3}x^3 + x^2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{x^3 + 3x^2 + 3c_1}{3x^2}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{x^3 + 3x^2 + 3c_1}{3x^2}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{x^3 + 3x^2 + 3c_1}{3x^2}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left(\frac{x^3 + 3x^2 + 3c_1}{3x^2}\right)$$
$$\mathrm{d}\left(\frac{y}{x}\right) = \left(\frac{x^3 + 3x^2 + 3c_1}{3x^3}\right) \mathrm{d}x$$

Integrating gives

$$\frac{y}{x} = \int \frac{x^3 + 3x^2 + 3c_1}{3x^3} \mathrm{d}x$$
$$\frac{y}{x} = \frac{x}{3} + \ln(x) - \frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(\frac{x}{3} + \ln(x) - \frac{c_1}{2x^2}\right) + c_2x$$

which simplifies to

$$y = x\left(\frac{x}{3} + \ln(x) - \frac{c_1}{2x^2} + c_2\right)$$

Summary

The solution(s) found are the following

$$y = x\left(\frac{x}{3} + \ln(x) - \frac{c_1}{2x^2} + c_2\right) \tag{1}$$

Verification of solutions

$$y = x\left(\frac{x}{3} + \ln(x) - \frac{c_1}{2x^2} + c_2\right)$$

Verified OK.

5.3.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)\, u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^2 \\
B &= x \\
C &= -1 \\
F &= x^2 + 2x
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= (x^2)\,(0) + (x)\,(1) + (-1)\,(x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$x^3 v'' + (3x^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$x^2 (u'(x)\, x + 3u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= -\frac{3u}{x}
\end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{3}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\
\ln(u) &= -3\ln(x) + c_1 \\
u &= e^{-3\ln(x) + c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1}{x^3}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1}{x^3}\, \mathrm{d}x \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (x)\left(-\frac{c_1}{2x^2} + c_2\right) \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \frac{1}{x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^2+2x}{x}}{-2x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{-x-2}{2x} dx$$

Hence

$$u_1 = \frac{x}{2} + \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x^2+2x)}{-2x}\, dx$$

Which simplifies to

$$u_2 = \int \left(-\frac{1}{2}x^2 - x\right) dx$$

Hence

$$u_2 = -\frac{1}{6}x^3 - \frac{1}{2}x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{1}{6}x^3 - \frac{1}{2}x^2}{x} + \left(\frac{x}{2} + \ln(x)\right) x$$

Which simplifies to

$$y_p(x) = \frac{x(2x - 3 + 6\ln(x))}{6}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x\right) + \left(\frac{x(2x - 3 + 6\ln(x))}{6}\right) \\ &= \frac{6\ln(x)\, x^2 + 2x^3 + (6c_2 - 3)\, x^2 - 3c_1}{6x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{6\ln(x)\, x^2 + 2x^3 + (6c_2 - 3)\, x^2 - 3c_1}{6x} \tag{1}$$

Verification of solutions

$$y = \frac{6\ln(x)\, x^2 + 2x^3 + (6c_2 - 3)\, x^2 - 3c_1}{6x}$$

Verified OK.

5.3.7 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' + xy' - y = x^2 + 2x$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int (x^2y'' + xy' - y)\, dx &= \int (x^2 + 2x)\, dx \\ x^2y' - xy &= \frac{1}{3}x^3 + x^2 + c_1\end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{1}{x} \\ q(x) &= \frac{x^3 + 3x^2 + 3c_1}{3x^2}\end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{x^3 + 3x^2 + 3c_1}{3x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^3 + 3x^2 + 3c_1}{3x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{x^3 + 3x^2 + 3c_1}{3x^2}\right) \\ \mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{x^3 + 3x^2 + 3c_1}{3x^3}\right) \mathrm{d}x\end{aligned}$$

Integrating gives

$$\frac{y}{x} = \int \frac{x^3 + 3x^2 + 3c_1}{3x^3} \mathrm{d}x$$
$$\frac{y}{x} = \frac{x}{3} + \ln(x) - \frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(\frac{x}{3} + \ln(x) - \frac{c_1}{2x^2}\right) + c_2 x$$

which simplifies to

$$y = x\left(\frac{x}{3} + \ln(x) - \frac{c_1}{2x^2} + c_2\right)$$

Summary

The solution(s) found are the following

$$y = x\left(\frac{x}{3} + \ln(x) - \frac{c_1}{2x^2} + c_2\right) \tag{1}$$

Verification of solutions

$$y = x\left(\frac{x}{3} + \ln(x) - \frac{c_1}{2x^2} + c_2\right)$$

Verified OK.

5.3.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + x y' - y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 69: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\,(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^2}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + xy' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x} + \frac{c_2 x}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{x}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x}{2} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x}{2} \\ -\frac{1}{x^2} & \frac{1}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{2}\right) - \left(\frac{x}{2}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x(x^2+2x)}{2}}{x}\,dx$$

Which simplifies to

$$u_1 = -\int \left(\frac{1}{2}x^2 + x\right) dx$$

Hence

$$u_1 = -\frac{1}{6}x^3 - \frac{1}{2}x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x^2+2x}{x}}{x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x+2}{x} dx$$

Hence

$$u_2 = x + 2\ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{1}{6}x^3 - \frac{1}{2}x^2}{x} + \frac{(x + 2\ln(x))\,x}{2}$$

Which simplifies to

$$y_p(x) = \frac{x(2x - 3 + 6\ln(x))}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2 x}{2}\right) + \left(\frac{x(2x - 3 + 6\ln(x))}{6}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} + \frac{x(2x - 3 + 6\ln(x))}{6} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} + \frac{x(2x - 3 + 6\ln(x))}{6}$$

Verified OK.

5.3.9 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 \\ q(x) &= x \\ r(x) &= -1 \\ s(x) &= x^2 + 2x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 1 \end{aligned}$$

Therefore (1) becomes

$$2 - (1) + (-1) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$x^2y' - xy = \int x^2 + 2x\,dx$$

We now have a first order ode to solve which is

$$x^2y' - xy = \frac{1}{3}x^3 + x^2 + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{x^3 + 3x^2 + 3c_1}{3x^2} \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{x^3 + 3x^2 + 3c_1}{3x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^3 + 3x^2 + 3c_1}{3x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{x^3 + 3x^2 + 3c_1}{3x^2}\right) \\ \mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{x^3 + 3x^2 + 3c_1}{3x^3}\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x} &= \int \frac{x^3 + 3x^2 + 3c_1}{3x^3} \mathrm{d}x \\ \frac{y}{x} &= \frac{x}{3} + \ln(x) - \frac{c_1}{2x^2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(\frac{x}{3} + \ln(x) - \frac{c_1}{2x^2}\right) + c_2 x$$

which simplifies to

$$y = x\left(\frac{x}{3} + \ln(x) - \frac{c_1}{2x^2} + c_2\right)$$

Summary
The solution(s) found are the following

$$y = x\left(\frac{x}{3} + \ln(x) - \frac{c_1}{2x^2} + c_2\right) \tag{1}$$

Verification of solutions

$$y = x\left(\frac{x}{3} + \ln(x) - \frac{c_1}{2x^2} + c_2\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 22

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=x^2+2*x,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x} + c_2 x + \frac{(x + 3\ln(x))\,x}{3}$$

✓ Solution by Mathematica
Time used: 0.017 (sec). Leaf size: 31

```
DSolve[x^2*y''[x]+x*y'[x]-y[x]==x^2+2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{3} + x\log(x) + \left(-\frac{1}{2} + c_2\right)x + \frac{c_1}{x}$$

5.4 problem 4

Internal problem ID [5825]
Internal file name [`OUTPUT/5073_Sunday_June_05_2022_03_20_17_PM_83693862/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.3.4 problems. page 104
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^3y'' + xy' - y = \cos\left(\frac{1}{x}\right)}$$

5.4.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^3, B = x, C = -1, f(x) = \cos\left(\frac{1}{x}\right)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^3y'' + xy' - y = 0$$

In normal form the ode

$$x^3y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x^2}$$
$$q(x) = -\frac{1}{x^3}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^3} - \frac{1}{x^3} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \left(\frac{2}{x} + \frac{1}{x^2}\right) v'(x) = 0$$
$$v''(x) + \frac{(1+2x)\, v'(x)}{x^2} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+2x)\, u(x)}{x^2} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{(1+2x)\, u}{x^2} \end{aligned}$$

Where $f(x) = -\frac{1+2x}{x^2}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1+2x}{x^2}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1+2x}{x^2}\, dx \\ \ln(u) &= -2\ln(x) + \frac{1}{x} + c_1 \\ u &= e^{-2\ln(x) + \frac{1}{x} + c_1} \\ &= c_1 e^{-2\ln(x) + \frac{1}{x}} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 e^{\frac{1}{x}}}{x^2}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -c_1 e^{\frac{1}{x}} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(-c_1 e^{\frac{1}{x}} + c_2\right) x \\ &= -\left(c_1 e^{\frac{1}{x}} - c_2\right) x \end{aligned}$$

Now the particular solution to this ODE is found

$$x^3y'' + xy' - y = \cos\left(\frac{1}{x}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = e^{\frac{1}{x}}x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & e^{\frac{1}{x}}x \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(e^{\frac{1}{x}}x\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & e^{\frac{1}{x}}x \\ 1 & -\frac{e^{\frac{1}{x}}}{x} + e^{\frac{1}{x}} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{e^{\frac{1}{x}}}{x} + e^{\frac{1}{x}}\right) - \left(e^{\frac{1}{x}} x\right)(1)$$

Which simplifies to

$$W = -e^{\frac{1}{x}}$$

Which simplifies to

$$W = -e^{\frac{1}{x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{\frac{1}{x}} x \cos\left(\frac{1}{x}\right)}{-x^3 e^{\frac{1}{x}}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{\cos\left(\frac{1}{x}\right)}{x^2} dx$$

Hence

$$u_1 = -\sin\left(\frac{1}{x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\frac{1}{x}\right) x}{-x^3 e^{\frac{1}{x}}}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(\frac{1}{x}\right) e^{-\frac{1}{x}}}{x^2} dx$$

Hence

$$u_2 = -\frac{\cos\left(\frac{1}{x}\right) e^{-\frac{1}{x}}}{2} + \frac{e^{-\frac{1}{x}} \sin\left(\frac{1}{x}\right)}{2}$$

Which simplifies to

$$u_1 = -\sin\left(\frac{1}{x}\right)$$

$$u_2 = -\frac{e^{-\frac{1}{x}}\left(\cos\left(\frac{1}{x}\right)-\sin\left(\frac{1}{x}\right)\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\sin\left(\frac{1}{x}\right)x - \frac{e^{-\frac{1}{x}}\left(\cos\left(\frac{1}{x}\right)-\sin\left(\frac{1}{x}\right)\right)e^{\frac{1}{x}}x}{2}$$

Which simplifies to

$$y_p(x) = -\frac{x\left(\sin\left(\frac{1}{x}\right)+\cos\left(\frac{1}{x}\right)\right)}{2}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(-c_1 e^{\frac{1}{x}} + c_2\right)x\right) + \left(-\frac{x\left(\sin\left(\frac{1}{x}\right)+\cos\left(\frac{1}{x}\right)\right)}{2}\right) \\
&= -\frac{x\left(\sin\left(\frac{1}{x}\right)+\cos\left(\frac{1}{x}\right)\right)}{2} + \left(-c_1 e^{\frac{1}{x}} + c_2\right)x
\end{aligned}$$

Which simplifies to

$$y = -\frac{x\left(2c_1 e^{\frac{1}{x}} + \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) - 2c_2\right)}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{x\left(2c_1 e^{\frac{1}{x}} + \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) - 2c_2\right)}{2} \tag{1}$$

Verification of solutions

$$y = -\frac{x\left(2c_1 e^{\frac{1}{x}} + \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) - 2c_2\right)}{2}$$

Verified OK.

5.4.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^3 \\ B &= x \\ C &= -1 \\ F &= \cos\left(\frac{1}{x}\right) \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (x^3)(0) + (x)(1) + (-1)(x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$x^4 v'' + (2x^3 + x^2) v' = 0$$

Now by applying $v' = u$ the above becomes

$$x^2 (u'(x) x^2 + 2u(x) x + u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{u(1+2x)}{x^2} \end{aligned}$$

Where $f(x) = -\frac{1+2x}{x^2}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u} \, du &= -\frac{1+2x}{x^2} \, dx \\ \int \frac{1}{u} \, du &= \int -\frac{1+2x}{x^2} \, dx \\ \ln(u) &= -2\ln(x) + \frac{1}{x} + c_1 \\ u &= \mathrm{e}^{-2\ln(x) + \frac{1}{x} + c_1} \\ &= c_1 \mathrm{e}^{-2\ln(x) + \frac{1}{x}} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 \mathrm{e}^{\frac{1}{x}}}{x^2}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1 \mathrm{e}^{\frac{1}{x}}}{x^2} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{c_1 \mathrm{e}^{\frac{1}{x}}}{x^2}\, \mathrm{d}x \\
&= -c_1 \mathrm{e}^{\frac{1}{x}} + c_2
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= Bv \\
&= (x) \left(-c_1 \mathrm{e}^{\frac{1}{x}} + c_2\right) \\
&= -\left(c_1 \mathrm{e}^{\frac{1}{x}} - c_2\right) x
\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x \\
y_2 &= \mathrm{e}^{\frac{1}{x}} x
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \mathrm{e}^{\frac{1}{x}} x \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\mathrm{e}^{\frac{1}{x}} x\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & e^{\frac{1}{x}} x \\ 1 & -\frac{e^{\frac{1}{x}}}{x} + e^{\frac{1}{x}} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{e^{\frac{1}{x}}}{x} + e^{\frac{1}{x}}\right) - \left(e^{\frac{1}{x}} x\right)(1)$$

Which simplifies to

$$W = -e^{\frac{1}{x}}$$

Which simplifies to

$$W = -e^{\frac{1}{x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{\frac{1}{x}} x \cos\left(\frac{1}{x}\right)}{-x^3 e^{\frac{1}{x}}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{\cos\left(\frac{1}{x}\right)}{x^2} dx$$

Hence

$$u_1 = -\sin\left(\frac{1}{x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\frac{1}{x}\right) x}{-x^3 e^{\frac{1}{x}}}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(\frac{1}{x}\right) e^{-\frac{1}{x}}}{x^2} dx$$

Hence

$$u_2 = -\frac{\cos\left(\frac{1}{x}\right) e^{-\frac{1}{x}}}{2} + \frac{e^{-\frac{1}{x}} \sin\left(\frac{1}{x}\right)}{2}$$

Which simplifies to

$$u_1 = -\sin\left(\frac{1}{x}\right)$$

$$u_2 = -\frac{e^{-\frac{1}{x}}\left(\cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right)\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\sin\left(\frac{1}{x}\right)x - \frac{e^{-\frac{1}{x}}\left(\cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right)\right)e^{\frac{1}{x}}x}{2}$$

Which simplifies to

$$y_p(x) = -\frac{x\left(\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)\right)}{2}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-\left(c_1 e^{\frac{1}{x}} - c_2\right)x\right) + \left(-\frac{x\left(\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)\right)}{2}\right) \\ &= -\frac{x\left(2c_1 e^{\frac{1}{x}} + \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) - 2c_2\right)}{2} \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{x\left(2c_1 e^{\frac{1}{x}} + \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) - 2c_2\right)}{2} \tag{1}$$

<u>Verification of solutions</u>

$$y = -\frac{x\left(2c_1 e^{\frac{1}{x}} + \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) - 2c_2\right)}{2}$$

Verified OK.

5.4.3 Solving using Kovacic algorithm

Writing the ode as

$$x^3y'' + xy' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3 \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 70: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 0\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned}\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a}+v\right)\\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a}+v\right)\end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{1}{4x^4}$$

There is pole in r at $x=0$ of order 4, hence $v=2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c=0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v=2$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c=0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned}b &= (0)-(0)\\ &= 0\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_c &= \frac{1}{2x^2}\\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a}+v\right) = \frac{1}{2}\left(\frac{0}{\frac{1}{2}}+2\right) = 1\\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a}+v\right) = \frac{1}{2}\left(-\frac{0}{\frac{1}{2}}+2\right) = 1\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= 0\\ \alpha_\infty^{-} &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{1}{2x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\ &= 1 - (1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x^2} + \frac{1}{x} + (-)\,(0) \\
&= -\frac{1}{2x^2} + \frac{1}{x} \\
&= \frac{2x - 1}{2x^2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{1}{2x^2} + \frac{1}{x}\right)^2 - \left(\frac{1}{4x^4}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x^2} + \frac{1}{x}\right) dx} \\
&= x\, \mathrm{e}^{\frac{1}{2x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^3}\, dx} \\
&= z_1 e^{\frac{1}{2x}} \\
&= z_1 \left(\mathrm{e}^{\frac{1}{2x}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{1}{x}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^3}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{1}{x}}}{(y_1)^2}\, dx \\ &= y_1 \left(\mathrm{e}^{-\frac{1}{x}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{\frac{1}{x}} x\right) + c_2 \left(\mathrm{e}^{\frac{1}{x}} x \left(\mathrm{e}^{-\frac{1}{x}}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^3 y'' + xy' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = x c_1 \mathrm{e}^{\frac{1}{x}} + c_2 x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{\frac{1}{x}} x \\ y_2 &= x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{\frac{1}{x}} x & x \\ \frac{d}{dx}\left(\mathrm{e}^{\frac{1}{x}} x\right) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{\frac{1}{x}} x & x \\ -\frac{\mathrm{e}^{\frac{1}{x}}}{x} + \mathrm{e}^{\frac{1}{x}} & 1 \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{\frac{1}{x}} x\right)(1) - (x)\left(-\frac{\mathrm{e}^{\frac{1}{x}}}{x} + \mathrm{e}^{\frac{1}{x}}\right)$$

Which simplifies to

$$W = \mathrm{e}^{\frac{1}{x}}$$

Which simplifies to

$$W = e^{\frac{1}{x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\cos\left(\frac{1}{x}\right) x}{x^3 e^{\frac{1}{x}}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\cos\left(\frac{1}{x}\right) e^{-\frac{1}{x}}}{x^2}\,dx$$

Hence

$$u_1 = -\frac{\cos\left(\frac{1}{x}\right) e^{-\frac{1}{x}}}{2} + \frac{e^{-\frac{1}{x}} \sin\left(\frac{1}{x}\right)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{\frac{1}{x}} x \cos\left(\frac{1}{x}\right)}{x^3 e^{\frac{1}{x}}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\cos\left(\frac{1}{x}\right)}{x^2}\,dx$$

Hence

$$u_2 = -\sin\left(\frac{1}{x}\right)$$

Which simplifies to

$$u_1 = -\frac{e^{-\frac{1}{x}}\left(\cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right)\right)}{2}$$

$$u_2 = -\sin\left(\frac{1}{x}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\sin\left(\frac{1}{x}\right) x - \frac{e^{-\frac{1}{x}}\left(\cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right)\right) e^{\frac{1}{x}} x}{2}$$

Which simplifies to

$$y_p(x) = -\frac{x\left(\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)\right)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x c_1 e^{\frac{1}{x}} + c_2 x\right) + \left(-\frac{x\left(\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)\right)}{2}\right) \end{aligned}$$

Which simplifies to

$$y = x\left(c_1 e^{\frac{1}{x}} + c_2\right) - \frac{x\left(\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)\right)}{2}$$

Summary

The solution(s) found are the following

$$y = x\left(c_1 e^{\frac{1}{x}} + c_2\right) - \frac{x\left(\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)\right)}{2} \tag{1}$$

Verification of solutions

$$y = x\left(c_1 e^{\frac{1}{x}} + c_2\right) - \frac{x\left(\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)\right)}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(x^3*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=cos(1/x),y(x), singsol=all)
```

$$y(x) = -\frac{x\left(-2\,\mathrm{e}^{\frac{1}{x}} c_2 + \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) - 2c_1\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.272 (sec). Leaf size: 32

```
DSolve[x^3*y''[x]+x*y'[x]-y[x]==Cos[1/x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}x\left(\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right) - 2\left(c_1 e^{\frac{1}{x}} + c_2\right)\right)$$

5.5 problem 5

Internal problem ID [5826]
Internal file name [`OUTPUT/5074_Sunday_June_05_2022_03_20_20_PM_74240442/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.3.4 problems. page 104
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x(1+x)\,y''+(x+2)\,y'-y=x+\frac{1}{x}}$$

5.5.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2 + x, B = x + 2, C = -1, f(x) = x + \frac{1}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$\left(x^2 + x\right) y'' + (x + 2)\, y' - y = 0$$

In normal form the ode

$$\left(x^2 + x\right) y'' + (x + 2)\, y' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x + 2}{x\,(1 + x)} \\ q(x) &= -\frac{1}{x\,(1 + x)} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(x + 2)}{x^2\,(1 + x)} - \frac{1}{x\,(1 + x)} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(-\frac{2}{x} + \frac{x+2}{x(1+x)}\right) v'(x) &= 0 \\ v''(x) - \frac{v'(x)}{1+x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) - \frac{u(x)}{1+x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u}{1+x} \end{aligned}$$

Where $f(x) = \frac{1}{1+x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= \frac{1}{1+x}\,dx \\ \int \frac{1}{u}\,du &= \int \frac{1}{1+x}\,dx \\ \ln(u) &= \ln(1+x) + c_1 \\ u &= e^{\ln(1+x)+c_1} \\ &= (1+x)\,c_1 \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= c_1\left(\frac{1}{2}x^2 + x\right) + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \frac{c_1\left(\frac{1}{2}x^2 + x\right) + c_2}{x} \\
&= \frac{c_1x^2 + 2c_1x + 2c_2}{2x}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
\left(x^2 + x\right) y'' + (x + 2)\, y' - y = x + \frac{1}{x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x} \\
y_2 &= \frac{x}{2} + 1
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \frac{1}{x} & \frac{x}{2} + 1 \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{x}{2} + 1\right) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x}{2}+1 \\ -\frac{1}{x^2} & \frac{1}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{2}\right) - \left(\frac{x}{2}+1\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1+x}{x^2}$$

Which simplifies to

$$W = \frac{1+x}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{x}{2}+1\right)\left(x+\frac{1}{x}\right)}{\frac{(x^2+x)(1+x)}{x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(x+2)\left(x^2+1\right)}{2\left(1+x\right)^2}dx$$

Hence

$$u_1 = -\frac{x^2}{4} + \frac{1}{1+x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x+\frac{1}{x}}{x}}{\frac{(x^2+x)(1+x)}{x^2}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x^2+1}{x\left(1+x\right)^2}dx$$

Hence

$$u_2 = \frac{2}{1+x} + \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{x^2}{4} + \frac{1}{1+x}}{x} + \left(\frac{2}{1+x} + \ln(x)\right)\left(\frac{x}{2} + 1\right)$$

Which simplifies to

$$y_p(x) = \frac{(2x^2 + 4x)\ln(x) - x^2 + 4x + 4}{4x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1\left(\frac{1}{2}x^2 + x\right) + c_2}{x}\right) + \left(\frac{(2x^2 + 4x)\ln(x) - x^2 + 4x + 4}{4x}\right) \\ &= \frac{(2x^2 + 4x)\ln(x) - x^2 + 4x + 4}{4x} + \frac{c_1\left(\frac{1}{2}x^2 + x\right) + c_2}{x} \end{aligned}$$

Which simplifies to

$$y = \frac{(2x^2 + 4x)\ln(x) + (2c_1 - 1)x^2 + (4c_1 + 4)x + 4c_2 + 4}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{(2x^2 + 4x)\ln(x) + (2c_1 - 1)x^2 + (4c_1 + 4)x + 4c_2 + 4}{4x} \tag{1}$$

Verification of solutions

$$y = \frac{(2x^2 + 4x)\ln(x) + (2c_1 - 1)x^2 + (4c_1 + 4)x + 4c_2 + 4}{4x}$$

Verified OK.

5.5.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(\left(x^2+x\right)y''+(x+2)\,y'-y\right)dx = \int \left(x+\frac{1}{x}\right)dx$$

$$(1-x)\,y+\left(x^2+x\right)y' = \frac{x^2}{2}+\ln\left(x\right)+c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{x-1}{x\left(1+x\right)}\\ q(x) &= \frac{x^2+2\ln\left(x\right)+2c_1}{2x\left(1+x\right)}\end{aligned}$$

Hence the ode is

$$y'-\frac{(x-1)\,y}{x\left(1+x\right)} = \frac{x^2+2\ln\left(x\right)+2c_1}{2x\left(1+x\right)}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{x-1}{x(1+x)}dx}\\ &= \mathrm{e}^{-2\ln(1+x)+\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{x}{\left(1+x\right)^2}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^2+2\ln\left(x\right)+2c_1}{2x\left(1+x\right)}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xy}{\left(1+x\right)^2}\right) &= \left(\frac{x}{\left(1+x\right)^2}\right)\left(\frac{x^2+2\ln\left(x\right)+2c_1}{2x\left(1+x\right)}\right)\\ \mathrm{d}\left(\frac{xy}{\left(1+x\right)^2}\right) &= \left(\frac{x^2+2\ln\left(x\right)+2c_1}{2\left(1+x\right)^3}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$
\begin{aligned}
\frac{xy}{(1+x)^2} &= \int \frac{x^2+2\ln(x)+2c_1}{2(1+x)^3}\, \mathrm{d}x \\
\frac{xy}{(1+x)^2} &= -\frac{2c_1+1}{4(1+x)^2}+\frac{3}{2(1+x)}+\frac{\ln(x)\, x(x+2)}{2(1+x)^2}+c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{x}{(1+x)^2}$ results in

$$
y = \frac{(1+x)^2\left(-\frac{2c_1+1}{4(1+x)^2}+\frac{3}{2(1+x)}+\frac{\ln(x)x(x+2)}{2(1+x)^2}\right)}{x}+\frac{c_2(1+x)^2}{x}
$$

which simplifies to

$$
y = \frac{2\ln(x)\, x^2+4c_2x^2+4\ln(x)\, x+8c_2x-2c_1+4c_2+6x+5}{4x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2\ln(x)\, x^2+4c_2x^2+4\ln(x)\, x+8c_2x-2c_1+4c_2+6x+5}{4x} \tag{1}
$$

Verification of solutions

$$
y = \frac{2\ln(x)\, x^2+4c_2x^2+4\ln(x)\, x+8c_2x-2c_1+4c_2+6x+5}{4x}
$$

Verified OK.

5.5.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay''+By'+Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v+v'B \\
y'' &= B''v+B'v'+v''B+v'B' \\
&= v''B+2v'+B'+B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x^2 + x \\
B &= x + 2 \\
C &= -1 \\
F &= x + \frac{1}{x}
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(x^2 + x\right)(0) + (x + 2)(1) + (-1)(x + 2) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
x(1 + x)(x + 2) v'' + \left(3x^2 + 6x + 4\right) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
\left(x^3 + 3x^2 + 2x\right) u'(x) + 3\left(x^2 + 2x + \frac{4}{3}\right) u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= -\frac{u(3x^2 + 6x + 4)}{x\left(x^2 + 3x + 2\right)}
\end{aligned}
$$

Where $f(x) = -\frac{3x^2+6x+4}{x(x^2+3x+2)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{3x^2+6x+4}{x\left(x^2+3x+2\right)}\,dx\\
\int \frac{1}{u}\,du &= \int -\frac{3x^2+6x+4}{x\left(x^2+3x+2\right)}\,dx\\
\ln\left(u\right) &= \ln\left(1+x\right) - 2\ln\left(x\right) - 2\ln\left(x+2\right) + c_1\\
u &= \mathrm{e}^{\ln(1+x)-2\ln(x)-2\ln(x+2)+c_1}\\
&= c_1\mathrm{e}^{\ln(1+x)-2\ln(x)-2\ln(x+2)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_1\left(\frac{1}{x^2\left(x+2\right)^2} + \frac{1}{x\left(x+2\right)^2}\right)
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u\\
&= c_1\left(\frac{1}{x^2\left(x+2\right)^2} + \frac{1}{x\left(x+2\right)^2}\right)
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{\left(1+x\right)c_1}{x^2\left(x+2\right)^2}\,\mathrm{d}x\\
&= c_1\left(-\frac{1}{4x} + \frac{1}{4x+8}\right) + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv\\
&= \left(x+2\right)\left(c_1\left(-\frac{1}{4x} + \frac{1}{4x+8}\right) + c_2\right)\\
&= \frac{2c_2x^2 + 4c_2x - c_1}{2x}
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = x + 2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & x+2 \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}(x+2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x+2 \\ -\frac{1}{x^2} & 1 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)(1) - (x+2)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{2+2x}{x^2}$$

Which simplifies to

$$W = \frac{2+2x}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x+2)\left(x+\frac{1}{x}\right)}{\frac{(x^2+x)(2+2x)}{x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(x+2)\,(x^2+1)}{2\,(1+x)^2}dx$$

Hence

$$u_1 = -\frac{x^2}{4} + \frac{1}{1+x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x+\frac{1}{x}}{x}}{\frac{(x^2+x)(2+2x)}{x^2}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x^2+1}{2x\,(1+x)^2}dx$$

Hence

$$u_2 = \frac{1}{1+x} + \frac{\ln(x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{x^2}{4}+\frac{1}{1+x}}{x} + \left(\frac{1}{1+x} + \frac{\ln(x)}{2}\right)(x+2)$$

Which simplifies to

$$y_p(x) = \frac{(2x^2+4x)\ln(x) - x^2 + 4x + 4}{4x}$$

Hence the complete solution is

$$\begin{aligned}
y(x) &= y_h + y_p \\
&= \left(\frac{2c_2x^2 + 4c_2x - c_1}{2x}\right) + \left(\frac{(2x^2+4x)\ln(x) - x^2 + 4x + 4}{4x}\right) \\
&= \frac{(2x^2+4x)\ln(x) + (4c_2-1)x^2 + (8c_2+4)x - 2c_1 + 4}{4x}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{(2x^2+4x)\ln(x) + (4c_2-1)x^2 + (8c_2+4)x - 2c_1 + 4}{4x} \tag{1}$$

Verification of solutions

$$y = \frac{(2x^2+4x)\ln(x) + (4c_2-1)x^2 + (8c_2+4)x - 2c_1 + 4}{4x}$$

Verified OK.

5.5.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(x^2+x)y'' + (x+2)y' - y = x + \frac{1}{x}$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}
\int \left((x^2+x)y'' + (x+2)y' - y\right)dx &= \int \left(x + \frac{1}{x}\right)dx \\
(1-x)y + (x^2+x)y' &= \frac{x^2}{2} + \ln(x) + c_1
\end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{x-1}{x(1+x)} \\
q(x) &= \frac{x^2 + 2\ln(x) + 2c_1}{2x(1+x)}
\end{aligned}$$

Hence the ode is

$$y' - \frac{(x-1)\,y}{x\,(1+x)} = \frac{x^2+2\ln(x)+2c_1}{2x\,(1+x)}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{x-1}{x(1+x)}dx}\\ &= \mathrm{e}^{-2\ln(1+x)+\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{x}{(1+x)^2}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^2+2\ln(x)+2c_1}{2x\,(1+x)}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xy}{(1+x)^2}\right) &= \left(\frac{x}{(1+x)^2}\right)\left(\frac{x^2+2\ln(x)+2c_1}{2x\,(1+x)}\right)\\ \mathrm{d}\left(\frac{xy}{(1+x)^2}\right) &= \left(\frac{x^2+2\ln(x)+2c_1}{2\,(1+x)^3}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{xy}{(1+x)^2} &= \int \frac{x^2+2\ln(x)+2c_1}{2\,(1+x)^3}\,\mathrm{d}x\\ \frac{xy}{(1+x)^2} &= -\frac{2c_1+1}{4\,(1+x)^2} + \frac{3}{2\,(1+x)} + \frac{\ln(x)\,x(x+2)}{2\,(1+x)^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{x}{(1+x)^2}$ results in

$$y = \frac{(1+x)^2\left(-\frac{2c_1+1}{4(1+x)^2} + \frac{3}{2(1+x)} + \frac{\ln(x)x(x+2)}{2(1+x)^2}\right)}{x} + \frac{c_2(1+x)^2}{x}$$

which simplifies to

$$y = \frac{2\ln(x)\,x^2+4c_2x^2+4\ln(x)\,x+8c_2x-2c_1+4c_2+6x+5}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{2\ln(x)\,x^2+4c_2x^2+4\ln(x)\,x+8c_2x-2c_1+4c_2+6x+5}{4x} \tag{1}$$

Verification of solutions

$$
y = \frac{2 \ln (x) x^2 + 4c_2x^2 + 4 \ln (x) x + 8c_2x - 2c_1 + 4c_2 + 6x + 5}{4x}
$$

Verified OK.

5.5.5 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
(x^2 + x) y'' + (x + 2) y' - y = 0 \tag{1} \\
Ay'' + By' + Cy = 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= x^2 + x \\
B &= x + 2 \tag{3} \\
C &= -1
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A} dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3}{4 (1 + x)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 3 \\
t &= 4(1 + x)^2
\end{align*}
$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4\left(1+x\right)^{2}}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 71: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(1+x)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4(1+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4(1+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2(1+x)} + (-)(0) \\ &= -\frac{1}{2(1+x)} \\ &= -\frac{1}{2(1+x)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(1+x)}\right)(0) + \left(\left(\frac{1}{2(1+x)^2}\right) + \left(-\frac{1}{2(1+x)}\right)^2 - \left(\frac{3}{4(1+x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= \mathrm{e}^{\int -\frac{1}{2(1+x)}dx} \\
&= \frac{1}{\sqrt{1+x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x+2}{x^2+x}\, dx} \\
&= z_1 e^{\frac{\ln(1+x)}{2} - \ln(x)} \\
&= z_1 \left(\frac{\sqrt{1+x}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x+2}{x^2+x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(1+x) - 2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x(x+2)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x(x+2)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$\left(x^2 + x\right) y'' + (x + 2)\, y' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x} + c_2\left(\frac{x}{2} + 1\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$

$$y_2 = \frac{x}{2} + 1$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x}{2} + 1 \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{x}{2} + 1\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x}{2}+1 \\ -\frac{1}{x^2} & \frac{1}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{2}\right) - \left(\frac{x}{2}+1\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1+x}{x^2}$$

Which simplifies to

$$W = \frac{1+x}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{x}{2}+1\right)\left(x+\frac{1}{x}\right)}{\frac{(x^2+x)(1+x)}{x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(x+2)\,(x^2+1)}{2\,(1+x)^2}dx$$

Hence

$$u_1 = -\frac{x^2}{4} + \frac{1}{1+x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x+\frac{1}{x}}{x}}{\frac{(x^2+x)(1+x)}{x^2}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x^2+1}{x\,(1+x)^2}dx$$

Hence

$$u_2 = \frac{2}{1+x} + \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{x^2}{4} + \frac{1}{1+x}}{x} + \left(\frac{2}{1+x} + \ln(x)\right)\left(\frac{x}{2} + 1\right)$$

Which simplifies to

$$y_p(x) = \frac{(2x^2 + 4x)\ln(x) - x^2 + 4x + 4}{4x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + c_2\left(\frac{x}{2} + 1\right)\right) + \left(\frac{(2x^2 + 4x)\ln(x) - x^2 + 4x + 4}{4x}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_1}{x} + \frac{(x+2)\,c_2}{2} + \frac{(2x^2 + 4x)\ln(x) - x^2 + 4x + 4}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{(x+2)\,c_2}{2} + \frac{(2x^2 + 4x)\ln(x) - x^2 + 4x + 4}{4x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{(x+2)\,c_2}{2} + \frac{(2x^2 + 4x)\ln(x) - x^2 + 4x + 4}{4x}$$

Verified OK.

5.5.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 + x \\ q(x) &= x + 2 \\ r(x) &= -1 \\ s(x) &= x + \frac{1}{x} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 1 \end{aligned}$$

Therefore (1) becomes

$$2 - (1) + (-1) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$(1 - x)\, y + \left(x^2 + x\right) y' = \int x + \frac{1}{x}\, dx$$

We now have a first order ode to solve which is

$$(1 - x)\, y + \left(x^2 + x\right) y' = \frac{x^2}{2} + \ln(x) + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{x-1}{x(1+x)} \\ q(x) &= \frac{x^2 + 2\ln(x) + 2c_1}{2x(1+x)}\end{aligned}$$

Hence the ode is

$$y' - \frac{(x-1)\,y}{x(1+x)} = \frac{x^2 + 2\ln(x) + 2c_1}{2x(1+x)}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{x-1}{x(1+x)}dx} \\ &= \mathrm{e}^{-2\ln(1+x)+\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{x}{(1+x)^2}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^2 + 2\ln(x) + 2c_1}{2x(1+x)}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xy}{(1+x)^2}\right) &= \left(\frac{x}{(1+x)^2}\right)\left(\frac{x^2 + 2\ln(x) + 2c_1}{2x(1+x)}\right) \\ \mathrm{d}\left(\frac{xy}{(1+x)^2}\right) &= \left(\frac{x^2 + 2\ln(x) + 2c_1}{2(1+x)^3}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{xy}{(1+x)^2} &= \int \frac{x^2 + 2\ln(x) + 2c_1}{2(1+x)^3}\,\mathrm{d}x \\ \frac{xy}{(1+x)^2} &= -\frac{2c_1+1}{4(1+x)^2} + \frac{3}{2(1+x)} + \frac{\ln(x)\,x(x+2)}{2(1+x)^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{x}{(1+x)^2}$ results in

$$y = \frac{(1+x)^2\left(-\frac{2c_1+1}{4(1+x)^2} + \frac{3}{2(1+x)} + \frac{\ln(x)x(x+2)}{2(1+x)^2}\right)}{x} + \frac{c_2(1+x)^2}{x}$$

which simplifies to

$$y = \frac{2\ln(x)\, x^2 + 4c_2x^2 + 4\ln(x)\, x + 8c_2x - 2c_1 + 4c_2 + 6x + 5}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{2\ln(x)\, x^2 + 4c_2x^2 + 4\ln(x)\, x + 8c_2x - 2c_1 + 4c_2 + 6x + 5}{4x} \tag{1}$$

Verification of solutions

$$y = \frac{2\ln(x)\, x^2 + 4c_2x^2 + 4\ln(x)\, x + 8c_2x - 2c_1 + 4c_2 + 6x + 5}{4x}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(x*(1+x)*diff(y(x),x$2)+(x+2)*diff(y(x),x)-y(x)=x+1/x,y(x), singsol=all)
```

$$y(x) = \frac{2\ln(x)\, x^2 + 4c_2x^2 + 4\ln(x)\, x + 8c_2x + 4c_1 + 4c_2 + 6x + 5}{4x}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 37

```
DSolve[x*(1+x)*y''[x]+(x+2)*y'[x]-y[x]==x+1/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(x+2)\log(x) + \frac{1+c_1}{x} + \frac{1}{4}(-1+2c_2)x + 1 + c_2$$

5.6 problem 6

Internal problem ID [5827]
Internal file name [`OUTPUT/5075_Sunday_June_05_2022_03_20_22_PM_50506697/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.3.4 problems. page 104
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + (-2 + x)\, y' - y = x^2 - 1}$$

5.6.1 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= 2x \\
B &= -2 + x \\
C &= -1 \\
F &= x^2 - 1
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (2x)\,(0) + (-2 + x)\,(1) + (-1)\,(-2 + x) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
2x(-2 + x)\, v'' + \left(x^2 + 4\right) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
\left(2x^2 - 4x\right) u'(x) + \left(x^2 + 4\right) u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{u(x^2 + 4)}{2x\,(-2 + x)}
\end{aligned}
$$

Where $f(x) = -\frac{x^2+4}{2(-2+x)x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{x^2+4}{2\left(-2+x\right)x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{x^2+4}{2\left(-2+x\right)x}\, dx \\
\ln\left(u\right) &= -\frac{x}{2} + \ln\left(x\right) - 2\ln\left(-2+x\right) + c_1 \\
u &= \mathrm{e}^{-\frac{x}{2}+\ln(x)-2\ln(-2+x)+c_1} \\
&= c_1 \mathrm{e}^{-\frac{x}{2}+\ln(x)-2\ln(-2+x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_1 \mathrm{e}^{-\frac{x}{2}} x}{\left(-2+x\right)^2}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1 \mathrm{e}^{-\frac{x}{2}} x}{\left(-2+x\right)^2}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1 \mathrm{e}^{-\frac{x}{2}} x}{\left(-2+x\right)^2}\, \mathrm{d}x \\
&= -\frac{2c_1 \mathrm{e}^{-\frac{x}{2}}}{-2+x} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (-2+x)\left(-\frac{2c_1 \mathrm{e}^{-\frac{x}{2}}}{-2+x} + c_2\right) \\
&= -2c_1 \mathrm{e}^{-\frac{x}{2}} + c_2(-2+x)
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = -2 + x$$
$$y_2 = e^{-\frac{x}{2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -2+x & e^{-\frac{x}{2}} \\ \frac{d}{dx}(-2+x) & \frac{d}{dx}\left(e^{-\frac{x}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -2+x & e^{-\frac{x}{2}} \\ 1 & -\frac{e^{-\frac{x}{2}}}{2} \end{vmatrix}$$

Therefore

$$W = (-2+x)\left(-\frac{e^{-\frac{x}{2}}}{2}\right) - \left(e^{-\frac{x}{2}}\right)(1)$$

Which simplifies to

$$W = -\frac{x\,e^{-\frac{x}{2}}}{2}$$

Which simplifies to

$$W = -\frac{x\,e^{-\frac{x}{2}}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{-\frac{x}{2}} (x^2 - 1)}{-x^2 \mathrm{e}^{-\frac{x}{2}}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{-x^2 + 1}{x^2} dx$$

Hence

$$u_1 = x + \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(-2 + x)(x^2 - 1)}{-x^2 \mathrm{e}^{-\frac{x}{2}}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{(-x^3 + 2x^2 + x - 2)\, \mathrm{e}^{\frac{x}{2}}}{x^2} dx$$

Hence

$$u_2 = -\frac{2\, \mathrm{e}^{\frac{x}{2}} (x^2 - 4x - 1)}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(x + \frac{1}{x}\right)(-2 + x) - \frac{2\, \mathrm{e}^{\frac{x}{2}} (x^2 - 4x - 1)\, \mathrm{e}^{-\frac{x}{2}}}{x}$$

Which simplifies to

$$y_p(x) = x^2 - 4x + 9$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-2c_1 \mathrm{e}^{-\frac{x}{2}} + c_2(-2 + x)\right) + \left(x^2 - 4x + 9\right) \\ &= -2c_1 \mathrm{e}^{-\frac{x}{2}} + x^2 + (c_2 - 4)\, x - 2c_2 + 9 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -2c_1 e^{-\frac{x}{2}} + x^2 + (c_2 - 4)\, x - 2c_2 + 9 \tag{1}$$

Verification of solutions

$$y = -2c_1 e^{-\frac{x}{2}} + x^2 + (c_2 - 4)\, x - 2c_2 + 9$$

Verified OK.

5.6.2 Solving using Kovacic algorithm

Writing the ode as

$$2xy'' + (-2 + x)\, y' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -2 + x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 12}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x + 12 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 12}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 72: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{1}{4x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{1}{2x} + \frac{1}{x^2} - \frac{2}{x^3} + \frac{2}{x^4} + \frac{4}{x^5} - \frac{24}{x^6} + \frac{48}{x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{4} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 12}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{4x + 12}{16x^2}\right) \\ &= \frac{1}{16} + \frac{4x + 12}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{4}} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{4}} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 12}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\left(\frac{1}{4}\right) \\ &= -\frac{1}{2x} - \frac{1}{4} \\ &= -\frac{x+2}{4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{1}{4}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x} - \frac{1}{4}\right)^2 - \left(\frac{x^2 + 4x + 12}{16x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{1}{4}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x}{4}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2+x}{2x} \, dx} \\
&= z_1 e^{-\frac{x}{4} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{-\frac{x}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2+x}{2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x}{2} + \ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(2(-2 + x)\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}}\left(2(-2+x)\,\mathrm{e}^{\frac{x}{2}}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
2xy'' + (-2+x)\,y' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2(2x-4)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-\frac{x}{2}} \\
y_2 &= 2x - 4
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-\frac{x}{2}} & 2x-4 \\ \frac{d}{dx}\left(e^{-\frac{x}{2}}\right) & \frac{d}{dx}(2x-4) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-\frac{x}{2}} & 2x-4 \\ -\frac{e^{-\frac{x}{2}}}{2} & 2 \end{vmatrix}$$

Therefore

$$W = \left(e^{-\frac{x}{2}}\right)(2) - (2x-4)\left(-\frac{e^{-\frac{x}{2}}}{2}\right)$$

Which simplifies to

$$W = x\,e^{-\frac{x}{2}}$$

Which simplifies to

$$W = x\,e^{-\frac{x}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(2x-4)(x^2-1)}{2x^2 e^{-\frac{x}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(-2+x)(x^2-1)\,e^{\frac{x}{2}}}{x^2}\,dx$$

Hence

$$u_1 = -\frac{2\,e^{\frac{x}{2}}(x^2-4x-1)}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-\frac{x}{2}}(x^2-1)}{2x^2 e^{-\frac{x}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x^2 - 1}{2x^2} dx$$

Hence

$$u_2 = \frac{x}{2} + \frac{1}{2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{2\,e^{\frac{x}{2}} (x^2 - 4x - 1)\,e^{-\frac{x}{2}}}{x} + \left(\frac{x}{2} + \frac{1}{2x}\right)(2x - 4)$$

Which simplifies to

$$y_p(x) = x^2 - 4x + 9$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-\frac{x}{2}} + c_2(2x - 4)\right) + \left(x^2 - 4x + 9\right) \end{aligned}$$

Which simplifies to

$$y = c_1 e^{-\frac{x}{2}} + 2c_2(-2 + x) + x^2 - 4x + 9$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-\frac{x}{2}} + 2c_2(-2 + x) + x^2 - 4x + 9 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-\frac{x}{2}} + 2c_2(-2 + x) + x^2 - 4x + 9$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve(2*x*diff(y(x),x$2)+(x-2)*diff(y(x),x)-y(x)=x^2-1,y(x), singsol=all)
```

$$y(x) = (-2 + x)\, c_2 + c_1 \mathrm{e}^{-\frac{x}{2}} + x^2 + 1$$

✓ Solution by Mathematica

Time used: 0.256 (sec). Leaf size: 30

```
DSolve[2*x*y''[x]+(x-2)*y'[x]-y[x]==x^2-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 - 4x + c_1 e^{-x/2} + 2c_2(x-2) + 9$$

5.7 problem 7

Internal problem ID [5828]
Internal file name [`OUTPUT/5076_Sunday_June_05_2022_03_20_25_PM_94345149/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.3.4 problems. page 104
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

Unable to solve or complete the solution.

$$\boxed{x^2(1+x)\,y'' + x(4x+3)\,y' - y = x + \frac{1}{x}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 640

```
dsolve(x^2*(x+1)*diff(y(x),x$2)+x*(4*x+3)*diff(y(x),x)-y(x)=x+1/x,y(x), singsol=all)
```

$$y(x) = \frac{-5x^{-\sqrt{2}}\left(\sqrt{2}-\frac{6}{5}\right)\operatorname{hypergeom}\left(\left[2-\sqrt{2},-1-\sqrt{2}\right],\left[1-2\sqrt{2}\right],-x\right)\left(\int \frac{}{\left(-7\sqrt{2}\operatorname{hypergeom}\left(\left[\sqrt{2}-1,\sqrt{2}-1\right],\left[1+2\cdots\right.\right.\right.}\right.}{}$$

✓ Solution by Mathematica

Time used: 7.882 (sec). Leaf size: 636

```
DSolve[x^2*(x+1)*y''[x]+x*(4*x+3)*y'[x]-y[x]==x+1/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^{-1-\sqrt{2}}\Bigg(x^{2\sqrt{2}}\,\text{Hypergeometric2F1}\left(-1+\sqrt{2},2+\sqrt{2},1+2\sqrt{2},\right.$$
$$\left.-x\right)\int_1^x \frac{}{(K[2]+1)\left(\left(4+\sqrt{2}\right)\text{Hypergeometric2F1}\left(-\sqrt{2},3-\sqrt{2},2-2\sqrt{2},-K[2]\right)\text{Hypergeometric2}\right.}$$
$$+\,\text{Hypergeometric2F1}\left(-1-\sqrt{2},2-\sqrt{2},1-2\sqrt{2},-x\right)\int_1^x$$
$$-\frac{}{(K[1]+1)\left(\left(4+\sqrt{2}\right)\text{Hypergeometric2F1}\left(-\sqrt{2},3-\sqrt{2},2-2\sqrt{2},-K[1]\right)\text{Hypergeometric2F1}\left(-1\right.\right.}$$
$$+\,c_2x^{2\sqrt{2}}\,\text{Hypergeometric2F1}\left(-1+\sqrt{2},2+\sqrt{2},1+2\sqrt{2},-x\right)$$
$$+\,c_1\,\text{Hypergeometric2F1}\left(-1-\sqrt{2},2-\sqrt{2},1-2\sqrt{2},-x\right)\Bigg)$$

5.8 problem 8

Internal problem ID [5829]
Internal file name [`OUTPUT/5077_Sunday_June_05_2022_03_20_43_PM_69128330/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.3.4 problems. page 104
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(\ln(x)-1)\,y''-xy'+y=x(-\ln(x)+1)^2}$$

5.8.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=x^2(\ln(x)-1)\,, B=-x, C=1, f(x)=x(\ln(x)-1)^2$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$x^2(\ln(x)-1)\,y''-xy'+y=0$$

In normal form the ode

$$x^2(\ln(x)-1)\, y''-x y'+y=0 \tag{1}$$

Becomes

$$y''+p(x)\, y'+q(x)\, y=0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= -\frac{1}{x\left(\ln\left(x\right)-1\right)}\\
q(x) &= \frac{1}{x^{2}\left(\ln\left(x\right)-1\right)}
\end{aligned}$$

Applying change of variables on the depndent variable $y=v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right) v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right) v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}-\frac{n}{x^{2}\left(\ln\left(x\right)-1\right)}+\frac{1}{x^{2}\left(\ln\left(x\right)-1\right)}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}
v''(x)+\left(\frac{2}{x}-\frac{1}{x\left(\ln\left(x\right)-1\right)}\right) v'(x) &= 0\\
v''(x)+\left(\frac{2}{x}-\frac{1}{x\left(\ln\left(x\right)-1\right)}\right) v'(x) &= 0
\end{aligned} \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x) + \left(\frac{2}{x} - \frac{1}{x(\ln(x) - 1)}\right) u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u(-3 + 2\ln(x))}{x(\ln(x) - 1)} \end{aligned}$$

Where $f(x) = -\frac{-3+2\ln(x)}{x(\ln(x)-1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u} du &= -\frac{-3 + 2\ln(x)}{x(\ln(x) - 1)} dx \\ \int \frac{1}{u} du &= \int -\frac{-3 + 2\ln(x)}{x(\ln(x) - 1)} dx \\ \ln(u) &= -2\ln(x) + \ln(\ln(x) - 1) + c_1 \\ u &= e^{-2\ln(x)+\ln(\ln(x)-1)+c_1} \\ &= c_1 e^{-2\ln(x)+\ln(\ln(x)-1)} \end{aligned}$$

Which simplifies to

$$u(x) = c_1 \left(\frac{\ln(x)}{x^2} - \frac{1}{x^2}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{c_1 \ln(x)}{x} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(-\frac{c_1 \ln(x)}{x} + c_2\right) x \\ &= -c_1 \ln(x) + c_2 x \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2(\ln(x)-1)\,y''-xy'+y=x(\ln(x)-1)^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= \ln(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1=-\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2=\int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W=\begin{vmatrix} x & \ln(x) \\ \frac{d}{dx}(x) & \frac{d}{dx}(\ln(x)) \end{vmatrix}$$

Which gives

$$W=\begin{vmatrix} x & \ln(x) \\ 1 & \frac{1}{x} \end{vmatrix}$$

Therefore

$$W=(x)\left(\frac{1}{x}\right)-(\ln(x))\,(1)$$

Which simplifies to

$$W = -\ln(x) + 1$$

Which simplifies to

$$W = -\ln(x) + 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\, x(\ln(x) - 1)^2}{x^2 (\ln(x) - 1)(-\ln(x) + 1)}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{\ln(x)}{x} dx$$

Hence

$$u_1 = \frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2(\ln(x) - 1)^2}{x^2 (\ln(x) - 1)(-\ln(x) + 1)}\, dx$$

Which simplifies to

$$u_2 = \int (-1)\, dx$$

Hence

$$u_2 = -x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2\, x}{2} - \ln(x)\, x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(-\frac{c_1 \ln (x)}{x} + c_2\right) x\right) + \left(\frac{\ln (x)^2 x}{2} - \ln (x) x\right) \\
&= \frac{\ln (x)^2 x}{2} - \ln (x) x + \left(-\frac{c_1 \ln (x)}{x} + c_2\right) x
\end{aligned}
$$

Which simplifies to

$$
y = \frac{\ln (x)^2 x}{2} + (-c_1 - x) \ln (x) + c_2 x
$$

Summary

The solution(s) found are the following

$$
y = \frac{\ln (x)^2 x}{2} + (-c_1 - x) \ln (x) + c_2 x \tag{1}
$$

Verification of solutions

$$
y = \frac{\ln (x)^2 x}{2} + (-c_1 - x) \ln (x) + c_2 x
$$

Verified OK.

5.8.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2(\ln(x) - 1) \\ B &= -x \\ C &= 1 \\ F &= x(\ln(x) - 1)^2 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2(\ln(x) - 1)\right)(0) + (-x)(-1) + (1)(-x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-x^3(\ln(x) - 1) v'' + \left(x^2(3 - 2\ln(x))\right) v' = 0$$

Now by applying $v' = u$ the above becomes

$$-x^2((\ln(x) - 1) xu'(x) + 2u(x)\ln(x) - 3u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u(-3 + 2\ln(x))}{x(\ln(x) - 1)} \end{aligned}$$

Where $f(x) = -\frac{-3+2\ln(x)}{x(\ln(x)-1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{-3+2\ln(x)}{x(\ln(x)-1)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{-3+2\ln(x)}{x(\ln(x)-1)}\,dx \\
\ln(u) &= -2\ln(x) + \ln(\ln(x)-1) + c_1 \\
u &= \mathrm{e}^{-2\ln(x)+\ln(\ln(x)-1)+c_1} \\
&= c_1 \mathrm{e}^{-2\ln(x)+\ln(\ln(x)-1)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_1\left(\frac{\ln(x)}{x^2} - \frac{1}{x^2}\right)
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1\left(\frac{\ln(x)}{x^2} - \frac{1}{x^2}\right)
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1(\ln(x)-1)}{x^2}\,\mathrm{d}x \\
&= -\frac{c_1\ln(x)}{x} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (-x)\left(-\frac{c_1\ln(x)}{x} + c_2\right) \\
&= c_1\ln(x) - c_2 x
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \ln(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \ln(x) \\ \frac{d}{dx}(x) & \frac{d}{dx}(\ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \ln(x) \\ 1 & \frac{1}{x} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{1}{x}\right) - (\ln(x))(1)$$

Which simplifies to

$$W = -\ln(x) + 1$$

Which simplifies to

$$W = -\ln(x) + 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\, x(\ln(x) - 1)^2}{x^2 (\ln(x) - 1)(-\ln(x) + 1)}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{\ln(x)}{x} dx$$

Hence

$$u_1 = \frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2(\ln(x)-1)^2}{x^2(\ln(x)-1)(-\ln(x)+1)} dx$$

Which simplifies to

$$u_2 = \int (-1)\, dx$$

Hence

$$u_2 = -x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2 x}{2} - \ln(x)\, x$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (c_1 \ln(x) - c_2 x) + \left(\frac{\ln(x)^2 x}{2} - \ln(x)\, x\right) \\ &= \frac{\ln(x)^2 x}{2} + (-x + c_1)\ln(x) - c_2 x \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\ln(x)^2 x}{2} + (-x + c_1)\ln(x) - c_2 x \tag{1}$$

Verification of solutions

$$y = \frac{\ln(x)^2 x}{2} + (-x + c_1)\ln(x) - c_2 x$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
Try integration with the canonical coordinates of the symmetry [0, x]
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (-2*_b(_a)*_a*ln(_a)+ln(_a)^2+3*_b(_a)*
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- differential order: 2; canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 25

```
dsolve(x^2*(ln(x)-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=x*(1-ln(x))^2,y(x), singsol=all)
```

$$y(x) = \frac{\ln(x)^2 x}{2} + (-x - c_1)\ln(x) + c_2 x$$

✓ Solution by Mathematica

Time used: 0.105 (sec). Leaf size: 27

```
DSolve[x^2*(Log[x]-1)*y''[x]-x*y'[x]+y[x]==x*(1-Log[x])^2,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{2} x \log^2(x) + c_1 x - (x + c_2)\log(x)$$

5.9 problem 9

Internal problem ID [5830]
Internal file name [`OUTPUT/5078_Sunday_June_05_2022_03_20_47_PM_75354217/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.3.4 problems. page 104
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{xy'' + 2y' + xy = \sec(x)}$$

5.9.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$xy'' + 2y' + xy = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= 1
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= 1 - \frac{\left(\frac{2}{x}\right)'}{2} - \frac{\left(\frac{2}{x}\right)^2}{4} \\
&= 1 - \frac{\left(-\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\
&= 1 - \left(-\frac{1}{x^2}\right) - \frac{1}{x^2} \\
&= 1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{2}{x}}{2}} \\
&= \frac{1}{x}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = \frac{v(x)}{x} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(x) + v(x) = \sec(x)
$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = f(x)
$$

Where $A = 1, B = 0, C = 1, f(x) = \sec(x)$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$v''(x) + v(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^{0}(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$v(x) = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution v_h is

$$v_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution v_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$v_p(x) = u_1 v_1 + u_2 v_2 \tag{1}$$

Where u_1, u_2 to be determined, and v_1, v_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$v_1 = \cos(x)$$
$$v_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{v_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{v_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of v'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)}{1} dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\, dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sec(x)\cos(x)}{1} dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$v_p(x) = \ln(\cos(x))\cos(x) + \sin(x)\, x$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= (\cos(x) c_1 + c_2 \sin(x)) + (\ln(\cos(x)) \cos(x) + \sin(x) x) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x) z(x) \\ &= (\cos(x) c_1 + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + \sin(x) x) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x}$$

Hence (7) becomes

$$y = \frac{\cos(x) c_1 + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + \sin(x) x}{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\cos(x) c_1 + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + \sin(x) x}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \frac{\cos(x)}{x} \\ y_2 &= \frac{\sin(x)}{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos(x)}{x} & \frac{\sin(x)}{x} \\ \frac{d}{dx}\left(\frac{\cos(x)}{x}\right) & \frac{d}{dx}\left(\frac{\sin(x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos(x)}{x} & \frac{\sin(x)}{x} \\ -\frac{\sin(x)}{x} - \frac{\cos(x)}{x^2} & \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos(x)}{x}\right)\left(\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}\right) - \left(\frac{\sin(x)}{x}\right)\left(-\frac{\sin(x)}{x} - \frac{\cos(x)}{x^2}\right)$$

Which simplifies to

$$W = \frac{\cos(x)^2 + \sin(x)^2}{x^2}$$

Which simplifies to

$$W = \frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(x)\sec(x)}{x}}{\frac{1}{x}}\, dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\, dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos(x)\sec(x)}{x}}{\frac{1}{x}} dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(\cos(x))\cos(x)}{x} + \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\cos(x)c_1 + c_2\sin(x) + \ln(\cos(x))\cos(x) + \sin(x)x}{x}\right) \\ &\quad + \left(\frac{\ln(\cos(x))\cos(x)}{x} + \sin(x)\right) \end{aligned}$$

Which simplifies to

$$y = \frac{\ln(\cos(x))\cos(x) + \cos(x)c_1 + \sin(x)(c_2 + x)}{x} + \frac{\ln(\cos(x))\cos(x)}{x} + \sin(x)$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(\cos(x))\cos(x) + \cos(x)c_1 + \sin(x)(c_2 + x)}{x} + \frac{\ln(\cos(x))\cos(x)}{x} + \sin(x) \tag{1}$$

Verification of solutions

$$y = \frac{\ln(\cos(x))\cos(x) + \cos(x)c_1 + \sin(x)(c_2 + x)}{x} + \frac{\ln(\cos(x))\cos(x)}{x} + \sin(x)$$

Verified OK.

5.9.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + 2xy' + yx^2 = x\sec(x) \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= -\frac{1}{2} \\
\beta &= 1 \\
n &= \frac{1}{2} \\
\gamma &= 1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{2}\, \sin(x)}{x\sqrt{\pi}} - \frac{c_2\sqrt{2}\, \cos(x)}{x\sqrt{\pi}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1\sqrt{2}\, \sin(x)}{x\sqrt{\pi}} - \frac{c_2\sqrt{2}\, \cos(x)}{x\sqrt{\pi}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\cos(x)}{x}$$
$$y_2 = \frac{\sin(x)}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos(x)}{x} & \frac{\sin(x)}{x} \\ \frac{d}{dx}\left(\frac{\cos(x)}{x}\right) & \frac{d}{dx}\left(\frac{\sin(x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos(x)}{x} & \frac{\sin(x)}{x} \\ -\frac{\sin(x)}{x} - \frac{\cos(x)}{x^2} & \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos(x)}{x}\right)\left(\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}\right) - \left(\frac{\sin(x)}{x}\right)\left(-\frac{\sin(x)}{x} - \frac{\cos(x)}{x^2}\right)$$

Which simplifies to

$$W = \frac{\cos(x)^2 + \sin(x)^2}{x^2}$$

Which simplifies to

$$W = \frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\,dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sec(x)\cos(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(\cos(x))\cos(x)}{x} + \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1\sqrt{2}\,\sin(x)}{x\sqrt{\pi}} - \frac{c_2\sqrt{2}\,\cos(x)}{x\sqrt{\pi}}\right) + \left(\frac{\ln(\cos(x))\cos(x)}{x} + \sin(x)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1\sqrt{2}\,\sin(x)}{x\sqrt{\pi}} - \frac{c_2\sqrt{2}\,\cos(x)}{x\sqrt{\pi}} + \frac{\ln(\cos(x))\cos(x)}{x} + \sin(x) \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{2}\,\sin(x)}{x\sqrt{\pi}} - \frac{c_2\sqrt{2}\,\cos(x)}{x\sqrt{\pi}} + \frac{\ln(\cos(x))\cos(x)}{x} + \sin(x)$$

Verified OK.

5.9.3 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 73: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(x)}{x}\right) + c_2\left(\frac{\cos(x)}{x}(\tan(x))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$xy'' + 2y' + xy = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\cos(x)}{x}$$
$$y_2 = \frac{\sin(x)}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos(x)}{x} & \frac{\sin(x)}{x} \\ \frac{d}{dx}\left(\frac{\cos(x)}{x}\right) & \frac{d}{dx}\left(\frac{\sin(x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos(x)}{x} & \frac{\sin(x)}{x} \\ -\frac{\sin(x)}{x} - \frac{\cos(x)}{x^2} & \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos(x)}{x}\right)\left(\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}\right) - \left(\frac{\sin(x)}{x}\right)\left(-\frac{\sin(x)}{x} - \frac{\cos(x)}{x^2}\right)$$

Which simplifies to

$$W = \frac{\cos(x)^2 + \sin(x)^2}{x^2}$$

Which simplifies to

$$W = \frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(x)\sec(x)}{x}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\,dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos(x)\sec(x)}{x}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(\cos(x))\cos(x)}{x} + \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x}\right) + \left(\frac{\ln(\cos(x))\cos(x)}{x} + \sin(x)\right) \end{aligned}$$

Which simplifies to

$$y = \frac{\cos(x) c_1 + c_2 \sin(x)}{x} + \frac{\ln(\cos(x))\cos(x)}{x} + \sin(x)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\cos(x) c_1 + c_2 \sin(x)}{x} + \frac{\ln(\cos(x))\cos(x)}{x} + \sin(x) \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{\cos(x) c_1 + c_2 \sin(x)}{x} + \frac{\ln(\cos(x))\cos(x)}{x} + \sin(x)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 26

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=sec(x),y(x), singsol=all)
```

$$y(x) = \frac{-\ln(\sec(x))\cos(x) + \cos(x)\, c_1 + \sin(x)\,(x + c_2)}{x}$$

✓ Solution by Mathematica
Time used: 0.077 (sec). Leaf size: 65

```
DSolve[x*y''[x]+2*y'[x]+x*y[x]==Sec[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(e^{2ix}\log(1 + e^{-2ix}) + \log(1 + e^{2ix}) - ic_2e^{2ix} + 2c_1)}{2x}$$

5.10 problem 10

Internal problem ID [5831]
Internal file name [`OUTPUT/5079_Sunday_June_05_2022_03_20_49_PM_27306556/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.3.4 problems. page 104
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{(-x^2+1)\,y''-xy'+\frac{y}{4}=-\frac{x^2}{2}+\frac{1}{2}}$$

5.10.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$(-x^2+1)\,y''-xy'+\frac{y}{4}=0$$

In normal form the ode

$$(-x^2+1)\,y''-xy'+\frac{y}{4}=0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x}{x^2-1} \\ q(x) &= -\frac{1}{4x^2-4} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\left(\int p(x)dx\right)}\,dx \\ &= \int e^{-\left(\int \frac{x}{x^2-1}dx\right)}\,dx \\ &= \int e^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}}\,dx \\ &= \int \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}dx \\ &= \frac{\sqrt{(x-1)(1+x)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{1+x}} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{4x^2-4}}{\frac{1}{(x-1)(1+x)}} \\ &= -\frac{1}{4} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{y(\tau)}{4} &= 0
\end{aligned}
$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Ay''(\tau) + By'(\tau) + Cy(\tau) = 0
$$

Where in the above $A = 1, B = 0, C = -\frac{1}{4}$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda\tau} - \frac{\mathrm{e}^{\lambda\tau}}{4} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$
\lambda^2 - \frac{1}{4} = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = 0, C = -\frac{1}{4}$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)\left(-\frac{1}{4}\right)} \\
&= \pm\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= +\frac{1}{2} \\
\lambda_2 &= -\frac{1}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2} \\
\lambda_2 &= -\frac{1}{2}
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{\left(\frac{1}{2}\right)\tau} + c_2 e^{\left(-\frac{1}{2}\right)\tau}$$

Or

$$y(\tau) = c_1 \mathrm{e}^{\frac{\tau}{2}} + c_2 \mathrm{e}^{-\frac{\tau}{2}}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(x + \sqrt{x^2 - 1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x + \sqrt{x^2 - 1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \left(x + \sqrt{x^2 - 1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x + \sqrt{x^2 - 1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x + \sqrt{x^2 - 1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}$$

$$y_2 = \left(x + \sqrt{x^2 - 1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}} & \left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}} \\ \frac{d}{dx}\left(\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\right) & \frac{d}{dx}\left(\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} & \left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}} \\ \left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\left(\left(-\frac{x}{2\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{1+x}}+\frac{\sqrt{x^2-1}}{4(x-1)^{\frac{3}{2}}\sqrt{1+x}}+\frac{\sqrt{x^2-1}}{4\sqrt{x-1}\,(1+x)^{\frac{3}{2}}}\right)\ln\left(x+\sqrt{x^2-1}\right)-\right. & \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\right)\left(\left(x\right.\right. \\ & +\sqrt{x^2-1}\Big)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\left(\left(\frac{x}{2\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{1+x}}-\frac{\sqrt{x^2-1}}{4\,(x-1)^{\frac{3}{2}}\sqrt{1+x}}-\frac{\sqrt{x^2-1}}{4\sqrt{x-1}\,(1+x)^{\frac{3}{2}}}\right)\ln\Big(x+\right. \\ & \left.\left.\left. +\frac{\sqrt{x^2-1}\left(1+\frac{x}{\sqrt{x^2-1}}\right)}{2\sqrt{x-1}\,\sqrt{1+x}\,\left(x+\sqrt{x^2-1}\right)}\right)\right)-\left(\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\right)\left(\left(x\right.\right. \\ & +\sqrt{x^2-1}\Big)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\left(\left(-\frac{x}{2\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{1+x}}+\frac{\sqrt{x^2-1}}{4\,(x-1)^{\frac{3}{2}}\sqrt{1+x}}+\frac{\sqrt{x^2-1}}{4\sqrt{x-1}\,(1+x)^{\frac{3}{2}}}\right)\ln\Big(x\right. \\ & \left.\left.\left. -\frac{\sqrt{x^2-1}\left(1+\frac{x}{\sqrt{x^2-1}}\right)}{2\sqrt{x-1}\,\sqrt{1+x}\,\left(x+\sqrt{x^2-1}\right)}\right)\right) \end{aligned}$$

Which simplifies to

$$W = \frac{\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\left(x^2-1\right)}{(x-1)^{\frac{3}{2}}\,(1+x)^{\frac{3}{2}}}$$

Which simplifies to

$$W = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}} \left(-\frac{x^2}{2}+\frac{1}{2}\right)}{\frac{-x^2+1}{\sqrt{x-1}\,\sqrt{1+x}}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{1+x}\,\sqrt{x-1}\,\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{\sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}}{2} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}} \left(-\frac{x^2}{2}+\frac{1}{2}\right)}{\frac{-x^2+1}{\sqrt{x-1}\,\sqrt{1+x}}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{1+x}\,\sqrt{x-1}\,\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2} dx$$

Hence

$$u_2 = \int_0^x \frac{\sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{-\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}}{2} d\alpha$$

Which simplifies to

$$u_1 = -\frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)}{2}$$

$$u_2 = \frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{-\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2} + \frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{-\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}} + c_2\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\right) \\ &+ \left(-\frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2}\right. \\ &\left.+ \frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{-\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}} + c_2\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}} \\
- \frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\left(\alpha+\sqrt{\alpha^2-1}\right)^{\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2} \\
+ \frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\left(\alpha+\sqrt{\alpha^2-1}\right)^{-\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}} + c_2\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}} \\
- \frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\left(\alpha+\sqrt{\alpha^2-1}\right)^{\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2} \\
+ \frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\left(\alpha+\sqrt{\alpha^2-1}\right)^{-\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2}
\end{aligned}
$$

Verified OK.

5.10.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = -x^2+1, B = -x, C = \frac{1}{4}, f(x) = -\frac{x^2}{2}+\frac{1}{2}$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
\left(-x^2+1\right)y'' - xy' + \frac{y}{4} = 0
$$

In normal form the ode

$$
\left(-x^2+1\right)y'' - xy' + \frac{y}{4} = 0 \tag{1}
$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x}{x^2 - 1} \\ q(x) &= -\frac{1}{4x^2 - 4} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-\frac{1}{4x^2-4}}}{c} \\ \tau'' &= \frac{4x}{c\sqrt{-\frac{1}{4x^2-4}}\, (4x^2 - 4)^2} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{4x}{c\sqrt{-\frac{1}{4x^2-4}}\,(4x^2-4)^2} + \frac{x}{x^2-1}\frac{\sqrt{-\frac{1}{4x^2-4}}}{c}}{\left(\frac{\sqrt{-\frac{1}{4x^2-4}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\, dx \\ &= \frac{\int \sqrt{-\frac{1}{4x^2-4}} dx}{c} \\ &= \frac{\sqrt{-\frac{1}{x^2-1}}\, \sqrt{x^2-1}\, \ln\left(x + \sqrt{x^2-1}\right)}{2c} \end{aligned}$$

Substituting the above into the solution obtained gives

$$\begin{aligned} y = {} & c_1 \cos\left(\frac{\sqrt{-\frac{1}{x^2-1}}\, \sqrt{x^2-1}\, \ln\left(x + \sqrt{x^2-1}\right)}{2}\right) \\ & + c_2 \sin\left(\frac{\sqrt{-\frac{1}{x^2-1}}\, \sqrt{x^2-1}\, \ln\left(x + \sqrt{x^2-1}\right)}{2}\right) \end{aligned}$$

Now the particular solution to this ODE is found

$$\left(-x^2+1\right) y'' - xy' + \frac{y}{4} = -\frac{x^2}{2} + \frac{1}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}$$

$$y_2 = \left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}} & \left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}} \\ \frac{d}{dx}\left(\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\right) & \frac{d}{dx}\left(\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} & \left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}} \\ \left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\left(\left(-\frac{x}{2\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{1+x}}+\frac{\sqrt{x^2-1}}{4(x-1)^{\frac{3}{2}}\sqrt{1+x}}+\frac{\sqrt{x^2-1}}{4\sqrt{x-1}\,(1+x)^{\frac{3}{2}}}\right)\ln\left(x+\sqrt{x^2-1}\right) - \right. & \end{vmatrix}$$

Therefore

$$
\begin{aligned}
W = {} & \left(\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\right)\Bigg(\Bigg(x \\
& +\sqrt{x^2-1}\Big)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\left(\left(\frac{x}{2\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{1+x}}-\frac{\sqrt{x^2-1}}{4\left(x-1\right)^{\frac{3}{2}}\sqrt{1+x}}-\frac{\sqrt{x^2-1}}{4\sqrt{x-1}\,\left(1+x\right)^{\frac{3}{2}}}\right)\ln\Big(x+ \right. \\
& \left. + \frac{\sqrt{x^2-1}\left(1+\frac{x}{\sqrt{x^2-1}}\right)}{2\sqrt{x-1}\,\sqrt{1+x}\,\left(x+\sqrt{x^2-1}\right)}\right)\Bigg) - \left(\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\right)\Bigg(\Bigg(x \\
& +\sqrt{x^2-1}\Big)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\left(\left(-\frac{x}{2\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{1+x}}+\frac{\sqrt{x^2-1}}{4\left(x-1\right)^{\frac{3}{2}}\sqrt{1+x}}+\frac{\sqrt{x^2-1}}{4\sqrt{x-1}\,\left(1+x\right)^{\frac{3}{2}}}\right)\ln\Big(x \right. \\
& \left. - \frac{\sqrt{x^2-1}\left(1+\frac{x}{\sqrt{x^2-1}}\right)}{2\sqrt{x-1}\,\sqrt{1+x}\,\left(x+\sqrt{x^2-1}\right)}\right)\Bigg)
\end{aligned}
$$

Which simplifies to

$$
W = \frac{\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\left(x^2-1\right)}{\left(x-1\right)^{\frac{3}{2}}\left(1+x\right)^{\frac{3}{2}}}
$$

Which simplifies to

$$
W = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\left(-\frac{x^2}{2}+\frac{1}{2}\right)}{\frac{-x^2+1}{\sqrt{x-1}\,\sqrt{1+x}}}\,dx
$$

Which simplifies to

$$
u_1 = -\int \frac{\sqrt{1+x}\,\sqrt{x-1}\,\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2}dx
$$

Hence

$$u_1 = -\left(\int_0^x \frac{\sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}}{2}d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}\left(-\frac{x^2}{2}+\frac{1}{2}\right)}{\frac{-x^2+1}{\sqrt{x-1}\,\sqrt{1+x}}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{1+x}\,\sqrt{x-1}\,\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2}dx$$

Hence

$$u_2 = \int_0^x \frac{\sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{-\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}}{2}d\alpha$$

Which simplifies to

$$u_1 = -\frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)}{2}$$

$$u_2 = \frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{-\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2}$$
$$+\frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{-\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos\left(\frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{2}\right)\right. \\
&\qquad \left.+ c_2 \sin\left(\frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{2}\right)\right) \\
&\quad + \left(-\frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2}\right. \\
&\qquad \left.+ \frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{-\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2}\right) \\
&= -\frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2} \\
&\quad + \frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{-\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2} \\
&\quad + c_1 \cos\left(\frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{2}\right) \\
&\quad + c_2 \sin\left(\frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{2}\right)
\end{aligned}
$$

Which simplifies to

$$y = -\frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2} + \frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{-\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2} + c_1\cos\left(\frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{2}\right) + c_2\sin\left(\frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{2}\right)$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2} + \frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{-\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2} + c_1\cos\left(\frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{2}\right) + c_2\sin\left(\frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{2}\right) \tag{1}$$

Verification of solutions

$$
\begin{aligned}
y = &-\frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2} \\
&+\frac{\left(\int_0^x \sqrt{1+\alpha}\,\sqrt{\alpha-1}\,\left(\alpha+\sqrt{\alpha^2-1}\right)^{-\frac{\sqrt{\alpha^2-1}}{2\sqrt{\alpha-1}\,\sqrt{1+\alpha}}}d\alpha\right)\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{x^2-1}}{2\sqrt{x-1}\,\sqrt{1+x}}}}{2} \\
&+c_1\cos\left(\frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{2}\right) \\
&+c_2\sin\left(\frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{2}\right)
\end{aligned}
$$

Verified OK.

5.10.3 Solving using Kovacic algorithm

Writing the ode as

$$\left(-x^2+1\right)y''-xy'+\frac{y}{4}=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= -x^2+1 \\
B &= -x \\
C &= \frac{1}{4}
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB'-2BA'+B^2-4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4\left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{4\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 74: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\,(x-1)^2} + \frac{3}{16\,(x-1)} - \frac{3}{16\,(1+x)} - \frac{3}{16\,(1+x)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{4\,(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{4x - 4} + \frac{3}{4(1 + x)} + (-)(0) \\ &= \frac{1}{4x - 4} + \frac{3}{4(1 + x)} \\ &= \frac{2x - 1}{2x^2 - 2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x-4} + \frac{3}{4(1+x)}\right)(0) + \left(\left(-\frac{1}{4(x-1)^2} - \frac{3}{4(1+x)^2}\right) + \left(\frac{1}{4x-4} + \frac{3}{4(1+x)}\right)^2 - \left(-\frac{3}{4(x^2}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{4x-4} + \frac{3}{4(1+x)}\right) dx} \\ &= (x-1)^{\frac{1}{4}} (1+x)^{\frac{3}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{-x^2+1}\, dx} \\ &= z_1 e^{-\frac{\ln(x-1)}{4} - \frac{\ln(1+x)}{4}} \\ &= z_1 \left(\frac{1}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{1+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{-x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\sqrt{x-1}}{\sqrt{1+x}}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{1+x}\right) + c_2\left(\sqrt{1+x}\left(\frac{\sqrt{x-1}}{\sqrt{1+x}}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
\left(-x^2+1\right) y'' - xy' + \frac{y}{4} = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\sqrt{1+x} + c_2\sqrt{x-1}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sqrt{1+x} \\
y_2 &= \sqrt{x-1}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{1+x} & \sqrt{x-1} \\ \frac{d}{dx}\left(\sqrt{1+x}\right) & \frac{d}{dx}\left(\sqrt{x-1}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{1+x} & \sqrt{x-1} \\ \frac{1}{2\sqrt{1+x}} & \frac{1}{2\sqrt{x-1}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{1+x}\right)\left(\frac{1}{2\sqrt{x-1}}\right) - \left(\sqrt{x-1}\right)\left(\frac{1}{2\sqrt{1+x}}\right)$$

Which simplifies to

$$W = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$$

Which simplifies to

$$W = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sqrt{x-1}\left(-\frac{x^2}{2}+\frac{1}{2}\right)}{\frac{-x^2+1}{\sqrt{x-1}\,\sqrt{1+x}}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{1+x}\,(x-1)}{2} dx$$

Hence

$$u_1 = -\frac{(1+x)^{\frac{3}{2}}(3x-7)}{15}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{1+x}\left(-\frac{x^2}{2}+\frac{1}{2}\right)}{\frac{-x^2+1}{\sqrt{x-1}\,\sqrt{1+x}}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{x-1}\,(1+x)}{2}dx$$

Hence

$$u_2 = \frac{(x-1)^{\frac{3}{2}}\,(3x+7)}{15}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{(1+x)^2\,(3x-7)}{15} + \frac{(x-1)^2\,(3x+7)}{15}$$

Which simplifies to

$$y_p(x) = \frac{2x^2}{15} + \frac{14}{15}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\sqrt{1+x} + c_2\sqrt{x-1}\right) + \left(\frac{2x^2}{15} + \frac{14}{15}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{1+x} + c_2\sqrt{x-1} + \frac{2x^2}{15} + \frac{14}{15} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{1+x} + c_2\sqrt{x-1} + \frac{2x^2}{15} + \frac{14}{15}$$

Verified OK.

5.10.4 Maple step by step solution

Let's solve

$\left(-x^2+1\right) y''-x y'+\frac{y}{4}=-\frac{x^2}{2}+\frac{1}{2}$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=\frac{y}{4(x^2-1)}-\frac{2x y'-x^2+1}{2(x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{x y'}{x^2-1}-\frac{y}{4(x^2-1)}=\frac{1}{2}$

- Multiply by denominators of ODE

$\left(-x^2+1\right) y''-x y'+\frac{y}{4}=0$

- Make a change of variables

$\theta=\arccos\left(x\right)$

- Calculate y' with change of variables

$y'=\left(\frac{d}{d\theta}y(\theta)\right)\theta'(x)$

- Compute 1st derivative y'

$y'=-\frac{\frac{d}{d\theta}y(\theta)}{\sqrt{-x^2+1}}$

- Calculate y'' with change of variables

$y''=\left(\frac{d^2}{d\theta^2}y(\theta)\right)\theta'(x)^2+\theta''(x)\left(\frac{d}{d\theta}y(\theta)\right)$

- Compute 2nd derivative y''

$y''=\frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1}-\frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}$

- Apply the change of variables to the ODE

$\left(-x^2+1\right)\left(\frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1}-\frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}\right)+\frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{\sqrt{-x^2+1}}+\frac{y}{4}=0$

- Multiply through

$-\frac{\left(\frac{d^2}{d\theta^2}y(\theta)\right)x^2}{-x^2+1}+\frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1}+\frac{x^3\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}-\frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}+\frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{\sqrt{-x^2+1}}+\frac{y}{4}=0$

- Simplify ODE

$\frac{y}{4} + \frac{d^2}{d\theta^2} y(\theta) = 0$

- ODE is that of a harmonic oscillator with given general solution

  $y(\theta) = c_1 \sin\left(\frac{\theta}{2}\right) + c_2 \cos\left(\frac{\theta}{2}\right)$

- Revert back to x

  $y = c_1 \sin\left(\frac{\arccos(x)}{2}\right) + c_2 \cos\left(\frac{\arccos(x)}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve((1-x^2)*diff(y(x),x$2)-x*diff(y(x),x)+1/4*y(x)=1/2*(1-x^2),y(x), singsol=all)
```

$$y(x) = \frac{2(x^2+7)\sqrt{x+\sqrt{x^2-1}} + 15c_1x + 15c_1\sqrt{x^2-1} + 15c_2}{15\sqrt{x+\sqrt{x^2-1}}}$$

✓ Solution by Mathematica

Time used: 19.346 (sec). Leaf size: 307

```
DSolve[(1-x^2)*y''[x]-x*y'[x]+1/4*y[x]==1/2*(1-x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) \to & \cosh\left(\frac{\sqrt{1-x^2}\arctan\left(\frac{\sqrt{1-x^2}}{x+1}\right)}{\sqrt{x^2-1}}\right)\int_1^x \sqrt{K[1]^2-1}\sinh\left(\frac{\arctan\left(\frac{\sqrt{1-K[1]^2}}{K[1]+1}\right)\sqrt{1-K[1]^2}}{\sqrt{K[1]^2-1}}\right)dK[1] \\
& - i\sinh\left(\frac{\sqrt{1-x^2}\arctan\left(\frac{\sqrt{1-x^2}}{x+1}\right)}{\sqrt{x^2-1}}\right)\int_1^x \\
& -i\cosh\left(\frac{\arctan\left(\frac{\sqrt{1-K[2]^2}}{K[2]+1}\right)\sqrt{1-K[2]^2}}{\sqrt{K[2]^2-1}}\right)\sqrt{K[2]^2-1}dK[2] \\
& + c_1\cosh\left(\frac{\sqrt{1-x^2}\arctan\left(\frac{\sqrt{1-x^2}}{x+1}\right)}{\sqrt{x^2-1}}\right) - ic_2\sinh\left(\frac{\sqrt{1-x^2}\arctan\left(\frac{\sqrt{1-x^2}}{x+1}\right)}{\sqrt{x^2-1}}\right)
\end{aligned}
$$

5.11 problem 11

Internal problem ID [5832]
Internal file name [OUTPUT/5080_Sunday_June_05_2022_03_20_51_PM_73272133/index.tex]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.3.4 problems. page 104
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{(\cos(x)+\sin(x))\,y''-2\cos(x)\,y'+(\cos(x)-\sin(x))\,y=(\cos(x)+\sin(x))^2\,\mathrm{e}^{2x}}$$

5.11.1 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay''+By'+Cy=F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y=Bv$$

This results in

$$\begin{aligned}
y' &= B'v+v'B \\
y'' &= B''v+B'v'+v''B+v'B' \\
&= v''B+2v'+B'+B''v
\end{aligned}$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= \cos(x) + \sin(x) \\
B &= -2\cos(x) \\
C &= \cos(x) - \sin(x) \\
F &= e^{2x}(1 + \sin(2x))
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (\cos(x) + \sin(x))\,(2\cos(x)) + (-2\cos(x))\,(2\sin(x)) + (\cos(x) - \sin(x))\,(-2\cos(\\
&= 2(\cos(x) + \sin(x))\cos(x) - 4\cos(x)\sin(x) - 2\cos(x)\,(\cos(x) - \sin(x)) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-\cos(2x) - 1 - \sin(2x)\, v'' + (4 + 2\sin(2x))\, v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
-(\cos(2x) + \sin(2x) + 1)\, u'(x) + 2(2 + \sin(2x))\, u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= \frac{2(2 + \sin(2x))\, u}{\cos(2x) + \sin(2x) + 1}
\end{aligned}
$$

Where $f(x) = \frac{4+2\sin(2x)}{\cos(2x)+\sin(2x)+1}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{4+2\sin(2x)}{\cos(2x)+\sin(2x)+1}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{4+2\sin(2x)}{\cos(2x)+\sin(2x)+1}\,dx \\
\ln(u) &= \frac{\ln(1+\tan(x)^2)}{2} + \ln(\tan(x)+1) + x + c_1 \\
u &= e^{\frac{\ln(1+\tan(x)^2)}{2}+\ln(\tan(x)+1)+x+c_1} \\
&= c_1 e^{\frac{\ln(1+\tan(x)^2)}{2}+\ln(\tan(x)+1)+x}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= c_1 e^{\frac{\ln(1+\tan(x)^2)}{2}+\ln(\tan(x)+1)+x}
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int c_1 e^{\frac{\ln(1+\tan(x)^2)}{2}+\ln(\tan(x)+1)+x}\,dx \\
&= c_1 e^{\frac{\ln(1+\tan(x)^2)}{2}+x} + c_2
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= Bv \\
&= (-2\cos(x))\left(c_1 e^{\frac{\ln(1+\tan(x)^2)}{2}+x} + c_2\right) \\
&= -2c_1 \operatorname{csgn}(\sec(x))\, e^{x} - 2c_2\cos(x)
\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \operatorname{csgn}(\sec(x))\, e^{x} \\
y_2 &= \cos(x)
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \operatorname{csgn}(\sec(x))\, e^{x} & \cos(x) \\ \frac{d}{dx}(\operatorname{csgn}(\sec(x))\, e^{x}) & \frac{d}{dx}(\cos(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \operatorname{csgn}(\sec(x))\, e^{x} & \cos(x) \\ \operatorname{csgn}(1, \sec(x)) \sec(x) \tan(x)\, e^{x} + \operatorname{csgn}(\sec(x))\, e^{x} & -\sin(x) \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & (\operatorname{csgn}(\sec(x))\, e^{x})(-\sin(x)) \\ & - (\cos(x))(\operatorname{csgn}(1, \sec(x)) \sec(x) \tan(x)\, e^{x} + \operatorname{csgn}(\sec(x))\, e^{x}) \end{aligned}$$

Which simplifies to

$$\begin{aligned} W = {} & -\sec(x) \cos(x) \tan(x)\, e^{x} \operatorname{csgn}(1, \sec(x)) \\ & - \cos(x) \operatorname{csgn}(\sec(x))\, e^{x} - \operatorname{csgn}(\sec(x))\, e^{x} \sin(x) \end{aligned}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\cos(x)\, e^{2x}(1 + \sin(2x))}{(\cos(x) + \sin(x))(-\sec(x) \cos(x) \tan(x)\, e^{x} \operatorname{csgn}(1, \sec(x)) - \cos(x) \operatorname{csgn}(\sec(x))\, e^{x} - \operatorname{csgn}(\text{se}}$$

Hence

$$u_1 = -\left(\int_0^x \frac{\cos(\alpha)\, e^{2\alpha}(1 + \sin(2\alpha))}{(\cos(\alpha) + \sin(\alpha))(-\sec(\alpha) \cos(\alpha) \tan(\alpha)\, e^{\alpha} \operatorname{csgn}(1, \sec(\alpha)) - \cos(\alpha) \operatorname{csgn}(\sec(\alpha))\, e^{\alpha} - \text{csgn}}\right.$$

And Eq. (3) becomes

$$u_2 = \int \frac{\operatorname{csgn}(\sec(x))\,e^{x}e^{2x}(1+\sin(2x))}{(\cos(x)+\sin(x))\,(-\sec(x)\cos(x)\tan(x)\,e^{x}\operatorname{csgn}(1,\sec(x)) - \cos(x)\operatorname{csgn}(\sec(x))\,e^{x} - \operatorname{csgn}(\sec(x$$

Hence

$$u_2 = \int_0^x \frac{\operatorname{csgn}(\sec(\alpha))\,e^{\alpha}e^{2\alpha}(1+\sin(2\alpha))}{(\cos(\alpha)+\sin(\alpha))\,(-\sec(\alpha)\cos(\alpha)\tan(\alpha)\,e^{\alpha}\operatorname{csgn}(1,\sec(\alpha)) - \cos(\alpha)\operatorname{csgn}(\sec(\alpha))\,e^{\alpha} - \operatorname{csgn}(\mathrm{se}$$

Which simplifies to

$$u_1 = \int_0^x \frac{e^{\alpha}(\cos(\alpha)+\sin(\alpha))\cos(\alpha)^2}{\sin(\alpha)\operatorname{csgn}(1,\sec(\alpha)) + \cos(\alpha)\operatorname{csgn}(\sec(\alpha))\,(\cos(\alpha)+\sin(\alpha))}d\alpha$$

$$u_2 = -\left(\int_0^x \frac{\operatorname{csgn}(\sec(\alpha))\cos(\alpha)\,(\cos(\alpha)+\sin(\alpha))\,e^{2\alpha}}{\sin(\alpha)\operatorname{csgn}(1,\sec(\alpha)) + (\cos(\alpha)+\sin(\alpha))\operatorname{csgn}(\sec(\alpha))\cos(\alpha)}d\alpha\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\int_0^x \frac{e^{\alpha}(\cos(\alpha)+\sin(\alpha))\cos(\alpha)^2}{\sin(\alpha)\operatorname{csgn}(1,\sec(\alpha)) + \cos(\alpha)\operatorname{csgn}(\sec(\alpha))\,(\cos(\alpha)+\sin(\alpha))}d\alpha\right)\operatorname{csgn}(\sec(x))\,e^{x} - \left(\int_0^x \frac{\operatorname{csgn}(\sec(\alpha))\cos(\alpha)\,(\cos(\alpha)+\sin(\alpha))\,e^{2\alpha}}{\sin(\alpha)\operatorname{csgn}(1,\sec(\alpha)) + (\cos(\alpha)+\sin(\alpha))\operatorname{csgn}(\sec(\alpha))\cos(\alpha)}d\alpha\right)\cos(x)$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (-2c_1\operatorname{csgn}(\sec(x))\,e^{x} - 2c_2\cos(x)) + \left(\left(\int_0^x \frac{e^{\alpha}(\cos(\alpha)+\sin(\alpha))\cos(\alpha)^2}{\sin(\alpha)\operatorname{csgn}(1,\sec(\alpha)) + \cos(\alpha)\operatorname{csgn}(\sec(\alpha))\,(\cos\right.\right. \\ &= -2c_1\operatorname{csgn}(\sec(x))\,e^{x} - 2c_2\cos(x) + \left(\int_0^x \frac{e^{\alpha}(\cos(\alpha)+\sin(\alpha))\cos(\alpha)^2}{\sin(\alpha)\operatorname{csgn}(1,\sec(\alpha)) + \cos(\alpha)\operatorname{csgn}(\sec(\alpha))\,(\cos(\alpha)\right. \end{aligned}$$

Simplifying the solution $y = -2c_1\operatorname{csgn}(\sec(x))\,e^{x} - 2c_2\cos(x) + \left(\int_0^x \frac{e^{\alpha}(\cos(\alpha)+\sin(\alpha))\cos(\alpha)^2}{\sin(\alpha)\operatorname{csgn}(1,\sec(\alpha))+\cos(\alpha)\operatorname{csgn}(\sec(\alpha))(\cos(\alpha)+\right.$ $\left(\int_0^x \frac{\operatorname{csgn}(\sec(\alpha))\cos(\alpha)(\cos(\alpha)+\sin(\alpha))e^{2\alpha}}{\sin(\alpha)\operatorname{csgn}(1,\sec(\alpha))+(\cos(\alpha)+\sin(\alpha))\operatorname{csgn}(\sec(\alpha))\cos(\alpha)}d\alpha\right)\cos(x)$ to $y = -2c_1 e^{x} - 2c_2\cos(x) +$

Summary

The solution(s) found are the

$$\left(\int_0^x \frac{e^{\alpha}(\cos(\alpha)+\sin(\alpha))\cos(\alpha)^2}{\sin(\alpha)+\cos(\alpha)(\cos(\alpha)+\sin(\alpha))}d\alpha\right)e^{x}-\left(\int_0^x \frac{\cos(\alpha)(\cos(\alpha)+\sin(\alpha))e^{2\alpha}}{\sin(\alpha)+(\cos(\alpha)+\sin(\alpha))\cos(\alpha)}d\alpha\right)\cos(x)$$

$$y = -2c_1 e^{x} - 2c_2 \cos(x$$

$$-\left(\int_0^x \frac{\cos(\alpha)\,(}{\sin(\alpha)+(c}\right.$$

Verification of solutions

$$y = -2c_1 e^{x} - 2c_2 \cos(x) + \left(\int_0^x \frac{e^{\alpha}(\cos(\alpha)+\sin(\alpha))\cos(\alpha)^2}{\sin(\alpha)+\cos(\alpha)(\cos(\alpha)+\sin(\alpha))}d\alpha\right)e^{x} - \left(\int_0^x \frac{\cos(\alpha)(\cos(\alpha)+\sin(\alpha))e^{2\alpha}}{\sin(\alpha)+(\cos(\alpha)+\sin(\alpha))\cos(\alpha)}d\alpha\right)\cos(x)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         <- linear symmetries successful
      Change of variables used:
         [x = arccos(t)]
      Linear ODE actually solved:
         (t-(-t^2+1)^(1/2))*u(t)+((-t^2+1)^(1/2)*t-t^2)*diff(u(t),t)+(-(-t^2+1)^(1/2)*t^2-t^
   <- change of variables successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.422 (sec). Leaf size: 322

```
dsolve((cos(x)+sin(x))*diff(y(x),x$2)-2*cos(x)*diff(y(x),x)+(cos(x)-sin(x))*y(x)=(cos(x)+sin
```

$$y(x) = -\cos(x)\left(\left(\left(\int e^{\int \frac{(-\cot(x)+1)\cos(x)+2\sin(x)(\tan(x)+1)}{\cos(x)+\sin(x)}dx}\sin(x)\,dx\right)c_1 - \left(\int e^{2x-2\left(\int \frac{\sin(x)}{\cos(x)+\sin(x)}dx\right)-2\left(\int \frac{\sin(x)\tan(x)}{\cos(x)+\sin(x)}dx\right)+\int \frac{\cos(x)\cot(x)}{\cos(x)+\sin(x)}dx-\left(\int \frac{\cos(x)}{\cos(x)+\sin(x)}dx\right)}(\csc(x)+\sec(x))\,dx\right)\left(\int e^{2(\cdots} + \int e^{2x-2\left(\int \frac{\sin(x)}{\cos(x)+\sin(x)}dx\right)-2\left(\int \frac{\sin(x)\tan(x)}{\cos(x)+\sin(x)}dx\right)+\int \frac{\cos(x)\cot(x)}{\cos(x)+\sin(x)}dx-\left(\int \frac{\cos(x)}{\cos(x)+\sin(x)}dx\right)}(\csc(x)+\sec(x))\left(\int e^{2\left(\int \frac{\sin(x)}{\cos(x)+\sin(x)}dx\right)+2\left(\int \frac{\sin(x)\tan(x)}{\cos(x)+\sin(x)}dx\right)-\left(\int \frac{\cos(x)\cot(x)}{\cos(x)+\sin(x)}dx\right)+\int \frac{\cos(x)}{\cos(x)+\sin(x)}dx}\sin(x)\,dx\right)dx - c_2\right)$$

✓ Solution by Mathematica

Time used: 4.817 (sec). Leaf size: 476

```
DSolve[(Cos[x]+Sin[x])*y''[x]-2*Cos[x]*y'[x]+(Cos[x]-Sin[x])*y[x]==(Cos[x]+Sin[x])^2*Exp[2*x
```

$$y(x) \to \frac{\left(\frac{1}{4}+\frac{i}{4}\right)\left(e^{-2ix}\right)^{\frac{1}{2}-\frac{i}{2}}\left(e^{ix}\right)^{1-2i}\left(-\frac{i\left(-1+e^{2i\arctan\left(e^{-2ix}\right)}\right)}{1+e^{2i\arctan\left(e^{-2ix}\right)}}\right)^{-\frac{1}{2}-\frac{i}{2}}\left(-i\left(e^{-2ix}\right)^{i}\sqrt{1+e^{-4ix}}\sqrt{1+e^{4ix}}e^{2i(2x+\arctan\cdots}\right.}{\sqrt{-e^{4ix}}\sqrt{-(1+\cdots}} + \frac{c_2e^{3ix}\left(e^{-2ix}\right)^{\frac{1}{2}+\frac{i}{2}}\sqrt{1+e^{-4ix}}\left(e^{2i\arctan\left(e^{-2ix}\right)}+i\right)\left(-\frac{i\left(-1+e^{2i\arctan\left(e^{-2ix}\right)}\right)}{1+e^{2i\arctan\left(e^{-2ix}\right)}}\right)^{\frac{1}{2}-\frac{i}{2}}}{\sqrt{1+e^{4ix}}\left(-1+e^{2i\arctan\left(e^{-2ix}\right)}\right)} + c_1\left(e^{ix}\right)^{-i}$$

5.12 problem 12

Internal problem ID [5833]
Internal file name [`OUTPUT/5081_Sunday_June_05_2022_03_21_06_PM_68043547/index.tex`]

Book: Ordinary differential equations and calculus of variations. Makarets and Reshetnyak. Wold Scientific. Singapore. 1995
Section: Chapter 2. Linear homogeneous equations. Section 2.3.4 problems. page 104
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$(\cos(x)-\sin(x))\,y''-2\sin(x)\,y'+(\cos(x)+\sin(x))\,y=(\cos(x)-\sin(x))^2$$

✗ Solution by Maple

```
dsolve((cos(x)-sin(x))*diff(y(x),x$2)-2*sin(x)*diff(y(x),x)+(cos(x)+sin(x))*y(x)=(cos(x)-sin
```

No solution found

✓ Solution by Mathematica
Time used: 15.918 (sec). Leaf size: 7186

```
DSolve[(Cos[x]-Sin[x])*y''[x]-2*Sin[x]*y'[x]+(Cos[x]+Sin[x])*y[x]==(Cos[x]-Sin[x])^2,y[x],x,
```

Too large to display