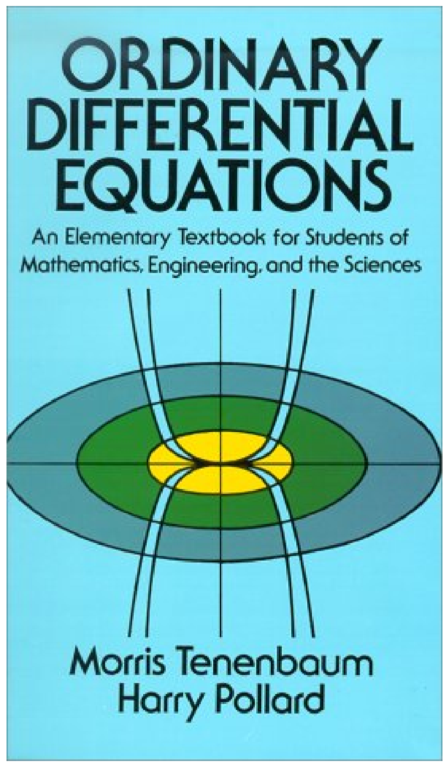
A Solution Manual For

Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963



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1 Chapter 2. Special types of differential equations of the first kind. Lesson 7

1.1 problem First order with homogeneous Coefficients. Exercise 7.2, page 61

Internal problem ID [4427]
Internal file name [`OUTPUT/3920_Sunday_June_05_2022_11_49_18_AM_38203277/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 7
Problem number: First order with homogeneous Coefficients. Exercise 7.2, page 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{2xy + \left(x^2 + y^2\right) y' = 0}$$

1.1.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2x^2u(x) + \left(x^2 + u(x)^2\, x^2\right)\left(u'(x)\, x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(u^2+3)}{x\,(u^2+1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u^2+3)}{u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u^2+3)}{u^2+1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u(u^2+3)}{u^2+1}}\, du = \int -\frac{1}{x}\, dx$$

$$\frac{\ln\left(u(u^2+3)\right)}{3} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$\left(u(u^2+3)\right)^{\frac{1}{3}} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\left(u(u^2+3)\right)^{\frac{1}{3}} = \frac{c_3}{x}$$

Which simplifies to

$$\left(u(x)\left(u(x)^2+3\right)\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\left(u(x)\left(u(x)^2+3\right)\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{y\left(\frac{y^2}{x^2}+3\right)}{x}\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

$$\left(\frac{y(y^2+3x^2)}{x^3}\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

<u>Summary</u>

The solution(s) found are the following

$$\left(\frac{y(y^2+3x^2)}{x^3}\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{x} \tag{1}$$

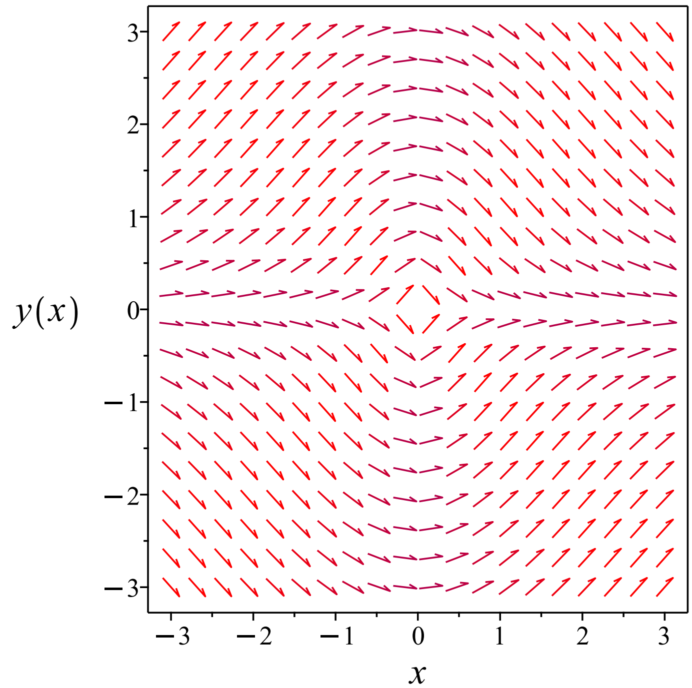


Figure 1: Slope field plot

Verification of solutions

$$\left(\frac{y(y^2+3x^2)}{x^3}\right)^{\frac{1}{3}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

1.1.2 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{2xy}{x^2+y^2} \tag{1}$$

Which becomes

$$(y^2)\,dy = (-x^2)\,dy + (-2xy)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x^2)\,dy + (-2xy)\,dx = d(-y\,x^2)$$

Hence (2) becomes

$$(y^2)\,dy = d(-y\,x^2)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(12c_1 + 4\sqrt{4x^6 + 9c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{2x^2}{\left(12c_1 + 4\sqrt{4x^6 + 9c_1^2}\right)^{\frac{1}{3}}} + c_1$$

$$y = -\frac{\left(12c_1 + 4\sqrt{4x^6 + 9c_1^2}\right)^{\frac{1}{3}}}{4} + \frac{x^2}{\left(12c_1 + 4\sqrt{4x^6 + 9c_1^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}{2} + \frac{2x^2}{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)}\right.}{2}$$

$$y = -\frac{\left(12c_1 + 4\sqrt{4x^6 + 9c_1^2}\right)^{\frac{1}{3}}}{4} + \frac{x^2}{\left(12c_1 + 4\sqrt{4x^6 + 9c_1^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}{2} + \frac{2x^2}{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)}\right.}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(12c_1 + 4\sqrt{4x^6 + 9c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{2x^2}{\left(12c_1 + 4\sqrt{4x^6 + 9c_1^2}\right)^{\frac{1}{3}}} + c_1 \tag{1}$$

$$y = -\frac{\left(12c_1 + 4\sqrt{4x^6 + 9c_1^2}\right)^{\frac{1}{3}}}{4} + \frac{x^2}{\left(12c_1 + 4\sqrt{4x^6 + 9c_1^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}{2} + \frac{2x^2}{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1 \tag{2}$$

$$y = -\frac{\left(12c_1 + 4\sqrt{4x^6 + 9c_1^2}\right)^{\frac{1}{3}}}{4} + \frac{x^2}{\left(12c_1 + 4\sqrt{4x^6 + 9c_1^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}{2} + \frac{2x^2}{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1 \tag{3}$$

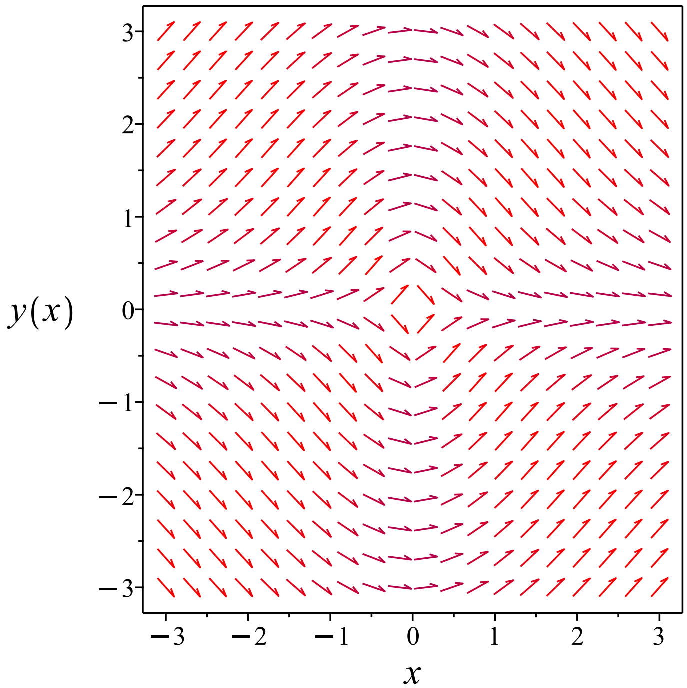


Figure 2: Slope field plot

Verification of solutions

$$y = \frac{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{2x^2}{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}} + c_1$$

Verified OK.

$$y = -\frac{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}{4} + \frac{x^2}{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}{2} + \frac{2x^2}{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1$$

Verified OK.

$$y = -\frac{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}{4} + \frac{x^2}{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}{2} + \frac{2x^2}{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1$$

Verified OK.

1.1.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2xy}{x^2+y^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{2xy(b_3 - a_2)}{x^2+y^2} - \frac{4x^2y^2a_3}{(x^2+y^2)^2} - \left(-\frac{2y}{x^2+y^2} + \frac{4x^2y}{(x^2+y^2)^2}\right)(xa_2 + ya_3 + a_1) \\ & - \left(-\frac{2x}{x^2+y^2} + \frac{4x\,y^2}{(x^2+y^2)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^4b_2 - 6x^2y^2a_3 + 4x\,y^3a_2 - 4x\,y^3b_3 + 2y^4a_3 + y^4b_2 + 2x^3b_1 - 2x^2ya_1 - 2x\,y^2b_1 + 2y^3a_1}{(x^2+y^2)^2} = 0$$

Setting the numerator to zero gives

$$3x^4b_2 - 6x^2y^2a_3 + 4x\,y^3a_2 - 4x\,y^3b_3 + 2y^4a_3 + y^4b_2 + 2x^3b_1 - 2x^2ya_1 - 2x\,y^2b_1 + 2y^3a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 4a_2v_1v_2^3 - 6a_3v_1^2v_2^2 + 2a_3v_2^4 + 3b_2v_1^4 + b_2v_2^4 - 4b_3v_1v_2^3 \\ & - 2a_1v_1^2v_2 + 2a_1v_2^3 + 2b_1v_1^3 - 2b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$3b_2v_1^4 + 2b_1v_1^3 - 6a_3v_1^2v_2^2 - 2a_1v_1^2v_2 + (4a_2 - 4b_3)\,v_1v_2^3 - 2b_1v_1v_2^2 + (2a_3 + b_2)\,v_2^4 + 2a_1v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_1 &= 0 \\ 2a_1 &= 0 \\ -6a_3 &= 0 \\ -2b_1 &= 0 \\ 2b_1 &= 0 \\ 3b_2 &= 0 \\ 4a_2 - 4b_3 &= 0 \\ 2a_3 + b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{2xy}{x^2+y^2}\right)(x) \\
&= \frac{3y\,x^2+y^3}{x^2+y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3y\,x^2+y^3}{x^2+y^2}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln\left(y(3x^2+y^2)\right)}{3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{2xy}{x^2+y^2}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2x}{3x^2 + y^2} \\
S_y &= \frac{x^2 + y^2}{3y\,x^2 + y^3}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

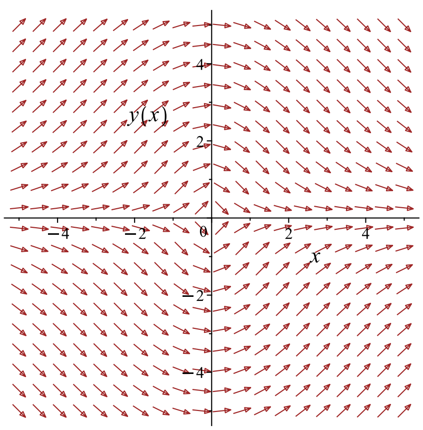
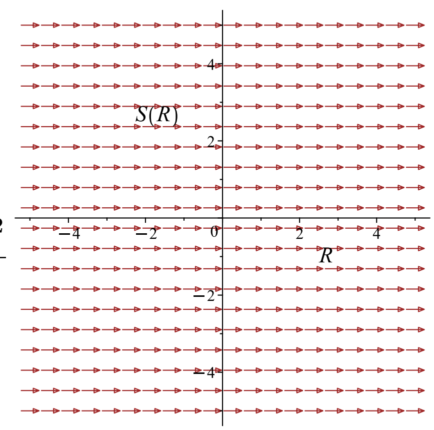
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{3} + \frac{\ln(y^2 + 3x^2)}{3} = c_1$$

Which simplifies to

$$\frac{\ln(y)}{3} + \frac{\ln(y^2 + 3x^2)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2xy}{x^2+y^2}$ 	$R = x$ $S = \frac{\ln(y)}{3} + \frac{\ln(3x^2 + y^2)}{3}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{3} + \frac{\ln(y^2 + 3x^2)}{3} = c_1 \tag{1}$$

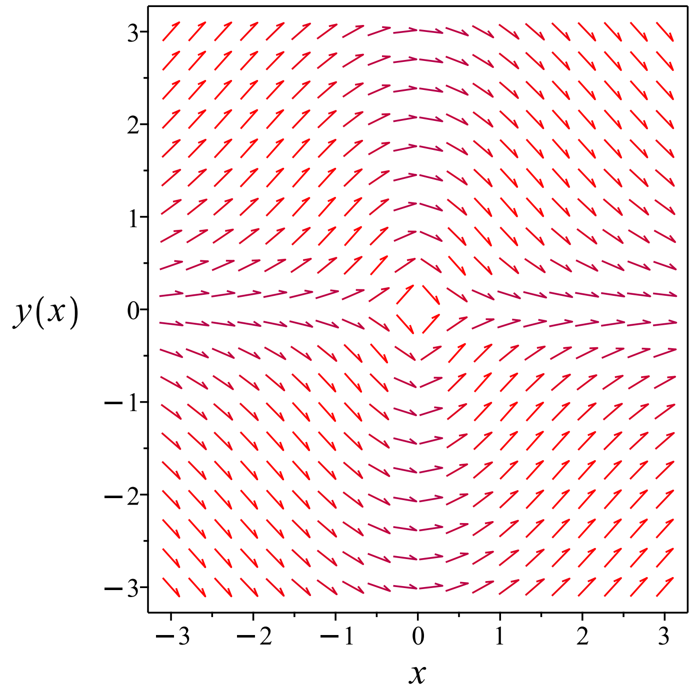


Figure 3: Slope field plot

Verification of solutions

$$\frac{\ln(y)}{3} + \frac{\ln(y^2 + 3x^2)}{3} = c_1$$

Verified OK.

1.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x^2 + y^2\right) \mathrm{d}y = (-2xy)\,\mathrm{d}x$$
$$(2xy)\,\mathrm{d}x + \left(x^2 + y^2\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = 2xy$$
$$N(x,y) = x^2 + y^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy)$$
$$= 2x$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2\right)$$
$$= 2x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2xy \, \mathrm{d}x$$

$$\phi = y x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 + y^2$. Therefore equation (4) becomes

$$x^2 + y^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y^2) \, \mathrm{d}y$$

$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y x^2 + \frac{1}{3} y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^2 + \frac{1}{3}y^3$$

Summary

The solution(s) found are the following

$$yx^2 + \frac{y^3}{3} = c_1 \tag{1}$$

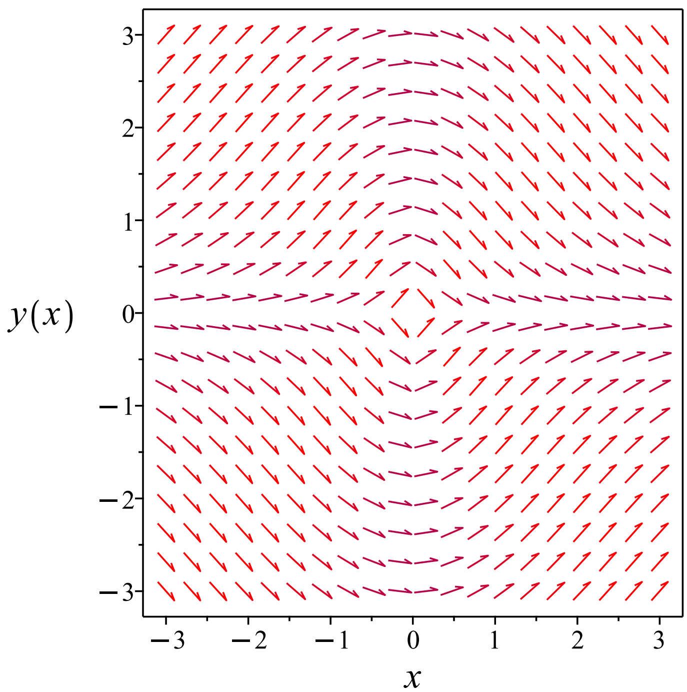


Figure 4: Slope field plot

Verification of solutions

$$yx^2 + \frac{y^3}{3} = c_1$$

Verified OK.

1.1.5 Maple step by step solution

Let's solve

$2xy + (x^2 + y^2)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $2x = 2x$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int 2xy dx + f_1(y)$

- Evaluate integral

  $F(x, y) = y\, x^2 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $x^2 + y^2 = x^2 + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = y^2$

- Solve for $f_1(y)$

  $f_1(y) = \frac{y^3}{3}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = y\,x^2 + \frac{1}{3}y^3$$

- Substitute $F(x,y)$ into the solution of the ODE

$$y\,x^2 + \frac{1}{3}y^3 = c_1$$

- Solve for y

$$\left\{y = \frac{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{2x^2}{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}, y = -\frac{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}{4} + \frac{x^2}{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}} - \frac{I\sqrt{3}\cdots}{\cdots}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 209

```
dsolve(2*x*y(x)+(x^2+y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{2\left(c_1 x^2 - \frac{\left(4+4\sqrt{4c_1^3 x^6+1}\right)^{\frac{2}{3}}}{4}\right)}{\left(4+4\sqrt{4c_1^3 x^6+1}\right)^{\frac{1}{3}}\sqrt{c_1}}$$

$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left(4+4\sqrt{4c_1^3 x^6+1}\right)^{\frac{1}{3}}}{4\sqrt{c_1}} - \frac{\sqrt{c_1}\left(i\sqrt{3}-1\right)x^2}{\left(4+4\sqrt{4c_1^3 x^6+1}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{4i\sqrt{3}\,c_1 x^2 + i\left(4+4\sqrt{4c_1^3 x^6+1}\right)^{\frac{2}{3}}\sqrt{3} + 4c_1 x^2 - \left(4+4\sqrt{4c_1^3 x^6+1}\right)^{\frac{2}{3}}}{4\left(4+4\sqrt{4c_1^3 x^6+1}\right)^{\frac{1}{3}}\sqrt{c_1}}$$

✓ Solution by Mathematica

Time used: 15.191 (sec). Leaf size: 401

```
DSolve[2*x*y[x]+(x^2+y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{\sqrt{4x^6+e^{6c_1}}+e^{3c_1}}}{\sqrt[3]{2}} - \frac{\sqrt[3]{2}x^2}{\sqrt[3]{\sqrt{4x^6+e^{6c_1}}+e^{3c_1}}}$$

$$y(x) \to \frac{i2^{2/3}\left(\sqrt{3}+i\right)\left(\sqrt{4x^6+e^{6c_1}}+e^{3c_1}\right)^{2/3}+\sqrt[3]{2}\left(2+2i\sqrt{3}\right)x^2}{4\sqrt[3]{\sqrt{4x^6+e^{6c_1}}+e^{3c_1}}}$$

$$y(x) \to \frac{\left(1-i\sqrt{3}\right)x^2}{2^{2/3}\sqrt[3]{\sqrt{4x^6+e^{6c_1}}+e^{3c_1}}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{\sqrt{4x^6+e^{6c_1}}+e^{3c_1}}}{2\sqrt[3]{2}}$$

$$y(x) \to 0$$

$$y(x) \to \frac{1}{2}\sqrt[6]{x^6}\left(\frac{\left(1-i\sqrt{3}\right)\left(x^6\right)^{2/3}}{x^4} - i\sqrt{3}-1\right)$$

$$y(x) \to \frac{1}{2}\sqrt[6]{x^6}\left(\frac{\left(1+i\sqrt{3}\right)\left(x^6\right)^{2/3}}{x^4} + i\sqrt{3}-1\right)$$

$$y(x) \to \sqrt[6]{x^6} - \frac{\left(x^6\right)^{5/6}}{x^4}$$

1.2 problem First order with homogeneous Coefficients. Exercise 7.3, page 61

Internal problem ID [4428]
Internal file name [`OUTPUT/3921_Sunday_June_05_2022_11_49_30_AM_82457950/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 7
Problem number: First order with homogeneous Coefficients. Exercise 7.3, page 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{\left(x+\sqrt{y^2-xy}\right)y'-y=0}$$

1.2.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y}{x+\sqrt{-xy+y^2}} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{y(b_3 - a_2)}{x + \sqrt{-xy + y^2}} - \frac{y^2 a_3}{\left(x + \sqrt{-xy + y^2}\right)^2} + \frac{y\left(1 - \frac{y}{2\sqrt{-xy+y^2}}\right)(x a_2 + y a_3 + a_1)}{\left(x + \sqrt{-xy + y^2}\right)^2} - \left(\frac{1}{x + \sqrt{-xy + y^2}} - \frac{y(-x + 2y)}{2\left(x + \sqrt{-xy + y^2}\right)^2 \sqrt{-xy + y^2}}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2(-xy + y^2)^{\frac{3}{2}} b_2 - 3x^2 y b_2 + x y^2 a_2 + 4x y^2 b_2 - x y^2 b_3 - 2y^3 a_2 - y^3 a_3 + 2y^3 b_3 - 2\sqrt{-xy + y^2}\, x b_1 + 2\sqrt{-x}}{2\left(x + \sqrt{-xy + y^2}\right)^2 \sqrt{-xy + y^2}} = 0$$

Setting the numerator to zero gives

$$2(-xy + y^2)^{\frac{3}{2}} b_2 - 3x^2 y b_2 + x y^2 a_2 + 4x y^2 b_2 - x y^2 b_3 - 2y^3 a_2 - y^3 a_3 + 2y^3 b_3 - 2\sqrt{-xy + y^2}\, x b_1 + 2\sqrt{-xy + y^2}\, y a_1 + x y b_1 - y^2 a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$2(-y(x - y))^{\frac{3}{2}} b_2 + 2(-xy + y^2)\, x b_2 - 2(-xy + y^2)\, y a_2 - x^2 y b_2 - x y^2 a_2 + 2x y^2 b_2 - x y^2 b_3 - y^3 a_3 + 2y^3 b_3 - 2(-xy + y^2)\, b_1 - 2\sqrt{-y(x - y)}\, x b_1 + 2\sqrt{-y(x - y)}\, y a_1 - x y b_1 - y^2 a_1 + 2y^2 b_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-3x^2 y b_2 - 2y\sqrt{-y(x - y)}\, b_2 x + x y^2 a_2 + 4x y^2 b_2 - x y^2 b_3 + 2y^2 \sqrt{-y(x - y)}\, b_2 - 2y^3 a_2 - y^3 a_3 + 2y^3 b_3 - 2\sqrt{-y(x - y)}\, x b_1 + x y b_1 + 2\sqrt{-y(x - y)}\, y a_1 - y^2 a_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-y\left(x-y\right)}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-y\left(x-y\right)} = v_3\right\}$$

The above PDE (6E) now becomes

$$v_1v_2^2a_2 - 2v_2^3a_2 - v_2^3a_3 - 3v_1^2v_2b_2 + 4v_1v_2^2b_2 - 2v_2v_3b_2v_1 + 2v_2^2v_3b_2 \\ - v_1v_2^2b_3 + 2v_2^3b_3 - v_2^2a_1 + 2v_3v_2a_1 + v_1v_2b_1 - 2v_3v_1b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$-3v_1^2v_2b_2 + (a_2 + 4b_2 - b_3)\, v_1v_2^2 - 2v_2v_3b_2v_1 + v_1v_2b_1 - 2v_3v_1b_1 \\ + (-2a_2 - a_3 + 2b_3)\, v_2^3 + 2v_2^2v_3b_2 - v_2^2a_1 + 2v_3v_2a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -a_1 &= 0 \\ 2a_1 &= 0 \\ -2b_1 &= 0 \\ -3b_2 &= 0 \\ -2b_2 &= 0 \\ 2b_2 &= 0 \\ -2a_2 - a_3 + 2b_3 &= 0 \\ a_2 + 4b_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{y}{x} \\
&= \frac{y}{x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x}
$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= \ln(x) \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{x + \sqrt{-xy + y^2}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{y}{x^2} \\ R_y &= \frac{1}{x} \\ S_x &= \frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\left(-x - \sqrt{-y(x-y)}\right)x}{\sqrt{-y(x-y)}\,y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{\sqrt{R-1}\,\sqrt{R}+1}{\sqrt{R-1}\,R^{\frac{3}{2}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\ln(R) - \frac{2\sqrt{R-1}}{\sqrt{R}} + c_1 \tag{4}$$

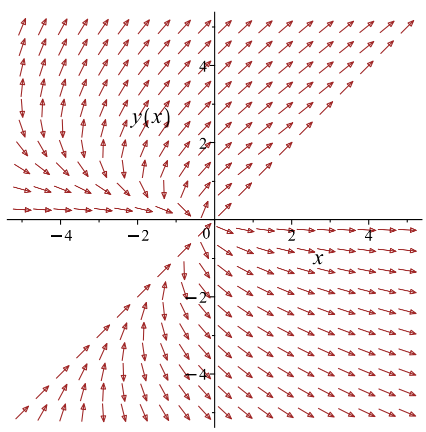
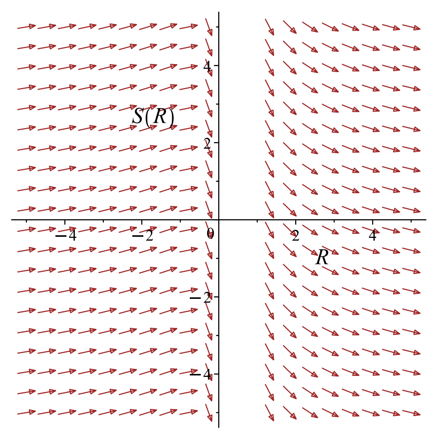
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = -\ln\left(\frac{y}{x}\right) - \frac{2\sqrt{\frac{y}{x}-1}}{\sqrt{\frac{y}{x}}} + c_1$$

Which simplifies to

$$\frac{2i\sqrt{x-y} + \ln(y)\sqrt{y} - c_1\sqrt{y}}{\sqrt{y}} = 0$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x+\sqrt{-xy+y^2}}$ 	$R = \frac{y}{x}$ $S = \ln(x)$	$\frac{dS}{dR} = -\frac{\sqrt{R-1}\sqrt{R}+1}{\sqrt{R-1}\,R^{\frac{3}{2}}}$ 

Summary

The solution(s) found are the following

$$\frac{2i\sqrt{x-y} + \ln(y)\sqrt{y} - c_1\sqrt{y}}{\sqrt{y}} = 0 \tag{1}$$

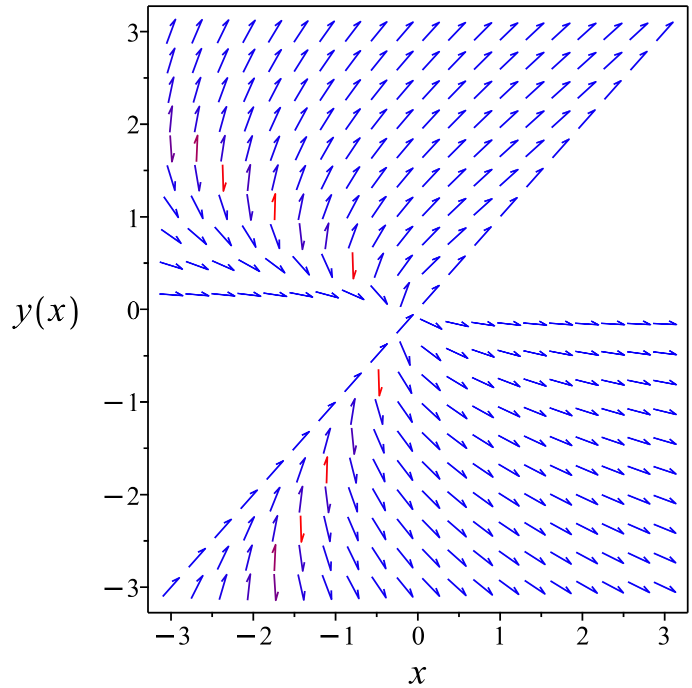


Figure 5: Slope field plot

Verification of solutions

$$\frac{2i\sqrt{x-y}+\ln\left(y\right)\sqrt{y}-c_1\sqrt{y}}{\sqrt{y}}=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve((x+sqrt(y(x)^2-x*y(x)))*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$\frac{\ln\left(y(x)\right)y(x)-c_1y(x)+2\sqrt{y\left(x\right)\left(y\left(x\right)-x\right)}}{y\left(x\right)}=0$$

✓ Solution by Mathematica

Time used: 0.291 (sec). Leaf size: 43

```
DSolve[(x+Sqrt[y[x]^2-x*y[x]])*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{2\sqrt{\frac{y(x)}{x}-1}}{\sqrt{\frac{y(x)}{x}}}+\log\left(\frac{y(x)}{x}\right)=-\log(x)+c_1,y(x)\right]$$

1.3 problem First order with homogeneous Coefficients. Exercise 7.4, page 61

Internal problem ID [4429]
Internal file name [`OUTPUT/3922_Sunday_June_05_2022_11_49_38_AM_95438003/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 7
Problem number: First order with homogeneous Coefficients. Exercise 7.4, page 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y-(x-y)\,y'=-x}$$

1.3.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x-(x-u(x)\,x)\,(u'(x)\,x+u(x))=-x$$

In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=-\frac{u^2+1}{x\,(u-1)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}}\,du = -\frac{1}{x}\,dx$$
$$\int \frac{1}{\frac{u^2+1}{u-1}}\,du = \int -\frac{1}{x}\,dx$$
$$\frac{\ln(u^2+1)}{2} - \arctan(u) = -\ln(x) + c_2$$

The solution is

$$\frac{\ln\left(u(x)^2+1\right)}{2} - \arctan(u(x)) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$
$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

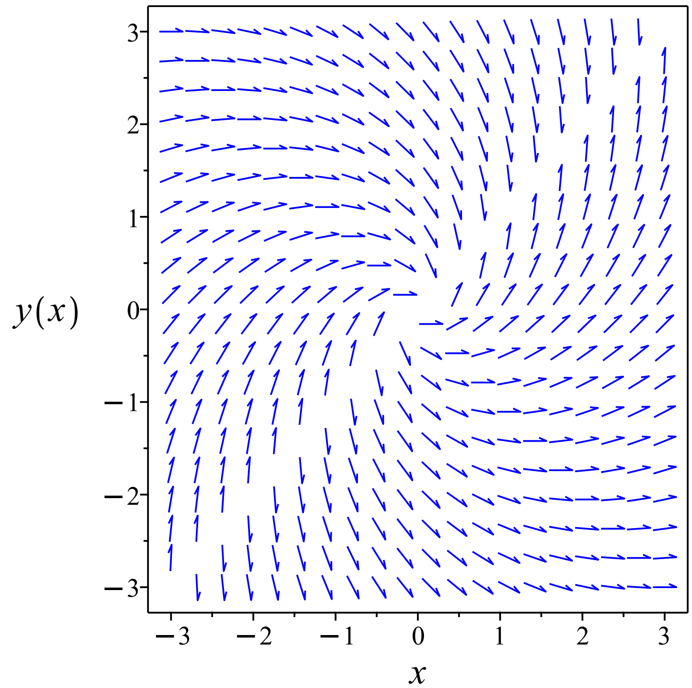


Figure 6: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2}-\arctan\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Verified OK.

1.3.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x+y}{-x+y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{(x+y)(b_3 - a_2)}{-x+y} - \frac{(x+y)^2 a_3}{(-x+y)^2} \\
&- \left(-\frac{1}{-x+y} - \frac{x+y}{(-x+y)^2}\right)(xa_2 + ya_3 + a_1) \\
&- \left(-\frac{1}{-x+y} + \frac{x+y}{(-x+y)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_2 + x^2a_3 + x^2b_2 - x^2b_3 - 2xya_2 + 2xya_3 + 2xyb_2 + 2xyb_3 - y^2a_2 - y^2a_3 - y^2b_2 + y^2b_3 + 2xb_1 - 2ya}{(x-y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-x^2a_2 - x^2a_3 - x^2b_2 + x^2b_3 + 2xya_2 - 2xya_3 - 2xyb_2 \\
&- 2xyb_3 + y^2a_2 + y^2a_3 + y^2b_2 - y^2b_3 - 2xb_1 + 2ya_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a_2v_1^2 + 2a_2v_1v_2 + a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 \\
&- 2b_2v_1v_2 + b_2v_2^2 + b_3v_1^2 - 2b_3v_1v_2 - b_3v_2^2 + 2a_1v_2 - 2b_1v_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_2 - a_3 - b_2 + b_3) v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3) v_1 v_2 - 2b_1 v_1 + (a_2 + a_3 + b_2 - b_3) v_2^2 + 2a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 &= 0 \\ -2b_1 &= 0 \\ -a_2 - a_3 - b_2 + b_3 &= 0 \\ a_2 + a_3 + b_2 - b_3 &= 0 \\ 2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= -b_2 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(-\frac{x+y}{-x+y}\right)(x) \\ &= \frac{-x^2 - y^2}{x - y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2 - y^2}{x - y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y}{-x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x + y}{x^2 + y^2} \\ S_y &= \frac{-x + y}{x^2 + y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

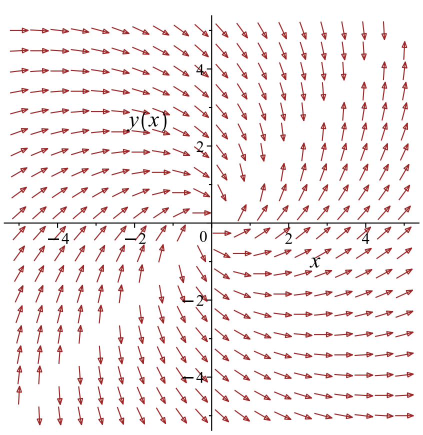
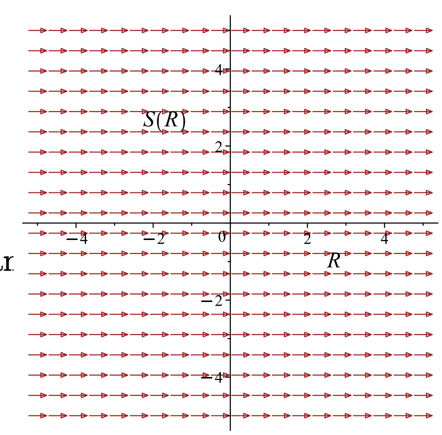
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

Which simplifies to

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y}{-x+y}$ 	$R = x$ $S = \frac{\ln(x^2+y^2)}{2} - \arctan$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1 \tag{1}$$

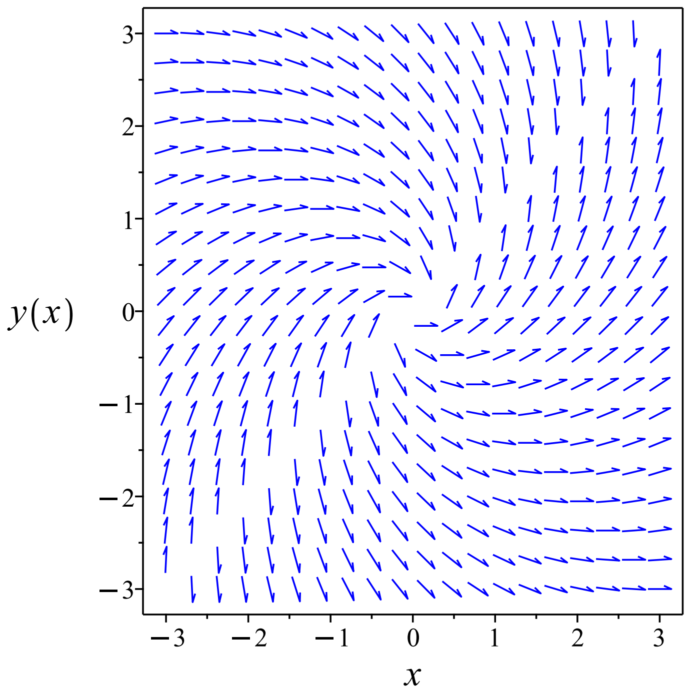


Figure 7: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

Verified OK.

1.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x+y)\,\mathrm{d}y&=(-y-x)\,\mathrm{d}x\\ (x+y)\,\mathrm{d}x+(-x+y)\,\mathrm{d}y&=0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x+y\\ N(x,y)&=-x+y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(x+y)\\ &=1\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-x+y)$$
$$= -1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = x + y$ and $N = -x + y$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{x+y}{x^2+y^2}$$
$$N = \frac{-x+y}{x^2+y^2}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-x+y}{x^2+y^2}\right)\mathrm{d}y = \left(-\frac{x+y}{x^2+y^2}\right)\mathrm{d}x$$
$$\left(\frac{x+y}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{-x+y}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{x+y}{x^2+y^2}$$
$$N(x,y) = \frac{-x+y}{x^2+y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x+y}{x^2+y^2}\right)\\ &= \frac{x^2-2xy-y^2}{\left(x^2+y^2\right)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x+y}{x^2+y^2}\right)\\ &= \frac{x^2-2xy-y^2}{\left(x^2+y^2\right)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \frac{x+y}{x^2+y^2} \, \mathrm{d}x \\
\phi &= \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{y}{x^2+y^2} - \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \qquad (4) \\
&= \frac{-x+y}{x^2+y^2} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x+y}{x^2+y^2}$. Therefore equation (4) becomes

$$
\frac{-x+y}{x^2+y^2} = \frac{-x+y}{x^2+y^2} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right)
$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(x^2+y^2\right)}{2}+\arctan\left(\frac{x}{y}\right)=c_1 \tag{1}$$

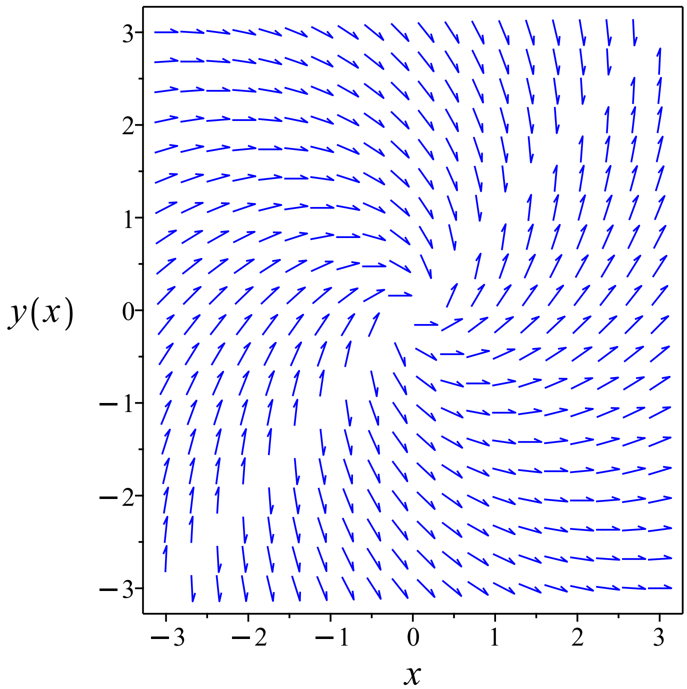


Figure 8: Slope field plot

Verification of solutions

$$\frac{\ln\left(x^2+y^2\right)}{2}+\arctan\left(\frac{x}{y}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve((x+y(x))-(x-y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \tan\left(\mathrm{RootOf}\left(-2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x\right) + 2c_1\right)\right)x$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 36

```
DSolve[(x+y[x])-(x-y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\frac{1}{2}\log\left(\frac{y(x)^2}{x^2}+1\right) - \arctan\left(\frac{y(x)}{x}\right) = -\log(x) + c_1, y(x)\right]$$

1.4 problem First order with homogeneous Coefficients. Exercise 7.5, page 61

Internal problem ID [4430]
Internal file name [`OUTPUT/3923_Sunday_June_05_2022_11_49_47_AM_84319316/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 7
Problem number: First order with homogeneous Coefficients. Exercise 7.5, page 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{xy' - y - x\sin\left(\frac{y}{x}\right) = 0}$$

1.4.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \sin\left(\frac{y}{x}\right) + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$
\begin{aligned}
\frac{du}{dx}x + u &= u + g(x)\, f(bu)^{\frac{n}{m}} \\
u' &= \frac{1}{x} g(x)\, f(bu)^{\frac{n}{m}}
\end{aligned} \tag{2}
$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$
\begin{aligned}
g(x) &= 1 \\
b &= 1 \\
f\left(\frac{bx}{y}\right) &= \sin\left(\frac{y}{x}\right)
\end{aligned}
$$

Substituting the above in (2) results in the $u(x)$ ode as

$$
u'(x) = \frac{\sin(u(x))}{x}
$$

Which is now solved as separable In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{\sin(u)}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \sin(u)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\sin(u)}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{\sin(u)}\, du &= \int \frac{1}{x}\, dx \\
\ln(\csc(u) - \cot(u)) &= \ln(x) + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
\csc(u) - \cot(u) = e^{\ln(x)+c_1}
$$

Which simplifies to

$$
\csc(u) - \cot(u) = c_2 x
$$

Therefore the solution is

$$
\begin{aligned}
y &= ux \\
&= x \arctan\left(\frac{2c_2 x\, \mathrm{e}^{c_1}}{c_2^2 x^2 \mathrm{e}^{2c_1}+1}, -\frac{c_2^2 x^2 \mathrm{e}^{2c_1}-1}{c_2^2 x^2 \mathrm{e}^{2c_1}+1}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x \arctan\left(\frac{2c_2 x\, \mathrm{e}^{c_1}}{c_2^2 x^2 \mathrm{e}^{2c_1}+1}, -\frac{c_2^2 x^2 \mathrm{e}^{2c_1}-1}{c_2^2 x^2 \mathrm{e}^{2c_1}+1}\right) \tag{1}
$$

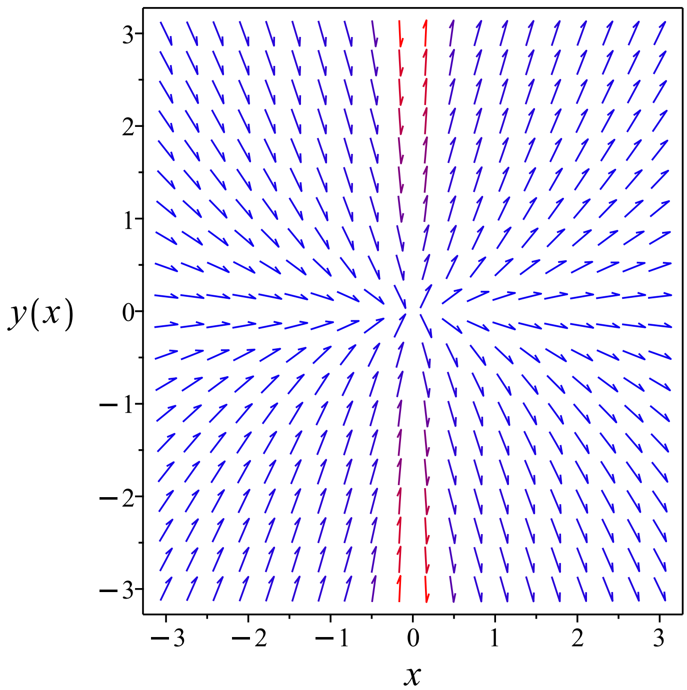


Figure 9: Slope field plot

Verification of solutions

$$
y = x \arctan\left(\frac{2c_2 x\, \mathrm{e}^{c_1}}{c_2^2 x^2 \mathrm{e}^{2c_1}+1}, -\frac{c_2^2 x^2 \mathrm{e}^{2c_1}-1}{c_2^2 x^2 \mathrm{e}^{2c_1}+1}\right)
$$

Verified OK.

1.4.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)\,x + u(x)) - u(x)\,x - x\sin(u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\sin(u)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \sin(u)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\sin(u)}\,du &= \frac{1}{x}\,dx \\ \int \frac{1}{\sin(u)}\,du &= \int \frac{1}{x}\,dx \\ \ln(\csc(u) - \cot(u)) &= \ln(x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\csc(u) - \cot(u) = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\csc(u) - \cot(u) = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x\arctan\left(\frac{2c_3 x\,\mathrm{e}^{c_2}}{\mathrm{e}^{2c_2}c_3^2x^2+1}, -\frac{\mathrm{e}^{2c_2}c_3^2x^2-1}{\mathrm{e}^{2c_2}c_3^2x^2+1}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\arctan\left(\frac{2c_3 x\,\mathrm{e}^{c_2}}{\mathrm{e}^{2c_2}c_3^2x^2+1}, -\frac{\mathrm{e}^{2c_2}c_3^2x^2-1}{\mathrm{e}^{2c_2}c_3^2x^2+1}\right) \tag{1}$$

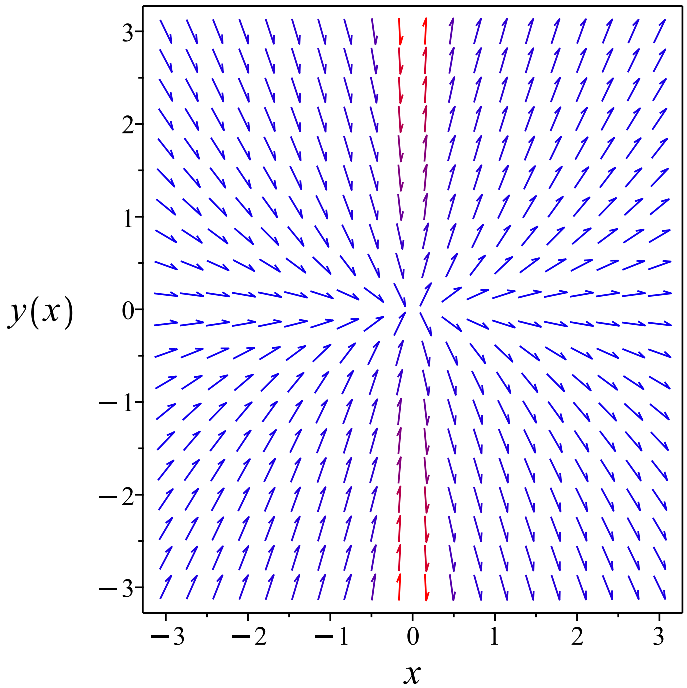


Figure 10: Slope field plot

Verification of solutions

$$y = x \arctan\left(\frac{2c_3 x\, e^{c_2}}{e^{2c_2} c_3^2 x^2 + 1}, -\frac{e^{2c_2} c_3^2 x^2 - 1}{e^{2c_2} c_3^2 x^2 + 1}\right)$$

Verified OK.

1.4.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + x \sin\left(\frac{y}{x}\right)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 2: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x^2 \\
\eta(x, y) &= xy
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{xy}{x^2} \\ &= \frac{y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x^2}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= -\frac{1}{x}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y + x\sin\left(\frac{y}{x}\right)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{y}{x^2} \\ R_y &= \frac{1}{x} \\ S_x &= \frac{1}{x^2} \\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\csc\left(\frac{y}{x}\right)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -S(R)\csc(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1(\csc(R) + \cot(R)) \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1\left(\csc\left(\frac{y}{x}\right) + \cot\left(\frac{y}{x}\right)\right)$$

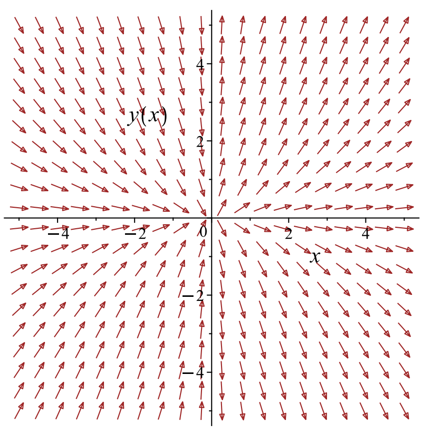
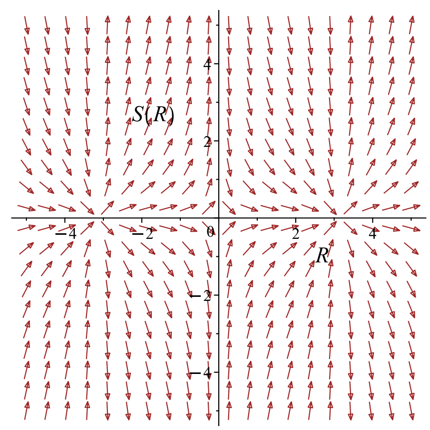
Which simplifies to

$$-\frac{1}{x} = c_1\left(\csc\left(\frac{y}{x}\right) + \cot\left(\frac{y}{x}\right)\right)$$

Which gives

$$y = \arctan\left(-\frac{2c_1x}{c_1^2x^2+1}, -\frac{c_1^2x^2-1}{c_1^2x^2+1}\right)x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+x\sin\left(\frac{y}{x}\right)}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -S(R)\csc(R)$ 

Summary
The solution(s) found are the following

$$y = \arctan\left(-\frac{2c_1x}{c_1^2x^2+1}, -\frac{c_1^2x^2-1}{c_1^2x^2+1}\right)x \tag{1}$$

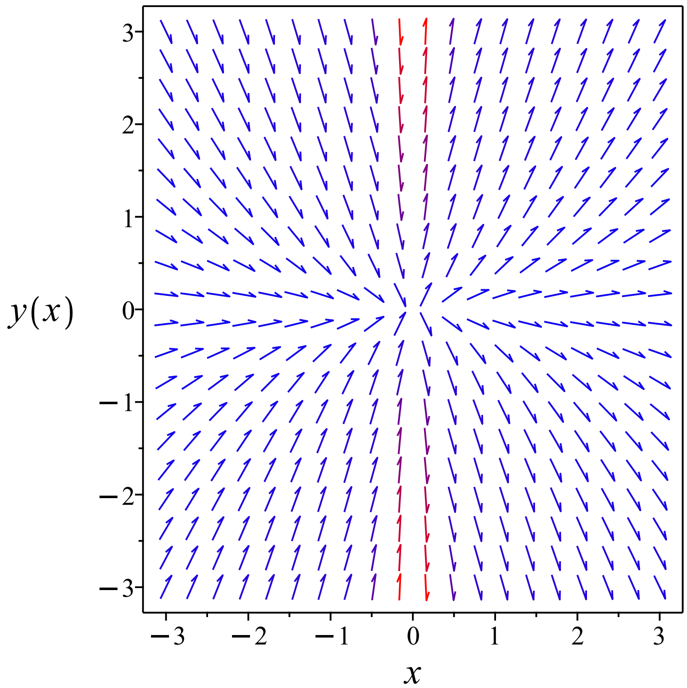


Figure 11: Slope field plot

Verification of solutions

$$y = \arctan\left(-\frac{2c_1 x}{c_1^2 x^2+1}, -\frac{c_1^2 x^2-1}{c_1^2 x^2+1}\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
dsolve(x*diff(y(x),x)-y(x)-x*sin(y(x)/x)=0,y(x), singsol=all)
```

$$y(x) = \arctan\left(\frac{2xc_1}{x^2c_1^2+1}, \frac{-x^2c_1^2+1}{x^2c_1^2+1}\right)x$$

✓ Solution by Mathematica

Time used: 0.325 (sec). Leaf size: 52

```
DSolve[x*y'[x]-y[x]-x*Sin[y[x]/x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x \arccos(-\tanh(\log(x) + c_1))$$
$$y(x) \to x \arccos(-\tanh(\log(x) + c_1))$$
$$y(x) \to 0$$
$$y(x) \to -\pi x$$
$$y(x) \to \pi x$$

1.5 problem First order with homogeneous Coefficients. Exercise 7.6, page 61

Internal problem ID [4431]
Internal file name [`OUTPUT/3924_Sunday_June_05_2022_11_49_57_AM_20984525/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 7
Problem number: First order with homogeneous Coefficients. Exercise 7.6, page 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{2yx^2 + y^3 + (xy^2 - 2x^3)\, y' = 0}$$

1.5.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$2u(x)\, x^3 + u(x)^3\, x^3 + \left(x^3 u(x)^2 - 2x^3\right)\left(u'(x)\, x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u^3}{x\,(u^2-2)} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u^3}{u^2-2}$. Integrating both sides gives

$$\frac{1}{\frac{u^3}{u^2-2}}\,du = -\frac{2}{x}\,dx$$
$$\int \frac{1}{\frac{u^3}{u^2-2}}\,du = \int -\frac{2}{x}\,dx$$
$$\ln(u) + \frac{1}{u^2} = -2\ln(x) + c_2$$

The solution is

$$\ln(u(x)) + \frac{1}{u(x)^2} + 2\ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\ln\left(\frac{y}{x}\right) + \frac{x^2}{y^2} + 2\ln(x) - c_2 = 0$$
$$\ln\left(\frac{y}{x}\right) + \frac{x^2}{y^2} + 2\ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\ln\left(\frac{y}{x}\right) + \frac{x^2}{y^2} + 2\ln(x) - c_2 = 0 \tag{1}$$

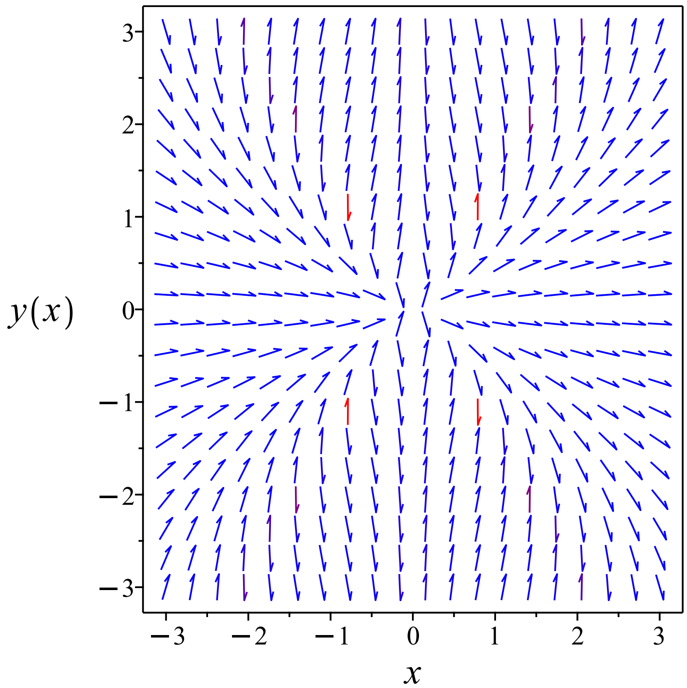


Figure 12: Slope field plot

Verification of solutions

$$\ln\left(\frac{y}{x}\right)+\frac{x^2}{y^2}+2\ln\left(x\right)-c_2=0$$

Verified OK.

1.5.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(2x^2+y^2)}{x\left(-2x^2+y^2\right)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{y(2x^2+y^2)\,(b_3-a_2)}{x\,(-2x^2+y^2)} - \frac{y^2(2x^2+y^2)^2\,a_3}{x^2\,(-2x^2+y^2)^2} \\
&- \left(-\frac{4y}{-2x^2+y^2} + \frac{y(2x^2+y^2)}{x^2\,(-2x^2+y^2)} - \frac{4y(2x^2+y^2)}{(-2x^2+y^2)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{2x^2+y^2}{x\,(-2x^2+y^2)} - \frac{2y^2}{x\,(-2x^2+y^2)} + \frac{2y^2(2x^2+y^2)}{x\,(-2x^2+y^2)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{12x^4y^2b_2 - 8x^3y^3a_2 + 8x^3y^3b_3 - 4x^2y^4a_3 - 2x^2y^4b_2 + 2y^6a_3 + 4x^5b_1 - 4x^4ya_1 + 8x^3y^2b_1 - 8x^2y^3a_1 - x}{(2x^2-y^2)^2\,x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-12x^4y^2b_2 + 8x^3y^3a_2 - 8x^3y^3b_3 + 4x^2y^4a_3 + 2x^2y^4b_2 - 2y^6a_3 \\
&- 4x^5b_1 + 4x^4ya_1 - 8x^3y^2b_1 + 8x^2y^3a_1 + x\,y^4b_1 - y^5a_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&8a_2v_1^3v_2^3 + 4a_3v_1^2v_2^4 - 2a_3v_2^6 - 12b_2v_1^4v_2^2 + 2b_2v_1^2v_2^4 - 8b_3v_1^3v_2^3 \\
&+ 4a_1v_1^4v_2 + 8a_1v_1^2v_2^3 - a_1v_2^5 - 4b_1v_1^5 - 8b_1v_1^3v_2^2 + b_1v_1v_2^4 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-4b_1 v_1^5 - 12b_2 v_1^4 v_2^2 + 4a_1 v_1^4 v_2 + (8a_2 - 8b_3) v_1^3 v_2^3 - 8b_1 v_1^3 v_2^2 \\
&+ (4a_3 + 2b_2) v_1^2 v_2^4 + 8a_1 v_1^2 v_2^3 + b_1 v_1 v_2^4 - 2a_3 v_2^6 - a_1 v_2^5 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_1 &= 0 \\
-a_1 &= 0 \\
4a_1 &= 0 \\
8a_1 &= 0 \\
-2a_3 &= 0 \\
-8b_1 &= 0 \\
-4b_1 &= 0 \\
-12b_2 &= 0 \\
8a_2 - 8b_3 &= 0 \\
4a_3 + 2b_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y(2x^2+y^2)}{x\,(-2x^2+y^2)}\right)(x) \\
&= -\frac{2y^3}{2x^2-y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{2y^3}{2x^2-y^2}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{2} + \frac{x^2}{2y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(2x^2+y^2)}{x\,(-2x^2+y^2)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x}{y^2} \\
S_y &= \frac{-2x^2+y^2}{2y^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{2R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(y)\, y^2 + x^2}{2y^2} = -\frac{\ln(x)}{2} + c_1
$$

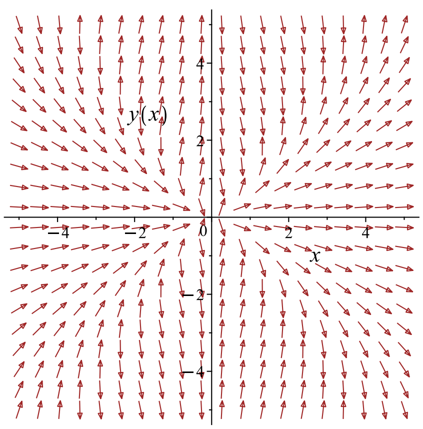
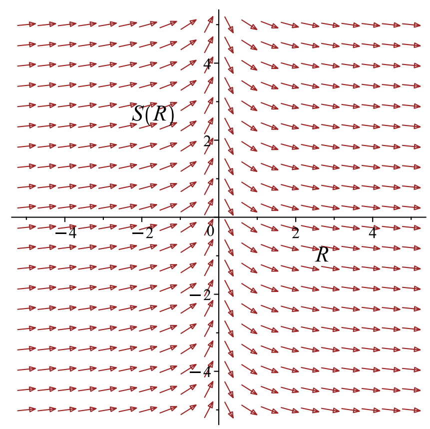
Which simplifies to

$$
\frac{\ln(y)\, y^2 + x^2}{2y^2} = -\frac{\ln(x)}{2} + c_1
$$

Which gives

$$
y = \frac{\mathrm{e}^{\frac{\mathrm{LambertW}\left(-2x^4 \mathrm{e}^{-4c_1}\right)}{2} + 2c_1}}{x}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(2x^2+y^2)}{x(-2x^2+y^2)}$  	$R = x$ $S = \frac{\ln(y)\, y^2 + x^2}{2y^2}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{\frac{\mathrm{LambertW}\left(-2x^4 e^{-4c_1}\right)}{2}+2c_1}}{x} \tag{1}$$

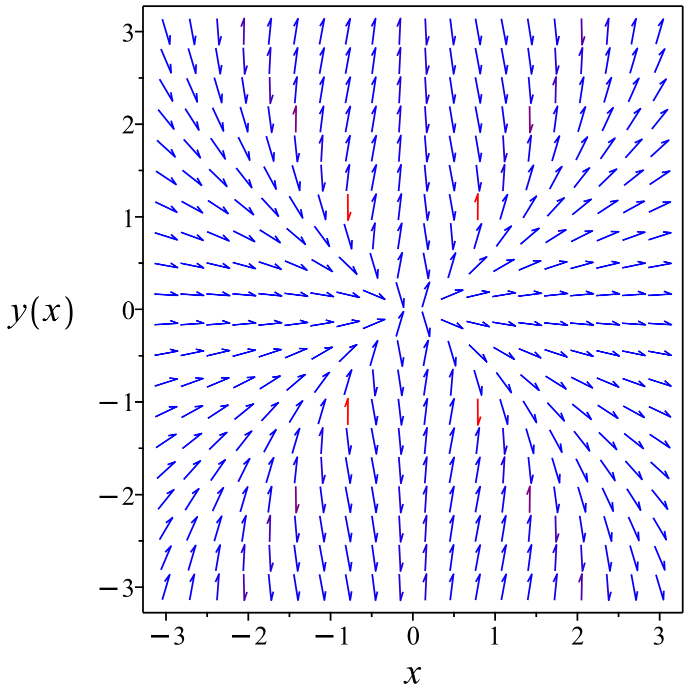


Figure 13: Slope field plot

Verification of solutions

$$y = \frac{e^{\frac{\mathrm{LambertW}\left(-2x^4 e^{-4c_1}\right)}{2}+2c_1}}{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve((2*x^2*y(x)+y(x)^3)+(x*y(x)^2-2*x^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{2}\,\sqrt{-\frac{1}{\text{LambertW}\left(-2c_1x^4\right)}}\,x$$

✓ Solution by Mathematica

Time used: 5.64 (sec). Leaf size: 66

```
DSolve[(2*x^2*y[x]+y[x]^3)+(x*y[x]^2-2*x^3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\frac{i\sqrt{2}x}{\sqrt{W\left(-2e^{-2c_1}x^4\right)}}$$
$$y(x) \to \frac{i\sqrt{2}x}{\sqrt{W\left(-2e^{-2c_1}x^4\right)}}$$
$$y(x) \to 0$$

1.6 problem First order with homogeneous Coefficients. Exercise 7.7, page 61

Internal problem ID [4432]
Internal file name [`OUTPUT/3925_Sunday_June_05_2022_11_50_05_AM_73863914/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 7
Problem number: First order with homogeneous Coefficients. Exercise 7.7, page 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _dAlembert]
```

$$\boxed{y^2 + \left(x\sqrt{y^2 - x^2} - xy\right) y' = 0}$$

1.6.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y^2}{\left(-\sqrt{-x^2 + y^2} + y\right) x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2+\frac{y^2(b_3-a_2)}{\left(-\sqrt{-x^2+y^2}+y\right)x}-\frac{y^4 a_3}{\left(-\sqrt{-x^2+y^2}+y\right)^2 x^2} \\ & -\left(-\frac{y^2}{\left(-\sqrt{-x^2+y^2}+y\right)^2\sqrt{-x^2+y^2}}-\frac{y^2}{\left(-\sqrt{-x^2+y^2}+y\right)x^2}\right)(x a_2+y a_3+a_1) \\ & -\left(\frac{2y}{\left(-\sqrt{-x^2+y^2}+y\right)x}-\frac{y^2\left(-\frac{y}{\sqrt{-x^2+y^2}}+1\right)}{\left(-\sqrt{-x^2+y^2}+y\right)^2 x}\right)(x b_2+y b_3+b_1)=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{\left(-x^2+y^2\right)^{\frac{3}{2}} x^2 b_2+x^3y^2a_2-x^3y^2b_3+2x^2y^3a_3-x^2y^3b_2-y^5a_3-\sqrt{-x^2+y^2}\,x\,y^2b_1+\sqrt{-x^2+y^2}\,y^3a_1-}{\left(\sqrt{-x^2+y^2}-y\right)^2\sqrt{-x^2+y^2}\,x^2}$$
$$=0$$

Setting the numerator to zero gives

$$\begin{aligned} & \left(-x^2+y^2\right)^{\frac{3}{2}} x^2 b_2+x^3y^2a_2-x^3y^2b_3+2x^2y^3a_3-x^2y^3b_2-y^5a_3 \\ & -\sqrt{-x^2+y^2}\,x\,y^2b_1+\sqrt{-x^2+y^2}\,y^3a_1-2x^3yb_1+2x^2y^2a_1+x\,y^3b_1-y^4a_1=0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} & \left(-x^2+y^2\right)^{\frac{3}{2}} x^2 b_2+\left(-x^2+y^2\right)x\,y^2b_3-\left(-x^2+y^2\right)y^3a_3+x^3y^2a_2 \\ & +x^2y^3a_3-x^2y^3b_2-x\,y^4b_3+2\left(-x^2+y^2\right)xyb_1-\left(-x^2+y^2\right)y^2a_1 \\ & -\sqrt{-x^2+y^2}\,x\,y^2b_1+\sqrt{-x^2+y^2}\,y^3a_1+x^2y^2a_1-x\,y^3b_1=0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} & -x^4\sqrt{-x^2+y^2}\,b_2+x^3y^2a_2-x^3y^2b_3+x^2\sqrt{-x^2+y^2}\,y^2b_2+2x^2y^3a_3-x^2y^3b_2-y^5a_3 \\ & -2x^3yb_1+2x^2y^2a_1-\sqrt{-x^2+y^2}\,x\,y^2b_1+x\,y^3b_1+\sqrt{-x^2+y^2}\,y^3a_1-y^4a_1=0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-x^2 + y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-x^2 + y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &v_1^3v_2^2a_2 + 2v_1^2v_2^3a_3 - v_2^5a_3 - v_1^4v_3b_2 - v_1^2v_2^3b_2 + v_1^2v_3v_2^2b_2 - v_1^3v_2^2b_3 \\ &+ 2v_1^2v_2^2a_1 - v_2^4a_1 + v_3v_2^3a_1 - 2v_1^3v_2b_1 + v_1v_2^3b_1 - v_3v_1v_2^2b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &-v_1^4v_3b_2 + (-b_3 + a_2)\, v_1^3v_2^2 - 2v_1^3v_2b_1 + (2a_3 - b_2)\, v_1^2v_2^3 + v_1^2v_3v_2^2b_2 \\ &+ 2v_1^2v_2^2a_1 + v_1v_2^3b_1 - v_3v_1v_2^2b_1 - v_2^5a_3 - v_2^4a_1 + v_3v_2^3a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ -a_1 &= 0 \\ 2a_1 &= 0 \\ -a_3 &= 0 \\ -2b_1 &= 0 \\ -b_1 &= 0 \\ -b_2 &= 0 \\ 2a_3 - b_2 &= 0 \\ -b_3 + a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x\\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi\\
&= y - \left(\frac{y^2}{\left(-\sqrt{-x^2+y^2}+y\right)x}\right)(x)\\
&= \frac{\sqrt{-x^2+y^2}\,y}{\sqrt{-x^2+y^2}-y}\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy\\
&= \int \frac{1}{\frac{\sqrt{-x^2+y^2}\,y}{\sqrt{-x^2+y^2}-y}} dy
\end{aligned}
$$

Which results in

$$S = \ln(y) - \ln\left(\sqrt{-x^2+y^2}+y\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2}{\left(-\sqrt{-x^2+y^2}+y\right)x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x}{\sqrt{-x^2+y^2}\,\left(\sqrt{-x^2+y^2}+y\right)} \\
S_y &= \frac{\sqrt{-x^2+y^2}-y}{\sqrt{-x^2+y^2}\,y}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x^2 - \sqrt{-x^2+y^2}\,y - y^2}{x\sqrt{-x^2+y^2}\,\left(\sqrt{-x^2+y^2}+y\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

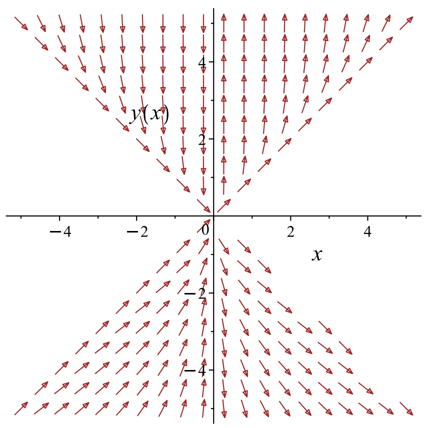
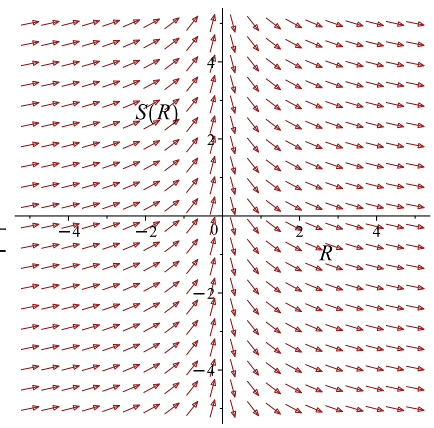
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) - \ln\left(\sqrt{y^2-x^2}+y\right) = -\ln(x) + c_1$$

Which simplifies to

$$\ln (y) - \ln \left(\sqrt{y^{2}-x^{2}}+y\right) = -\ln (x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^{2}}{\left(-\sqrt{-x^{2}+y^{2}}+y\right) x}$ 	$R = x$ $S = \ln (y) - \ln \left(\sqrt{-x^{2}+}\right.$	$\frac{dS}{dR} = -\frac{1}{R}$ 

<u>Summary</u>

The solution(s) found are the following

$$\ln (y) - \ln \left(\sqrt{y^{2}-x^{2}}+y\right) = -\ln (x) + c_1 \tag{1}$$

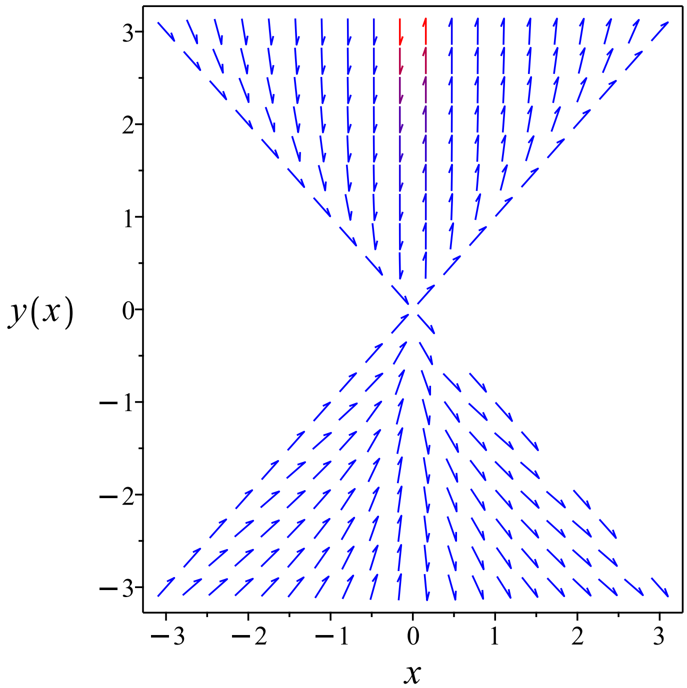


Figure 14: Slope field plot

Verification of solutions

$$\ln(y) - \ln\left(\sqrt{y^2 - x^2} + y\right) = -\ln(x) + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
dsolve(y(x)^2+(x*sqrt(y(x)^2-x^2)-x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{-c_1 x y(x) + y(x) + \sqrt{y(x)^2 - x^2}}{x y(x)} = 0$$

✓ Solution by Mathematica

Time used: 2.247 (sec). Leaf size: 111

```
DSolve[y[x]^2+(x*Sqrt[y[x]^2-x^2]-x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{\sqrt{\frac{y(x)^2}{x^2}-1}\left(\log\left(\sqrt{\frac{y(x)}{x}+1}-1\right)+\log\left(\sqrt{\frac{y(x)}{x}+1}+1\right)\right)}{\sqrt{\frac{y(x)}{x}-1}\sqrt{\frac{y(x)}{x}+1}} - 2\log\left(\sqrt{\frac{y(x)}{x}-1}-\sqrt{\frac{y(x)}{x}+1}\right) = \log(x) + c_1, y(x)\right]$$

1.7 problem First order with homogeneous Coefficients. Exercise 7.8, page 61

Internal problem ID [4433]
Internal file name [`OUTPUT/3926_Sunday_June_05_2022_11_50_16_AM_88267990/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 7
Problem number: First order with homogeneous Coefficients. Exercise 7.8, page 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{\frac{y\cos\left(\frac{y}{x}\right)}{x}-\left(\frac{x\sin\left(\frac{y}{x}\right)}{y}+\cos\left(\frac{y}{x}\right)\right)y'=0}$$

1.7.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\cos\left(u(x)\right)-\left(\frac{\sin\left(u(x)\right)}{u\left(x\right)}+\cos\left(u(x)\right)\right)\left(u'(x)\,x+u(x)\right)=0$$

In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=-\frac{u\sin\left(u\right)}{\left(u\cos\left(u\right)+\sin\left(u\right)\right)x}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u \sin(u)}{u \cos(u)+\sin(u)}$. Integrating both sides gives

$$\frac{1}{\frac{u \sin(u)}{u \cos(u)+\sin(u)}} du = -\frac{1}{x} dx$$
$$\int \frac{1}{\frac{u \sin(u)}{u \cos(u)+\sin(u)}} du = \int -\frac{1}{x} dx$$
$$\ln(\sin(u)) + \ln(u) = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{\ln(\sin(u))+\ln(u)} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$u \sin(u) = \frac{c_3}{x}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= \text{RootOf}(-\_Zx \sin(\_Z) + c_3)\, x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \text{RootOf}(-\_Zx \sin(\_Z) + c_3)\, x \tag{1}$$

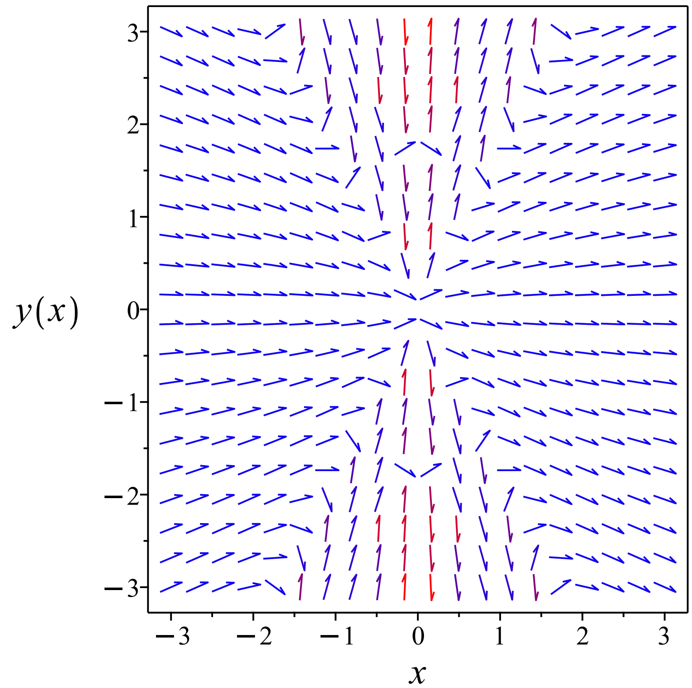


Figure 15: Slope field plot

Verification of solutions

$$y = \text{RootOf}\left(-\_Zx\sin\left(\_Z\right) + c_3\right)x$$

Verified OK.

1.7.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y^2\cos\left(\frac{y}{x}\right)}{x\left(x\sin\left(\frac{y}{x}\right) + y\cos\left(\frac{y}{x}\right)\right)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{y^2 \cos\left(\frac{y}{x}\right)(b_3 - a_2)}{x\left(x\sin\left(\frac{y}{x}\right) + y\cos\left(\frac{y}{x}\right)\right)} - \frac{y^4 \cos\left(\frac{y}{x}\right)^2 a_3}{x^2\left(x\sin\left(\frac{y}{x}\right) + y\cos\left(\frac{y}{x}\right)\right)^2} \\ &- \left(-\frac{y^2\cos\left(\frac{y}{x}\right)}{x^2\left(x\sin\left(\frac{y}{x}\right) + y\cos\left(\frac{y}{x}\right)\right)} + \frac{y^3\sin\left(\frac{y}{x}\right)}{x^3\left(x\sin\left(\frac{y}{x}\right) + y\cos\left(\frac{y}{x}\right)\right)}\right. \\ &\left. - \frac{y^2\cos\left(\frac{y}{x}\right)\left(\sin\left(\frac{y}{x}\right) - \frac{y\cos\left(\frac{y}{x}\right)}{x} + \frac{y^2\sin\left(\frac{y}{x}\right)}{x^2}\right)}{x\left(x\sin\left(\frac{y}{x}\right) + y\cos\left(\frac{y}{x}\right)\right)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(\frac{2y\cos\left(\frac{y}{x}\right)}{x\left(x\sin\left(\frac{y}{x}\right) + y\cos\left(\frac{y}{x}\right)\right)} - \frac{y^2\sin\left(\frac{y}{x}\right)}{x^2\left(x\sin\left(\frac{y}{x}\right) + y\cos\left(\frac{y}{x}\right)\right)}\right. \\ &\left. - \frac{y^2\cos\left(\frac{y}{x}\right)\left(2\cos\left(\frac{y}{x}\right) - \frac{y\sin\left(\frac{y}{x}\right)}{x}\right)}{x\left(x\sin\left(\frac{y}{x}\right) + y\cos\left(\frac{y}{x}\right)\right)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{\sin\left(\frac{y}{x}\right)^2 x^4 b_2 + \sin\left(\frac{y}{x}\right)^2 x^2y^2b_2 - \sin\left(\frac{y}{x}\right)^2 x\,y^3a_2 + \sin\left(\frac{y}{x}\right)^2 x\,y^3b_3 - \sin\left(\frac{y}{x}\right)^2 y^4a_3 + \sin\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right) x^2y^2a_2}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &\sin\left(\frac{y}{x}\right)^2 x^4 b_2 + \sin\left(\frac{y}{x}\right)^2 x^2y^2b_2 - \sin\left(\frac{y}{x}\right)^2 x\,y^3a_2 + \sin\left(\frac{y}{x}\right)^2 x\,y^3b_3 \\ &- \sin\left(\frac{y}{x}\right)^2 y^4a_3 + \sin\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right) x^2y^2a_2 - \sin\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right) x^2y^2b_3 \\ &+ 2\sin\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right) x\,y^3a_3 + \cos\left(\frac{y}{x}\right)^2 x^2y^2b_2 - \cos\left(\frac{y}{x}\right)^2 x\,y^3a_2 \\ &+ \cos\left(\frac{y}{x}\right)^2 x\,y^3b_3 - y^4\cos\left(\frac{y}{x}\right)^2 a_3 + \sin\left(\frac{y}{x}\right)^2 x\,y^2b_1 - \sin\left(\frac{y}{x}\right)^2 y^3a_1 \\ &- 2\sin\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right) x^2yb_1 + 2\sin\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right) x\,y^2a_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$-\frac{x\left(-x^2y^2a_2\sin\left(\frac{2y}{x}\right)+x^2y^2b_3\sin\left(\frac{2y}{x}\right)-2x\,y^3a_3\sin\left(\frac{2y}{x}\right)+x^4b_2\cos\left(\frac{2y}{x}\right)+2x^2yb_1\sin\left(\frac{2y}{x}\right)-2x\,y^2a_1\sin\left(\right.\right.}{2} = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \cos\left(\frac{2y}{x}\right), \sin\left(\frac{2y}{x}\right)\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \cos\left(\frac{2y}{x}\right) = v_3, \sin\left(\frac{2y}{x}\right) = v_4\right\}$$

The above PDE (6E) now becomes

$$-\frac{v_1(-v_1^2v_2^2a_2v_4-2v_1v_2^3a_3v_4+v_1^4b_2v_3+v_1^2v_2^2b_3v_4-2v_1v_2^2a_1v_4-v_2^3a_1v_3+2v_1v_2^3a_2+2v_2^4a_3+2v_1^2v_2b_1v_4+}{2} = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned}&\frac{b_2v_1^5}{2}-\frac{b_2v_3v_1^5}{2}+b_2v_2^2v_1^3+\left(\frac{a_2}{2}-\frac{b_3}{2}\right)v_4v_2^2v_1^3-b_1v_4v_2v_1^3+(b_3-a_2)\,v_2^3v_1^2\\&+\frac{b_1v_2^2v_1^2}{2}+a_1v_4v_2^2v_1^2+a_3v_4v_2^3v_1^2-\frac{b_1v_3v_2^2v_1^2}{2}+\frac{a_1v_2^3v_3v_1}{2}-\frac{a_1v_2^3v_1}{2}-a_3v_2^4v_1=0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_1 &= 0 \\
a_3 &= 0 \\
b_2 &= 0 \\
-\frac{a_1}{2} &= 0 \\
\frac{a_1}{2} &= 0 \\
-a_3 &= 0 \\
-b_1 &= 0 \\
-\frac{b_1}{2} &= 0 \\
\frac{b_1}{2} &= 0 \\
-\frac{b_2}{2} &= 0 \\
\frac{b_2}{2} &= 0 \\
\frac{a_2}{2} - \frac{b_3}{2} &= 0 \\
b_3 - a_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y^2 \cos\left(\frac{y}{x}\right)}{x\left(x\sin\left(\frac{y}{x}\right) + y\cos\left(\frac{y}{x}\right)\right)}\right)(x) \\
&= \frac{\sin\left(\frac{y}{x}\right) xy}{x\sin\left(\frac{y}{x}\right) + y\cos\left(\frac{y}{x}\right)} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{\sin\left(\frac{y}{x}\right)xy}{x\sin\left(\frac{y}{x}\right)+y\cos\left(\frac{y}{x}\right)}} dy
\end{aligned}$$

Which results in

$$S = \ln\left(\sin\left(\frac{y}{x}\right)\right) + \ln\left(\frac{y}{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2 \cos\left(\frac{y}{x}\right)}{x\left(x\sin\left(\frac{y}{x}\right) + y\cos\left(\frac{y}{x}\right)\right)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-y \cot\left(\frac{y}{x}\right) - x}{x^2} \\
S_y &= \frac{\cot\left(\frac{y}{x}\right)}{x} + \frac{1}{y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

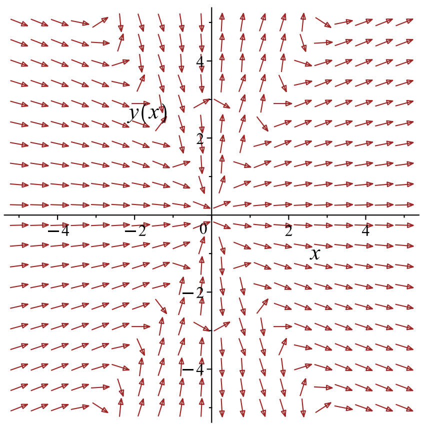
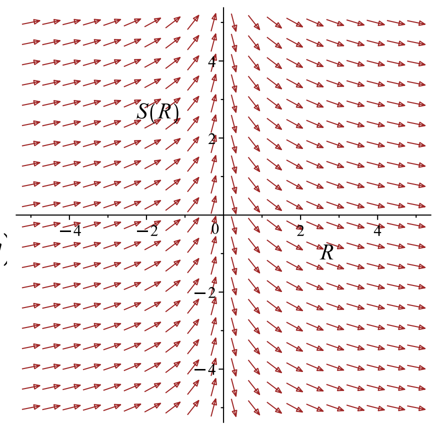
We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(R) + c_1 \tag{4}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2 \cos\left(\frac{y}{x}\right)}{x\left(x \sin\left(\frac{y}{x}\right)+y \cos\left(\frac{y}{x}\right)\right)}$ 	$R = x$ $S = \ln\left(\sin\left(\frac{y}{x}\right)\right) + \ln(y$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$\ln\left(\sin\left(\frac{y}{x}\right)\right) + \ln(y) - \ln(x) = -\ln(x) + c_1 \tag{1}$$

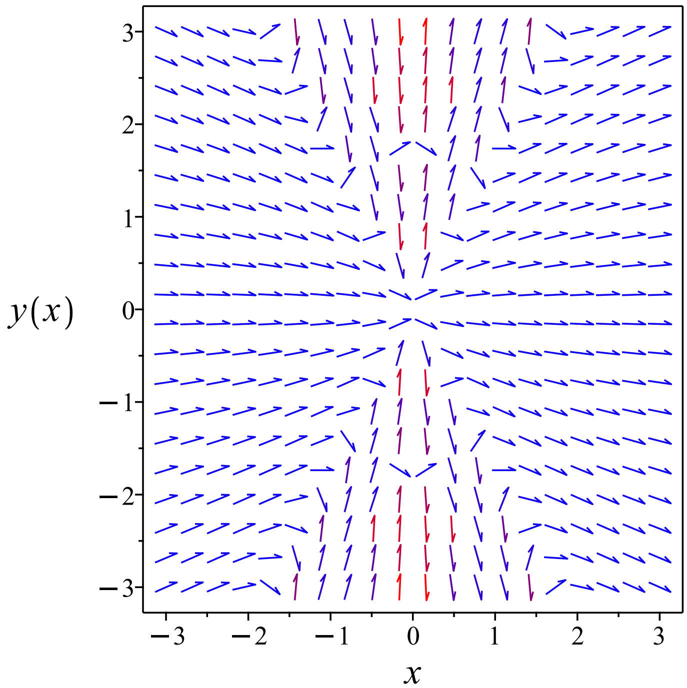


Figure 16: Slope field plot

Verification of solutions

$$\ln\left(\sin\left(\frac{y}{x}\right)\right)+\ln(y)-\ln(x)=-\ln(x)+c_1$$

Verified OK.

1.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x\left(x \sin\left(\frac{y}{x}\right) + y \cos\left(\frac{y}{x}\right)\right)\right) \mathrm{d}y = \left(y^2 \cos\left(\frac{y}{x}\right)\right) \mathrm{d}x$$
$$\left(-y^2 \cos\left(\frac{y}{x}\right)\right) \mathrm{d}x + \left(x\left(x \sin\left(\frac{y}{x}\right) + y \cos\left(\frac{y}{x}\right)\right)\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -y^2 \cos\left(\frac{y}{x}\right)$$
$$N(x, y) = x\left(x \sin\left(\frac{y}{x}\right) + y \cos\left(\frac{y}{x}\right)\right)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y^2 \cos\left(\frac{y}{x}\right)\right) \\
&= \frac{y\left(y \sin\left(\frac{y}{x}\right) - 2x \cos\left(\frac{y}{x}\right)\right)}{x}
\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(x\sin\left(\frac{y}{x}\right)+y\cos\left(\frac{y}{x}\right)\right)\right)\\ &= \frac{\sin\left(\frac{y}{x}\right)(2x^2+y^2)}{x}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x\left(x\sin\left(\frac{y}{x}\right)+y\cos\left(\frac{y}{x}\right)\right)}\left(\left(-2y\cos\left(\frac{y}{x}\right)+\frac{y^2\sin\left(\frac{y}{x}\right)}{x}\right)-\left(x\sin\left(\frac{y}{x}\right)+y\cos\left(\frac{y}{x}\right)+x\left(\sin\left(\frac{y}{x}\right)-\right.\right.\right.\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}\left(-y^2\cos\left(\frac{y}{x}\right)\right)\\ &= -\frac{y^2\cos\left(\frac{y}{x}\right)}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^2}\left(x\left(x\sin\left(\frac{y}{x}\right)+y\cos\left(\frac{y}{x}\right)\right)\right)\\ &= \frac{x\sin\left(\frac{y}{x}\right)+y\cos\left(\frac{y}{x}\right)}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-\frac{y^2 \cos\left(\frac{y}{x}\right)}{x^2}\right) + \left(\frac{x \sin\left(\frac{y}{x}\right) + y \cos\left(\frac{y}{x}\right)}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{y^2 \cos\left(\frac{y}{x}\right)}{x^2}\,\mathrm{d}x$$
$$\phi = y \sin\left(\frac{y}{x}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin\left(\frac{y}{x}\right) + \frac{y \cos\left(\frac{y}{x}\right)}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x \sin\left(\frac{y}{x}\right) + y \cos\left(\frac{y}{x}\right)}{x}$. Therefore equation (4) becomes

$$\frac{x \sin\left(\frac{y}{x}\right) + y \cos\left(\frac{y}{x}\right)}{x} = \sin\left(\frac{y}{x}\right) + \frac{y \cos\left(\frac{y}{x}\right)}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y \sin\left(\frac{y}{x}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y \sin\left(\frac{y}{x}\right)$$

Summary

The solution(s) found are the following

$$y \sin\left(\frac{y}{x}\right) = c_1 \tag{1}$$

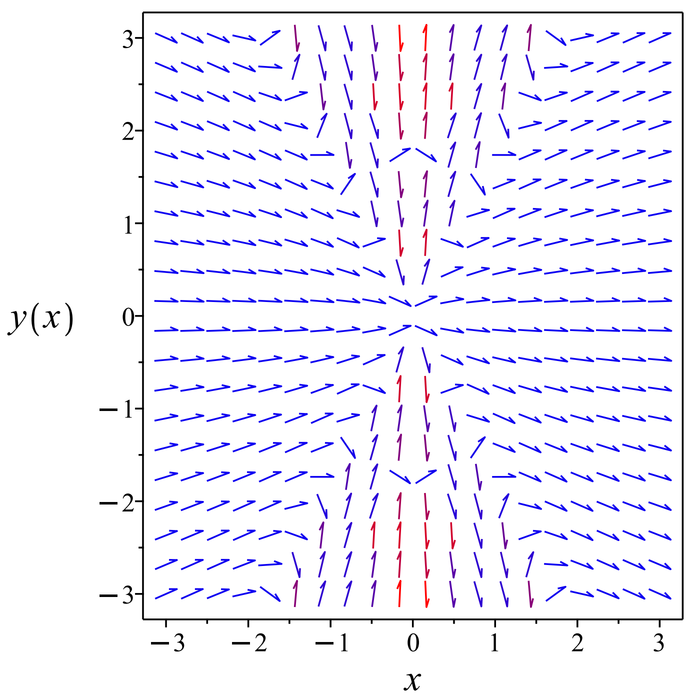


Figure 17: Slope field plot

Verification of solutions

$$y \sin\left(\frac{y}{x}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve(y(x)/x*cos(y(x)/x)-(x/y(x)*sin(y(x)/x)+cos(y(x)/x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \text{RootOf}\left(\_Zxc_1 \sin\left(\_Z\right) - 1\right) x$$

✓ Solution by Mathematica

Time used: 0.247 (sec). Leaf size: 27

```
DSolve[y[x]/x*Cos[y[x]/x]-(x/y[x]*Sin[y[x]/x]+Cos[y[x]/x])*y'[x]==0,y[x],x,IncludeSingularSo
```

$$\text{Solve}\left[\log\left(\frac{y(x)}{x}\right) + \log\left(\sin\left(\frac{y(x)}{x}\right)\right) = -\log(x) + c_1, y(x)\right]$$

1.8 problem First order with homogeneous Coefficients. Exercise 7.9, page 61

Internal problem ID [4434]
Internal file name [`OUTPUT/3927_Sunday_June_05_2022_11_50_26_AM_73917441/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 7
Problem number: First order with homogeneous Coefficients. Exercise 7.9, page 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y+x\ln\left(\frac{y}{x}\right)y'-2xy'=0}$$

1.8.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x+x\ln\left(u(x)\right)\left(u'(x)\,x+u(x)\right)-2x(u'(x)\,x+u(x))=0$$

In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=-\frac{u(\ln\left(u\right)-1)}{x\left(\ln\left(u\right)-2\right)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(\ln(u)-1)}{\ln(u)-2}$. Integrating both sides gives

$$\frac{1}{\frac{u(\ln(u)-1)}{\ln(u)-2}} du = -\frac{1}{x} dx$$

$$\int \frac{1}{\frac{u(\ln(u)-1)}{\ln(u)-2}} du = \int -\frac{1}{x} dx$$

$$\ln(u) - \ln(\ln(u) - 1) = -\ln(x) + c_2$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(u)-\ln(\ln(u)-1)} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{u}{\ln(u) - 1} = \frac{c_3}{x}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x\,\mathrm{e}^{-\operatorname{LambertW}\left(-\frac{x\,\mathrm{e}}{c_3}\right)+1} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\,\mathrm{e}^{-\operatorname{LambertW}\left(-\frac{x\,\mathrm{e}}{c_3}\right)+1} \tag{1}$$

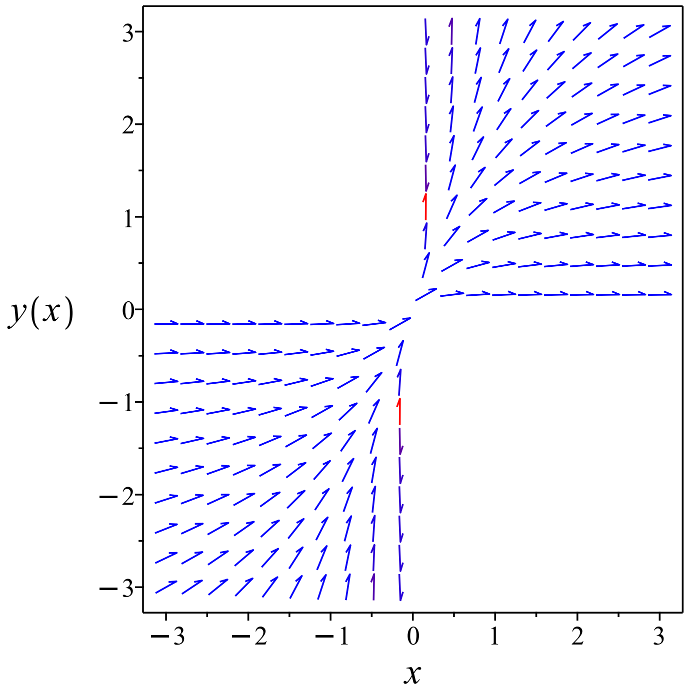


Figure 18: Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}^{-\operatorname{LambertW}\left(-\frac{x\,\mathrm{e}}{c_3}\right)+1}$$

Verified OK.

1.8.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y}{x\left(\ln\left(\frac{y}{x}\right)-2\right)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{y(b_3 - a_2)}{x\left(\ln\left(\frac{y}{x}\right) - 2\right)} - \frac{y^2 a_3}{x^2\left(\ln\left(\frac{y}{x}\right) - 2\right)^2} \\
&- \left(\frac{y}{x^2\left(\ln\left(\frac{y}{x}\right) - 2\right)} - \frac{y}{x^2\left(\ln\left(\frac{y}{x}\right) - 2\right)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(-\frac{1}{\left(\ln\left(\frac{y}{x}\right) - 2\right) x} + \frac{1}{x\left(\ln\left(\frac{y}{x}\right) - 2\right)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$\frac{\ln\left(\frac{y}{x}\right)^2 x^2 b_2 - 3\ln\left(\frac{y}{x}\right) x^2 b_2 - \ln\left(\frac{y}{x}\right) y^2 a_3 + \ln\left(\frac{y}{x}\right) x b_1 - \ln\left(\frac{y}{x}\right) y a_1 + b_2 x^2 + x y a_2 - x y b_3 + 2y^2 a_3 - 3x b_1 +}{x^2\left(\ln\left(\frac{y}{x}\right) - 2\right)^2}$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&\ln\left(\frac{y}{x}\right)^2 x^2 b_2 - 3\ln\left(\frac{y}{x}\right) x^2 b_2 - \ln\left(\frac{y}{x}\right) y^2 a_3 + \ln\left(\frac{y}{x}\right) x b_1 \\
&- \ln\left(\frac{y}{x}\right) y a_1 + b_2 x^2 + x y a_2 - x y b_3 + 2y^2 a_3 - 3x b_1 + 3y a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \ln\left(\frac{y}{x}\right)\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \ln\left(\frac{y}{x}\right) = v_3\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&v_3^2 v_1^2 b_2 - v_3 v_2^2 a_3 - 3v_3 v_1^2 b_2 - v_3 v_2 a_1 + v_1 v_2 a_2 + 2v_2^2 a_3 \\
&+ v_3 v_1 b_1 + b_2 v_1^2 - v_1 v_2 b_3 + 3v_2 a_1 - 3v_1 b_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &v_3^2 v_1^2 b_2 - 3v_3 v_1^2 b_2 + b_2 v_1^2 + (-b_3 + a_2) v_1 v_2 + v_3 v_1 b_1 \\ &- 3v_1 b_1 - v_3 v_2^2 a_3 + 2v_2^2 a_3 - v_3 v_2 a_1 + 3v_2 a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ b_2 &= 0 \\ -a_1 &= 0 \\ 3a_1 &= 0 \\ -a_3 &= 0 \\ 2a_3 &= 0 \\ -3b_1 &= 0 \\ -3b_2 &= 0 \\ -b_3 + a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y}{x\left(\ln\left(\frac{y}{x}\right)-2\right)}\right)(x) \\
&= \frac{-y + y\ln\left(\frac{y}{x}\right)}{\ln\left(\frac{y}{x}\right)-2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-y+y\ln\left(\frac{y}{x}\right)}{\ln\left(\frac{y}{x}\right)-2}} dy
\end{aligned}
$$

Which results in

$$
S = \ln\left(\frac{y}{x}\right) - \ln\left(-1 + \ln\left(\frac{y}{x}\right)\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y}{x\left(\ln\left(\frac{y}{x}\right)-2\right)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{x} + \frac{1}{x(-1+\ln(y)-\ln(x))} \\
S_y &= \frac{1}{y} + \frac{1}{y(1-\ln(y)+\ln(x))}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{\left(-1+\ln\left(\frac{y}{x}\right)\right)(2-\ln(y)+\ln(x))}{(1-\ln(y)+\ln(x))\, x \left(\ln\left(\frac{y}{x}\right)-2\right)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

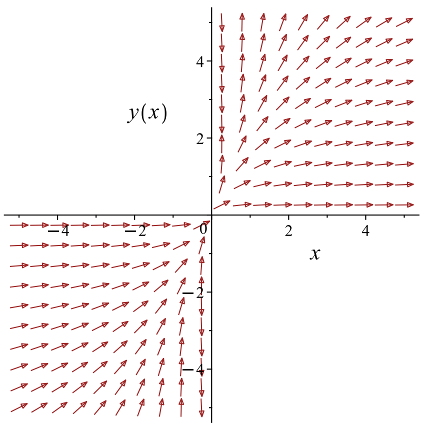
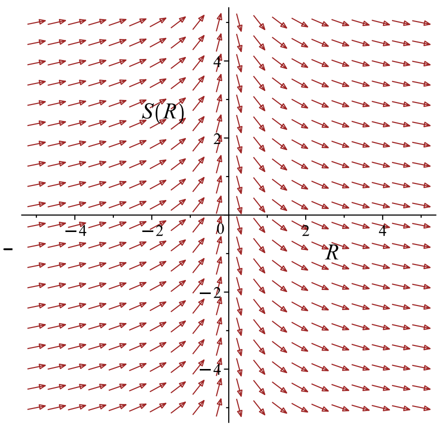
The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(R) + c_1 \tag{4}
$$

Which gives

$$
y = \mathrm{e}^{-\operatorname{LambertW}\left(-\mathrm{e}^{1-c_1} x\right)+1} x
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{x\left(\ln\left(\frac{y}{x}\right)-2\right)}$ 	$R = x$ $S = \ln(y) - \ln(x) - \ln($-	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\mathrm{e}^{1-c_1}x\right)+1}x \tag{1}$$

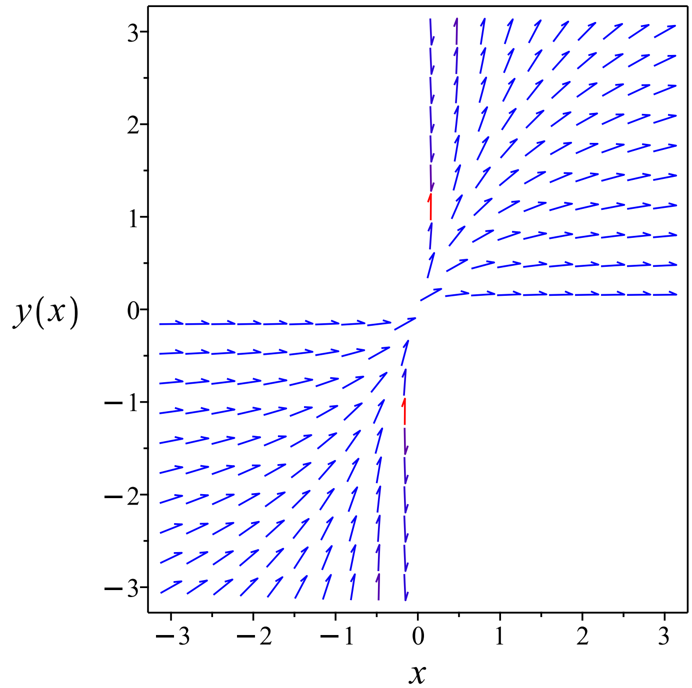


Figure 19: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\mathrm{e}^{1-c_1}x\right)+1}x$$

Verified OK.

1.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\ln\left(\frac{y}{x}\right) x - 2x\right) \mathrm{d}y = (-y)\,\mathrm{d}x$$
$$(y)\,\mathrm{d}x + \left(\ln\left(\frac{y}{x}\right) x - 2x\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y$$
$$N(x, y) = \ln\left(\frac{y}{x}\right) x - 2x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y)$$
$$= 1$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\ln\left(\frac{y}{x}\right)x - 2x\right) \\
&= \ln\left(\frac{y}{x}\right) - 3
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = y$ and $N = \ln\left(\frac{y}{x}\right)x - 2x$ by this integrating factor the ode becomes exact. The new M, N are

$$
\begin{aligned}
M &= \frac{1}{xy} \\
N &= \frac{\ln\left(\frac{y}{x}\right)x - 2x}{xy^2}
\end{aligned}
$$

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{\ln\left(\frac{y}{x}\right)x - 2x}{x\,y^2}\right)\mathrm{d}y = \left(-\frac{1}{yx}\right)\mathrm{d}x$$

$$\left(\frac{1}{yx}\right)\mathrm{d}x + \left(\frac{\ln\left(\frac{y}{x}\right)x - 2x}{x\,y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{1}{yx}$$
$$N(x,y) = \frac{\ln\left(\frac{y}{x}\right)x - 2x}{x\,y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{yx}\right)$$
$$= -\frac{1}{y^2 x}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\ln\left(\frac{y}{x}\right)x - 2x}{x\,y^2}\right)$$
$$= -\frac{1}{y^2 x}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \frac{1}{yx} \, \mathrm{d}x \\
\phi &= \frac{\ln(x)}{y} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{\ln(x)}{y^2} + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\ln\left(\frac{y}{x}\right)x - 2x}{x\,y^2}$. Therefore equation (4) becomes

$$
\frac{\ln\left(\frac{y}{x}\right)x - 2x}{x\,y^2} = -\frac{\ln(x)}{y^2} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{\ln(x) + \ln\left(\frac{y}{x}\right) - 2}{y^2}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{\ln(x) + \ln\left(\frac{y}{x}\right) - 2}{y^2}\right) \mathrm{d}y \\
f(y) &= -\frac{\ln(x)}{y} - \frac{\ln\left(\frac{y}{x}\right)}{y} + \frac{1}{y} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{\ln\left(\frac{y}{x}\right)}{y} + \frac{1}{y} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln\left(\frac{y}{x}\right)}{y} + \frac{1}{y}$$

The solution becomes

$$y = \frac{\text{LambertW}\left(c_1 x\, \mathrm{e}\right)}{c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{\text{LambertW}\left(c_1 x\, \mathrm{e}\right)}{c_1} \tag{1}$$

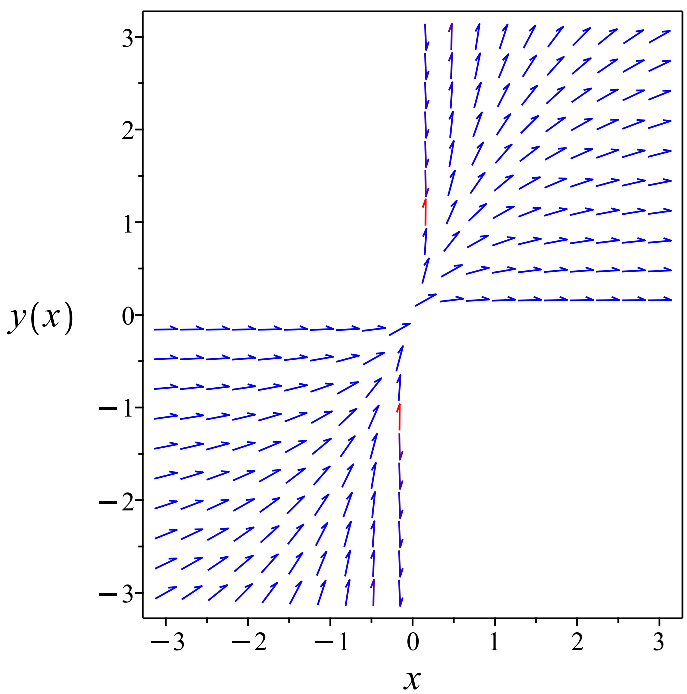


Figure 20: Slope field plot

Verification of solutions

$$y = \frac{\text{LambertW}\left(c_1 x\, \mathrm{e}\right)}{c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 16

```
dsolve(y(x)+x*ln(y(x)/x)*diff(y(x),x)-2*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\text{LambertW}\left(-\mathrm{e}x c_1\right)}{c_1}$$

✓ Solution by Mathematica
Time used: 5.502 (sec). Leaf size: 35

```
DSolve[y[x]+x*Log[y[x]/x]*y'[x]-2*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned}
y(x) &\to -e^{c_1} W\left(-e^{1-c_1} x\right) \\
y(x) &\to 0 \\
y(x) &\to e x
\end{aligned}$$

1.9 problem First order with homogeneous Coefficients. Exercise 7.10, page 61

Internal problem ID [4435]
Internal file name [`OUTPUT/3928_Sunday_June_05_2022_11_50_35_AM_32186788/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 7
Problem number: First order with homogeneous Coefficients. Exercise 7.10, page 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$
\boxed{2y\,\mathrm{e}^{\frac{x}{y}}+\left(y-2x\,\mathrm{e}^{\frac{x}{y}}\right)y'=0}
$$

1.9.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$
2u(x)\,x\,\mathrm{e}^{\frac{1}{u(x)}}+\left(u(x)\,x-2x\,\mathrm{e}^{\frac{1}{u(x)}}\right)\left(u'(x)\,x+u(x)\right)=0
$$

In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{u^2}{\left(-2\,\mathrm{e}^{\frac{1}{u}}+u\right)x}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2}{-2\,\mathrm{e}^{\frac{1}{u}}+u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{-2\,\mathrm{e}^{\frac{1}{u}}+u}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2}{-2\,\mathrm{e}^{\frac{1}{u}}+u}}\,du = \int -\frac{1}{x}\,dx$$

$$-\ln\left(\frac{1}{u}\right) + 2\,\mathrm{e}^{\frac{1}{u}} = -\ln(x) + c_2$$

The solution is

$$-\ln\left(\frac{1}{u(x)}\right) + 2\,\mathrm{e}^{\frac{1}{u(x)}} + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\ln\left(\frac{x}{y}\right) + 2\,\mathrm{e}^{\frac{x}{y}} + \ln(x) - c_2 = 0$$

$$-\ln\left(\frac{x}{y}\right) + 2\,\mathrm{e}^{\frac{x}{y}} + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$-\ln\left(\frac{x}{y}\right) + 2\,\mathrm{e}^{\frac{x}{y}} + \ln(x) - c_2 = 0 \tag{1}$$

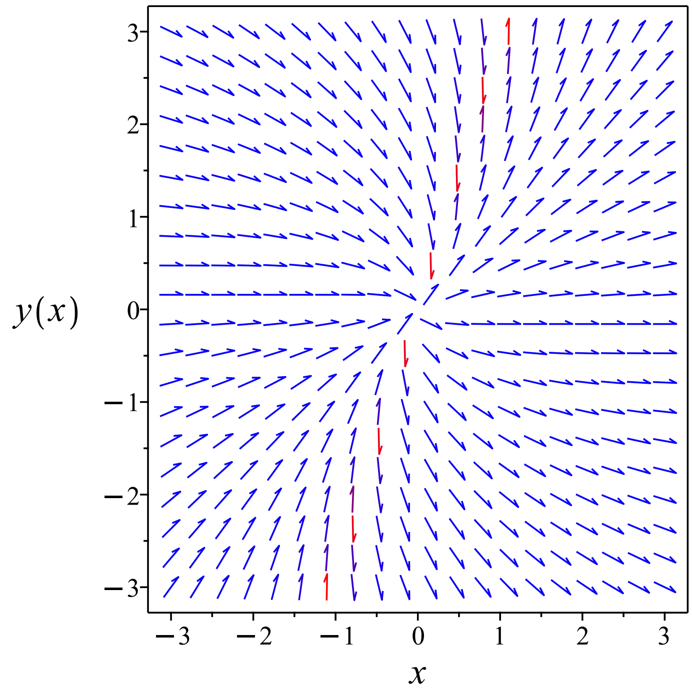


Figure 21: Slope field plot

Verification of solutions

$$-\ln\left(\frac{x}{y}\right)+2\,\mathrm{e}^{\frac{x}{y}}+\ln(x)-c_2=0$$

Verified OK.

1.9.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{2y\,\mathrm{e}^{\frac{x}{y}}}{2x\,\mathrm{e}^{\frac{x}{y}}-y}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{2y\,\mathrm{e}^{\frac{x}{y}}(b_3 - a_2)}{2x\,\mathrm{e}^{\frac{x}{y}} - y} - \frac{4y^2\mathrm{e}^{\frac{2x}{y}}a_3}{\left(2x\,\mathrm{e}^{\frac{x}{y}} - y\right)^2} \\ &- \left(\frac{2\,\mathrm{e}^{\frac{x}{y}}}{2x\,\mathrm{e}^{\frac{x}{y}} - y} - \frac{2y\,\mathrm{e}^{\frac{x}{y}}\left(2\,\mathrm{e}^{\frac{x}{y}} + \frac{2x\,\mathrm{e}^{\frac{x}{y}}}{y}\right)}{\left(2x\,\mathrm{e}^{\frac{x}{y}} - y\right)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(\frac{2\,\mathrm{e}^{\frac{x}{y}}}{2x\,\mathrm{e}^{\frac{x}{y}} - y} - \frac{2x\,\mathrm{e}^{\frac{x}{y}}}{y\left(2x\,\mathrm{e}^{\frac{x}{y}} - y\right)} - \frac{2y\,\mathrm{e}^{\frac{x}{y}}\left(-\frac{2x^2\mathrm{e}^{\frac{x}{y}}}{y^2} - 1\right)}{\left(2x\,\mathrm{e}^{\frac{x}{y}} - y\right)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4\,\mathrm{e}^{\frac{2x}{y}}xb_1 - 4\,\mathrm{e}^{\frac{2x}{y}}ya_1 + 2\,\mathrm{e}^{\frac{x}{y}}x^2b_2 - 2\,\mathrm{e}^{\frac{x}{y}}xya_2 + 4\,\mathrm{e}^{\frac{x}{y}}xyb_2 + 2\,\mathrm{e}^{\frac{x}{y}}xyb_3 - 2\,\mathrm{e}^{\frac{x}{y}}y^2a_2 - 2\,\mathrm{e}^{\frac{x}{y}}y^2a_3 + 2\,\mathrm{e}^{\frac{x}{y}}y^2b_3 + 2\,\mathrm{e}}{\left(2x\,\mathrm{e}^{\frac{x}{y}} - y\right)^2}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-4\,\mathrm{e}^{\frac{2x}{y}}xb_1 + 4\,\mathrm{e}^{\frac{2x}{y}}ya_1 - 2\,\mathrm{e}^{\frac{x}{y}}x^2b_2 + 2\,\mathrm{e}^{\frac{x}{y}}xya_2 - 4\,\mathrm{e}^{\frac{x}{y}}xyb_2 - 2\,\mathrm{e}^{\frac{x}{y}}xyb_3 \\ &+ 2\,\mathrm{e}^{\frac{x}{y}}y^2a_2 + 2\,\mathrm{e}^{\frac{x}{y}}y^2a_3 - 2\,\mathrm{e}^{\frac{x}{y}}y^2b_3 - 2\,\mathrm{e}^{\frac{x}{y}}xb_1 + 2\,\mathrm{e}^{\frac{x}{y}}ya_1 + y^2b_2 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-y^2\Big(4\,\mathrm{e}^{\frac{2x}{y}}xb_1 - 4\,\mathrm{e}^{\frac{2x}{y}}ya_1 + 2\,\mathrm{e}^{\frac{x}{y}}x^2b_2 - 2\,\mathrm{e}^{\frac{x}{y}}xya_2 + 4\,\mathrm{e}^{\frac{x}{y}}xyb_2 + 2\,\mathrm{e}^{\frac{x}{y}}xyb_3 \\ &- 2\,\mathrm{e}^{\frac{x}{y}}y^2a_2 - 2\,\mathrm{e}^{\frac{x}{y}}y^2a_3 + 2\,\mathrm{e}^{\frac{x}{y}}y^2b_3 + 2\,\mathrm{e}^{\frac{x}{y}}xb_1 - 2\,\mathrm{e}^{\frac{x}{y}}ya_1 - y^2b_2\Big) = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \mathrm{e}^{\frac{x}{y}}, \mathrm{e}^{\frac{2x}{y}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \mathrm{e}^{\frac{x}{y}} = v_3, \mathrm{e}^{\frac{2x}{y}} = v_4\right\}$$

The above PDE (6E) now becomes

$$-v_2^2\left(-2v_3v_1v_2a_2 - 2v_3v_2^2a_2 - 2v_3v_2^2a_3 + 2v_3v_1^2b_2 + 4v_3v_1v_2b_2 + 2v_3v_1v_2b_3 + 2v_3v_2^2b_3 - 2v_3v_2a_1 - 4v_4v_2a_1 + 2v_3v_1b_1 + 4v_4v_1b_1 - v_2^2b_2\right) = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$-2v_2^2b_2v_3v_1^2 + (2a_2 - 4b_2 - 2b_3)\, v_3v_1v_2^3 - 2b_1v_3v_1v_2^2 - 4b_1v_4v_1v_2^2 + (2a_2 + 2a_3 - 2b_3)\, v_3v_2^4 + b_2v_2^4 + 2a_1v_3v_2^3 + 4a_1v_4v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ 2a_1 &= 0 \\ 4a_1 &= 0 \\ -4b_1 &= 0 \\ -2b_1 &= 0 \\ -2b_2 &= 0 \\ 2a_2 + 2a_3 - 2b_3 &= 0 \\ 2a_2 - 4b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{2y\,\mathrm{e}^{\frac{x}{y}}}{2x\,\mathrm{e}^{\frac{x}{y}} - y}\right)(x) \\
&= -\frac{y^2}{2x\,\mathrm{e}^{\frac{x}{y}} - y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{y^2}{2x\,\mathrm{e}^{\frac{x}{y}} - y}} dy
\end{aligned}$$

Which results in

$$S = -\ln\left(\frac{1}{y}\right) + 2\,\mathrm{e}^{\frac{x}{y}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2y\,\mathrm{e}^{\frac{x}{y}}}{2x\,\mathrm{e}^{\frac{x}{y}} - y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2\,\mathrm{e}^{\frac{x}{y}}}{y} \\ S_y &= \frac{y - 2x\,\mathrm{e}^{\frac{x}{y}}}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) + 2\,\mathrm{e}^{\frac{x}{y}} = c_1$$

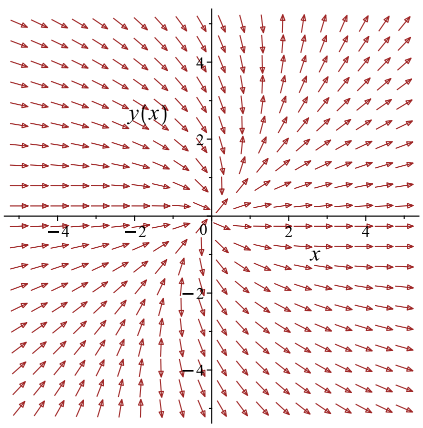
Which simplifies to

$$\ln (y) + 2\,\mathrm{e}^{\frac{x}{y}} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{2y\,\mathrm{e}^{\frac{x}{y}}}{2x\,\mathrm{e}^{\frac{x}{y}} - y}$



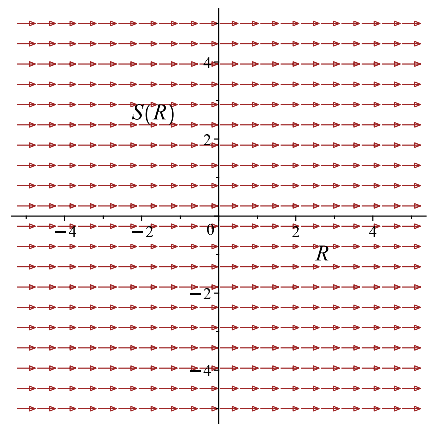
</td><td>

$R = x$

$S = \ln (y) + 2\,\mathrm{e}^{\frac{x}{y}}$

</td><td>

$\frac{dS}{dR} = 0$



</td></tr>
</table>

<u>Summary</u>

The solution(s) found are the following

$$\ln (y) + 2\,\mathrm{e}^{\frac{x}{y}} = c_1 \tag{1}$$

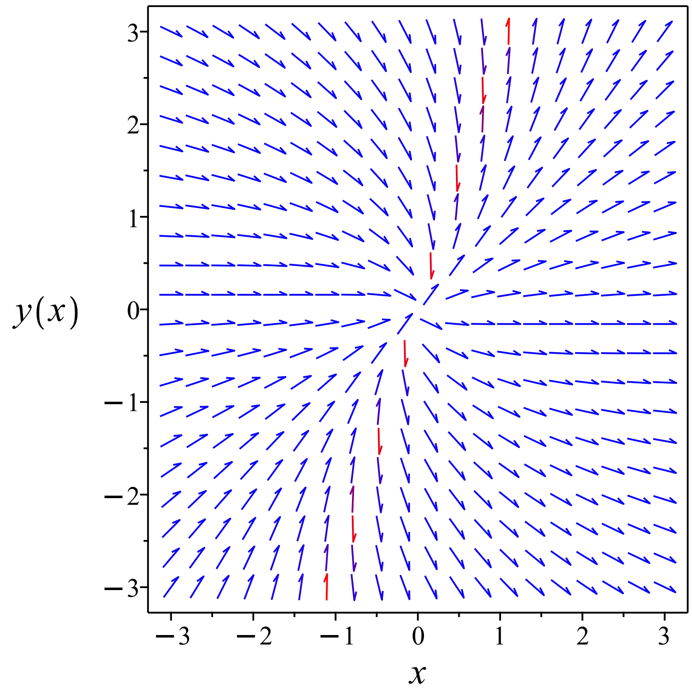


Figure 22: Slope field plot

Verification of solutions

$$\ln(y) + 2\,\mathrm{e}^{\frac{x}{y}} = c_1$$

Verified OK.

1.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(y - 2x\,\mathrm{e}^{\frac{x}{y}}\right)\mathrm{d}y &= \left(-2y\,\mathrm{e}^{\frac{x}{y}}\right)\mathrm{d}x\\ \left(2y\,\mathrm{e}^{\frac{x}{y}}\right)\mathrm{d}x + \left(y - 2x\,\mathrm{e}^{\frac{x}{y}}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2y\,\mathrm{e}^{\frac{x}{y}}\\ N(x, y) &= y - 2x\,\mathrm{e}^{\frac{x}{y}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y\,\mathrm{e}^{\frac{x}{y}}\right)\\ &= -\frac{2\,\mathrm{e}^{\frac{x}{y}}(x - y)}{y}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y - 2x\,\mathrm{e}^{\frac{x}{y}}\right) \\
&= -\frac{2\,\mathrm{e}^{\frac{x}{y}}(x+y)}{y}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{2x\,\mathrm{e}^{\frac{x}{y}} - y}\left(\left(2\,\mathrm{e}^{\frac{x}{y}} - \frac{2x\,\mathrm{e}^{\frac{x}{y}}}{y}\right) - \left(-2\,\mathrm{e}^{\frac{x}{y}} - \frac{2x\,\mathrm{e}^{\frac{x}{y}}}{y}\right)\right) \\
&= -\frac{4\,\mathrm{e}^{\frac{x}{y}}}{2x\,\mathrm{e}^{\frac{x}{y}} - y}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{\mathrm{e}^{-\frac{x}{y}}}{2y}\left(\left(-2\,\mathrm{e}^{\frac{x}{y}} - \frac{2x\,\mathrm{e}^{\frac{x}{y}}}{y}\right) - \left(2\,\mathrm{e}^{\frac{x}{y}} - \frac{2x\,\mathrm{e}^{\frac{x}{y}}}{y}\right)\right) \\
&= -\frac{2}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int -\frac{2}{y} \,\mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(y)} \\
&= \frac{1}{y^2}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^2}\left(2y\,\mathrm{e}^{\frac{x}{y}}\right) \\
&= \frac{2\,\mathrm{e}^{\frac{x}{y}}}{y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}\left(y - 2x\,\mathrm{e}^{\frac{x}{y}}\right) \\
&= \frac{y - 2x\,\mathrm{e}^{\frac{x}{y}}}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{2\,\mathrm{e}^{\frac{x}{y}}}{y}\right) + \left(\frac{y - 2x\,\mathrm{e}^{\frac{x}{y}}}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x
$$

$$
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{2\,\mathrm{e}^{\frac{x}{y}}}{y}\,\mathrm{d}x
$$

$$
\phi = 2\,\mathrm{e}^{\frac{x}{y}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{2x\,\mathrm{e}^{\frac{x}{y}}}{y^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{y-2x\,\mathrm{e}^{\frac{x}{y}}}{y^2}$. Therefore equation (4) becomes

$$\frac{y-2x\,\mathrm{e}^{\frac{x}{y}}}{y^2} = -\frac{2x\,\mathrm{e}^{\frac{x}{y}}}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(y) + 2\,\mathrm{e}^{\frac{x}{y}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(y) + 2\,\mathrm{e}^{\frac{x}{y}}$$

Summary

The solution(s) found are the following

$$\ln(y) + 2\,\mathrm{e}^{\frac{x}{y}} = c_1 \tag{1}$$

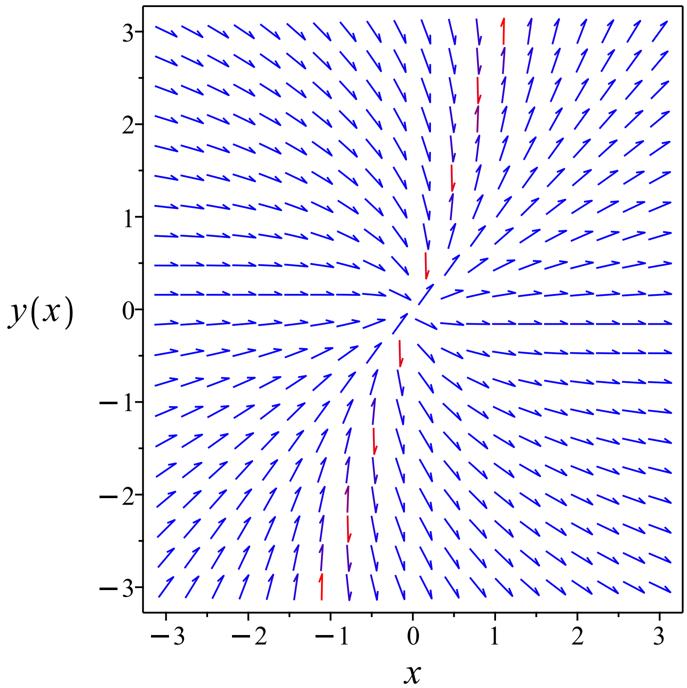


Figure 23: Slope field plot

Verification of solutions

$$\ln \left(y\right)+2\,{\mathrm e}^{\frac{x}{y}} = c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 21

```
dsolve(2*y(x)*exp(x/y(x))+(y(x)-2*x*exp(x/y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{\text{RootOf}\left(-\_Z\mathrm{e}^{-2\,\mathrm{e}^{\_Z}} + c_1 x\right)}$$

✓ Solution by Mathematica

Time used: 0.247 (sec). Leaf size: 29

```
DSolve[2*y[x]*Exp[x/y[x]]+(y[x]-2*x*Exp[x/y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$\text{Solve}\left[-2e^{\frac{x}{y(x)}} - \log\left(\frac{y(x)}{x}\right) = \log(x) + c_1, y(x)\right]$$

1.10 problem First order with homogeneous Coefficients. Exercise 7.11, page 61

Internal problem ID [4436]
Internal file name [`OUTPUT/3929_Sunday_June_05_2022_11_50_46_AM_16135110/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 7
Problem number: First order with homogeneous Coefficients. Exercise 7.11, page 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{x\,\mathrm{e}^{\frac{y}{x}}-y\sin\left(\frac{y}{x}\right)+x\sin\left(\frac{y}{x}\right)y'=0}$$

1.10.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x\,\mathrm{e}^{u(x)}-u(x)\,x\sin\left(u(x)\right)+x\sin\left(u(x)\right)\left(u'(x)\,x+u(x)\right)=0$$

In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=-\frac{\mathrm{e}^{u}\csc\left(u\right)}{x}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = e^u \csc(u)$. Integrating both sides gives

$$\frac{1}{e^u \csc(u)}\, du = -\frac{1}{x}\, dx$$
$$\int \frac{1}{e^u \csc(u)}\, du = \int -\frac{1}{x}\, dx$$
$$\frac{e^{-u}(-\sin(u) - \cos(u))}{2} = -\ln(x) + c_2$$

The solution is

$$\frac{e^{-u(x)}(-\sin(u(x)) - \cos(u(x)))}{2} + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{e^{-\frac{y}{x}}\left(-\sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right)\right)}{2} + \ln(x) - c_2 = 0$$
$$\frac{e^{-\frac{y}{x}}\left(-\sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right)\right)}{2} + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{e^{-\frac{y}{x}}\left(-\sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right)\right)}{2} + \ln(x) - c_2 = 0 \tag{1}$$

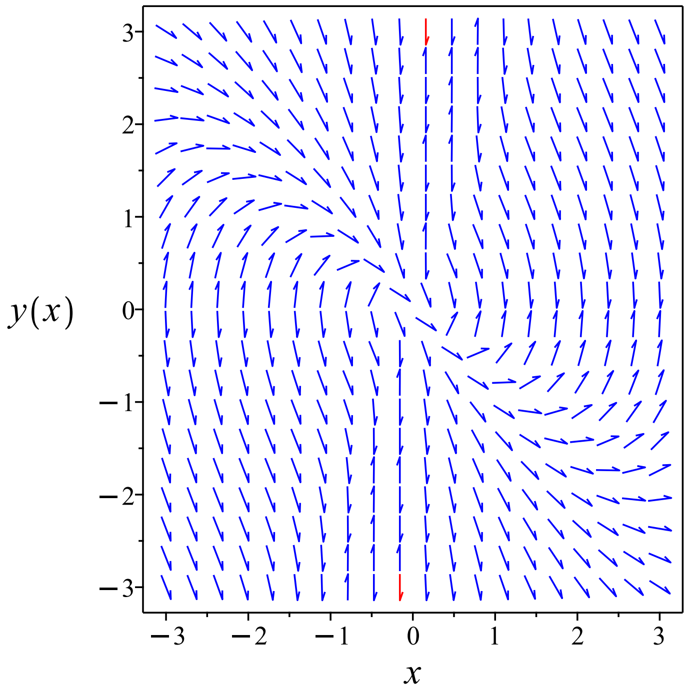


Figure 24: Slope field plot

Verification of solutions

$$\frac{\mathrm{e}^{-\frac{y}{x}}\left(-\sin\left(\frac{y}{x}\right)-\cos\left(\frac{y}{x}\right)\right)}{2}+\ln(x)-c_2=0$$

Verified OK.

1.10.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x\,\mathrm{e}^{\frac{y}{x}}-y\sin\left(\frac{y}{x}\right)}{x\sin\left(\frac{y}{x}\right)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{\left(x\,e^{\frac{y}{x}} - y\sin\left(\frac{y}{x}\right)\right)(b_3 - a_2)}{x\sin\left(\frac{y}{x}\right)} - \frac{\left(x\,e^{\frac{y}{x}} - y\sin\left(\frac{y}{x}\right)\right)^2 a_3}{x^2\sin\left(\frac{y}{x}\right)^2} - \left(-\frac{e^{\frac{y}{x}} - \frac{y\,e^{\frac{y}{x}}}{x} + \frac{y^2\cos\left(\frac{y}{x}\right)}{x^2}}{x\sin\left(\frac{y}{x}\right)} + \frac{x\,e^{\frac{y}{x}} - y\sin\left(\frac{y}{x}\right)}{x^2\sin\left(\frac{y}{x}\right)} - \frac{\left(x\,e^{\frac{y}{x}} - y\sin\left(\frac{y}{x}\right)\right) y\cos\left(\frac{y}{x}\right)}{x^3\sin\left(\frac{y}{x}\right)^2}\right)(x a_2 + y a_3 + a_1) - \left(-\frac{e^{\frac{y}{x}} - \sin\left(\frac{y}{x}\right) - \frac{y\cos\left(\frac{y}{x}\right)}{x}}{x\sin\left(\frac{y}{x}\right)} + \frac{\left(x\,e^{\frac{y}{x}} - y\sin\left(\frac{y}{x}\right)\right)\cos\left(\frac{y}{x}\right)}{x^2\sin\left(\frac{y}{x}\right)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-\sin\left(\frac{y}{x}\right) e^{\frac{y}{x}} x^2 a_2 - \sin\left(\frac{y}{x}\right) e^{\frac{y}{x}} x^2 b_2 + \sin\left(\frac{y}{x}\right) e^{\frac{y}{x}} x^2 b_3 + \sin\left(\frac{y}{x}\right) e^{\frac{y}{x}} x y a_2 - 2\sin\left(\frac{y}{x}\right) e^{\frac{y}{x}} x y a_3 - \sin\left(\frac{y}{x}\right) e^{\frac{y}{x}} x y b_3 \cdots}{\quad} = 0$$

Setting the numerator to zero gives

$$\sin\left(\frac{y}{x}\right) e^{\frac{y}{x}} x^2 a_2 + \sin\left(\frac{y}{x}\right) e^{\frac{y}{x}} x^2 b_2 - \sin\left(\frac{y}{x}\right) e^{\frac{y}{x}} x^2 b_3 - \sin\left(\frac{y}{x}\right) e^{\frac{y}{x}} x y a_2 + 2\sin\left(\frac{y}{x}\right) e^{\frac{y}{x}} x y a_3 + \sin\left(\frac{y}{x}\right) e^{\frac{y}{x}} x y b_3 - \sin\left(\frac{y}{x}\right) e^{\frac{y}{x}} y^2 a_3 - e^{\frac{2y}{x}} x^2 a_3 - e^{\frac{y}{x}}\cos\left(\frac{y}{x}\right) x^2 b_2 + e^{\frac{y}{x}}\cos\left(\frac{y}{x}\right) x y a_2 - e^{\frac{y}{x}}\cos\left(\frac{y}{x}\right) x y b_3 + e^{\frac{y}{x}}\cos\left(\frac{y}{x}\right) y^2 a_3 - \sin\left(\frac{y}{x}\right)^2 x b_1 + \sin\left(\frac{y}{x}\right)^2 y a_1 + \sin\left(\frac{y}{x}\right) e^{\frac{y}{x}} x b_1 - \sin\left(\frac{y}{x}\right) e^{\frac{y}{x}} y a_1 - e^{\frac{y}{x}}\cos\left(\frac{y}{x}\right) x b_1 + e^{\frac{y}{x}}\cos\left(\frac{y}{x}\right) y a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-\frac{x\left(2\,\mathrm{e}^{\frac{2y}{x}} x^2 a_3 + x b_1 - y a_1 - 2\sin\left(\frac{y}{x}\right)\mathrm{e}^{\frac{y}{x}} x^2 a_2 - 2\sin\left(\frac{y}{x}\right)\mathrm{e}^{\frac{y}{x}} x^2 b_2 + 2\sin\left(\frac{y}{x}\right)\mathrm{e}^{\frac{y}{x}} x^2 b_3 + 2\sin\left(\frac{y}{x}\right)\mathrm{e}^{\frac{y}{x}} x y a_2 - 4}{} \tag{6E}$$
$$= 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \cos\left(\frac{y}{x}\right), \cos\left(\frac{2y}{x}\right), \mathrm{e}^{\frac{y}{x}}, \mathrm{e}^{\frac{2y}{x}}, \sin\left(\frac{y}{x}\right)\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \cos\left(\frac{y}{x}\right) = v_3, \cos\left(\frac{2y}{x}\right) = v_4, \mathrm{e}^{\frac{y}{x}} = v_5, \mathrm{e}^{\frac{2y}{x}} = v_6, \sin\left(\frac{y}{x}\right) = v_7\right\}$$

The above PDE (6E) now becomes

$$-\frac{v_1(-2v_7v_5v_1^2a_2 - 2v_5v_3v_1v_2a_2 + 2v_7v_5v_1v_2a_2 - 4v_7v_5v_1v_2a_3 - 2v_5v_3v_2^2a_3 + 2v_7v_5v_2^2a_3 + 2v_5v_3v_1^2b_2 - 2v_7v}{} \tag{7E}$$
$$= 0$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-b_2v_3v_5v_1^3 + (a_2 + b_2 - b_3)\,v_5v_7v_1^3 - a_3v_6v_1^3 - \frac{b_1v_1^2}{2} + \frac{b_1v_4v_1^2}{2} - b_1v_3v_5v_1^2 \\
&+ b_1v_5v_7v_1^2 + (-b_3 + a_2)\,v_2v_3v_5v_1^2 + (-a_2 + 2a_3 + b_3)\,v_2v_5v_7v_1^2 + \frac{a_1v_2v_1}{2} \\
&- \frac{a_1v_2v_4v_1}{2} + a_3v_2^2v_3v_5v_1 - a_3v_2^2v_5v_7v_1 + a_1v_2v_3v_5v_1 - a_1v_2v_5v_7v_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0\\
a_3 &= 0\\
b_1 &= 0\\
-a_1 &= 0\\
-\frac{a_1}{2} &= 0\\
\frac{a_1}{2} &= 0\\
-a_3 &= 0\\
-b_1 &= 0\\
-\frac{b_1}{2} &= 0\\
\frac{b_1}{2} &= 0\\
-b_2 &= 0\\
-b_3 + a_2 &= 0\\
-a_2 + 2a_3 + b_3 &= 0\\
a_2 + b_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x\\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{x\,\mathrm{e}^{\frac{y}{x}} - y\sin\left(\frac{y}{x}\right)}{x\sin\left(\frac{y}{x}\right)}\right)(x) \\
&= \frac{\mathrm{e}^{\frac{y}{x}}x}{\sin\left(\frac{y}{x}\right)} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{\mathrm{e}^{\frac{y}{x}}x}{\sin\left(\frac{y}{x}\right)}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{\mathrm{e}^{-\frac{y}{x}}\cos\left(\frac{y}{x}\right)}{2} - \frac{\mathrm{e}^{-\frac{y}{x}}\sin\left(\frac{y}{x}\right)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x\,\mathrm{e}^{\frac{y}{x}} - y\sin\left(\frac{y}{x}\right)}{x\sin\left(\frac{y}{x}\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y\,\mathrm{e}^{-\frac{y}{x}}\sin\left(\frac{y}{x}\right)}{x^2} \\
S_y &= \frac{\mathrm{e}^{-\frac{y}{x}}\sin\left(\frac{y}{x}\right)}{x}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

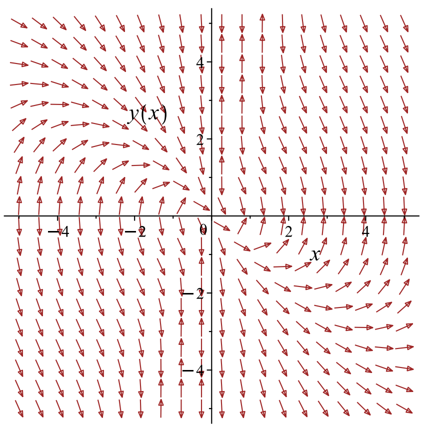
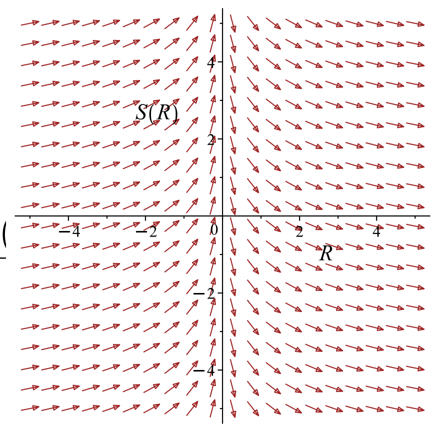
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\mathrm{e}^{-\frac{y}{x}}\left(\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right)}{2} = -\ln(x) + c_1$$

Which simplifies to

$$-\frac{\mathrm{e}^{-\frac{y}{x}}\left(\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right)}{2} = -\ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x\,e^{\frac{y}{x}} - y \sin\left(\frac{y}{x}\right)}{x \sin\left(\frac{y}{x}\right)}$ 	$R = x$ $S = -\frac{e^{-\frac{y}{x}}\left(\sin\left(\frac{y}{x}\right) + \cos(\right)}{2}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$-\frac{e^{-\frac{y}{x}}\left(\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right)}{2} = -\ln(x) + c_1 \tag{1}$$

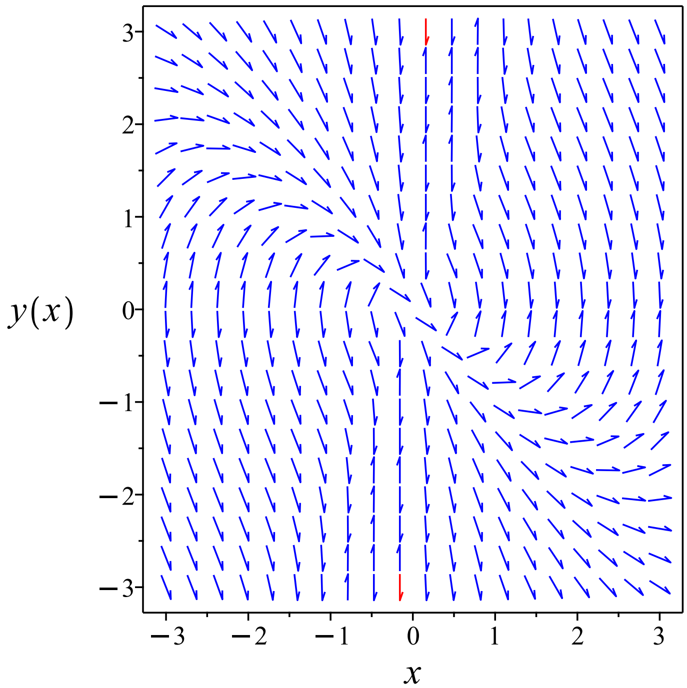


Figure 25: Slope field plot

Verification of solutions

$$-\frac{e^{-\frac{y}{x}}\left(\sin\left(\frac{y}{x}\right)+\cos\left(\frac{y}{x}\right)\right)}{2} = -\ln(x)+c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 63

```
dsolve((x*exp(y(x)/x)-y(x)*sin(y(x)/x))+x*sin(y(x)/x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{RootOf}\left(\mathrm{e}^{2\_Z}\left(4\ln\left(x\right)^{2}\mathrm{e}^{2\_Z} + 8\ln\left(x\right)\mathrm{e}^{2\_Z}c_{1} + 4\,\mathrm{e}^{2\_Z}c_{1}^{2} - 4\ln\left(x\right)\sin\left(\_Z\right)\mathrm{e}^{-\_Z} - 4\sin\left(\_Z\right)\mathrm{e}^{-\_Z}c_{1} + 2\sin\left(\_Z\right)^{2} - 1\right)\right)x$$

✓ Solution by Mathematica

Time used: 0.328 (sec). Leaf size: 39

```
DSolve[(x*Exp[y[x]/x]-y[x]*Sin[y[x]/x])+x*Sin[y[x]/x]*y'[x]==0,y[x],x,IncludeSingularSolutio
```

$$\mathrm{Solve}\left[-\frac{1}{2}e^{-\frac{y(x)}{x}}\left(\sin\left(\frac{y(x)}{x}\right)+\cos\left(\frac{y(x)}{x}\right)\right) = -\log(x)+c_1, y(x)\right]$$

1.11 problem First order with homogeneous Coefficients. Exercise 7.12, page 61

Internal problem ID [4437]
Internal file name [`OUTPUT/3930_Sunday_June_05_2022_11_50_57_AM_85380893/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 7
Problem number: First order with homogeneous Coefficients. Exercise 7.12, page 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 - 2xyy' = -x^2}$$

With initial conditions

$$[y(-1) = 0]$$

1.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{x^2 + y^2}{2xy} \end{aligned}$$

$f(x,y)$ is not defined at $y = 0$ therefore existence and uniqueness theorem do not apply.

1.11.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 - 2x^2 u(x)\left(u'(x)\,x + u(x)\right) = -x^2$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{u^2-1}{2ux}
\end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{u^2-1}{u}}\,du &= -\frac{1}{2x}\,dx\\
\int \frac{1}{\frac{u^2-1}{u}}\,du &= \int -\frac{1}{2x}\,dx\\
\frac{\ln\left(u-1\right)}{2} + \frac{\ln\left(u+1\right)}{2} &= -\frac{\ln\left(x\right)}{2} + c_2
\end{aligned}$$

The above can be written as

$$\begin{aligned}
\left(\frac{1}{2}\right)\left(\ln\left(u-1\right) + \ln\left(u+1\right)\right) &= -\frac{\ln\left(x\right)}{2} + 2c_2\\
\ln\left(u-1\right) + \ln\left(u+1\right) &= (2)\left(-\frac{\ln\left(x\right)}{2} + 2c_2\right)\\
&= -\ln\left(x\right) + 4c_2
\end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(u-1)+\ln(u+1)} = \mathrm{e}^{-\ln(x)+2c_2}$$

Which simplifies to

$$\begin{aligned}
u^2 - 1 &= \frac{2c_2}{x}\\
&= \frac{c_3}{x}
\end{aligned}$$

The solution is

$$u(x)^2 - 1 = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x}$$
$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x}$$

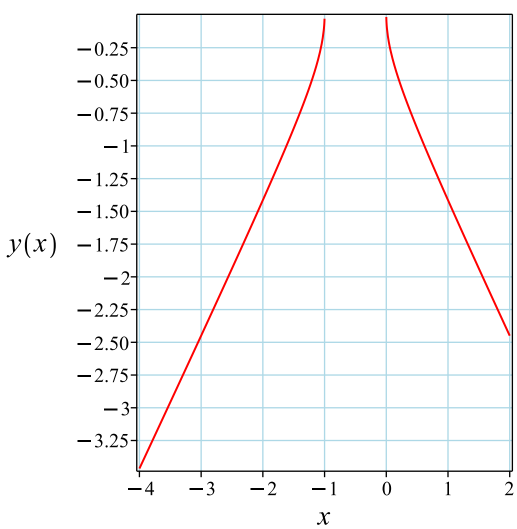
Substituting initial conditions and solving for c_3 gives $c_3 = 1$. Hence the solution becomes Solving for y from the above gives

$$y = \sqrt{x(x+1)}$$
$$y = -\sqrt{x(x+1)}$$

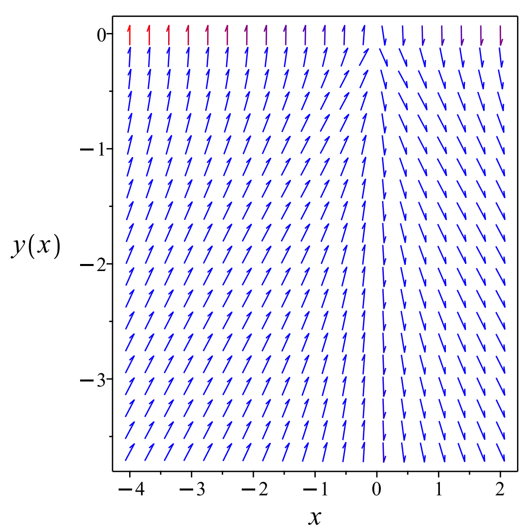
Summary

The solution(s) found are the following

$$y = \sqrt{x(x+1)} \tag{1}$$
$$y = -\sqrt{x(x+1)} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{x(x+1)}$$

Verified OK.

$$y = -\sqrt{x(x+1)}$$

Verified OK.

1.11.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 + y^2}{2xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 4: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x}{y}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x}{y}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{y^2}{2x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{x^2 + y^2}{2xy}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y^2}{2x^2} \\
S_y &= \frac{y}{x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R}{2} + c_1 \tag{4}
$$

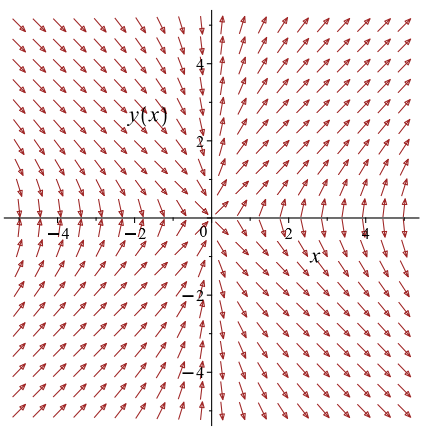
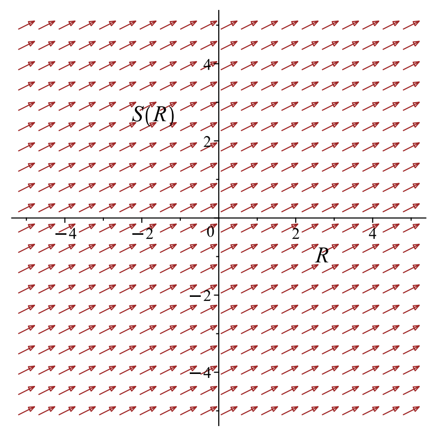
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y^2}{2x} = \frac{x}{2} + c_1
$$

Which simplifies to

$$
\frac{y^2}{2x} = \frac{x}{2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ 	$R = x$ $S = \frac{y^2}{2x}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{2} + c_1$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{y^2}{2x} = \frac{x}{2} + \frac{1}{2}$$

The above simplifies to

$$-x^2 + y^2 - x = 0$$

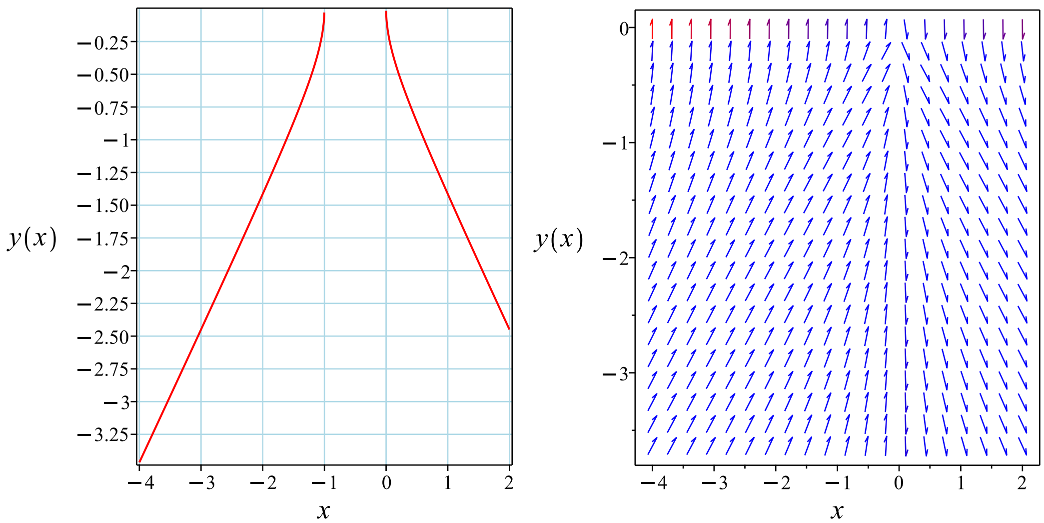
Solving for y from the above gives

$$y = \sqrt{x(x+1)}$$
$$y = -\sqrt{x(x+1)}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x(x+1)} \tag{1}$$
$$y = -\sqrt{x(x+1)} \tag{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sqrt{x(x+1)}$$

Verified OK.

$$y = -\sqrt{x(x+1)}$$

Verified OK.

1.11.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^2+y^2}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y + \frac{x}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{2x} \\ f_1(x) &= \frac{x}{2} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{y^2}{2x} + \frac{x}{2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{w(x)}{2x} + \frac{x}{2} \\ w' &= \frac{w}{x} + x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= x
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{w(x)}{x} = x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(x) \\
\mathrm{d}\left(\frac{w}{x}\right) &= \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x} &= \int \mathrm{d}x \\
\frac{w}{x} &= x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
w(x) = c_1 x + x^2
$$

which simplifies to

$$
w(x) = x(x + c_1)
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$
y^2 = x(x + c_1)
$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = 1 - c_1
$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y^2 = x(x+1)$$

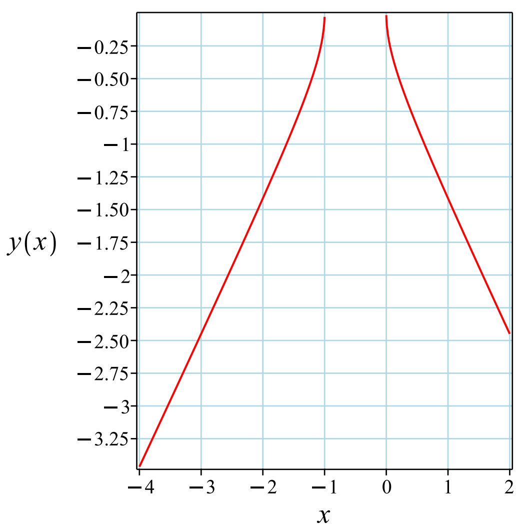
Solving for y from the above gives

$$y = \sqrt{x\left(x+1\right)}$$
$$y = -\sqrt{x\left(x+1\right)}$$

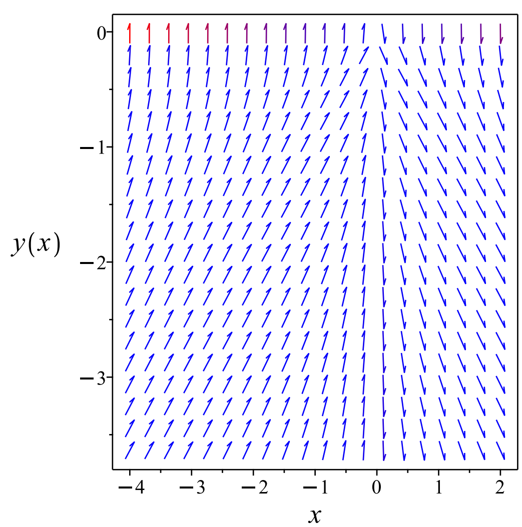
Summary

The solution(s) found are the following

$$y = \sqrt{x\left(x+1\right)} \tag{1}$$
$$y = -\sqrt{x\left(x+1\right)} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{x\left(x+1\right)}$$

Verified OK.

$$y = -\sqrt{x\left(x+1\right)}$$

Verified OK.

1.11.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2xy)\,\mathrm{d}y&=\left(-x^2-y^2\right)\mathrm{d}x\\ \left(x^2+y^2\right)\mathrm{d}x+(-2xy)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x^2+y^2\\ N(x,y)&=-2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2+y^2\right)\\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2xy)\\ &= -2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{2yx}((2y)-(-2y))\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}\left(x^2+y^2\right)\\ &= \frac{x^2+y^2}{x^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(-2xy) \\
&= -\frac{2y}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^2+y^2}{x^2}\right) + \left(-\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x^2+y^2}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = x - \frac{y^2}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{2y}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{x}$. Therefore equation (4) becomes

$$
-\frac{2y}{x} = -\frac{2y}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x - \frac{y^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x - \frac{y^2}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$x - \frac{y^2}{x} = -1$$

The above simplifies to

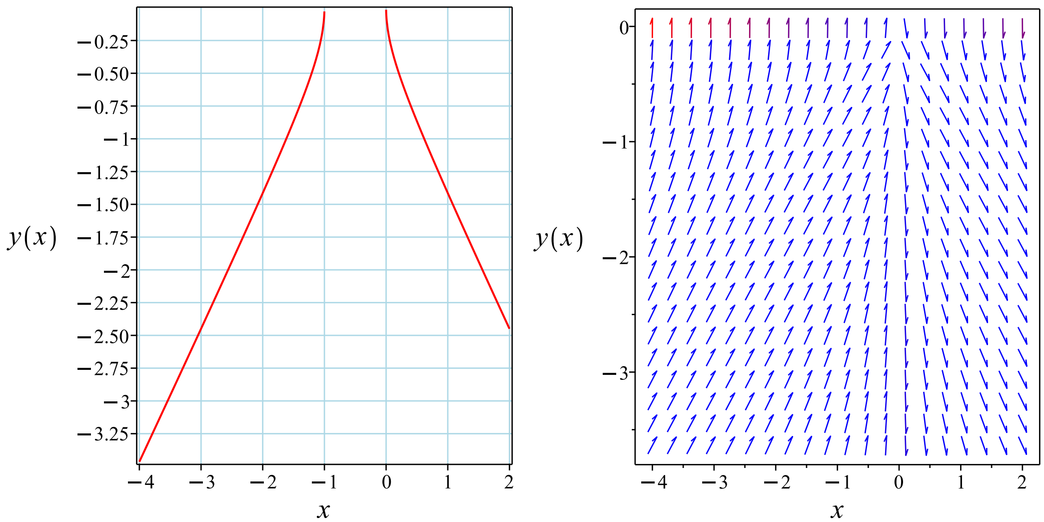
$$x^2 - y^2 + x = 0$$

Solving for y from the above gives

$$y = \sqrt{x(x+1)}$$
$$y = -\sqrt{x(x+1)}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \sqrt{x(x+1)} \tag{1}$$
$$y = -\sqrt{x(x+1)} \tag{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sqrt{x(x+1)}$$

Verified OK.

$$y = -\sqrt{x(x+1)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 23

```
dsolve([(x^2+y(x)^2)=2*x*y(x)*diff(y(x),x),y(-1) = 0],y(x), singsol=all)
```

$$y(x) = \sqrt{x(1+x)}$$
$$y(x) = -\sqrt{x(1+x)}$$

✓ Solution by Mathematica

Time used: 0.19 (sec). Leaf size: 36

```
DSolve[{(x^2+y[x]^2)==2*x*y[x]*y'[x],y[-1]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x}\sqrt{x+1}$$
$$y(x) \to \sqrt{x}\sqrt{x+1}$$

1.12 problem First order with homogeneous Coefficients. Exercise 7.13, page 61

Internal problem ID [4438]
Internal file name [`OUTPUT/3931_Sunday_June_05_2022_11_51_12_AM_72984459/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 7
Problem number: First order with homogeneous Coefficients. Exercise 7.13, page 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{x\,\mathrm{e}^{\frac{y}{x}}+y-xy'=0}$$

With initial conditions

$$[y(1)=0]$$

1.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned}y' &= f(x,y)\\ &= \frac{x\,\mathrm{e}^{\frac{y}{x}}+y}{x}\end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{-\infty<x<\infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x\,\mathrm{e}^{\frac{y}{x}} + y}{x}\right) \\ &= \frac{\mathrm{e}^{\frac{y}{x}} + 1}{x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

1.12.2 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \mathrm{e}^{\frac{y}{x}} + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$\begin{aligned}\frac{du}{dx}x + u &= u + g(x)\, f(bu)^{\frac{n}{m}} \\ u' &= \frac{1}{x}g(x)\, f(bu)^{\frac{n}{m}} \end{aligned} \tag{2}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$
\begin{aligned}
g(x) &= 1 \\
b &= 1 \\
f\left(\frac{bx}{y}\right) &= e^{\frac{y}{x}}
\end{aligned}
$$

Substituting the above in (2) results in the $u(x)$ ode as

$$
u'(x) = \frac{e^{u(x)}}{x}
$$

Which is now solved as separable In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{e^{u}}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = e^{u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{e^{u}}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{e^{u}}\,du &= \int \frac{1}{x}\,dx \\
-e^{-u} &= \ln(x) + c_1
\end{aligned}
$$

The solution is

$$
-e^{-u(x)} - \ln(x) - c_1 = 0
$$

Therefore the solution is found using $y = ux$. Hence

$$
-e^{-\frac{y}{x}} - \ln(x) - c_1 = 0
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
-1 - c_1 = 0
$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$-e^{-\frac{y}{x}} - \ln(x) + 1 = 0$$

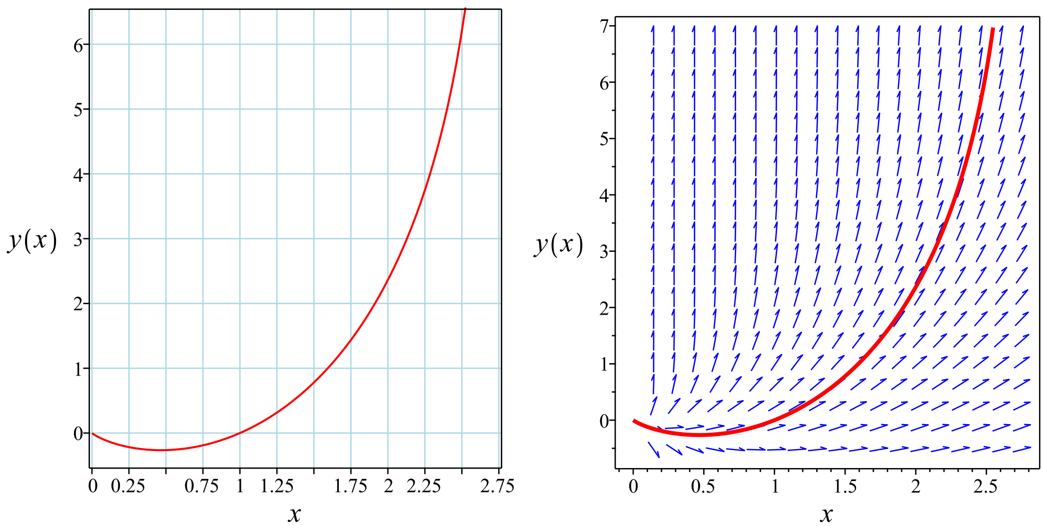
Solving for y from the above gives

$$y = -\ln(-\ln(x) + 1)\, x$$

Summary

The solution(s) found are the following

$$y = -\ln(-\ln(x) + 1)\, x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\ln(-\ln(x) + 1)\, x$$

Verified OK.

1.12.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x\,e^{u(x)} + u(x)\,x - x(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{e^{u}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = e^{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{e^{u}}\,du &= \frac{1}{x}\,dx \\ \int \frac{1}{e^{u}}\,du &= \int \frac{1}{x}\,dx \\ -e^{-u} &= \ln(x) + c_2 \end{aligned}$$

The solution is

$$-e^{-u(x)} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

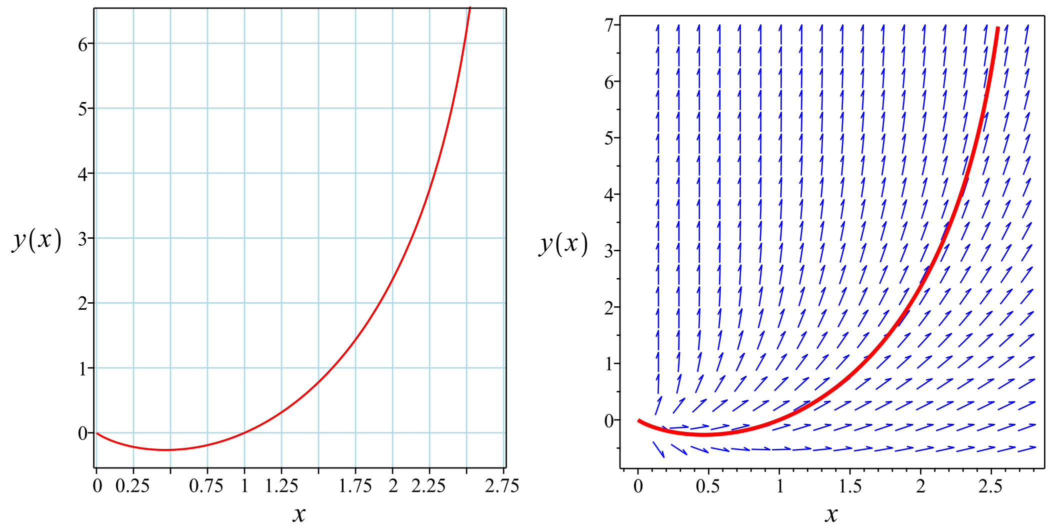
$$\begin{aligned} -e^{-\frac{y}{x}} - \ln(x) - c_2 &= 0 \\ -e^{-\frac{y}{x}} - \ln(x) - c_2 &= 0 \end{aligned}$$

Substituting initial conditions and solving for c_2 gives $c_2 = -1$. Hence the solution becomes Solving for y from the above gives

$$y = -\ln(-\ln(x) + 1)\,x$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\ln(-\ln(x) + 1)\,x \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\ln\left(-\ln\left(x\right)+1\right)x$$

Verified OK.

1.12.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x\,\mathrm{e}^{\frac{y}{x}}+y}{x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 6: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= x^2 \\ \eta(x,y) &= xy \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{xy}{x^2}\\ &= \frac{y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{x^2}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= -\frac{1}{x}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x\,\mathrm{e}^{\frac{y}{x}} + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{y}{x^2}\\ R_y &= \frac{1}{x}\\ S_x &= \frac{1}{x^2}\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{-\frac{y}{x}}}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -S(R)\, e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 e^{e^{-R}} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 e^{e^{-\frac{y}{x}}}$$

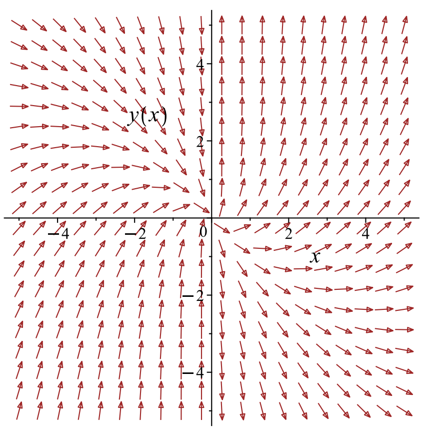
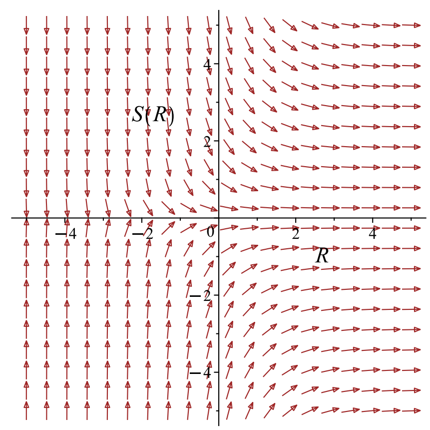
Which simplifies to

$$-\frac{1}{x} = c_1 e^{e^{-\frac{y}{x}}}$$

Which gives

$$y = -\ln\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x\,\mathrm{e}^{\frac{y}{x}}+y}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -S(R)\,\mathrm{e}^{-R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\ln\left(\ln\left(-\frac{1}{c_1}\right)\right)$$

$$c_1 = -\mathrm{e}^{-1}$$

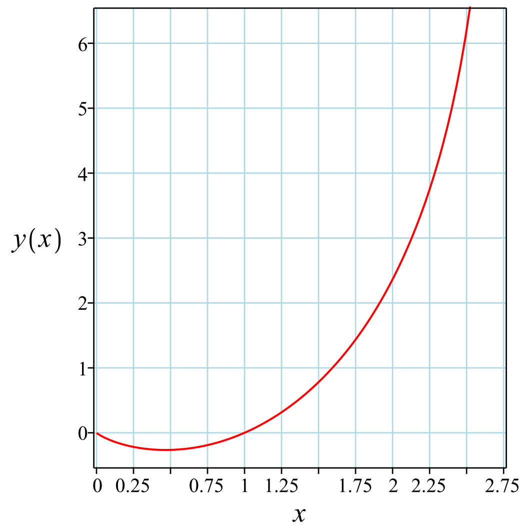
Substituting c_1 found above in the general solution gives

$$y = -\ln\left(1+\ln\left(\frac{1}{x}\right)\right)x$$

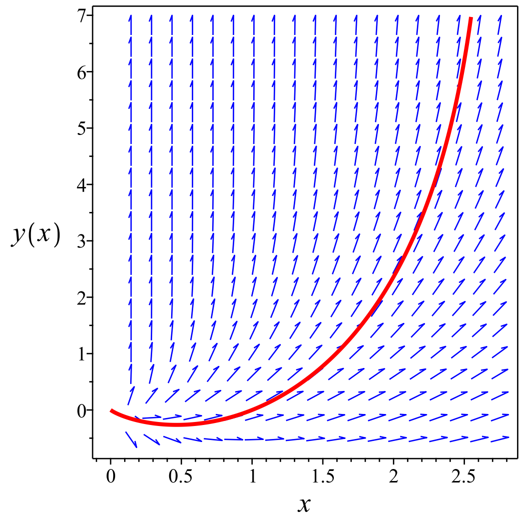
Summary

The solution(s) found are the following

$$y = -\ln\left(1+\ln\left(\frac{1}{x}\right)\right)x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\ln\left(1 + \ln\left(\frac{1}{x}\right)\right)x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve([(x*exp(y(x)/x)+y(x))=x*diff(y(x),x),y(1) = 0],y(x), singsol=all)
```

$$y(x) = \ln\left(-\frac{1}{\ln(x) - 1}\right) x$$

✓ Solution by Mathematica

Time used: 0.316 (sec). Leaf size: 15

```
DSolve[{(x*Exp[y[x]/x]+y[x])==x*y'[x],y[1]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x \log(1 - \log(x))$$

1.13 problem First order with homogeneous Coefficients. Exercise 7.14, page 61

Internal problem ID [4439]
Internal file name [`OUTPUT/3932_Sunday_June_05_2022_11_51_22_AM_40645187/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 7
Problem number: First order with homogeneous Coefficients. Exercise 7.14, page 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y' - \frac{y}{x} + \csc\left(\frac{y}{x}\right) = 0}$$

With initial conditions

$$[y(1) = 0]$$

1.13.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{\csc\left(\frac{y}{x}\right)x - y}{x} \end{aligned}$$

$f(x,y)$ is not defined at $y = 0$ therefore existence and uniqueness theorem do not apply.

1.13.2 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \frac{y}{x} - \csc\left(\frac{y}{x}\right) \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$\begin{aligned} \frac{du}{dx}x + u &= u + g(x)\, f(bu)^{\frac{n}{m}} \\ u' &= \frac{1}{x}g(x)\, f(bu)^{\frac{n}{m}} \end{aligned} \tag{2}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$\begin{aligned} g(x) &= -1 \\ b &= 1 \\ f\left(\frac{bx}{y}\right) &= \csc\left(\frac{y}{x}\right) \end{aligned}$$

Substituting the above in (2) results in the $u(x)$ ode as

$$u'(x) = -\frac{\csc(u(x))}{x}$$

Which is now solved as separable In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{\csc(u)}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \csc(u)$. Integrating both sides gives

$$\frac{1}{\csc(u)}\, du = -\frac{1}{x}\, dx$$
$$\int \frac{1}{\csc(u)}\, du = \int -\frac{1}{x}\, dx$$
$$-\cos(u) = -\ln(x) + c_1$$

The solution is

$$-\cos(u(x)) + \ln(x) - c_1 = 0$$

Therefore the solution is found using $y = ux$. Hence

$$-\cos\left(\frac{y}{x}\right) + \ln(x) - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-1 - c_1 = 0$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

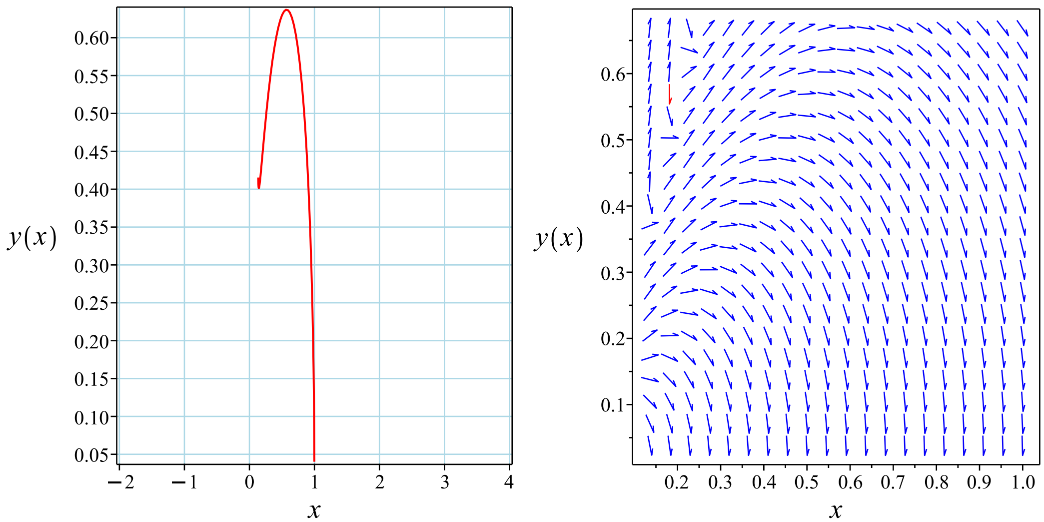
$$-\cos\left(\frac{y}{x}\right) + \ln(x) + 1 = 0$$

Solving for y from the above gives

$$y = \arccos(1 + \ln(x))\, x$$

Summary
The solution(s) found are the following

$$y = \arccos(1 + \ln(x))\, x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \arccos(1 + \ln(x))\, x$$

Verified OK.

1.13.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + \csc(u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{\csc(u)}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \csc(u)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\csc(u)}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{\csc(u)}\, du &= \int -\frac{1}{x}\, dx \\ -\cos(u) &= -\ln(x) + c_2 \end{aligned}$$

The solution is

$$-\cos(u(x)) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\cos\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$
$$-\cos\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

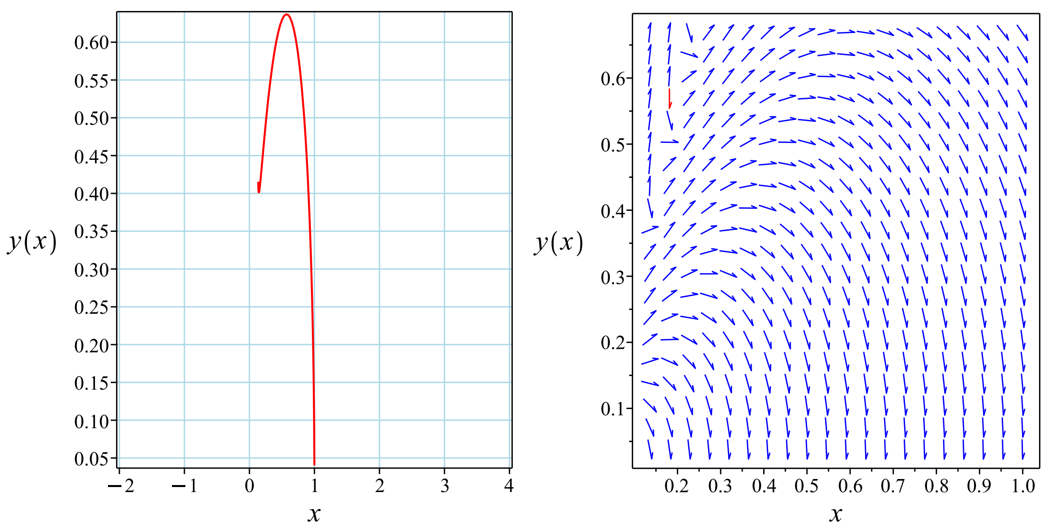
Substituting initial conditions and solving for c_2 gives $c_2 = -1$. Hence the solution becomes Solving for y from the above gives

$$y = \arccos(1 + \ln(x))\, x$$

Summary

The solution(s) found are the following

$$y = \arccos(1 + \ln(x))\, x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \arccos(1 + \ln(x))\, x$$

Verified OK.

1.13.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\csc\left(\frac{y}{x}\right)x - y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 8: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= x^2 \\ \eta(x, y) &= xy \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned} \frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{xy}{x^2} \\ &= \frac{y}{x} \end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x^2} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= -\frac{1}{x} \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\csc\left(\frac{y}{x}\right)x - y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= -\frac{y}{x^2} \\
R_y &= \frac{1}{x} \\
S_x &= \frac{1}{x^2} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\sin\left(\frac{y}{x}\right)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = S(R)\sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 e^{-\cos(R)} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 e^{-\cos\left(\frac{y}{x}\right)}$$

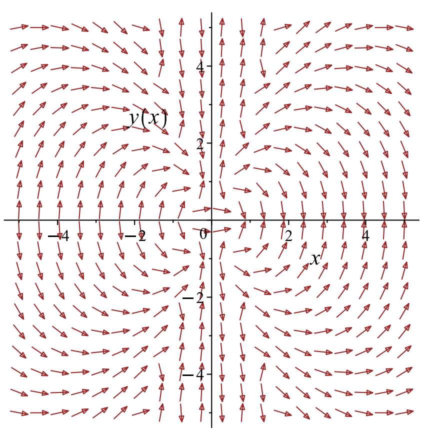
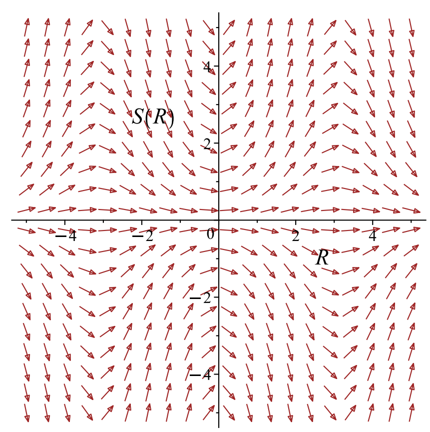
Which simplifies to

$$-\frac{1}{x} = c_1 \mathrm{e}^{-\cos\left(\frac{y}{x}\right)}$$

Which gives

$$y = -\left(-\pi + \arccos\left(\ln\left(-\frac{1}{c_1 x}\right)\right)\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\csc\left(\frac{y}{x}\right)x - y}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = S(R)\sin(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\pi}{2} + \arcsin\left(\ln\left(-\frac{1}{c_1}\right)\right)$$

$$c_1 = -\mathrm{e}$$

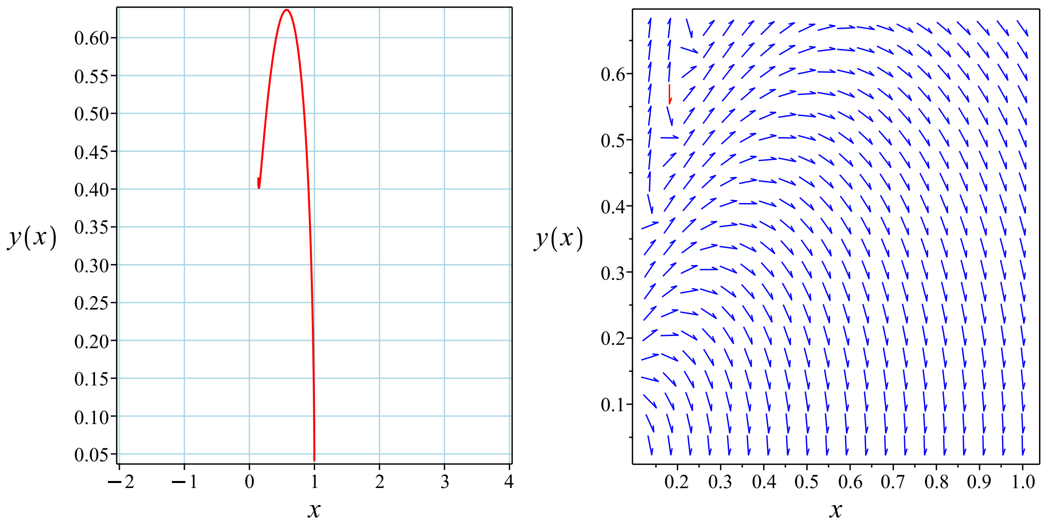
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi x}{2} + x \arcsin\left(-1 + \ln\left(\frac{1}{x}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{\pi x}{2} + x \arcsin\left(-1 + \ln\left(\frac{1}{x}\right)\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\pi x}{2} + x \arcsin\left(-1 + \ln\left(\frac{1}{x}\right)\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 22

```
dsolve([diff(y(x),x)-y(x)/x+csc(y(x)/x)=0,y(1) = 0],y(x), singsol=all)
```

$$y(x) = \arccos(\ln(x) + 1) x$$
$$y(x) = -\arccos(\ln(x) + 1) x$$

✓ Solution by Mathematica

Time used: 0.394 (sec). Leaf size: 24

```
DSolve[{y'[x]-y[x]/x+Csc[y[x]/x]==0,y[1]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x \arccos(\log(x) + 1)$$
$$y(x) \to x \arccos(\log(x) + 1)$$

1.14 problem First order with homogeneous Coefficients. Exercise 7.15, page 61

Internal problem ID [4440]
Internal file name [`OUTPUT/3933_Sunday_June_05_2022_11_51_35_AM_72031574/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 7
Problem number: First order with homogeneous Coefficients. Exercise 7.15, page 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{xy - y^2 - x^2y' = 0}$$

With initial conditions

$$[y(1) = 1]$$

1.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{y(-x + y)}{x^2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y(-x+y)}{x^2}\right)\\ &= -\frac{-x+y}{x^2} - \frac{y}{x^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.14.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^2u(x) - u(x)^2\,x^2 - x^2(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{u^2}{x}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u^2}\,du &= -\frac{1}{x}\,dx\\ \int \frac{1}{u^2}\,du &= \int -\frac{1}{x}\,dx\\ -\frac{1}{u} &= -\ln(x) + c_2\end{aligned}$$

The solution is

$$-\frac{1}{u(x)}+\ln(x)-c_2=0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{x}{y}+\ln(x)-c_2=0$$
$$-\frac{x}{y}+\ln(x)-c_2=0$$

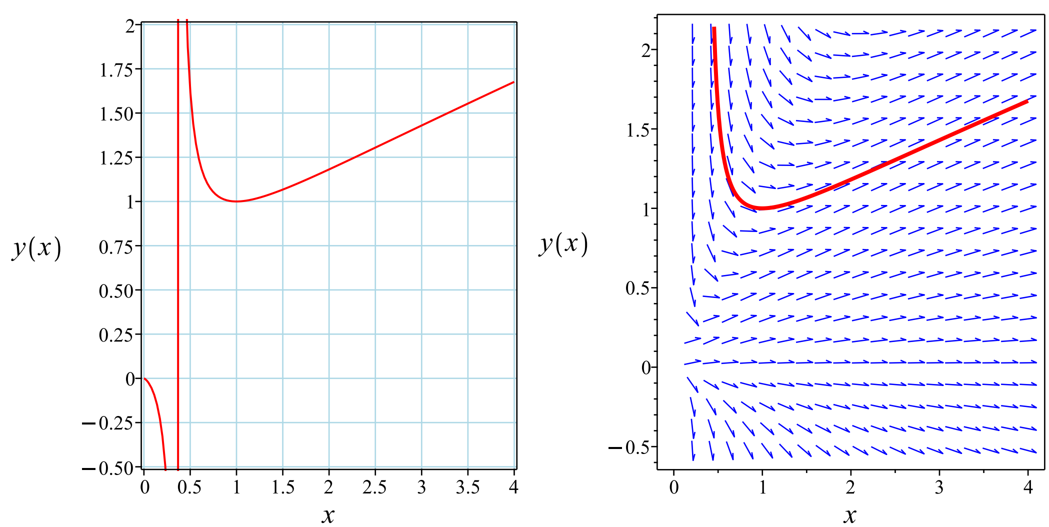
Substituting initial conditions and solving for c_2 gives $c_2=-1$. Hence the solution becomes Solving for y from the above gives

$$y=\frac{x}{1+\ln(x)}$$

Summary

The solution(s) found are the following

$$y=\frac{x}{1+\ln(x)} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y=\frac{x}{1+\ln(x)}$$

Verified OK.

1.14.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y(-x+y)}{x^2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 10: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \frac{y^2}{x} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(-x+y)}{x^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{y} \\
S_y &= \frac{x}{y^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = -\ln(x) + c_1$$

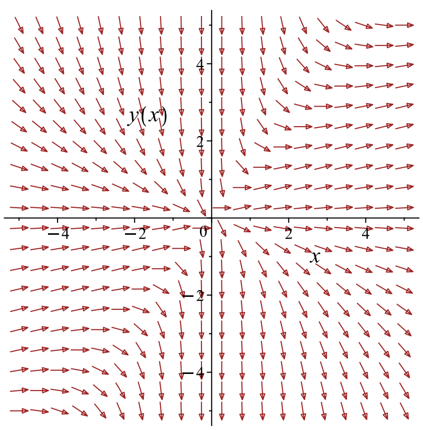
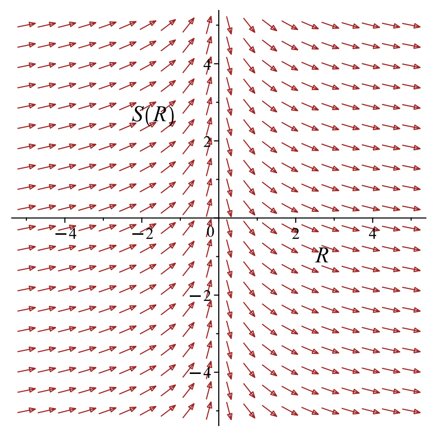
Which simplifies to

$$-\frac{x}{y} = -\ln(x) + c_1$$

Which gives

$$y = \frac{x}{\ln(x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(-x+y)}{x^2}$ 	$R = x$ $S = -\frac{x}{y}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{c_1}$$

$$c_1 = -1$$

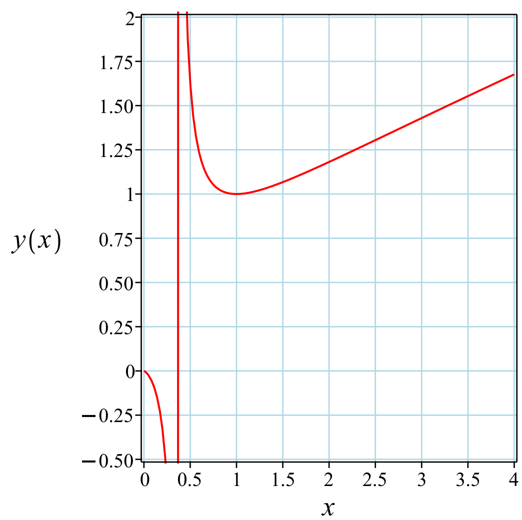
Substituting c_1 found above in the general solution gives

$$y = \frac{x}{1 + \ln(x)}$$

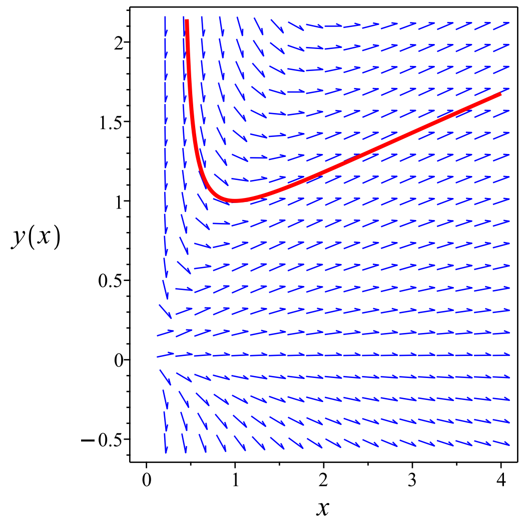
Summary

The solution(s) found are the following

$$y = \frac{x}{1 + \ln(x)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x}{1 + \ln(x)}$$

Verified OK.

1.14.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(-x + y)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y - \frac{1}{x^2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{x} \\
f_1(x) &= -\frac{1}{x^2} \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$
y' \frac{1}{y^2} = \frac{1}{yx} - \frac{1}{x^2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{1}{y^2} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= \frac{w(x)}{x} - \frac{1}{x^2} \\
w' &= -\frac{w}{x} + \frac{1}{x^2}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{w(x)}{x} = \frac{1}{x^2}
$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{1}{x^2}\right) \\ \frac{d}{dx}(xw) &= (x)\left(\frac{1}{x^2}\right) \\ d(xw) &= \frac{1}{x}\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}xw &= \int \frac{1}{x}\,dx \\ xw &= \ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = \frac{\ln(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$w(x) = \frac{\ln(x) + c_1}{x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{\ln(x) + c_1}{x}$$

Or

$$y = \frac{x}{\ln(x) + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{c_1}$$

$$c_1 = 1$$

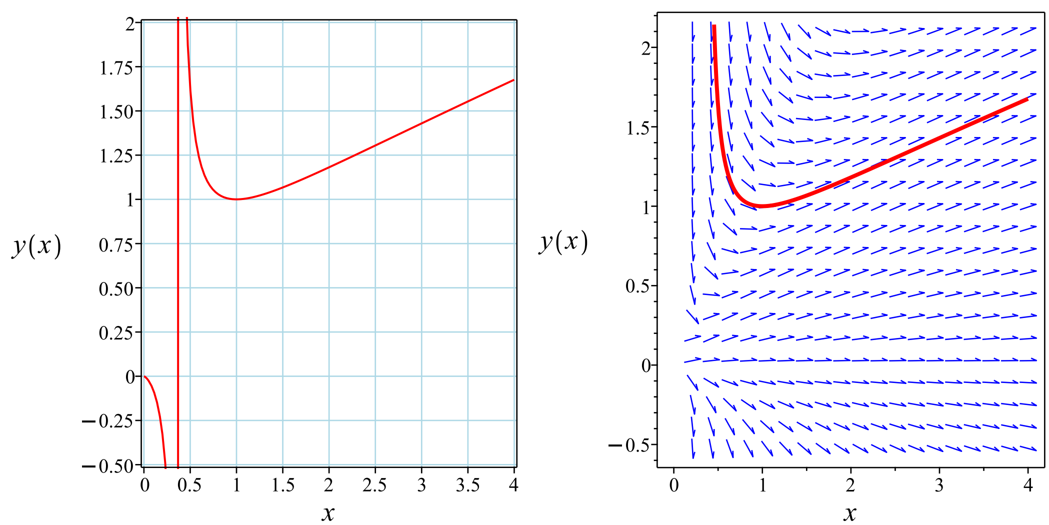
Substituting c_1 found above in the general solution gives

$$y = \frac{x}{1 + \ln(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{1 + \ln(x)} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{x}{1 + \ln(x)}$$

Verified OK.

1.14.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-x^2\right)\mathrm{d}y&=\left(-xy+y^2\right)\mathrm{d}x\\ \left(xy-y^2\right)\mathrm{d}x+\left(-x^2\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=xy-y^2\\ N(x,y)&=-x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(xy-y^2\right)\\ &=x-2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^2)\\ &= -2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = -y^2 + xy$ and $N = -x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{-y^2 + xy}{xy^2}\\ N &= -\frac{x}{y^2}\end{aligned}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{x}{y^2}\right)\mathrm{d}y = \left(-\frac{xy-y^2}{x\,y^2}\right)\mathrm{d}x$$
$$\left(\frac{xy-y^2}{x\,y^2}\right)\mathrm{d}x + \left(-\frac{x}{y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{xy-y^2}{x\,y^2}$$
$$N(x,y) = -\frac{x}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{xy-y^2}{x\,y^2}\right)$$
$$= -\frac{1}{y^2}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{x}{y^2}\right)$$
$$= -\frac{1}{y^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{xy - y^2}{x\,y^2} \, \mathrm{d}x$$

$$\phi = -\ln(x) + \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x}{y^2}$. Therefore equation (4) becomes

$$-\frac{x}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{x}{y}$$

The solution becomes

$$y = \frac{x}{\ln(x) + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{c_1}$$

$$c_1 = 1$$

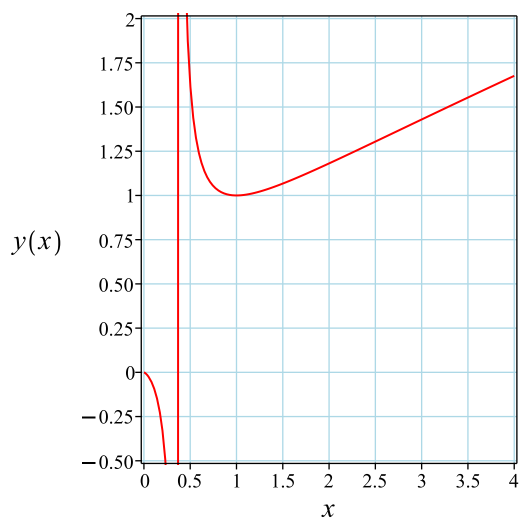
Substituting c_1 found above in the general solution gives

$$y = \frac{x}{1 + \ln(x)}$$

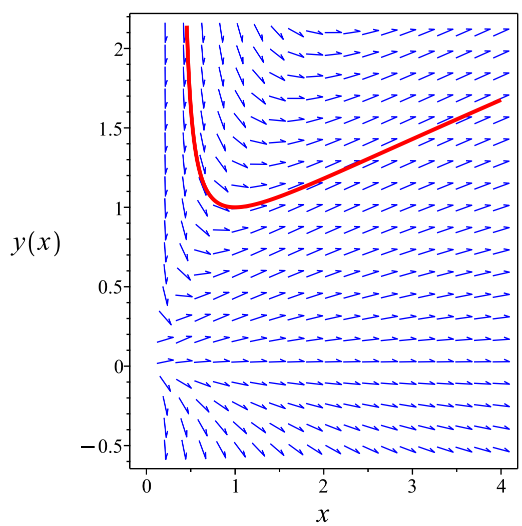
Summary

The solution(s) found are the following

$$y = \frac{x}{1 + \ln(x)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x}{1 + \ln(x)}$$

Verified OK.

1.14.6 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= -\frac{y(-x+y)}{x^2}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = \frac{y}{x} - \frac{y^2}{x^2}
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = -\frac{1}{x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{u}{x^2}} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= \frac{2}{x^3} \\
f_1 f_2 &= -\frac{1}{x^3} \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-\frac{u''(x)}{x^2} - \frac{u'(x)}{x^3} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_2 \ln(x) + c_1
$$

The above shows that

$$u'(x) = \frac{c_2}{x}$$

Using the above in (1) gives the solution

$$y = \frac{c_2 x}{c_2 \ln(x) + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x}{\ln(x) + c_3}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{c_3}$$

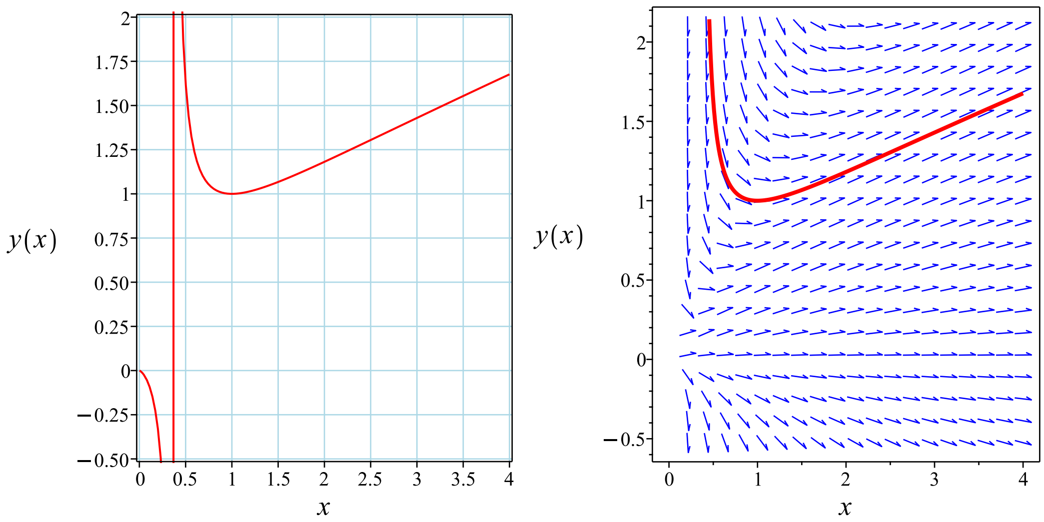
$$c_3 = 1$$

Substituting c_3 found above in the general solution gives

$$y = \frac{x}{1 + \ln(x)}$$

Summary
The solution(s) found are the following

$$y = \frac{x}{1 + \ln(x)} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{x}{1 + \ln(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve([(x*y(x)-y(x)^2)-x^2*diff(y(x),x)=0,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{x}{\ln(x) + 1}$$

✓ Solution by Mathematica

Time used: 0.139 (sec). Leaf size: 13

```
DSolve[{(x*y[x]-y[x]^2)-x^2*y'[x]==0,y[1]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{\log(x) + 1}$$

2 Chapter 2. Special types of differential equations of the first kind. Lesson 8

2.1 problem Differential equations with Linear Coefficients. Exercise 8.1, page 69

Internal problem ID [4441]
Internal file name [`OUTPUT/3934_Sunday_June_05_2022_11_51_46_AM_44108514/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 8
Problem number: Differential equations with Linear Coefficients. Exercise 8.1, page 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{2y - (2x - 4y)\, y' = -x + 4}$$

2.1.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{X + x_0 + 2Y(X) + 2y_0 - 4}{2\left(-X - x_0 + 2Y(X) + 2y_0\right)}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 2$$
$$y_0 = 1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{X + 2Y(X)}{2\left(-X + 2Y(X)\right)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X,Y) \\
&= -\frac{X+2Y}{2(-X+2Y)}
\end{aligned}
\tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X,Y)
$$

In this case, it can be seen that both $M = X + 2Y$ and $N = 2X - 4Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-1-2u}{4u-2} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-1-2u(X)}{4u(X)-2} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{\frac{-1-2u(X)}{4u(X)-2} - u(X)}{X} = 0
$$

Or

$$
4\left(\frac{d}{dX}u(X)\right)Xu(X) - 2\left(\frac{d}{dX}u(X)\right)X + 4u(X)^2 + 1 = 0
$$

Or

$$
1 + 2X(2u(X) - 1)\left(\frac{d}{dX}u(X)\right) + 4u(X)^2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X,u) \\
&= f(X)g(u) \\
&= -\frac{4u^2+1}{2X(2u-1)}
\end{aligned}
$$

Where $f(X) = -\frac{1}{2X}$ and $g(u) = \frac{4u^2+1}{2u-1}$. Integrating both sides gives

$$\frac{1}{\frac{4u^2+1}{2u-1}}\,du = -\frac{1}{2X}\,dX$$

$$\int \frac{1}{\frac{4u^2+1}{2u-1}}\,du = \int -\frac{1}{2X}\,dX$$

$$\frac{\ln(4u^2+1)}{4} - \frac{\arctan(2u)}{2} = -\frac{\ln(X)}{2} + c_2$$

The solution is

$$\frac{\ln\left(4u(X)^2+1\right)}{4} - \frac{\arctan(2u(X))}{2} + \frac{\ln(X)}{2} - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{4Y(X)^2}{X^2}+1\right)}{4} - \frac{\arctan\left(\frac{2Y(X)}{X}\right)}{2} + \frac{\ln(X)}{2} - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{4Y(X)^2}{X^2}+1\right)}{4} - \frac{\arctan\left(\frac{2Y(X)}{X}\right)}{2} + \frac{\ln(X)}{2} - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = 1 + y$$
$$X = x + 2$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{4(y-1)^2}{(-2+x)^2}+1\right)}{4} - \frac{\arctan\left(\frac{2y-2}{-2+x}\right)}{2} + \frac{\ln(-2+x)}{2} - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{4(y-1)^2}{(-2+x)^2}+1\right)}{4}-\frac{\arctan\left(\frac{2y-2}{-2+x}\right)}{2}+\frac{\ln\left(-2+x\right)}{2}-c_2=0 \tag{1}$$

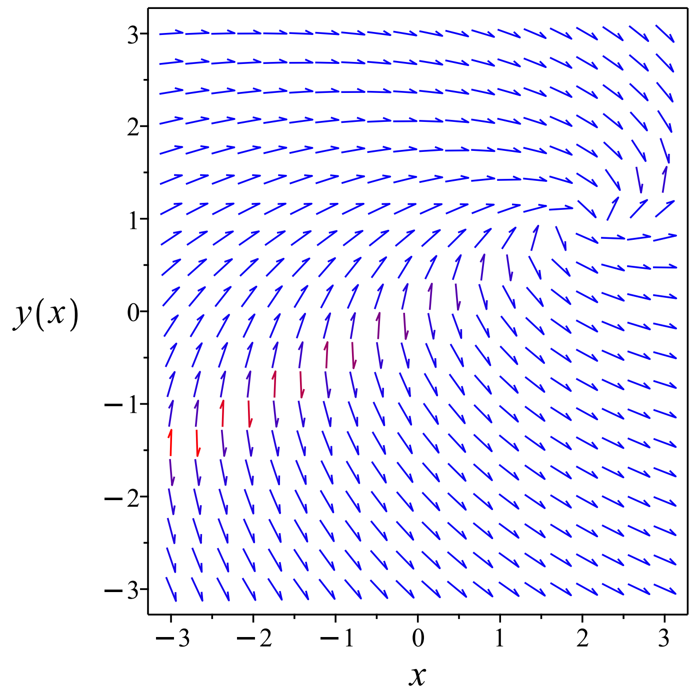


Figure 41: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{4(y-1)^2}{(-2+x)^2}+1\right)}{4}-\frac{\arctan\left(\frac{2y-2}{-2+x}\right)}{2}+\frac{\ln\left(-2+x\right)}{2}-c_2=0$$

Verified OK.

2.1.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{x+2y-4}{2\left(-x+2y\right)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(x+2y-4)(b_3-a_2)}{2(-x+2y)} - \frac{(x+2y-4)^2 a_3}{4(-x+2y)^2} \\ & - \left(-\frac{1}{2(-x+2y)} - \frac{x+2y-4}{2(-x+2y)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{1}{-x+2y} + \frac{x+2y-4}{(-x+2y)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^2a_2 + x^2a_3 + 4x^2b_2 - 2x^2b_3 - 8xya_2 + 4xya_3 + 16xyb_2 + 8xyb_3 - 8y^2a_2 - 4y^2a_3 - 16y^2b_2 + 8y^2b_3 -}{4(x-2y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -2x^2a_2 - x^2a_3 - 4x^2b_2 + 2x^2b_3 + 8xya_2 - 4xya_3 - 16xyb_2 \\ & - 8xyb_3 + 8y^2a_2 + 4y^2a_3 + 16y^2b_2 - 8y^2b_3 + 8xa_3 - 8xb_1 + 16xb_2 \\ & - 8xb_3 + 8ya_1 - 16ya_2 + 8ya_3 + 32yb_3 - 8a_1 - 16a_3 + 16b_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2a_2v_1^2 + 8a_2v_1v_2 + 8a_2v_2^2 - a_3v_1^2 - 4a_3v_1v_2 + 4a_3v_2^2 - 4b_2v_1^2 - 16b_2v_1v_2 \\
&+ 16b_2v_2^2 + 2b_3v_1^2 - 8b_3v_1v_2 - 8b_3v_2^2 + 8a_1v_2 - 16a_2v_2 + 8a_3v_1 \\
&+ 8a_3v_2 - 8b_1v_1 + 16b_2v_1 - 8b_3v_1 + 32b_3v_2 - 8a_1 - 16a_3 + 16b_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-2a_2 - a_3 - 4b_2 + 2b_3)\, v_1^2 + (8a_2 - 4a_3 - 16b_2 - 8b_3)\, v_1v_2 \\
&+ (8a_3 - 8b_1 + 16b_2 - 8b_3)\, v_1 + (8a_2 + 4a_3 + 16b_2 - 8b_3)\, v_2^2 \\
&+ (8a_1 - 16a_2 + 8a_3 + 32b_3)\, v_2 - 8a_1 - 16a_3 + 16b_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-8a_1 - 16a_3 + 16b_1 &= 0 \\
8a_1 - 16a_2 + 8a_3 + 32b_3 &= 0 \\
-2a_2 - a_3 - 4b_2 + 2b_3 &= 0 \\
8a_2 - 4a_3 - 16b_2 - 8b_3 &= 0 \\
8a_2 + 4a_3 + 16b_2 - 8b_3 &= 0 \\
8a_3 - 8b_1 + 16b_2 - 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 4b_2 - 2b_3 \\
a_2 &= b_3 \\
a_3 &= -4b_2 \\
b_1 &= -2b_2 - b_3 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2 + x \\
\eta &= y - 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - 1 - \left(-\frac{x + 2y - 4}{2(-x + 2y)}\right)(-2 + x) \\
&= \frac{-x^2 - 4y^2 + 4x + 8y - 8}{2x - 4y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2 - 4y^2 + 4x + 8y - 8}{2x - 4y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + 4y^2 - 4x - 8y + 8)}{2} + \frac{4(2 - x)\arctan\left(\frac{8y - 8}{-8 + 4x}\right)}{-8 + 4x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + 2y - 4}{2(-x + 2y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x + 2y - 4}{x^2 + 4y^2 - 4x - 8y + 8} \\
S_y &= \frac{-2x + 4y}{x^2 + 4y^2 - 4x - 8y + 8}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

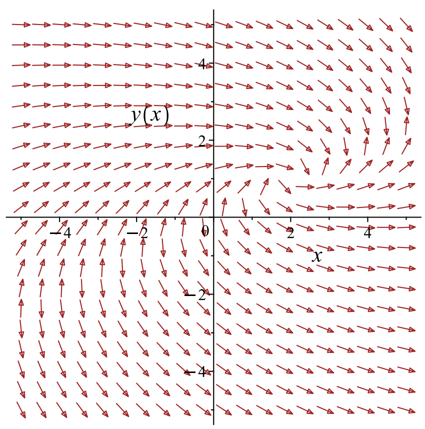
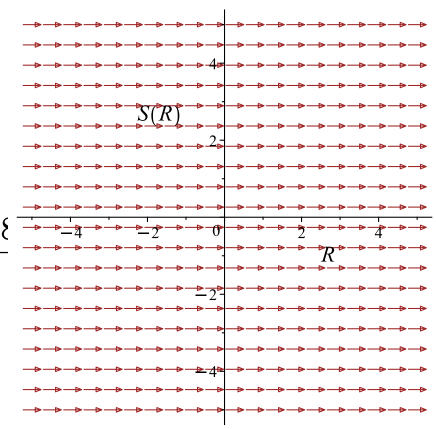
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(4y^2 + x^2 - 8y - 4x + 8)}{2} - \arctan\left(\frac{2y - 2}{-2 + x}\right) = c_1
$$

Which simplifies to

$$
\frac{\ln(4y^2 + x^2 - 8y - 4x + 8)}{2} - \arctan\left(\frac{2y - 2}{-2 + x}\right) = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+2y-4}{2(-x+2y)}$ 	$R = x$ $S = \frac{\ln (x^2 + 4y^2 - 4x - 8}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln (4y^2 + x^2 - 8y - 4x + 8)}{2} - \arctan\left(\frac{2y - 2}{-2 + x}\right) = c_1 \tag{1}$$

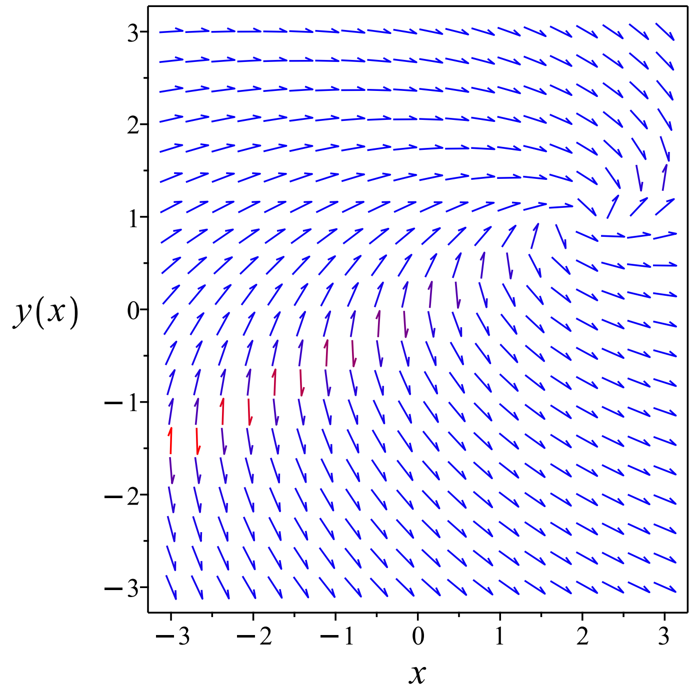


Figure 42: Slope field plot

Verification of solutions

$$\frac{\ln\left(4y^2+x^2-8y-4x+8\right)}{2}-\arctan\left(\frac{2y-2}{-2+x}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 31

```
dsolve((x+2*y(x)-4)-(2*x-4*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 1 - \frac{\tan\left(\mathrm{RootOf}\left(2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x-2\right) + 2c_1\right)\right)\left(x-2\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 63

```
DSolve[(x+2*y[x]-4)-(2*x-4*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[2\arctan\left(\frac{-2y(x)-x+4}{x-2y(x)}\right) + \log\left(\frac{x^2+4y(x)^2-8y(x)-4x+8}{2(x-2)^2}\right) + 2\log(x-2) + c_1 = 0, y(x)\right]$$

2.2 problem Differential equations with Linear Coefficients. Exercise 8.2, page 69

Internal problem ID [4442]
Internal file name [`OUTPUT/3935_Sunday_June_05_2022_11_51_58_AM_89538891/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 8
Problem number: Differential equations with Linear Coefficients. Exercise 8.2, page 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{2y - (3x + 2y - 1)\, y' = -3x - 1}$$

2.2.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{3x + 2y + 1}{3x + 2y - 1} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{(3x+2y+1)(b_3-a_2)}{3x+2y-1} - \frac{(3x+2y+1)^2 a_3}{(3x+2y-1)^2} \\ &- \left(\frac{3}{3x+2y-1} - \frac{3(3x+2y+1)}{(3x+2y-1)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(\frac{2}{3x+2y-1} - \frac{2(3x+2y+1)}{(3x+2y-1)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{9x^2a_2 + 9x^2a_3 - 9x^2b_2 - 9x^2b_3 + 12xya_2 + 12xya_3 - 12xyb_2 - 12xyb_3 + 4y^2a_2 + 4y^2a_3 - 4y^2b_2 - 4y^2b_3}{(3x+2y-1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-9x^2a_2 - 9x^2a_3 + 9x^2b_2 + 9x^2b_3 - 12xya_2 - 12xya_3 + 12xyb_2 \\ &+ 12xyb_3 - 4y^2a_2 - 4y^2a_3 + 4y^2b_2 + 4y^2b_3 + 6xa_2 - 6xa_3 - 2xb_2 \\ &+ 2ya_3 - 4yb_2 + 4yb_3 + 6a_1 + a_2 - a_3 + 4b_1 + b_2 - b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-9a_2v_1^2 - 12a_2v_1v_2 - 4a_2v_2^2 - 9a_3v_1^2 - 12a_3v_1v_2 - 4a_3v_2^2 + 9b_2v_1^2 \\ &+ 12b_2v_1v_2 + 4b_2v_2^2 + 9b_3v_1^2 + 12b_3v_1v_2 + 4b_3v_2^2 + 6a_2v_1 - 6a_3v_1 \\ &+ 2a_3v_2 - 2b_2v_1 - 4b_2v_2 + 4b_3v_2 + 6a_1 + a_2 - a_3 + 4b_1 + b_2 - b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(-9a_2 - 9a_3 + 9b_2 + 9b_3) v_1^2 + (-12a_2 - 12a_3 + 12b_2 + 12b_3) v_1 v_2 \\ &+ (6a_2 - 6a_3 - 2b_2) v_1 + (-4a_2 - 4a_3 + 4b_2 + 4b_3) v_2^2 \\ &+ (2a_3 - 4b_2 + 4b_3) v_2 + 6a_1 + a_2 - a_3 + 4b_1 + b_2 - b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}6a_2 - 6a_3 - 2b_2 &= 0 \\ 2a_3 - 4b_2 + 4b_3 &= 0 \\ -12a_2 - 12a_3 + 12b_2 + 12b_3 &= 0 \\ -9a_2 - 9a_3 + 9b_2 + 9b_3 &= 0 \\ -4a_2 - 4a_3 + 4b_2 + 4b_3 &= 0 \\ 6a_1 + a_2 - a_3 + 4b_1 + b_2 - b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= a_1 \\ a_2 &= -9a_1 - 6b_1 \\ a_3 &= -6a_1 - 4b_1 \\ b_1 &= b_1 \\ b_2 &= -9a_1 - 6b_1 \\ b_3 &= -6a_1 - 4b_1\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -6x - 4y \\ \eta &= -6x - 4y + 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -6x - 4y + 1 - \left(\frac{3x+2y+1}{3x+2y-1}\right)(-6x-4y) \\
&= \frac{15x+10y-1}{3x+2y-1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{15x+10y-1}{3x+2y-1}} dy
\end{aligned}$$

Which results in

$$S = \frac{y}{5} - \frac{2\ln(15x+10y-1)}{25}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{3x+2y+1}{3x+2y-1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{6}{75x + 50y - 5} \\
S_y &= \frac{3x + 2y - 1}{15x + 10y - 1}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{5} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{5}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R}{5} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y}{5} - \frac{2\ln\left(15x + 10y - 1\right)}{25} = \frac{x}{5} + c_1
$$

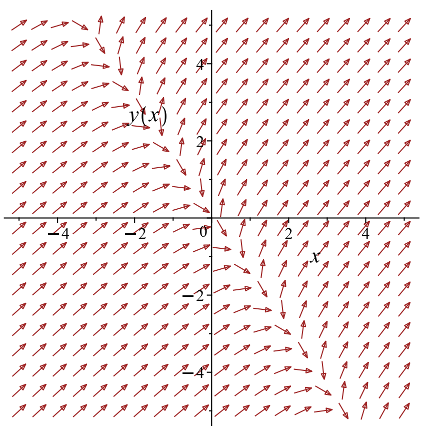
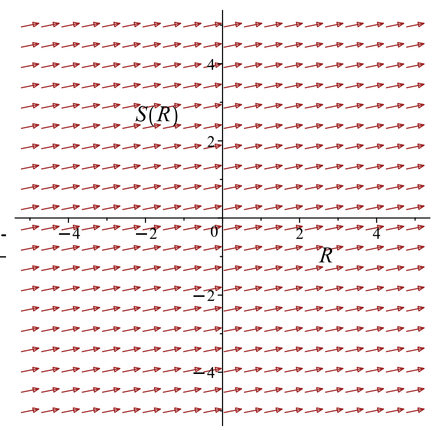
Which simplifies to

$$
\frac{y}{5} - \frac{2\ln\left(15x + 10y - 1\right)}{25} = \frac{x}{5} + c_1
$$

Which gives

$$
y = -\frac{3x}{2} - \frac{2\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-\frac{25x}{4}+\frac{1}{4}-\frac{25c_1}{2}}}{4}\right)}{5} + \frac{1}{10}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x+2y+1}{3x+2y-1}$ 	$R = x$ $S = \frac{y}{5} - \frac{2\ln(15x + 10y -}{25}$	$\frac{dS}{dR} = \frac{1}{5}$ 

Summary
The solution(s) found are the following

$$y = -\frac{3x}{2} - \frac{2\,\mathrm{LambertW}\left(-\frac{e^{-\frac{25x}{4}+\frac{1}{4}-\frac{25c_1}{2}}}{4}\right)}{5} + \frac{1}{10} \tag{1}$$

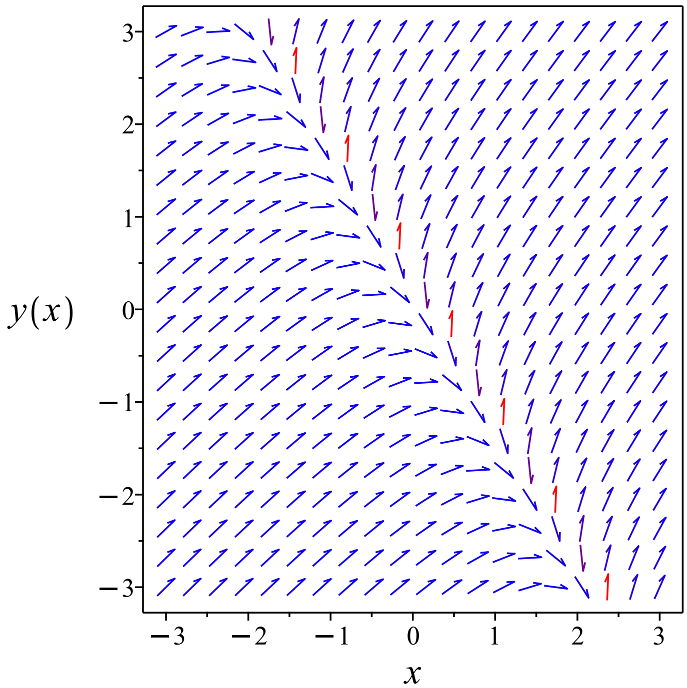


Figure 43: Slope field plot

Verification of solutions

$$y = -\frac{3x}{2} - \frac{2\,\mathrm{LambertW}\left(-\frac{e^{-\frac{25x}{4}+\frac{1}{4}-\frac{25c_1}{2}}}{4}\right)}{5} + \frac{1}{10}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -3/2, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve((3*x+2*y(x)+1)-(3*x+2*y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{3x}{2} - \frac{2\,\mathrm{LambertW}\left(-\frac{c_1 e^{\frac{1}{4}-\frac{25x}{4}}}{4}\right)}{5} + \frac{1}{10}$$

✓ Solution by Mathematica

Time used: 4.816 (sec). Leaf size: 43

```
DSolve[(3*x+2*y[x]+1)-(3*x+2*y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{10}\left(-4W\left(-e^{-\frac{25x}{4}-1+c_1}\right) - 15x + 1\right)$$
$$y(x) \to \frac{1}{10} - \frac{3x}{2}$$

2.3 problem Differential equations with Linear Coefficients. Exercise 8.3, page 69

Internal problem ID [4443]
Internal file name [`OUTPUT/3936_Sunday_June_05_2022_11_52_07_AM_25632392/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 8
Problem number: Differential equations with Linear Coefficients. Exercise 8.3, page 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y + (2x + 2y + 2)\, y' = -1 - x}$$

2.3.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{1}{2}\, \mathrm{d}x \\ &= -\frac{x}{2} + c_1 \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{x}{2} + c_1 \tag{1}$$

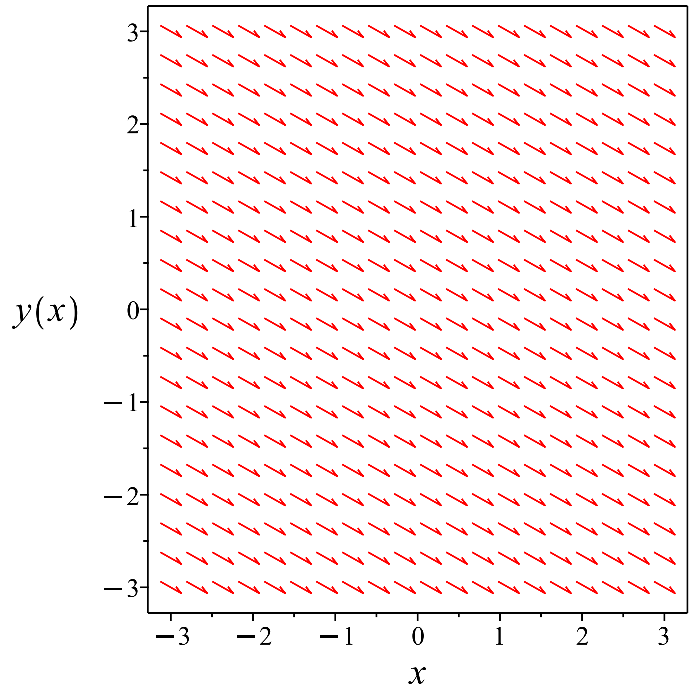


Figure 44: Slope field plot

Verification of solutions

$$y = -\frac{x}{2} + c_1$$

Verified OK.

2.3.2 Maple step by step solution

Let's solve

$y + (2x + 2y + 2)\, y' = -1 - x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' = -\frac{1}{2}$

- Integrate both sides with respect to x

  $\int y' dx = \int -\frac{1}{2} dx + c_1$

- Evaluate integral

$y = -\frac{x}{2} + c_1$

- Solve for y

$y = -\frac{x}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve((x+y(x)+1)+(2*x+2*y(x)+2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -1 - x$$
$$y(x) = -\frac{x}{2} + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 22

```
DSolve[(x+y[x]+1)+(2*x+2*y[x]+2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x - 1$$
$$y(x) \to -\frac{x}{2} + c_1$$

2.4 problem Differential equations with Linear Coefficients. Exercise 8.4, page 69

Internal problem ID [4444]
Internal file name [`OUTPUT/3937_Sunday_June_05_2022_11_52_15_AM_51595077/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 8
Problem number: Differential equations with Linear Coefficients. Exercise 8.4, page 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y + (2x + 2y - 3)\, y' = 1 - x}$$

2.4.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x - 1 + y}{2x + 2y - 3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{(x-1+y)(b_3-a_2)}{2x+2y-3} - \frac{(x-1+y)^2 a_3}{(2x+2y-3)^2} \\
&- \left(-\frac{1}{2x+2y-3} + \frac{2x-2+2y}{(2x+2y-3)^2}\right)(xa_2 + ya_3 + a_1) \\
&- \left(-\frac{1}{2x+2y-3} + \frac{2x-2+2y}{(2x+2y-3)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2a_2 - x^2a_3 + 4x^2b_2 - 2x^2b_3 + 4xya_2 - 2xya_3 + 8xyb_2 - 4xyb_3 + 2y^2a_2 - y^2a_3 + 4y^2b_2 - 2y^2b_3 - 6xa_2}{(2x+2y-3)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&2x^2a_2 - x^2a_3 + 4x^2b_2 - 2x^2b_3 + 4xya_2 - 2xya_3 + 8xyb_2 - 4xyb_3 \\
&+ 2y^2a_2 - y^2a_3 + 4y^2b_2 - 2y^2b_3 - 6xa_2 + 2xa_3 - 13xb_2 + 5xb_3 \\
&- 5ya_2 + ya_3 - 12yb_2 + 4yb_3 - a_1 + 3a_2 - a_3 - b_1 + 9b_2 - 3b_3 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&2a_2v_1^2 + 4a_2v_1v_2 + 2a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - a_3v_2^2 + 4b_2v_1^2 + 8b_2v_1v_2 \\
&+ 4b_2v_2^2 - 2b_3v_1^2 - 4b_3v_1v_2 - 2b_3v_2^2 - 6a_2v_1 - 5a_2v_2 + 2a_3v_1 + a_3v_2 \\
&- 13b_2v_1 - 12b_2v_2 + 5b_3v_1 + 4b_3v_2 - a_1 + 3a_2 - a_3 - b_1 + 9b_2 - 3b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(2a_2 - a_3 + 4b_2 - 2b_3)\, v_1^2 + (4a_2 - 2a_3 + 8b_2 - 4b_3)\, v_1 v_2 \\ &+ (-6a_2 + 2a_3 - 13b_2 + 5b_3)\, v_1 + (2a_2 - a_3 + 4b_2 - 2b_3)\, v_2^2 \\ &+ (-5a_2 + a_3 - 12b_2 + 4b_3)\, v_2 - a_1 + 3a_2 - a_3 - b_1 + 9b_2 - 3b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-6a_2 + 2a_3 - 13b_2 + 5b_3 &= 0\\ -5a_2 + a_3 - 12b_2 + 4b_3 &= 0\\ 2a_2 - a_3 + 4b_2 - 2b_3 &= 0\\ 4a_2 - 2a_3 + 8b_2 - 4b_3 &= 0\\ -a_1 + 3a_2 - a_3 - b_1 + 9b_2 - 3b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 2b_2 - b_1\\ a_2 &= -2b_2\\ a_3 &= -2b_2\\ b_1 &= b_1\\ b_2 &= b_2\\ b_3 &= b_2\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -1\\ \eta &= 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi\\ &= 1 - \left(-\frac{x - 1 + y}{2x + 2y - 3}\right)(-1)\\ &= \frac{x + y - 2}{2x + 2y - 3}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x+y-2}{2x+2y-3}} dy \end{aligned}$$

Which results in

$$S = 2y + \ln(x + y - 2)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x - 1 + y}{2x + 2y - 3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x + y - 2} \\ S_y &= 2 + \frac{1}{x + y - 2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2y + \ln(x + y - 2) = -x + c_1$$

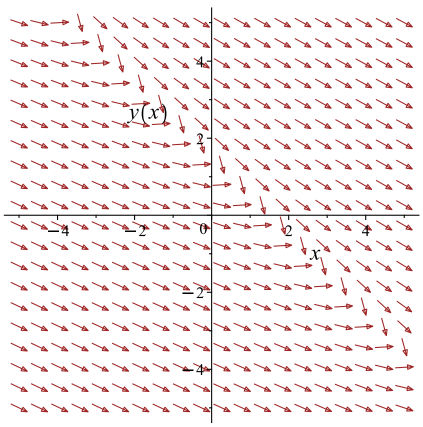
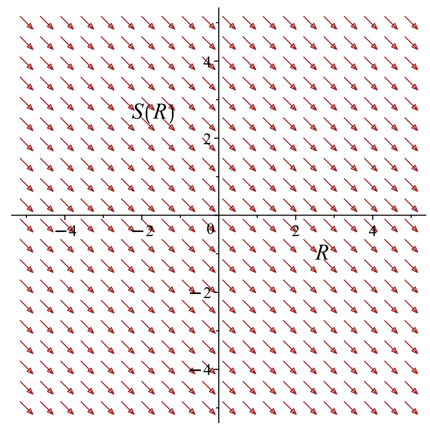
Which simplifies to

$$2y + \ln(x + y - 2) = -x + c_1$$

Which gives

$$y = \frac{\text{LambertW}\left(2\,e^{x-4+c_1}\right)}{2} - x + 2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x-1+y}{2x+2y-3}$ 	$R = x$ $S = 2y + \ln(x + y - 2)$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = \frac{\text{LambertW}\left(2\,e^{x-4+c_1}\right)}{2} - x + 2 \tag{1}$$

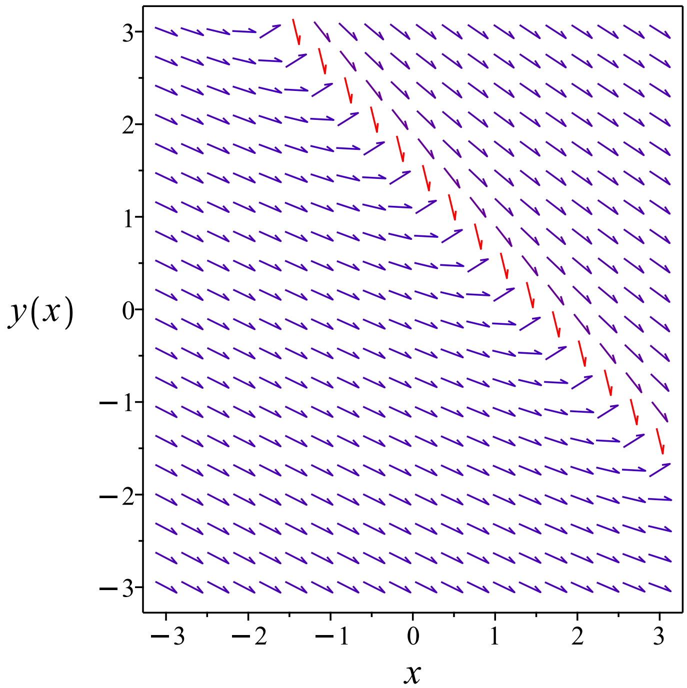


Figure 45: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\text{LambertW}\left(2\,e^{x-4+c_1}\right)}{2} - x + 2$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -1, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve((x+y(x)-1)+(2*x+2*y(x)-3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{LambertW}\left(2\,\mathrm{e}^{x-4-c_1}\right)}{2} + 2 - x$$

✓ Solution by Mathematica

Time used: 4.725 (sec). Leaf size: 33

```
DSolve[(x+y[x]-1)+(2*x+2*y[x]-3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(W\left(-e^{x-1+c_1}\right) - 2x + 4\right)$$
$$y(x) \to 2 - x$$

2.5 problem Differential equations with Linear Coefficients. Exercise 8.5, page 69

Internal problem ID [4445]
Internal file name [`OUTPUT/3938_Sunday_June_05_2022_11_52_24_AM_11158087/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 8
Problem number: Differential equations with Linear Coefficients. Exercise 8.5, page 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y-(x-y-1)\,y'=1-x}$$

2.5.1 Solving as homogeneousTypeMapleC ode

Let $Y=y+y_0$ and $X=x+x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X)=-\frac{X+x_0-1+Y(X)+y_0}{-X-x_0+Y(X)+y_0+1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0=1$$
$$y_0=0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X)=-\frac{X+Y(X)}{-X+Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X,Y) \\
&= -\frac{X+Y}{-X+Y}
\end{aligned}
\tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X,Y)
$$

In this case, it can be seen that both $M = X+Y$ and $N = X-Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-u-1}{u-1} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX}u(X)\right) X u(X) - \left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 1 = 0
$$

Or

$$
(u(X)-1)\,X\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 1 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X,u) \\
&= f(X)g(u) \\
&= -\frac{u^2+1}{(u-1)\,X}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+1}{u-1}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(u^2+1)}{2} - \arctan(u) = -\ln(X) + c_2$$

The solution is

$$\frac{\ln\left(u(X)^2+1\right)}{2} - \arctan(u(X)) + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x + 1$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{y^2}{(x-1)^2}+1\right)}{2} - \arctan\left(\frac{y}{x-1}\right) + \ln(x-1) - c_2 = 0$$

Summary
The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{(x-1)^2}+1\right)}{2}-\arctan\left(\frac{y}{x-1}\right)+\ln(x-1)-c_2=0 \tag{1}$$

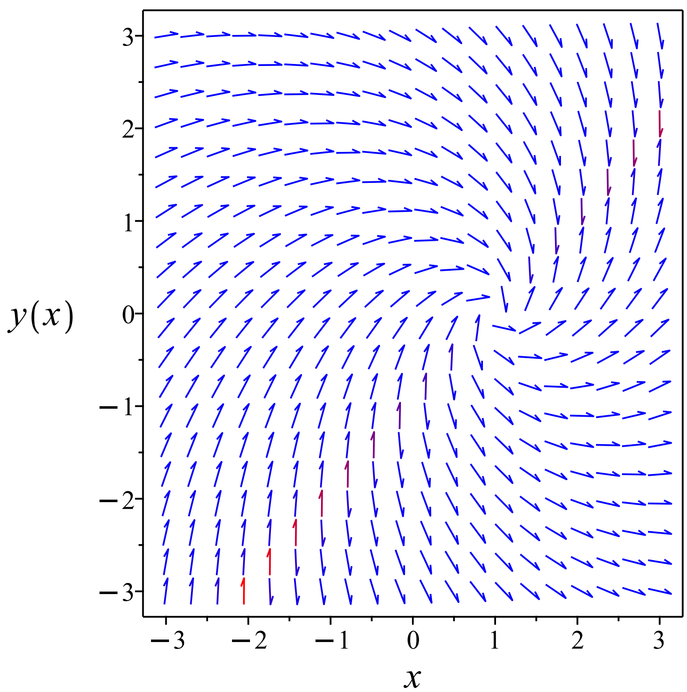


Figure 46: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{(x-1)^2}+1\right)}{2}-\arctan\left(\frac{y}{x-1}\right)+\ln(x-1)-c_2=0$$

Verified OK.

2.5.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{x-1+y}{-x+y+1} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(x-1+y)(b_3-a_2)}{-x+y+1} - \frac{(x-1+y)^2 a_3}{(-x+y+1)^2} \\ &- \left(-\frac{1}{-x+y+1} - \frac{x-1+y}{(-x+y+1)^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(-\frac{1}{-x+y+1} + \frac{x-1+y}{(-x+y+1)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2 a_2 + x^2 a_3 + x^2 b_2 - x^2 b_3 - 2xy a_2 + 2xy a_3 + 2xy b_2 + 2xy b_3 - y^2 a_2 - y^2 a_3 - y^2 b_2 + y^2 b_3 - 2x a_2 - 2x a}{(x-y-1)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-x^2 a_2 - x^2 a_3 - x^2 b_2 + x^2 b_3 + 2xy a_2 - 2xy a_3 - 2xy b_2 - 2xy b_3 \\ &+ y^2 a_2 + y^2 a_3 + y^2 b_2 - y^2 b_3 + 2x a_2 + 2x a_3 - 2x b_1 - 2x b_3 \\ &+ 2y a_1 + 2y a_3 + 2y b_2 + 2y b_3 - a_2 - a_3 + 2b_1 + b_2 + b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a_2v_1^2 + 2a_2v_1v_2 + a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 - 2b_2v_1v_2 \\
&+ b_2v_2^2 + b_3v_1^2 - 2b_3v_1v_2 - b_3v_2^2 + 2a_1v_2 + 2a_2v_1 + 2a_3v_1 + 2a_3v_2 \\
&- 2b_1v_1 + 2b_2v_2 - 2b_3v_1 + 2b_3v_2 - a_2 - a_3 + 2b_1 + b_2 + b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-a_2 - a_3 - b_2 + b_3)\, v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3)\, v_1v_2 + (2a_2 + 2a_3 - 2b_1 - 2b_3)\, v_1 \\
&+ (a_2 + a_3 + b_2 - b_3)\, v_2^2 + (2a_1 + 2a_3 + 2b_2 + 2b_3)\, v_2 - a_2 - a_3 + 2b_1 + b_2 + b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
2a_1 + 2a_3 + 2b_2 + 2b_3 &= 0 \\
-a_2 - a_3 - b_2 + b_3 &= 0 \\
a_2 + a_3 + b_2 - b_3 &= 0 \\
2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0 \\
2a_2 + 2a_3 - 2b_1 - 2b_3 &= 0 \\
-a_2 - a_3 + 2b_1 + b_2 + b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -b_3 \\
a_2 &= b_3 \\
a_3 &= -b_2 \\
b_1 &= -b_2 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = -y$$
$$\eta = x - 1$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= x - 1 - \left(-\frac{x - 1 + y}{-x + y + 1}\right)(-y) \\
&= \frac{x^2 + y^2 - 2x + 1}{x - y - 1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2+y^2-2x+1}{x-y-1}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln(x^2 + y^2 - 2x + 1)}{2} + \frac{2(x-1)\arctan\left(\frac{2y}{2x-2}\right)}{2x - 2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x-1+y}{-x+y+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-x-y+1}{x^2+y^2-2x+1} \\ S_y &= \frac{x-y-1}{x^2+y^2-2x+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

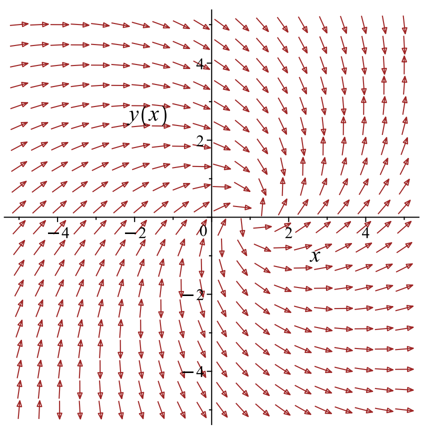
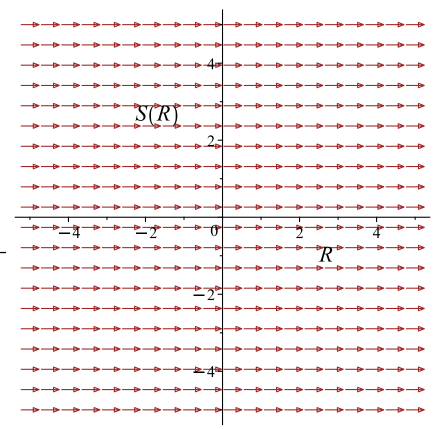
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y^2+x^2-2x+1)}{2} + \arctan\left(\frac{y}{x-1}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y^2+x^2-2x+1)}{2} + \arctan\left(\frac{y}{x-1}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x-1+y}{-x+y+1}$ 	$R = x$ $S = -\frac{\ln\left(x^2+y^2-2x+\right.}{2}$	$\frac{dS}{dR} = 0$ 

<u>Summary</u>

The solution(s) found are the following

$$-\frac{\ln\left(y^2+x^2-2x+1\right)}{2}+\arctan\left(\frac{y}{x-1}\right)=c_1 \tag{1}$$

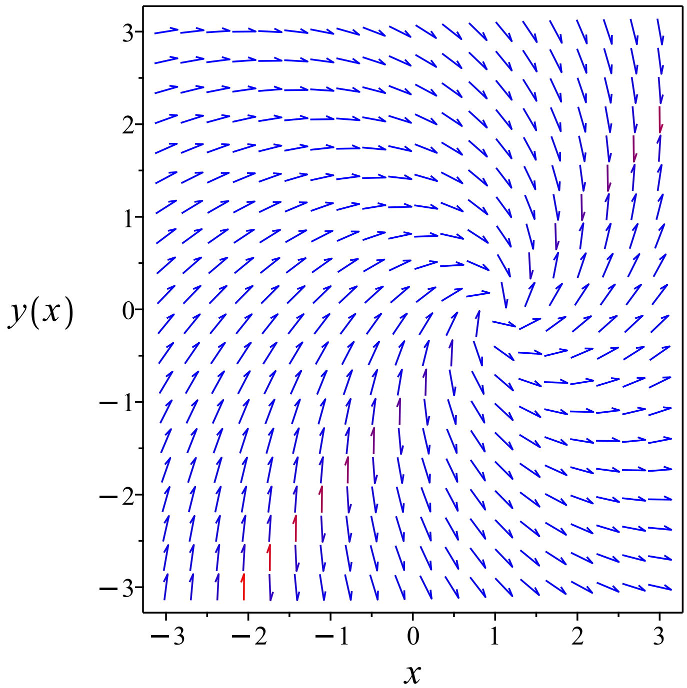


Figure 47: Slope field plot

<u>Verification of solutions</u>

$$-\frac{\ln\left(y^2+x^2-2x+1\right)}{2}+\arctan\left(\frac{y}{x-1}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 30

```
dsolve((x+y(x)-1)-(x-y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \tan\left(\operatorname{RootOf}\left(2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x-1\right) + 2c_1\right)\right)(1-x)$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 48

```
DSolve[(x+y[x]-1)-(x-y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[2\arctan\left(\frac{y(x)+x-1}{-y(x)+x-1}\right) = \log\left(\frac{1}{2}\left(\frac{y(x)^2}{(x-1)^2}+1\right)\right) + 2\log(x-1) + c_1, y(x)\right]$$

2.6 problem Differential equations with Linear Coefficients. Exercise 8.6, page 69

Internal problem ID [4446]
Internal file name [`OUTPUT/3939_Sunday_June_05_2022_11_52_34_AM_65010050/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 8
Problem number: Differential equations with Linear Coefficients. Exercise 8.6, page 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y + (2x + 2y - 1)\, y' = -x}$$

2.6.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x + y}{2x + 2y - 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(x+y)(b_3-a_2)}{2x+2y-1} - \frac{(x+y)^2 a_3}{(2x+2y-1)^2} - \left(-\frac{1}{2x+2y-1} + \frac{2x+2y}{(2x+2y-1)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{1}{2x+2y-1} + \frac{2x+2y}{(2x+2y-1)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2a_2 - x^2a_3 + 4x^2b_2 - 2x^2b_3 + 4xya_2 - 2xya_3 + 8xyb_2 - 4xyb_3 + 2y^2a_2 - y^2a_3 + 4y^2b_2 - 2y^2b_3 - 2xa_2}{(2x+2y-1)^2} = 0$$

Setting the numerator to zero gives

$$2x^2a_2 - x^2a_3 + 4x^2b_2 - 2x^2b_3 + 4xya_2 - 2xya_3 + 8xyb_2 - 4xyb_3 + 2y^2a_2 - y^2a_3 + 4y^2b_2 - 2y^2b_3 - 2xa_2 - 5xb_2 + xb_3 - ya_2 - ya_3 - 4yb_2 - a_1 - b_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$2a_2v_1^2 + 4a_2v_1v_2 + 2a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - a_3v_2^2 + 4b_2v_1^2 + 8b_2v_1v_2 + 4b_2v_2^2 - 2b_3v_1^2 - 4b_3v_1v_2 - 2b_3v_2^2 - 2a_2v_1 - a_2v_2 - a_3v_2 - 5b_2v_1 - 4b_2v_2 + b_3v_1 - a_1 - b_1 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(2a_2 - a_3 + 4b_2 - 2b_3)\, v_1^2 + (4a_2 - 2a_3 + 8b_2 - 4b_3)\, v_1 v_2 + (-2a_2 - 5b_2 + b_3)\, v_1 + (2a_2 - a_3 + 4b_2 - 2b_3)\, v_2^2 + (-a_2 - a_3 - 4b_2)\, v_2 - a_1 - b_1 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a_1 - b_1 + b_2 &= 0 \\ -2a_2 - 5b_2 + b_3 &= 0 \\ -a_2 - a_3 - 4b_2 &= 0 \\ 2a_2 - a_3 + 4b_2 - 2b_3 &= 0 \\ 4a_2 - 2a_3 + 8b_2 - 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -b_1 + b_2 \\ a_2 &= -2b_2 \\ a_3 &= -2b_2 \\ b_1 &= b_1 \\ b_2 &= b_2 \\ b_3 &= b_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -1 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x,y)\,\xi \\ &= 1 - \left(-\frac{x+y}{2x+2y-1}\right)(-1) \\ &= \frac{x-1+y}{2x+2y-1} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x-1+y}{2x+2y-1}} dy \end{aligned}$$

Which results in

$$S = 2y + \ln(x - 1 + y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y}{2x + 2y - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x - 1 + y} \\ S_y &= 2 + \frac{1}{x - 1 + y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2y + \ln(x - 1 + y) = -x + c_1$$

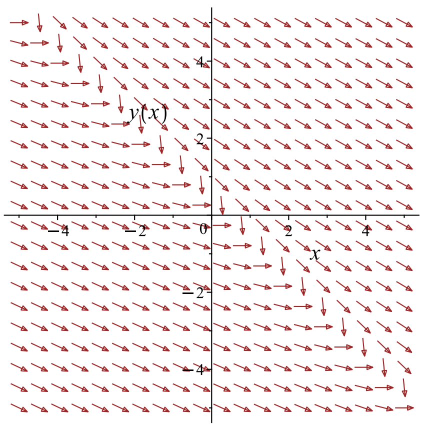
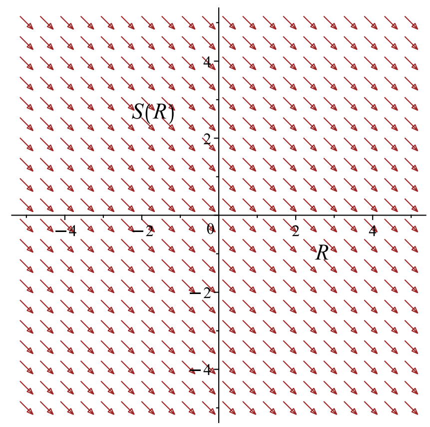
Which simplifies to

$$2y + \ln(x - 1 + y) = -x + c_1$$

Which gives

$$y = \frac{\text{LambertW}\left(2\,\mathrm{e}^{x-2+c_1}\right)}{2} - x + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y}{2x+2y-1}$ 	$R = x$ $S = 2y + \ln(x - 1 + y)$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = \frac{\text{LambertW}\left(2\,\mathrm{e}^{x-2+c_1}\right)}{2} - x + 1 \tag{1}$$

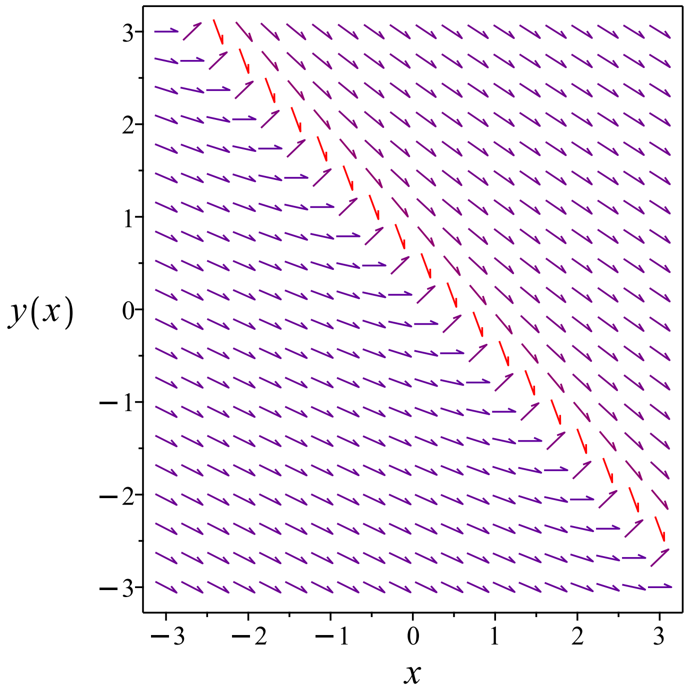


Figure 48: Slope field plot

Verification of solutions

$$y = \frac{\mathrm{LambertW}\left(2\,\mathrm{e}^{x-2+c_1}\right)}{2} - x + 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 21

```
dsolve((x+y(x))+(2*x+2*y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{LambertW}\left(2\,\mathrm{e}^{x-2-c_1}\right)}{2} - x + 1$$

✓ Solution by Mathematica

Time used: 1.056 (sec). Leaf size: 33

```
DSolve[(x+y[x])+(2*x+2*y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(W\left(-e^{x-1+c_1}\right) - 2x + 2\right)$$
$$y(x) \to 1 - x$$

2.7 problem Differential equations with Linear Coefficients. Exercise 8.7, page 69

Internal problem ID [4447]
Internal file name [`OUTPUT/3940_Sunday_June_05_2022_11_52_43_AM_32271147/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 8
Problem number: Differential equations with Linear Coefficients. Exercise 8.7, page 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{7y + (2x+1)\,y' = 3}$$

2.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-7y+3}{2x+1} \end{aligned}$$

Where $f(x) = \frac{1}{2x+1}$ and $g(y) = -7y + 3$. Integrating both sides gives

$$\frac{1}{-7y+3}\,dy = \frac{1}{2x+1}\,dx$$

$$\int \frac{1}{-7y+3}\,dy = \int \frac{1}{2x+1}\,dx$$

$$-\frac{\ln(-7y+3)}{7} = \frac{\ln(2x+1)}{2} + c_1$$

Raising both side to exponential gives

$$\frac{1}{(-7y+3)^{\frac{1}{7}}} = e^{\frac{\ln(2x+1)}{2}+c_1}$$

Which simplifies to

$$\frac{1}{(-7y+3)^{\frac{1}{7}}} = c_2\sqrt{2x+1}$$

Which simplifies to

$$y = \frac{\left(3c_2^7 e^{7c_1}(2x+1)^{\frac{7}{2}} - 1\right)e^{-7c_1}}{7c_2^7(2x+1)^{\frac{7}{2}}}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{\left(3c_2^7 e^{7c_1}(2x+1)^{\frac{7}{2}} - 1\right)e^{-7c_1}}{7c_2^7(2x+1)^{\frac{7}{2}}} \tag{1}$$

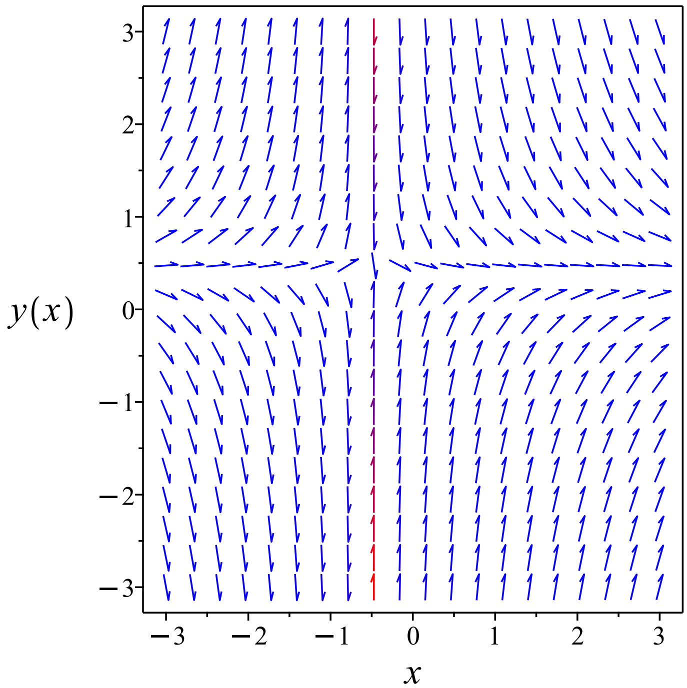


Figure 49: Slope field plot

Verification of solutions

$$y = \frac{\left(3c_2^7 e^{7c_1} (2x+1)^{\frac{7}{2}} - 1\right) e^{-7c_1}}{7c_2^7 (2x+1)^{\frac{7}{2}}}$$

Verified OK.

2.7.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{7}{2x+1}$$
$$q(x) = \frac{3}{2x+1}$$

Hence the ode is

$$y' + \frac{7y}{2x+1} = \frac{3}{2x+1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{7}{2x+1}dx} \\ &= (2x+1)^{\frac{7}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{3}{2x+1}\right) \\ \frac{d}{dx}\left((2x+1)^{\frac{7}{2}} y\right) &= \left((2x+1)^{\frac{7}{2}}\right)\left(\frac{3}{2x+1}\right) \\ d\left((2x+1)^{\frac{7}{2}} y\right) &= \left(3(2x+1)^{\frac{5}{2}}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}(2x+1)^{\frac{7}{2}} y &= \int 3(2x+1)^{\frac{5}{2}}\, dx \\ (2x+1)^{\frac{7}{2}} y &= \frac{3(2x+1)^{\frac{7}{2}}}{7} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = (2x+1)^{\frac{7}{2}}$ results in

$$y = \frac{3}{7} + \frac{c_1}{(2x+1)^{\frac{7}{2}}}$$

Summary

The solution(s) found are the following

$$y = \frac{3}{7} + \frac{c_1}{(2x+1)^{\frac{7}{2}}} \tag{1}$$

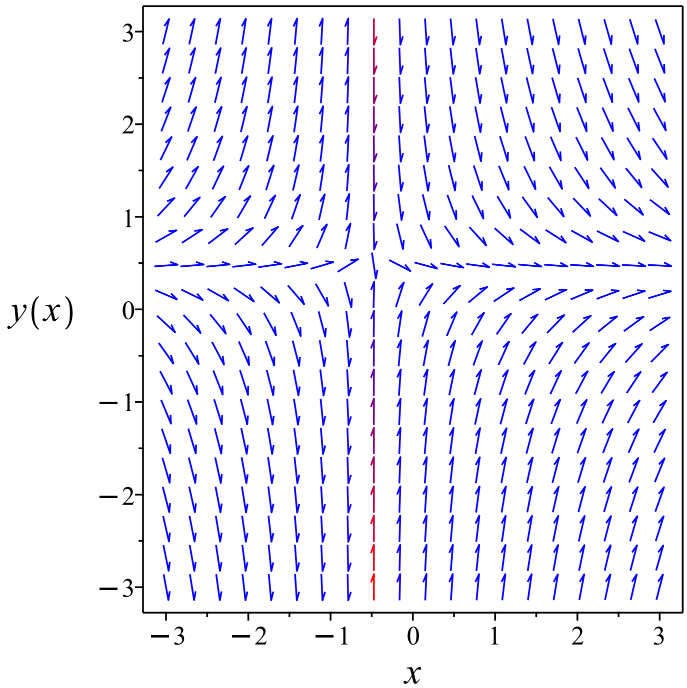


Figure 50: Slope field plot

Verification of solutions

$$y = \frac{3}{7} + \frac{c_1}{(2x+1)^{\frac{7}{2}}}$$

Verified OK.

2.7.3 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{7Y(X) + 7y_0 - 3}{2X + 2x_0 + 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -\frac{1}{2}$$
$$y_0 = \frac{3}{7}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{7Y(X)}{2X}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{7Y}{2X}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -7Y$ and $N = 2X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= -\frac{7u}{2} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= -\frac{9u(X)}{2X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) + \frac{9u(X)}{2X} = 0
$$

Or

$$
2\left(\frac{d}{dX}u(X)\right)X + 9u(X) = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X)g(u) \\
&= -\frac{9u}{2X}
\end{aligned}
$$

Where $f(X) = -\frac{9}{2X}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{9}{2X}\, dX \\
\int \frac{1}{u}\, du &= \int -\frac{9}{2X}\, dX \\
\ln(u) &= -\frac{9\ln(X)}{2} + c_2 \\
u &= \mathrm{e}^{-\frac{9\ln(X)}{2}+c_2} \\
&= \frac{c_2}{X^{\frac{9}{2}}}
\end{aligned}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \frac{c_2}{X^{\frac{7}{2}}}$$

Using the solution for $Y(X)$

$$Y(X) = \frac{c_2}{X^{\frac{7}{2}}}$$

And replacing back terms in the above solution using

$$\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}$$

Or

$$\begin{aligned}
Y &= y + \frac{3}{7} \\
X &= x - \frac{1}{2}
\end{aligned}$$

Then the solution in y becomes

$$y - \frac{3}{7} = \frac{c_2}{\left(\frac{1}{2} + x\right)^{\frac{7}{2}}}$$

Summary

The solution(s) found are the following

$$y - \frac{3}{7} = \frac{c_2}{\left(\frac{1}{2} + x\right)^{\frac{7}{2}}} \tag{1}$$

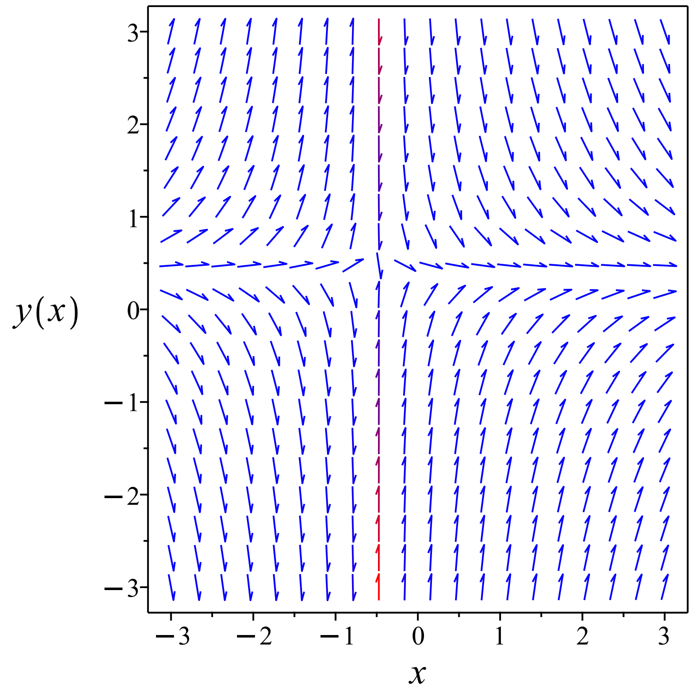


Figure 51: Slope field plot

Verification of solutions

$$y-\frac{3}{7}=\frac{c_2}{\left(\frac{1}{2}+x\right)^{\frac{7}{2}}}$$

Verified OK.

2.7.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{7y-3}{2x+1}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 13: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \frac{1}{(2x+1)^{\frac{7}{2}}} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(2x+1)^{\frac{7}{2}}}} dy \end{aligned}$$

Which results in

$$S = (2x+1)^{\frac{7}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{7y - 3}{2x + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 7(2x+1)^{\frac{5}{2}} y \\ S_y &= (2x+1)^{\frac{7}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3(2x+1)^{\frac{5}{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3(2R+1)^{\frac{5}{2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{3(2R+1)^{\frac{7}{2}}}{7} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(2x+1)^{\frac{7}{2}} y = \frac{3(2x+1)^{\frac{7}{2}}}{7} + c_1$$

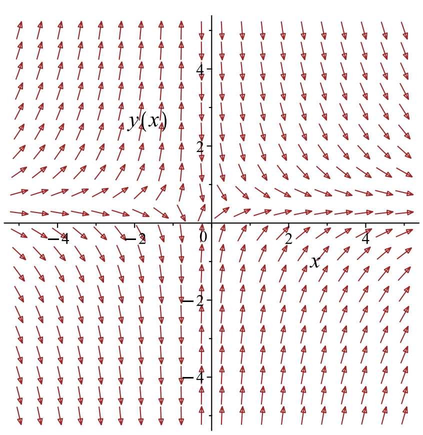
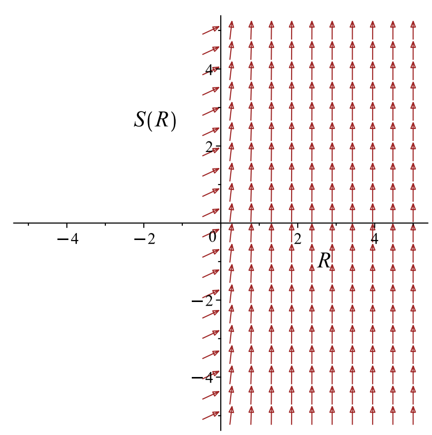
Which simplifies to

$$(2x+1)^{\frac{7}{2}} y = \frac{3(2x+1)^{\frac{7}{2}}}{7} + c_1$$

Which gives

$$y = \frac{3(2x+1)^{\frac{7}{2}} + 7c_1}{7\,(2x+1)^{\frac{7}{2}}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{7y-3}{2x+1}$ 	$R = x$ $S = (2x+1)^{\frac{7}{2}} y$	$\frac{dS}{dR} = 3(2R+1)^{\frac{5}{2}}$ 

Summary
The solution(s) found are the following

$$y = \frac{3(2x+1)^{\frac{7}{2}} + 7c_1}{7\,(2x+1)^{\frac{7}{2}}} \tag{1}$$

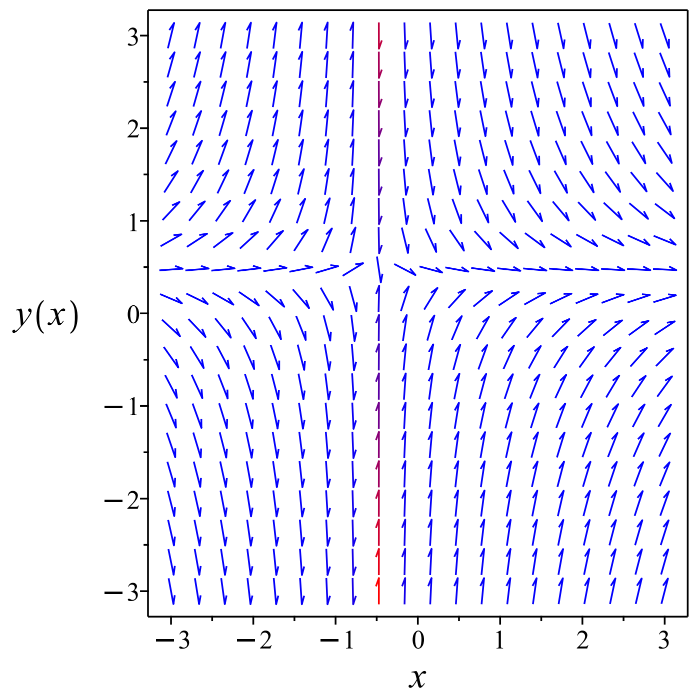


Figure 52: Slope field plot

Verification of solutions

$$y = \frac{3(2x+1)^{\frac{7}{2}} + 7c_1}{7\,(2x+1)^{\frac{7}{2}}}$$

Verified OK.

2.7.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{-7y+3}\right)\mathrm{d}y&=\left(\frac{1}{2x+1}\right)\mathrm{d}x\\ \left(-\frac{1}{2x+1}\right)\mathrm{d}x+\left(\frac{1}{-7y+3}\right)\mathrm{d}y&=0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{2x+1}\\ N(x,y)&=\frac{1}{-7y+3}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{2x+1}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-7y+3}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{2x+1}\,\mathrm{d}x\\ \phi &= -\frac{\ln(2x+1)}{2} + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-7y+3}$. Therefore equation (4) becomes

$$\frac{1}{-7y+3} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{7y-3}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{7y-3}\right)\mathrm{d}y$$

$$f(y) = -\frac{\ln(7y-3)}{7} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(2x+1)}{2} - \frac{\ln(7y-3)}{7} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(2x+1)}{2} - \frac{\ln(7y-3)}{7}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-7c_1 - \frac{7\ln(2x+1)}{2}}}{7} + \frac{3}{7}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-7c_1 - \frac{7\ln(2x+1)}{2}}}{7} + \frac{3}{7} \tag{1}$$

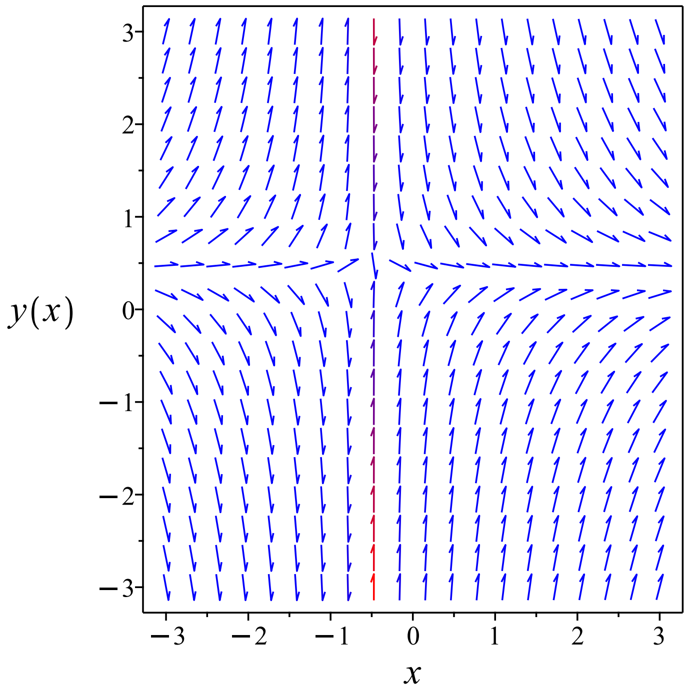


Figure 53: Slope field plot

Verification of solutions

$$y = \frac{e^{-7c_1 - \frac{7\ln(2x+1)}{2}}}{7} + \frac{3}{7}$$

Verified OK.

2.7.6 Maple step by step solution

Let's solve

$7y + (2x+1)\,y' = 3$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-7y+3} = \frac{1}{2x+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-7y+3}dx = \int \frac{1}{2x+1}dx + c_1$

- Evaluate integral

$$-\frac{\ln(-7y+3)}{7} = \frac{\ln(2x+1)}{2} + c_1$$

- Solve for y

$$y = -\frac{e^{-7c_1 - \frac{7\ln(2x+1)}{2}}}{7} + \frac{3}{7}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((7*y(x)-3)+(2*x+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{3}{7} + \frac{c_1}{(1+2x)^{\frac{7}{2}}}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 28

```
DSolve[(7*y[x]-3)+(2*x+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3}{7} + \frac{c_1}{(2x+1)^{7/2}}$$
$$y(x) \to \frac{3}{7}$$

2.8 problem Differential equations with Linear Coefficients. Exercise 8.8, page 69

Internal problem ID [4448]
Internal file name [`OUTPUT/3941_Sunday_June_05_2022_11_52_52_AM_55242867/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 8
Problem number: Differential equations with Linear Coefficients. Exercise 8.8, page 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{2y + (3x + 6y + 3)\, y' = -x}$$

2.8.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{2y + x}{3\,(x + 2y + 1)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(2y+x)(b_3-a_2)}{3(x+2y+1)} - \frac{(2y+x)^2 a_3}{9(x+2y+1)^2} - \left(-\frac{1}{3(x+2y+1)} + \frac{2y+x}{3(x+2y+1)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{2}{3(x+2y+1)} + \frac{\frac{4y}{3}+\frac{2x}{3}}{(x+2y+1)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^2a_2 - x^2a_3 + 9x^2b_2 - 3x^2b_3 + 12xya_2 - 4xya_3 + 36xyb_2 - 12xyb_3 + 12y^2a_2 - 4y^2a_3 + 36y^2b_2 - 12y^2b_3}{9(x+2y+1)^2} = 0$$

Setting the numerator to zero gives

$$3x^2a_2 - x^2a_3 + 9x^2b_2 - 3x^2b_3 + 12xya_2 - 4xya_3 + 36xyb_2 - 12xyb_3 + 12y^2a_2 - 4y^2a_3 + 36y^2b_2 - 12y^2b_3 + 6xa_2 + 24xb_2 - 3xb_3 + 6ya_2 + 3ya_3 + 36yb_2 + 3a_1 + 6b_1 + 9b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$3a_2v_1^2 + 12a_2v_1v_2 + 12a_2v_2^2 - a_3v_1^2 - 4a_3v_1v_2 - 4a_3v_2^2 + 9b_2v_1^2 + 36b_2v_1v_2 + 36b_2v_2^2 - 3b_3v_1^2 - 12b_3v_1v_2 - 12b_3v_2^2 + 6a_2v_1 + 6a_2v_2 + 3a_3v_2 + 24b_2v_1 + 36b_2v_2 - 3b_3v_1 + 3a_1 + 6b_1 + 9b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(3a_2-a_3+9b_2-3b_3)\,v_1^2+(12a_2-4a_3+36b_2-12b_3)\,v_1v_2+(6a_2+24b_2-3b_3)\,v_1 \\ +(12a_2-4a_3+36b_2-12b_3)\,v_2^2+(6a_2+3a_3+36b_2)\,v_2+3a_1+6b_1+9b_2=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 3a_1+6b_1+9b_2&=0\\ 6a_2+3a_3+36b_2&=0\\ 6a_2+24b_2-3b_3&=0\\ 3a_2-a_3+9b_2-3b_3&=0\\ 12a_2-4a_3+36b_2-12b_3&=0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1&=-2b_1-3b_2\\ a_2&=-3b_2\\ a_3&=-6b_2\\ b_1&=b_1\\ b_2&=b_2\\ b_3&=2b_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi&=-2\\ \eta&=1 \end{aligned}$$

Shifting is now applied to make $\xi=0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta&=\eta-\omega(x,y)\,\xi\\ &=1-\left(-\frac{2y+x}{3\,(x+2y+1)}\right)(-2)\\ &=\frac{x+2y+3}{3x+6y+3}\\ \xi&=0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x+2y+3}{3x+6y+3}} dy \end{aligned}$$

Which results in

$$S = 3y - 3 \ln (x + 2y + 3)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2y + x}{3 (x + 2y + 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{3}{x + 2y + 3} \\ S_y &= 3 - \frac{6}{x + 2y + 3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$3y - 3\ln(x + 2y + 3) = -x + c_1$$

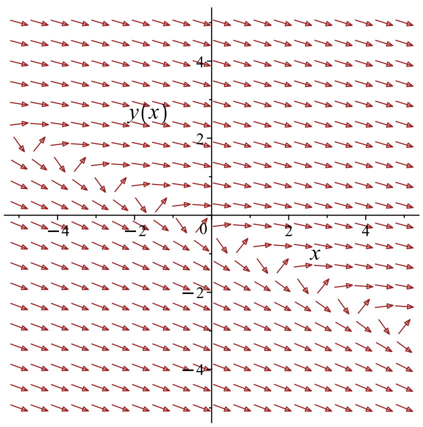
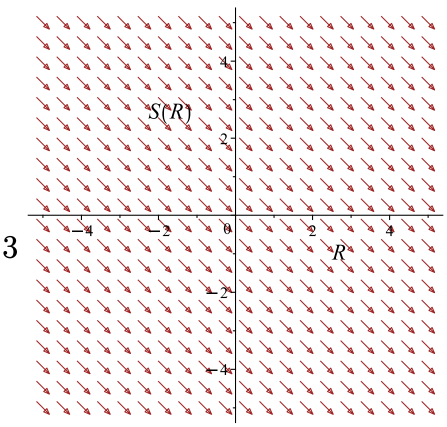
Which simplifies to

$$3y - 3\ln(x + 2y + 3) = -x + c_1$$

Which gives

$$y = -\frac{x}{2} - \operatorname{LambertW}\left(-\frac{e^{-\frac{x}{6}-\frac{c_1}{3}-\frac{3}{2}}}{2}\right) - \frac{3}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y+x}{3(x+2y+1)}$ 	$R = x$ $S = 3y - 3\ln\left(x + 2y + 3\right.$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = -\frac{x}{2} - \mathrm{LambertW}\left(-\frac{\mathrm{e}^{-\frac{x}{6}-\frac{c_1}{3}-\frac{3}{2}}}{2}\right) - \frac{3}{2} \tag{1}$$

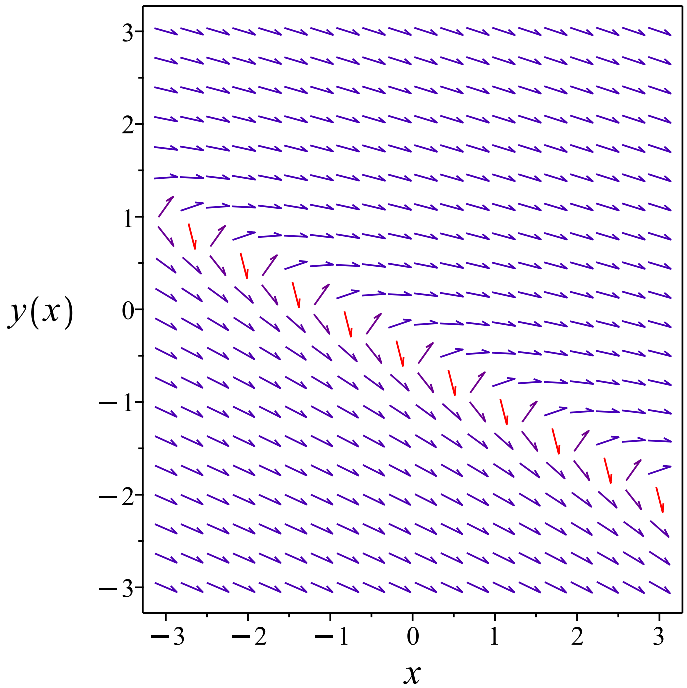


Figure 54: Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{x}{2} - \text{LambertW}\left(-\frac{e^{-\frac{x}{6}-\frac{c_1}{3}-\frac{3}{2}}}{2}\right) - \frac{3}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -1/2, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve((x+2*y(x))+(3*x+6*y(x)+3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\operatorname{LambertW}\left(-\frac{e^{-\frac{3}{2}-\frac{x}{6}+\frac{c_1}{6}}}{2}\right) - \frac{3}{2} - \frac{x}{2}$$

✓ Solution by Mathematica

Time used: 4.834 (sec). Leaf size: 43

```
DSolve[(x+2*y[x])+(3*x+6*y[x]+3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-2W\left(-e^{-\frac{x}{6}-1+c_1}\right) - x - 3\right)$$
$$y(x) \to \frac{1}{2}(-x-3)$$

2.9 problem Differential equations with Linear Coefficients. Exercise 8.9, page 69

Internal problem ID [4449]
Internal file name [`OUTPUT/3942_Sunday_June_05_2022_11_53_01_AM_4367037/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 8
Problem number: Differential equations with Linear Coefficients. Exercise 8.9, page 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{2y + (y - 1)\, y' = -x}$$

2.9.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{2Y(X) + 2y_0 + X + x_0}{Y(X) + y_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -2$$
$$y_0 = 1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{2Y(X) + X}{Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{2Y + X}{Y}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -2Y - X$ and $N = Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= -2 - \frac{1}{u} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{-2 - \frac{1}{u(X)} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) - \frac{-2 - \frac{1}{u(X)} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX} u(X)\right) u(X)\, X + u(X)^2 + 2u(X) + 1 = 0
$$

Or

$$
\left(\frac{d}{dX} u(X)\right) u(X)\, X + (1 + u(X))^2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= -\frac{(1 + u)^2}{uX}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{(1+u)^2}{u}$. Integrating both sides gives

$$\frac{1}{\frac{(1+u)^2}{u}} du = -\frac{1}{X} dX$$

$$\int \frac{1}{\frac{(1+u)^2}{u}} du = \int -\frac{1}{X} dX$$

$$\ln(1+u) + \frac{1}{1+u} = -\ln(X) + c_2$$

The solution is

$$\ln(1+u(X)) + \frac{1}{1+u(X)} + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\ln\left(1 + \frac{Y(X)}{X}\right) + \frac{1}{1 + \frac{Y(X)}{X}} + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\ln\left(\frac{X + Y(X)}{X}\right) + \frac{X}{X + Y(X)} + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = 1 + y$$
$$X = -2 + x$$

Then the solution in y becomes

$$\ln\left(\frac{y + x + 1}{x + 2}\right) + \frac{x + 2}{y + x + 1} + \ln(x + 2) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\ln\left(\frac{y + x + 1}{x + 2}\right) + \frac{x + 2}{y + x + 1} + \ln(x + 2) - c_2 = 0 \tag{1}$$

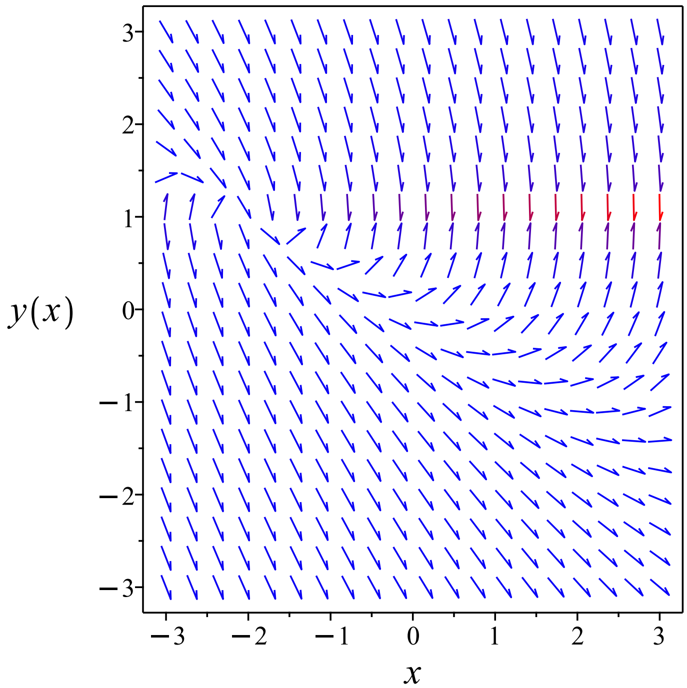


Figure 55: Slope field plot

Verification of solutions

$$\ln\left(\frac{y+x+1}{x+2}\right)+\frac{x+2}{y+x+1}+\ln\left(x+2\right)-c_2=0$$

Verified OK.

2.9.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{2y+x}{y-1}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(2y+x)(b_3-a_2)}{y-1} - \frac{(2y+x)^2 a_3}{(y-1)^2} + \frac{xa_2+ya_3+a_1}{y-1} - \left(-\frac{2}{y-1} + \frac{2y+x}{(y-1)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_3 + x^2b_2 - 2xya_2 + 4xya_3 + 2xyb_3 - 2y^2a_2 + 3y^2a_3 - y^2b_2 + 2y^2b_3 + 2xa_2 + xb_1 + 2xb_2 - xb_3 - ya_1}{(y-1)^2} = 0$$

Setting the numerator to zero gives

$$-x^2a_3 - x^2b_2 + 2xya_2 - 4xya_3 - 2xyb_3 + 2y^2a_2 - 3y^2a_3 + y^2b_2 - 2y^2b_3 - 2xa_2 - xb_1 - 2xb_2 + xb_3 + ya_1 - 2ya_2 - ya_3 - 2yb_2 - a_1 - 2b_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$2a_2v_1v_2 + 2a_2v_2^2 - a_3v_1^2 - 4a_3v_1v_2 - 3a_3v_2^2 - b_2v_1^2 + b_2v_2^2 - 2b_3v_1v_2 - 2b_3v_2^2 + a_1v_2 - 2a_2v_1 - 2a_2v_2 - a_3v_2 - b_1v_1 - 2b_2v_1 - 2b_2v_2 + b_3v_1 - a_1 - 2b_1 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_3 - b_2) v_1^2 + (2a_2 - 4a_3 - 2b_3) v_1 v_2 + (-2a_2 - b_1 - 2b_2 + b_3) v_1 \\ + (2a_2 - 3a_3 + b_2 - 2b_3) v_2^2 + (a_1 - 2a_2 - a_3 - 2b_2) v_2 - a_1 - 2b_1 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a_3 - b_2 &= 0 \\ -a_1 - 2b_1 + b_2 &= 0 \\ 2a_2 - 4a_3 - 2b_3 &= 0 \\ a_1 - 2a_2 - a_3 - 2b_2 &= 0 \\ -2a_2 - b_1 - 2b_2 + b_3 &= 0 \\ 2a_2 - 3a_3 + b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -3b_2 + 2b_3 \\ a_2 &= -2b_2 + b_3 \\ a_3 &= -b_2 \\ b_1 &= 2b_2 - b_3 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x + 2 \\ \eta &= y - 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - 1 - \left(-\frac{2y + x}{y - 1}\right) (x + 2) \\ &= \frac{x^2 + 2xy + y^2 + 2x + 2y + 1}{y - 1} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2+2xy+y^2+2x+2y+1}{y-1}} dy \end{aligned}$$

Which results in

$$S = -\frac{-x-2}{x+y+1} + \ln(x+y+1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2y + x}{y - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2y + x}{(x + y + 1)^2} \\ S_y &= \frac{y - 1}{(x + y + 1)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(y+x+1)\ln(y+x+1)+x+2}{y+x+1} = c_1$$

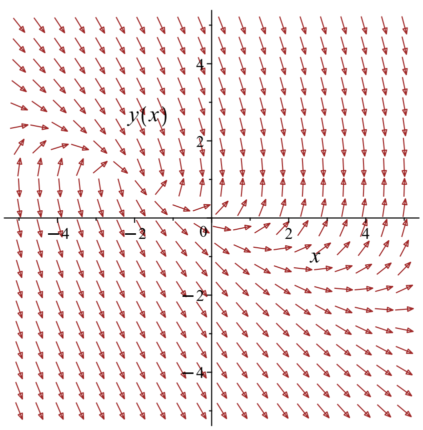
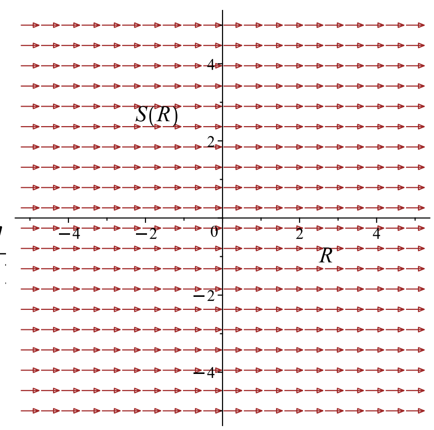
Which simplifies to

$$\frac{(y+x+1)\ln(y+x+1)+x+2}{y+x+1} = c_1$$

Which gives

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-(x+2)\mathrm{e}^{-c_1}\right)+c_1} - x - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y+x}{y-1}$ 	$R = x$ $S = \frac{(x + y + 1)\ln(x + y}{x + y +}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = e^{\text{LambertW}(-(x+2)e^{-c_1})+c_1} - x - 1 \tag{1}$$

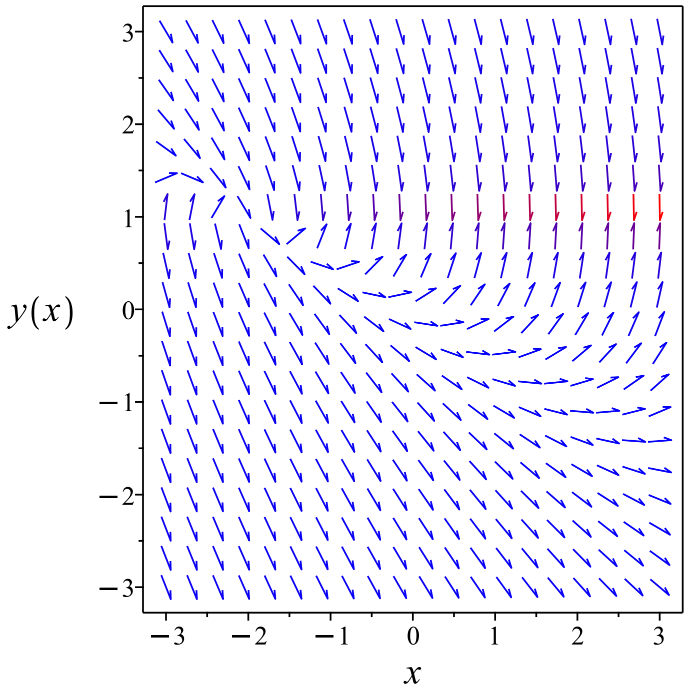


Figure 56: Slope field plot

Verification of solutions

$$y = e^{\mathrm{LambertW}\left(-(x+2)\,e^{-c_1}\right)+c_1} - x - 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 30

```
dsolve((x+2*y(x))+(y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{(-1-x)\,\mathrm{LambertW}\left(c_1(2+x)\right)-2-x}{\mathrm{LambertW}\left(c_1\left(2+x\right)\right)}$$

✓ Solution by Mathematica

Time used: 1.178 (sec). Leaf size: 143

```
DSolve[(x+2*y[x])+(y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[-\frac{(-2)^{2/3}\left(-\left((x+1)\log\left(-\frac{3(-2)^{2/3}(x+2)}{y(x)-1}\right)\right)+x\log\left(\frac{3(-2)^{2/3}(y(x)+x+1)}{y(x)-1}\right)+\log\left(\frac{3(-2)^{2/3}(y(x)+x+1)}{y(x)-1}\right)+\right.}{9(y(x)+x+1)}\right.$$

2.10 problem Differential equations with Linear Coefficients. Exercise 8.10, page 69

Internal problem ID [4450]
Internal file name [`OUTPUT/3943_Sunday_June_05_2022_11_53_21_AM_12144779/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 8
Problem number: Differential equations with Linear Coefficients. Exercise 8.10, page 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{-2y-(2x+7y-1)\,y'=-3x-4}$$

2.10.1 Solving as differentialType ode

Writing the ode as

$$y'=\frac{-3x+2y-4}{-2x-7y+1} \tag{1}$$

Which becomes

$$(-1+7y)\,dy=(-2x)\,dy+(3x-2y+4)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-2x)\,dy+(3x-2y+4)\,dx=d\left(\frac{3}{2}x^2-2xy+4x\right)$$

Hence (2) becomes

$$(-1+7y)\,dy = d\left(\frac{3}{2}x^2 - 2xy + 4x\right)$$

Integrating both sides gives gives these solutions

$$y = -\frac{2x}{7} + \frac{1}{7} + \frac{\sqrt{25x^2 + 14c_1 + 52x + 1}}{7} + c_1$$
$$y = -\frac{2x}{7} + \frac{1}{7} - \frac{\sqrt{25x^2 + 14c_1 + 52x + 1}}{7} + c_1$$

Summary

The solution(s) found are the following

$$y = -\frac{2x}{7} + \frac{1}{7} + \frac{\sqrt{25x^2 + 14c_1 + 52x + 1}}{7} + c_1 \tag{1}$$
$$y = -\frac{2x}{7} + \frac{1}{7} - \frac{\sqrt{25x^2 + 14c_1 + 52x + 1}}{7} + c_1 \tag{2}$$

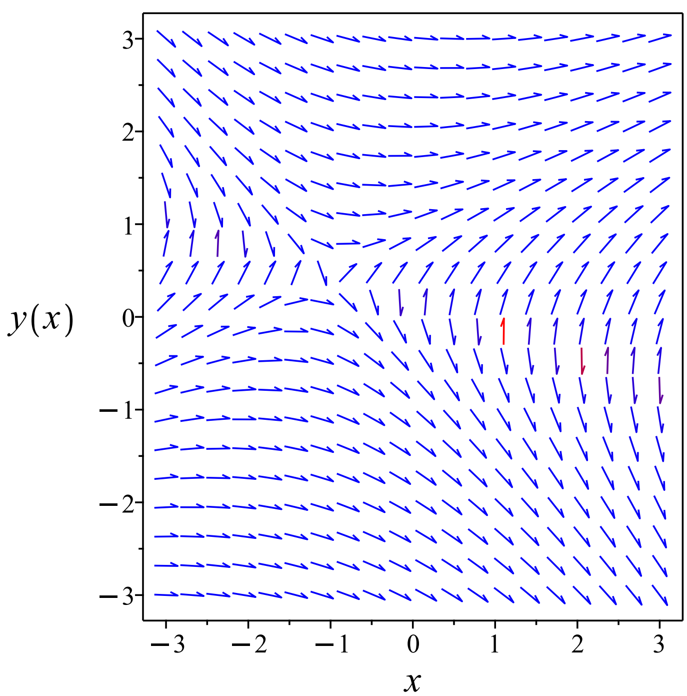


Figure 57: Slope field plot

Verification of solutions

$$y = -\frac{2x}{7} + \frac{1}{7} + \frac{\sqrt{25x^2 + 14c_1 + 52x + 1}}{7} + c_1$$

Verified OK.

$$y = -\frac{2x}{7} + \frac{1}{7} - \frac{\sqrt{25x^2 + 14c_1 + 52x + 1}}{7} + c_1$$

Verified OK.

2.10.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{-3X - 3x_0 + 2Y(X) + 2y_0 - 4}{2X + 2x_0 + 7Y(X) + 7y_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -\frac{26}{25}$$
$$y_0 = \frac{11}{25}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{-3X + 2Y(X)}{2X + 7Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{-3X + 2Y}{2X + 7Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = 3X - 2Y$ and $N = 2X + 7Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}X}X + u = \frac{-2u+3}{7u+2}$$
$$\frac{\mathrm{d}u}{\mathrm{d}X} = \frac{\frac{-2u(X)+3}{7u(X)+2} - u(X)}{X}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-2u(X)+3}{7u(X)+2} - u(X)}{X} = 0$$

Or

$$7\left(\frac{d}{dX}u(X)\right)Xu(X) + 2\left(\frac{d}{dX}u(X)\right)X + 7u(X)^2 + 4u(X) - 3 = 0$$

Or

$$-3 + X(7u(X)+2)\left(\frac{d}{dX}u(X)\right) + 7u(X)^2 + 4u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{7u^2+4u-3}{X(7u+2)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{7u^2+4u-3}{7u+2}$. Integrating both sides gives

$$\frac{1}{\frac{7u^2+4u-3}{7u+2}}\,du = -\frac{1}{X}\,dX$$
$$\int \frac{1}{\frac{7u^2+4u-3}{7u+2}}\,du = \int -\frac{1}{X}\,dX$$
$$\frac{\ln(7u^2+4u-3)}{2} = -\ln(X) + c_2$$

Raising both side to exponential gives

$$\sqrt{7u^2+4u-3} = \mathrm{e}^{-\ln(X)+c_2}$$

Which simplifies to

$$\sqrt{7u^2+4u-3} = \frac{c_3}{X}$$

Which simplifies to

$$\sqrt{7u(X)^2+4u(X)-3} = \frac{c_3 e^{c_2}}{X}$$

The solution is

$$\sqrt{7u(X)^2+4u(X)-3} = \frac{c_3 e^{c_2}}{X}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\sqrt{\frac{7Y(X)^2}{X^2}+\frac{4Y(X)}{X}-3} = \frac{c_3 e^{c_2}}{X}$$

Which simplifies to

$$\sqrt{-\frac{(Y(X)+X)(3X-7Y(X))}{X^2}} = \frac{c_3 e^{c_2}}{X}$$

Using the solution for $Y(X)$

$$\sqrt{-\frac{(Y(X)+X)(3X-7Y(X))}{X^2}} = \frac{c_3 e^{c_2}}{X}$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y+y_0 \\ X &= x+x_0 \end{aligned}$$

Or

$$\begin{aligned} Y &= y+\frac{11}{25} \\ X &= x-\frac{26}{25} \end{aligned}$$

Then the solution in y becomes

$$\sqrt{-\frac{\left(y+\frac{3}{5}+x\right)\left(3x+\frac{31}{5}-7y\right)}{\left(x+\frac{26}{25}\right)^2}} = \frac{c_3 e^{c_2}}{x+\frac{26}{25}}$$

Summary

The solution(s) found are the following

$$\sqrt{-\frac{\left(y+\frac{3}{5}+x\right)\left(3x+\frac{31}{5}-7y\right)}{\left(x+\frac{26}{25}\right)^{2}}} = \frac{c_3 \mathrm{e}^{c_2}}{x+\frac{26}{25}} \tag{1}$$

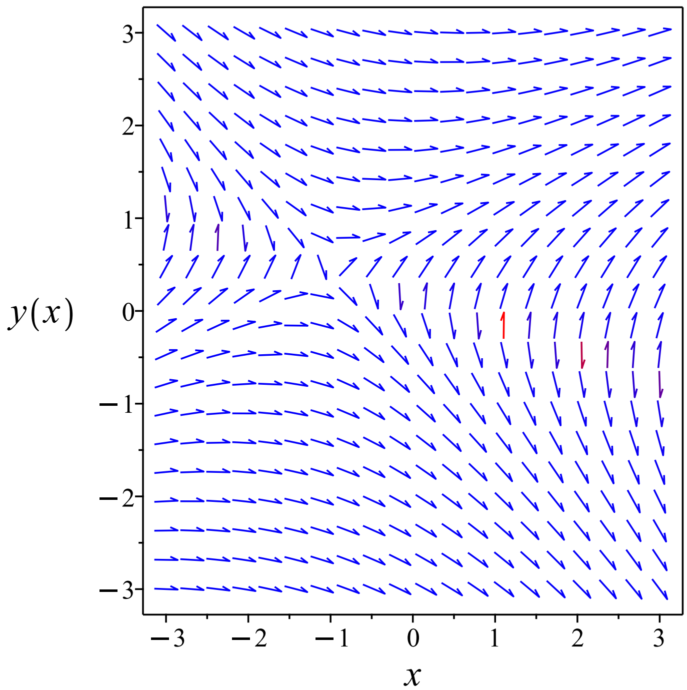


Figure 58: Slope field plot

Verification of solutions

$$\sqrt{-\frac{\left(y+\frac{3}{5}+x\right)\left(3x+\frac{31}{5}-7y\right)}{\left(x+\frac{26}{25}\right)^{2}}} = \frac{c_3 \mathrm{e}^{c_2}}{x+\frac{26}{25}}$$

Verified OK.

2.10.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{-3x+2y-4}{2x+7y-1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 - \frac{(-3x+2y-4)(b_3-a_2)}{2x+7y-1} - \frac{(-3x+2y-4)^2 a_3}{(2x+7y-1)^2} \\
& - \left(\frac{3}{2x+7y-1} + \frac{-6x+4y-8}{(2x+7y-1)^2}\right)(x a_2 + y a_3 + a_1) \\
& - \left(-\frac{2}{2x+7y-1} + \frac{-21x+14y-28}{(2x+7y-1)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{6x^2a_2 + 9x^2a_3 - 29x^2b_2 - 6x^2b_3 + 42xya_2 - 12xya_3 - 28xyb_2 - 42xyb_3 - 14y^2a_2 + 29y^2a_3 - 49y^2b_2 +}{}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& -6x^2a_2 - 9x^2a_3 + 29x^2b_2 + 6x^2b_3 - 42xya_2 + 12xya_3 + 28xyb_2 \\
& + 42xyb_3 + 14y^2a_2 - 29y^2a_3 + 49y^2b_2 - 14y^2b_3 + 6xa_2 \\
& - 24xa_3 + 25xb_1 + 22xb_2 + 5xb_3 - 25ya_1 - 30ya_2 + 27ya_3 \\
& - 14yb_2 + 56yb_3 + 11a_1 + 4a_2 - 16a_3 + 26b_1 + b_2 - 4b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-6a_2v_1^2 - 42a_2v_1v_2 + 14a_2v_2^2 - 9a_3v_1^2 + 12a_3v_1v_2 - 29a_3v_2^2 + 29b_2v_1^2 \\
&+ 28b_2v_1v_2 + 49b_2v_2^2 + 6b_3v_1^2 + 42b_3v_1v_2 - 14b_3v_2^2 - 25a_1v_2 \\
&+ 6a_2v_1 - 30a_2v_2 - 24a_3v_1 + 27a_3v_2 + 25b_1v_1 + 22b_2v_1 - 14b_2v_2 \\
&+ 5b_3v_1 + 56b_3v_2 + 11a_1 + 4a_2 - 16a_3 + 26b_1 + b_2 - 4b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-6a_2 - 9a_3 + 29b_2 + 6b_3)\, v_1^2 + (-42a_2 + 12a_3 + 28b_2 + 42b_3)\, v_1v_2 \\
&+ (6a_2 - 24a_3 + 25b_1 + 22b_2 + 5b_3)\, v_1 + (14a_2 - 29a_3 + 49b_2 - 14b_3)\, v_2^2 \\
&+ (-25a_1 - 30a_2 + 27a_3 - 14b_2 + 56b_3)\, v_2 \\
&+ 11a_1 + 4a_2 - 16a_3 + 26b_1 + b_2 - 4b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-42a_2 + 12a_3 + 28b_2 + 42b_3 &= 0 \\
-6a_2 - 9a_3 + 29b_2 + 6b_3 &= 0 \\
14a_2 - 29a_3 + 49b_2 - 14b_3 &= 0 \\
-25a_1 - 30a_2 + 27a_3 - 14b_2 + 56b_3 &= 0 \\
6a_2 - 24a_3 + 25b_1 + 22b_2 + 5b_3 &= 0 \\
11a_1 + 4a_2 - 16a_3 + 26b_1 + b_2 - 4b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= \frac{100a_1}{27} - \frac{77b_3}{27} \\
a_3 &= \frac{175a_1}{27} - \frac{182b_3}{27} \\
b_1 &= \frac{26a_1}{9} - \frac{31b_3}{9} \\
b_2 &= \frac{25a_1}{9} - \frac{26b_3}{9} \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -\frac{77x}{27} - \frac{182y}{27} \\
\eta &= -\frac{31}{9} - \frac{26x}{9} + y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -\frac{31}{9} - \frac{26x}{9} + y - \left(-\frac{-3x + 2y - 4}{2x + 7y - 1}\right)\left(-\frac{77x}{27} - \frac{182y}{27}\right) \\
&= \frac{75x^2 - 100xy - 175y^2 + 200x + 50y + 93}{54x + 189y - 27} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{75x^2-100xy-175y^2+200x+50y+93}{54x+189y-27}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{27 \ln \left(-75x^2+100xy+175y^2-200x-50y-93\right)}{50}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{-3x+2y-4}{2x+7y-1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-81x+54y-108}{(5x+5y+3)(15x-35y+31)} \\
S_y &= \frac{54x+189y-27}{(5x+5y+3)(15x-35y+31)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

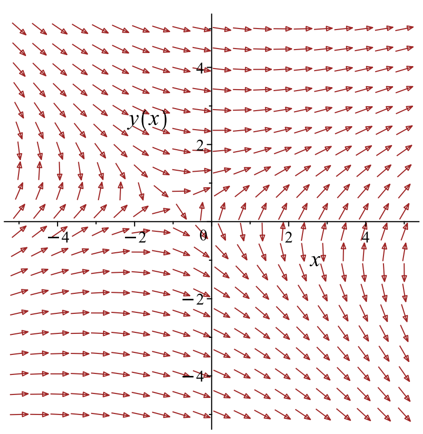
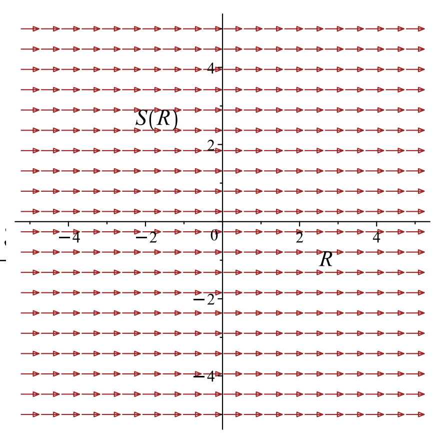
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{27 \ln(-5y - 3 - 5x)}{50} - \frac{27 \ln(15x + 31 - 35y)}{50} = c_1$$

Which simplifies to

$$-\frac{27 \ln(-5y - 3 - 5x)}{50} - \frac{27 \ln(15x + 31 - 35y)}{50} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-3x+2y-4}{2x+7y-1}$ 	$R = x$ $S = -\frac{27 \ln(-5x - 5y -}{50}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$-\frac{27 \ln(-5y - 3 - 5x)}{50} - \frac{27 \ln(15x + 31 - 35y)}{50} = c_1 \tag{1}$$

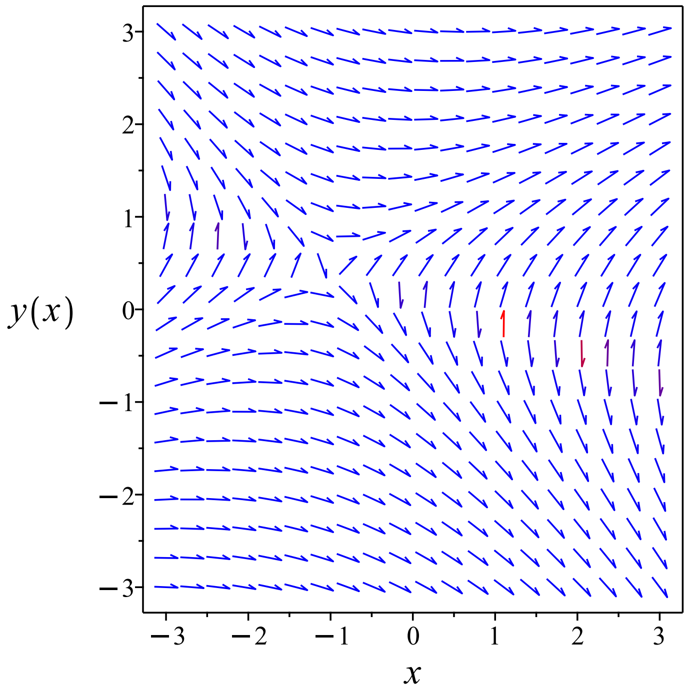


Figure 59: Slope field plot

Verification of solutions

$$
-\frac{27 \ln \left(-5 y-3-5 x\right)}{50}-\frac{27 \ln \left(15 x+31-35 y\right)}{50}=c_{1}
$$

Verified OK.

2.10.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x, y)+N(x, y) \frac{d y}{d x}=0 \tag{A}
$$

We assume there exists a function $\phi(x, y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{d x} \phi(x, y)=0
$$

Hence

$$
\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}
$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2x - 7y + 1)\, \mathrm{d}y &= (-3x + 2y - 4)\, \mathrm{d}x\\ (3x - 2y + 4)\, \mathrm{d}x + (-2x - 7y + 1)\, \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 3x - 2y + 4\\ N(x, y) &= -2x - 7y + 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x - 2y + 4)\\ &= -2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2x - 7y + 1)\\ &= -2\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 3x - 2y + 4 \, \mathrm{d}x$$
$$\phi = \frac{x(3x - 4y + 8)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -2x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -2x - 7y + 1$. Therefore equation (4) becomes

$$-2x - 7y + 1 = -2x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1 - 7y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (1 - 7y) \, \mathrm{d}y$$
$$f(y) = y - \frac{7}{2}y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(3x - 4y + 8)}{2} + y - \frac{7y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(3x - 4y + 8)}{2} + y - \frac{7y^2}{2}$$

Summary

The solution(s) found are the following

$$\frac{x(3x - 4y + 8)}{2} + y - \frac{7y^2}{2} = c_1 \tag{1}$$

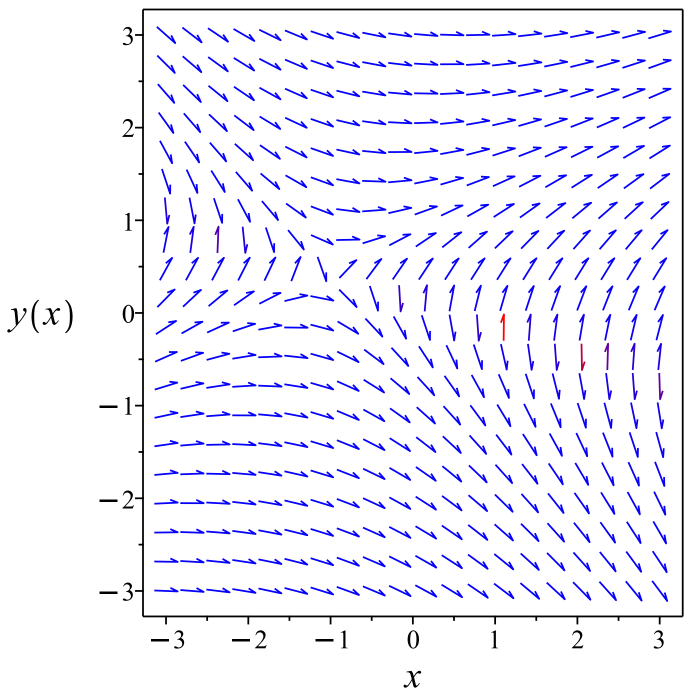


Figure 60: Slope field plot

Verification of solutions

$$\frac{x(3x - 4y + 8)}{2} + y - \frac{7y^2}{2} = c_1$$

Verified OK.

2.10.5 Maple step by step solution

Let's solve

$-2y - (2x + 7y - 1)\,y' = -3x - 4$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

  ◦ Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

  ◦ Evaluate derivatives

  $-2 = -2$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (3x - 2y + 4)\,dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{3x^2}{2} - 2xy + 4x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $-2x - 7y + 1 = -2x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 1 - 7y$

- Solve for $f_1(y)$

  $f_1(y) = y - \frac{7}{2}y^2$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = \frac{3}{2}x^2 - 2xy - \frac{7}{2}y^2 + 4x + y$$

- Substitute $F(x,y)$ into the solution of the ODE

  $$\frac{3}{2}x^2 - 2xy - \frac{7}{2}y^2 + 4x + y = c_1$$

- Solve for y

  $$\left\{y = -\frac{2x}{7} + \frac{1}{7} - \frac{\sqrt{25x^2 - 14c_1 + 52x + 1}}{7}, y = -\frac{2x}{7} + \frac{1}{7} + \frac{\sqrt{25x^2 - 14c_1 + 52x + 1}}{7}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 33

```
dsolve((3*x-2*y(x)+4)-(2*x+7*y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-\sqrt{7 + 15625\left(x + \frac{26}{25}\right)^2 c_1^2} + (-50x + 25)\,c_1}{175c_1}$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 65

```
DSolve[(3*x-2*y[x]+4)-(2*x+7*y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{7}\left(-\sqrt{25x^2+52x+1+49c_1}-2x+1\right)$$
$$y(x) \to \frac{1}{7}\left(\sqrt{25x^2+52x+1+49c_1}-2x+1\right)$$

2.11 problem Differential equations with Linear Coefficients. Exercise 8.11, page 69

Internal problem ID [4451]
Internal file name [`OUTPUT/3944_Sunday_June_05_2022_11_53_32_AM_79593616/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 8
Problem number: Differential equations with Linear Coefficients. Exercise 8.11, page 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y + (3x + 3y - 4)\, y' = -x}$$

With initial conditions

$$[y(1) = 0]$$

2.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{x + y}{3x + 3y - 4} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\left\{x < \frac{4}{3} \vee \frac{4}{3} < x\right\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\left\{y < \frac{1}{3} \vee \frac{1}{3} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x+y}{3x+3y-4}\right)\\ &= -\frac{1}{3x+3y-4} + \frac{3x+3y}{(3x+3y-4)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\left\{x < \frac{4}{3} \vee \frac{4}{3} < x\right\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\left\{y < \frac{1}{3} \vee \frac{1}{3} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

2.11.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{x+y}{3x+3y-4}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(x+y)(b_3 - a_2)}{3x+3y-4} - \frac{(x+y)^2 a_3}{(3x+3y-4)^2} - \left(-\frac{1}{3x+3y-4} + \frac{3x+3y}{(3x+3y-4)^2}\right)(xa_2 + ya_3 + a_1) - \left(-\frac{1}{3x+3y-4} + \frac{3x+3y}{(3x+3y-4)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^2a_2 - x^2a_3 + 9x^2b_2 - 3x^2b_3 + 6xya_2 - 2xya_3 + 18xyb_2 - 6xyb_3 + 3y^2a_2 - y^2a_3 + 9y^2b_2 - 3y^2b_3 - 8xa_2}{(3x+3y-4)^2} = 0$$

Setting the numerator to zero gives

$$3x^2a_2 - x^2a_3 + 9x^2b_2 - 3x^2b_3 + 6xya_2 - 2xya_3 + 18xyb_2 - 6xyb_3 + 3y^2a_2 - y^2a_3 + 9y^2b_2 - 3y^2b_3 - 8xa_2 - 28xb_2 + 4xb_3 - 4ya_2 - 4ya_3 - 24yb_2 - 4a_1 - 4b_1 + 16b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$3a_2v_1^2 + 6a_2v_1v_2 + 3a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - a_3v_2^2 + 9b_2v_1^2 + 18b_2v_1v_2 + 9b_2v_2^2 - 3b_3v_1^2 - 6b_3v_1v_2 - 3b_3v_2^2 - 8a_2v_1 - 4a_2v_2 - 4a_3v_2 - 28b_2v_1 - 24b_2v_2 + 4b_3v_1 - 4a_1 - 4b_1 + 16b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(3a_2 - a_3 + 9b_2 - 3b_3) v_1^2 + (6a_2 - 2a_3 + 18b_2 - 6b_3) v_1 v_2 + (-8a_2 - 28b_2 + 4b_3) v_1 + (3a_2 - a_3 + 9b_2 - 3b_3) v_2^2 + (-4a_2 - 4a_3 - 24b_2) v_2 - 4a_1 - 4b_1 + 16b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -4a_1 - 4b_1 + 16b_2 &= 0 \\ -8a_2 - 28b_2 + 4b_3 &= 0 \\ -4a_2 - 4a_3 - 24b_2 &= 0 \\ 3a_2 - a_3 + 9b_2 - 3b_3 &= 0 \\ 6a_2 - 2a_3 + 18b_2 - 6b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -b_1 + 4b_2 \\ a_2 &= -3b_2 \\ a_3 &= -3b_2 \\ b_1 &= b_1 \\ b_2 &= b_2 \\ b_3 &= b_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -1 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= 1 - \left(-\frac{x + y}{3x + 3y - 4}\right)(-1) \\ &= \frac{2x + 2y - 4}{3x + 3y - 4} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2x+2y-4}{3x+3y-4}} dy \end{aligned}$$

Which results in

$$S = \frac{3y}{2} + \ln(x + y - 2)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y}{3x + 3y - 4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x + y - 2} \\ S_y &= \frac{3}{2} + \frac{1}{x + y - 2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3y}{2} + \ln(x+y-2) = -\frac{x}{2} + c_1$$

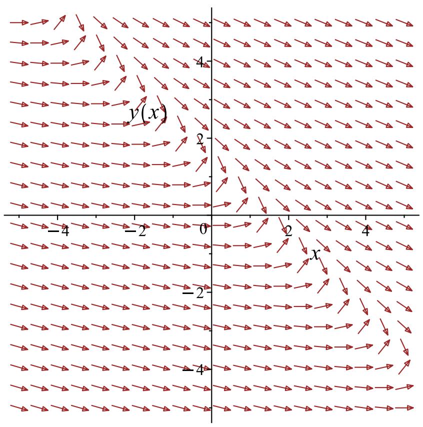
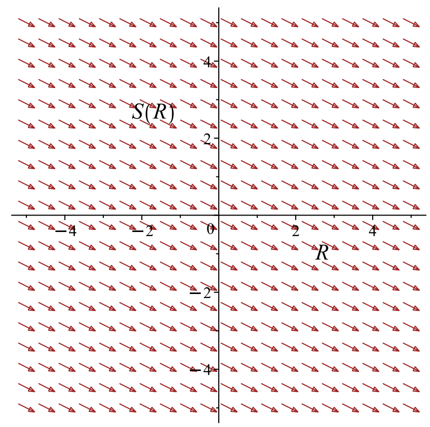
Which simplifies to

$$\frac{3y}{2} + \ln(x+y-2) = -\frac{x}{2} + c_1$$

Which gives

$$y = \frac{2\,\mathrm{LambertW}\left(\frac{3\,\mathrm{e}^{x-3+c_1}}{2}\right)}{3} - x + 2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y}{3x+3y-4}$ 	$R = x$ $S = \frac{3y}{2} + \ln(x+y-2)$	$\frac{dS}{dR} = -\frac{1}{2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{2\,\text{LambertW}\left(\frac{3\,e^{-2+c_1}}{2}\right)}{3} + 1$$

$$c_1 = i\pi + \frac{1}{2}$$

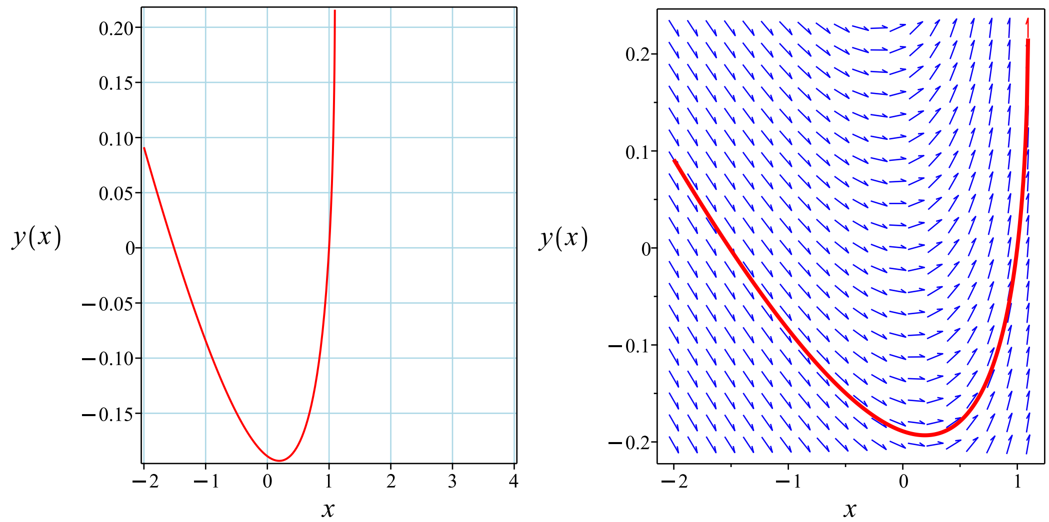
Substituting c_1 found above in the general solution gives

$$y = \frac{2\,\text{LambertW}\left(-\frac{3\,e^{x-\frac{5}{2}}}{2}\right)}{3} - x + 2$$

Summary

The solution(s) found are the following

$$y = \frac{2\,\text{LambertW}\left(-1, -\frac{3\,e^{x-\frac{5}{2}}}{2}\right)}{3} - x + 2 \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{2\,\text{LambertW}\left(-1, -\frac{3\,e^{x-\frac{5}{2}}}{2}\right)}{3} - x + 2$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 19

```
dsolve([(x+y(x))+(3*x+3*y(x)-4)*diff(y(x),x)=0,y(1) = 0],y(x), singsol=all)
```

$$y(x) = \frac{2\,\mathrm{LambertW}\left(-1, -\frac{3\,\mathrm{e}^{-\frac{5}{2}+x}}{2}\right)}{3} + 2 - x$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{(x+y[x])+(3*x+3*y[x]-4)*y'[x]==0,y[1]==0},y[x],x,IncludeSingularSolutions -> True]
```

{}

2.12 problem Differential equations with Linear Coefficients. Exercise 8.12, page 69

Internal problem ID [4452]
Internal file name [`OUTPUT/3945_Sunday_June_05_2022_11_53_43_AM_27976780/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 8
Problem number: Differential equations with Linear Coefficients. Exercise 8.12, page 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{2y-(x+2y-1)\,y' = -3x-3}$$

2.12.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{3X + 3x_0 + 2Y(X) + 2y_0 + 3}{X + x_0 + 2Y(X) + 2y_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -2$$
$$y_0 = \frac{3}{2}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{3X + 2Y(X)}{X + 2Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{3X + 2Y}{X + 2Y}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = 3X + 2Y$ and $N = X + 2Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= \frac{2u + 3}{2u + 1} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{2u(X)+3}{2u(X)+1} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) - \frac{\frac{2u(X)+3}{2u(X)+1} - u(X)}{X} = 0
$$

Or

$$
2\left(\frac{d}{dX} u(X)\right) X u(X) + \left(\frac{d}{dX} u(X)\right) X + 2u(X)^2 - u(X) - 3 = 0
$$

Or

$$
-3 + X(2u(X) + 1) \left(\frac{d}{dX} u(X)\right) + 2u(X)^2 - u(X) = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= -\frac{2u^2 - u - 3}{X (2u + 1)}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{2u^2-u-3}{2u+1}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{2u^2-u-3}{2u+1}}\, du &= -\frac{1}{X}\, dX \\
\int \frac{1}{\frac{2u^2-u-3}{2u+1}}\, du &= \int -\frac{1}{X}\, dX \\
\frac{\ln(u+1)}{5} + \frac{4\ln(2u-3)}{5} &= -\ln(X) + c_2
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
\frac{\ln(u+1) + 4\ln(2u-3)}{5} &= -\ln(X) + c_2 \\
\ln(u+1) + 4\ln(2u-3) &= (5)(-\ln(X) + c_2) \\
&= -5\ln(X) + 5c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{\ln(u+1)+4\ln(2u-3)} = \mathrm{e}^{-5\ln(X)+5c_2}
$$

Which simplifies to

$$
\begin{aligned}
(u+1)(2u-3)^4 &= \frac{5c_2}{X^5} \\
&= \frac{c_3}{X^5}
\end{aligned}
$$

Which simplifies to

$$
u(X) = \mathrm{RootOf}\left(16\_Z^5 - 80\_Z^4 + 120\_Z^3 - \frac{c_3 \mathrm{e}^{5c_2}}{X^5} - 135\_Z + 81\right)
$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$
Y(X) = X \,\mathrm{RootOf}\left(16\_Z^5 X^5 - 80\_Z^4 X^5 + 120\_Z^3 X^5 - c_3 \mathrm{e}^{5c_2} - 135\_Z X^5 + 81X^5\right)
$$

Using the solution for $Y(X)$

$$
Y(X) = X \,\mathrm{RootOf}\left(16\_Z^5 X^5 - 80\_Z^4 X^5 + 120\_Z^3 X^5 - c_3 \mathrm{e}^{5c_2} - 135\_Z X^5 + 81X^5\right)
$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + \frac{3}{2}$$
$$X = -2 + x$$

Then the solution in y becomes

$$y - \frac{3}{2} = (x+2)\,\mathrm{RootOf}\left(\left(16x^5 + 160x^4 + 640x^3 + 1280x^2 + 1280x + 512\right)\_Z^5 + \left(-80x^5 - 800x^4 - 320\right.\right.$$

Summary

The solution(s) found are the following

$$\begin{aligned} y - \frac{3}{2} = (x+2)\,\mathrm{RootOf}\Big(&\left(16x^5 + 160x^4 + 640x^3 + 1280x^2 + 1280x + 512\right)\_Z^5 \\ &+ \left(-80x^5 - 800x^4 - 3200x^3 - 6400x^2 - 6400x - 2560\right)\_Z^4 \\ &+ \left(120x^5 + 1200x^4 + 4800x^3 + 9600x^2 + 9600x + 3840\right)\_Z^3 \\ &+ \left(-135x^5 - 1350x^4 - 5400x^3 - 10800x^2 - 10800x - 4320\right)\_Z - c_3 e^{5c_2} \\ &+ 81x^5 + 810x^4 + 3240x^3 + 6480x^2 + 6480x + 2592\Big) \end{aligned} \tag{1}$$

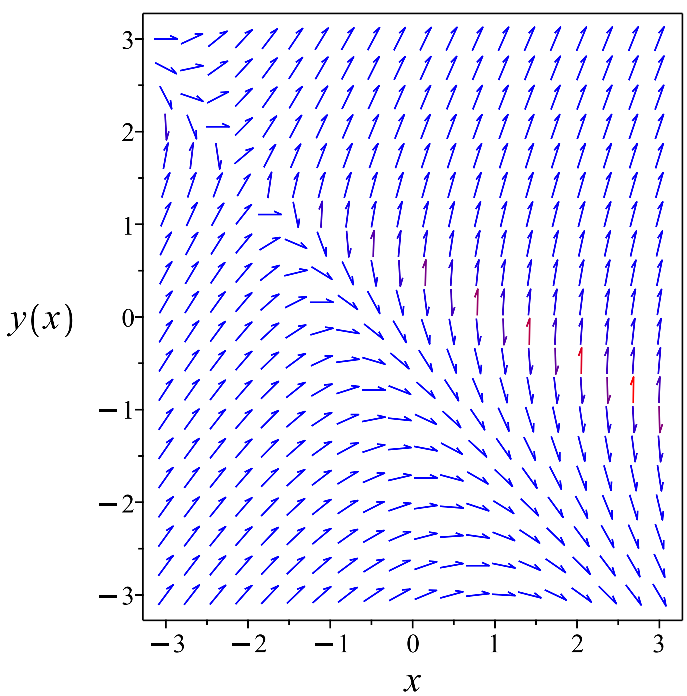


Figure 62: Slope field plot

Verification of solutions

$$
\begin{aligned}
y-\frac{3}{2} = (x+2)\operatorname{RootOf}\Big(&\left(16x^{5}+160x^{4}+640x^{3}+1280x^{2}+1280x+512\right)\_Z^{5} \\
&+\left(-80x^{5}-800x^{4}-3200x^{3}-6400x^{2}-6400x-2560\right)\_Z^{4} \\
&+\left(120x^{5}+1200x^{4}+4800x^{3}+9600x^{2}+9600x+3840\right)\_Z^{3} \\
&+\left(-135x^{5}-1350x^{4}-5400x^{3}-10800x^{2}-10800x-4320\right)\_Z-c_{3}\mathrm{e}^{5c_{2}} \\
&+81x^{5}+810x^{4}+3240x^{3}+6480x^{2}+6480x+2592\Big)
\end{aligned}
$$

Verified OK.

2.12.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{3x+2y+3}{x-1+2y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_{x}+\omega(\eta_{y}-\xi_{x})-\omega^{2}\xi_{y}-\omega_{x}\xi-\omega_{y}\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_{2}+ya_{3}+a_{1} \tag{1E}
$$

$$
\eta = xb_{2}+yb_{3}+b_{1} \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_{1},a_{2},a_{3},b_{1},b_{2},b_{3}\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_{2}+\frac{(3x+2y+3)(b_{3}-a_{2})}{x-1+2y}-\frac{(3x+2y+3)^{2}a_{3}}{(x-1+2y)^{2}} \\
-\left(\frac{3}{x-1+2y}-\frac{3x+2y+3}{(x-1+2y)^{2}}\right)(xa_{2}+ya_{3}+a_{1}) \\
-\left(\frac{2}{x-1+2y}-\frac{2(3x+2y+3)}{(x-1+2y)^{2}}\right)(xb_{2}+yb_{3}+b_{1})=0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$-\frac{3x^2a_2 + 9x^2a_3 - 5x^2b_2 - 3x^2b_3 + 12xya_2 + 12xya_3 - 4xyb_2 - 12xyb_3 + 4y^2a_2 + 8y^2a_3 - 4y^2b_2 - 4y^2b_3}{(x - 1}$$

$= 0$

Setting the numerator to zero gives

$$\begin{aligned} &-3x^2a_2 - 9x^2a_3 + 5x^2b_2 + 3x^2b_3 - 12xya_2 - 12xya_3 + 4xyb_2 + 12xyb_3 \\ &- 4y^2a_2 - 8y^2a_3 + 4y^2b_2 + 4y^2b_3 + 6xa_2 - 18xa_3 + 4xb_1 + 6xb_2 - 4ya_1 \\ &- 4ya_2 - 6ya_3 - 4yb_2 + 12yb_3 + 6a_1 + 3a_2 - 9a_3 + 8b_1 + b_2 - 3b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-3a_2v_1^2 - 12a_2v_1v_2 - 4a_2v_2^2 - 9a_3v_1^2 - 12a_3v_1v_2 - 8a_3v_2^2 + 5b_2v_1^2 + 4b_2v_1v_2 \\ &+ 4b_2v_2^2 + 3b_3v_1^2 + 12b_3v_1v_2 + 4b_3v_2^2 - 4a_1v_2 + 6a_2v_1 - 4a_2v_2 - 18a_3v_1 - 6a_3v_2 \\ &+ 4b_1v_1 + 6b_2v_1 - 4b_2v_2 + 12b_3v_2 + 6a_1 + 3a_2 - 9a_3 + 8b_1 + b_2 - 3b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-3a_2 - 9a_3 + 5b_2 + 3b_3)\, v_1^2 + (-12a_2 - 12a_3 + 4b_2 + 12b_3)\, v_1v_2 \\ &+ (6a_2 - 18a_3 + 4b_1 + 6b_2)\, v_1 + (-4a_2 - 8a_3 + 4b_2 + 4b_3)\, v_2^2 \\ &+ (-4a_1 - 4a_2 - 6a_3 - 4b_2 + 12b_3)\, v_2 + 6a_1 + 3a_2 - 9a_3 + 8b_1 + b_2 - 3b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-12a_2 - 12a_3 + 4b_2 + 12b_3 &= 0 \\
-4a_2 - 8a_3 + 4b_2 + 4b_3 &= 0 \\
-3a_2 - 9a_3 + 5b_2 + 3b_3 &= 0 \\
6a_2 - 18a_3 + 4b_1 + 6b_2 &= 0 \\
-4a_1 - 4a_2 - 6a_3 - 4b_2 + 12b_3 &= 0 \\
6a_1 + 3a_2 - 9a_3 + 8b_1 + b_2 - 3b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -\frac{5a_3}{2} + 2b_3 \\
a_2 &= -\frac{a_3}{2} + b_3 \\
a_3 &= a_3 \\
b_1 &= 3a_3 - \frac{3b_3}{2} \\
b_2 &= \frac{3a_3}{2} \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x + 2 \\
\eta &= -\frac{3}{2} + y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -\frac{3}{2} + y - \left(\frac{3x + 2y + 3}{x - 1 + 2y}\right)(x + 2) \\
&= \frac{-6x^2 - 2xy + 4y^2 - 21x - 16y - 9}{2x - 2 + 4y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-6x^2-2xy+4y^2-21x-16y-9}{2x-2+4y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(2x+2y+1)}{5} + \frac{4\ln(2y-3x-9)}{5}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{3x+2y+3}{x-1+2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{6x+4y+6}{(2x+2y+1)(3x-2y+9)} \\ S_y &= \frac{-2x-4y+2}{(2x+2y+1)(3x-2y+9)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

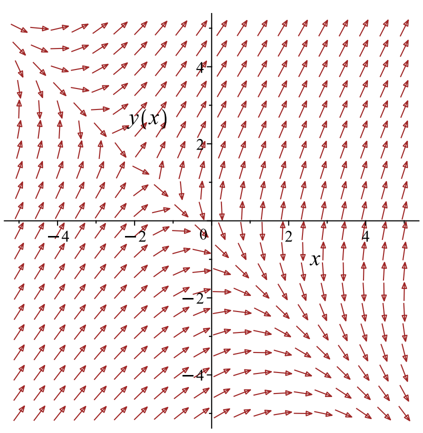
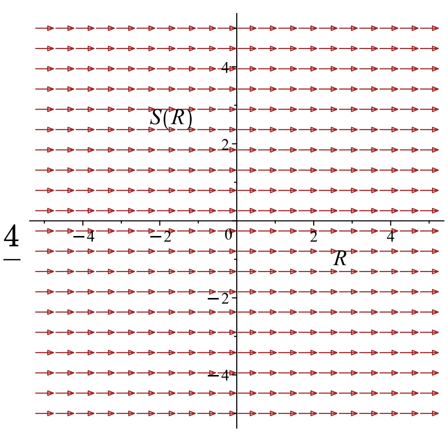
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(2x+2y+1)}{5} + \frac{4\ln(2y-3x-9)}{5} = c_1$$

Which simplifies to

$$\frac{\ln(2x+2y+1)}{5} + \frac{4\ln(2y-3x-9)}{5} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x+2y+3}{x-1+2y}$ 	$R = x$ $S = \frac{\ln(2x+2y+1)}{5} + \frac{4}{}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln (2x+2y+1)}{5}+\frac{4\ln (2y-3x-9)}{5}=c_1 \tag{1}$$

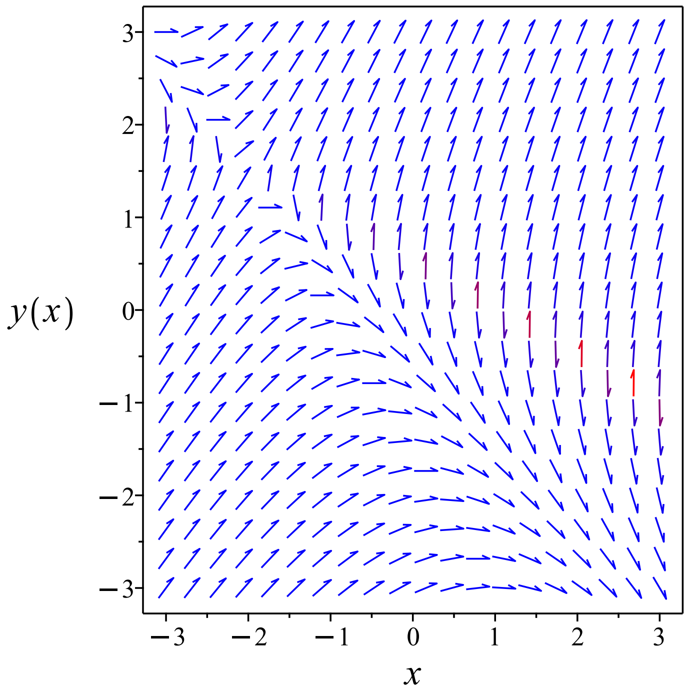


Figure 63: Slope field plot

Verification of solutions

$$\frac{\ln (2x+2y+1)}{5}+\frac{4\ln (2y-3x-9)}{5}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.187 (sec). Leaf size: 93

```
dsolve((3*x+2*y(x)+3)-(x+2*y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{(-2-x)\,\mathrm{RootOf}\left(-1+(16c_1x^5+160c_1x^4+640c_1x^3+1280c_1x^2+1280c_1x+512c_1)\_Z^{25}+(-80c_1x^5\right.}{2} + \frac{3x}{2} + \frac{9}{2}$$

✓ Solution by Mathematica

Time used: 60.094 (sec). Leaf size: 3081

```
DSolve[(3*x+2*y[x]+3)-(x+2*y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

2.13 problem Differential equations with Linear Coefficients. Exercise 8.13, page 69

Internal problem ID [4453]
Internal file name [`OUTPUT/3946_Sunday_June_05_2022_11_53_54_AM_96816161/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 8
Problem number: Differential equations with Linear Coefficients. Exercise 8.13, page 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y + (2x + y + 3)\, y' = -7}$$

With initial conditions

$$[y(0) = 1]$$

2.13.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{y+7}{2x+y+3} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{x < -2 \vee -2 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{y < -3 \lor -3 < y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y+7}{2x+y+3}\right)\\ &= -\frac{1}{2x+y+3} + \frac{y+7}{(2x+y+3)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{y < -3 \lor -3 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.13.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{Y(X) + y_0 + 7}{2X + 2x_0 + Y(X) + y_0 + 3}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= 2\\ y_0 &= -7\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{Y(X)}{2X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned}Y' &= F(X, Y)\\ &= -\frac{Y}{2X + Y}\end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -Y$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{du}{dX}X + u = -\frac{u}{u+2}$$
$$\frac{du}{dX} = \frac{-\frac{u(X)}{u(X)+2} - u(X)}{X}$$

Or

$$\frac{d}{dX}u(X) - \frac{-\frac{u(X)}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) Xu(X) + 2\left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 3u(X) = 0$$

Or

$$X(u(X) + 2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 3u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{u(u+3)}{X(u+2)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u(u+3)}{u+2}$. Integrating both sides gives

$$\frac{1}{\frac{u(u+3)}{u+2}}\, du = -\frac{1}{X}\, dX$$

$$\int \frac{1}{\frac{u(u+3)}{u+2}}\, du = \int -\frac{1}{X}\, dX$$

$$\frac{2\ln(u)}{3} + \frac{\ln(u+3)}{3} = -\ln(X) + c_2$$

The above can be written as

$$
\begin{aligned}
\frac{2\ln(u)+\ln(u+3)}{3} &= -\ln(X)+c_2 \\
2\ln(u)+\ln(u+3) &= (3)(-\ln(X)+c_2) \\
&= -3\ln(X)+3c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{2\ln(u)+\ln(u+3)} = e^{-3\ln(X)+3c_2}
$$

Which simplifies to

$$
\begin{aligned}
u^2(u+3) &= \frac{3c_2}{X^3} \\
&= \frac{c_3}{X^3}
\end{aligned}
$$

Which simplifies to

$$
u(X)^2(u(X)+3) = \frac{c_3 e^{3c_2}}{X^3}
$$

The solution is

$$
u(X)^2(u(X)+3) = \frac{c_3 e^{3c_2}}{X^3}
$$

Now u in the above solution is replaced back by Y using $u=\frac{Y}{X}$ which results in the solution

$$
\frac{Y(X)^2\left(\frac{Y(X)}{X}+3\right)}{X^2} = \frac{c_3 e^{3c_2}}{X^3}
$$

Which simplifies to

$$
Y(X)^2(Y(X)+3X) = c_3 e^{3c_2}
$$

Using the solution for $Y(X)$

$$
Y(X)^2(Y(X)+3X) = c_3 e^{3c_2}
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y+y_0 \\
X &= x+x_0
\end{aligned}
$$

Or

$$Y = y - 7$$
$$X = x + 2$$

Then the solution in y becomes

$$(y+7)^2 (y+1+3x) = c_3 e^{3c_2}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$128 = c_3 e^{3c_2}$$

$$c_2 = \frac{\ln\left(\frac{128}{c_3}\right)}{3}$$

Substituting c_2 found above in the general solution gives

$$(y+7)^2 (y+3x+1) = 128$$

<u>Summary</u>
The solution(s) found are the following

$$(y+7)^2 (y+1+3x) = 128 \tag{1}$$

<u>Verification of solutions</u>

$$(y+7)^2 (y+1+3x) = 128$$

Verified OK.

2.13.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y+7}{2x+y+3}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(y+7)(b_3 - a_2)}{2x+y+3} - \frac{(y+7)^2 a_3}{(2x+y+3)^2} - \frac{2(y+7)(xa_2 + ya_3 + a_1)}{(2x+y+3)^2} \\ &- \left(-\frac{1}{2x+y+3} + \frac{y+7}{(2x+y+3)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{6x^2b_2 + 4xyb_2 + y^2a_2 - 3y^2a_3 + y^2b_2 - y^2b_3 + 2xb_1 + 8xb_2 - 14xb_3 - 2ya_1 + 10ya_2 - 28ya_3 + 6yb_2 - 14y}{(2x+y+3)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &6x^2b_2 + 4xyb_2 + y^2a_2 - 3y^2a_3 + y^2b_2 - y^2b_3 + 2xb_1 + 8xb_2 - 14xb_3 - 2ya_1 \\ &+ 10ya_2 - 28ya_3 + 6yb_2 - 14yb_3 - 14a_1 + 21a_2 - 49a_3 - 4b_1 + 9b_2 - 21b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&a_2v_2^2 - 3a_3v_2^2 + 6b_2v_1^2 + 4b_2v_1v_2 + b_2v_2^2 - b_3v_2^2 - 2a_1v_2 + 10a_2v_2 - 28a_3v_2 + 2b_1v_1 \\
&+ 8b_2v_1 + 6b_2v_2 - 14b_3v_1 - 14b_3v_2 - 14a_1 + 21a_2 - 49a_3 - 4b_1 + 9b_2 - 21b_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&6b_2v_1^2 + 4b_2v_1v_2 + (2b_1 + 8b_2 - 14b_3)\, v_1 + (a_2 - 3a_3 + b_2 - b_3)\, v_2^2 \\
&+ (-2a_1 + 10a_2 - 28a_3 + 6b_2 - 14b_3)\, v_2 - 14a_1 + 21a_2 - 49a_3 - 4b_1 + 9b_2 - 21b_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
4b_2 &= 0 \\
6b_2 &= 0 \\
2b_1 + 8b_2 - 14b_3 &= 0 \\
a_2 - 3a_3 + b_2 - b_3 &= 0 \\
-2a_1 + 10a_2 - 28a_3 + 6b_2 - 14b_3 &= 0 \\
-14a_1 + 21a_2 - 49a_3 - 4b_1 + 9b_2 - 21b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_3 - 2b_3 \\
a_2 &= 3a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= 7b_3 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2 + x \\
\eta &= y + 7
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y + 7 - \left(-\frac{y+7}{2x+y+3}\right)(-2+x) \\
&= \frac{3xy + y^2 + 21x + 8y + 7}{2x+y+3} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3xy+y^2+21x+8y+7}{2x+y+3}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{2\ln(y+7)}{3} + \frac{\ln(y+3x+1)}{3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y+7}{2x+y+3}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{y+3x+1} \\
S_y &= \frac{2x+y+3}{(y+7)(y+3x+1)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

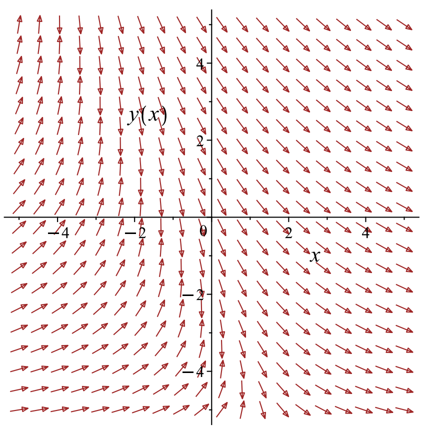
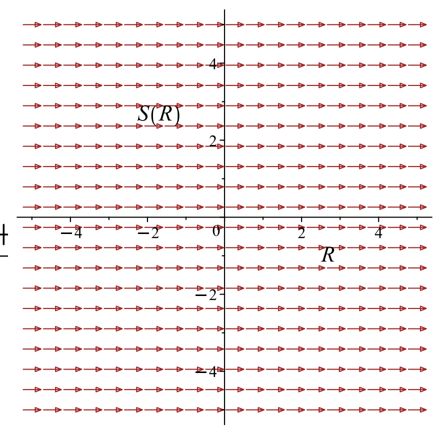
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{2\ln(y+7)}{3} + \frac{\ln(y+1+3x)}{3} = c_1
$$

Which simplifies to

$$
\frac{2\ln(y+7)}{3} + \frac{\ln(y+1+3x)}{3} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y+7}{2x+y+3}$ 	$R = x$ $S = \frac{2\ln(y+7)}{3} + \frac{\ln(y+}{}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{7\ln(2)}{3} = c_1$$

$$c_1 = \frac{7\ln(2)}{3}$$

Substituting c_1 found above in the general solution gives

$$\frac{2\ln(y+7)}{3} + \frac{\ln(y+3x+1)}{3} = \frac{7\ln(2)}{3}$$

Summary

The solution(s) found are the following

$$\frac{2\ln(y+7)}{3} + \frac{\ln(y+1+3x)}{3} = \frac{7\ln(2)}{3} \tag{1}$$

Verification of solutions

$$\frac{2\ln(y+7)}{3} + \frac{\ln(y+1+3x)}{3} = \frac{7\ln(2)}{3}$$

Verified OK.

2.13.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x+y+3)\,\mathrm{d}y&=(-y-7)\,\mathrm{d}x\\ (y+7)\,\mathrm{d}x+(2x+y+3)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y+7\\ N(x,y)&=2x+y+3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y+7)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x+y+3)\\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2x+y+3}((1)-(2))\\ &= -\frac{1}{2x+y+3}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= \frac{1}{y+7}((2)-(1))\\ &= \frac{1}{y+7}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y}\\ &= e^{\int \frac{1}{y+7} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(y+7)} \\ &= y+7\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= y+7(y+7) \\ &= (y+7)^2\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= y+7(2x+y+3) \\ &= (y+7)(2x+y+3)\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left((y+7)^2\right)+\left((y+7)(2x+y+3)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int(y+7)^2\,\mathrm{d}x\end{aligned}$$

$$\phi = (y+7)^2 x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2(y+7)\,x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (y+7)\,(2x+y+3)$. Therefore equation (4) becomes

$$(y+7)\,(2x+y+3) = 2(y+7)\,x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2 + 10y + 21$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(y^2 + 10y + 21\right) \mathrm{d}y$$

$$f(y) = \frac{1}{3}y^3 + 5y^2 + 21y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = (y+7)^2\,x + \frac{y^3}{3} + 5y^2 + 21y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y+7)^2\,x + \frac{y^3}{3} + 5y^2 + 21y$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{79}{3} = c_1$$

$$c_1 = \frac{79}{3}$$

Substituting c_1 found above in the general solution gives

$$(y+7)^2 x+\frac{y^3}{3}+5y^2+21y=\frac{79}{3}$$

Summary

The solution(s) found are the following

$$(y+7)^2 x+\frac{y^3}{3}+5y^2+21y=\frac{79}{3} \tag{1}$$

Verification of solutions

$$(y+7)^2 x+\frac{y^3}{3}+5y^2+21y=\frac{79}{3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 87

```
dsolve([(y(x)+7)+(2*x+y(x)+3)*diff(y(x),x)=0,y(0) = 1],y(x), singsol=all)
```

$$y(x)=\left(-x^3+6x^2-12x+72+8\sqrt{-2x^3+12x^2-24x+80}\right)^{\frac{1}{3}}+\frac{(x-2)^2}{\left(-x^3+6x^2-12x+72+8\sqrt{-2x^3+12x^2-24x+80}\right)^{\frac{1}{3}}}-x-5$$

✓ Solution by Mathematica

Time used: 6.783 (sec). Leaf size: 198

```
DSolve[{(y[x]+7)+(2*x+y[x]+3)*y'[x]==0,y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x^2 - \left(\sqrt[3]{-x^3+6x^2+8\sqrt{2}\sqrt{-x^3+6x^2-12x+40}-12x+72}+4\right)x + \left(-x^3+6x^2+8\sqrt{2}\sqrt{-x^3+6\right.}{\sqrt[3]{-x^3+6x^2+8\sqrt{2}\sqrt{-x^3+6x^}}}$$

2.14 problem Differential equations with Linear Coefficients. Exercise 8.14, page 69

Internal problem ID [4454]
Internal file name [OUTPUT/3947_Sunday_June_05_2022_11_54_03_AM_78134721/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 8
Problem number: Differential equations with Linear Coefficients. Exercise 8.14, page 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y - (x - y - 4)\, y' = -x - 2}$$

2.14.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{X + x_0 + Y(X) + y_0 + 2}{-X - x_0 + Y(X) + y_0 + 4}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = -3$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{X + Y(X)}{-X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= -\frac{X+Y}{-X+Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = X+Y$ and $N = X-Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-u-1}{u-1} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) X u(X) - \left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 1 = 0$$

Or

$$X(u(X)-1)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{u^2+1}{X(u-1)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}} \, du = -\frac{1}{X} \, dX$$

$$\int \frac{1}{\frac{u^2+1}{u-1}} \, du = \int -\frac{1}{X} \, dX$$

$$\frac{\ln (u^2+1)}{2} - \arctan (u) = -\ln (X) + c_2$$

The solution is

$$\frac{\ln \left(u(X)^2+1\right)}{2} - \arctan (u(X)) + \ln (X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln \left(\frac{Y(X)^2}{X^2}+1\right)}{2} - \arctan \left(\frac{Y(X)}{X}\right) + \ln (X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln \left(\frac{Y(X)^2}{X^2}+1\right)}{2} - \arctan \left(\frac{Y(X)}{X}\right) + \ln (X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 3$$
$$X = x + 1$$

Then the solution in y becomes

$$\frac{\ln \left(\frac{(y+3)^2}{(x-1)^2}+1\right)}{2} - \arctan \left(\frac{y+3}{x-1}\right) + \ln (x-1) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{(y+3)^2}{(x-1)^2}+1\right)}{2}-\arctan\left(\frac{y+3}{x-1}\right)+\ln(x-1)-c_2=0 \tag{1}$$

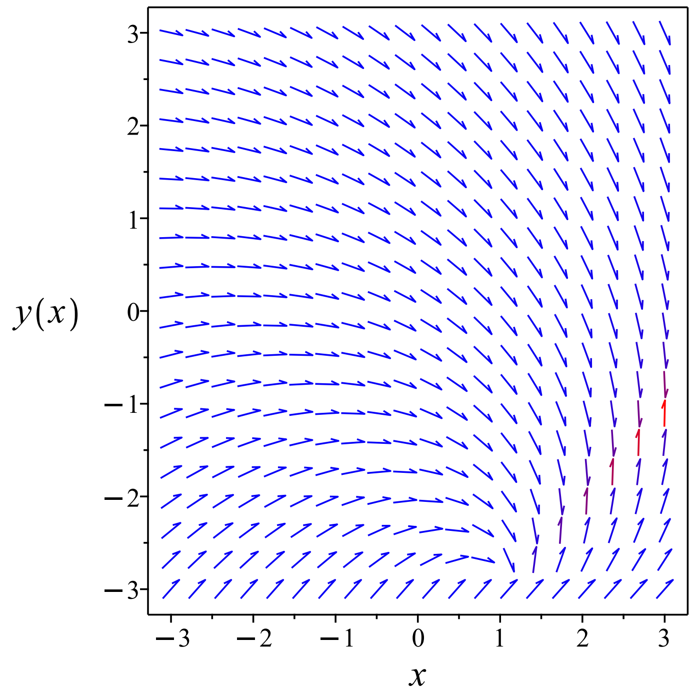


Figure 64: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{(y+3)^2}{(x-1)^2}+1\right)}{2}-\arctan\left(\frac{y+3}{x-1}\right)+\ln(x-1)-c_2=0$$

Verified OK.

2.14.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{x+y+2}{-x+y+4}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(x+y+2)(b_3-a_2)}{-x+y+4} - \frac{(x+y+2)^2 a_3}{(-x+y+4)^2} \\ &- \left(-\frac{1}{-x+y+4} - \frac{x+y+2}{(-x+y+4)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{1}{-x+y+4} + \frac{x+y+2}{(-x+y+4)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_2 + x^2a_3 + x^2b_2 - x^2b_3 - 2xya_2 + 2xya_3 + 2xyb_2 + 2xyb_3 - y^2a_2 - y^2a_3 - y^2b_2 + y^2b_3 - 8xa_2 + 4xa}{(x-y-}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-x^2a_2 - x^2a_3 - x^2b_2 + x^2b_3 + 2xya_2 - 2xya_3 - 2xyb_2 - 2xyb_3 + y^2a_2 \\ &+ y^2a_3 + y^2b_2 - y^2b_3 + 8xa_2 - 4xa_3 - 2xb_1 - 6xb_2 - 2xb_3 + 2ya_1 \\ &+ 6ya_2 + 2ya_3 + 8yb_2 - 4yb_3 + 6a_1 + 8a_2 - 4a_3 + 2b_1 + 16b_2 - 8b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a_2 v_1^2 + 2a_2 v_1 v_2 + a_2 v_2^2 - a_3 v_1^2 - 2a_3 v_1 v_2 + a_3 v_2^2 - b_2 v_1^2 - 2b_2 v_1 v_2 + b_2 v_2^2 \\
&+ b_3 v_1^2 - 2b_3 v_1 v_2 - b_3 v_2^2 + 2a_1 v_2 + 8a_2 v_1 + 6a_2 v_2 - 4a_3 v_1 + 2a_3 v_2 - 2b_1 v_1 \\
&- 6b_2 v_1 + 8b_2 v_2 - 2b_3 v_1 - 4b_3 v_2 + 6a_1 + 8a_2 - 4a_3 + 2b_1 + 16b_2 - 8b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-a_2 - a_3 - b_2 + b_3) v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3) v_1 v_2 \\
&+ (8a_2 - 4a_3 - 2b_1 - 6b_2 - 2b_3) v_1 + (a_2 + a_3 + b_2 - b_3) v_2^2 \\
&+ (2a_1 + 6a_2 + 2a_3 + 8b_2 - 4b_3) v_2 + 6a_1 + 8a_2 - 4a_3 + 2b_1 + 16b_2 - 8b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-a_2 - a_3 - b_2 + b_3 &= 0 \\
a_2 + a_3 + b_2 - b_3 &= 0 \\
2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0 \\
2a_1 + 6a_2 + 2a_3 + 8b_2 - 4b_3 &= 0 \\
8a_2 - 4a_3 - 2b_1 - 6b_2 - 2b_3 &= 0 \\
6a_1 + 8a_2 - 4a_3 + 2b_1 + 16b_2 - 8b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -3b_2 - b_3 \\
a_2 &= b_3 \\
a_3 &= -b_2 \\
b_1 &= -b_2 + 3b_3 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -3 - y \\
\eta &= x - 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= x - 1 - \left(-\frac{x+y+2}{-x+y+4}\right)(-3-y) \\
&= \frac{x^2+y^2-2x+6y+10}{x-y-4} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2+y^2-2x+6y+10}{x-y-4}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{\ln(x^2+y^2-2x+6y+10)}{2} + \frac{2(x-1)\arctan\left(\frac{2y+6}{2x-2}\right)}{2x-2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x+y+2}{-x+y+4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-x-y-2}{x^2+y^2-2x+6y+10} \\
S_y &= \frac{x-y-4}{x^2+y^2-2x+6y+10}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

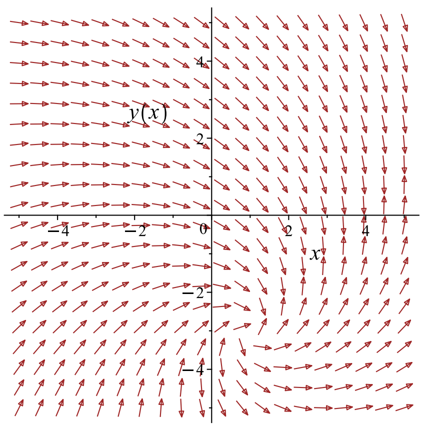
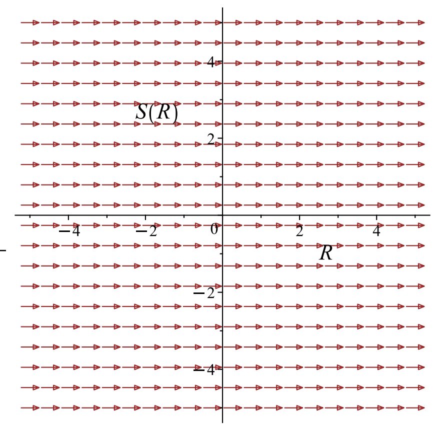
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y^2+x^2+6y-2x+10)}{2} + \arctan\left(\frac{y+3}{x-1}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y^2+x^2+6y-2x+10)}{2} + \arctan\left(\frac{y+3}{x-1}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y+2}{-x+y+4}$ 	$R = x$ $S = -\frac{\ln\left(x^2+y^2-2x+\right.}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$-\frac{\ln\left(y^2+x^2+6y-2x+10\right)}{2}+\arctan\left(\frac{y+3}{x-1}\right)=c_1 \tag{1}$$

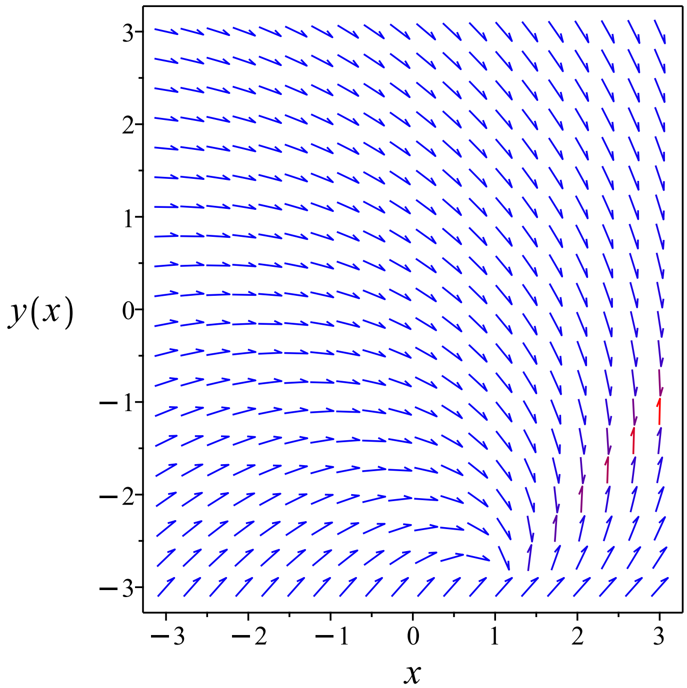


Figure 65: Slope field plot

Verification of solutions

$$-\frac{\ln\left(y^2+x^2+6y-2x+10\right)}{2}+\arctan\left(\frac{y+3}{x-1}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.204 (sec). Leaf size: 31

```
dsolve((x+y(x)+2)-(x-y(x)-4)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -3 - \tan\left(\operatorname{RootOf}\left(2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x-1\right) + 2c_1\right)\right)\left(x-1\right)$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 58

```
DSolve[(x+y[x]+2)-(x-y[x]-4)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[2\arctan\left(\frac{y(x)+x+2}{y(x)-x+4}\right) + \log\left(\frac{x^2+y(x)^2+6y(x)-2x+10}{2(x-1)^2}\right) + 2\log(x-1) + c_1 = 0, y(x)\right]$$

3 Chapter 2. Special types of differential equations of the first kind. Lesson 9

3.1 problem Exact Differential equations. Exercise 9.4, page 79

Internal problem ID [4455]
Internal file name [`OUTPUT/3948_Sunday_June_05_2022_11_54_13_AM_42012435/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 9
Problem number: Exact Differential equations. Exercise 9.4, page 79.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact, _rational]`

$$\boxed{3yx^2+8xy^2+\left(x^3+8yx^2+12y^2\right)y'=0}$$

3.1.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x^3 + 8y\,x^2 + 12y^2)\,\mathrm{d}y &= (-3y\,x^2 - 8y^2x)\,\mathrm{d}x \\
(3y\,x^2 + 8y^2x)\,\mathrm{d}x + (x^3 + 8y\,x^2 + 12y^2)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= 3y\,x^2 + 8y^2x \\
N(x, y) &= x^3 + 8y\,x^2 + 12y^2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3y\,x^2 + 8y^2x) \\
&= x(3x + 16y)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^3 + 8y\,x^2 + 12y^2) \\
&= x(3x + 16y)
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int 3y\,x^2 + 8y^2 x \, \mathrm{d}x \\
\phi &= y\,x^2(x+4y) + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}
\frac{\partial \phi}{\partial y} &= x^2(x+4y) + 4y\,x^2 + f'(y) \qquad (4) \\
&= x^2(x+8y) + f'(y)
\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3 + 8y\,x^2 + 12y^2$. Therefore equation (4) becomes

$$x^3 + 8y\,x^2 + 12y^2 = x^2(x+8y) + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 12y^2$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int (12y^2) \, \mathrm{d}y \\
f(y) &= 4y^3 + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,x^2(x+4y) + 4y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^2(x+4y) + 4y^3$$

Summary

The solution(s) found are the following

$$yx^2(x+4y)+4y^3=c_1 \tag{1}$$

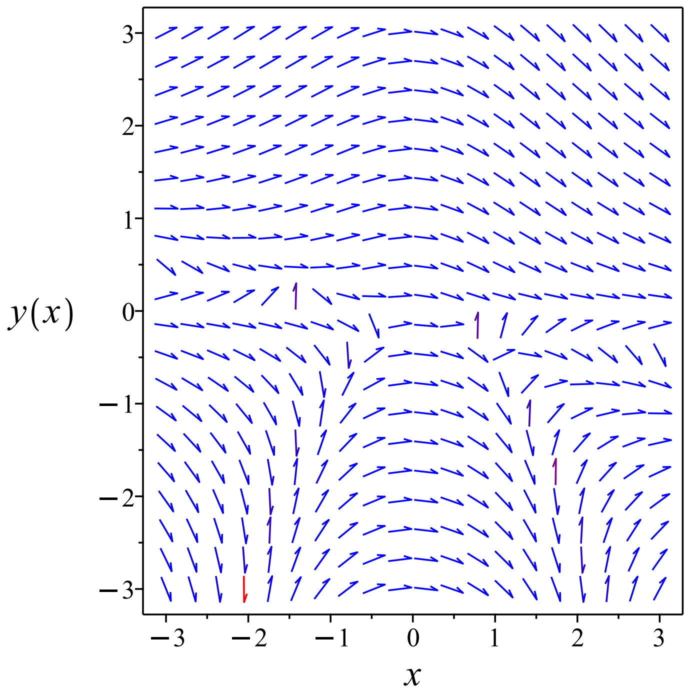


Figure 66: Slope field plot

Verification of solutions

$$yx^2(x+4y)+4y^3=c_1$$

Verified OK.

3.1.2 Maple step by step solution

Let's solve

$3yx^2+8xy^2+(x^3+8yx^2+12y^2)\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $3x^2 + 16xy = 3x^2 + 16xy$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (3y\,x^2 + 8y^2x)\,dx + f_1(y)$

- Evaluate integral

  $F(x, y) = y(x^3 + 4y\,x^2) + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $x^3 + 8y\,x^2 + 12y^2 = x^3 + 8y\,x^2 + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 12y^2$

- Solve for $f_1(y)$

  $f_1(y) = 4y^3$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = y(x^3 + 4y\,x^2) + 4y^3$

- Substitute $F(x, y)$ into the solution of the ODE

  $y(x^3 + 4y\,x^2) + 4y^3 = c_1$

- Solve for y

$$\left\{ y = \frac{\left(9x^5+27c_1-8x^6+3\sqrt{-3x^{10}+3x^9-48c_1x^6+54c_1x^5+81c_1^2}\right)^{\frac{1}{3}}}{6} - \frac{6\left(\frac{1}{12}x^3-\frac{1}{9}x^4\right)}{\left(9x^5+27c_1-8x^6+3\sqrt{-3x^{10}+3x^9-48c_1x^6+54c_1x^5+81c\right.}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 475

```
dsolve((3*x^2*y(x)+8*x*y(x)^2)+(x^3+8*x^2*y(x)+12*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(9x^5 - 27c_1 - 8x^6 + 3\sqrt{-3x^{10} + 3x^9 + 48c_1x^6 - 54c_1x^5 + 81c_1^2}\right)^{\frac{1}{3}}}{6} + \frac{x^3(-3 + 4x)}{6\left(9x^5 - 27c_1 - 8x^6 + 3\sqrt{-3x^{10} + 3x^9 + 48c_1x^6 - 54c_1x^5 + 81c_1^2}\right)^{\frac{1}{3}}} - \frac{x^2}{3}$$

$$y(x) = \frac{\frac{\left(-i\sqrt{3}-1\right)\left(9x^5-27c_1-8x^6+3\sqrt{-3x^{10}+3x^9+48c_1x^6-54c_1x^5+81c_1^2}\right)^{\frac{2}{3}}}{4} + \left(-\left(9x^5 - 27c_1 - 8x^6 + 3\sqrt{-3x^{10} + 3x^9 + 48\right.\right.}{3\left(9x^5 - 27c_1 - 8x^6 + 3\sqrt{-3x^{10} + 3x^9 + 48c_1x^6 - 54c_1x^5 +\right.}$$

$$y(x) = -\frac{\left(-\frac{i\sqrt{3}}{4} + \frac{1}{4}\right)\left(9x^5 - 27c_1 - 8x^6 + 3\sqrt{-3x^{10} + 3x^9 + 48c_1x^6 - 54c_1x^5 + 81c_1^2}\right)^{\frac{2}{3}} + \left(\left(9x^5 - 27c_1 - 8x\right.\right.}{3\left(9x^5 - 27c_1 - 8x^6 + 3\sqrt{-3x^{10} + 3x^9 + 48c\right.}$$

✓ Solution by Mathematica

Time used: 1.703 (sec). Leaf size: 474

```
DSolve[(3*x^2*y[x]+8*x*y[x]^2)+(x^3+8*x^2*y[x]+12*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{1}{6}\left(-2x^2 + \sqrt[3]{-8x^6 + 9x^5 + 3\sqrt{3}\sqrt{-x^{10} + x^9 - 16c_1x^6 + 18c_1x^5 + 27c_1{}^2} + 27c_1} + \frac{(4x-3)x^3}{\sqrt[3]{-8x^6 + 9x^5 + 3\sqrt{3}\sqrt{-x^{10} + x^9 - 16c_1x^6 + 18c_1x^5 + 27c_1{}^2} + 27c_1}}\right)$$

$$y(x) \to \frac{1}{48}\left(-16x^2 + 4i\left(\sqrt{3} + i\right)\sqrt[3]{-8x^6 + 9x^5 + 3\sqrt{3}\sqrt{-x^{10} + x^9 - 16c_1x^6 + 18c_1x^5 + 27c_1{}^2} + 27c_1} - \frac{4i\left(\sqrt{3} - i\right)(4x-3)x^3}{\sqrt[3]{-8x^6 + 9x^5 + 3\sqrt{3}\sqrt{-x^{10} + x^9 - 16c_1x^6 + 18c_1x^5 + 27c_1{}^2} + 27c_1}}\right)$$

$$y(x) \to \frac{1}{48}\left(-16x^2 - 4\left(1 + i\sqrt{3}\right)\sqrt[3]{-8x^6 + 9x^5 + 3\sqrt{3}\sqrt{-x^{10} + x^9 - 16c_1x^6 + 18c_1x^5 + 27c_1{}^2} + 27c_1} + \frac{4i\left(\sqrt{3} + i\right)(4x-3)x^3}{\sqrt[3]{-8x^6 + 9x^5 + 3\sqrt{3}\sqrt{-x^{10} + x^9 - 16c_1x^6 + 18c_1x^5 + 27c_1{}^2} + 27c_1}}\right)$$

3.2 problem Exact Differential equations. Exercise 9.5, page 79

Internal problem ID [4456]
Internal file name [`OUTPUT/3949_Sunday_June_05_2022_11_54_22_AM_38184820/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 9
Problem number: Exact Differential equations. Exercise 9.5, page 79.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{\frac{2xy+1}{y}+\frac{(-x+y)\,y'}{y^2}=0}$$

3.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{-x + y}{y^2}\right) \mathrm{d}y &= \left(-\frac{2xy + 1}{y}\right) \mathrm{d}x \\ \left(\frac{2xy + 1}{y}\right) \mathrm{d}x + \left(\frac{-x + y}{y^2}\right) \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \frac{2xy + 1}{y} \\ N(x, y) &= \frac{-x + y}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2xy + 1}{y}\right) \\ &= -\frac{1}{y^2}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x+y}{y^2}\right) \\
&= -\frac{1}{y^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \frac{2xy+1}{y} \, \mathrm{d}x \\
\phi &= \frac{x(xy+1)}{y} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{x^2}{y} - \frac{x(xy+1)}{y^2} + f'(y) \qquad (4) \\
&= -\frac{x}{y^2} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x+y}{y^2}$. Therefore equation (4) becomes

$$
\frac{-x+y}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right)\mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(xy+1)}{y} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(xy+1)}{y} + \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{-x^2+\mathrm{LambertW}\left(-x\,\mathrm{e}^{x^2-c_1}\right)+c_1}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-x^2+\mathrm{LambertW}\left(-x\,\mathrm{e}^{x^2-c_1}\right)+c_1} \tag{1}$$

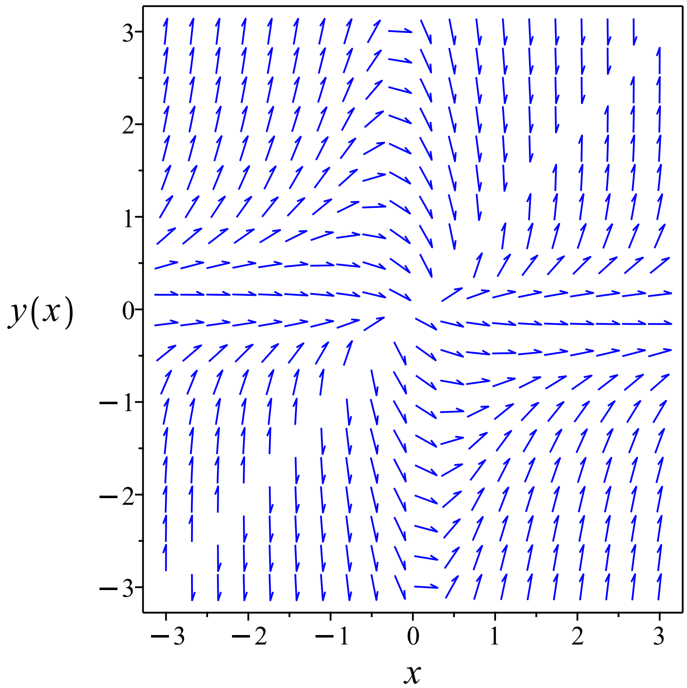


Figure 67: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-x^2+\mathrm{LambertW}\left(-x\,\mathrm{e}^{x^2-c_1}\right)+c_1}$$

Verified OK.

3.2.2 Maple step by step solution

Let's solve

$\frac{2xy+1}{y} + \frac{(-x+y)y'}{y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  ◦ Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$

- Evaluate derivatives

  $\frac{2x}{y} - \frac{2xy+1}{y^2} = -\frac{1}{y^2}$

- Simplify

  $-\frac{1}{y^2} = -\frac{1}{y^2}$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int \frac{2xy+1}{y}dx + f_1(y)$

- Evaluate integral

  $F(x,y) = \frac{y\,x^2+x}{y} + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $\frac{-x+y}{y^2} = -\frac{y\,x^2+x}{y^2} + \frac{x^2}{y} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = \frac{-x+y}{y^2} + \frac{y\,x^2+x}{y^2} - \frac{x^2}{y}$

- Solve for $f_1(y)$

  $f_1(y) = \ln(y)$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = \frac{y\,x^2+x}{y} + \ln(y)$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{y\,x^2+x}{y} + \ln(y) = c_1$

- Solve for y

  $y = \mathrm{e}^{-x^2+\mathit{LambertW}\left(-x\,\mathrm{e}^{x^2-c_1}\right)+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve((2*x*y(x)+1)/y(x)+(y(x)-x)/y(x)^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{\text{LambertW}\left(-e^{x^2}c_1x\right)}$$

✓ Solution by Mathematica

Time used: 5.208 (sec). Leaf size: 29

```
DSolve[(2*x*y[x]+1)/y[x]+(y[x]-x)/y[x]^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{W\left(x\left(-e^{x^2-c_1}\right)\right)}$$
$$y(x) \to 0$$

3.3 problem Exact Differential equations. Exercise 9.6, page 79

Internal problem ID [4457]
Internal file name [`OUTPUT/3950_Sunday_June_05_2022_11_54_29_AM_16225991/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 9
Problem number: Exact Differential equations. Exercise 9.6, page 79.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{2xy + (x^2 + y^2)\, y' = 0}$$

3.3.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2 + y^2)\,\mathrm{d}y &= (-2xy)\,\mathrm{d}x\\ (2xy)\,\mathrm{d}x + (x^2 + y^2)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2xy\\ N(x, y) &= x^2 + y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2xy)\\ &= 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + y^2)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int 2xy \mathrm{d}x \\
\phi &= y x^2 + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x^2 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 + y^2$. Therefore equation (4) becomes

$$
x^2 + y^2 = x^2 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y^2
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int (y^2) \mathrm{d}y \\
f(y) &= \frac{y^3}{3} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = y x^2 + \frac{1}{3} y^3 + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = y x^2 + \frac{1}{3} y^3
$$

Summary
The solution(s) found are the following

$$yx^2 + \frac{y^3}{3} = c_1 \tag{1}$$

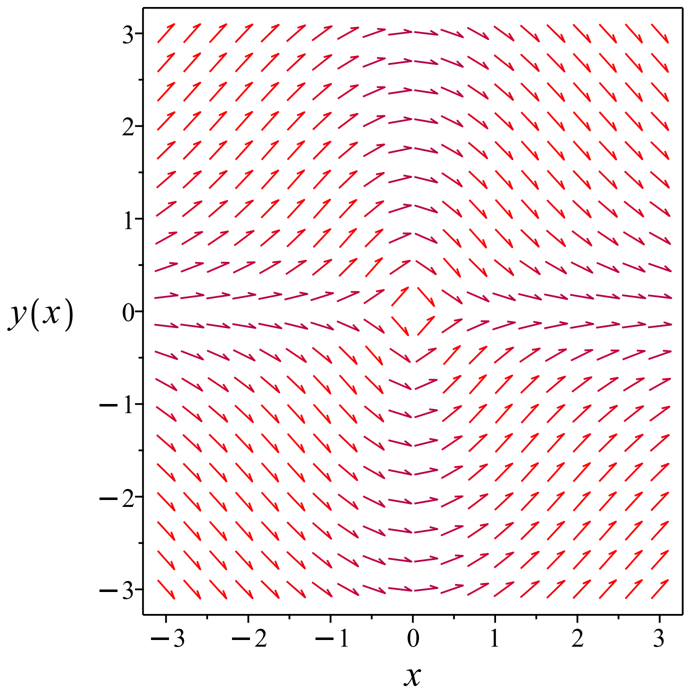


Figure 68: Slope field plot

Verification of solutions

$$yx^2 + \frac{y^3}{3} = c_1$$

Verified OK.

3.3.2 Maple step by step solution

Let's solve

$2xy + (x^2 + y^2)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$
- Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$
- Evaluate derivatives

  $2x=2x$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int 2xydx+f_1(y)$
- Evaluate integral

  $F(x,y)=y\,x^2+f_1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$
- Compute derivative

  $x^2+y^2=x^2+\frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=y^2$
- Solve for $f_1(y)$

  $f_1(y)=\frac{y^3}{3}$
- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=y\,x^2+\frac{1}{3}y^3$
- Substitute $F(x,y)$ into the solution of the ODE

  $y\,x^2+\frac{1}{3}y^3=c_1$
- Solve for y

$$\left\{ y = \frac{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{2x^2}{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}, y = -\frac{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}}{4} + \frac{x^2}{\left(12c_1+4\sqrt{4x^6+9c_1^2}\right)^{\frac{1}{3}}} - \frac{I\sqrt{3}\cdots}{\cdots} \right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 209

```
dsolve(2*x*y(x)+(x^2+y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{2\left(c_1x^2 - \frac{\left(4+4\sqrt{4c_1^3x^6+1}\right)^{\frac{2}{3}}}{4}\right)}{\left(4+4\sqrt{4c_1^3x^6+1}\right)^{\frac{1}{3}}\sqrt{c_1}}$$

$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left(4+4\sqrt{4c_1^3x^6+1}\right)^{\frac{1}{3}}}{4\sqrt{c_1}} - \frac{\sqrt{c_1}\left(i\sqrt{3}-1\right)x^2}{\left(4+4\sqrt{4c_1^3x^6+1}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{4i\sqrt{3}\,c_1x^2 + i\left(4+4\sqrt{4c_1^3x^6+1}\right)^{\frac{2}{3}}\sqrt{3} + 4c_1x^2 - \left(4+4\sqrt{4c_1^3x^6+1}\right)^{\frac{2}{3}}}{4\left(4+4\sqrt{4c_1^3x^6+1}\right)^{\frac{1}{3}}\sqrt{c_1}}$$

✓ Solution by Mathematica

Time used: 15.514 (sec). Leaf size: 401

```
DSolve[2*x*y[x]+(x^2+y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{\sqrt{4x^6+e^{6c_1}}+e^{3c_1}}}{\sqrt[3]{2}} - \frac{\sqrt[3]{2}x^2}{\sqrt[3]{\sqrt{4x^6+e^{6c_1}}+e^{3c_1}}}$$

$$y(x) \to \frac{i2^{2/3}\left(\sqrt{3}+i\right)\left(\sqrt{4x^6+e^{6c_1}}+e^{3c_1}\right)^{2/3}+\sqrt[3]{2}\left(2+2i\sqrt{3}\right)x^2}{4\sqrt[3]{\sqrt{4x^6+e^{6c_1}}+e^{3c_1}}}$$

$$y(x) \to \frac{\left(1-i\sqrt{3}\right)x^2}{2^{2/3}\sqrt[3]{\sqrt{4x^6+e^{6c_1}}+e^{3c_1}}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{\sqrt{4x^6+e^{6c_1}}+e^{3c_1}}}{2\sqrt[3]{2}}$$

$$y(x) \to 0$$

$$y(x) \to \frac{1}{2}\sqrt[6]{x^6}\left(\frac{\left(1-i\sqrt{3}\right)\left(x^6\right)^{2/3}}{x^4} - i\sqrt{3} - 1\right)$$

$$y(x) \to \frac{1}{2}\sqrt[6]{x^6}\left(\frac{\left(1+i\sqrt{3}\right)\left(x^6\right)^{2/3}}{x^4} + i\sqrt{3} - 1\right)$$

$$y(x) \to \sqrt[6]{x^6} - \frac{\left(x^6\right)^{5/6}}{x^4}$$

3.4 problem Exact Differential equations. Exercise 9.7, page 79

Internal problem ID [4458]
Internal file name [`OUTPUT/3951_Sunday_June_05_2022_11_54_35_AM_77214017/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 9
Problem number: Exact Differential equations. Exercise 9.7, page 79.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{e^{x} \sin(y) + e^{-y} - \left(x\, e^{-y} - e^{x} \cos(y)\right) y' = 0}$$

3.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\mathrm{e}^x \cos(y) - x\,\mathrm{e}^{-y}\right)\mathrm{d}y &= \left(-\mathrm{e}^x \sin(y) - \mathrm{e}^{-y}\right)\mathrm{d}x \\
\left(\mathrm{e}^x \sin(y) + \mathrm{e}^{-y}\right)\mathrm{d}x + \left(\mathrm{e}^x \cos(y) - x\,\mathrm{e}^{-y}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= \mathrm{e}^x \sin(y) + \mathrm{e}^{-y} \\
N(x,y) &= \mathrm{e}^x \cos(y) - x\,\mathrm{e}^{-y}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^x \sin(y) + \mathrm{e}^{-y}\right) \\
&= \mathrm{e}^x \cos(y) - \mathrm{e}^{-y}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\mathrm{e}^x \cos(y) - x\,\mathrm{e}^{-y}\right) \\
&= \mathrm{e}^x \cos(y) - \mathrm{e}^{-y}
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \mathrm{e}^{x} \sin(y) + \mathrm{e}^{-y} \mathrm{d}x$$
$$\phi = \mathrm{e}^{x} \sin(y) + x\,\mathrm{e}^{-y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} \cos(y) - x\,\mathrm{e}^{-y} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} \cos(y) - x\,\mathrm{e}^{-y}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} \cos(y) - x\,\mathrm{e}^{-y} = \mathrm{e}^{x} \cos(y) - x\,\mathrm{e}^{-y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{x} \sin(y) + x\,\mathrm{e}^{-y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{x} \sin(y) + x\,\mathrm{e}^{-y}$$

Summary
The solution(s) found are the following

$$\mathrm{e}^{x} \sin(y) + x\,\mathrm{e}^{-y} = c_1 \tag{1}$$

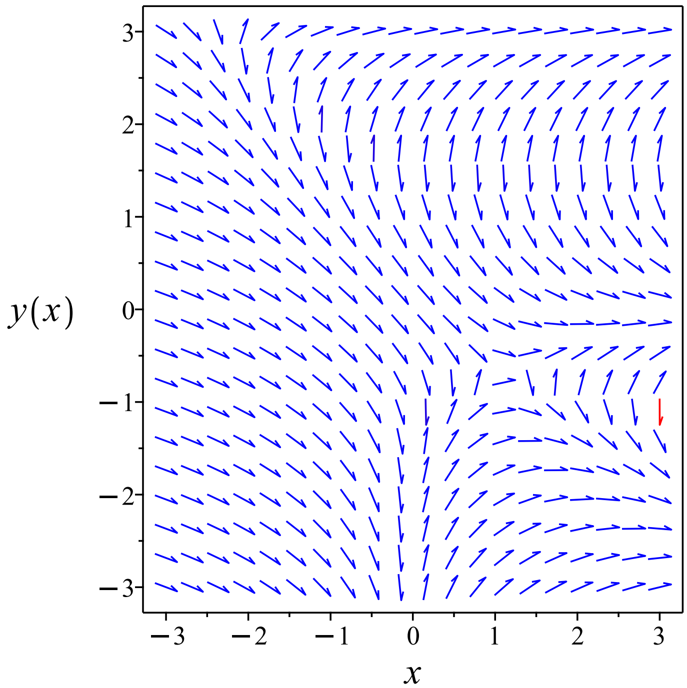


Figure 69: Slope field plot

Verification of solutions

$$e^x \sin(y) + x e^{-y} = c_1$$

Verified OK.

3.4.2 Maple step by step solution

Let's solve

$e^x \sin(y) + e^{-y} - (x e^{-y} - e^x \cos(y)) y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $e^x \cos(y) - e^{-y} = e^x \cos(y) - e^{-y}$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (e^x \sin(y) + e^{-y})\, dx + f_1(y)$
- Evaluate integral

  $F(x,y) = e^x \sin(y) + x\, e^{-y} + f_1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$
- Compute derivative

  $e^x \cos(y) - x\, e^{-y} = e^x \cos(y) - x\, e^{-y} + \frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$
- Solve for $f_1(y)$

  $f_1(y) = 0$
- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = e^x \sin(y) + x\, e^{-y}$
- Substitute $F(x,y)$ into the solution of the ODE

  $e^x \sin(y) + x\, e^{-y} = c_1$
- Solve for y

  $y = \mathit{RootOf}\left(-\sin(\_Z)\, e^{-\_Z} e^x + c_1 e^{-\_Z} - x\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve((exp(x)*sin(y(x))+exp(-y(x)))-(x*exp(-y(x))-exp(x)*cos(y(x)))*diff(y(x),x)=0,y(x), si
```

$$e^{x} \sin \left(y(x)\right)+x\,e^{-y(x)}+c_{1}=0$$

✓ Solution by Mathematica

Time used: 0.389 (sec). Leaf size: 24

```
DSolve[(Exp[x]*Sin[y[x]]+Exp[-y[x]])-(x*Exp[-y[x]]-Exp[x]*Cos[y[x]])*y'[x]==0,y[x],x,Include
```

$$\text{Solve}\left[x\left(-e^{-y(x)}\right)-e^{x}\sin(y(x))=c_{1},y(x)\right]$$

3.5 problem Exact Differential equations. Exercise 9.8, page 79

Internal problem ID [4459]
Internal file name [`OUTPUT/3952_Sunday_June_05_2022_11_54_45_AM_83113949/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 9
Problem number: Exact Differential equations. Exercise 9.8, page 79.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{\cos(y) - (x\sin(y) - y^2)\, y' = 0}$$

3.5.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\sin(y)\,x+y^2\right)\mathrm{d}y&=(-\cos(y))\,\mathrm{d}x\\ (\cos(y))\,\mathrm{d}x+\left(-\sin(y)\,x+y^2\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\cos(y)\\ N(x,y)&=-\sin(y)\,x+y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(\cos(y))\\ &=-\sin(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(-\sin(y)\,x+y^2\right)\\ &=-\sin(y)\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \cos(y) \, \mathrm{d}x$$

$$\phi = \cos(y) x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\sin(y) x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\sin(y) x + y^2$. Therefore equation (4) becomes

$$-\sin(y) x + y^2 = -\sin(y) x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y^2) \, \mathrm{d}y$$

$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \cos(y) x + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(y) x + \frac{y^3}{3}$$

Summary

The solution(s) found are the following

$$\cos(y)\, x + \frac{y^3}{3} = c_1 \tag{1}$$

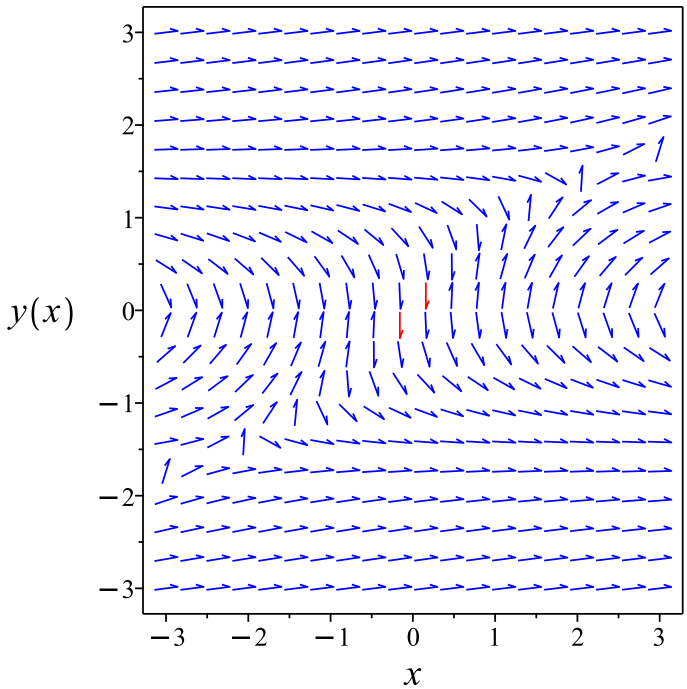


Figure 70: Slope field plot

Verification of solutions

$$\cos(y)\, x + \frac{y^3}{3} = c_1$$

Verified OK.

3.5.2 Maple step by step solution

Let's solve

$\cos(y) - (x \sin(y) - y^2)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$
- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$
- Evaluate derivatives

  $-\sin(y) = -\sin(y)$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \cos(y) \, dx + f_1(y)$
- Evaluate integral

  $F(x, y) = \cos(y) x + f_1(y)$
- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$
- Compute derivative

  $-\sin(y) x + y^2 = -\sin(y) x + \frac{d}{dy} f_1(y)$
- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = y^2$
- Solve for $f_1(y)$

  $f_1(y) = \frac{y^3}{3}$
- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = \cos(y) x + \frac{y^3}{3}$
- Substitute $F(x, y)$ into the solution of the ODE

  $\cos(y) x + \frac{y^3}{3} = c_1$
- Solve for y

  $y = \mathit{RootOf}\left(-\_Z^3 - 3\cos(\_Z) x + 3c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve(cos(y(x))-(x*sin(y(x))-y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$x+\frac{\sec\left(y(x)\right)\left(y(x)^{3}-3c_{1}\right)}{3}=0$$

✓ Solution by Mathematica

Time used: 0.124 (sec). Leaf size: 23

```
DSolve[Cos[y[x]]-(x*Sin[y[x]]-y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x=-\frac{1}{3}y(x)^{3}\sec(y(x))+c_{1}\sec(y(x)),y(x)\right]$$

3.6 problem Exact Differential equations. Exercise 9.9, page 79

Internal problem ID [4460]
Internal file name [`OUTPUT/3953_Sunday_June_05_2022_11_54_53_AM_74718260/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 9
Problem number: Exact Differential equations. Exercise 9.9, page 79.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{-2xy + \mathrm{e}^{y} + \left(y - x^{2} + x\,\mathrm{e}^{y}\right) y' = -x}$$

3.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(y - x^2 + x\,\mathrm{e}^y\right)\mathrm{d}y &= \left(-x + 2xy - \mathrm{e}^y\right)\mathrm{d}x \\
\left(-2xy + \mathrm{e}^y + x\right)\mathrm{d}x + \left(y - x^2 + x\,\mathrm{e}^y\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -2xy + \mathrm{e}^y + x \\
N(x, y) &= y - x^2 + x\,\mathrm{e}^y
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2xy + \mathrm{e}^y + x\right) \\
&= -2x + \mathrm{e}^y
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y - x^2 + x\,\mathrm{e}^y\right) \\
&= -2x + \mathrm{e}^y
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -2xy + \mathrm{e}^y + x\,\mathrm{d}x$$
$$\phi = x\,\mathrm{e}^y - \left(y - \frac{1}{2}\right)x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x\,\mathrm{e}^y - x^2 + f'(y) \tag{4}$$
$$= x(\mathrm{e}^y - x) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y - x^2 + x\,\mathrm{e}^y$. Therefore equation (4) becomes

$$y - x^2 + x\,\mathrm{e}^y = x(\mathrm{e}^y - x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y)\,\mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x\,\mathrm{e}^y - \left(y - \frac{1}{2}\right)x^2 + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\,e^y - \left(y - \frac{1}{2}\right)x^2 + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$x\,e^y - \left(y - \frac{1}{2}\right)x^2 + \frac{y^2}{2} = c_1 \tag{1}$$

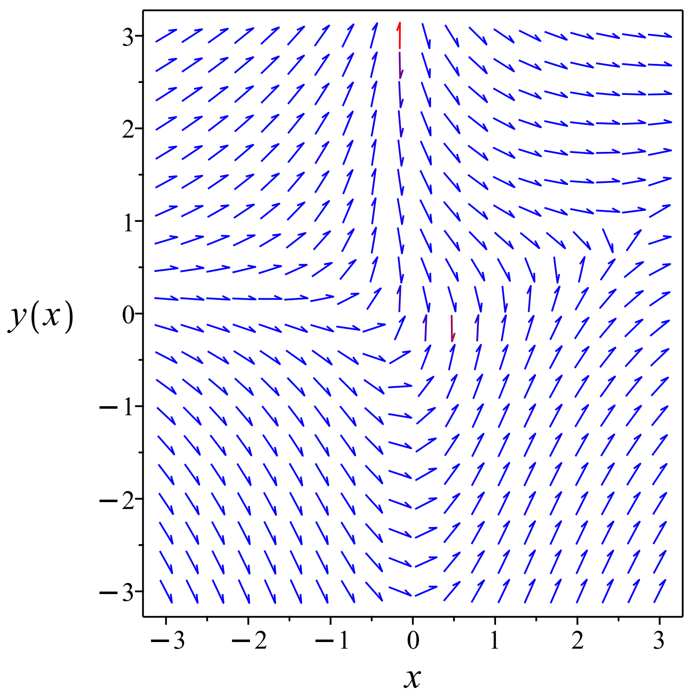


Figure 71: Slope field plot

Verification of solutions

$$x\,e^y - \left(y - \frac{1}{2}\right)x^2 + \frac{y^2}{2} = c_1$$

Verified OK.

3.6.2 Maple step by step solution

Let's solve

$-2xy+\mathrm{e}^{y}+\left(y-x^{2}+x\,\mathrm{e}^{y}\right)y'=-x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $-2x+\mathrm{e}^{y}=-2x+\mathrm{e}^{y}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_{1},M(x,y)=F'(x,y),N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\left(-2xy+\mathrm{e}^{y}+x\right)dx+f_{1}(y)$

- Evaluate integral

  $F(x,y)=-y\,x^{2}+x\,\mathrm{e}^{y}+\frac{x^{2}}{2}+f_{1}(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $y-x^{2}+x\,\mathrm{e}^{y}=-x^{2}+x\,\mathrm{e}^{y}+\frac{d}{dy}f_{1}(y)$

- Isolate for $\frac{d}{dy}f_{1}(y)$

  $\frac{d}{dy}f_{1}(y)=y$

- Solve for $f_{1}(y)$

  $f_{1}(y)=\frac{y^{2}}{2}$

- Substitute $f_{1}(y)$ into equation for $F(x,y)$

$F(x,y) = -y\,x^2 + x\,\mathrm{e}^y + \frac{x^2}{2} + \frac{y^2}{2}$

- Substitute $F(x,y)$ into the solution of the ODE

  $-y\,x^2 + x\,\mathrm{e}^y + \frac{x^2}{2} + \frac{y^2}{2} = c_1$

- Solve for y

  $y = RootOf\left(2x^2\_Z - 2\,\mathrm{e}^{\_Z}x - \_Z^2 - x^2 + 2c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve((x-2*x*y(x)+exp(y(x)))+(y(x)-x^2+x*exp(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$-y(x)\,x^2 + x\,\mathrm{e}^{y(x)} + \frac{x^2}{2} + \frac{y(x)^2}{2} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.315 (sec). Leaf size: 35

```
DSolve[(x-2*x*y[x]+Exp[y[x]])+(y[x]-x^2+x*Exp[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolution
```

$$\text{Solve}\left[x^2(-y(x)) + \frac{x^2}{2} + xe^{y(x)} + \frac{y(x)^2}{2} = c_1, y(x)\right]$$

3.7 problem Exact Differential equations. Exercise 9.10, page 79

Internal problem ID [4461]
Internal file name [`OUTPUT/3954_Sunday_June_05_2022_11_55_04_AM_50263305/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 9
Problem number: Exact Differential equations. Exercise 9.10, page 79.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{y^2 - (\mathrm{e}^y - 2xy)\, y' = -x^2 + x}$$

3.7.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(-\mathrm{e}^y + 2xy)\,\mathrm{d}y &= \left(-x^2 - y^2 + x\right)\mathrm{d}x \\
\left(x^2 + y^2 - x\right)\mathrm{d}x + (-\mathrm{e}^y + 2xy)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= x^2 + y^2 - x \\
N(x, y) &= -\mathrm{e}^y + 2xy
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + y^2 - x\right) \\
&= 2y
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-\mathrm{e}^y + 2xy) \\
&= 2y
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int x^2 + y^2 - x \,\mathrm{d}x$$

$$\phi = \frac{1}{3}x^3 + y^2 x - \frac{1}{2}x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2xy + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\mathrm{e}^y + 2xy$. Therefore equation (4) becomes

$$-\mathrm{e}^y + 2xy = 2xy + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\mathrm{e}^y$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (-\mathrm{e}^y) \,\mathrm{d}y$$

$$f(y) = -\mathrm{e}^y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^3}{3} + y^2 x - \frac{x^2}{2} - \mathrm{e}^y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^3}{3} + y^2 x - \frac{x^2}{2} - \mathrm{e}^y$$

Summary

The solution(s) found are the following

$$\frac{x^3}{3} + x y^2 - \frac{x^2}{2} - e^y = c_1 \tag{1}$$

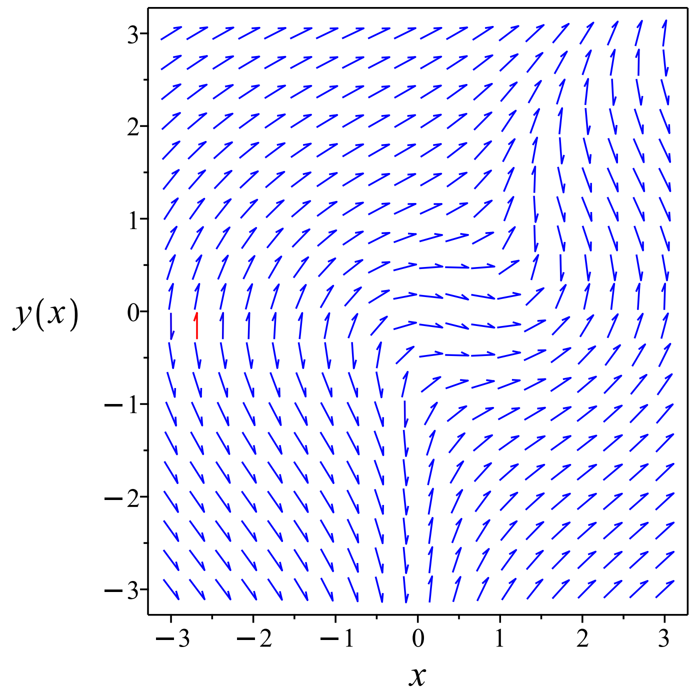


Figure 72: Slope field plot

Verification of solutions

$$\frac{x^3}{3} + x y^2 - \frac{x^2}{2} - e^y = c_1$$

Verified OK.

3.7.2 Maple step by step solution

Let's solve

$y^2 - (e^y - 2xy)\, y' = -x^2 + x$

- Highest derivative means the order of the ODE is 1

y'

□ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$
- Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$
- Evaluate derivatives

  $2y=2y$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int (x^2+y^2-x)\,dx+f_1(y)$
- Evaluate integral

  $F(x,y)=\frac{x^3}{3}+y^2x-\frac{x^2}{2}+f_1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$
- Compute derivative

  $-e^y+2xy=2xy+\frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=-e^y$
- Solve for $f_1(y)$

  $f_1(y)=-e^y$
- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=\frac{x^3}{3}+y^2x-\frac{x^2}{2}-e^y$
- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{x^3}{3}+y^2x-\frac{x^2}{2}-e^y=c_1$
- Solve for y

  $y=\mathit{RootOf}\left(-6\_Z^2x-2x^3+3x^2+6\,e^{\_Z}+6c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve((x^2-x+y(x)^2)-(exp(y(x))-2*x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{x^3}{3} + x y(x)^2 - \frac{x^2}{2} - \mathrm{e}^{y(x)} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.198 (sec). Leaf size: 32

```
DSolve[(x^2-x+y[x]^2)-(Exp[y[x]]-2*x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{x^3}{3} + \frac{x^2}{2} - x y(x)^2 + e^{y(x)} = c_1, y(x)\right]$$

3.8 problem Exact Differential equations. Exercise 9.11, page 79

Internal problem ID [4462]
Internal file name [`OUTPUT/3955_Sunday_June_05_2022_11_55_13_AM_93234884/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 9
Problem number: Exact Differential equations. Exercise 9.11, page 79.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{y\cos(x)+(2y+\sin(x)-\sin(y))\,y'=-2x}$$

3.8.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2y+\sin(x)-\sin(y))\,\mathrm{d}y&=(-2x-y\cos(x))\,\mathrm{d}x\\ (2x+y\cos(x))\,\mathrm{d}x+(2y+\sin(x)-\sin(y))\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=2x+y\cos(x)\\ N(x,y)&=2y+\sin(x)-\sin(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(2x+y\cos(x))\\ &=\cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(2y+\sin(x)-\sin(y))\\ &=\cos(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int 2x + y\cos(x) \, dx$$

$$\phi = y\sin(x) + x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y + \sin(x) - \sin(y)$. Therefore equation (4) becomes

$$2y + \sin(x) - \sin(y) = \sin(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y - \sin(y)$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int (2y - \sin(y)) \, dy$$

$$f(y) = y^2 + \cos(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y\sin(x) + x^2 + y^2 + \cos(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\sin(x) + x^2 + y^2 + \cos(y)$$

Summary

The solution(s) found are the following

$$\sin(x) y + x^2 + y^2 + \cos(y) = c_1 \tag{1}$$

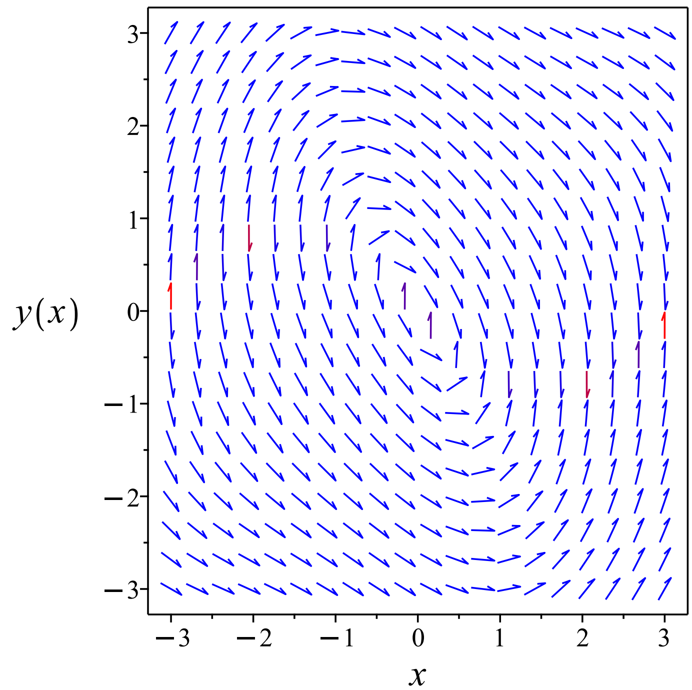


Figure 73: Slope field plot

Verification of solutions

$$\sin(x) y + x^2 + y^2 + \cos(y) = c_1$$

Verified OK.

3.8.2 Maple step by step solution

Let's solve

$y\cos(x) + (2y + \sin(x) - \sin(y)) y' = -2x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $\cos(x) = \cos(x)$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (2x + y\cos(x))\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = y\sin(x) + x^2 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $2y + \sin(x) - \sin(y) = \sin(x) + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 2y - \sin(y)$

- Solve for $f_1(y)$

  $f_1(y) = y^2 + \cos(y)$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = y\sin(x) + x^2 + y^2 + \cos(y)$

- Substitute $F(x, y)$ into the solution of the ODE

  $y\sin(x) + x^2 + y^2 + \cos(y) = c_1$

- Solve for y

  $y = \mathit{RootOf}\left(-\_Z\sin(x) - x^2 - \_Z^2 - \cos(\_Z) + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve((2*x+y(x)*cos(x))+(2*y(x)+sin(x)-sin(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\sin(x)\, y(x) + x^2 + y(x)^2 + \cos(y(x)) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.198 (sec). Leaf size: 22

```
DSolve[(2*x+y[x]*Cos[x])+(2*y[x]+Sin[x]-Sin[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions
```

$$\text{Solve}\left[x^2 + y(x)^2 + y(x)\sin(x) + \cos(y(x)) = c_1, y(x)\right]$$

3.9 problem Exact Differential equations. Exercise 9.12, page 79

Internal problem ID [4463]
Internal file name [`OUTPUT/3956_Sunday_June_05_2022_11_55_23_AM_92089622/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 9
Problem number: Exact Differential equations. Exercise 9.12, page 79.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _dAlembert]
```

$$\boxed{x\sqrt{x^2+y^2}-\frac{x^2yy'}{y-\sqrt{x^2+y^2}}=0}$$

3.9.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{x^2 y}{y - \sqrt{x^2+y^2}}\right)\mathrm{d}y &= \left(-x\sqrt{x^2+y^2}\right)\mathrm{d}x \\
\left(x\sqrt{x^2+y^2}\right)\mathrm{d}x + \left(-\frac{x^2 y}{y - \sqrt{x^2+y^2}}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= x\sqrt{x^2+y^2} \\
N(x,y) &= -\frac{x^2 y}{y - \sqrt{x^2+y^2}}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x\sqrt{x^2+y^2}\right) \\
&= \frac{yx}{\sqrt{x^2+y^2}}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{x^2 y}{y-\sqrt{x^2+y^2}}\right) \\
&= \frac{xy\left(x^2+2y^2-2\sqrt{x^2+y^2}\, y\right)}{\sqrt{x^2+y^2}\,\left(y-\sqrt{x^2+y^2}\right)^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1)\\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int x\sqrt{x^2+y^2}\,\mathrm{d}x \\
\phi &= \frac{(x^2+y^2)^{\frac{3}{2}}}{3} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \sqrt{x^2+y^2}\, y + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x^2 y}{y-\sqrt{x^2+y^2}}$. Therefore equation (4) becomes

$$
-\frac{x^2 y}{y-\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}\, y + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y^2
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y^2)\,\mathrm{d}y$$

$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(x^2+y^2)^{\frac{3}{2}}}{3} + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(x^2+y^2)^{\frac{3}{2}}}{3} + \frac{y^3}{3}$$

Summary

The solution(s) found are the following

$$\frac{(x^2+y^2)^{\frac{3}{2}}}{3} + \frac{y^3}{3} = c_1 \tag{1}$$

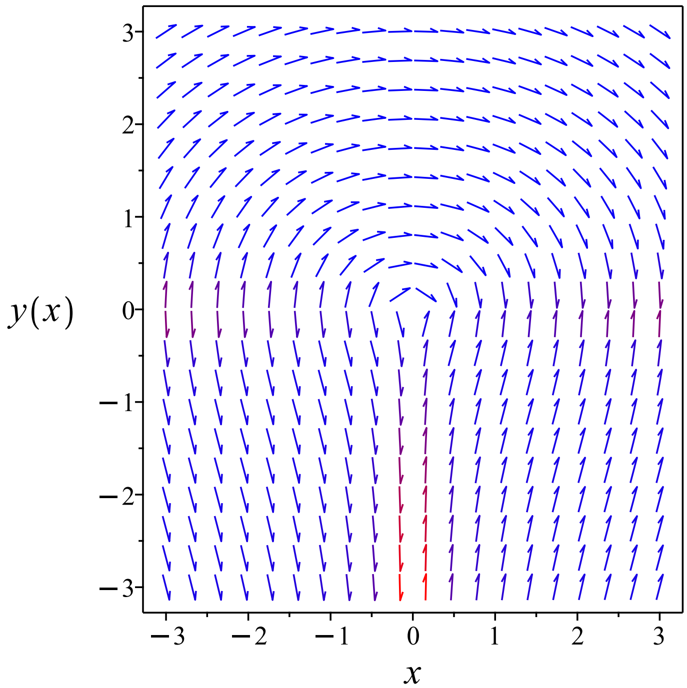


Figure 74: Slope field plot

Verification of solutions

$$\frac{(x^2+y^2)^{\frac{3}{2}}}{3}+\frac{y^3}{3}=c_1$$

Verified OK.

3.9.2 Maple step by step solution

Let's solve

$x\sqrt{x^2+y^2}-\frac{x^2yy'}{y-\sqrt{x^2+y^2}}=0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  ◦ Compute derivative of lhs

$F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $\frac{yx}{\sqrt{x^2+y^2}} = -\frac{2xy}{y-\sqrt{x^2+y^2}} - \frac{x^3 y}{\left(y-\sqrt{x^2+y^2}\right)^2 \sqrt{x^2+y^2}}$
- Simplify

  $\frac{yx}{\sqrt{x^2+y^2}} = \frac{xy\left(x^2+2y^2-2\sqrt{x^2+y^2}\, y\right)}{\sqrt{x^2+y^2}\left(y-\sqrt{x^2+y^2}\right)^2}$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int x\sqrt{x^2 + y^2} dx + f_1(y)$
- Evaluate integral

  $F(x, y) = \frac{(x^2+y^2)^{\frac{3}{2}}}{3} + f_1(y)$
- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$
- Compute derivative

  $-\frac{x^2 y}{y-\sqrt{x^2+y^2}} = \sqrt{x^2 + y^2}\, y + \frac{d}{dy} f_1(y)$
- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = -\frac{x^2 y}{y-\sqrt{x^2+y^2}} - \sqrt{x^2 + y^2}\, y$
- Solve for $f_1(y)$

  $f_1(y) = -\frac{(x^2+y^2)^{\frac{3}{2}}}{3} - x^2\left(-\frac{y^3}{3x^2} - \frac{(x^2+y^2)^{\frac{3}{2}}}{3x^2}\right)$
- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = -x^2\left(-\frac{y^3}{3x^2} - \frac{(x^2+y^2)^{\frac{3}{2}}}{3x^2}\right)$
- Substitute $F(x, y)$ into the solution of the ODE

  $-x^2\left(-\frac{y^3}{3x^2} - \frac{(x^2+y^2)^{\frac{3}{2}}}{3x^2}\right) = c_1$
- Solve for y

$$y = RootOf\left(3x^2\_Z^4 + 3x^4\_Z^2 + x^6 + 6c_1\_Z^3 - 9c_1^2\right)$$

Maple trace

```
`Classification methods on request
Methods to be used are: [exact]
----------------------------
* Tackling ODE using method: exact
--- Trying classification methods ---
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*sqrt(x^2+y(x)^2)-(x^2*y(x))/(y(x)- sqrt(x^2+y(x)^2))*diff(y(x),x)=0,y(x), singsol=a
```

$$c_1 + \left(x^2 + y(x)^2\right)^{\frac{3}{2}} + y(x)^3 = 0$$

✓ Solution by Mathematica

Time used: 60.259 (sec). Leaf size: 2125

```
DSolve[x*Sqrt[x^2+y[x]^2]-(x^2*y[x])/(y[x]- Sqrt[x^2+y[x]^2])*y'[x]==0,y[x],x,IncludeSingula
```

$y(x) \rightarrow$

$$x^2\sqrt{\frac{e^{6c_1}}{x^4} - 6x^2 + \frac{3\left(5x^6-4e^{6c_1}\right)}{\sqrt[3]{-11x^{12}+14e^{6c_1}x^6+2\sqrt{\left(-x^6+e^{6c_1}\right)\left(x^6+e^{6c_1}\right)^3}-2e^{12c_1}}} + \frac{3\sqrt[3]{-11x^{12}+14e^{6}}}{}}$$

$$-\frac{}{}$$

$y(x)$

$$x^2\left(-\sqrt{\frac{e^{6c_1}}{x^4} - 6x^2 + \frac{3\left(5x^6-4e^{6c_1}\right)}{\sqrt[3]{-11x^{12}+14e^{6c_1}x^6+2\sqrt{\left(-x^6+e^{6c_1}\right)\left(x^6+e^{6c_1}\right)^3}-2e^{12c_1}}} + \frac{3\sqrt[3]{-11x^{12}+14}}{}}\right.$$

$$\rightarrow \frac{}{}$$

$y(x)$

$$x^2\sqrt{\frac{e^{6c_1}}{x^4} - 6x^2 + \frac{3\left(5x^6-4e^{6c_1}\right)}{\sqrt[3]{-11x^{12}+14e^{6c_1}x^6+2\sqrt{\left(-x^6+e^{6c_1}\right)\left(x^6+e^{6c_1}\right)^3}-2e^{12c_1}}} + \frac{3\sqrt[3]{-11x^{12}+14e^{6c_1}}}{}}$$

$$\rightarrow \frac{}{}$$

$y(x)$

$$x^2\sqrt{\frac{e^{6c_1}}{x^4} - 6x^2 + \frac{3\left(5x^6-4e^{6c_1}\right)}{\sqrt[3]{-11x^{12}+14e^{6c_1}x^6+2\sqrt{\left(-x^6+e^{6c_1}\right)\left(x^6+e^{6c_1}\right)^3}-2e^{12c_1}}} + \frac{3\sqrt[3]{-11x^{12}+14e^{6c_1}}}{}}$$

$$\rightarrow \frac{}{}$$

3.10 problem Exact Differential equations. Exercise 9.13, page 79

Internal problem ID [4464]
Internal file name [`OUTPUT/3957_Sunday_June_05_2022_11_55_30_AM_31017462/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 9
Problem number: Exact Differential equations. Exercise 9.13, page 79.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{y^3 - (y^2 + 1 - 3xy^2)\, y' = -4x^3 + \sin(x)}$$

3.10.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(3y^2x - y^2 - 1)\,\mathrm{d}y &= (-4x^3 + \sin(x) - y^3)\,\mathrm{d}x \\
(4x^3 - \sin(x) + y^3)\,\mathrm{d}x + (3y^2x - y^2 - 1)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= 4x^3 - \sin(x) + y^3 \\
N(x, y) &= 3y^2x - y^2 - 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(4x^3 - \sin(x) + y^3) \\
&= 3y^2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3y^2x - y^2 - 1) \\
&= 3y^2
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int 4x^3 - \sin(x) + y^3 \mathrm{d}x \\
\phi &= x^4 + x y^3 + \cos(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 3y^2 x + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3y^2 x - y^2 - 1$. Therefore equation (4) becomes

$$
3y^2 x - y^2 - 1 = 3y^2 x + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -y^2 - 1
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(-y^2 - 1\right) \mathrm{d}y \\
f(y) &= -\frac{1}{3}y^3 - y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = x^4 + x y^3 + \cos(x) - \frac{y^3}{3} - y + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = x^4 + x y^3 + \cos(x) - \frac{y^3}{3} - y
$$

Summary
The solution(s) found are the following

$$x^4 + y^3 x + \cos(x) - \frac{y^3}{3} - y = c_1 \tag{1}$$

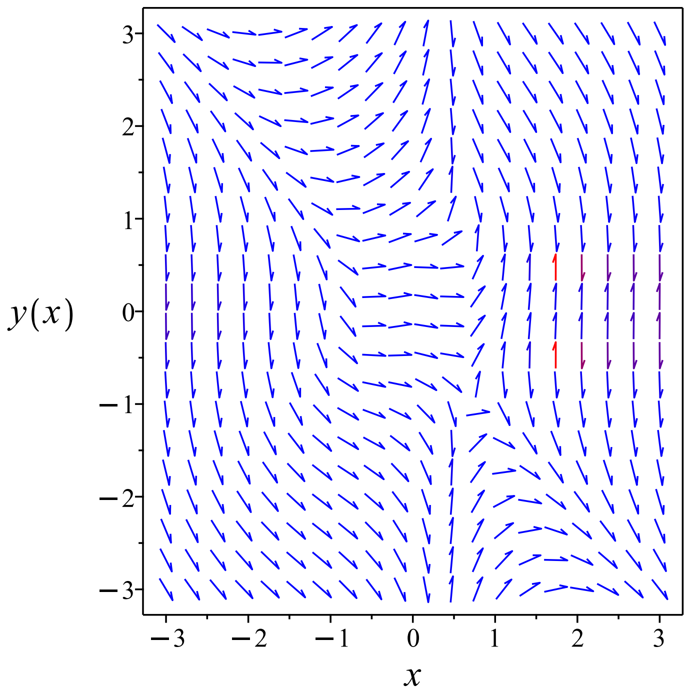


Figure 75: Slope field plot

Verification of solutions

$$x^4 + y^3 x + \cos(x) - \frac{y^3}{3} - y = c_1$$

Verified OK.

3.10.2 Maple step by step solution

Let's solve

$y^3 - (y^2 + 1 - 3xy^2) y' = -4x^3 + \sin(x)$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$

- Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

- Evaluate derivatives

  $3y^2=3y^2$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(4x^3-\sin(x)+y^3)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=x^4+x\,y^3+\cos(x)+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $3y^2x-y^2-1=3y^2x+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=-y^2-1$

- Solve for $f_1(y)$

  $f_1(y)=-\frac{1}{3}y^3-y$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=x^4+x\,y^3+\cos(x)-\frac{y^3}{3}-y$

- Substitute $F(x,y)$ into the solution of the ODE

  $x^4+x\,y^3+\cos(x)-\frac{y^3}{3}-y=c_1$

- Solve for y

$$\left\{ y = \frac{\left(\left(-12x^4 - 12\cos(x) + 4\sqrt{\frac{27x^9 - 9x^8 + 54\cos(x)x^5 - 54c_1x^5 - 18\cos(x)x^4 + 18c_1x^4 + 27\cos(x)^2x - 54\cos(x)c_1x + 27c_1^2x - 9\cos(x)^2 + 18\cos(}{3x-1}}\right.\right.}{2(3x-1)}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 658

```
dsolve((4*x^3-sin(x)+y(x)^3)-(y(x)^2+1-3*x*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(2^{\frac{1}{3}}\left(\left(-3x^4 - 3\cos(x) + \sqrt{\frac{(27x-9)\cos(x)^2+54\left(x-\frac{1}{3}\right)(x^4+c_1)\cos(x)+27x^9-9x^8+54c_1x^5-18c_1x^4+27c_1^2x-9c_1^2-4}{3x-1}} - 3c_1\right)\right.\right.}{\left(\left(-3x^4 - 3\cos(x) + \sqrt{\frac{(27x-9)\cos(x)^2+54\left(x-\frac{1}{3}\right)(x^4+c_1)\cos(x)+27x^9-9x^8+54c_1x^5-18c_1x^4+27c_1^2x-9c_1^2-4}{3x-1}} - 3c_1\right)\right.}$$

$$y(x) = -\frac{\left(2^{\frac{1}{3}}\left(1+i\sqrt{3}\right)\left(-\left(3x^4 + 3\cos(x) - \sqrt{\frac{(27x-9)\cos(x)^2+54\left(x-\frac{1}{3}\right)(x^4+c_1)\cos(x)+27x^9-9x^8+54c_1x^5-18c_1x^4+27c_1^2x-9}{3x-1}}\right.\right.\right.}{4\left(-\left(3x^4 + 3\cos(x) - \sqrt{\frac{(27x-9)\cos(x)^2+54\left(x-\frac{1}{3}\right)(x^4+c_1)\cos(x)+27x^9-9x^8+54c_1x^5-18c_1x^4+27c_1^2x}{3x-1}}\right.\right.}$$

$$y(x) = \frac{\left(2^{\frac{1}{3}}\left(i\sqrt{3}-1\right)\left(-\left(3x^4 + 3\cos(x) - \sqrt{\frac{(27x-9)\cos(x)^2+54\left(x-\frac{1}{3}\right)(x^4+c_1)\cos(x)+27x^9-9x^8+54c_1x^5-18c_1x^4+27c_1^2x-9c_1^2-}{3x-1}}\right.\right.\right.}{4\left(-\left(3x^4 + 3\cos(x) - \sqrt{\frac{(27x-9)\cos(x)^2+54\left(x-\frac{1}{3}\right)(x^4+c_1)\cos(x)+27x^9-9x^8+54c_1x^5-18c_1x^4+27c_1^2x-9}{3x-1}}\right.\right.}$$

✓ Solution by Mathematica

Time used: 60.207 (sec). Leaf size: 682

```
DSolve[(4*x^3-Sin[x]+y[x]^3)-(y[x]^2+1-3*x*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{\sqrt[3]{2}\left(-27x^6+18x^5-3x^4+\frac{1}{27}\sqrt{4(9-27x)^3+6561(1-3x)^4\left(x^4+\cos(x)-c_1\right)^2}-27x^2\cos(x)+27c_1\right.}{2^{2/3}(3x-1)\sqrt[3]{-27x^6+18x^5-3x^4+\frac{1}{27}\sqrt{4(9-27x)^3+6561(1-3x)^4\left(x^4+\cos(x)-c_1\right)^2}-27x^2\,c}}$$

$$y(x) \to \frac{9i\sqrt[3]{2}\left(\sqrt{3}+i\right)\left(-27x^6+18x^5-3x^4+\frac{1}{27}\sqrt{4(9-27x)^3+6561(1-3x)^4\left(x^4+\cos(x)-c_1\right)^2}-27x^2\,co\right.}{18\;2^{2/3}(3x-1)\sqrt[3]{-27x^6+18x^5-3x^4+\frac{1}{27}\sqrt{4(9-27x)^3+6561(1-3x)^4\left(x^4+\cos(x)\right.}}}$$

$$y(x) \to \frac{i\left(\sqrt{3}+i\right)}{2^{2/3}\sqrt[3]{-27x^6+18x^5-3x^4+\frac{1}{27}\sqrt{4(9-27x)^3+6561(1-3x)^4\left(x^4+\cos(x)-c_1\right)^2}-27x^2\cos(x)+27}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{-54x^6+36x^5-6x^4+\frac{2}{27}\sqrt{4(9-27x)^3+6561(1-3x)^4\left(x^4+\cos(x)-c_1\right)^2}-54x^2\cos}}{2\;2^{2/3}(3x-1)}$$

3.11 problem Exact Differential equations. Exercise 9.15, page 79

Internal problem ID [4465]
Internal file name [`OUTPUT/3958_Sunday_June_05_2022_11_55_38_AM_16775353/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 9
Problem number: Exact Differential equations. Exercise 9.15, page 79.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_exact, _Bernoulli]
```

$$\boxed{\mathrm{e}^{x}\left(y^{3}+y^{3}x+1\right)+3y^{2}(\mathrm{e}^{x}x-6)\,y'=0}$$

With initial conditions

$$[y(0)=1]$$

3.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{\mathrm{e}^{x}(x\,y^{3}+y^{3}+1)}{3y^{2}\left(\mathrm{e}^{x}x-6\right)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{x<\mathrm{LambertW}\left(\_Z28,6\right)\vee \mathrm{LambertW}\left(\_Z28,6\right)<x\}$$

But the point $x_0=0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

3.11.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(3y^2(\mathrm{e}^x x-6)\right)\mathrm{d}y&=\left(-\mathrm{e}^x\left(x\,y^3+y^3+1\right)\right)\mathrm{d}x\\ \left(\mathrm{e}^x\left(x\,y^3+y^3+1\right)\right)\mathrm{d}x+\left(3y^2(\mathrm{e}^x x-6)\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\mathrm{e}^x\left(x\,y^3+y^3+1\right)\\ N(x,y)&=3y^2(\mathrm{e}^x x-6)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{x}\left(x\,y^{3}+y^{3}+1\right)\right)\\ &= 3\,\mathrm{e}^{x} y^{2}(x+1)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(3y^{2}(\mathrm{e}^{x}x-6)\right)\\ &= 3\,\mathrm{e}^{x} y^{2}(x+1)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \mathrm{e}^{x}\left(x\,y^{3}+y^{3}+1\right)\mathrm{d}x\\ \phi &= \left(x\,y^{3}+1\right)\mathrm{e}^{x}+f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3y^{2}\mathrm{e}^{x}x+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3y^{2}(\mathrm{e}^{x}x-6)$. Therefore equation (4) becomes

$$3y^{2}(\mathrm{e}^{x}x-6) = 3y^{2}\mathrm{e}^{x}x+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -18y^2$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int (-18y^2)\, \mathrm{d}y$$
$$f(y) = -6y^3 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = (x\, y^3 + 1)\, \mathrm{e}^x - 6y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (x\, y^3 + 1)\, \mathrm{e}^x - 6y^3$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-5 = c_1$$

$$c_1 = -5$$

Substituting c_1 found above in the general solution gives

$$(x\, y^3 + 1)\, \mathrm{e}^x - 6y^3 = -5$$

Summary
The solution(s) found are the following

$$\mathrm{e}^x y^3 x - 6y^3 + \mathrm{e}^x = -5 \tag{1}$$

Verification of solutions

$$\mathrm{e}^x y^3 x - 6y^3 + \mathrm{e}^x = -5$$

Verified OK.

3.11.3 Maple step by step solution

Let's solve

$[e^x(y^3 + y^3x + 1) + 3y^2(e^x x - 6)\, y' = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

  ◦ Evaluate derivatives

    $e^x(3y^2x + 3y^2) = 3y^2(e^x x + e^x)$

  ◦ Simplify

    $3\, e^x y^2(x + 1) = 3\, e^x y^2(x + 1)$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\, , N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int e^x(x\, y^3 + y^3 + 1)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = (x\, y^3 + 1)\, e^x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $3y^2(e^x x - 6) = 3y^2 e^x x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 3y^2(e^x x - 6) - 3y^2 e^x x$

- Solve for $f_1(y)$

$f_1(y) = -6y^3$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$F(x, y) = (x\,y^3 + 1)\,e^x - 6y^3$

- Substitute $F(x, y)$ into the solution of the ODE

$(x\,y^3 + 1)\,e^x - 6y^3 = c_1$

- Solve for y

$$\left\{y = \frac{\left((c_1 - e^x)(e^x x - 6)^2\right)^{\frac{1}{3}}}{e^x x - 6}, y = -\frac{\left((c_1 - e^x)(e^x x - 6)^2\right)^{\frac{1}{3}}}{2(e^x x - 6)} - \frac{I\sqrt{3}\left((c_1 - e^x)(e^x x - 6)^2\right)^{\frac{1}{3}}}{2(e^x x - 6)}, y = -\frac{\left((c_1 - e^x)(e^x x - 6)^2\right)^{\frac{1}{3}}}{2(e^x x - 6)} +\right.$$

- Use initial condition $y(0) = 1$

$1 = -\frac{(-36+36c_1)^{\frac{1}{3}}}{6}$

- Solution does not satisfy initial condition
- Use initial condition $y(0) = 1$

$1 = \frac{(-36+36c_1)^{\frac{1}{3}}}{12} + \frac{I\sqrt{3}\,(-36+36c_1)^{\frac{1}{3}}}{12}$

- Solution does not satisfy initial condition
- Use initial condition $y(0) = 1$

$1 = \frac{(-36+36c_1)^{\frac{1}{3}}}{12} - \frac{I\sqrt{3}\,(-36+36c_1)^{\frac{1}{3}}}{12}$

- Solve for c_1

$c_1 = -5$

- Substitute $c_1 = -5$ into general solution and simplify

$y = \frac{\left(I\sqrt{3}-1\right)\left(-(e^x+5)(e^x x-6)^2\right)^{\frac{1}{3}}}{2\,e^x x - 12}$

- Solution to the IVP

$y = \frac{\left(I\sqrt{3}-1\right)\left(-(e^x+5)(e^x x-6)^2\right)^{\frac{1}{3}}}{2\,e^x x - 12}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 38

```
dsolve([exp(x)*(y(x)^3+x*y(x)^3+1)+3*y(x)^2*(x*exp(x)-6)*diff(y(x),x)=0,y(0) = 1],y(x), sing
```

$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left(-\left(\mathrm{e}^{x}+5\right)\left(x\,\mathrm{e}^{x}-6\right)^{2}\right)^{\frac{1}{3}}}{2x\,\mathrm{e}^{x}-12}$$

✓ Solution by Mathematica

Time used: 1.114 (sec). Leaf size: 28

```
DSolve[{Exp[x]*(y[x]^3+x*y[x]^3+1)+3*y[x]^2*(x*Exp[x]-6)*y'[x]==0,y[0]==1},y[x],x,IncludeSin
```

$$y(x) \to \frac{\sqrt[3]{-e^x-5}}{\sqrt[3]{e^x x-6}}$$

3.12 problem Exact Differential equations. Exercise 9.16, page 79

Internal problem ID [4466]
Internal file name [`OUTPUT/3959_Sunday_June_05_2022_11_55_48_AM_90518544/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 9
Problem number: Exact Differential equations. Exercise 9.16, page 79.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\sin(x)\cos(y) + \cos(x)\sin(y)\,y' = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\right]$$

3.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{\sin(x)\cos(y)}{\cos(x)\sin(y)} \end{aligned}$$

The x domain of $f(x,y)$ when $y = \frac{\pi}{4}$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z29 \vee \frac{1}{2}\pi + \pi\_Z29 < x\right\}$$

And the point $x_0 = \frac{\pi}{4}$ is inside this domain. The y domain of $f(x, y)$ when $x = \frac{\pi}{4}$ is

$$\{y < \pi\_Z30 \vee \pi\_Z30 < y\}$$

And the point $y_0 = \frac{\pi}{4}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sin(x)\cos(y)}{\cos(x)\sin(y)}\right) \\ &= \frac{\sin(x)}{\cos(x)} + \frac{\sin(x)\cos(y)^2}{\cos(x)\sin(y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{\pi}{4}$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z29 \vee \frac{1}{2}\pi + \pi\_Z29 < x\right\}$$

And the point $x_0 = \frac{\pi}{4}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \frac{\pi}{4}$ is

$$\{y < \pi\_Z30 \vee \pi\_Z30 < y\}$$

And the point $y_0 = \frac{\pi}{4}$ is inside this domain. Therefore solution exists and is unique.

3.12.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{\sin(y)}{\cos(y)}\right)\mathrm{d}y = \left(\frac{\sin(x)}{\cos(x)}\right)\mathrm{d}x$$
$$\left(-\frac{\sin(x)}{\cos(x)}\right)\mathrm{d}x + \left(-\frac{\sin(y)}{\cos(y)}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\frac{\sin(x)}{\cos(x)} \\ N(x,y) &= -\frac{\sin(y)}{\cos(y)} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sin(x)}{\cos(x)}\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\sin(y)}{\cos(y)}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x=\int M\, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x=\int -\frac{\sin (x)}{\cos (x)}\, \mathrm{d}x$$
$$\phi=\ln (\cos (x))+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=-\frac{\sin(y)}{\cos(y)}$. Therefore equation (4) becomes

$$-\frac{\sin (y)}{\cos (y)}=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) &= -\frac{\sin (y)}{\cos (y)} \\ &= -\tan (y) \end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y)\, \mathrm{d}y=\int (-\tan (y))\, \mathrm{d}y$$
$$f(y)=\ln (\cos (y))+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(\cos(x)) + \ln(\cos(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(\cos(x)) + \ln(\cos(y))$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$-\ln(2) = c_1$$

$$c_1 = -\ln(2)$$

Substituting c_1 found above in the general solution gives

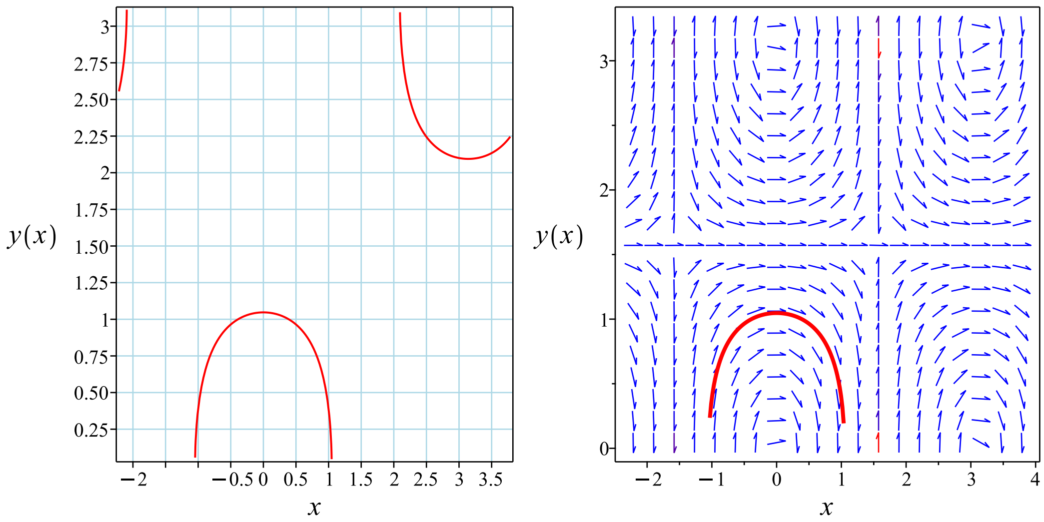
$$\ln(\cos(x)) + \ln(\cos(y)) = -\ln(2)$$

Solving for y from the above gives

$$y = \arccos\left(\frac{\sec(x)}{2}\right)$$

Summary
The solution(s) found are the following

$$y = \arccos\left(\frac{\sec(x)}{2}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \arccos\left(\frac{\sec(x)}{2}\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 9

```
dsolve([sin(x)*cos(y(x))+cos(x)*sin(y(x))*diff(y(x),x)=0,y(1/4*Pi) = 1/4*Pi],y(x), singsol=a
```

$$y(x) = \frac{\pi}{2} - \arcsin\left(\frac{\sec(x)}{2}\right)$$

✓ Solution by Mathematica

Time used: 6.111 (sec). Leaf size: 12

```
DSolve[{Sin[x]*Cos[y[x]]+Cos[x]*Sin[y[x]]*y'[x]==0,y[Pi/4]==Pi/4},y[x],x,IncludeSingularSolu
```

$$y(x) \to \arccos\left(\frac{\sec(x)}{2}\right)$$

3.13 problem Exact Differential equations. Exercise 9.17, page 79

Internal problem ID [4467]
Internal file name [`OUTPUT/3960_Sunday_June_05_2022_11_55_57_AM_22271646/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 9
Problem number: Exact Differential equations. Exercise 9.17, page 79.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{y^2 e^{x y^2}+\left(2 x y\, e^{x y^2}-3 y^2\right) y'=-4 x^3}$$

With initial conditions

$$[y(1)=0]$$

3.13.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y^2 e^{y^2 x}+4 x^3}{y\left(-2\, e^{y^2 x} x+3 y\right)} \end{aligned}$$

$f(x,y)$ is not defined at $y=0$ therefore existence and uniqueness theorem do not apply.

3.13.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(2xy\,\mathrm{e}^{y^2x}-3y^2\right)\mathrm{d}y=\left(-y^2\mathrm{e}^{y^2x}-4x^3\right)\mathrm{d}x$$

$$\left(y^2\mathrm{e}^{y^2x}+4x^3\right)\mathrm{d}x+\left(2xy\,\mathrm{e}^{y^2x}-3y^2\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y^2\mathrm{e}^{y^2x}+4x^3\\ N(x,y)&=2xy\,\mathrm{e}^{y^2x}-3y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(y^{2}e^{y^{2}x}+4x^{3}\right)\\&=2y\,e^{y^{2}x}\left(y^{2}x+1\right)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(2xy\,e^{y^{2}x}-3y^{2}\right)\\&=2y\,e^{y^{2}x}\left(y^{2}x+1\right)\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$

$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial \phi}{\partial x}\,\mathrm{d}x&=\int M\,\mathrm{d}x\\\int\frac{\partial \phi}{\partial x}\,\mathrm{d}x&=\int y^{2}e^{y^{2}x}+4x^{3}\,\mathrm{d}x\\\phi&=e^{y^{2}x}+x^{4}+f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=2xy\,e^{y^{2}x}+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=2xy\,e^{y^{2}x}-3y^{2}$. Therefore equation (4) becomes

$$2xy\,e^{y^{2}x}-3y^{2}=2xy\,e^{y^{2}x}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -3y^2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-3y^2)\,\mathrm{d}y$$
$$f(y) = -y^3 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{y^2 x} + x^4 - y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{y^2 x} + x^4 - y^3$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$\mathrm{e}^{y^2 x} + x^4 - y^3 = 2$$

Summary

The solution(s) found are the following

$$\mathrm{e}^{x y^2} + x^4 - y^3 = 2 \tag{1}$$

Verification of solutions

$$\mathrm{e}^{x y^2} + x^4 - y^3 = 2$$

Verified OK.

3.13.3 Maple step by step solution

Let's solve

$\left[y^2 e^{x y^2}+\left(2 x y\, e^{x y^2}-3 y^2\right) y'=-4 x^3, y(1)=0\right]$

- Highest derivative means the order of the ODE is 1

y'

□ Check if ODE is exact

◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x,y)=0$

◦ Compute derivative of lhs

$F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right) y'=0$

◦ Evaluate derivatives

$2y\, e^{y^2 x}+2y^3 x\, e^{y^2 x}=2y\, e^{y^2 x}+2y^3 x\, e^{y^2 x}$

◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

$\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

$F(x,y)=\int\left(y^2 e^{y^2 x}+4x^3\right)dx+f_1(y)$

- Evaluate integral

$F(x,y)=e^{y^2 x}+x^4+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

$N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

$2xy\, e^{y^2 x}-3y^2=2xy\, e^{y^2 x}+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

$\frac{d}{dy}f_1(y)=-3y^2$

- Solve for $f_1(y)$

$f_1(y)=-y^3$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = e^{y^2x} + x^4 - y^3$$

- Substitute $F(x,y)$ into the solution of the ODE

$$e^{y^2x} + x^4 - y^3 = c_1$$

- Solve for y

$$y = RootOf\left(-e^{\_Z^2x} - x^4 + \_Z^3 + c_1\right)$$

- Use initial condition $y(1) = 0$

$$0 = RootOf\left(-e^{\_Z^2} - 1 + \_Z^3 + c_1\right)$$

- Solve for c_1

$$c_1 = 2$$

- Substitute $c_1 = 2$ into general solution and simplify

$$y = RootOf\left(-e^{\_Z^2x} - x^4 + \_Z^3 + 2\right)$$

- Solution to the IVP

$$y = RootOf\left(-e^{\_Z^2x} - x^4 + \_Z^3 + 2\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 23

```
dsolve([(y(x)^2*exp(x*y(x)^2)+4*x^3)+(2*x*y(x)*exp(x*y(x)^2)-3*y(x)^2)*diff(y(x),x)=0,y(1) =
```

$$y(x) = \mathrm{RootOf}\left(-\mathrm{e}^{x\_Z^2} - x^4 + \_Z^3 + 2\right)$$

✓ Solution by Mathematica

Time used: 0.332 (sec). Leaf size: 23

```
DSolve[{(y[x]^2*Exp[x*y[x]^2]+4*x^3)+(2*x*y[x]*Exp[x*y[x]^2]-3*y[x]^2)*y'[x]==0,y[1]==0},y[x
```

$$\mathrm{Solve}\left[x^4 + e^{xy(x)^2} - y(x)^3 = 2, y(x)\right]$$

4 Chapter 2. Special types of differential equations of the first kind. Lesson 10

4.1 problem Recognizable Exact Differential equations. Integrating factors. Example 10.51, page 90

Internal problem ID [4468]
Internal file name [`OUTPUT/3961_Sunday_June_05_2022_11_56_04_AM_50885699/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Example 10.51, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y^2 + y - xy' = 0}$$

4.1.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y(1+y)}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x \\
\left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{1}{y(1+y)}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x} \\
N(x,y) &= \frac{1}{y(1+y)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y\left(1+y\right)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln\left(x\right) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y(1+y)}$. Therefore equation (4) becomes

$$
\frac{1}{y\left(1+y\right)} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y\left(1+y\right)}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y\left(1+y\right)}\right) \mathrm{d}y \\
f(y) &= -\ln\left(1+y\right) + \ln\left(y\right) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \ln(1+y) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \ln(1+y) + \ln(y)$$

The solution becomes

$$y = -\frac{x\,e^{c_1}}{-1 + x\,e^{c_1}}$$

Summary
The solution(s) found are the following

$$y = -\frac{x\,e^{c_1}}{-1 + x\,e^{c_1}} \tag{1}$$

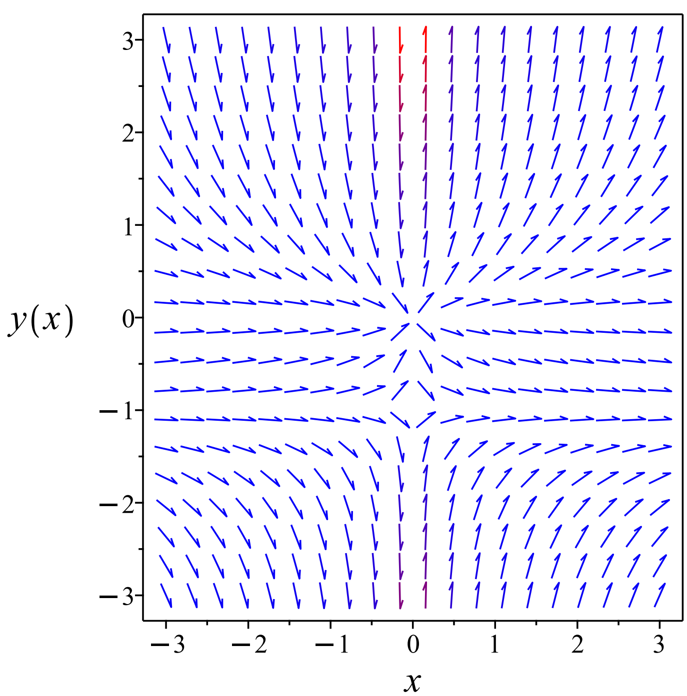


Figure 77: Slope field plot

Verification of solutions

$$y = -\frac{x\,e^{c_1}}{-1 + x\,e^{c_1}}$$

Verified OK.

4.1.2 Maple step by step solution

Let's solve

$y^2 + y - xy' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2+y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2+y}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $-\ln(1+y) + \ln(y) = \ln(x) + c_1$

- Solve for y

  $y = -\frac{x\,e^{c_1}}{-1+x\,e^{c_1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve((y(x)^2+y(x))-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{-x + c_1}$$

✓ Solution by Mathematica

Time used: 0.274 (sec). Leaf size: 32

```
DSolve[(y[x]^2+y[x])-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{c_1} x}{1 - e^{c_1} x}$$
$$y(x) \to -1$$
$$y(x) \to 0$$

4.2 problem Recognizable Exact Differential equations. Integrating factors. Example 10.52, page 90

Internal problem ID [4469]
Internal file name [`OUTPUT/3962_Sunday_June_05_2022_11_56_10_AM_32245722/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Example 10.52, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y \sec(x) + \sin(x) y' = 0}$$

4.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right) \mathrm{d}y = \left(\frac{\sec(x)}{\sin(x)}\right) \mathrm{d}x$$
$$\left(-\frac{\sec(x)}{\sin(x)}\right) \mathrm{d}x + \left(-\frac{1}{y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{\sec(x)}{\sin(x)}$$
$$N(x, y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\sec(x)}{\sin(x)}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{\sec(x)}{\sin(x)} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(\tan(x)) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$
-\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= -\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(\tan(x)) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\tan(x)) - \ln(y)$$

The solution becomes

$$y = \frac{e^{-c_1}}{\tan(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-c_1}}{\tan(x)} \tag{1}$$

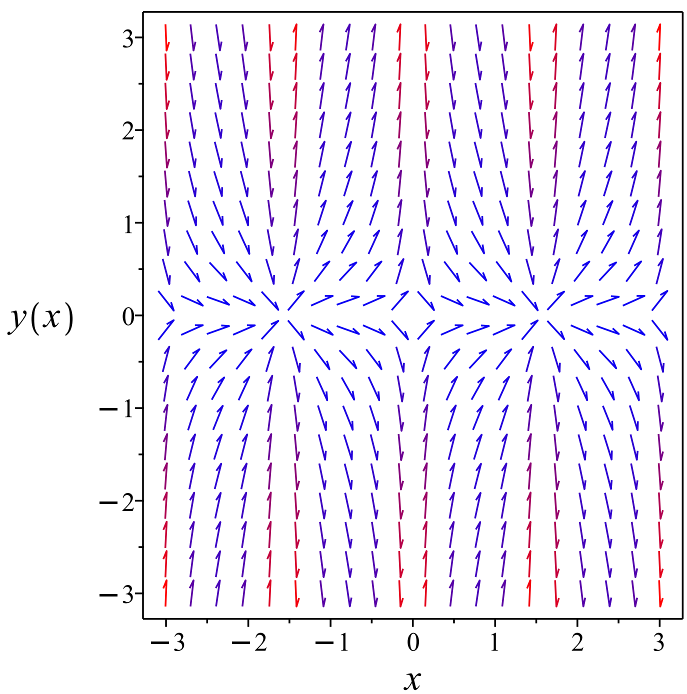


Figure 78: Slope field plot

Verification of solutions

$$y = \frac{e^{-c_1}}{\tan(x)}$$

Verified OK.

4.2.2 Maple step by step solution

Let's solve

$y \sec(x) + \sin(x) y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{\sec(x)}{\sin(x)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{\sec(x)}{\sin(x)} dx + c_1$

- Evaluate integral

  $\ln(y) = -\ln(\tan(x)) + c_1$

- Solve for y

  $y = \frac{e^{c_1}}{\tan(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve((y(x)*sec(x))+sin(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \cot(x) c_1$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 15

```
DSolve[(y[x]*Sec[x])+Sin[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cot(x)$$
$$y(x) \to 0$$

4.3 problem Recognizable Exact Differential equations. Integrating factors. Example 10.661, page 90

Internal problem ID [4470]
Internal file name [`OUTPUT/3963_Sunday_June_05_2022_11_56_17_AM_94206111/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Example 10.661, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{-\sin(y) + \cos(y)\, y' = -e^{x}}$$

4.3.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(\cos(y))\,\mathrm{d}y &= (-\mathrm{e}^x + \sin(y))\,\mathrm{d}x \\
(\mathrm{e}^x - \sin(y))\,\mathrm{d}x + (\cos(y))\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= \mathrm{e}^x - \sin(y) \\
N(x, y) &= \cos(y)
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\mathrm{e}^x - \sin(y)) \\
&= -\cos(y)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(y)) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \sec(y)\,((-\cos(y)) - (0)) \\
&= -1
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, dx} \\ &= e^{\int -1 \, dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-x} \\ &= \mathrm{e}^{-x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-x}(\mathrm{e}^{x} - \sin(y)) \\ &= 1 - \sin(y)\,\mathrm{e}^{-x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(\cos(y)) \\ &= \cos(y)\,\mathrm{e}^{-x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(1 - \sin(y)\,\mathrm{e}^{-x}\right) + \left(\cos(y)\,\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int 1 - \sin(y)\,\mathrm{e}^{-x} \, \mathrm{d}x\end{aligned}$$

$$\phi = x + \sin(y)\,\mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cos(y)\,e^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(y)\,e^{-x}$. Therefore equation (4) becomes

$$\cos(y)\,e^{-x} = \cos(y)\,e^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x + \sin(y)\,e^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x + \sin(y)\,e^{-x}$$

Summary

The solution(s) found are the following

$$x + \sin(y)\,e^{-x} = c_1 \tag{1}$$

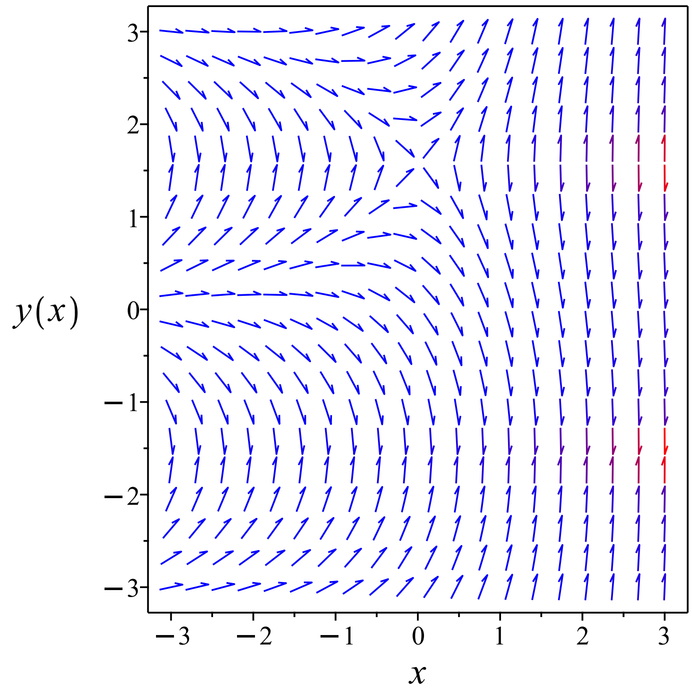


Figure 79: Slope field plot

Verification of solutions

$$x + \sin(y)\,e^{-x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve((exp(x)-sin(y(x)))+cos(y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\arcsin\left(\left(x+c_1\right)\mathrm{e}^{x}\right)$$

✓ Solution by Mathematica

Time used: 11.754 (sec). Leaf size: 16

```
DSolve[(Exp[x]-Sin[y[x]])+Cos[y[x]]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\arcsin\left(e^{x}(x+c_1)\right)$$

4.4 problem Recognizable Exact Differential equations. Integrating factors. Example 10.701, page 90

Internal problem ID [4471]
Internal file name [`OUTPUT/3964_Sunday_June_05_2022_11_56_27_AM_69786947/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Example 10.701, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{xy + (x^2 + 1) y' = 0}$$

4.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right) \mathrm{d}y = \left(\frac{x}{x^2+1}\right) \mathrm{d}x$$
$$\left(-\frac{x}{x^2+1}\right) \mathrm{d}x + \left(-\frac{1}{y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x}{x^2+1}$$
$$N(x, y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{x^2+1}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln(x^2+1)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$
-\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{y}\right)\mathrm{d}y \\
f(y) &= -\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} - \ln(y)$$

The solution becomes

$$y = e^{-\frac{\ln(x^2+1)}{2} - c_1}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{\ln(x^2+1)}{2} - c_1} \tag{1}$$

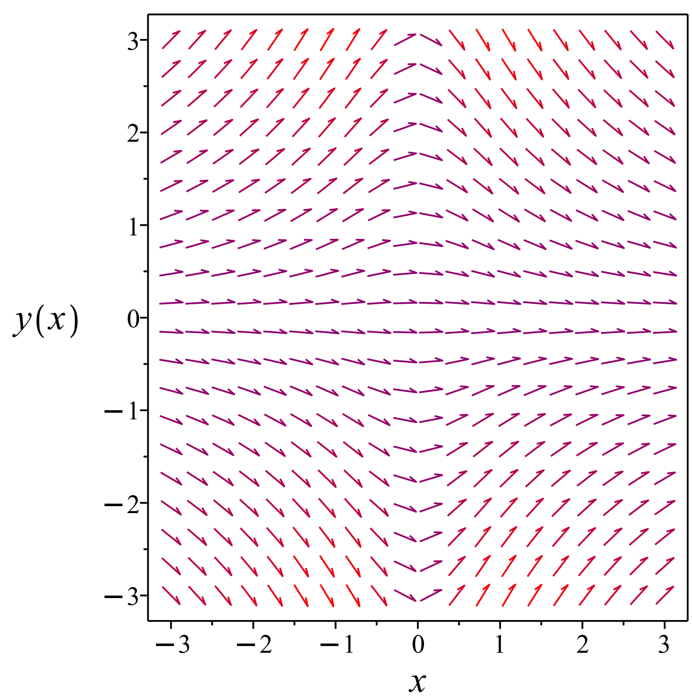


Figure 80: Slope field plot

Verification of solutions

$$y = e^{-\frac{\ln(x^2+1)}{2}-c_1}$$

Verified OK.

4.4.2 Maple step by step solution

Let's solve

$xy + (x^2+1)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{x}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{x}{x^2+1} dx + c_1$

- Evaluate integral

  $\ln(y) = -\frac{\ln(x^2+1)}{2} + c_1$

- Solve for y

  $y = e^{-\frac{\ln(x^2+1)}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve((x*y(x))+(1+x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x^2+1}}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 22

```
DSolve[(x*y[x])+(1+x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{\sqrt{x^2+1}}$$
$$y(x) \to 0$$

4.5 problem Recognizable Exact Differential equations. Integrating factors. Example 10.741, page 90

Internal problem ID [4472]
Internal file name [`OUTPUT/3965_Sunday_June_05_2022_11_56_33_AM_66923797/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Example 10.741, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class C`]]
```

$$\boxed{y^3 + xy^2 + y + (x^3 + yx^2 + x)\, y' = 0}$$

4.5.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^3 + y\,x^2 + x)\,\mathrm{d}y &= (-y^2x - y^3 - y)\,\mathrm{d}x\\ (y^2x + y^3 + y)\,\mathrm{d}x + (x^3 + y\,x^2 + x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y^2x + y^3 + y\\ N(x,y) &= x^3 + y\,x^2 + x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y^2x + y^3 + y)\\ &= 2xy + 3y^2 + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^3 + y\,x^2 + x)\\ &= 3x^2 + 2xy + 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^3 + y\,x^2 + x}\left(\left(2xy + 3y^2 + 1\right) - \left(3x^2 + 2xy + 1\right)\right) \\
&= \frac{-3x^2 + 3y^2}{x\left(x^2 + xy + 1\right)}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y^2x + y^3 + y}\left(\left(3x^2 + 2xy + 1\right) - \left(2xy + 3y^2 + 1\right)\right) \\
&= \frac{3x^2 - 3y^2}{y\left(xy + y^2 + 1\right)}
\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(3x^2 + 2xy + 1) - (2xy + 3y^2 + 1)}{x\left(y^2x + y^3 + y\right) - y\left(x^3 + y\,x^2 + x\right)} \\
&= -\frac{3}{yx}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{3}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{3}{t}\right) \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(t)}\\ &= \frac{1}{t^3}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^3y^3}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^3y^3}\left(y^2x+y^3+y\right)\\ &= \frac{xy+y^2+1}{y^2x^3}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^3y^3}\left(x^3+y\,x^2+x\right)\\ &= \frac{x^2+xy+1}{x^2y^3}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{xy+y^2+1}{y^2x^3}\right)+\left(\frac{x^2+xy+1}{x^2y^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \frac{xy + y^2 + 1}{y^2 x^3} \, \mathrm{d}x \\
\phi &= \frac{-2xy - y^2 - 1}{2y^2 x^2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{-2x - 2y}{2y^2 x^2} - \frac{-2xy - y^2 - 1}{y^3 x^2} + f'(y) \qquad (4) \\
&= \frac{xy + 1}{x^2 y^3} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2 + xy + 1}{x^2 y^3}$. Therefore equation (4) becomes

$$
\frac{x^2 + xy + 1}{x^2 y^3} = \frac{xy + 1}{x^2 y^3} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^3}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y^3} \right) \mathrm{d}y \\
f(y) &= -\frac{1}{2y^2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{-2xy - y^2 - 1}{2y^2 x^2} - \frac{1}{2y^2} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-2xy - y^2 - 1}{2y^2x^2} - \frac{1}{2y^2}$$

Summary

The solution(s) found are the following

$$\frac{-2xy - y^2 - 1}{2y^2x^2} - \frac{1}{2y^2} = c_1 \tag{1}$$

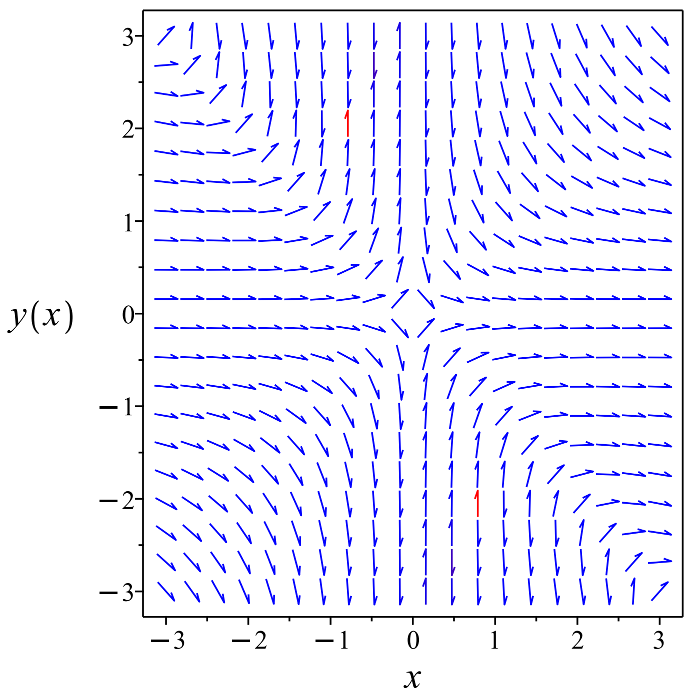


Figure 81: Slope field plot

Verification of solutions

$$\frac{-2xy - y^2 - 1}{2y^2x^2} - \frac{1}{2y^2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 99

```
dsolve((y(x)^3+x*y(x)^2+y(x))+(x^3+x^2*y(x)+x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2+1}{\left(\sqrt{x^2+1}\,\sqrt{\frac{-1+(x^4+x^2)c_1}{x^2(x^2+1)}}-1\right)x}$$
$$y(x) = \frac{-x^2-1}{\left(\sqrt{x^2+1}\,\sqrt{\frac{-1+(x^4+x^2)c_1}{x^2(x^2+1)}}+1\right)x}$$

✓ Solution by Mathematica

Time used: 3.726 (sec). Leaf size: 114

```
DSolve[(y[x]^3+x*y[x]^2+y[x])+(x^3+x^2*y[x]+x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -\frac{\sqrt{\frac{1}{x^3}}x(x^2+1)}{\sqrt{\frac{1}{x^3}}x^2 - \sqrt{c_1x^3 - \frac{1}{x} + c_1x}}$$

$$y(x) \to -\frac{\sqrt{\frac{1}{x^3}}x(x^2+1)}{\sqrt{\frac{1}{x^3}}x^2 + \sqrt{c_1x^3 - \frac{1}{x} + c_1x}}$$

$$y(x) \to 0$$

4.6 problem Recognizable Exact Differential equations. Integrating factors. Example 10.781, page 90

Internal problem ID [4473]
Internal file name [`OUTPUT/3966_Sunday_June_05_2022_11_56_41_AM_11651003/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Example 10.781, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{3y - xy' = 0}$$

4.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{3y}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{1}{3y}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{1}{x}\\ N(x,y) &= \frac{1}{3y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{3y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x \\
\phi &= -\ln(x) + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{3y}$. Therefore equation (4) becomes

$$
\frac{1}{3y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{3y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{3y}\right)\mathrm{d}y \\
f(y) &= \frac{\ln(y)}{3} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{\ln(y)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{\ln(y)}{3}$$

The solution becomes

$$y = e^{3c_1}x^3$$

Summary

The solution(s) found are the following

$$y = e^{3c_1}x^3 \tag{1}$$

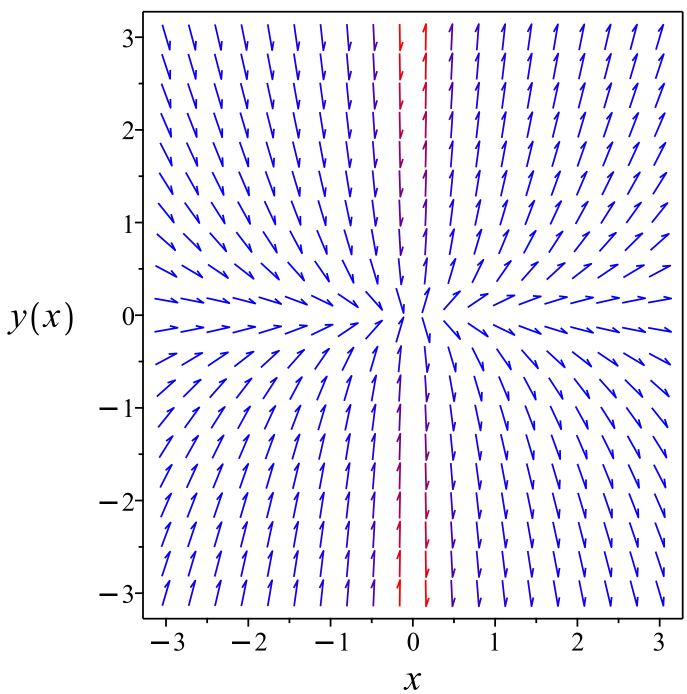


Figure 82: Slope field plot

Verification of solutions

$$y = e^{3c_1}x^3$$

Verified OK.

4.6.2 Maple step by step solution

Let's solve

$3y - xy' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{3}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{3}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = 3\ln(x) + c_1$

- Solve for y

  $y = e^{c_1}x^3$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve((3*y(x))-(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1x^3$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 16

```
DSolve[(3*y[x])-(x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x^3$$
$$y(x) \to 0$$

4.7 problem Recognizable Exact Differential equations. Integrating factors. Example 10.81, page 90

Internal problem ID [4474]
Internal file name [`OUTPUT/3967_Sunday_June_05_2022_11_56_48_AM_35981178/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Example 10.81, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y - 3xy' = 0}$$

4.7.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{3}{y}\right) \mathrm{d}y &= \left(\frac{1}{x}\right) \mathrm{d}x\\ \left(-\frac{1}{x}\right) \mathrm{d}x + \left(\frac{3}{y}\right) \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x}\\ N(x, y) &= \frac{3}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{3}{y}\right) \\ = 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{1}{x}\,\mathrm{d}x$$

$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{3}{y}$. Therefore equation (4) becomes

$$\frac{3}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{3}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{3}{y}\right)\mathrm{d}y$$

$$f(y) = 3\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + 3\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + 3\ln(y)$$

The solution becomes

$$y = e^{\frac{\ln(x)}{3} + \frac{c_1}{3}}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{\ln(x)}{3} + \frac{c_1}{3}} \tag{1}$$

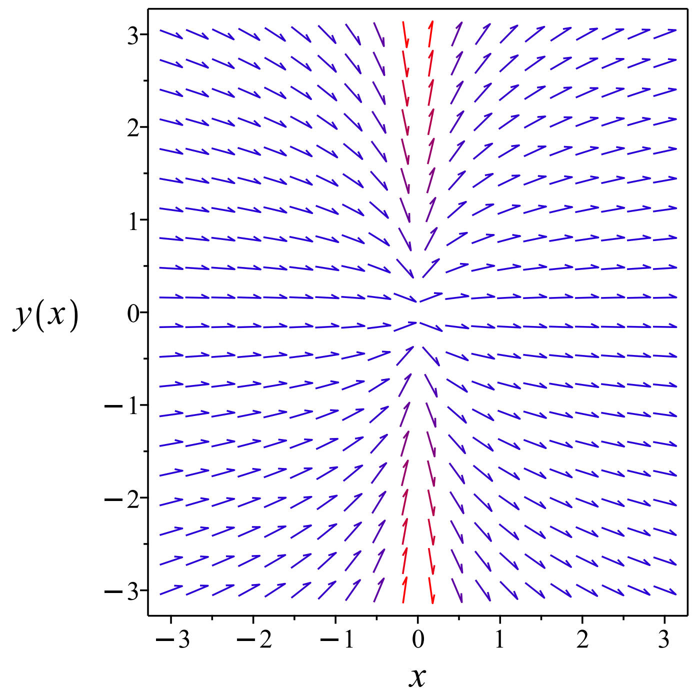


Figure 83: Slope field plot

Verification of solutions

$$y = e^{\frac{\ln(x)}{3} + \frac{c_1}{3}}$$

Verified OK.

4.7.2 Maple step by step solution

Let's solve

$y - 3xy' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{3x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{3x}dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{\ln(x)}{3} + c_1$

- Solve for y

  $y = \frac{\left(x\left(\mathrm{e}^{-3c_1}\right)^2\right)^{\frac{1}{3}}}{\mathrm{e}^{-3c_1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve((y(x))-(3*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^{\frac{1}{3}}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 18

```
DSolve[(y[x])-(3*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \sqrt[3]{x}$$
$$y(x) \to 0$$

4.8 problem Recognizable Exact Differential equations. Integrating factors. Example 10.83, page 90

Internal problem ID [4475]
Internal file name [`OUTPUT/3968_Sunday_June_05_2022_11_56_54_AM_66373173/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Example 10.83, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y(2y^3x^2+3)+x(y^3x^2-1)\,y'=0}$$

4.8.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(2y^3x^2+3)}{x\,(y^3x^2-1)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{y(2y^3x^2+3)(b_3-a_2)}{x(y^3x^2-1)} - \frac{y^2(2y^3x^2+3)^2 a_3}{x^2(y^3x^2-1)^2} \\ &- \left(-\frac{4y^4}{y^3x^2-1} + \frac{y(2y^3x^2+3)}{x^2(y^3x^2-1)} + \frac{2y^4(2y^3x^2+3)}{(y^3x^2-1)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{2y^3x^2+3}{x(y^3x^2-1)} - \frac{6y^3x}{y^3x^2-1} + \frac{3y^3(2y^3x^2+3)x}{(y^3x^2-1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^6y^6b_2 - 6x^4y^8a_3 + 2x^5y^6b_1 - 2x^4y^7a_1 - 16x^4y^3b_2 - 10x^3y^4a_2 - 15x^3y^4b_3 - 23x^2y^5a_3 - 14x^3y^3b_1 - 11x}{(y^3x^2-1)^2 x^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &3x^6y^6b_2 - 6x^4y^8a_3 + 2x^5y^6b_1 - 2x^4y^7a_1 - 16x^4y^3b_2 - 10x^3y^4a_2 - 15x^3y^4b_3 \\ &- 23x^2y^5a_3 - 14x^3y^3b_1 - 11x^2y^4a_1 - 2b_2x^2 - 6y^2a_3 - 3xb_1 + 3ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-6a_3v_1^4v_2^8 + 3b_2v_1^6v_2^6 - 2a_1v_1^4v_2^7 + 2b_1v_1^5v_2^6 - 10a_2v_1^3v_2^4 - 23a_3v_1^2v_2^5 - 16b_2v_1^4v_2^3 \\ &- 15b_3v_1^3v_2^4 - 11a_1v_1^2v_2^4 - 14b_1v_1^3v_2^3 - 6a_3v_2^2 - 2b_2v_1^2 + 3a_1v_2 - 3b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$3b_2 v_1^6 v_2^6 + 2b_1 v_1^5 v_2^6 - 6a_3 v_1^4 v_2^8 - 2a_1 v_1^4 v_2^7 - 16b_2 v_1^4 v_2^3 + (-10a_2 - 15b_3) v_1^3 v_2^4 \\ - 14b_1 v_1^3 v_2^3 - 23a_3 v_1^2 v_2^5 - 11a_1 v_1^2 v_2^4 - 2b_2 v_1^2 - 3b_1 v_1 - 6a_3 v_2^2 + 3a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -11a_1 &= 0 \\ -2a_1 &= 0 \\ 3a_1 &= 0 \\ -23a_3 &= 0 \\ -6a_3 &= 0 \\ -14b_1 &= 0 \\ -3b_1 &= 0 \\ 2b_1 &= 0 \\ -16b_2 &= 0 \\ -2b_2 &= 0 \\ 3b_2 &= 0 \\ -10a_2 - 15b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -\frac{3b_3}{2} \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -\frac{3x}{2} \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y(2y^3x^2+3)}{x\,(y^3x^2-1)}\right)\left(-\frac{3x}{2}\right) \\
&= \frac{-4y^4x^2-11y}{2y^3x^2-2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-4y^4x^2-11y}{2y^3x^2-2}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{2\ln(y)}{11} - \frac{5\ln(4y^3x^2+11)}{22}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y(2y^3x^2+3)}{x\,(y^3x^2-1)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{20x\,y^3}{44y^3x^2 + 121} \\
S_y &= \frac{-2y^3x^2 + 2}{4y^4x^2 + 11y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{6}{11x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{6}{11R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{6\ln(R)}{11} + c_1 \tag{4}
$$

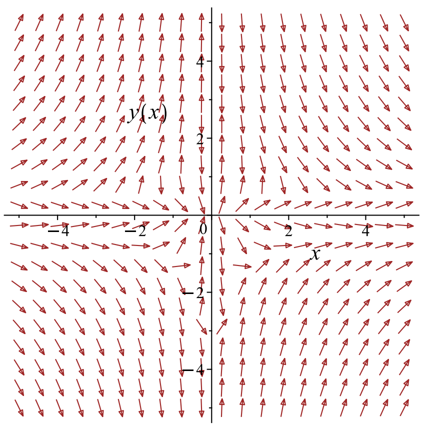
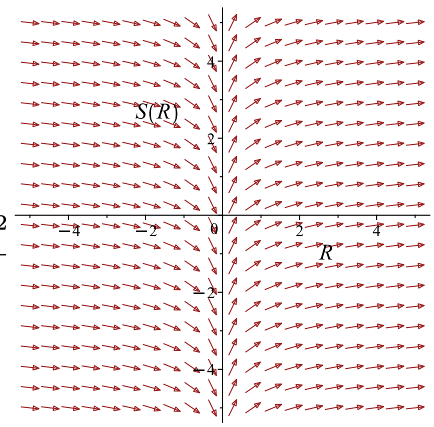
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{2\ln(y)}{11} - \frac{5\ln(4y^3x^2 + 11)}{22} = \frac{6\ln(x)}{11} + c_1
$$

Which simplifies to

$$
\frac{2\ln(y)}{11} - \frac{5\ln(4y^3x^2 + 11)}{22} = \frac{6\ln(x)}{11} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(2y^3x^2+3)}{x(y^3x^2-1)}$ 	$R = x$ $S = \frac{2\ln(y)}{11} - \frac{5\ln(4y^3x^2}{22}$	$\frac{dS}{dR} = \frac{6}{11R}$ 

Summary
The solution(s) found are the following

$$\frac{2\ln(y)}{11} - \frac{5\ln(4y^3x^2+11)}{22} = \frac{6\ln(x)}{11} + c_1 \tag{1}$$

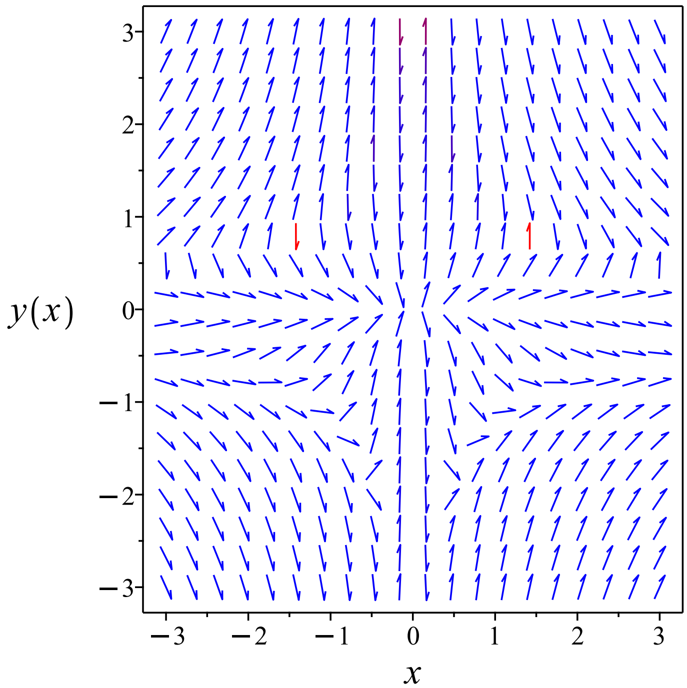


Figure 84: Slope field plot

Verification of solutions

$$\frac{2\ln(y)}{11} - \frac{5\ln(4y^3x^2+11)}{22} = \frac{6\ln(x)}{11} + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 39

```
dsolve((y(x)*(2*x^2*y(x)^3+3))+(x*(x^2*y(x)^3-1))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{-\frac{11c_1}{3}}x^3}{\operatorname{RootOf}\left(11\,e^{11c_1}\_Z^{15} - e^{11c_1}\_Z^{11} + 4x^{11}\right)^5}$$

✓ Solution by Mathematica

Time used: 10.635 (sec). Leaf size: 1081

```
DSolve[(y[x]*(2*x^2*y[x]^3+3))+(x*(x^2*y[x]^3-1))*y'[x]==0,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \text{Root}\left[1024\#1^{15}x^{22} + 14080\#1^{12}x^{20} + 77440\#1^{9}x^{18} + 212960\#1^{6}x^{16} - \#1^{4}e^{\frac{44c_1}{3}} + 292820\#1^{3}x^{14} + 161051x^{12}\&, 1\right]$$

$$y(x) \to \text{Root}\left[1024\#1^{15}x^{22} + 14080\#1^{12}x^{20} + 77440\#1^{9}x^{18} + 212960\#1^{6}x^{16} - \#1^{4}e^{\frac{44c_1}{3}} + 292820\#1^{3}x^{14} + 161051x^{12}\&, 2\right]$$

$$y(x) \to \text{Root}\left[1024\#1^{15}x^{22} + 14080\#1^{12}x^{20} + 77440\#1^{9}x^{18} + 212960\#1^{6}x^{16} - \#1^{4}e^{\frac{44c_1}{3}} + 292820\#1^{3}x^{14} + 161051x^{12}\&, 3\right]$$

$$y(x) \to \text{Root}\left[1024\#1^{15}x^{22} + 14080\#1^{12}x^{20} + 77440\#1^{9}x^{18} + 212960\#1^{6}x^{16} - \#1^{4}e^{\frac{44c_1}{3}} + 292820\#1^{3}x^{14} + 161051x^{12}\&, 4\right]$$

$$y(x) \to \text{Root}\left[1024\#1^{15}x^{22} + 14080\#1^{12}x^{20} + 77440\#1^{9}x^{18} + 212960\#1^{6}x^{16} - \#1^{4}e^{\frac{44c_1}{3}} + 292820\#1^{3}x^{14} + 161051x^{12}\&, 5\right]$$

$$y(x) \to \text{Root}\left[1024\#1^{15}x^{22} + 14080\#1^{12}x^{20} + 77440\#1^{9}x^{18} + 212960\#1^{6}x^{16} - \#1^{4}e^{\frac{44c_1}{3}} + 292820\#1^{3}x^{14} + 161051x^{12}\&, 6\right]$$

$$y(x) \to \text{Root}\left[1024\#1^{15}x^{22} + 14080\#1^{12}x^{20} + 77440\#1^{9}x^{18} + 212960\#1^{6}x^{16} - \#1^{4}e^{\frac{44c_1}{3}} + 292820\#1^{3}x^{14} + 161051x^{12}\&, 7\right]$$

$$y(x) \to \text{Root}\left[1024\#1^{15}x^{22} + 14080\#1^{12}x^{20} + 77440\#1^{9}x^{18} + 212960\#1^{6}x^{16} - \#1^{4}e^{\frac{44c_1}{3}} + 292820\#1^{3}x^{14} + 161051x^{12}\&, 8\right]$$

$$y(x) \to \text{Root}\left[1024\#1^{15}x^{22} + 14080\#1^{12}x^{20} + 77440\#1^{9}x^{18} + 212960\#1^{6}x^{16} - \#1^{4}e^{\frac{44c_1}{3}} + 292820\#1^{3}x^{14} + 161051x^{12}\&, 9\right]$$

$$y(x) \to \text{Root}\left[1024\#1^{15}x^{22} + 14080\#1^{12}x^{20} + 77440\#1^{9}x^{18} + 212960\#1^{6}x^{16} - \#1^{4}e^{\frac{44c_1}{3}} + 292820\#1^{3}x^{14} + 161051x^{12}\&, 10\right]$$

$$y(x) \to \text{Root}\left[1024\#1^{15}x^{22} + 14080\#1^{12}x^{20} + 77440\#1^{9}x^{18} + 212960\#1^{6}x^{16} - \#1^{4}e^{\frac{44c_1}{3}} + 292820\#1^{3}x^{14} + 161051x^{12}\&, 11\right]$$

$$y(x) \to \text{Root}\left[1024\#1^{15}x^{22} + 14080\#1^{12}x^{20} + 77440\#1^{9}x^{18} + 212960\#1^{6}x^{16} - \#1^{4}e^{\frac{44c_1}{3}} + 292820\#1^{3}x^{14} + 161051x^{12}\&, 12\right]$$

$$y(x) \to \text{Root}\left[1024\#1^{15}x^{22} + 14080\#1^{12}x^{20} + 77440\#1^{9}x^{18} + 212960\#1^{6}x^{16} - \#1^{4}e^{\frac{44c_1}{3}}\right.$$

4.9 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.1, page 90

Internal problem ID [4476]
Internal file name [`OUTPUT/3969_Sunday_June_05_2022_11_57_07_AM_22264960/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.1, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{2xy + \left(x^2 + y^2\right) y' = -x^2}$$

4.9.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2 + y^2)\,\mathrm{d}y &= (-x^2 - 2xy)\,\mathrm{d}x \\ (x^2 + 2xy)\,\mathrm{d}x + (x^2 + y^2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = x^2 + 2xy$$
$$N(x,y) = x^2 + y^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^2 + 2xy) \\ &= 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + y^2) \\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int x^2 + 2xy\,\mathrm{d}x$$
$$\phi = \frac{x^2(x+3y)}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 + y^2$. Therefore equation (4) becomes

$$x^2 + y^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y^2)\,\mathrm{d}y$$
$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2(x+3y)}{3} + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2(x+3y)}{3} + \frac{y^3}{3}$$

Summary

The solution(s) found are the following

$$\frac{x^2(x+3y)}{3} + \frac{y^3}{3} = c_1 \tag{1}$$

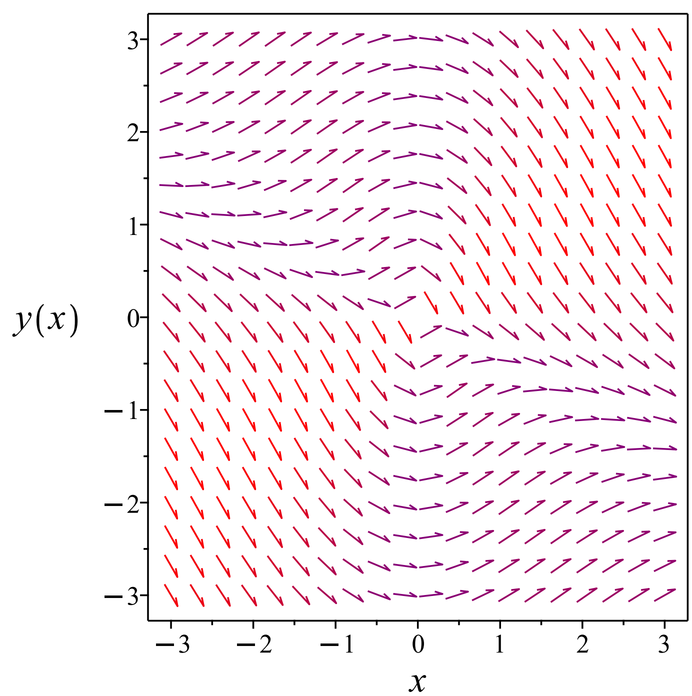


Figure 85: Slope field plot

Verification of solutions

$$\frac{x^2(x+3y)}{3} + \frac{y^3}{3} = c_1$$

Verified OK.

4.9.2 Maple step by step solution

Let's solve

$2xy + (x^2 + y^2)\, y' = -x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $2x = 2x$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (x^2 + 2xy)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{x^3}{3} + y\, x^2 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $x^2 + y^2 = x^2 + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = y^2$

- Solve for $f_1(y)$

  $f_1(y) = \frac{y^3}{3}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$F(x,y) = \frac{1}{3}x^3 + y\,x^2 + \frac{1}{3}y^3$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{1}{3}x^3 + y\,x^2 + \frac{1}{3}y^3 = c_1$

- Solve for y

$$\left\{y = \frac{\left(-4x^3+12c_1+4\sqrt{5x^6-6c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{2x^2}{\left(-4x^3+12c_1+4\sqrt{5x^6-6c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}, y = -\frac{\left(-4x^3+12c_1+4\sqrt{5x^6-6c_1x^3\right.}}{4}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 321

```
dsolve((2*x*y(x)+x^2)+(x^2+y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{2\left(c_1x^2 - \frac{\left(4-4x^3c_1^{\frac{3}{2}}+4\sqrt{5x^6c_1^3-2x^3c_1^{\frac{3}{2}}+1}\right)^{\frac{2}{3}}}{4}\right)}{\sqrt{c_1}\left(4-4x^3c_1^{\frac{3}{2}}+4\sqrt{5x^6c_1^3-2x^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left(4-4x^3c_1^{\frac{3}{2}}+4\sqrt{5x^6c_1^3-2x^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}}{4\sqrt{c_1}} - \frac{\left(i\sqrt{3}-1\right)\sqrt{c_1}\,x^2}{\left(4-4x^3c_1^{\frac{3}{2}}+4\sqrt{5x^6c_1^3-2x^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{4i\sqrt{3}\,c_1x^2 + i\left(4-4x^3c_1^{\frac{3}{2}}+4\sqrt{5x^6c_1^3-2x^3c_1^{\frac{3}{2}}+1}\right)^{\frac{2}{3}}\sqrt{3} + 4c_1x^2 - \left(4-4x^3c_1^{\frac{3}{2}}+4\sqrt{5x^6c_1^3-2x^3c_1^{\frac{3}{2}}+\right.}{4\left(4-4x^3c_1^{\frac{3}{2}}+4\sqrt{5x^6c_1^3-2x^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}\sqrt{c_1}}$$

✓ Solution by Mathematica

Time used: 23.867 (sec). Leaf size: 597

```
DSolve[(2*x*y[x]+x^2)+(x^2+y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{\sqrt[3]{-x^3+\sqrt{5x^6-2e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}}{\sqrt[3]{2}}-\frac{\sqrt[3]{2}x^2}{\sqrt[3]{-x^3+\sqrt{5x^6-2e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}}$$

$$y(x)\to \frac{\sqrt[3]{2}\left(2+2i\sqrt{3}\right)x^2+i2^{2/3}\left(\sqrt{3}+i\right)\left(-x^3+\sqrt{5x^6-2e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}\right)^{2/3}}{4\sqrt[3]{-x^3+\sqrt{5x^6-2e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}}$$

$$y(x)\to \frac{\left(1-i\sqrt{3}\right)x^2}{2^{2/3}\sqrt[3]{-x^3+\sqrt{5x^6-2e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}}-\frac{\left(1+i\sqrt{3}\right)\sqrt[3]{-x^3+\sqrt{5x^6-2e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}}{2\sqrt[3]{2}}$$

$$y(x)\to \frac{2\sqrt[3]{-2}x^2+(-2)^{2/3}\left(\sqrt{5}\sqrt{x^6}-x^3\right)^{2/3}}{2\sqrt[3]{\sqrt{5}\sqrt{x^6}-x^3}}$$

$$y(x)\to \frac{\left(2\sqrt{5}\sqrt{x^6}-2x^3\right)^{2/3}-2\sqrt[3]{2}x^2}{2\sqrt[3]{\sqrt{5}\sqrt{x^6}-x^3}}$$

$$y(x)\to \frac{\sqrt[3]{2}\left(2-2i\sqrt{3}\right)x^2+\left(-1-i\sqrt{3}\right)\left(2\sqrt{5}\sqrt{x^6}-2x^3\right)^{2/3}}{4\sqrt[3]{\sqrt{5}\sqrt{x^6}-x^3}}$$

4.10 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.2, page 90

Internal problem ID [4477]
Internal file name [`OUTPUT/3970_Sunday_June_05_2022_11_57_13_AM_64019506/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.2, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{y\cos(x) + \left(y^3 + \sin(x)\right) y' = -x^2}$$

4.10.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(y^3 + \sin(x)\right) \mathrm{d}y &= \left(-y\cos(x) - x^2\right) \mathrm{d}x \\ \left(x^2 + y\cos(x)\right) \mathrm{d}x + \left(y^3 + \sin(x)\right) \mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x^2 + y\cos(x) \\ N(x, y) &= y^3 + \sin(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + y\cos(x)\right) \\ &= \cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y^3 + \sin(x)\right) \\ &= \cos(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int x^2 + y \cos(x) \, \mathrm{d}x$$

$$\phi = \frac{x^3}{3} + y \sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^3 + \sin(x)$. Therefore equation (4) becomes

$$y^3 + \sin(x) = \sin(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^3$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y^3) \, \mathrm{d}y$$

$$f(y) = \frac{y^4}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^3}{3} + y \sin(x) + \frac{y^4}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^3}{3} + y \sin(x) + \frac{y^4}{4}$$

Summary

The solution(s) found are the following

$$\frac{x^3}{3} + \sin(x) y + \frac{y^4}{4} = c_1 \tag{1}$$

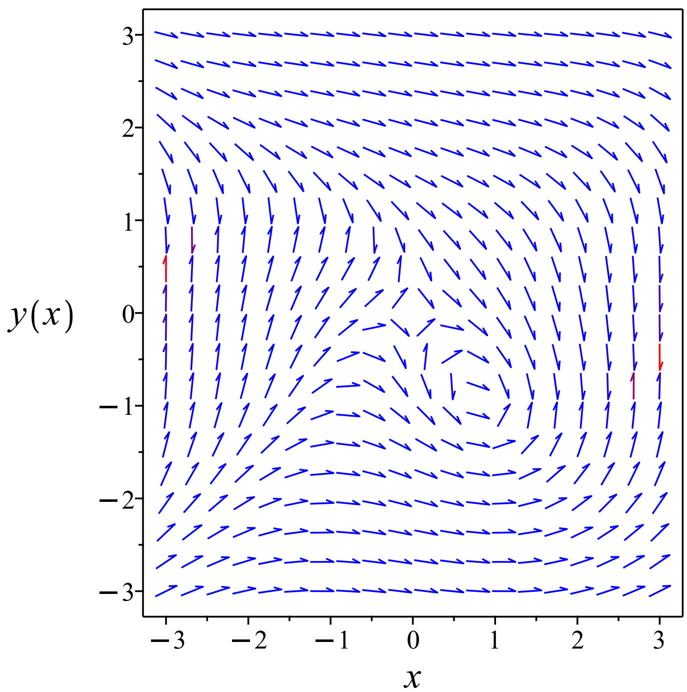


Figure 86: Slope field plot

Verification of solutions

$$\frac{x^3}{3} + \sin(x) y + \frac{y^4}{4} = c_1$$

Verified OK.

4.10.2 Maple step by step solution

Let's solve

$y\cos(x)+(y^3+\sin(x))\,y'=-x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$

  ◦ Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  ◦ Evaluate derivatives

  $\cos(x)=\cos(x)$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(x^2+y\cos(x))\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{x^3}{3}+y\sin(x)+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $y^3+\sin(x)=\sin(x)+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=y^3$

- Solve for $f_1(y)$

  $f_1(y)=\frac{y^4}{4}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$F(x,y) = \frac{x^3}{3} + y\sin(x) + \frac{y^4}{4}$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{x^3}{3} + y\sin(x) + \frac{y^4}{4} = c_1$

- Solve for y

  $y = RootOf\left(3\_Z^4 + 4x^3 + 12\_Z\sin(x) - 12c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve((x^2+y(x)*cos(x))+(y(x)^3+sin(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{x^3}{3} + \sin(x)\,y(x) + \frac{y(x)^4}{4} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 60.198 (sec). Leaf size: 1119

```
DSolve[(x^2+y[x]*Cos[x])+(y[x]^3+Sin[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{\frac{4x^3+\left(27\sin^2(x)+\sqrt{729\sin^4(x)-64(x^3-3c_1)^3}\right)^{2/3}-12c_1}{\sqrt[3]{27\sin^2(x)+\sqrt{729\sin^4(x)-64\left(x^3-3c_1\right)^3}}}}}{\sqrt{6}}$$

$$-\frac{1}{2}\sqrt{-\frac{8\left(x^3-3c_1\right)}{3\sqrt[3]{27\sin^2(x)+\sqrt{729\sin^4(x)-64\left(x^3-3c_1\right)^3}}}-\frac{2}{3}\sqrt[3]{27\sin^2(x)+\sqrt{729\sin^4(x)-64\left(x^3-3c_1\right.}}}$$

$$y(x) \to \frac{\sqrt{\frac{4x^3+\left(27\sin^2(x)+\sqrt{729\sin^4(x)-64(x^3-3c_1)^3}\right)^{2/3}-12c_1}{\sqrt[3]{27\sin^2(x)+\sqrt{729\sin^4(x)-64\left(x^3-3c_1\right)^3}}}}}{\sqrt{6}}$$

$$+\frac{1}{2}\sqrt{-\frac{8\left(x^3-3c_1\right)}{3\sqrt[3]{27\sin^2(x)+\sqrt{729\sin^4(x)-64\left(x^3-3c_1\right)^3}}}-\frac{2}{3}\sqrt[3]{27\sin^2(x)+\sqrt{729\sin^4(x)-64\left(x^3-3c_1\right.}}}$$

$$y(x) \to -\frac{\sqrt{\frac{4x^3+\left(27\sin^2(x)+\sqrt{729\sin^4(x)-64(x^3-3c_1)^3}\right)^{2/3}-12c_1}{\sqrt[3]{27\sin^2(x)+\sqrt{729\sin^4(x)-64\left(x^3-3c_1\right)^3}}}}}{\sqrt{6}}$$

$$-\frac{1}{2}\sqrt{-\frac{8\left(x^3-3c_1\right)}{3\sqrt[3]{27\sin^2(x)+\sqrt{729\sin^4(x)-64\left(x^3-3c_1\right)^3}}}-\frac{2}{3}\sqrt[3]{27\sin^2(x)+\sqrt{729\sin^4(x)-64\left(x^3-3c_1\right.}}}$$

$$y(x) \to \frac{1}{2}\sqrt{-\frac{8\left(x^3-3c_1\right)}{3\sqrt[3]{27\sin^2(x)+\sqrt{729\sin^4(x)-64\left(x^3-3c_1\right)^3}}}-\frac{2}{3}\sqrt[3]{27\sin^2(x)+\sqrt{729\sin^4(x)-64\left(x^3-3c_1\right)}}}$$

$$\sqrt{\frac{4x^3+\left(27\sin^2(x)+\sqrt{729\sin^4(x)-64(x^3-3c_1)^3}\right)^{2/3}-12c_1}{\sqrt[3]{27\sin^2(x)+\sqrt{729\sin^4(x)-64\left(x^3-3c_1\right)^3}}}}$$

4.11 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.3, page 90

Internal problem ID [4478]
Internal file name [`OUTPUT/3971_Sunday_June_05_2022_11_57_21_AM_490435/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.3, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{y^2 + xyy' = -x^2 - x}$$

4.11.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(xy)\,\mathrm{d}y &= \left(-x^2 - y^2 - x\right)\mathrm{d}x\\ \left(x^2 + y^2 + x\right)\mathrm{d}x + (xy)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= x^2 + y^2 + x\\ N(x, y) &= xy \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + y^2 + x\right)\\ &= 2y \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xy)\\ &= y \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{yx}((2y) - (y))\\ &= \frac{1}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int \frac{1}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)} \\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x\left(x^2+y^2+x\right) \\ &= x\left(x^2+y^2+x\right)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x(xy) \\ &= y\,x^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(x\left(x^2+y^2+x\right)\right)+\left(y\,x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int x\left(x^2+y^2+x\right)\mathrm{d}x\end{aligned}$$

$$\phi = \frac{1}{4}x^4+\frac{1}{2}y^2x^2+\frac{1}{3}x^3+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = y\,x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y\,x^2$. Therefore equation (4) becomes

$$y\,x^2 = y\,x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{4}x^4 + \frac{1}{2}y^2x^2 + \frac{1}{3}x^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{4}x^4 + \frac{1}{2}y^2x^2 + \frac{1}{3}x^3$$

<u>Summary</u>
The solution(s) found are the following

$$\frac{x^4}{4} + \frac{y^2x^2}{2} + \frac{x^3}{3} = c_1 \tag{1}$$

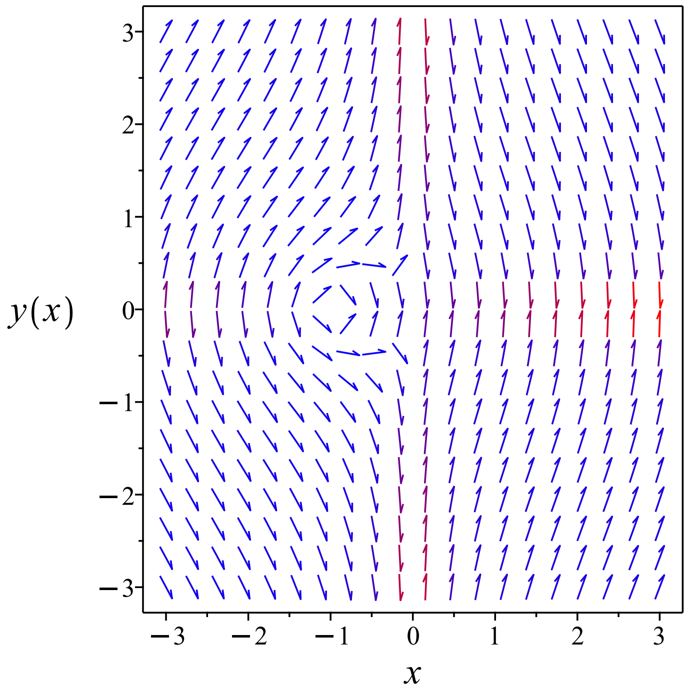


Figure 87: Slope field plot

Verification of solutions

$$\frac{x^4}{4} + \frac{y^2x^2}{2} + \frac{x^3}{3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
dsolve((x^2+y(x)^2+x)+(x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-18x^4 - 24x^3 + 36c_1}}{6x}$$
$$y(x) = \frac{\sqrt{-18x^4 - 24x^3 + 36c_1}}{6x}$$

✓ Solution by Mathematica

Time used: 0.242 (sec). Leaf size: 60

```
DSolve[(x^2+y[x]^2+x)+(x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-\frac{x^4}{2} - \frac{2x^3}{3} + c_1}}{x}$$
$$y(x) \to \frac{\sqrt{-\frac{x^4}{2} - \frac{2x^3}{3} + c_1}}{x}$$

4.12 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.4, page 90

Internal problem ID [4479]
Internal file name [`OUTPUT/3972_Sunday_June_05_2022_11_57_27_AM_75706137/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.4, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{-2xy + e^{y} + \left(y - x^{2} + x\,e^{y}\right) y' = -x}$$

4.12.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(y - x^2 + x\,\mathrm{e}^y\right)\mathrm{d}y &= \left(-x + 2xy - \mathrm{e}^y\right)\mathrm{d}x\\ \left(-2xy + \mathrm{e}^y + x\right)\mathrm{d}x + \left(y - x^2 + x\,\mathrm{e}^y\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2xy + \mathrm{e}^y + x\\ N(x,y) &= y - x^2 + x\,\mathrm{e}^y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2xy + \mathrm{e}^y + x\right)\\ &= -2x + \mathrm{e}^y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y - x^2 + x\,\mathrm{e}^y\right)\\ &= -2x + \mathrm{e}^y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -2xy + \mathrm{e}^y + x \,\mathrm{d}x$$
$$\phi = x\,\mathrm{e}^y - x^2\left(y - \frac{1}{2}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x\,\mathrm{e}^y - x^2 + f'(y) \tag{4}$$
$$= x(\mathrm{e}^y - x) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y - x^2 + x\,\mathrm{e}^y$. Therefore equation (4) becomes

$$y - x^2 + x\,\mathrm{e}^y = x(\mathrm{e}^y - x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y)\,\mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x\,\mathrm{e}^{y} - x^{2}\left(y - \frac{1}{2}\right) + \frac{y^{2}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\,\mathrm{e}^{y} - x^{2}\left(y - \frac{1}{2}\right) + \frac{y^{2}}{2}$$

Summary
The solution(s) found are the following

$$x\,\mathrm{e}^{y} - \left(y - \frac{1}{2}\right)x^{2} + \frac{y^{2}}{2} = c_1 \tag{1}$$

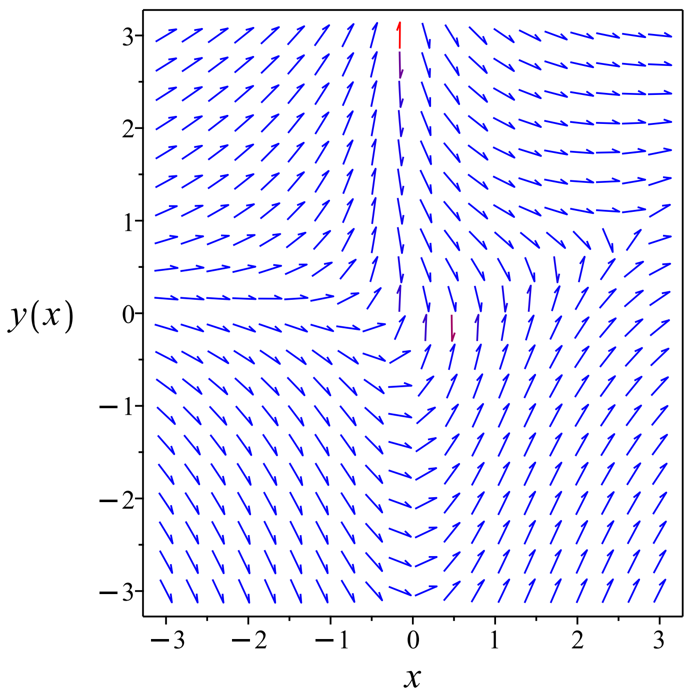


Figure 88: Slope field plot

Verification of solutions

$$x\,\mathrm{e}^{y} - \left(y - \frac{1}{2}\right)x^{2} + \frac{y^{2}}{2} = c_1$$

Verified OK.

4.12.2 Maple step by step solution

Let's solve

$-2xy + e^{y} + (y - x^2 + x\,e^{y})\,y' = -x$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$
  - Evaluate derivatives

    $-2x + e^{y} = -2x + e^{y}$
  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (-2xy + e^{y} + x)\,dx + f_1(y)$

- Evaluate integral

  $F(x, y) = -y\,x^2 + x\,e^{y} + \frac{x^2}{2} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $y - x^2 + x\,e^{y} = -x^2 + x\,e^{y} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = y$

- Solve for $f_1(y)$

  $f_1(y) = \frac{y^2}{2}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$F(x,y) = -y\,x^2 + x\,e^y + \frac{x^2}{2} + \frac{y^2}{2}$

- Substitute $F(x,y)$ into the solution of the ODE

  $-y\,x^2 + x\,e^y + \frac{x^2}{2} + \frac{y^2}{2} = c_1$

- Solve for y

  $y = \mathit{RootOf}\left(2x^2\_Z - 2\,e^{\_Z}x - \_Z^2 - x^2 + 2c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve((x-2*x*y(x)+exp(y(x)))+(y(x)-x^2+x*exp(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$-y(x)\,x^2 + x\,e^{y(x)} + \frac{x^2}{2} + \frac{y(x)^2}{2} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.316 (sec). Leaf size: 35

```
DSolve[(x-2*x*y[x]+Exp[y[x]])+(y[x]-x^2+x*Exp[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolution
```

$$\text{Solve}\left[x^2(-y(x)) + \frac{x^2}{2} + xe^{y(x)} + \frac{y(x)^2}{2} = c_1, y(x)\right]$$

4.13 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.5, page 90

Internal problem ID [4480]
Internal file name [`OUTPUT/3973_Sunday_June_05_2022_11_57_35_AM_13135549/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.5, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{e^{x} \sin \left(y\right)+e^{-y}-\left(x\,e^{-y}-e^{x} \cos \left(y\right)\right) y^{\prime}=0}$$

4.13.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\mathrm{e}^{x} \cos(y) - x\, \mathrm{e}^{-y}\right) \mathrm{d}y = \left(-\mathrm{e}^{x} \sin(y) - \mathrm{e}^{-y}\right) \mathrm{d}x$$
$$\left(\mathrm{e}^{x} \sin(y) + \mathrm{e}^{-y}\right) \mathrm{d}x + \left(\mathrm{e}^{x} \cos(y) - x\, \mathrm{e}^{-y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \mathrm{e}^{x} \sin(y) + \mathrm{e}^{-y}$$
$$N(x, y) = \mathrm{e}^{x} \cos(y) - x\, \mathrm{e}^{-y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{x} \sin(y) + \mathrm{e}^{-y}\right) \\ &= \mathrm{e}^{x} \cos(y) - \mathrm{e}^{-y}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\mathrm{e}^{x} \cos(y) - x\, \mathrm{e}^{-y}\right) \\ &= \mathrm{e}^{x} \cos(y) - \mathrm{e}^{-y}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int e^x \sin(y) + e^{-y} \, dx$$

$$\phi = e^x \sin(y) + x\, e^{-y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^x \cos(y) - x\, e^{-y} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^x \cos(y) - x\, e^{-y}$. Therefore equation (4) becomes

$$e^x \cos(y) - x\, e^{-y} = e^x \cos(y) - x\, e^{-y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = e^x \sin(y) + x\, e^{-y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^x \sin(y) + x\, e^{-y}$$

Summary

The solution(s) found are the following

$$e^{x} \sin(y) + x\,e^{-y} = c_1 \tag{1}$$

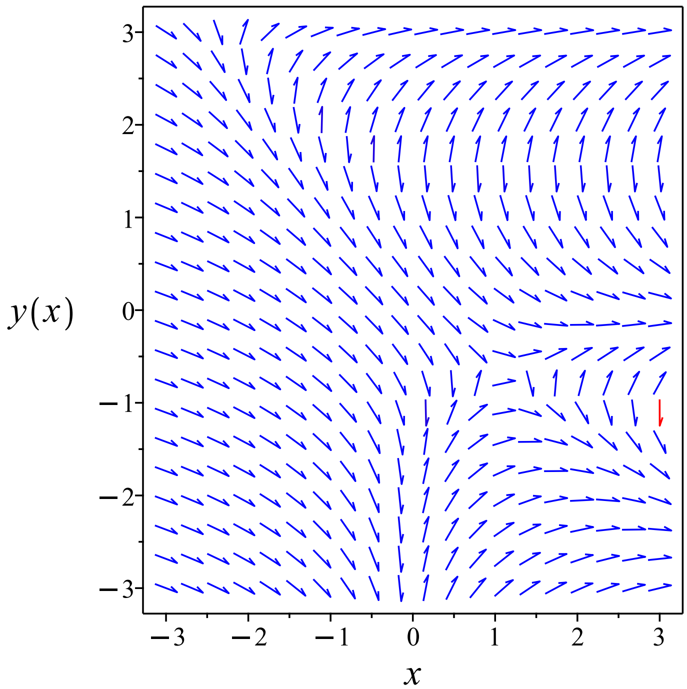


Figure 89: Slope field plot

Verification of solutions

$$e^{x} \sin(y) + x\,e^{-y} = c_1$$

Verified OK.

4.13.2 Maple step by step solution

Let's solve

$e^{x} \sin(y) + e^{-y} - (x\,e^{-y} - e^{x} \cos(y))\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x,y)=0$

- Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

- Evaluate derivatives

  $e^{x}\cos(y)-e^{-y}=e^{x}\cos(y)-e^{-y}$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(e^{x}\sin(y)+e^{-y})dx+f_1(y)$

- Evaluate integral

  $F(x,y)=e^{x}\sin(y)+x\,e^{-y}+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $e^{x}\cos(y)-x\,e^{-y}=e^{x}\cos(y)-x\,e^{-y}+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=0$

- Solve for $f_1(y)$

  $f_1(y)=0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=e^{x}\sin(y)+x\,e^{-y}$

- Substitute $F(x,y)$ into the solution of the ODE

  $e^{x}\sin(y)+x\,e^{-y}=c_1$

- Solve for y

  $y=\mathit{RootOf}\left(-\sin(\_Z)\,e^{-\_Z}e^{x}+c_1e^{-\_Z}-x\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve((exp(x)*sin(y(x))+exp(-y(x)))-(x*exp(-y(x))-exp(x)*cos(y(x)))*diff(y(x),x)=0,y(x), si
```

$$e^{x} \sin \left(y(x)\right)+x\, e^{-y(x)}+c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.377 (sec). Leaf size: 24

```
DSolve[(Exp[x]*Sin[y[x]]+Exp[-y[x]])-(x*Exp[-y[x]]-Exp[x]*Cos[y[x]])*y'[x]==0,y[x],x,Include
```

$$\text{Solve}\left[x\left(-e^{-y(x)}\right)-e^x\sin(y(x))=c_1,y(x)\right]$$

4.14 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.6, page 90

Internal problem ID [4481]
Internal file name [`OUTPUT/3974_Sunday_June_05_2022_11_57_45_AM_56729080/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.6, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _rational]
```

$$\boxed{-y^2 - y - \left(x^2 - y^2 - x\right) y' = -x^2}$$

4.14.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-x^2 + y^2 + y}{-x^2 + y^2 + x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2+\frac{(-x^2+y^2+y)(b_3-a_2)}{-x^2+y^2+x}-\frac{(-x^2+y^2+y)^2 a_3}{(-x^2+y^2+x)^2} \\ & -\left(-\frac{2x}{-x^2+y^2+x}-\frac{(-x^2+y^2+y)(1-2x)}{(-x^2+y^2+x)^2}\right)(xa_2+ya_3+a_1) \\ & -\left(\frac{2y+1}{-x^2+y^2+x}-\frac{2(-x^2+y^2+y)\,y}{(-x^2+y^2+x)^2}\right)(xb_2+yb_3+b_1)=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4a_2+x^4a_3-x^4b_2-x^4b_3-2x^2y^2a_2-2x^2y^2a_3+2x^2y^2b_2+2x^2y^2b_3+y^4a_2+y^4a_3-y^4b_2-y^4b_3-2x^3}{}$$

$$=0$$

Setting the numerator to zero gives

$$\begin{aligned} & -x^4a_2-x^4a_3+x^4b_2+x^4b_3+2x^2y^2a_2+2x^2y^2a_3-2x^2y^2b_2-2x^2y^2b_3 \\ & -y^4a_2-y^4a_3+y^4b_2+y^4b_3+2x^3a_2-x^3b_2-x^3b_3-x^2ya_2+3x^2ya_3 \\ & -2x^2yb_2-2x\,y^2a_3+3x\,y^2b_2-x\,y^2b_3-y^3a_2-y^3a_3+2y^3b_3 \\ & +x^2a_1+x^2b_1-2xya_1-2xyb_1+y^2a_1+y^2b_1-xb_1+ya_1=0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -a_2v_1^4+2a_2v_1^2v_2^2-a_2v_2^4-a_3v_1^4+2a_3v_1^2v_2^2-a_3v_2^4+b_2v_1^4-2b_2v_1^2v_2^2 \\ & +b_2v_2^4+b_3v_1^4-2b_3v_1^2v_2^2+b_3v_2^4+2a_2v_1^3-a_2v_1^2v_2-a_2v_2^3+3a_3v_1^2v_2 \\ & -2a_3v_1v_2^2-a_3v_2^3-b_2v_1^3-2b_2v_1^2v_2+3b_2v_1v_2^2-b_3v_1^3-b_3v_1v_2^2+2b_3v_2^3 \\ & +a_1v_1^2-2a_1v_1v_2+a_1v_2^2+b_1v_1^2-2b_1v_1v_2+b_1v_2^2+a_1v_2-b_1v_1=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(-a_2 - a_3 + b_2 + b_3)\, v_1^4 + (2a_2 - b_2 - b_3)\, v_1^3 + (2a_2 + 2a_3 - 2b_2 - 2b_3)\, v_1^2 v_2^2 \\&+ (-a_2 + 3a_3 - 2b_2)\, v_1^2 v_2 + (a_1 + b_1)\, v_1^2 + (-2a_3 + 3b_2 - b_3)\, v_1 v_2^2 \\&+ (-2a_1 - 2b_1)\, v_1 v_2 - b_1 v_1 + (-a_2 - a_3 + b_2 + b_3)\, v_2^4 \\&+ (-a_2 - a_3 + 2b_3)\, v_2^3 + (a_1 + b_1)\, v_2^2 + a_1 v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}a_1 &= 0 \\-b_1 &= 0 \\-2a_1 - 2b_1 &= 0 \\a_1 + b_1 &= 0 \\-a_2 - a_3 + 2b_3 &= 0 \\-a_2 + 3a_3 - 2b_2 &= 0 \\2a_2 - b_2 - b_3 &= 0 \\-2a_3 + 3b_2 - b_3 &= 0 \\-a_2 - a_3 + b_2 + b_3 &= 0 \\2a_2 + 2a_3 - 2b_2 - 2b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0 \\a_2 &= b_3 \\a_3 &= b_3 \\b_1 &= 0 \\b_2 &= b_3 \\b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x + y \\\eta &= x + y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= x + y - \left(\frac{-x^2+y^2+y}{-x^2+y^2+x}\right)(x+y) \\
&= \frac{-x^2+y^2}{x^2-y^2-x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2+y^2}{x^2-y^2-x}} dy
\end{aligned}
$$

Which results in

$$
S = -y + \frac{\ln(x+y)}{2} - \frac{\ln(-x+y)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{-x^2+y^2+y}{-x^2+y^2+x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y}{x^2 - y^2} \\
S_y &= -1 + \frac{1}{2y + 2x} + \frac{1}{-2y + 2x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -1 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -1
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -R + c_1 \tag{4}
$$

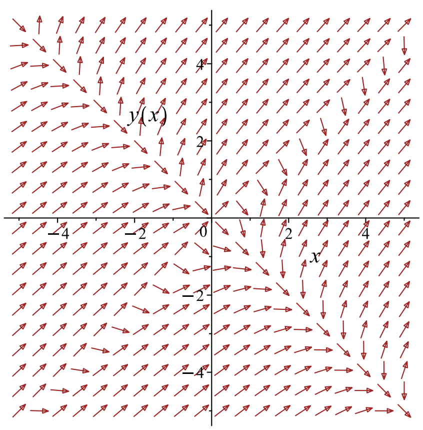
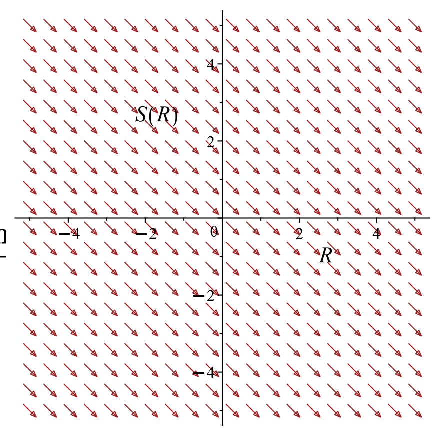
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-y + \frac{\ln(x + y)}{2} - \frac{\ln(-x + y)}{2} = -x + c_1
$$

Which simplifies to

$$
-y + \frac{\ln(x + y)}{2} - \frac{\ln(-x + y)}{2} = -x + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x^2+y^2+y}{-x^2+y^2+x}$ 	$R = x$ $S = -y + \frac{\ln(x+y)}{2} - \ln$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$-y + \frac{\ln(x+y)}{2} - \frac{\ln(-x+y)}{2} = -x + c_1 \tag{1}$$

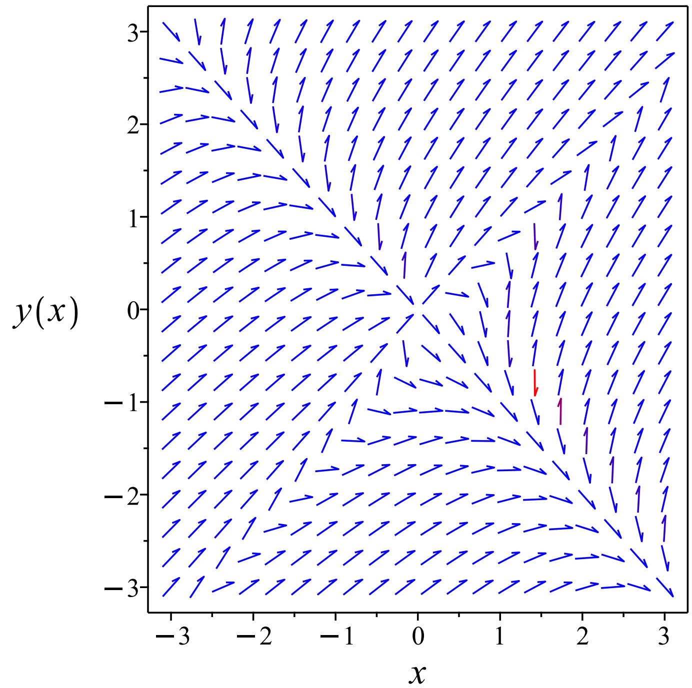


Figure 90: Slope field plot

Verification of solutions

$$-y+\frac{\ln (x+y)}{2}-\frac{\ln (-x+y)}{2}=-x+c_1$$

Verified OK.

4.14.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (-x^2 + y^2 + x)\,\mathrm{d}y &= (-x^2 + y^2 + y)\,\mathrm{d}x \\ (x^2 - y^2 - y)\,\mathrm{d}x + (-x^2 + y^2 + x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= x^2 - y^2 - y \\ N(x, y) &= -x^2 + y^2 + x \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^2 - y^2 - y) \\ &= -2y - 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^2 + y^2 + x) \\ &= 1 - 2x \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2-y^2}$ is an integrating factor. Therefore by multiplying $M = x^2 - y^2 - y$ and $N = -x^2 + y^2 + x$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{x^2 - y^2 - y}{x^2 - y^2}$$

$$N = \frac{-x^2 + y^2 + x}{x^2 - y^2}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-x^2+y^2+x}{x^2-y^2}\right) \mathrm{d}y = \left(-\frac{x^2-y^2-y}{x^2-y^2}\right) \mathrm{d}x$$

$$\left(\frac{x^2-y^2-y}{x^2-y^2}\right) \mathrm{d}x + \left(\frac{-x^2+y^2+x}{x^2-y^2}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= \frac{x^2-y^2-y}{x^2-y^2} \\ N(x,y) &= \frac{-x^2+y^2+x}{x^2-y^2} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x^2-y^2-y}{x^2-y^2}\right) \\ &= \frac{-x^2-y^2}{\left(x^2-y^2\right)^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x^2+y^2+x}{x^2-y^2}\right) \\ &= \frac{-x^2-y^2}{\left(x^2-y^2\right)^2} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathop{\mathrm{d}x} = \int M \mathop{\mathrm{d}x}$$

$$\int \frac{\partial \phi}{\partial x} \mathop{\mathrm{d}x} = \int \frac{x^2 - y^2 - y}{x^2 - y^2} \mathop{\mathrm{d}x}$$

$$\phi = x + \frac{\ln(x+y)}{2} - \frac{\ln(x-y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{2y + 2x} + \frac{1}{-2y + 2x} + f'(y) \tag{4}$$

$$= \frac{x}{x^2 - y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x^2 + y^2 + x}{x^2 - y^2}$. Therefore equation (4) becomes

$$\frac{-x^2 + y^2 + x}{x^2 - y^2} = \frac{x}{x^2 - y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -1$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathop{\mathrm{d}y} = \int (-1) \mathop{\mathrm{d}y}$$

$$f(y) = -y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x + \frac{\ln(x+y)}{2} - \frac{\ln(x-y)}{2} - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x + \frac{\ln(x+y)}{2} - \frac{\ln(x-y)}{2} - y$$

Summary
The solution(s) found are the following

$$x + \frac{\ln(x+y)}{2} - \frac{\ln(x-y)}{2} - y = c_1 \tag{1}$$

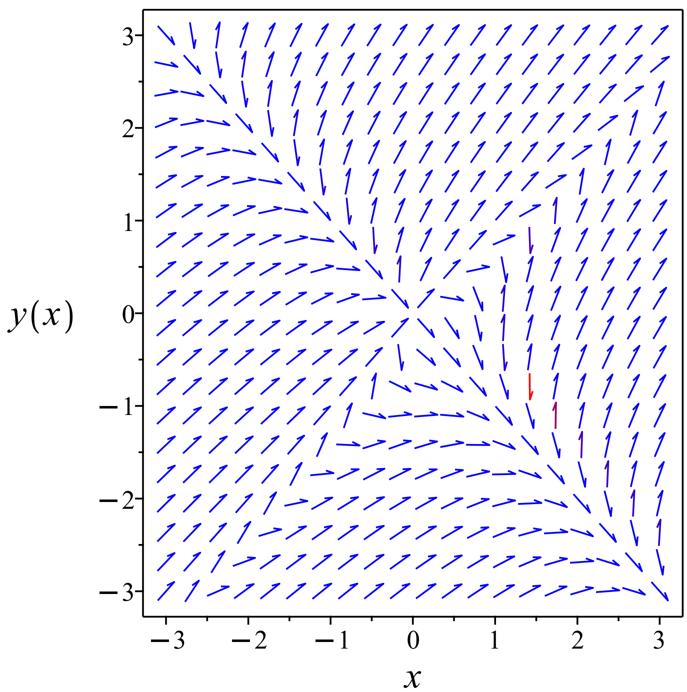


Figure 91: Slope field plot

Verification of solutions

$$x + \frac{\ln(x+y)}{2} - \frac{\ln(x-y)}{2} - y = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = 1, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 28

```
dsolve((x^2-y(x)^2-y(x))-(x^2-y(x)^2-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$2y(x) - \ln(y(x) + x) + \ln(y(x) - x) - 2x - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.242 (sec). Leaf size: 32

```
DSolve[(x^2-y[x]^2-y[x])-(x^2-y[x]^2-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{e^{2x-2y(x)}(y(x)+x)}{2(x-y(x))} = c_1, y(x)\right]$$

4.15 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.7, page 90

Internal problem ID [4482]
Internal file name [`OUTPUT/3975_Sunday_June_05_2022_11_57_54_AM_10593353/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.7, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{y^2x^4 - y + (y^4x^2 - x)\, y' = 0}$$

4.15.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y^4x^2 - x)\,\mathrm{d}y &= (-y^2x^4 + y)\,\mathrm{d}x\\ (y^2x^4 - y)\,\mathrm{d}x + (y^4x^2 - x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y^2x^4 - y\\ N(x, y) &= y^4x^2 - x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y^2x^4 - y)\\ &= 2y\,x^4 - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^4x^2 - x)\\ &= 2y^4x - 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x\,(y^4x-1)}\left(\left(2y\,x^4-1\right) - \left(2y^4x-1\right)\right) \\
&= \frac{2y\,x^3-2y^4}{y^4x-1}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y\,(y\,x^4-1)}\left(\left(2y^4x-1\right) - \left(2y\,x^4-1\right)\right) \\
&= \frac{-2x^4+2x\,y^3}{y\,x^4-1}
\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(2y^4x-1) - (2y\,x^4-1)}{x\,(y^2x^4-y) - y\,(y^4x^2-x)} \\
&= -\frac{2}{yx}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{x^2 y^2}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2 y^2}\left(y^2 x^4 - y\right) \\
&= \frac{y\,x^4 - 1}{y\,x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2 y^2}\left(y^4 x^2 - x\right) \\
&= \frac{y^4 x - 1}{x\,y^2}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y\,x^4 - 1}{y\,x^2}\right) + \left(\frac{y^4 x - 1}{x\,y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{y\,x^4 - 1}{y\,x^2} \,\mathrm{d}x$$
$$\phi = \frac{y\,x^4 + 3}{3xy} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x^3}{3y} - \frac{y\,x^4 + 3}{3x\,y^2} + f'(y) \tag{4}$$
$$= -\frac{1}{y^2 x} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^4 x - 1}{x\,y^2}$. Therefore equation (4) becomes

$$\frac{y^4 x - 1}{x\,y^2} = -\frac{1}{y^2 x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (y^2) \,\mathrm{d}y$$
$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y\,x^4 + 3}{3xy} + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y\,x^4+3}{3xy} + \frac{y^3}{3}$$

Summary
The solution(s) found are the following

$$\frac{yx^4+3}{3xy} + \frac{y^3}{3} = c_1 \tag{1}$$

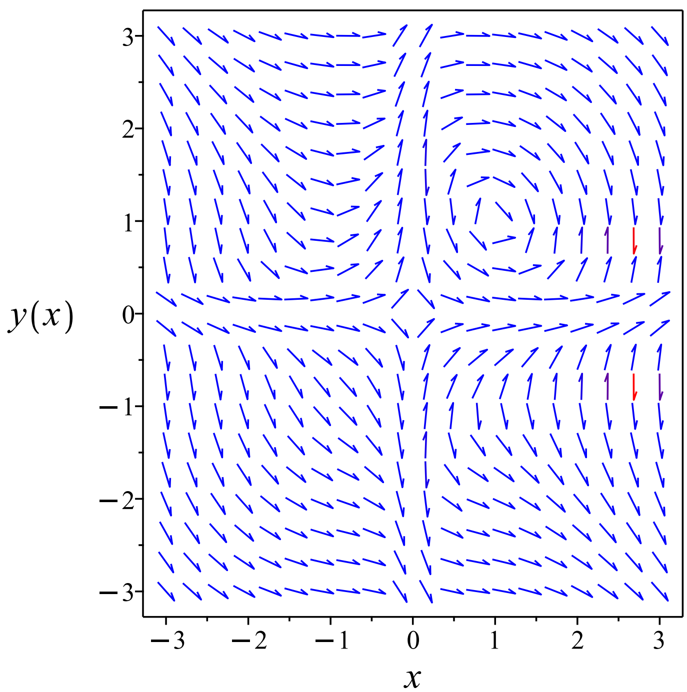


Figure 92: Slope field plot

Verification of solutions

$$\frac{yx^4+3}{3xy} + \frac{y^3}{3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2`[0, x*y^2/(x*y^4-1)], [0, (x^5*y^2+x^2*y^5+3*x*y)
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve((x^4*y(x)^2-y(x))+(x^2*y(x)^4-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$-\frac{x^3}{3} - \frac{1}{y(x)\,x} - \frac{y(x)^3}{3} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 60.131 (sec). Leaf size: 1507

```
DSolve[(x^4*y[x]^2-y[x])+(x^2*y[x]^4-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(\sqrt{2}\sqrt{\frac{8\sqrt[3]{2}x+2^{2/3}\left(x^9-6c_1x^6+9c_1^2x^3+\sqrt{x^2\left(-256x+\left(x^4-3c_1x\right)^4\right)}\right)^{2/3}}{x\sqrt[3]{x^9-6c_1x^6+9c_1^2x^3+\sqrt{x^2\left(-256x+\left(x^4-3c_1x\right)^4\right)}}}}\right.$$

$$-2\sqrt{-\frac{\sqrt[3]{x\left(x^4-3c_1x\right)^2+\sqrt{x^2\left(-256x+\left(x^4-3c_1x\right)^4\right)}}}{\sqrt[3]{2}x}-\frac{2\sqrt{2}\left(x^3-3c_1\right)}{\sqrt{\frac{8\sqrt[3]{2}x+2^{2/3}\left(x^9-6c_1x^6+9c_1^2x^3+\sqrt{x^2\left(-256x+\right.}\right.}{x\sqrt[3]{x^9-6c_1x^6+9c_1^2x^3+\sqrt{x^2\left(-256x\right.}}}}}}$$

$$y(x) \to \frac{1}{4}\left(\sqrt{2}\sqrt{\frac{8\sqrt[3]{2}x+2^{2/3}\left(x^9-6c_1x^6+9c_1^2x^3+\sqrt{x^2\left(-256x+\left(x^4-3c_1x\right)^4\right)}\right)^{2/3}}{x\sqrt[3]{x^9-6c_1x^6+9c_1^2x^3+\sqrt{x^2\left(-256x+\left(x^4-3c_1x\right)^4\right)}}}}\right.$$

$$+2\sqrt{-\frac{\sqrt[3]{x\left(x^4-3c_1x\right)^2+\sqrt{x^2\left(-256x+\left(x^4-3c_1x\right)^4\right)}}}{\sqrt[3]{2}x}-\frac{2\sqrt{2}\left(x^3-3c_1\right)}{\sqrt{\frac{8\sqrt[3]{2}x+2^{2/3}\left(x^9-6c_1x^6+9c_1^2x^3+\sqrt{x^2\left(-256x+\right.}\right.}{x\sqrt[3]{x^9-6c_1x^6+9c_1^2x^3+\sqrt{x^2\left(-256x\right.}}}}}}$$

$$y(x) \to \frac{1}{4}\left(-\sqrt{2}\sqrt{\frac{8\sqrt[3]{2}x+2^{2/3}\left(x^9-6c_1x^6+9c_1^2x^3+\sqrt{x^2\left(-256x+\left(x^4-3c_1x\right)^4\right)}\right)^{2/3}}{x\sqrt[3]{x^9-6c_1x^6+9c_1^2x^3+\sqrt{x^2\left(-256x+\left(x^4-3c_1x\right)^4\right)}}}}\right.$$

$$-2\sqrt{-\frac{\sqrt[3]{x\left(x^4-3c_1x\right)^2+\sqrt{x^2\left(-256x+\left(x^4-3c_1x\right)^4\right)}}}{\sqrt[3]{2}x}+\frac{2\sqrt{2}\left(x^3-3c_1\right)}{\sqrt{\frac{8\sqrt[3]{2}x+2^{2/3}\left(x^9-6c_1x^6+9c_1^2x^3+\sqrt{x^2\left(-256x+\right.}\right.}{x\sqrt[3]{x^9-6c_1x^6+9c_1^2x^3+\sqrt{x^2\left(-256x\right.}}}}}}$$

$y(x)$

4.16 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.8, page 90

Internal problem ID [4483]
Internal file name [`OUTPUT/3976_Sunday_June_05_2022_11_58_01_AM_29687728/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.8, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y\left(2x+y^{3}\right)-x\left(2x-y^{3}\right)y'=0}$$

4.16.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x(-y^3 + 2x))\,\mathrm{d}y &= (-y(y^3 + 2x))\,\mathrm{d}x\\ (y(y^3 + 2x))\,\mathrm{d}x + (-x(-y^3 + 2x))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y(y^3 + 2x)\\ N(x, y) &= -x(-y^3 + 2x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(y^3 + 2x))\\ &= 4y^3 + 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x(-y^3 + 2x))\\ &= y^3 - 4x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{-x\,y^3 + 2x^2}((4y^3 + 2x) - (y^3 - 4x))\\ &= \frac{-3y^3 - 6x}{-x\,y^3 + 2x^2}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y^4 + 2xy}\left(\left(y^3 - 4x\right) - \left(4y^3 + 2x\right)\right) \\
&= -\frac{3}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{3}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(y)} \\
&= \frac{1}{y^3}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^3}\left(y\left(y^3 + 2x\right)\right) \\
&= \frac{y^3 + 2x}{y^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^3}\left(-x\left(-y^3 + 2x\right)\right) \\
&= \frac{x\,y^3 - 2x^2}{y^3}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y^3 + 2x}{y^2}\right) + \left(\frac{x\,y^3 - 2x^2}{y^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \overline{M} \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int \frac{y^3 + 2x}{y^2} \, dx$$
$$\phi = \frac{x(y^3 + x)}{y^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3x - \frac{2x(y^3 + x)}{y^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x\,y^3 - 2x^2}{y^3}$. Therefore equation (4) becomes

$$\frac{x\,y^3 - 2x^2}{y^3} = \frac{x\,y^3 - 2x^2}{y^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(y^3 + x)}{y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(y^3 + x)}{y^2}$$

Summary
The solution(s) found are the following

$$\frac{x(y^3 + x)}{y^2} = c_1 \tag{1}$$

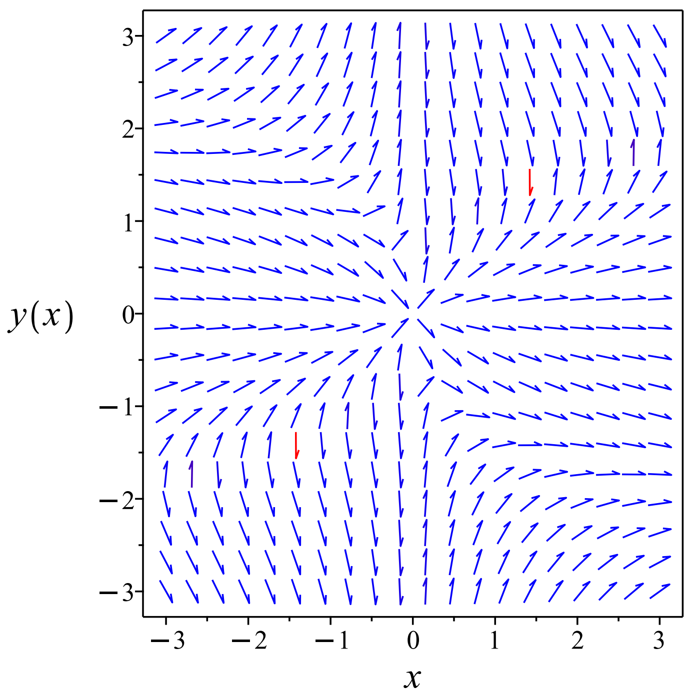


Figure 93: Slope field plot

Verification of solutions

$$\frac{x(y^3 + x)}{y^2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 330

```
dsolve((y(x)*(2*x+y(x)^3))-(x*(2*x-y(x)^3))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\frac{\left(-108x^4+12\sqrt{81x^4-12c_1^3}\,x^2+8c_1^3\right)^{\frac{1}{3}}}{2} + \frac{2c_1^2}{\left(-108x^4+12\sqrt{81x^4-12c_1^3}\,x^2+8c_1^3\right)^{\frac{1}{3}}} + c_1}{3x}$$

$$y(x) = \frac{\left(-i\sqrt{3}-1\right)\left(-108x^4+12\sqrt{81x^4-12c_1^3}\,x^2+8c_1^3\right)^{\frac{2}{3}} + 4\left(ic_1\sqrt{3}-c_1+\left(-108x^4+12\sqrt{81x^4-12c_1^3}\,x^2\right.\right.}{12\left(-108x^4+12\sqrt{81x^4-12c_1^3}\,x^2+8c_1^3\right)^{\frac{1}{3}}x}$$

$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left(-108x^4+12\sqrt{81x^4-12c_1^3}\,x^2+8c_1^3\right)^{\frac{2}{3}} + 4\left(-ic_1\sqrt{3}-c_1+\left(-108x^4+12\sqrt{81x^4-12c_1^3}\,x^2\right.\right.}{12\left(-108x^4+12\sqrt{81x^4-12c_1^3}\,x^2+8c_1^3\right)^{\frac{1}{3}}x}$$

✓ Solution by Mathematica

Time used: 11.386 (sec). Leaf size: 371

```
DSolve[(y[x]*(2*x+y[x]^3))-(x*(2*x-y[x]^3))*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\frac{\frac{2\sqrt[3]{2}c_1{}^2}{\sqrt[3]{27x^4+3\sqrt{81x^8+12c_1{}^3x^4}+2c_1{}^3}}+2^{2/3}\sqrt[3]{27x^4+3\sqrt{81x^8+12c_1{}^3x^4}+2c_1{}^3}+2c_1}{6x}$$

$$y(x) \to \frac{\frac{2\sqrt[3]{2}\left(1+i\sqrt{3}\right)c_1{}^2}{\sqrt[3]{27x^4+3\sqrt{81x^8+12c_1{}^3x^4}+2c_1{}^3}}+2^{2/3}\left(1-i\sqrt{3}\right)\sqrt[3]{27x^4+3\sqrt{81x^8+12c_1{}^3x^4}+2c_1{}^3}-4c_1}{12x}$$

$$y(x) \to \frac{\frac{2\sqrt[3]{2}\left(1-i\sqrt{3}\right)c_1{}^2}{\sqrt[3]{27x^4+3\sqrt{81x^8+12c_1{}^3x^4}+2c_1{}^3}}+2^{2/3}\left(1+i\sqrt{3}\right)\sqrt[3]{27x^4+3\sqrt{81x^8+12c_1{}^3x^4}+2c_1{}^3}-4c_1}{12x}$$

$$y(x) \to 0$$

4.17 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.9, page 90

Internal problem ID [4484]
Internal file name [`OUTPUT/3977_Sunday_June_05_2022_11_58_11_AM_15250014/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.9, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{\arctan(xy) + \frac{xy - 2xy^2}{y^2x^2 + 1} + \frac{(x^2 - 2yx^2)\, y'}{y^2x^2 + 1} = 0}$$

4.17.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-2y\,x^2 + x^2}{y^2x^2 + 1}\right) \mathrm{d}y = \left(-\arctan(xy) - \frac{-2y^2x + xy}{y^2x^2 + 1}\right) \mathrm{d}x$$
$$\left(\arctan(xy) + \frac{-2y^2x + xy}{y^2x^2 + 1}\right) \mathrm{d}x + \left(\frac{-2y\,x^2 + x^2}{y^2x^2 + 1}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \arctan(xy) + \frac{-2y^2x + xy}{y^2x^2 + 1}$$
$$N(x, y) = \frac{-2y\,x^2 + x^2}{y^2x^2 + 1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\arctan(xy) + \frac{-2y^2x + xy}{y^2x^2 + 1}\right)\\ &= \frac{(-4y + 2)\,x}{(y^2x^2 + 1)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-2y\,x^2+x^2}{y^2x^2+1}\right)\\ &= \frac{(-4y+2)\,x}{(y^2x^2+1)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$
$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int\arctan(xy)+\frac{-2y^2x+xy}{y^2x^2+1}\,\mathrm{d}x$$
$$\phi=-\ln\left(y^2x^2+1\right)+x\arctan(xy)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= -\frac{2y\,x^2}{y^2x^2+1}+\frac{x^2}{y^2x^2+1}+f'(y) \qquad (4)\\ &= \frac{(-2y+1)\,x^2}{y^2x^2+1}+f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\frac{-2y\,x^2+x^2}{y^2x^2+1}$. Therefore equation (4) becomes

$$\frac{-2y\,x^2+x^2}{y^2x^2+1}=\frac{(-2y+1)\,x^2}{y^2x^2+1}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln\left(y^2x^2 + 1\right) + x\arctan(xy) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln\left(y^2x^2 + 1\right) + x\arctan(xy)$$

Summary

The solution(s) found are the following

$$-\ln\left(y^2x^2 + 1\right) + x\arctan(xy) = c_1 \tag{1}$$

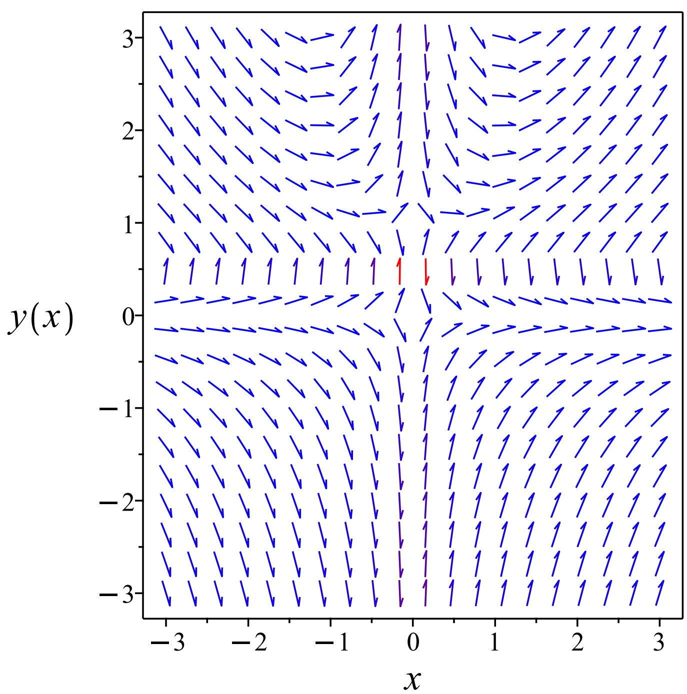


Figure 94: Slope field plot

Verification of solutions

$$-\ln\left(y^2x^2 + 1\right) + x\arctan(xy) = c_1$$

Verified OK.

4.17.2 Maple step by step solution

Let's solve

$\arctan(xy) + \frac{xy-2xy^2}{y^2x^2+1} + \frac{(x^2-2yx^2)y'}{y^2x^2+1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  - Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$

  - Evaluate derivatives

    $\frac{x}{y^2x^2+1} + \frac{-4xy+x}{y^2x^2+1} - \frac{2(-2y^2x+xy)y\,x^2}{(y^2x^2+1)^2} = \frac{-4xy+2x}{y^2x^2+1} - \frac{2(-2y\,x^2+x^2)y^2x}{(y^2x^2+1)^2}$

  - Simplify

    $\frac{(-4y+2)x}{(y^2x^2+1)^2} = \frac{(-4y+2)x}{(y^2x^2+1)^2}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int\left(\arctan(xy) + \frac{-2y^2x+xy}{y^2x^2+1}\right)dx + f_1(y)$

- Evaluate integral

  $F(x,y) = -\ln(y^2x^2+1) + x\arctan(xy) + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $\frac{-2y\,x^2+x^2}{y^2x^2+1} = -\frac{2y\,x^2}{y^2x^2+1} + \frac{x^2}{y^2x^2+1} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = \frac{-2y\,x^2+x^2}{y^2x^2+1} + \frac{2y\,x^2}{y^2x^2+1} - \frac{x^2}{y^2x^2+1}$

- Solve for $f_1(y)$

  $f_1(y) = 0$
- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = -\ln(y^2x^2+1) + x\arctan(xy)$
- Substitute $F(x,y)$ into the solution of the ODE

  $-\ln(y^2x^2+1) + x\arctan(xy) = c_1$
- Solve for y

  $$y = \frac{\tan\left(RootOf\left(-x\_Z + \ln\left(\tan(\_Z)^2+1\right)+c_1\right)\right)}{x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 22

```
dsolve((arctan(x*y(x))+(x*y(x)-2*x*y(x)^2)/(1+x^2*y(x)^2))+((x^2-2*x^2*y(x))/(1+x^2*y(x)^2))
```

$$y(x) = \frac{\tan\left(\operatorname{RootOf}\left(x\_Z - \ln\left(\sec(\_Z)^2\right) + c_1\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.173 (sec). Leaf size: 26

```
DSolve[(ArcTan[x*y[x]]+(x*y[x]-2*x*y[x]^2)/(1+x^2*y[x]^2))+((x^2-2*x^2*y[x])/(1+x^2*y[x]^2))
```

$$\text{Solve}\left[\log\left(x^2y(x)^2+1\right)-x\arctan(xy(x))=c_1,y(x)\right]$$

4.18 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.10, page 90

Internal problem ID [4485]
Internal file name [`OUTPUT/3978_Sunday_June_05_2022_11_58_18_AM_19816982/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.10, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`` [`y=_G(x,y')`] ``

$$\boxed{(\mathrm{e}^{y} y-\mathrm{e}^{x} x)\, y'=-\mathrm{e}^{x} (x+1)}$$

4.18.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^y y - \mathrm{e}^x x)\,\mathrm{d}y &= (-\mathrm{e}^x(x+1))\,\mathrm{d}x\\ (\mathrm{e}^x(x+1))\,\mathrm{d}x + (\mathrm{e}^y y - \mathrm{e}^x x)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \mathrm{e}^x(x+1)\\ N(x,y) &= \mathrm{e}^y y - \mathrm{e}^x x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\mathrm{e}^x(x+1))\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^y y - \mathrm{e}^x x)\\ &= \mathrm{e}^x(-1-x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{\mathrm{e}^x x - \mathrm{e}^y y}((0) - (-\mathrm{e}^x x - \mathrm{e}^x))\\ &= -\frac{\mathrm{e}^x(x+1)}{\mathrm{e}^x x - \mathrm{e}^y y}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{\mathrm{e}^{-x}}{x+1}((-\mathrm{e}^{x} x - \mathrm{e}^{x}) - (0)) \\
&= -1
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int -1 \,\mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-y} \\
&= \mathrm{e}^{-y}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-y}(\mathrm{e}^{x}(x+1)) \\
&= (x+1)\,\mathrm{e}^{x-y}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-y}(\mathrm{e}^{y} y - \mathrm{e}^{x} x) \\
&= y - x\,\mathrm{e}^{x-y}
\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left((x+1)\,\mathrm{e}^{x-y}\right) + \left(y - x\,\mathrm{e}^{x-y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int \overline{M}\, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int (x+1)\,\mathrm{e}^{x-y}\, \mathrm{d}x$$

$$\phi = x\,\mathrm{e}^{x-y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x\,\mathrm{e}^{x-y} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y - x\,\mathrm{e}^{x-y}$. Therefore equation (4) becomes

$$y - x\,\mathrm{e}^{x-y} = -x\,\mathrm{e}^{x-y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int (y)\, \mathrm{d}y$$

$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x\,\mathrm{e}^{x-y} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\,\mathrm{e}^{x-y} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$x\,\mathrm{e}^{x-y} + \frac{y^2}{2} = c_1 \tag{1}$$

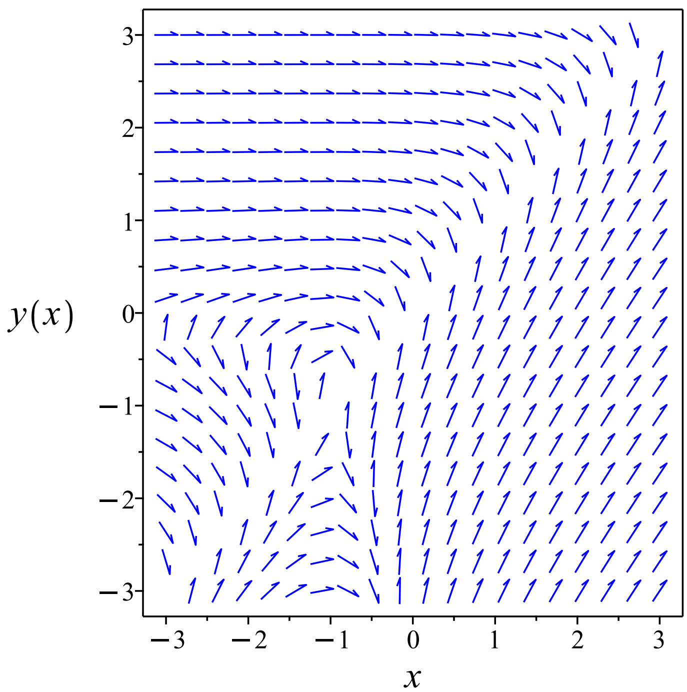


Figure 95: Slope field plot

Verification of solutions

$$x\,\mathrm{e}^{x-y} + \frac{y^2}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve((exp(x)*(x+1))+(y(x)*exp(y(x))-x*exp(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$x\,\mathrm{e}^{-y(x)+x}+\frac{y(x)^2}{2}+c_1=0$$

✓ Solution by Mathematica

Time used: 0.291 (sec). Leaf size: 26

```
DSolve[(Exp[x]*(x+1))+(y[x]*Exp[y[x]]-x*Exp[x])*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$\text{Solve}\left[-\frac{1}{2}y(x)^2-xe^{x-y(x)}=c_1,y(x)\right]$$

4.19 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.11, page 90

Internal problem ID [4486]
Internal file name [`OUTPUT/3979_Sunday_June_05_2022_11_58_27_AM_86151855/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.11, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _exact, _rational, [_Abel, `2nd
  type`, `class A`]]
```

$$\boxed{\frac{xy+1}{y}+\frac{(-x+2y)\,y'}{y^2}=0}$$

4.19.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{-x + 2y}{y^2}\right)\mathrm{d}y &= \left(-\frac{xy + 1}{y}\right)\mathrm{d}x\\ \left(\frac{xy + 1}{y}\right)\mathrm{d}x + \left(\frac{-x + 2y}{y^2}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \frac{xy + 1}{y}\\ N(x, y) &= \frac{-x + 2y}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{xy + 1}{y}\right)\\ &= -\frac{1}{y^2}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x+2y}{y^2}\right) \\
&= -\frac{1}{y^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{xy+1}{y} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x(xy+2)}{2y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{x^2}{2y} - \frac{x(xy+2)}{2y^2} + f'(y) \tag{4}
$$

$$
= -\frac{x}{y^2} + f'(y)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x+2y}{y^2}$. Therefore equation (4) becomes

$$
\frac{-x+2y}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{2}{y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{2}{y}\right)\mathrm{d}y$$
$$f(y) = 2\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(xy+2)}{2y} + 2\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(xy+2)}{2y} + 2\ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\operatorname{LambertW}\left(-\frac{x\,\mathrm{e}^{\frac{x^2}{4}-\frac{c_1}{2}}}{2}\right)-\frac{x^2}{4}+\frac{c_1}{2}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\operatorname{LambertW}\left(-\frac{x\,\mathrm{e}^{\frac{x^2}{4}-\frac{c_1}{2}}}{2}\right)-\frac{x^2}{4}+\frac{c_1}{2}} \tag{1}$$

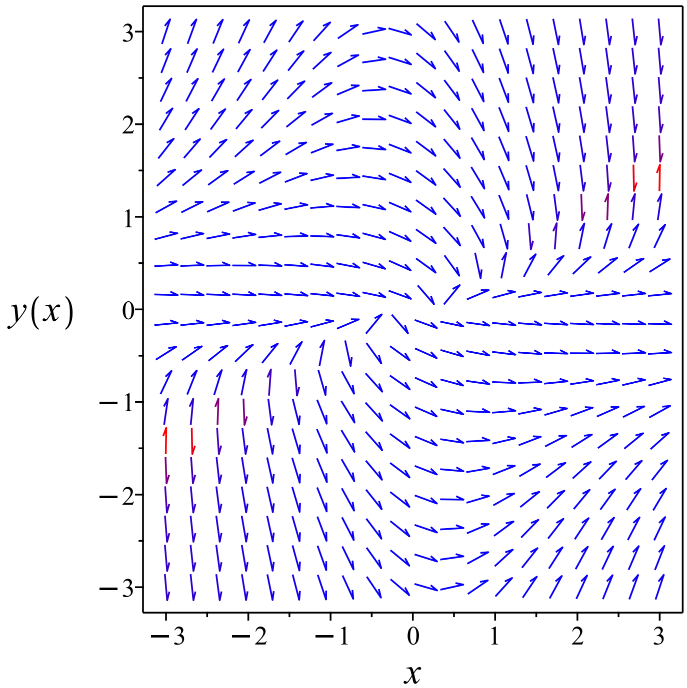


Figure 96: Slope field plot

Verification of solutions

$$y = e^{\mathrm{LambertW}\left(-\frac{x\,e^{\frac{x^2}{4}-\frac{c_1}{2}}}{2}\right)-\frac{x^2}{4}+\frac{c_1}{2}}$$

Verified OK.

4.19.2 Maple step by step solution

Let's solve

$\frac{xy+1}{y} + \frac{(-x+2y)y'}{y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ◦ Compute derivative of lhs

$F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $-\frac{xy+1}{y^2} + \frac{x}{y} = -\frac{1}{y^2}$

- Simplify

  $-\frac{1}{y^2} = -\frac{1}{y^2}$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \frac{xy+1}{y} dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{\frac{1}{2} y\, x^2 + x}{y} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $\frac{-x+2y}{y^2} = -\frac{\frac{1}{2} y\, x^2 + x}{y^2} + \frac{x^2}{2y} + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = \frac{-x+2y}{y^2} + \frac{\frac{1}{2} y\, x^2 + x}{y^2} - \frac{x^2}{2y}$

- Solve for $f_1(y)$

  $f_1(y) = 2 \ln(y)$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = \frac{\frac{1}{2} y\, x^2 + x}{y} + 2 \ln(y)$

- Substitute $F(x, y)$ into the solution of the ODE

  $\frac{\frac{1}{2} y\, x^2 + x}{y} + 2 \ln(y) = c_1$

- Solve for y

  $y = \mathrm{e}^{\mathit{LambertW}\left(-\frac{x\, \mathrm{e}^{\frac{x^2}{4} - \frac{c_1}{2}}}{2}\right) - \frac{x^2}{4} + \frac{c_1}{2}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(((x*y(x)+1)/y(x))+((2*y(x)-x)/y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{2 \operatorname{LambertW}\left(-\frac{e^{\frac{x^2}{4}} c_1 x}{2}\right)}$$

✓ Solution by Mathematica

Time used: 3.618 (sec). Leaf size: 37

```
DSolve[((x*y[x]+1)/y[x])+((2*y[x]-x)/y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to -\frac{x}{2W\left(-\frac{1}{2}xe^{\frac{1}{4}(x^2-2c_1)}\right)}$$
$$y(x) \to 0$$

4.20 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.12, page 90

Internal problem ID [4487]
Internal file name [`OUTPUT/3980_Sunday_June_05_2022_11_58_34_AM_44323298/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.12, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y^2 - 3xy + (xy - x^2)\, y' = 2x^2}$$

4.20.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x^2 + xy)\,\mathrm{d}y &= (2x^2 + 3xy - y^2)\,\mathrm{d}x \\ (-2x^2 - 3xy + y^2)\,\mathrm{d}x + (-x^2 + xy)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -2x^2 - 3xy + y^2 \\ N(x, y) &= -x^2 + xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2x^2 - 3xy + y^2) \\ &= -3x + 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^2 + xy) \\ &= -2x + y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{x(x-y)}((-3x+2y)-(-2x+y)) \\
&= \frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int \frac{1}{x} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(x)} \\
&= x
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x\left(-2x^2 - 3xy + y^2\right) \\
&= -2x^3 - 3y\,x^2 + y^2 x
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x\left(-x^2 + xy\right) \\
&= -x^2(x-y)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-2x^3 - 3y\,x^2 + y^2 x\right) + \left(-x^2(x-y)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -2x^3 - 3y\,x^2 + y^2 x \,\mathrm{d}x$$
$$\phi = -\frac{1}{2}x^4 - y\,x^3 + \frac{1}{2}y^2x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x^3 + y\,x^2 + f'(y) \tag{4}$$
$$= -x^2(x-y) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -x^2(x-y)$. Therefore equation (4) becomes

$$-x^2(x-y) = -x^2(x-y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{2}x^4 - y\,x^3 + \frac{1}{2}y^2x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{2}x^4 - y\,x^3 + \frac{1}{2}y^2x^2$$

Summary

The solution(s) found are the following

$$-\frac{x^4}{2} - yx^3 + \frac{y^2x^2}{2} = c_1 \tag{1}$$

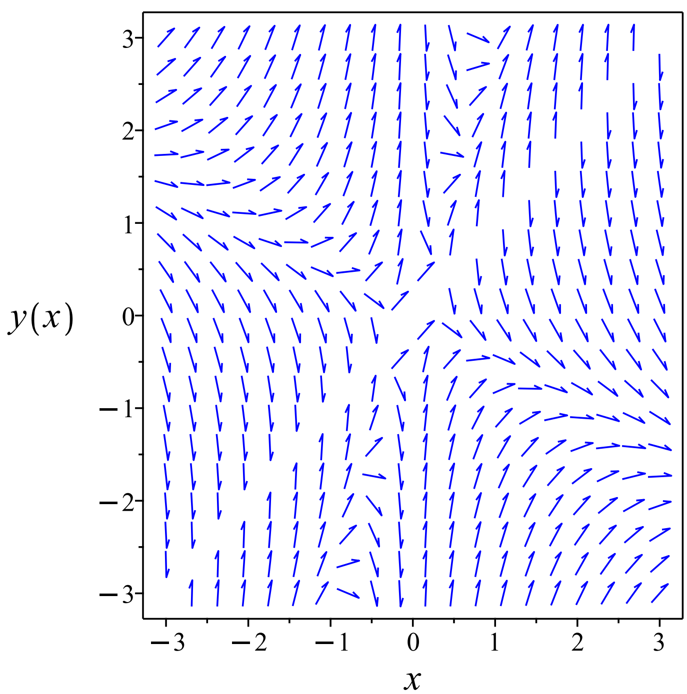


Figure 97: Slope field plot

Verification of solutions

$$-\frac{x^4}{2} - yx^3 + \frac{y^2x^2}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 59

```
dsolve((y(x)^2-3*x*y(x)-2*x^2)+(x*y(x)-x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1x^2 - \sqrt{2c_1^2x^4 + 1}}{c_1x}$$
$$y(x) = \frac{c_1x^2 + \sqrt{2c_1^2x^4 + 1}}{c_1x}$$

✓ Solution by Mathematica

Time used: 0.657 (sec). Leaf size: 99

```
DSolve[(y[x]^2-3*x*y[x]-2*x^2)+(x*y[x]-x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to x - \frac{\sqrt{2x^4 + e^{2c_1}}}{x}$$
$$y(x) \to x + \frac{\sqrt{2x^4 + e^{2c_1}}}{x}$$
$$y(x) \to x - \frac{\sqrt{2}\sqrt{x^4}}{x}$$
$$y(x) \to \frac{\sqrt{2}\sqrt{x^4}}{x} + x$$

4.21 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.13, page 90

Internal problem ID [4488]
Internal file name [`OUTPUT/3981_Sunday_June_05_2022_11_58_41_AM_19643663/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.13, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{y(y+2x+1)-x(x+2y-1)\,y'=0}$$

4.21.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(-x(x - 1 + 2y))\,\mathrm{d}y &= (-y(y + 2x + 1))\,\mathrm{d}x \\
(y(y + 2x + 1))\,\mathrm{d}x + (-x(x - 1 + 2y))\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= y(y + 2x + 1) \\
N(x, y) &= -x(x - 1 + 2y)
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(y + 2x + 1)) \\
&= 2x + 2y + 1
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x(x - 1 + 2y)) \\
&= -2x - 2y + 1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{(x - 1 + 2y)\,x}((2x + 2y + 1) - (-2x - 2y + 1)) \\
&= \frac{-4x - 4y}{x\,(x - 1 + 2y)}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y\,(y+2x+1)}((-2x-2y+1)-(2x+2y+1)) \\
&= \frac{-4x-4y}{y\,(y+2x+1)}
\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(-2x-2y+1)-(2x+2y+1)}{x\,(y\,(y+2x+1)) - y\,(-x\,(x-1+2y))} \\
&= -\frac{4}{3yx}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{4}{3t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R\,\mathrm{d}t} \\
&= e^{\int \left(-\frac{4}{3t}\right)\mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-\frac{4\ln(t)}{3}} \\
&= \frac{1}{t^{\frac{4}{3}}}
\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{(xy)^{\frac{4}{3}}}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{(xy)^{\frac{4}{3}}}(y(y+2x+1)) \\
&= \frac{y+2x+1}{x\,(xy)^{\frac{1}{3}}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(xy)^{\frac{4}{3}}}(-x(x-1+2y)) \\
&= -\frac{x-1+2y}{y\,(xy)^{\frac{1}{3}}}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y+2x+1}{x\,(xy)^{\frac{1}{3}}}\right) + \left(-\frac{x-1+2y}{y\,(xy)^{\frac{1}{3}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{y+2x+1}{x\,(xy)^{\frac{1}{3}}}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{3x-3y-3}{(xy)^{\frac{1}{3}}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= -\frac{3}{(xy)^{\frac{1}{3}}} - \frac{(x-y-1)\,x}{(xy)^{\frac{4}{3}}} + f'(y) \\
&= -\frac{x-1+2y}{y\,(xy)^{\frac{1}{3}}} + f'(y)
\end{aligned} \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x-1+2y}{y(xy)^{\frac{1}{3}}}$. Therefore equation (4) becomes

$$
-\frac{x-1+2y}{y\,(xy)^{\frac{1}{3}}} = -\frac{x-1+2y}{y\,(xy)^{\frac{1}{3}}} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{3x-3y-3}{(xy)^{\frac{1}{3}}} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{3x-3y-3}{(xy)^{\frac{1}{3}}}
$$

Summary

The solution(s) found are the following

$$
\frac{3x-3y-3}{(xy)^{\frac{1}{3}}} = c_1 \tag{1}
$$

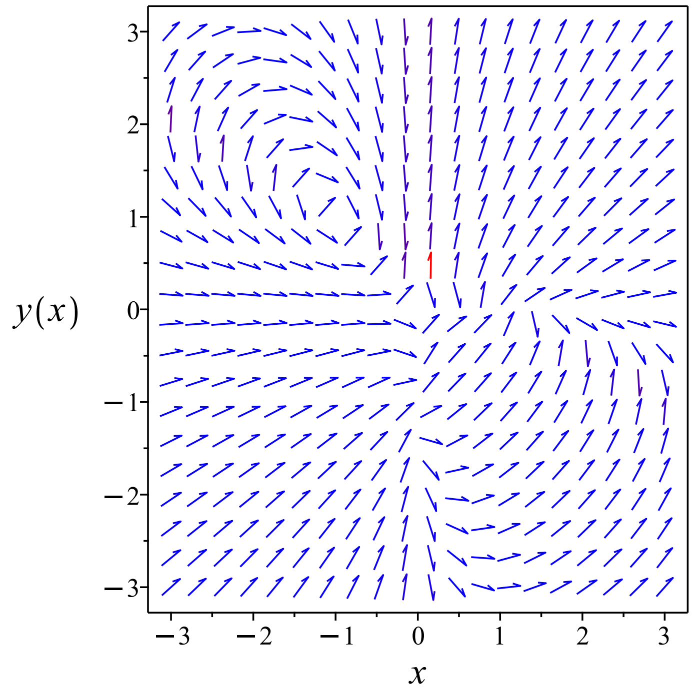


Figure 98: Slope field plot

Verification of solutions

$$\frac{3x - 3y - 3}{(xy)^{\frac{1}{3}}} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 389

```
dsolve((y(x)*(y(x)+2*x+1))-(x*(2*y(x)+x-1))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{3\,5^{\frac{1}{3}}\left(x\left(\sqrt{5}\,\sqrt{\frac{80c_1x^2-160c_1x+80c_1-x}{c_1}}+20x-20\right)c_1^2\right)^{\frac{1}{3}}}{40c_1} + \frac{3x5^{\frac{2}{3}}}{40\left(x\left(\sqrt{5}\,\sqrt{\frac{80c_1x^2-160c_1x+80c_1-x}{c_1}}+20x-20\right)c_1^2\right)^{\frac{1}{3}}} + x - 1$$

$$y(x) = \frac{\frac{3\,5^{\frac{1}{3}}\left(-i\sqrt{3}-1\right)\left(x\left(\sqrt{5}\,\sqrt{\frac{80(x-1)^2c_1-x}{c_1}}+20x-20\right)c_1^2\right)^{\frac{2}{3}}}{80} + \frac{3c_1\left(\frac{80(x-1)\left(x\left(\sqrt{5}\,\sqrt{\frac{80(x-1)^2c_1-x}{c_1}}+20x-20\right)c_1^2\right)^{\frac{1}{3}}}{3}+\left(i\sqrt{3}-1\right)5^{\frac{2}{3}}x\right)}{80}}{c_1\left(x\left(\sqrt{5}\,\sqrt{\frac{80(x-1)^2c_1-x}{c_1}}+20x-20\right)c_1^2\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{\frac{3\left(i\sqrt{3}-1\right)5^{\frac{1}{3}}\left(x\left(\sqrt{5}\,\sqrt{\frac{80(x-1)^2c_1-x}{c_1}}+20x-20\right)c_1^2\right)^{\frac{2}{3}}}{80} + \frac{3\left(-\frac{80(1-x)\left(x\left(\sqrt{5}\,\sqrt{\frac{80(x-1)^2c_1-x}{c_1}}+20x-20\right)c_1^2\right)^{\frac{1}{3}}}{3}+\left(-i\sqrt{3}-1\right)5^{\frac{2}{3}}x\right)c_1}{80}}{c_1\left(x\left(\sqrt{5}\,\sqrt{\frac{80(x-1)^2c_1-x}{c_1}}+20x-20\right)c_1^2\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 41.715 (sec). Leaf size: 463

```
DSolve[(y[x]*(y[x]+2*x+1))-(x*(2*y[x]+x-1))*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\frac{\sqrt[3]{2}x}{\sqrt[3]{-27c_1{}^2x^2+\sqrt{108c_1{}^3x^3+(27c_1{}^2x-27c_1{}^2x^2)^2}+27c_1{}^2x}} + \frac{\sqrt[3]{-27c_1{}^2x^2+\sqrt{108c_1{}^3x^3+(27c_1{}^2x-27c_1{}^2x^2)^2}+27c_1{}^2x}}{3\sqrt[3]{2}c_1} + x - 1$$

$$y(x) \to \frac{(1+i\sqrt{3})\,x}{2^{2/3}\sqrt[3]{-27c_1{}^2x^2+\sqrt{108c_1{}^3x^3+(27c_1{}^2x-27c_1{}^2x^2)^2}+27c_1{}^2x}} - \frac{(1-i\sqrt{3})\sqrt[3]{-27c_1{}^2x^2+\sqrt{108c_1{}^3x^3+(27c_1{}^2x-27c_1{}^2x^2)^2}+27c_1{}^2x}}{6\sqrt[3]{2}c_1} + x - 1$$

$$y(x) \to \frac{(1-i\sqrt{3})\,x}{2^{2/3}\sqrt[3]{-27c_1{}^2x^2+\sqrt{108c_1{}^3x^3+(27c_1{}^2x-27c_1{}^2x^2)^2}+27c_1{}^2x}} - \frac{(1+i\sqrt{3})\sqrt[3]{-27c_1{}^2x^2+\sqrt{108c_1{}^3x^3+(27c_1{}^2x-27c_1{}^2x^2)^2}+27c_1{}^2x}}{6\sqrt[3]{2}c_1} + x - 1$$

$$y(x) \to \text{Indeterminate}$$

$$y(x) \to x - 1$$

4.22 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.14, page 90

Internal problem ID [4489]
Internal file name [`OUTPUT/3982_Sunday_June_05_2022_11_58_48_AM_98876911/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.14, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{y(2x-y-1)+x(2y-x-1)\,y'=0}$$

4.22.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x(2y - x - 1))\, \mathrm{d}y &= (-y(2x - y - 1))\, \mathrm{d}x \\ (y(2x - y - 1))\, \mathrm{d}x + (x(2y - x - 1))\, \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y(2x - y - 1) \\ N(x, y) &= x(2y - x - 1)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(2x - y - 1)) \\ &= 2x - 2y - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x(2y - x - 1)) \\ &= 2y - 2x - 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{(-2y + x + 1)\, x}((2x - 2y - 1) - (2y - 2x - 1)) \\ &= \frac{-4x + 4y}{x\,(-2y + x + 1)}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y(2x-y-1)}((2y-2x-1)-(2x-2y-1)) \\
&= \frac{-4x+4y}{y(2x-y-1)}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(2y-2x-1)-(2x-2y-1)}{x(y(2x-y-1)) - y(x(2y-x-1))} \\
&= -\frac{4}{3yx}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{4}{3t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{4}{3t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{4\ln(t)}{3}} \\
&= \frac{1}{t^{\frac{4}{3}}}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{(xy)^{\frac{4}{3}}}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{(xy)^{\frac{4}{3}}}(y(2x - y - 1)) \\
&= \frac{2x - y - 1}{x\,(xy)^{\frac{1}{3}}}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(xy)^{\frac{4}{3}}}(x(2y - x - 1)) \\
&= -\frac{-2y + x + 1}{y\,(xy)^{\frac{1}{3}}}
\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{2x - y - 1}{x\,(xy)^{\frac{1}{3}}}\right) + \left(-\frac{-2y + x + 1}{y\,(xy)^{\frac{1}{3}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2x - y - 1}{x\,(xy)^{\frac{1}{3}}}\,\mathrm{d}x
\end{aligned}$$

$$\phi = \frac{3x + 3y + 3}{(xy)^{\frac{1}{3}}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= \frac{3}{(xy)^{\frac{1}{3}}} - \frac{(x+y+1)\,x}{(xy)^{\frac{4}{3}}} + f'(y) \\ &= -\frac{-2y+x+1}{y\,(xy)^{\frac{1}{3}}} + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{-2y+x+1}{y(xy)^{\frac{1}{3}}}$. Therefore equation (4) becomes

$$-\frac{-2y+x+1}{y\,(xy)^{\frac{1}{3}}} = -\frac{-2y+x+1}{y\,(xy)^{\frac{1}{3}}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3x+3y+3}{(xy)^{\frac{1}{3}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3x+3y+3}{(xy)^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$\frac{3y+3x+3}{(xy)^{\frac{1}{3}}} = c_1 \tag{1}$$

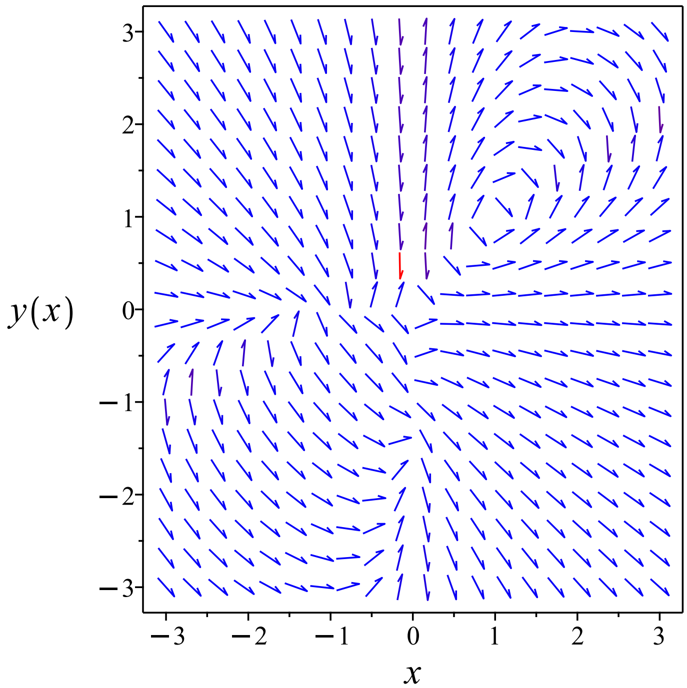


Figure 99: Slope field plot

<u>Verification of solutions</u>

$$\frac{3y + 3x + 3}{(xy)^{\frac{1}{3}}} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 391

```
dsolve((y(x)*(2*x-y(x)-1))+(x*(2*y(x)-x-1))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{3\,5^{\frac{1}{3}}\left(x\left(\sqrt{5}\,\sqrt{\frac{80c_1x^2+160c_1x+80c_1-x}{c_1}}-20x-20\right)c_1^2\right)^{\frac{1}{3}}}{40c_1} + \frac{3x5^{\frac{2}{3}}}{40\left(x\left(\sqrt{5}\,\sqrt{\frac{80c_1x^2+160c_1x+80c_1-x}{c_1}}-20x-20\right)c_1^2\right)^{\frac{1}{3}}}-1-x$$

$$y(x) = \frac{\frac{3\,5^{\frac{1}{3}}\left(-i\sqrt{3}-1\right)\left(-20\left(-\frac{\sqrt{5}\,\sqrt{\frac{80(1+x)^2c_1-x}{c_1}}}{20}+x+1\right)c_1^2x\right)^{\frac{2}{3}}}{80}+\frac{3c_1\left(\frac{80(-1-x)\left(-20\left(-\frac{\sqrt{5}\,\sqrt{\frac{80(1+x)^2c_1-x}{c_1}}}{20}+x+1\right)c_1^2x\right)^{\frac{1}{3}}}{3}+\left(i\sqrt{3}-1\right)5^{\frac{2}{3}}x\right)}{80}}{\left(-20\left(-\frac{\sqrt{5}\,\sqrt{\frac{80(1+x)^2c_1-x}{c_1}}}{20}+x+1\right)c_1^2x\right)^{\frac{1}{3}}c_1}$$

$$y(x) = \frac{\frac{3\left(i\sqrt{3}-1\right)5^{\frac{1}{3}}\left(-20\left(-\frac{\sqrt{5}\,\sqrt{\frac{80(1+x)^2c_1-x}{c_1}}}{20}+x+1\right)c_1^2x\right)^{\frac{2}{3}}}{80}+\frac{3\left(-\frac{80(1+x)\left(-20\left(-\frac{\sqrt{5}\,\sqrt{\frac{80(1+x)^2c_1-x}{c_1}}}{20}+x+1\right)c_1^2x\right)^{\frac{1}{3}}}{3}+\left(-i\sqrt{3}-1\right)5^{\frac{2}{3}}x\right)c_1}{80}}{\left(-20\left(-\frac{\sqrt{5}\,\sqrt{\frac{80(1+x)^2c_1-x}{c_1}}}{20}+x+1\right)c_1^2x\right)^{\frac{1}{3}}c_1}$$

✓ Solution by Mathematica

Time used: 40.285 (sec). Leaf size: 471

```
DSolve[(y[x]*(2*x-y[x]-1))+(x*(2*y[x]-x-1))*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\frac{\sqrt[3]{2}x}{\sqrt[3]{27c_1{}^2x^2+\sqrt{(27c_1{}^2x^2+27c_1{}^2x)^2-108c_1{}^3x^3}+27c_1{}^2x}} - \frac{\sqrt[3]{27c_1{}^2x^2+\sqrt{(27c_1{}^2x^2+27c_1{}^2x)^2-108c_1{}^3x^3}+27c_1{}^2x}}{3\sqrt[3]{2}c_1} - x - 1$$

$$y(x) \to \frac{(1+i\sqrt{3})\,x}{2^{2/3}\sqrt[3]{27c_1{}^2x^2+\sqrt{(27c_1{}^2x^2+27c_1{}^2x)^2-108c_1{}^3x^3}+27c_1{}^2x}} + \frac{(1-i\sqrt{3})\,\sqrt[3]{27c_1{}^2x^2+\sqrt{(27c_1{}^2x^2+27c_1{}^2x)^2-108c_1{}^3x^3}+27c_1{}^2x}}{6\sqrt[3]{2}c_1} - x - 1$$

$$y(x) \to \frac{(1-i\sqrt{3})\,x}{2^{2/3}\sqrt[3]{27c_1{}^2x^2+\sqrt{(27c_1{}^2x^2+27c_1{}^2x)^2-108c_1{}^3x^3}+27c_1{}^2x}} + \frac{(1+i\sqrt{3})\,\sqrt[3]{27c_1{}^2x^2+\sqrt{(27c_1{}^2x^2+27c_1{}^2x)^2-108c_1{}^3x^3}+27c_1{}^2x}}{6\sqrt[3]{2}c_1} - x - 1$$

$$y(x) \to \text{Indeterminate}$$

$$y(x) \to -x - 1$$

4.23 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.15, page 90

Internal problem ID [4490]
Internal file name [`OUTPUT/3983_Sunday_June_05_2022_11_58_55_AM_25237562/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.15, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _exact, _rational, [_Abel, `2nd
   type`, `class B`]]
```

$$\boxed{y^2 + 12yx^2 + (2xy + 4x^3)\, y' = 0}$$

4.23.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(4x^3 + 2xy\right)\mathrm{d}y = \left(-12y\,x^2 - y^2\right)\mathrm{d}x$$
$$\left(12y\,x^2 + y^2\right)\mathrm{d}x + \left(4x^3 + 2xy\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 12y\,x^2 + y^2$$
$$N(x, y) = 4x^3 + 2xy$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(12y\,x^2 + y^2\right)\\ &= 12x^2 + 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(4x^3 + 2xy\right)\\ &= 12x^2 + 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$
$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int 12y\,x^2+y^2\,\mathrm{d}x$$
$$\phi=4y\,x^3+y^2x+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=4x^3+2xy+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=4x^3+2xy$. Therefore equation (4) becomes

$$4x^3+2xy=4x^3+2xy+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=4y\,x^3+y^2x+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=4y\,x^3+y^2x$$

Summary

The solution(s) found are the following

$$4yx^3 + xy^2 = c_1 \tag{1}$$

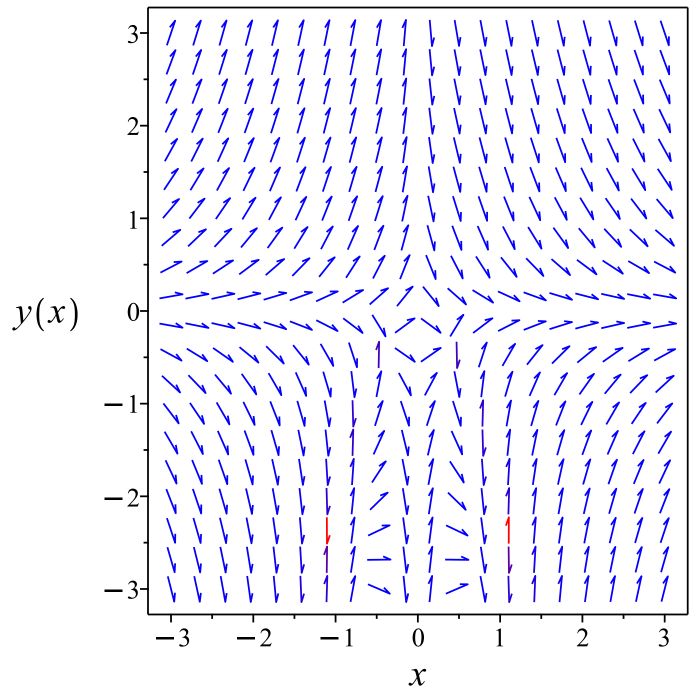


Figure 100: Slope field plot

Verification of solutions

$$4yx^3 + xy^2 = c_1$$

Verified OK.

4.23.2 Maple step by step solution

Let's solve

$y^2 + 12yx^2 + (2xy + 4x^3)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x,y) = 0$

- Compute derivative of lhs

  $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$
- Evaluate derivatives

  $12x^2 + 2y = 12x^2 + 2y$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (12y\,x^2 + y^2)\,dx + f_1(y)$
- Evaluate integral

  $F(x,y) = y(4x^3 + xy) + f_1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$
- Compute derivative

  $4x^3 + 2xy = 4x^3 + 2xy + \frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$
- Solve for $f_1(y)$

  $f_1(y) = 0$
- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = y(4x^3 + xy)$
- Substitute $F(x,y)$ into the solution of the ODE

  $y(4x^3 + xy) = c_1$
- Solve for y

  $\left\{y = \frac{-2x^3+\sqrt{4x^6+c_1x}}{x}, y = -\frac{2x^3+\sqrt{4x^6+c_1x}}{x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 51

```
dsolve((y(x)^2+12*x^2*y(x))+(2*x*y(x)+4*x^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-2x^3 + \sqrt{4x^6 + c_1 x}}{x}$$
$$y(x) = \frac{-2x^3 - \sqrt{4x^6 + c_1 x}}{x}$$

✓ Solution by Mathematica

Time used: 0.431 (sec). Leaf size: 58

```
DSolve[(y[x]^2+12*x^2*y[x])+(2*x*y[x]+4*x^3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to -\frac{2x^3 + \sqrt{x\left(4x^5 + c_1\right)}}{x}$$
$$y(x) \to \frac{-2x^3 + \sqrt{x\left(4x^5 + c_1\right)}}{x}$$

4.24 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.16, page 90

Internal problem ID [4491]
Internal file name [`OUTPUT/3984_Sunday_June_05_2022_11_59_02_AM_60172897/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.16, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{3(x+y)^2 + x(3y+2x)\, y' = 0}$$

4.24.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x(3y + 2x))\,\mathrm{d}y = \left(-3(x + y)^2\right)\mathrm{d}x$$
$$\left(3(x + y)^2\right)\mathrm{d}x + (x(3y + 2x))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 3(x + y)^2$$
$$N(x, y) = x(3y + 2x)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3(x + y)^2\right) \\ &= 6y + 6x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x(3y + 2x)) \\ &= 3y + 4x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x\,(3y+2x)}((6y+6x)-(3y+4x)) \\ &= \frac{1}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int \frac{1}{x}\,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(x)} \\ &= x \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= x\left(3(x+y)^2\right) \\ &= 3(x+y)^2\,x \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= x(x(3y+2x)) \\ &= x^2(3y+2x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(3(x+y)^2\,x\right) + \left(x^2(3y+2x)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int 3(x+y)^2 x \mathrm{d}x$$

$$\phi = \frac{3}{4}x^4 + 2y x^3 + \frac{3}{2}y^2x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x^3 + 3y x^2 + f'(y) \tag{4}$$

$$= x^2(3y + 2x) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2(3y + 2x)$. Therefore equation (4) becomes

$$x^2(3y + 2x) = x^2(3y + 2x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3}{4}x^4 + 2y x^3 + \frac{3}{2}y^2x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3}{4}x^4 + 2y\,x^3 + \frac{3}{2}y^2x^2$$

Summary

The solution(s) found are the following

$$\frac{3x^4}{4} + 2yx^3 + \frac{3y^2x^2}{2} = c_1 \tag{1}$$

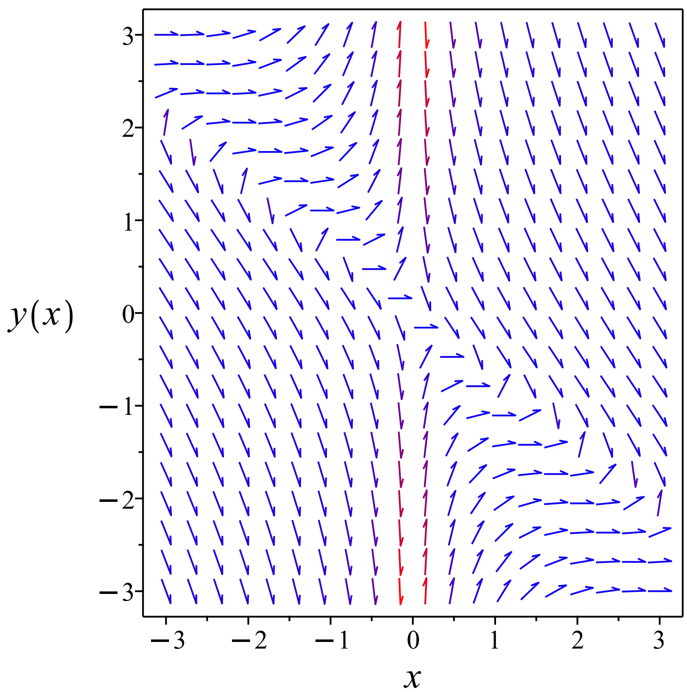


Figure 101: Slope field plot

Verification of solutions

$$\frac{3x^4}{4} + 2yx^3 + \frac{3y^2x^2}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 63

```
dsolve((3*(y(x)+x)^2)+(x*(3*y(x)+2*x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-4c_1x^2 - \sqrt{-2c_1^2x^4 + 6}}{6c_1x}$$
$$y(x) = \frac{-4c_1x^2 + \sqrt{-2c_1^2x^4 + 6}}{6c_1x}$$

✓ Solution by Mathematica

Time used: 1.741 (sec). Leaf size: 135

```
DSolve[(3*(y[x]+x)^2)+(x*(3*y[x]+2*x))*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{4x^2 + \sqrt{-2x^4 + 6e^{4c_1}}}{6x}$$
$$y(x) \to \frac{-4x^2 + \sqrt{-2x^4 + 6e^{4c_1}}}{6x}$$
$$y(x) \to -\frac{\sqrt{2}\sqrt{-x^4} + 4x^2}{6x}$$
$$y(x) \to \frac{\sqrt{2}\sqrt{-x^4} - 4x^2}{6x}$$

4.25 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.17, page 90

Internal problem ID [4492]
Internal file name [`OUTPUT/3985_Sunday_June_05_2022_11_59_08_AM_91385406/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.17, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{y-\left(x^{2}+y^{2}+x\right) y'=0}$$

4.25.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x^2 - y^2 - x)\,\mathrm{d}y &= (-y)\,\mathrm{d}x\\ (y)\,\mathrm{d}x + (-x^2 - y^2 - x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y\\ N(x,y) &= -x^2 - y^2 - x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^2 - y^2 - x)\\ &= -2x - 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = y$ and $N = -x^2 - y^2 - x$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{y}{x^2+y^2}$$
$$N = \frac{-x^2-y^2-x}{x^2+y^2}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-x^2-y^2-x}{x^2+y^2}\right)\mathrm{d}y = \left(-\frac{y}{x^2+y^2}\right)\mathrm{d}x$$
$$\left(\frac{y}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{-x^2-y^2-x}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= \frac{y}{x^2+y^2} \\ N(x,y) &= \frac{-x^2-y^2-x}{x^2+y^2} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) \\ &= \frac{x^2-y^2}{(x^2+y^2)^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x^2-y^2-x}{x^2+y^2}\right) \\ &= \frac{x^2-y^2}{(x^2+y^2)^2} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{y}{x^2+y^2}\,\mathrm{d}x \end{aligned}$$

$$\phi = \arctan\left(\frac{x}{y}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= -\frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \\ &= -\frac{x}{x^2+y^2} + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x^2-y^2-x}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{-x^2-y^2-x}{x^2+y^2} = -\frac{x}{x^2+y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -1$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int (-1)\,\mathrm{d}y \\ f(y) &= -y + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \arctan\left(\frac{x}{y}\right) - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \arctan\left(\frac{x}{y}\right) - y$$

Summary

The solution(s) found are the following

$$\arctan\left(\frac{x}{y}\right) - y = c_1 \tag{1}$$

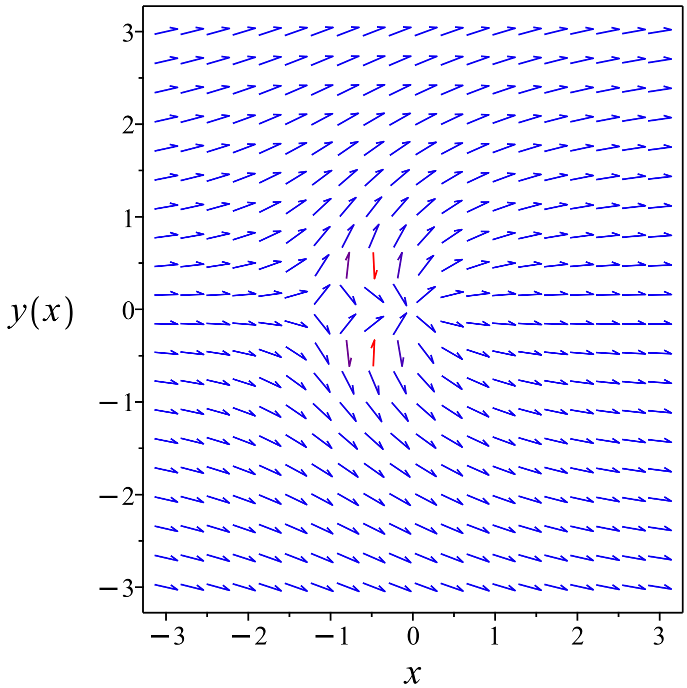


Figure 102: Slope field plot

Verification of solutions

$$\arctan\left(\frac{x}{y}\right) - y = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying Riccati sub-methods:
   <- Riccati particular polynomial solution successful
<- inverse_Riccati successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 40

```
dsolve((y(x))-(y(x)^2+x^2+x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{e^{-2iy(x)}(ix+y(x))+2(iy(x)+x)\,c_1}{2iy\left(x\right)+2x}=0$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 18

```
DSolve[(y[x])-(y[x]^2+x^2+x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[y(x)-\arctan\left(\frac{x}{y(x)}\right)=c_1,y(x)\right]$$

4.26 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.18, page 90

Internal problem ID [4493]
Internal file name [`OUTPUT/3986_Sunday_June_05_2022_11_59_15_AM_38712530/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.18, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational, [_1st_order, `_with_symmetry_[F(x)*G(y)
   ,0]`]]
```

$$\boxed{2xy + (x^2 + y^2 + a)\, y' = 0}$$

4.26.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (x^2 + y^2 + a) \,\mathrm{d}y &= (-2xy) \,\mathrm{d}x \\ (2xy) \,\mathrm{d}x + (x^2 + y^2 + a) \,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 2xy \\ N(x, y) &= x^2 + y^2 + a \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2xy) \\ &= 2x \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + y^2 + a) \\ &= 2x \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2xy \, \mathrm{d}x$$
$$\phi = y\,x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 + y^2 + a$. Therefore equation (4) becomes

$$x^2 + y^2 + a = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2 + a$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y^2 + a) \, \mathrm{d}y$$
$$f(y) = \frac{1}{3} y^3 + ya + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,x^2 + \frac{1}{3} y^3 + ya + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^2 + \frac{1}{3}y^3 + ya$$

Summary
The solution(s) found are the following

$$yx^2 + \frac{y^3}{3} + ya = c_1 \tag{1}$$

Verification of solutions

$$yx^2 + \frac{y^3}{3} + ya = c_1$$

Verified OK.

4.26.2 Maple step by step solution

Let's solve

$2xy + (x^2 + y^2 + a)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- ☐ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$
  - Evaluate derivatives

    $2x = 2x$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

$F(x, y) = \int 2xydx + f_1(y)$

- Evaluate integral

  $F(x, y) = y\,x^2 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $x^2 + y^2 + a = x^2 + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = y^2 + a$

- Solve for $f_1(y)$

  $f_1(y) = \frac{1}{3} y^3 + ya$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = y\,x^2 + \frac{1}{3} y^3 + ya$

- Substitute $F(x, y)$ into the solution of the ODE

  $y\,x^2 + \frac{1}{3} y^3 + ya = c_1$

- Solve for y

$$\left\{ y = \frac{\left(12c_1 + 4\sqrt{4x^6 + 12a\,x^4 + 12a^2x^2 + 4a^3 + 9c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{2(x^2 + a)}{\left(12c_1 + 4\sqrt{4x^6 + 12a\,x^4 + 12a^2x^2 + 4a^3 + 9c_1^2}\right)^{\frac{1}{3}}}, y = -\frac{\left(12c_1 + 4\sqrt{4x^6\right.}}{}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 313

```
dsolve((2*x*y(x))+(x^2+y(x)^2+a)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(-12c_1+4\sqrt{4x^6+12a\,x^4+12x^2a^2+4a^3+9c_1^2}\right)^{\frac{2}{3}}-4x^2-4a}{2\left(-12c_1+4\sqrt{4x^6+12a\,x^4+12x^2a^2+4a^3+9c_1^2}\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{\left(\frac{i\sqrt{3}}{4}+\frac{1}{4}\right)\left(-12c_1+4\sqrt{4x^6+12a\,x^4+12x^2a^2+4a^3+9c_1^2}\right)^{\frac{2}{3}}+(x^2+a)\left(i\sqrt{3}-1\right)}{\left(-12c_1+4\sqrt{4x^6+12a\,x^4+12x^2a^2+4a^3+9c_1^2}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{\frac{\left(i\sqrt{3}-1\right)\left(-12c_1+4\sqrt{4x^6+12a\,x^4+12x^2a^2+4a^3+9c_1^2}\right)^{\frac{2}{3}}}{4}+(x^2+a)\left(1+i\sqrt{3}\right)}{\left(-12c_1+4\sqrt{4x^6+12a\,x^4+12x^2a^2+4a^3+9c_1^2}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 4.319 (sec). Leaf size: 299

```
DSolve[(2*x*y[x])+(x^2+y[x]^2+a)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{2}\left(\sqrt{4(a+x^2)^3+9c_1{}^2}+3c_1\right)^{2/3}-2a-2x^2}{2^{2/3}\sqrt[3]{\sqrt{4(a+x^2)^3+9c_1{}^2}+3c_1}}$$

$$y(x) \to \frac{(1+i\sqrt{3})(a+x^2)}{2^{2/3}\sqrt[3]{\sqrt{4(a+x^2)^3+9c_1{}^2}+3c_1}}+\frac{i(\sqrt{3}+i)\sqrt[3]{\sqrt{4(a+x^2)^3+9c_1{}^2}+3c_1}}{2\sqrt[3]{2}}$$

$$y(x) \to \frac{(1-i\sqrt{3})(a+x^2)}{2^{2/3}\sqrt[3]{\sqrt{4(a+x^2)^3+9c_1{}^2}+3c_1}}-\frac{i(\sqrt{3}-i)\sqrt[3]{\sqrt{4(a+x^2)^3+9c_1{}^2}+3c_1}}{2\sqrt[3]{2}}$$

$$y(x) \to 0$$

4.27 problem Recognizable Exact Differential equations. Integrating factors. Exercise 10.19, page 90

Internal problem ID [4494]
Internal file name [`OUTPUT/3987_Sunday_June_05_2022_11_59_22_AM_37486158/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 10
Problem number: Recognizable Exact Differential equations. Integrating factors. Exercise 10.19, page 90.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{2xy + \left(x^2 + y^2 + a\right) y' = -x^2 - b}$$

4.27.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x^2 + y^2 + a)\,\mathrm{d}y &= (-x^2 - 2xy - b)\,\mathrm{d}x \\
(x^2 + 2xy + b)\,\mathrm{d}x + (x^2 + y^2 + a)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= x^2 + 2xy + b \\
N(x, y) &= x^2 + y^2 + a
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^2 + 2xy + b) \\
&= 2x
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + y^2 + a) \\
&= 2x
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int x^2 + 2xy + b \,\mathrm{d}x$$
$$\phi = \frac{1}{3}x^3 + y\,x^2 + bx + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 + y^2 + a$. Therefore equation (4) becomes

$$x^2 + y^2 + a = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2 + a$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (y^2 + a) \,\mathrm{d}y$$
$$f(y) = \frac{1}{3}y^3 + ya + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{3}x^3 + y\,x^2 + bx + \frac{1}{3}y^3 + ya + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{3}x^3 + y\,x^2 + bx + \frac{1}{3}y^3 + ya$$

Summary

The solution(s) found are the following

$$\frac{x^3}{3} + yx^2 + bx + \frac{y^3}{3} + ya = c_1 \tag{1}$$

Verification of solutions

$$\frac{x^3}{3} + yx^2 + bx + \frac{y^3}{3} + ya = c_1$$

Verified OK.

4.27.2 Maple step by step solution

Let's solve

$2xy + (x^2 + y^2 + a)\,y' = -x^2 - b$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $2x = 2x$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

$F(x,y) = \int (x^2 + 2xy + b)\, dx + f_1(y)$

- Evaluate integral

$F(x,y) = \frac{x^3}{3} + y\, x^2 + bx + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

$N(x,y) = \frac{\partial}{\partial y} F(x,y)$

- Compute derivative

$x^2 + y^2 + a = x^2 + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

$\frac{d}{dy} f_1(y) = y^2 + a$

- Solve for $f_1(y)$

$f_1(y) = \frac{1}{3}y^3 + ya$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$F(x,y) = \frac{1}{3}x^3 + y\, x^2 + bx + \frac{1}{3}y^3 + ya$

- Substitute $F(x,y)$ into the solution of the ODE

$\frac{1}{3}x^3 + y\, x^2 + bx + \frac{1}{3}y^3 + ya = c_1$

- Solve for y

$$\left\{ y = \frac{\left(-4x^3 - 12bx + 12c_1 + 4\sqrt{5x^6 + 12a\, x^4 + 6x^4 b - 6c_1 x^3 + 12a^2 x^2 + 9b^2 x^2 - 18bxc_1 + 4a^3 + 9c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{}{\left(-4x^3 - 12bx + 12c_1 + 4\sqrt{5x^6}\right.} \right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 505

```
dsolve((2*x*y(x)+x^2+b)+(y(x)^2+x^2+a)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-4x^2 - 4a + \left(-4x^3 - 12xb - 12c_1 + 4\sqrt{5x^6 + 6(2a+b)x^4 + 6c_1x^3 + 3(4a^2+3b^2)x^2 + 18xbc_1 + 4a^3}\right.}{2\left(-4x^3 - 12xb - 12c_1 + 4\sqrt{5x^6 + 6(2a+b)x^4 + 6c_1x^3 + 3(4a^2+3b^2)x^2 + 18xbc_1 + 4a^3 + 9c_1^2}\right.}$$

$$y(x) = -\frac{\left(\frac{i\sqrt{3}}{4} + \frac{1}{4}\right)\left(-4x^3 - 12xb - 12c_1 + 4\sqrt{5x^6 + 6(2a+b)x^4 + 6c_1x^3 + 3(4a^2+3b^2)x^2 + 18xbc_1 + 4a^3}\right.}{\left(-4x^3 - 12xb - 12c_1 + 4\sqrt{5x^6 + 6(2a+b)x^4 + 6c_1x^3 + 3(4a^2+3b^2)x^2 + 18xbc_1}\right.}$$

$$y(x) = \frac{\frac{(i\sqrt{3}-1)\left(-4x^3-12xb-12c_1+4\sqrt{5x^6+6(2a+b)x^4+6c_1x^3+3(4a^2+3b^2)x^2+18xbc_1+4a^3+9c_1^2}\right)^{\frac{2}{3}}}{4} + (x^2+a)\left(1+i\sqrt{3}\right)}{\left(-4x^3 - 12xb - 12c_1 + 4\sqrt{5x^6 + 6(2a+b)x^4 + 6c_1x^3 + 3(4a^2+3b^2)x^2 + 18xbc_1 + 4a^3 + 9c_1^2}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 6.558 (sec). Leaf size: 396

```
DSolve[(2*x*y[x]+x^2+b)+(y[x]^2+x^2+a)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{2}\left(\sqrt{4\left(a+x^2\right)^3+\left(3bx+x^3-3c_1\right)^2}-3bx-x^3+3c_1\right)^{2/3}-2a-2x^2}{2^{2/3}\sqrt[3]{\sqrt{4\left(a+x^2\right)^3+\left(3bx+x^3-3c_1\right)^2}-3bx-x^3+3c_1}}$$

$$y(x) \to \frac{\left(1+i\sqrt{3}\right)\left(a+x^2\right)}{2^{2/3}\sqrt[3]{\sqrt{4\left(a+x^2\right)^3+\left(3bx+x^3-3c_1\right)^2}-3bx-x^3+3c_1}} + \frac{i\left(\sqrt{3}+i\right)\sqrt[3]{\sqrt{4\left(a+x^2\right)^3+\left(3bx+x^3-3c_1\right)^2}-3bx-x^3+3c_1}}{2\sqrt[3]{2}}$$

$$y(x) \to \frac{\left(1-i\sqrt{3}\right)\left(a+x^2\right)}{2^{2/3}\sqrt[3]{\sqrt{4\left(a+x^2\right)^3+\left(3bx+x^3-3c_1\right)^2}-3bx-x^3+3c_1}} - \frac{i\left(\sqrt{3}-i\right)\sqrt[3]{\sqrt{4\left(a+x^2\right)^3+\left(3bx+x^3-3c_1\right)^2}-3bx-x^3+3c_1}}{2\sqrt[3]{2}}$$

5 Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations

5.1 problem Exercise 11.1, page 97

Internal problem ID [4495]
Internal file name [`OUTPUT/3988_Sunday_June_05_2022_11_59_31_AM_58016000/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.1, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' + y = x^3}$$

5.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = x^2$$

Hence the ode is

$$y' + \frac{y}{x} = x^2$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x}dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(x^2\right) \\
\frac{d}{dx}(xy) &= (x)\left(x^2\right) \\
d(xy) &= x^3\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
xy &= \int x^3\, dx \\
xy &= \frac{x^4}{4} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
y = \frac{x^3}{4} + \frac{c_1}{x}
$$

Summary
The solution(s) found are the following

$$
y = \frac{x^3}{4} + \frac{c_1}{x} \tag{1}
$$

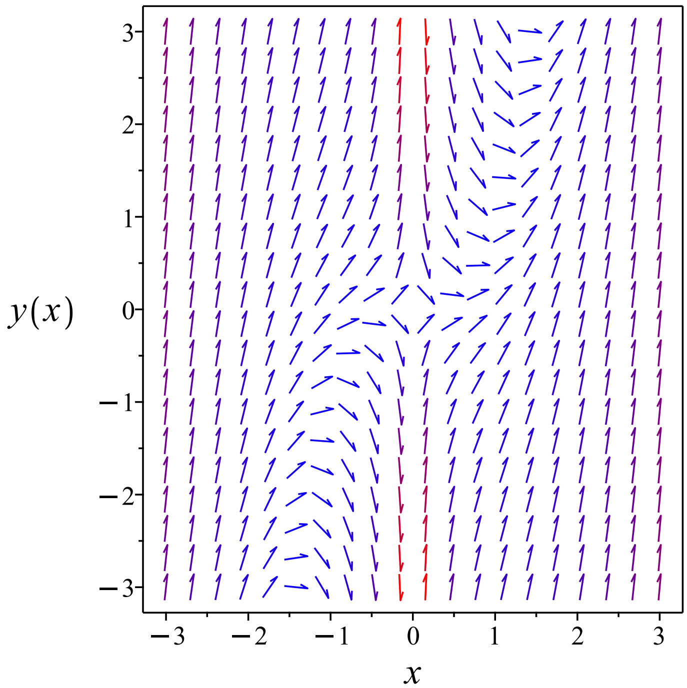


Figure 103: Slope field plot

Verification of solutions

$$y = \frac{x^3}{4} + \frac{c_1}{x}$$

Verified OK.

5.1.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-y + x^3}{x} \tag{1}$$

Which becomes

$$0 = (-x)\, dy + (x^3 - y)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + (x^3 - y)\, dx = d\left(\frac{1}{4}x^4 - xy\right)$$

Hence (2) becomes

$$0 = d\left(\frac{1}{4}x^4 - xy\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{x^4 + 4c_1}{4x} + c_1$$

Summary

The solution(s) found are the following

$$y = \frac{x^4 + 4c_1}{4x} + c_1 \tag{1}$$

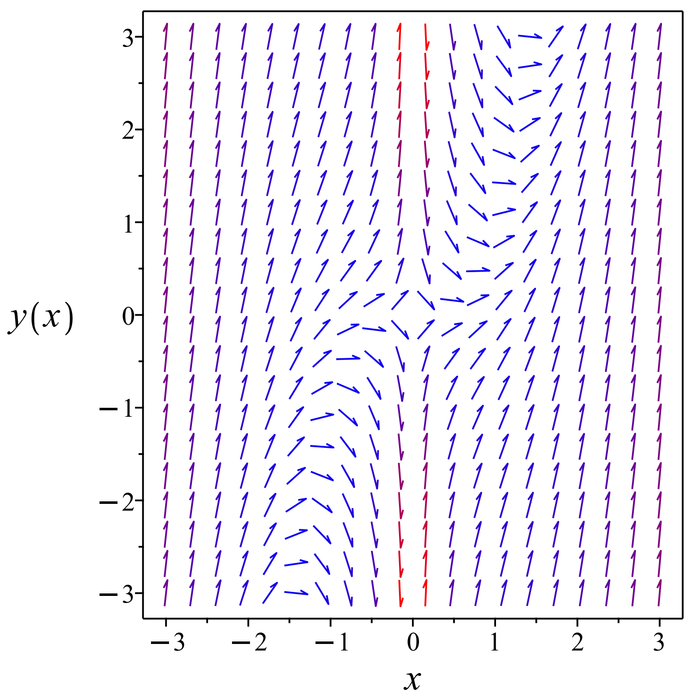


Figure 104: Slope field plot

Verification of solutions

$$y = \frac{x^4 + 4c_1}{4x} + c_1$$

Verified OK.

5.1.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-x^3 + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 43: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \frac{1}{x} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = xy$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-x^3 + y}{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= y \\
S_y &= x
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^4}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$xy = \frac{x^4}{4} + c_1$$

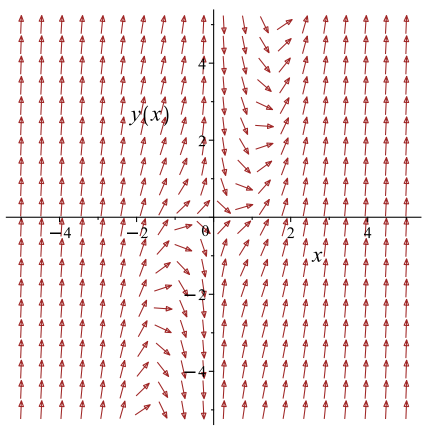
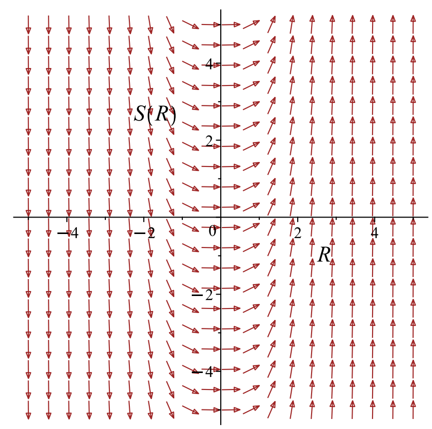
Which simplifies to

$$xy = \frac{x^4}{4} + c_1$$

Which gives

$$y = \frac{x^4 + 4c_1}{4x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{-x^3+y}{x}$
</td><td>$R = x$
$S = xy$</td><td>$\frac{dS}{dR} = R^3$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{x^4 + 4c_1}{4x} \tag{1}$$

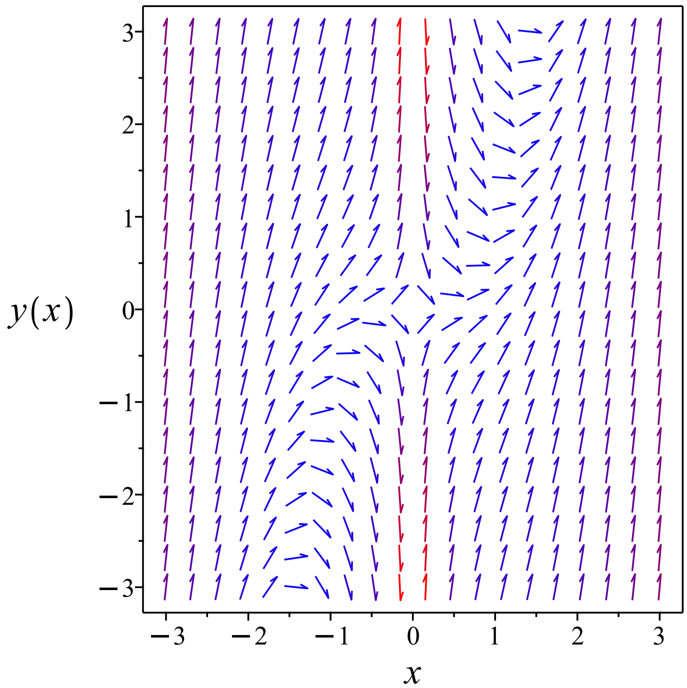


Figure 105: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{x^4 + 4c_1}{4x}$$

Verified OK.

5.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (x)\,\mathrm{d}y &= \left(x^3 - y\right)\mathrm{d}x \\ \left(-x^3 + y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x^3 + y$$
$$N(x, y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3 + y\right) \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x^3 + y \, \mathrm{d}x$$
$$\phi = -\frac{1}{4}x^4 + xy + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x$. Therefore equation (4) becomes

$$x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{4}x^4 + xy + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{4}x^4 + xy$$

The solution becomes

$$y = \frac{x^4 + 4c_1}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^4 + 4c_1}{4x} \tag{1}$$

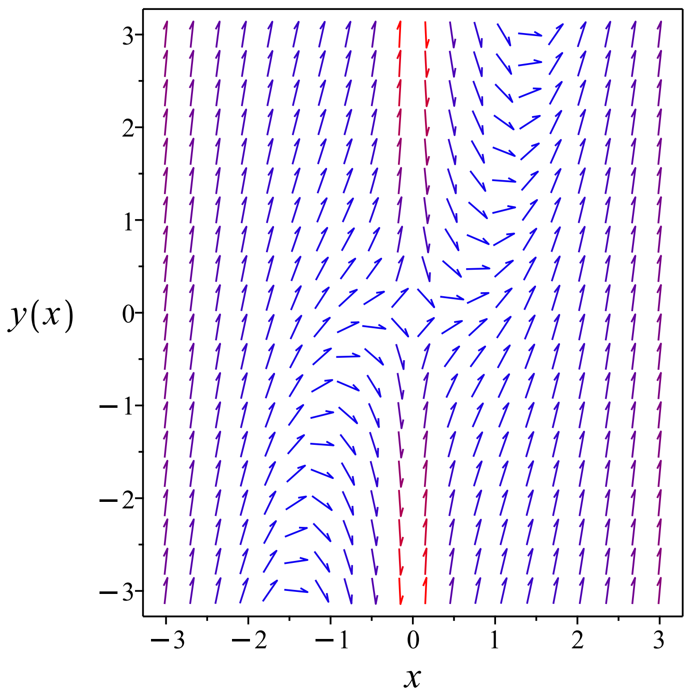


Figure 106: Slope field plot

Verification of solutions

$$y = \frac{x^4 + 4c_1}{4x}$$

Verified OK.

5.1.5 Maple step by step solution

Let's solve

$xy' + y = x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x} + x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu(x)\, x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^2 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int x^3 dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{x^4}{4} + c_1}{x}$

- Simplify

$y = \frac{x^4+4c_1}{4x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x)+y(x)=x^3,y(x), singsol=all)
```

$$y(x) = \frac{x^4 + 4c_1}{4x}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 19

```
DSolve[x*y'[x]+y[x]==x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{4} + \frac{c_1}{x}$$

5.2 problem Exercise 11.2, page 97

Internal problem ID [4496]
Internal file name [`OUTPUT/3989_Sunday_June_05_2022_11_59_40_AM_1736555/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.2, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' + ya = b}$$

5.2.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-ya+b} dy = \int dx$$
$$-\frac{\ln(-ya+b)}{a} = x + c_1$$

Raising both side to exponential gives

$$e^{-\frac{\ln(-ya+b)}{a}} = e^{x+c_1}$$

Which simplifies to

$$(-ya+b)^{-\frac{1}{a}} = c_2 e^x$$

Summary
The solution(s) found are the following

$$y = -\frac{(c_2 e^x)^{-a} - b}{a} \tag{1}$$

Verification of solutions

$$y = -\frac{(c_2 e^x)^{-a} - b}{a}$$

Verified OK.

5.2.2 Maple step by step solution

Let's solve

$y' + ya = b$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-ya+b} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-ya+b} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\frac{\ln(-ya+b)}{a} = x + c_1$

- Solve for y

  $y = -\frac{e^{-ac_1 - ax} - b}{a}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x)+a*y(x)=b,y(x), singsol=all)
```

$$y(x) = \frac{e^{-ax}c_1a + b}{a}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 29

```
DSolve[y'[x]+a*y[x]==b,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{b}{a} + c_1e^{-ax}$$
$$y(x) \to \frac{b}{a}$$

5.3 problem Exercise 11.3, page 97

Internal problem ID [4497]
Internal file name [`OUTPUT/3990_Sunday_June_05_2022_11_59_49_AM_49852744/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.3, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{xy' + y - \ln(x)\, y^2 = 0}$$

5.3.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(\ln(x)\, y - 1)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 47: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^2 x
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^2 x} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{yx}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(\ln(x) y - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{y x^2} \\ S_y &= \frac{1}{y^2 x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\ln(x)}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\ln(R)}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{R} - \frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{yx} = -\frac{\ln(x)}{x} - \frac{1}{x} + c_1$$

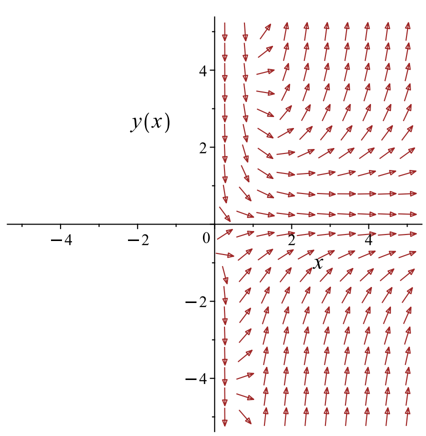
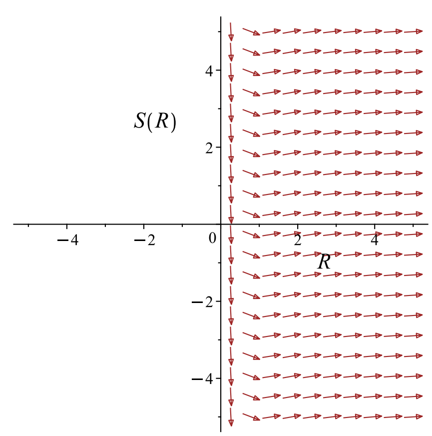
Which simplifies to

$$\frac{-yc_1x + \ln(x)\,y + y - 1}{xy} = 0$$

Which gives

$$y = \frac{1}{-c_1x + \ln(x) + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(\ln(x)y-1)}{x}$ 	$R = x$ $S = -\frac{1}{yx}$	$\frac{dS}{dR} = \frac{\ln(R)}{R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{1}{-c_1 x + \ln(x) + 1} \tag{1}$$

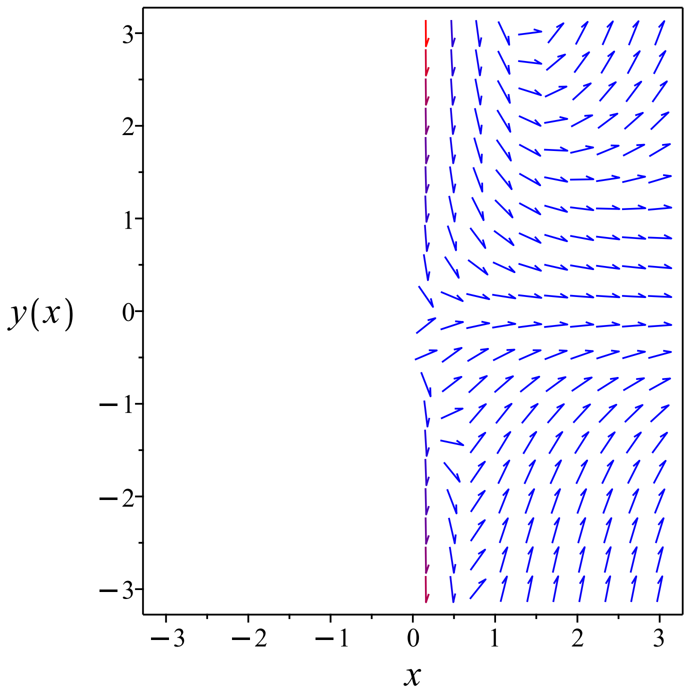


Figure 107: Slope field plot

Verification of solutions

$$y = \frac{1}{-c_1 x + \ln(x) + 1}$$

Verified OK.

5.3.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(\ln(x)\, y - 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x} y + \frac{\ln(x)}{x} y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= \frac{\ln(x)}{x} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{yx} + \frac{\ln(x)}{x} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{w(x)}{x} + \frac{\ln(x)}{x} \\ w' &= \frac{w}{x} - \frac{\ln(x)}{x} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -\frac{\ln(x)}{x} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = -\frac{\ln(x)}{x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-\frac{\ln(x)}{x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(-\frac{\ln(x)}{x}\right) \\ \mathrm{d}\left(\frac{w}{x}\right) &= \left(-\frac{\ln(x)}{x^2}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{w}{x} &= \int -\frac{\ln(x)}{x^2}\,\mathrm{d}x \\ \frac{w}{x} &= \frac{\ln(x)}{x} + \frac{1}{x} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = x\left(\frac{\ln(x)}{x} + \frac{1}{x}\right) + c_1 x$$

which simplifies to

$$w(x) = c_1 x + \ln(x) + 1$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = c_1 x + \ln(x) + 1$$

Or

$$y = \frac{1}{c_1 x + \ln(x) + 1}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{c_1 x + \ln(x) + 1} \tag{1}$$

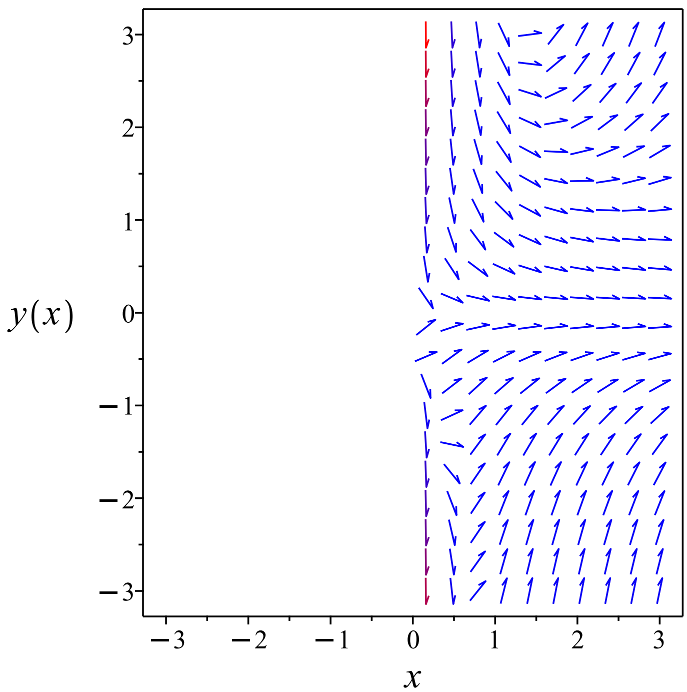


Figure 108: Slope field plot

Verification of solutions

$$y = \frac{1}{c_1 x + \ln(x) + 1}$$

Verified OK.

5.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(-y + \ln(x)\,y^2\right)\mathrm{d}x\\ \left(-\ln(x)\,y^2 + y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\ln(x)\,y^2 + y\\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\ln\left(x\right)y^2 + y\right)\\ &= -2\ln\left(x\right)y + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((-2\ln\left(x\right)y + 1) - (1))\\ &= -\frac{2\ln\left(x\right)y}{x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -\frac{1}{y\left(\ln\left(x\right)y - 1\right)}((1) - (-2\ln\left(x\right)y + 1))\\ &= -\frac{2\ln\left(x\right)}{\ln\left(x\right)y - 1}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - (-2\ln(x)\,y + 1)}{x\left(-\ln(x)\,y^2 + y\right) - y\,(x)} \\
&= -\frac{2}{yx}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^2 y^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2 y^2}\left(-\ln(x)\,y^2 + y\right) \\
&= \frac{-\ln(x)\,y + 1}{y\,x^2}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2 y^2}(x) \\
&= \frac{1}{y^2 x}
\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-\ln(x)\, y + 1}{y\, x^2}\right) + \left(\frac{1}{y^2 x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-\ln(x)\, y + 1}{y\, x^2}\,\mathrm{d}x$$
$$\phi = \frac{\ln(x)\, y + y - 1}{xy} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1 + \ln(x)}{xy} - \frac{\ln(x)\, y + y - 1}{x\, y^2} + f'(y) \tag{4}$$
$$= \frac{1}{y^2 x} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2 x}$. Therefore equation (4) becomes

$$\frac{1}{y^2 x} = \frac{1}{y^2 x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(x)\, y + y - 1}{xy} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x)\, y + y - 1}{xy}$$

The solution becomes

$$y = \frac{1}{-c_1 x + \ln(x) + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{-c_1 x + \ln(x) + 1} \tag{1}$$

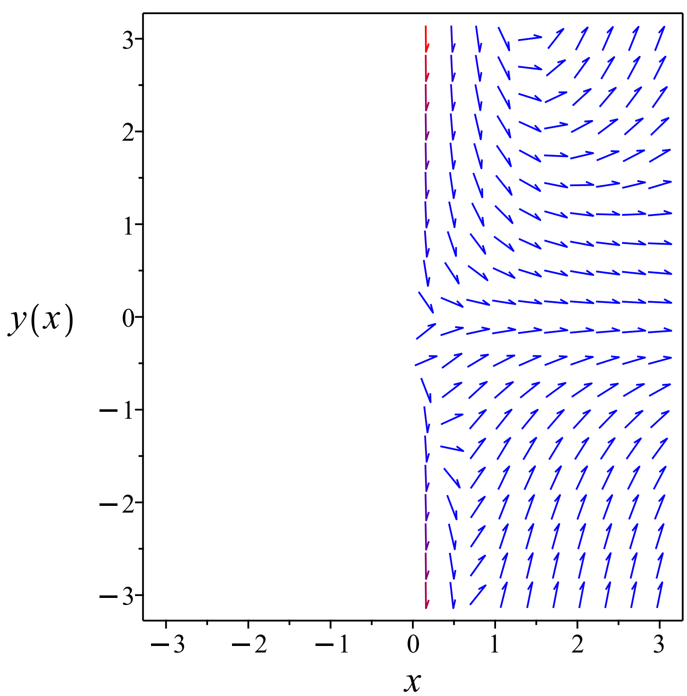


Figure 109: Slope field plot

Verification of solutions

$$y = \frac{1}{-c_1 x + \ln(x) + 1}$$

Verified OK.

5.3.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(\ln(x)\,y - 1)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{\ln(x)\,y^2}{x} - \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = \frac{\ln(x)}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{\ln(x)u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{\ln(x)}{x^2} + \frac{1}{x^2} \\ f_1 f_2 &= -\frac{\ln(x)}{x^2} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\ln(x)\,u''(x)}{x} - \left(-\frac{2\ln(x)}{x^2} + \frac{1}{x^2}\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{-c_2 \ln(x) + c_1 x - c_2}{x}$$

The above shows that

$$u'(x) = \frac{c_2 \ln(x)}{x^2}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{-c_2 \ln(x) + c_1 x - c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{1}{-c_3 x + \ln(x) + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{-c_3 x + \ln(x) + 1} \tag{1}$$

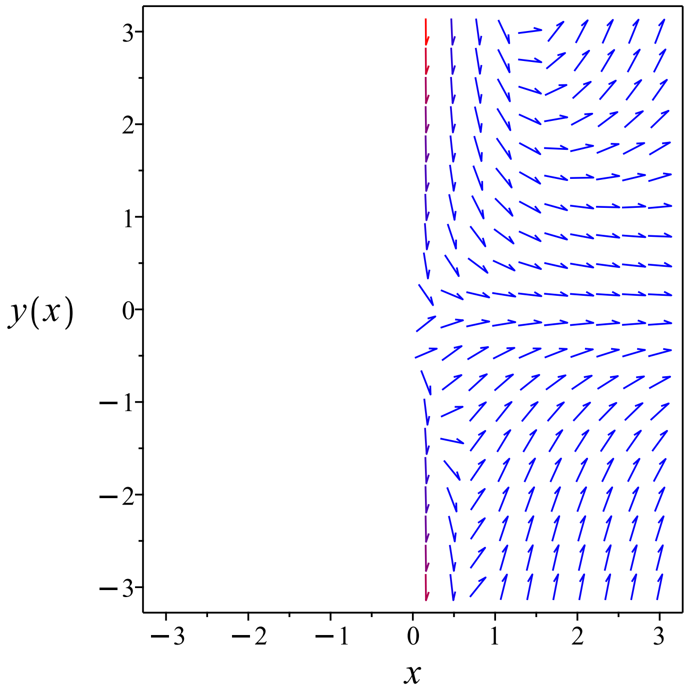


Figure 110: Slope field plot

Verification of solutions

$$y = \frac{1}{-c_3 x + \ln(x) + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(x*diff(y(x),x)+y(x)=y(x)^2*ln(x),y(x), singsol=all)
```

$$y(x) = \frac{1}{1 + c_1 x + \ln(x)}$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 20

```
DSolve[x*y'[x]+y[x]==y[x]^2*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{\log(x) + c_1 x + 1}$$
$$y(x) \to 0$$

5.4 problem Exercise 11.4, page 97

Internal problem ID [4498]
Internal file name [`OUTPUT/3991_Sunday_June_05_2022_11_59_58_AM_88742621/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.4, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{x' + 2yx = \mathrm{e}^{-y^2}}$$

5.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(y)x = q(y)$$

Where here

$$p(y) = 2y$$
$$q(y) = \mathrm{e}^{-y^2}$$

Hence the ode is

$$x' + 2yx = \mathrm{e}^{-y^2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 2y dy} \\
&= \mathrm{e}^{y^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}y}(\mu x) &= (\mu)\left(\mathrm{e}^{-y^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}y}\left(\mathrm{e}^{y^2} x\right) &= \left(\mathrm{e}^{y^2}\right)\left(\mathrm{e}^{-y^2}\right) \\
\mathrm{d}\left(\mathrm{e}^{y^2} x\right) &= \mathrm{d}y
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{y^2} x &= \int \mathrm{d}y \\
\mathrm{e}^{y^2} x &= y + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{y^2}$ results in

$$
x = \mathrm{e}^{-y^2} y + c_1 \mathrm{e}^{-y^2}
$$

which simplifies to

$$
x = \mathrm{e}^{-y^2}(y + c_1)
$$

Summary
The solution(s) found are the following

$$
x = \mathrm{e}^{-y^2}(y + c_1) \tag{1}
$$

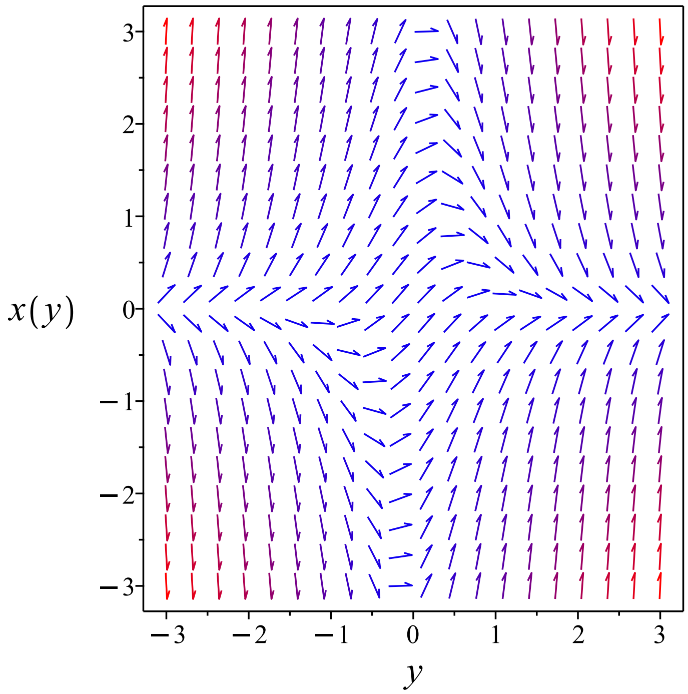


Figure 111: Slope field plot

Verification of solutions

$$x = e^{-y^2}(y + c_1)$$

Verified OK.

5.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = -2xy + e^{-y^2}$$
$$x' = \omega(y, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_x - \xi_y) - \omega^2 \xi_x - \omega_y \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 49: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(y, x) &= 0 \\
\eta(y, x) &= \mathrm{e}^{-y^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial x}\right) S(y, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{-y^2}} dy
\end{aligned}$$

Which results in

$$S = \mathrm{e}^{y^2} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y, x) S_x}{R_y + \omega(y, x) R_x} \tag{2}$$

Where in the above R_y, R_x, S_y, S_x are all partial derivatives and $\omega(y, x)$ is the right hand side of the original ode given by

$$\omega(y, x) = -2xy + \mathrm{e}^{-y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_y &= 1 \\
R_x &= 0 \\
S_y &= 2y\,\mathrm{e}^{y^2} x \\
S_x &= \mathrm{e}^{y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for y, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to y, x coordinates. This results in

$$x\,\mathrm{e}^{y^2} = y + c_1$$

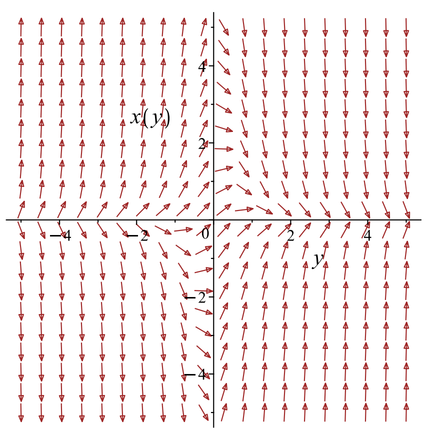
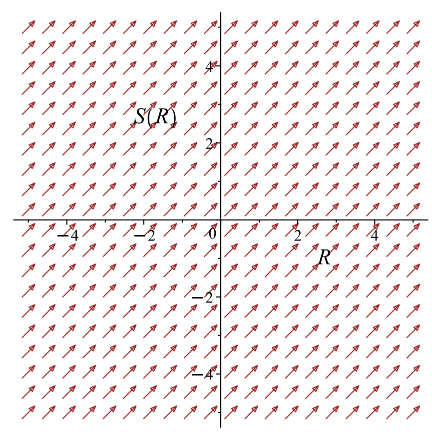
Which simplifies to

$$x\,\mathrm{e}^{y^2} = y + c_1$$

Which gives

$$x = \mathrm{e}^{-y^2}(y + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in y, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dy} = -2xy + \mathrm{e}^{-y^2}$ 	$R = y$ $S = \mathrm{e}^{y^2}x$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{-y^2}(y + c_1) \tag{1}$$

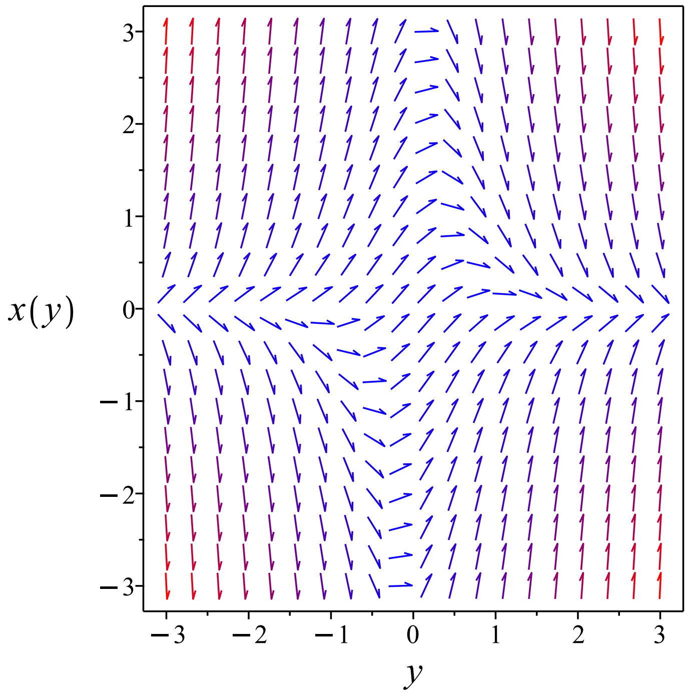


Figure 112: Slope field plot

<u>Verification of solutions</u>

$$x = e^{-y^2}(y + c_1)$$

Verified OK.

5.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(y, x)\,\mathrm{d}y + N(y, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\mathrm{d}x = \left(-2xy + \mathrm{e}^{-y^2}\right) \mathrm{d}y$$
$$\left(2xy - \mathrm{e}^{-y^2}\right) \mathrm{d}y + \mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(y, x) = 2xy - \mathrm{e}^{-y^2}$$
$$N(y, x) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

Using result found above gives

$$\frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left(2xy - \mathrm{e}^{-y^2}\right)$$
$$= 2y$$

And

$$\frac{\partial N}{\partial y} = \frac{\partial}{\partial y}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial y}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) \\ &= 1((2y) - (0)) \\ &= 2y \end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \,\mathrm{d}y} \\ &= e^{\int 2y \,\mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{y^2} \\ &= \mathrm{e}^{y^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{y^2}\left(2xy - \mathrm{e}^{-y^2}\right) \\ &= 2y\,\mathrm{e}^{y^2} x - 1 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{y^2}(1) \\ &= \mathrm{e}^{y^2} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}y} &= 0 \\ \left(2y\,\mathrm{e}^{y^2} x - 1\right) + \left(\mathrm{e}^{y^2}\right)\frac{\mathrm{d}x}{\mathrm{d}y} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(y,x)$

$$\frac{\partial\phi}{\partial y} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. y gives

$$\int \frac{\partial\phi}{\partial y}\,\mathrm{d}y = \int \overline{M}\,\mathrm{d}y$$

$$\int \frac{\partial\phi}{\partial y}\,\mathrm{d}y = \int 2y\,\mathrm{e}^{y^2}x - 1\,\mathrm{d}y$$

$$\phi = -y + \mathrm{e}^{y^2}x + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both y and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial\phi}{\partial x} = \mathrm{e}^{y^2} + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial x} = \mathrm{e}^{y^2}$. Therefore equation (4) becomes

$$\mathrm{e}^{y^2} = \mathrm{e}^{y^2} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = -y + \mathrm{e}^{y^2}x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -y + \mathrm{e}^{y^2}x$$

The solution becomes

$$x = e^{-y^2}(y + c_1)$$

Summary

The solution(s) found are the following

$$x = e^{-y^2}(y + c_1) \tag{1}$$

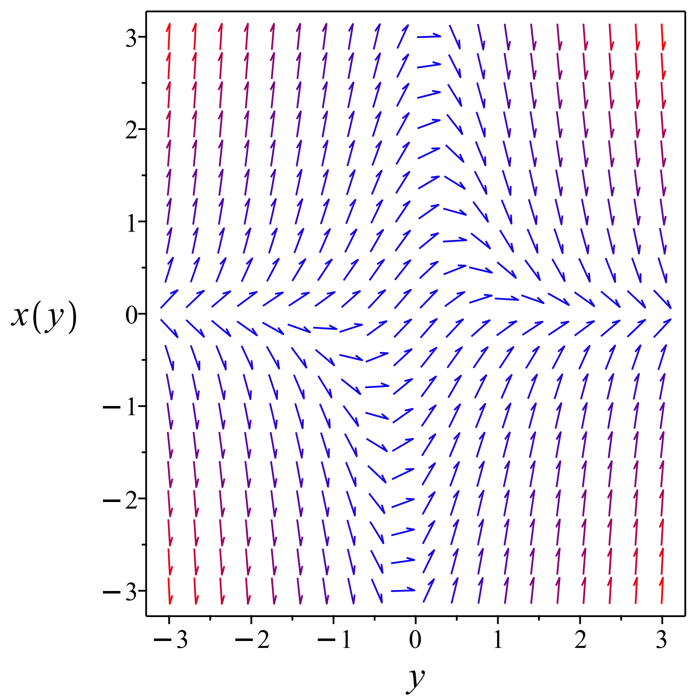


Figure 113: Slope field plot

Verification of solutions

$$x = e^{-y^2}(y + c_1)$$

Verified OK.

5.4.4 Maple step by step solution

Let's solve

$x' + 2yx = \mathrm{e}^{-y^2}$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -2yx + \mathrm{e}^{-y^2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + 2yx = \mathrm{e}^{-y^2}$

- The ODE is linear; multiply by an integrating factor $\mu(y)$

  $\mu(y)\,(x' + 2yx) = \mu(y)\,\mathrm{e}^{-y^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dy}(\mu(y)\,x)$

  $\mu(y)\,(x' + 2yx) = \mu'(y)\,x + \mu(y)\,x'$

- Isolate $\mu'(y)$

  $\mu'(y) = 2\mu(y)\,y$

- Solve to find the integrating factor

  $\mu(y) = \mathrm{e}^{y^2}$

- Integrate both sides with respect to y

  $\int \left(\frac{d}{dy}(\mu(y)\,x)\right) dy = \int \mu(y)\,\mathrm{e}^{-y^2} dy + c_1$

- Evaluate the integral on the lhs

  $\mu(y)\,x = \int \mu(y)\,\mathrm{e}^{-y^2} dy + c_1$

- Solve for x

  $x = \frac{\int \mu(y)\mathrm{e}^{-y^2} dy + c_1}{\mu(y)}$

- Substitute $\mu(y) = \mathrm{e}^{y^2}$

  $x = \frac{\int \mathrm{e}^{-y^2}\mathrm{e}^{y^2} dy + c_1}{\mathrm{e}^{y^2}}$

- Evaluate the integrals on the rhs

  $x = \frac{y + c_1}{\mathrm{e}^{y^2}}$

- Simplify

$$x = e^{-y^2}(y + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(x(y),y)+2*y*x(y)=exp(-y^2),x(y), singsol=all)
```

$$x(y) = (y + c_1) e^{-y^2}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 17

```
DSolve[x'[y]+2*y*x[y]==Exp[-y^2],x[y],y,IncludeSingularSolutions -> True]
```

$$x(y) \to e^{-y^2}(y + c_1)$$

5.5 problem Exercise 11.5, page 97

Internal problem ID [4499]
Internal file name [`OUTPUT/3992_Sunday_June_05_2022_12_00_08_PM_37636606/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.5, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{r' - (r + e^{-\theta}) \tan(\theta) = 0}$$

5.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r' + p(\theta)r = q(\theta)$$

Where here

$$p(\theta) = -\tan(\theta)$$
$$q(\theta) = \tan(\theta)\, e^{-\theta}$$

Hence the ode is

$$r' - \tan(\theta)\, r = \tan(\theta)\, e^{-\theta}$$

The integrating factor μ is

$$\mu = e^{\int -\tan(\theta)d\theta}$$
$$= \cos(\theta)$$

The ode becomes

$$\frac{d}{d\theta}(\mu r) = (\mu)\left(\tan(\theta)\,e^{-\theta}\right)$$
$$\frac{d}{d\theta}(\cos(\theta)\,r) = (\cos(\theta))\left(\tan(\theta)\,e^{-\theta}\right)$$
$$d(\cos(\theta)\,r) = \left(\sin(\theta)\,e^{-\theta}\right)\,d\theta$$

Integrating gives

$$\cos(\theta)\,r = \int \sin(\theta)\,e^{-\theta}\,d\theta$$
$$\cos(\theta)\,r = -\frac{e^{-\theta}\cos(\theta)}{2} - \frac{\sin(\theta)\,e^{-\theta}}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = \cos(\theta)$ results in

$$r = \sec(\theta)\left(-\frac{e^{-\theta}\cos(\theta)}{2} - \frac{\sin(\theta)\,e^{-\theta}}{2}\right) + c_1\sec(\theta)$$

which simplifies to

$$r = \frac{(-\tan(\theta) - 1)\,e^{-\theta}}{2} + c_1\sec(\theta)$$

Summary

The solution(s) found are the following

$$r = \frac{(-\tan(\theta) - 1)\,e^{-\theta}}{2} + c_1\sec(\theta) \tag{1}$$

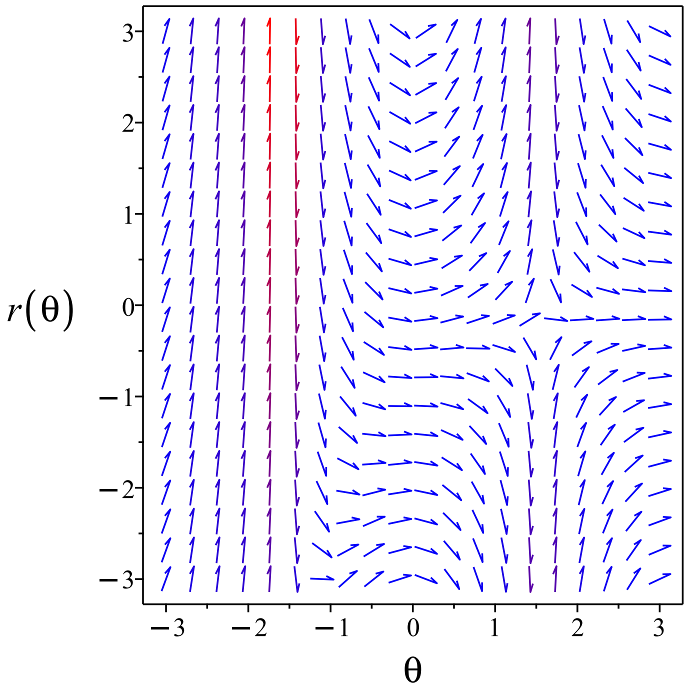


Figure 114: Slope field plot

Verification of solutions

$$r = \frac{(-\tan(\theta) - 1)\,e^{-\theta}}{2} + c_1 \sec(\theta)$$

Verified OK.

5.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$r' = \left(r + e^{-\theta}\right)\tan(\theta)$$
$$r' = \omega(\theta, r)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_\theta + \omega(\eta_r - \xi_\theta) - \omega^2 \xi_r - \omega_\theta \xi - \omega_r \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 52: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(\theta, r) &= 0 \\
\eta(\theta, r) &= \frac{1}{\cos(\theta)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(\theta, r) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{d\theta}{\xi} = \frac{dr}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial r}\right) S(\theta, r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = \theta$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\cos(\theta)}} dy \end{aligned}$$

Which results in

$$S = \cos(\theta) r$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_\theta + \omega(\theta, r) S_r}{R_\theta + \omega(\theta, r) R_r} \tag{2}$$

Where in the above $R_\theta, R_r, S_\theta, S_r$ are all partial derivatives and $\omega(\theta, r)$ is the right hand side of the original ode given by

$$\omega(\theta, r) = (r + e^{-\theta}) \tan(\theta)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_\theta &= 1 \\ R_r &= 0 \\ S_\theta &= -\sin(\theta) r \\ S_r &= \cos(\theta) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(\theta) e^{-\theta} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for θ, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(R) e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 - \frac{e^{-R}(\cos(R) + \sin(R))}{2} \tag{4}$$

To complete the solution, we just need to transform (4) back to θ, r coordinates. This results in

$$r \cos(\theta) = c_1 - \frac{e^{-\theta}(\cos(\theta) + \sin(\theta))}{2}$$

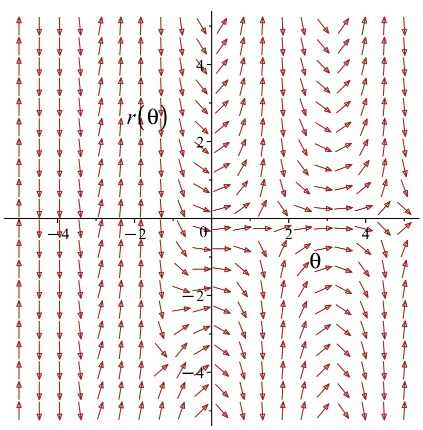
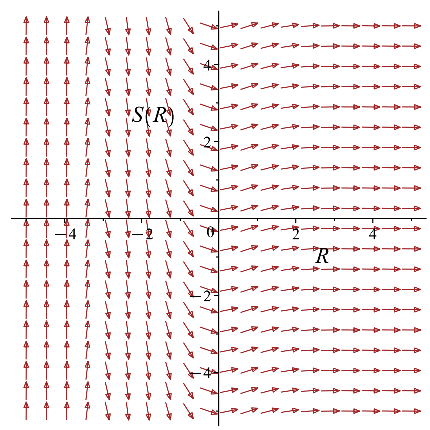
Which simplifies to

$$r \cos(\theta) = c_1 - \frac{e^{-\theta}(\cos(\theta) + \sin(\theta))}{2}$$

Which gives

$$r = -\frac{\sin(\theta) e^{-\theta} + e^{-\theta} \cos(\theta) - 2c_1}{2 \cos(\theta)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in θ, r coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dr}{d\theta} = (r + e^{-\theta}) \tan(\theta)$
</td><td>$R = \theta$
$S = \cos(\theta) r$</td><td>$\frac{dS}{dR} = \sin(R) e^{-R}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$r = -\frac{\sin(\theta) e^{-\theta} + e^{-\theta} \cos(\theta) - 2c_1}{2 \cos(\theta)} \tag{1}$$

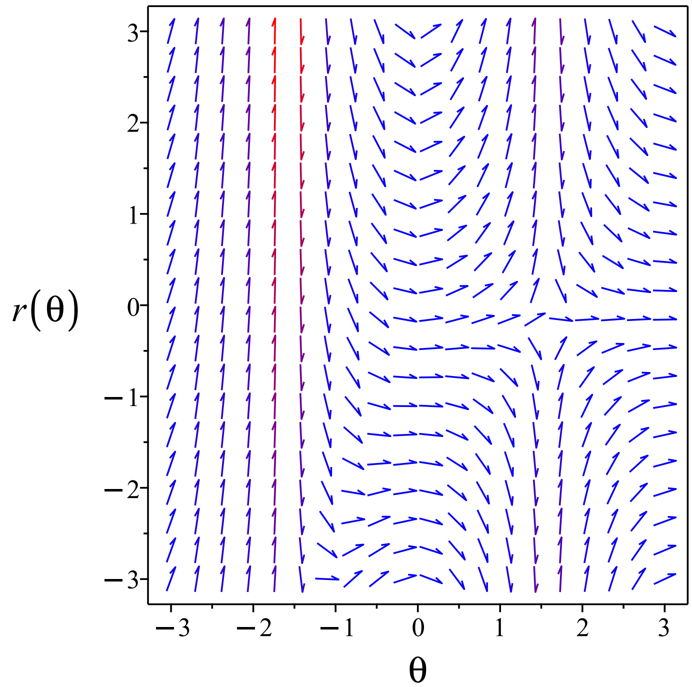


Figure 115: Slope field plot

Verification of solutions

$$r = -\frac{\sin(\theta)\,e^{-\theta} + e^{-\theta}\cos(\theta) - 2c_1}{2\cos(\theta)}$$

Verified OK.

5.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(\theta, r)\,\mathrm{d}\theta + N(\theta, r)\,\mathrm{d}r = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}r &= \left(\left(r + \mathrm{e}^{-\theta}\right)\tan(\theta)\right)\mathrm{d}\theta \\ \left(-\left(r + \mathrm{e}^{-\theta}\right)\tan(\theta)\right)\mathrm{d}\theta + \mathrm{d}r &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(\theta, r) &= -\left(r + \mathrm{e}^{-\theta}\right)\tan(\theta) \\ N(\theta, r) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}\left(-\left(r + \mathrm{e}^{-\theta}\right)\tan(\theta)\right) \\ &= -\tan(\theta)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial r} \neq \frac{\partial N}{\partial \theta}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial r} - \frac{\partial N}{\partial \theta}\right) \\ &= 1((-\tan(\theta)) - (0)) \\ &= -\tan(\theta) \end{aligned}$$

Since A does not depend on r, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}\theta} \\ &= e^{\int -\tan(\theta) \, \mathrm{d}\theta} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(\cos(\theta))} \\ &= \cos(\theta) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \cos(\theta)\left(-\left(r + \mathrm{e}^{-\theta}\right)\tan(\theta)\right) \\ &= -\left(r + \mathrm{e}^{-\theta}\right)\sin(\theta) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \cos(\theta)(1) \\ &= \cos(\theta) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0 \\ \left(-\left(r + \mathrm{e}^{-\theta}\right)\sin(\theta)\right) + (\cos(\theta))\frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(\theta, r)$

$$\frac{\partial \phi}{\partial \theta} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial r} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. θ gives

$$\int \frac{\partial \phi}{\partial \theta}\, \mathrm{d}\theta = \int \overline{M}\, \mathrm{d}\theta$$

$$\int \frac{\partial \phi}{\partial \theta}\, \mathrm{d}\theta = \int -\left(r + \mathrm{e}^{-\theta}\right) \sin\left(\theta\right) \mathrm{d}\theta$$

$$\phi = \frac{\mathrm{e}^{-\theta}(\cos\left(\theta\right) + \sin\left(\theta\right))}{2} + \cos\left(\theta\right) r + f(r) \tag{3}$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both θ and r. Taking derivative of equation (3) w.r.t r gives

$$\frac{\partial \phi}{\partial r} = \cos\left(\theta\right) + f'(r) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = \cos\left(\theta\right)$. Therefore equation (4) becomes

$$\cos\left(\theta\right) = \cos\left(\theta\right) + f'(r) \tag{5}$$

Solving equation (5) for $f'(r)$ gives

$$f'(r) = 0$$

Therefore

$$f(r) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(r)$ into equation (3) gives ϕ

$$\phi = \frac{\mathrm{e}^{-\theta}(\cos\left(\theta\right) + \sin\left(\theta\right))}{2} + \cos\left(\theta\right) r + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\mathrm{e}^{-\theta}(\cos\left(\theta\right) + \sin\left(\theta\right))}{2} + \cos\left(\theta\right) r$$

The solution becomes

$$r = -\frac{\sin\left(\theta\right) \mathrm{e}^{-\theta} + \mathrm{e}^{-\theta} \cos\left(\theta\right) - 2c_1}{2\cos\left(\theta\right)}$$

Summary

The solution(s) found are the following

$$r = -\frac{\sin(\theta)\,e^{-\theta} + e^{-\theta}\cos(\theta) - 2c_1}{2\cos(\theta)} \tag{1}$$

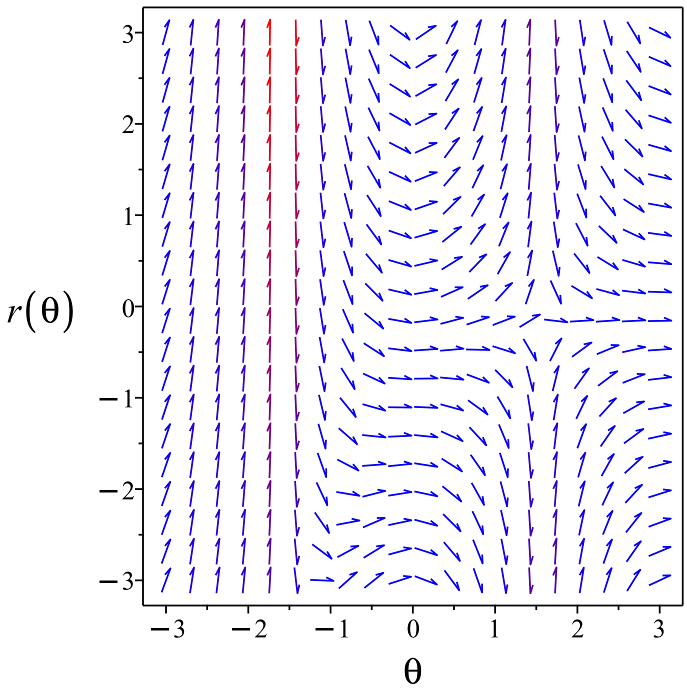


Figure 116: Slope field plot

Verification of solutions

$$r = -\frac{\sin(\theta)\,e^{-\theta} + e^{-\theta}\cos(\theta) - 2c_1}{2\cos(\theta)}$$

Verified OK.

5.5.4 Maple step by step solution

Let's solve

$r' - \left(r + e^{-\theta}\right)\tan(\theta) = 0$

- Highest derivative means the order of the ODE is 1

  r'

- Isolate the derivative

$$r' = \tan(\theta) r + \tan(\theta) e^{-\theta}$$

- Group terms with r on the lhs of the ODE and the rest on the rhs of the ODE

$$r' - \tan(\theta) r = \tan(\theta) e^{-\theta}$$

- The ODE is linear; multiply by an integrating factor $\mu(\theta)$

$$\mu(\theta) (r' - \tan(\theta) r) = \mu(\theta) \tan(\theta) e^{-\theta}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{d\theta}(\mu(\theta) r)$

$$\mu(\theta) (r' - \tan(\theta) r) = \mu'(\theta) r + \mu(\theta) r'$$

- Isolate $\mu'(\theta)$

$$\mu'(\theta) = -\mu(\theta) \tan(\theta)$$

- Solve to find the integrating factor

$$\mu(\theta) = \cos(\theta)$$

- Integrate both sides with respect to θ

$$\int \left(\frac{d}{d\theta}(\mu(\theta) r)\right) d\theta = \int \mu(\theta) \tan(\theta) e^{-\theta} d\theta + c_1$$

- Evaluate the integral on the lhs

$$\mu(\theta) r = \int \mu(\theta) \tan(\theta) e^{-\theta} d\theta + c_1$$

- Solve for r

$$r = \frac{\int \mu(\theta) \tan(\theta) e^{-\theta} d\theta + c_1}{\mu(\theta)}$$

- Substitute $\mu(\theta) = \cos(\theta)$

$$r = \frac{\int \tan(\theta) e^{-\theta} \cos(\theta) d\theta + c_1}{\cos(\theta)}$$

- Evaluate the integrals on the rhs

$$r = \frac{\frac{e^{-\theta}(-\cos(\theta) - \sin(\theta))}{2} + c_1}{\cos(\theta)}$$

- Simplify

$$r = \frac{(-\tan(\theta) - 1) e^{-\theta}}{2} + c_1 \sec(\theta)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(r(theta),theta)=(r(theta)+exp(-theta))*tan(theta),r(theta), singsol=all)
```

$$r(\theta) = \frac{(-\tan(\theta) - 1)\,\mathrm{e}^{-\theta}}{2} + \sec(\theta)\,c_1$$

✓ Solution by Mathematica

Time used: 0.093 (sec). Leaf size: 24

```
DSolve[r'[\[Theta]]==(r[\[Theta]]+Exp[-\[Theta]])*Tan[\[Theta]],r[\[Theta]],\[Theta],Include
```

$$r(\theta) \to -\frac{1}{2}e^{-\theta}(\tan(\theta) + 1) + c_1 \sec(\theta)$$

5.6 problem Exercise 11.6, page 97

Internal problem ID [4500]
Internal file name [`OUTPUT/3993_Sunday_June_05_2022_12_00_18_PM_70670884/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.6, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' - \frac{2xy}{x^2+1} = 1}$$

5.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2x}{x^2+1}$$
$$q(x) = 1$$

Hence the ode is

$$y' - \frac{2xy}{x^2+1} = 1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{2x}{x^2+1}dx} = \frac{1}{x^2+1}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2+1}\right) = \frac{1}{x^2+1}$$
$$\mathrm{d}\left(\frac{y}{x^2+1}\right) = \frac{1}{x^2+1}\mathrm{d}x$$

Integrating gives

$$\frac{y}{x^2+1} = \int \frac{1}{x^2+1}\,\mathrm{d}x$$
$$\frac{y}{x^2+1} = \arctan(x) + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2+1}$ results in

$$y = (x^2+1)\arctan(x) + c_1(x^2+1)$$

which simplifies to

$$y = (x^2+1)(\arctan(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = (x^2+1)(\arctan(x) + c_1) \tag{1}$$

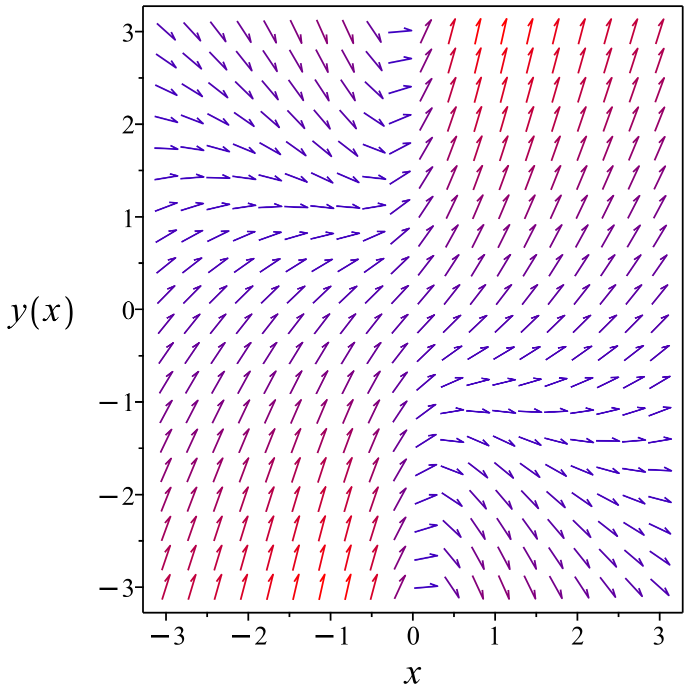


Figure 117: Slope field plot

Verification of solutions

$$y = (x^2 + 1)(\arctan(x) + c_1)$$

Verified OK.

5.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 + 2xy + 1}{x^2 + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 55: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^2 + 1
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2 + 1} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2 + 1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 + 2xy + 1}{x^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2yx}{(x^2 + 1)^2} \\ S_y &= \frac{1}{x^2 + 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan (R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2 + 1} = \arctan (x) + c_1$$

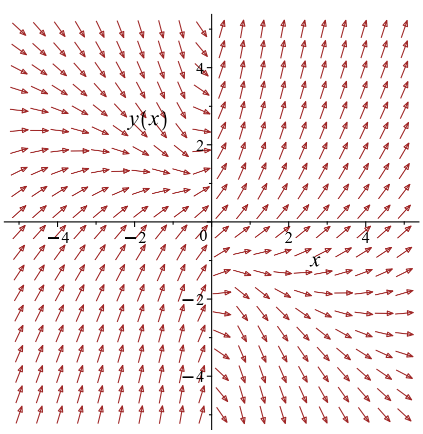
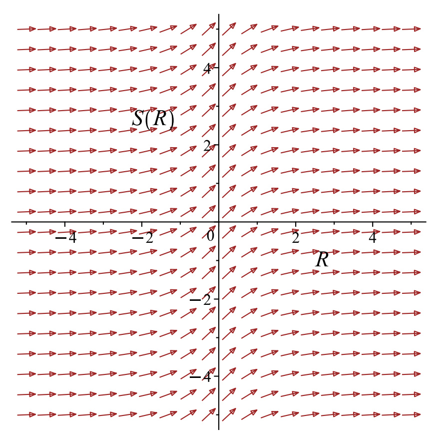
Which simplifies to

$$\frac{y}{x^2 + 1} = \arctan (x) + c_1$$

Which gives

$$y = (x^2 + 1) (\arctan (x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dy}{dx} = \frac{x^2+2xy+1}{x^2+1}$
</td><td>$R = x$
$S = \frac{y}{x^2 + 1}$</td><td>$\frac{dS}{dR} = \frac{1}{R^2+1}$
</td></tr>
</table>

<ins>Summary</ins>

The solution(s) found are the following

$$y = (x^2 + 1) (\arctan (x) + c_1) \tag{1}$$

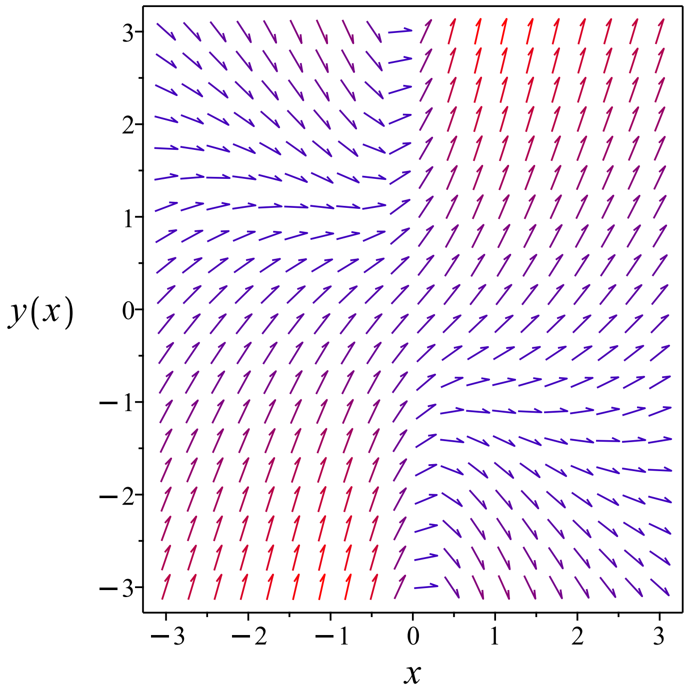


Figure 118: Slope field plot

Verification of solutions

$$y = (x^2 + 1)(\arctan(x) + c_1)$$

Verified OK.

5.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(\frac{2xy}{x^2+1} + 1\right)\mathrm{d}x \\ \left(-\frac{2xy}{x^2+1} - 1\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{2xy}{x^2+1} - 1 \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2xy}{x^2+1} - 1\right) \\ &= -\frac{2x}{x^2+1}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(-\frac{2x}{x^2+1}\right) - (0)\right)\\ &= -\frac{2x}{x^2+1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{2x}{x^2+1} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(x^2+1)}\\ &= \frac{1}{x^2+1}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2+1}\left(-\frac{2xy}{x^2+1} - 1\right)\\ &= \frac{-x^2 - 2xy - 1}{(x^2+1)^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^2+1}(1)\\ &= \frac{1}{x^2+1}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-x^2 - 2xy - 1}{(x^2+1)^2}\right) + \left(\frac{1}{x^2+1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-x^2 - 2xy - 1}{(x^2+1)^2}\,\mathrm{d}x$$
$$\phi = \frac{y}{x^2+1} - \arctan(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^2+1} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2+1}$. Therefore equation (4) becomes

$$\frac{1}{x^2+1} = \frac{1}{x^2+1} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y}{x^2+1} - \arctan(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y}{x^2+1} - \arctan(x)$$

Summary

The solution(s) found are the following

$$\frac{y}{x^2+1} - \arctan(x) = c_1 \tag{1}$$

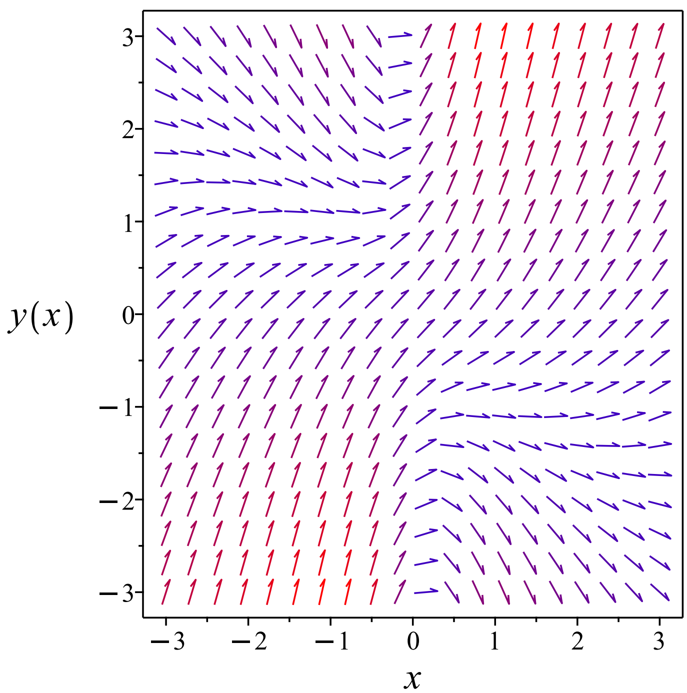


Figure 119: Slope field plot

Verification of solutions

$$\frac{y}{x^2+1} - \arctan(x) = c_1$$

Verified OK.

5.6.4 Maple step by step solution

Let's solve

$y' - \frac{2xy}{x^2+1} = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2xy}{x^2+1} + 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2xy}{x^2+1} = 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{2xy}{x^2+1}\right) = \mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{2xy}{x^2+1}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{2\mu(x)x}{x^2+1}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^2+1}$

  $y = (x^2+1)\left(\int \frac{1}{x^2+1}dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = (x^2+1)(\arctan(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)-(2*x*y(x))/(x^2+1)=1,y(x), singsol=all)
```

$$y(x) = (\arctan(x) + c_1)(x^2 + 1)$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 16

```
DSolve[y'[x]-2*x*y[x]/(x^2+1)==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x^2 + 1)(\arctan(x) + c_1)$$

5.7 problem Exercise 11.7, page 97

Internal problem ID [4501]
Internal file name [`OUTPUT/3994_Sunday_June_05_2022_12_00_27_PM_25304574/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.7, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y' + y - y^3 x = 0}$$

5.7.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= x\,y^3 - y \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -y + xy^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -1 \\
f_1(x) &= x \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y' \frac{1}{y^3} = -\frac{1}{y^2} + x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{2}{y^3} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= -w(x) + x \\
w' &= 2w - 2x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -2 \\
q(x) &= -2x
\end{aligned}
$$

Hence the ode is

$$
w'(x) - 2w(x) = -2x
$$

The integrating factor μ is

$$\mu = e^{\int (-2)dx}$$
$$= e^{-2x}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)(-2x)$$
$$\frac{d}{dx}(e^{-2x}w) = (e^{-2x})(-2x)$$
$$d(e^{-2x}w) = (-2x\,e^{-2x})\ dx$$

Integrating gives

$$e^{-2x}w = \int -2x\,e^{-2x}\,dx$$
$$e^{-2x}w = \frac{(2x+1)\,e^{-2x}}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-2x}$ results in

$$w(x) = \frac{e^{2x}(2x+1)\,e^{-2x}}{2} + c_1 e^{2x}$$

which simplifies to

$$w(x) = \frac{1}{2} + x + c_1 e^{2x}$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = \frac{1}{2} + x + c_1 e^{2x}$$

Solving for y gives

$$y(x) = \frac{2}{\sqrt{2 + 4c_1 e^{2x} + 4x}}$$
$$y(x) = -\frac{2}{\sqrt{2 + 4c_1 e^{2x} + 4x}}$$

Summary

The solution(s) found are the following

$$y = \frac{2}{\sqrt{2 + 4c_1 e^{2x} + 4x}} \tag{1}$$
$$y = -\frac{2}{\sqrt{2 + 4c_1 e^{2x} + 4x}} \tag{2}$$

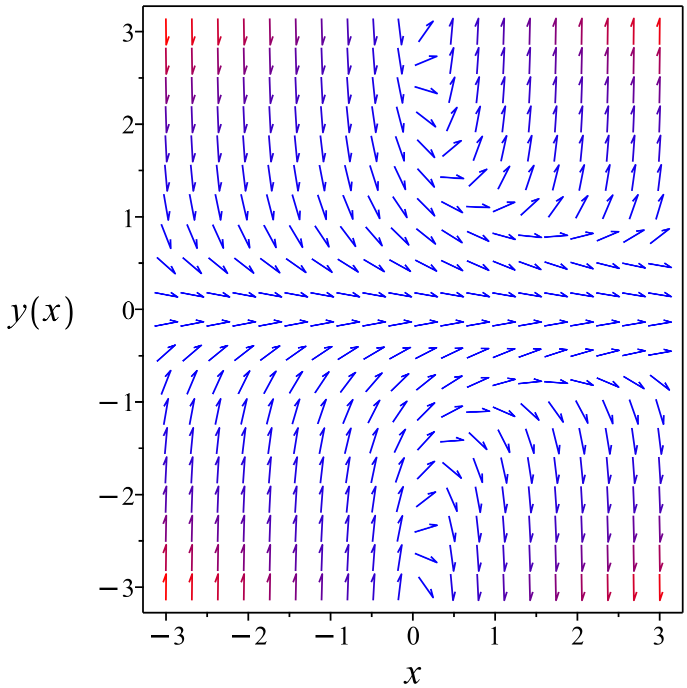


Figure 120: Slope field plot

Verification of solutions

$$y = \frac{2}{\sqrt{2 + 4c_1 e^{2x} + 4x}}$$

Verified OK.

$$y = -\frac{2}{\sqrt{2 + 4c_1 e^{2x} + 4x}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(x),x)+y(x)=x*y(x)^3,y(x), singsol=all)
```

$$y(x) = -\frac{2}{\sqrt{2 + 4\,\mathrm{e}^{2x} c_1 + 4x}}$$
$$y(x) = \frac{2}{\sqrt{2 + 4\,\mathrm{e}^{2x} c_1 + 4x}}$$

✓ Solution by Mathematica

Time used: 2.606 (sec). Leaf size: 50

```
DSolve[y'[x]+y[x]==x*y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{x + c_1 e^{2x} + \frac{1}{2}}}$$
$$y(x) \to \frac{1}{\sqrt{x + c_1 e^{2x} + \frac{1}{2}}}$$
$$y(x) \to 0$$

5.8 problem Exercise 11.8, page 97

Internal problem ID [4502]
Internal file name [OUTPUT/3995_Sunday_June_05_2022_12_00_39_PM_37364080/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.8, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{(-x^3+1)\,y'-2(x+1)\,y-y^{\frac{5}{2}}=0}$$

5.8.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y^{\frac{5}{2}}+2xy+2y}{x^3-1} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2+2x}{x^3-1}y-\frac{1}{x^3-1}y^{\frac{5}{2}} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y+f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n}+f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{2+2x}{x^3-1} \\ f_1(x) &= -\frac{1}{x^3-1} \\ n &= \frac{5}{2} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^{\frac{5}{2}}$ gives

$$y' \frac{1}{y^{\frac{5}{2}}} = -\frac{2+2x}{(x^3-1)\, y^{\frac{3}{2}}} - \frac{1}{x^3-1} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^{\frac{3}{2}}} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{3}{2y^{\frac{5}{2}}} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{2w'(x)}{3} &= -\frac{(2+2x)\, w(x)}{x^3-1} - \frac{1}{x^3-1} \\ w' &= \frac{3(2+2x)\, w}{2\,(x^3-1)} + \frac{3}{2\,(x^3-1)} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{3+3x}{x^3-1} \\ q(x) &= \frac{3}{2x^3-2} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{(3+3x)\,w(x)}{x^3-1} = \frac{3}{2x^3-2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{3+3x}{x^3-1}dx} \\ &= \mathrm{e}^{-2\ln(x-1)+\ln(x^2+x+1)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{x^2+x+1}{(x-1)^2}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(\frac{3}{2x^3-2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(x^2+x+1)\,w}{(x-1)^2}\right) &= \left(\frac{x^2+x+1}{(x-1)^2}\right)\left(\frac{3}{2x^3-2}\right) \\ \mathrm{d}\left(\frac{(x^2+x+1)\,w}{(x-1)^2}\right) &= \left(\frac{3}{2\,(x-1)^3}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{(x^2+x+1)\,w}{(x-1)^2} &= \int \frac{3}{2\,(x-1)^3}\,\mathrm{d}x \\ \frac{(x^2+x+1)\,w}{(x-1)^2} &= -\frac{3}{4\,(x-1)^2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{x^2+x+1}{(x-1)^2}$ results in

$$w(x) = -\frac{3}{4\,(x^2+x+1)} + \frac{c_1(x-1)^2}{x^2+x+1}$$

which simplifies to

$$w(x) = \frac{-3+4c_1(x-1)^2}{4x^2+4x+4}$$

Replacing w in the above by $\frac{1}{y^{\frac{3}{2}}}$ using equation (5) gives the final solution.

$$\frac{1}{y^{\frac{3}{2}}} = \frac{-3+4c_1(x-1)^2}{4x^2+4x+4}$$

Solving for y gives

$$y(x) = \frac{2^{\frac{1}{3}} \left(\left(x^2+x+1\right) \left(4c_1x^2-8c_1x+4c_1-3\right)^2\right)^{\frac{2}{3}}}{8 \left(-\frac{3}{4}+c_1 \left(x-1\right)^2\right)^2}$$

$$y(x) = -\frac{2^{\frac{1}{3}} \left(\left(x^2+x+1\right) \left(4c_1x^2-8c_1x+4c_1-3\right)^2\right)^{\frac{2}{3}} \left(\sqrt{3}+i\right)^2}{32 \left(-\frac{3}{4}+c_1 \left(x-1\right)^2\right)^2}$$

$$y(x) = -\frac{2^{\frac{1}{3}} \left(\left(x^2+x+1\right) \left(4c_1x^2-8c_1x+4c_1-3\right)^2\right)^{\frac{2}{3}} \left(i-\sqrt{3}\right)^2}{32 \left(-\frac{3}{4}+c_1 \left(x-1\right)^2\right)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{2^{\frac{1}{3}} \left(\left(x^2+x+1\right) \left(4c_1x^2-8c_1x+4c_1-3\right)^2\right)^{\frac{2}{3}}}{8 \left(-\frac{3}{4}+c_1 \left(x-1\right)^2\right)^2} \tag{1}$$

$$y = -\frac{2^{\frac{1}{3}} \left(\left(x^2+x+1\right) \left(4c_1x^2-8c_1x+4c_1-3\right)^2\right)^{\frac{2}{3}} \left(\sqrt{3}+i\right)^2}{32 \left(-\frac{3}{4}+c_1 \left(x-1\right)^2\right)^2} \tag{2}$$

$$y = -\frac{2^{\frac{1}{3}} \left(\left(x^2+x+1\right) \left(4c_1x^2-8c_1x+4c_1-3\right)^2\right)^{\frac{2}{3}} \left(i-\sqrt{3}\right)^2}{32 \left(-\frac{3}{4}+c_1 \left(x-1\right)^2\right)^2} \tag{3}$$

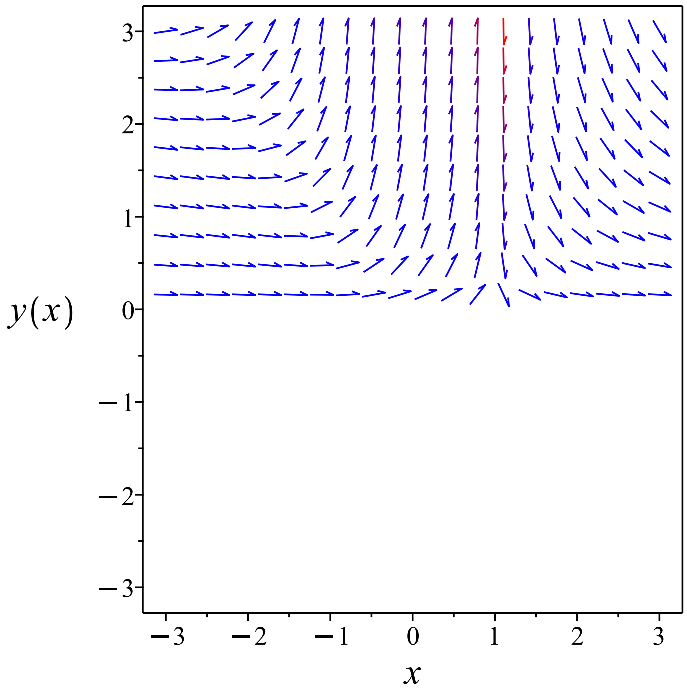


Figure 121: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{2^{\frac{1}{3}} \left((x^2 + x + 1) \left(4c_1 x^2 - 8c_1 x + 4c_1 - 3\right)^2 \right)^{\frac{2}{3}}}{8 \left(-\frac{3}{4} + c_1 (x-1)^2\right)^2}$$

Verified OK.

$$y = -\frac{2^{\frac{1}{3}} \left((x^2 + x + 1) \left(4c_1 x^2 - 8c_1 x + 4c_1 - 3\right)^2 \right)^{\frac{2}{3}} \left(\sqrt{3} + i\right)^2}{32 \left(-\frac{3}{4} + c_1 (x-1)^2\right)^2}$$

Verified OK.

$$y = -\frac{2^{\frac{1}{3}} \left((x^2 + x + 1) \left(4c_1 x^2 - 8c_1 x + 4c_1 - 3\right)^2 \right)^{\frac{2}{3}} \left(i - \sqrt{3}\right)^2}{32 \left(-\frac{3}{4} + c_1 (x-1)^2\right)^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 38

```
dsolve((1-x^3)*diff(y(x),x)-2*(1+x)*y(x)=y(x)^(5/2),y(x), singsol=all)
```

$$-\frac{(x-1)^2 c_1}{x^2+x+1}+\frac{1}{y(x)^{\frac{3}{2}}}+\frac{3}{4x^2+4x+4}=0$$

✓ Solution by Mathematica

Time used: 3.024 (sec). Leaf size: 41

```
DSolve[(1-x^3)*y'[x]-2*(1+x)*y[x]==y[x]^(5/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{2\sqrt[3]{2}}{\left(\frac{-3+4c_1(x-1)^2}{x^2+x+1}\right)^{2/3}}$$
$$y(x)\to 0$$

5.9 problem Exercise 11.9, page 97

Internal problem ID [4503]
Internal file name [`OUTPUT/3996_Sunday_June_05_2022_12_02_19_PM_30932864/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.9, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\tan(\theta)\, r' - r = \tan(\theta)^2}$$

5.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r' + p(\theta)r = q(\theta)$$

Where here

$$p(\theta) = -\cot(\theta)$$
$$q(\theta) = \tan(\theta)$$

Hence the ode is

$$r' - r\cot(\theta) = \tan(\theta)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\cot(\theta)d\theta} \\
&= \frac{1}{\sin(\theta)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \csc(\theta)
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{d\theta}(\mu r) &= (\mu)(\tan(\theta)) \\
\frac{d}{d\theta}(\csc(\theta) r) &= (\csc(\theta))(\tan(\theta)) \\
d(\csc(\theta) r) &= \sec(\theta)\ d\theta
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\csc(\theta) r &= \int \sec(\theta)\ d\theta \\
\csc(\theta) r &= \ln(\sec(\theta) + \tan(\theta)) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \csc(\theta)$ results in

$$
r = \sin(\theta)\ln(\sec(\theta) + \tan(\theta)) + c_1 \sin(\theta)
$$

which simplifies to

$$
r = \sin(\theta)(\ln(\sec(\theta) + \tan(\theta)) + c_1)
$$

<u>Summary</u>

The solution(s) found are the following

$$
r = \sin(\theta)(\ln(\sec(\theta) + \tan(\theta)) + c_1) \tag{1}
$$

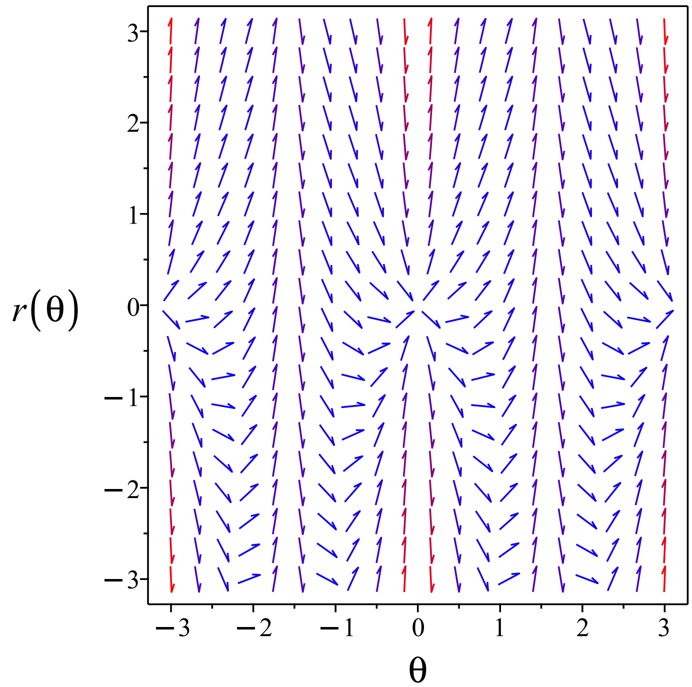


Figure 122: Slope field plot

Verification of solutions

$$r = \sin(\theta)(\ln(\sec(\theta) + \tan(\theta)) + c_1)$$

Verified OK.

5.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$r' = \frac{r + \tan(\theta)^2}{\tan(\theta)}$$
$$r' = \omega(\theta, r)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_\theta + \omega(\eta_r - \xi_\theta) - \omega^2 \xi_r - \omega_\theta \xi - \omega_r \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 58: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(\theta, r) &= 0 \\ \eta(\theta, r) &= \sin(\theta) \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(\theta, r) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{d\theta}{\xi} = \frac{dr}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial r}\right) S(\theta, r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = \theta$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sin(\theta)} dy \end{aligned}$$

Which results in

$$S = \frac{r}{\sin(\theta)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_\theta + \omega(\theta, r) S_r}{R_\theta + \omega(\theta, r) R_r} \tag{2}$$

Where in the above $R_\theta, R_r, S_\theta, S_r$ are all partial derivatives and $\omega(\theta, r)$ is the right hand side of the original ode given by

$$\omega(\theta, r) = \frac{r + \tan(\theta)^2}{\tan(\theta)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_\theta &= 1 \\ R_r &= 0 \\ S_\theta &= -\csc(\theta) \cot(\theta) r \\ S_r &= \csc(\theta) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sec(\theta) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for θ, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sec(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(\sec(R) + \tan(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to θ, r coordinates. This results in

$$\csc(\theta)\, r = \ln(\sec(\theta) + \tan(\theta)) + c_1$$

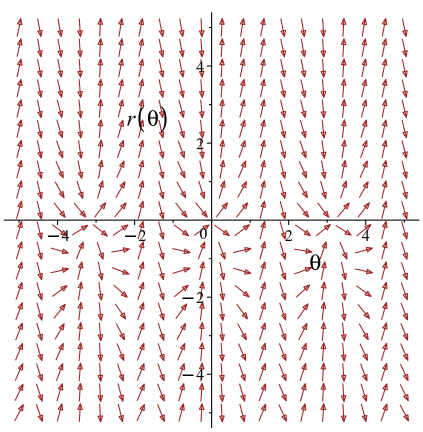
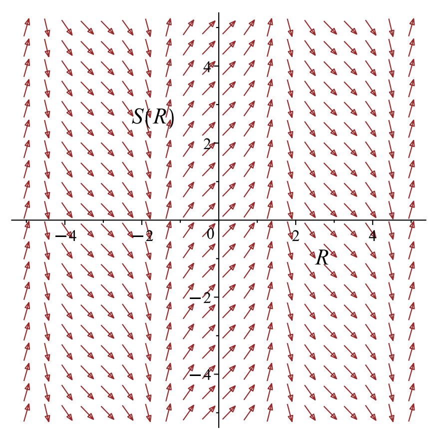
Which simplifies to

$$\csc(\theta)\, r = \ln(\sec(\theta) + \tan(\theta)) + c_1$$

Which gives

$$r = \frac{\ln(\sec(\theta) + \tan(\theta)) + c_1}{\csc(\theta)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in θ, r coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dr}{d\theta} = \frac{r+\tan(\theta)^2}{\tan(\theta)}$ 	$R = \theta$ $S = \csc(\theta)\, r$	$\frac{dS}{dR} = \sec(R)$ 

Summary

The solution(s) found are the following

$$r = \frac{\ln(\sec(\theta) + \tan(\theta)) + c_1}{\csc(\theta)} \tag{1}$$

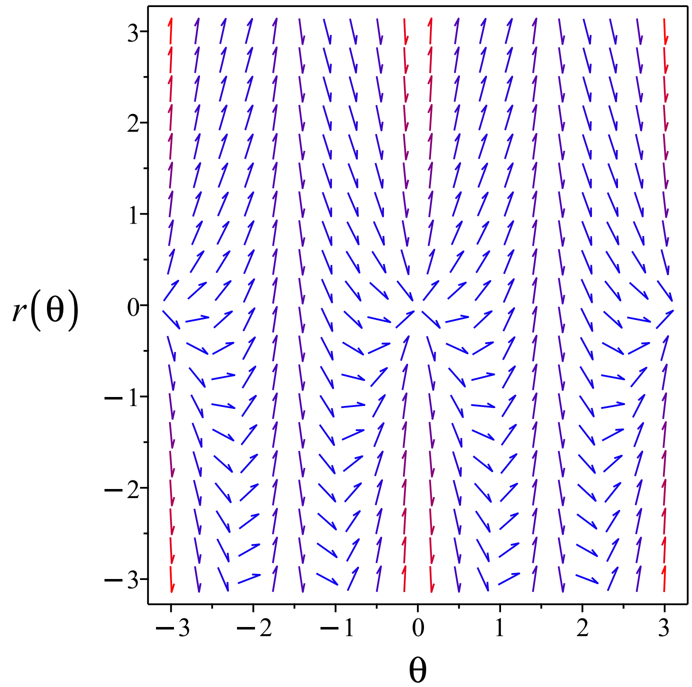


Figure 123: Slope field plot

Verification of solutions

$$r = \frac{\ln (\sec (\theta) + \tan (\theta)) + c_1}{\csc (\theta)}$$

Verified OK.

5.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(\theta, r)\, \mathrm{d}\theta + N(\theta, r)\, \mathrm{d}r = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(\tan(\theta))\, \mathrm{d}r &= \left(r + \tan(\theta)^2\right) \mathrm{d}\theta \\
\left(-r - \tan(\theta)^2\right) \mathrm{d}\theta + (\tan(\theta))\, \mathrm{d}r &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(\theta, r) &= -r - \tan(\theta)^2 \\
N(\theta, r) &= \tan(\theta)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}\left(-r - \tan(\theta)^2\right) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}(\tan(\theta)) \\
&= \sec(\theta)^2
\end{aligned}
$$

Since $\frac{\partial M}{\partial r} \neq \frac{\partial N}{\partial \theta}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial r} - \frac{\partial N}{\partial \theta}\right) \\
&= \cot(\theta)\left((-1) - \left(1 + \tan(\theta)^2\right)\right) \\
&= -2\cot(\theta) - \tan(\theta)
\end{aligned}
$$

Since A does not depend on r, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}\theta} \\
&= e^{\int -2\cot(\theta)-\tan(\theta) \, \mathrm{d}\theta}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(\sin(\theta))+\ln(\cos(\theta))} \\
&= \frac{\cos(\theta)}{\sin(\theta)^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\cos(\theta)}{\sin(\theta)^2}\left(-r - \tan(\theta)^2\right) \\
&= -\csc(\theta)\cot(\theta)\, r - \sec(\theta)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\cos(\theta)}{\sin(\theta)^2}(\tan(\theta)) \\
&= \csc(\theta)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0 \\
(-\csc(\theta)\cot(\theta)\, r - \sec(\theta)) + (\csc(\theta))\frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(\theta, r)$

$$\frac{\partial \phi}{\partial \theta} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial r} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. θ gives

$$\int \frac{\partial \phi}{\partial \theta} \, \mathrm{d}\theta = \int \overline{M} \, \mathrm{d}\theta$$
$$\int \frac{\partial \phi}{\partial \theta} \, \mathrm{d}\theta = \int -\csc(\theta)\cot(\theta)\, r - \sec(\theta) \, \mathrm{d}\theta$$
$$\phi = \csc(\theta)\, r - \ln(\sec(\theta) + \tan(\theta)) + f(r) \tag{3}$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both θ and r. Taking derivative of equation (3) w.r.t r gives

$$\frac{\partial \phi}{\partial r} = \csc(\theta) + f'(r) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = \csc(\theta)$. Therefore equation (4) becomes

$$\csc(\theta) = \csc(\theta) + f'(r) \tag{5}$$

Solving equation (5) for $f'(r)$ gives

$$f'(r) = 0$$

Therefore

$$f(r) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(r)$ into equation (3) gives ϕ

$$\phi = \csc(\theta)\, r - \ln(\sec(\theta) + \tan(\theta)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \csc(\theta)\, r - \ln(\sec(\theta) + \tan(\theta))$$

The solution becomes

$$r = \frac{\ln(\sec(\theta) + \tan(\theta)) + c_1}{\csc(\theta)}$$

Summary

The solution(s) found are the following

$$r = \frac{\ln(\sec(\theta) + \tan(\theta)) + c_1}{\csc(\theta)} \tag{1}$$

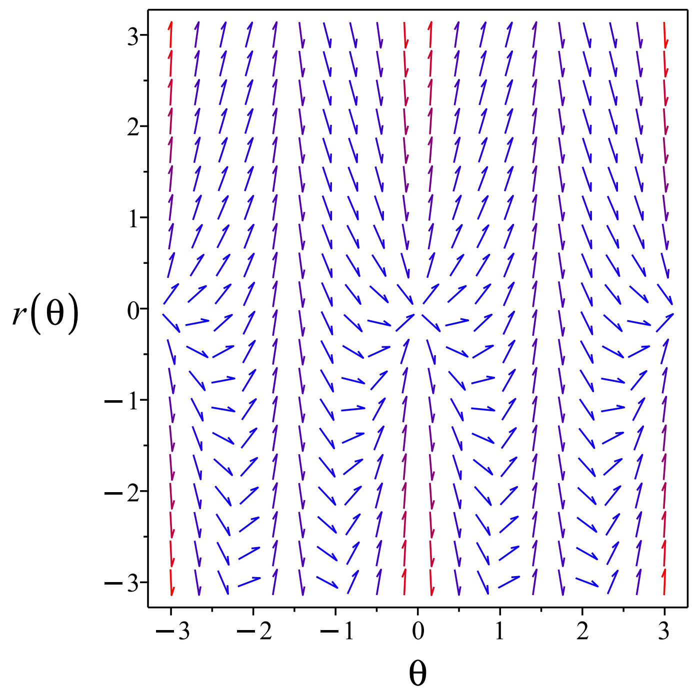


Figure 124: Slope field plot

Verification of solutions

$$r = \frac{\ln(\sec(\theta) + \tan(\theta)) + c_1}{\csc(\theta)}$$

Verified OK.

5.9.4 Maple step by step solution

Let's solve

$\tan(\theta) r' - r = \tan(\theta)^2$

- Highest derivative means the order of the ODE is 1

  r'

- Isolate the derivative

  $r' = \frac{r}{\tan(\theta)} + \tan(\theta)$

- Group terms with r on the lhs of the ODE and the rest on the rhs of the ODE

  $r' - \frac{r}{\tan(\theta)} = \tan(\theta)$

- The ODE is linear; multiply by an integrating factor $\mu(\theta)$

  $\mu(\theta)\left(r' - \frac{r}{\tan(\theta)}\right) = \mu(\theta)\tan(\theta)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{d\theta}(\mu(\theta) r)$

  $\mu(\theta)\left(r' - \frac{r}{\tan(\theta)}\right) = \mu'(\theta) r + \mu(\theta) r'$

- Isolate $\mu'(\theta)$

  $\mu'(\theta) = -\frac{\mu(\theta)}{\tan(\theta)}$

- Solve to find the integrating factor

  $\mu(\theta) = \frac{1}{\sin(\theta)}$

- Integrate both sides with respect to θ

  $\int \left(\frac{d}{d\theta}(\mu(\theta) r)\right) d\theta = \int \mu(\theta)\tan(\theta) d\theta + c_1$

- Evaluate the integral on the lhs

  $\mu(\theta) r = \int \mu(\theta)\tan(\theta) d\theta + c_1$

- Solve for r

  $r = \frac{\int \mu(\theta)\tan(\theta)d\theta + c_1}{\mu(\theta)}$

- Substitute $\mu(\theta) = \frac{1}{\sin(\theta)}$

  $r = \sin(\theta)\left(\int \frac{\tan(\theta)}{\sin(\theta)} d\theta + c_1\right)$

- Evaluate the integrals on the rhs

  $r = \sin(\theta)\left(\ln(\sec(\theta) + \tan(\theta)) + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(tan(theta)*diff(r(theta),theta)-r(theta)=tan(theta)^2,r(theta), singsol=all)
```

$$r(\theta) = (\ln (\sec (\theta) + \tan (\theta)) + c_1) \sin (\theta)$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 14

```
DSolve[Tan[\[Theta]]*r'[\[Theta]]-r[\[Theta]]==Tan[\[Theta]]^2,r[\[Theta]],\[Theta],IncludeS
```

$$r(\theta) \to \sin(\theta) \left(\coth^{-1}(\sin(\theta)) + c_1\right)$$

5.10 problem Exercise 11.11, page 97

Internal problem ID [4504]
Internal file name [`OUTPUT/3997_Sunday_June_05_2022_12_02_29_PM_58167478/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.11, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 2y = 3\,e^{-2x}}$$

5.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2$$
$$q(x) = 3\,e^{-2x}$$

Hence the ode is

$$y' + 2y = 3\,e^{-2x}$$

The integrating factor μ is

$$\mu = e^{\int 2dx}$$
$$= e^{2x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(3\,e^{-2x}\right)$$
$$\frac{d}{dx}\left(e^{2x}y\right) = \left(e^{2x}\right)\left(3\,e^{-2x}\right)$$
$$d\left(e^{2x}y\right) = 3\,dx$$

Integrating gives

$$e^{2x}y = \int 3\,dx$$
$$e^{2x}y = 3x + c_1$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = 3x\,e^{-2x} + c_1e^{-2x}$$

which simplifies to

$$y = e^{-2x}(3x + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(3x + c_1) \tag{1}$$

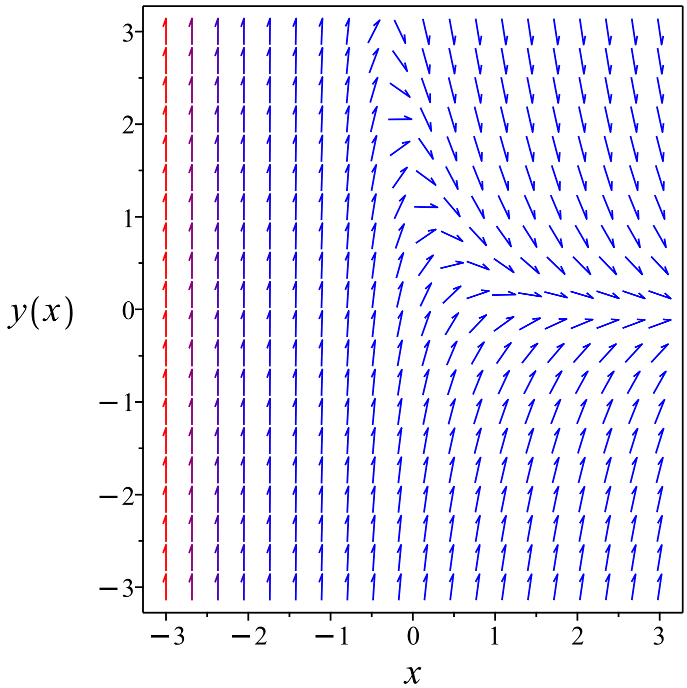


Figure 125: Slope field plot

Verification of solutions

$$y = e^{-2x}(3x + c_1)$$

Verified OK.

5.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2y + 3e^{-2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 61: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{-2x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-2x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{2x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2y + 3\,\mathrm{e}^{-2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2\,\mathrm{e}^{2x} y \\ S_y &= \mathrm{e}^{2x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 3R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{2x} y = 3x + c_1$$

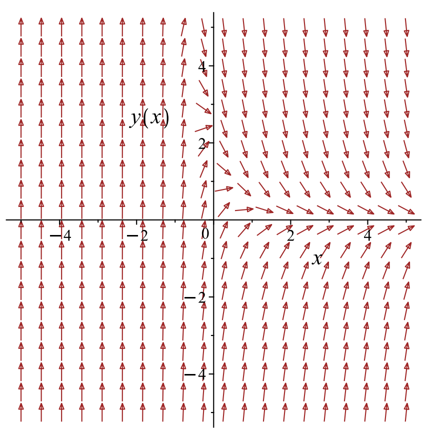
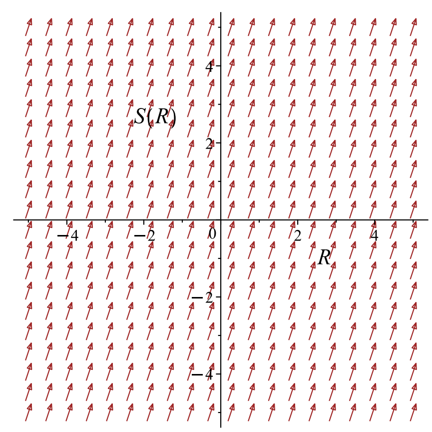
Which simplifies to

$$e^{2x} y = 3x + c_1$$

Which gives

$$y = e^{-2x}(3x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2y + 3\,e^{-2x}$ 	$R = x$ $S = e^{2x} y$	$\frac{dS}{dR} = 3$ 

Summary
The solution(s) found are the following

$$y = e^{-2x}(3x + c_1) \tag{1}$$

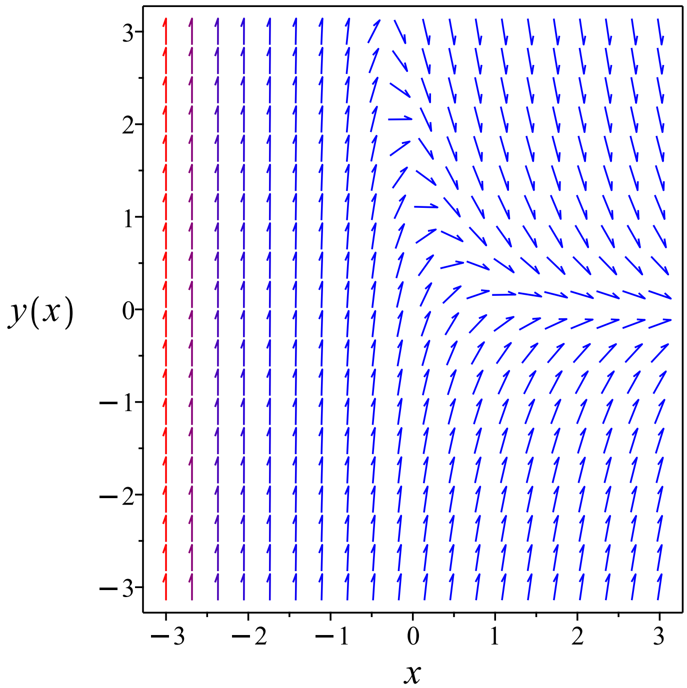


Figure 126: Slope field plot

Verification of solutions

$$y = e^{-2x}(3x + c_1)$$

Verified OK.

5.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(-2y + 3\,\mathrm{e}^{-2x}\right) \mathrm{d}x \\
\left(2y - 3\,\mathrm{e}^{-2x}\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 2y - 3\,\mathrm{e}^{-2x} \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y - 3\,\mathrm{e}^{-2x}\right) \\
&= 2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((2) - (0)) \\
&= 2
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 2 \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{2x} \\
&= \mathrm{e}^{2x}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{2x}\left(2y - 3\,\mathrm{e}^{-2x}\right) \\
&= 2\,\mathrm{e}^{2x} y - 3
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{2x}(1) \\
&= \mathrm{e}^{2x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(2\,\mathrm{e}^{2x} y - 3\right) + \left(\mathrm{e}^{2x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int 2\,\mathrm{e}^{2x} y - 3 \,\mathrm{d}x$$
$$\phi = -3x + \mathrm{e}^{2x} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x}$. Therefore equation (4) becomes

$$\mathrm{e}^{2x} = \mathrm{e}^{2x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -3x + \mathrm{e}^{2x} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -3x + \mathrm{e}^{2x} y$$

The solution becomes

$$y = \mathrm{e}^{-2x}(3x + c_1)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2x}(3x + c_1) \tag{1}$$

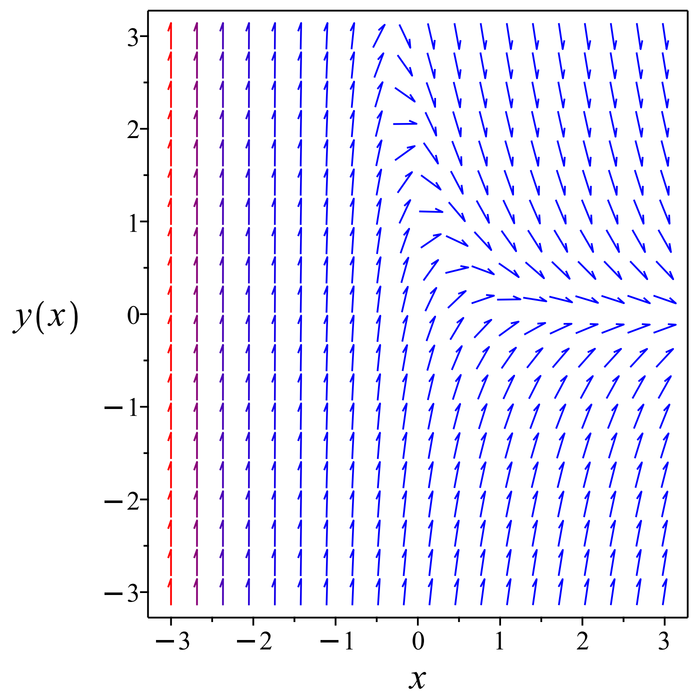


Figure 127: Slope field plot

Verification of solutions

$$y = e^{-2x}(3x + c_1)$$

Verified OK.

5.10.4 Maple step by step solution

Let's solve

$y' + 2y = 3\,e^{-2x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2y + 3\,e^{-2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2y = 3\,e^{-2x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + 2y) = 3\mu(x)\,e^{-2x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\,(y'+2y) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = 2\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{2x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 3\mu(x)\, \mathrm{e}^{-2x} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 3\mu(x)\, \mathrm{e}^{-2x} dx + c_1$
- Solve for y

  $y = \frac{\int 3\mu(x)\mathrm{e}^{-2x} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{2x}$

  $y = \frac{\int 3\,\mathrm{e}^{-2x}\mathrm{e}^{2x} dx + c_1}{\mathrm{e}^{2x}}$
- Evaluate the integrals on the rhs

  $y = \frac{3x+c_1}{\mathrm{e}^{2x}}$
- Simplify

  $y = \mathrm{e}^{-2x}(3x + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)+2*y(x)=3*exp(-2*x),y(x), singsol=all)
```

$$y(x) = (3x + c_1) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 17

```
DSolve[y'[x]+2*y[x]==3*Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}(3x + c_1)$$

5.11 problem Exercise 11.12, page 97

Internal problem ID [4505]
Internal file name [`OUTPUT/3998_Sunday_June_05_2022_12_02_38_PM_62524552/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.12, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 2y = \frac{3\,\mathrm{e}^{-2x}}{4}}$$

5.11.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2$$
$$q(x) = \frac{3\,\mathrm{e}^{-2x}}{4}$$

Hence the ode is

$$y' + 2y = \frac{3\,\mathrm{e}^{-2x}}{4}$$

The integrating factor μ is

$$\mu = e^{\int 2dx}$$
$$= e^{2x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{3\,e^{-2x}}{4}\right)$$
$$\frac{d}{dx}\left(e^{2x}y\right) = \left(e^{2x}\right)\left(\frac{3\,e^{-2x}}{4}\right)$$
$$d\left(e^{2x}y\right) = \frac{3}{4}\,dx$$

Integrating gives

$$e^{2x}y = \int \frac{3}{4}\,dx$$
$$e^{2x}y = \frac{3x}{4} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = \frac{3x\,e^{-2x}}{4} + c_1 e^{-2x}$$

which simplifies to

$$y = e^{-2x}\left(\frac{3x}{4} + c_1\right)$$

Summary
The solution(s) found are the following

$$y = e^{-2x}\left(\frac{3x}{4} + c_1\right) \tag{1}$$

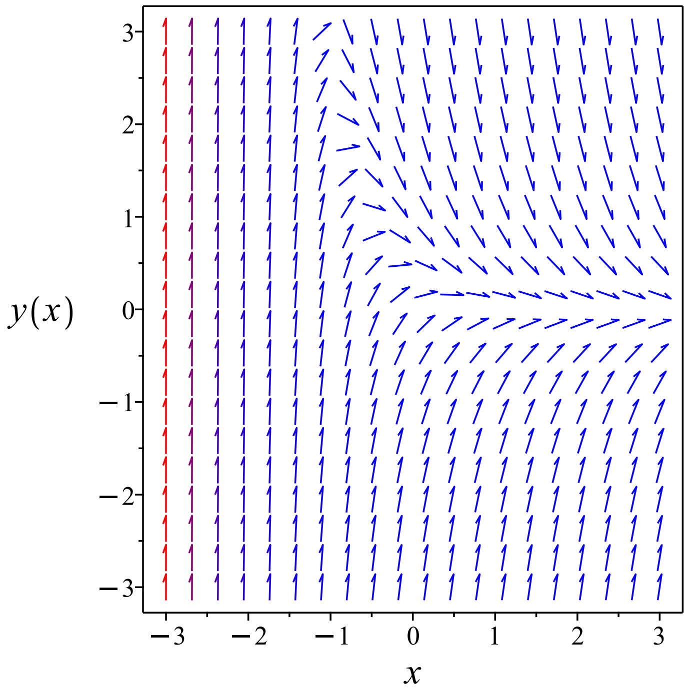


Figure 128: Slope field plot

Verification of solutions

$$y = e^{-2x}\left(\frac{3x}{4} + c_1\right)$$

Verified OK.

5.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2y + \frac{3\,e^{-2x}}{4}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 64: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= e^{-2x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-2x}} dy \end{aligned}$$

Which results in

$$S = e^{2x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -2y + \frac{3\,e^{-2x}}{4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2\,e^{2x} y \\ S_y &= e^{2x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3}{4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3}{4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{3R}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{2x} y = \frac{3x}{4} + c_1$$

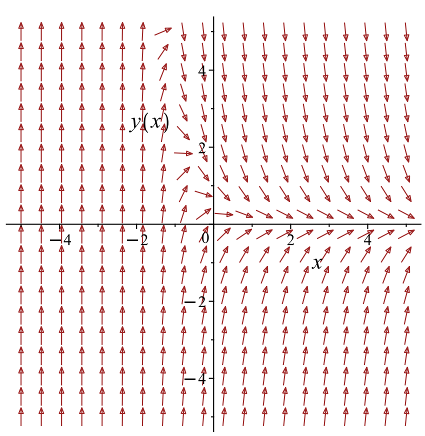
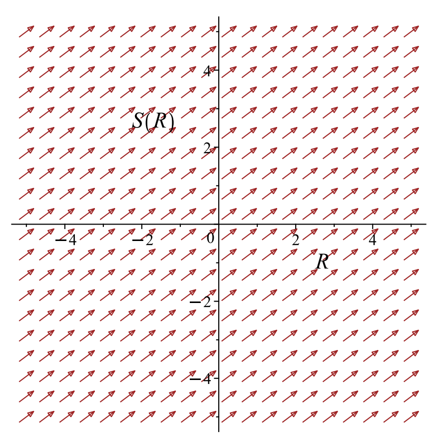
Which simplifies to

$$e^{2x} y = \frac{3x}{4} + c_1$$

Which gives

$$y = \frac{e^{-2x}(3x + 4c_1)}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2y + \frac{3\,e^{-2x}}{4}$ 	$R = x$ $S = e^{2x} y$	$\frac{dS}{dR} = \frac{3}{4}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{-2x}(3x + 4c_1)}{4} \tag{1}$$

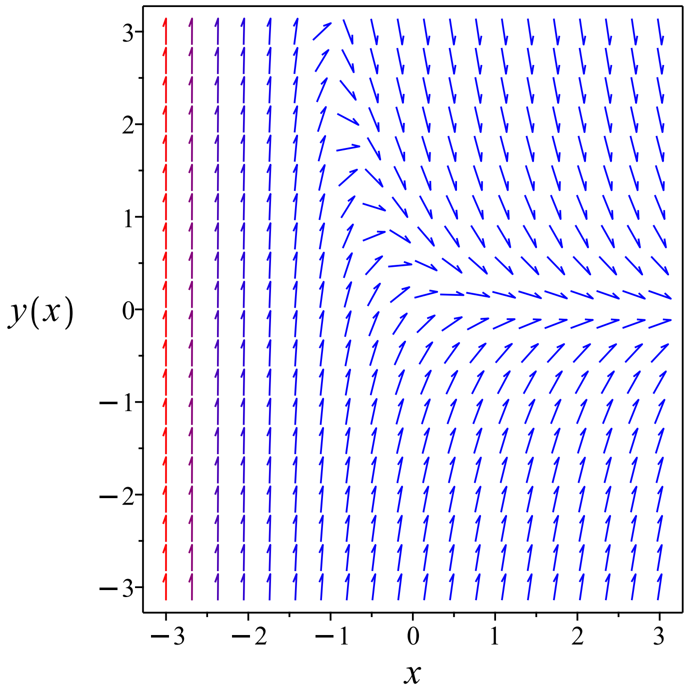


Figure 129: Slope field plot

Verification of solutions

$$y = \frac{e^{-2x}(3x + 4c_1)}{4}$$

Verified OK.

5.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-2y + \frac{3\,\mathrm{e}^{-2x}}{4}\right)\mathrm{d}x$$
$$\left(2y - \frac{3\,\mathrm{e}^{-2x}}{4}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 2y - \frac{3\,\mathrm{e}^{-2x}}{4}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y - \frac{3\,\mathrm{e}^{-2x}}{4}\right)\\ &= 2\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((2) - (0)) \\
&= 2
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 2 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2x} \\
&= \mathrm{e}^{2x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{2x}\left(2y - \frac{3\,\mathrm{e}^{-2x}}{4}\right) \\
&= 2\,\mathrm{e}^{2x} y - \frac{3}{4}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{2x}(1) \\
&= \mathrm{e}^{2x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(2\,\mathrm{e}^{2x} y - \frac{3}{4}\right) + \left(\mathrm{e}^{2x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \overline{M} \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int 2 \, e^{2x} y - \frac{3}{4} \, dx$$
$$\phi = -\frac{3x}{4} + e^{2x} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{2x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{2x}$. Therefore equation (4) becomes

$$e^{2x} = e^{2x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{3x}{4} + e^{2x} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{3x}{4} + e^{2x} y$$

The solution becomes

$$y = \frac{e^{-2x}(3x + 4c_1)}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-2x}(3x + 4c_1)}{4} \tag{1}$$

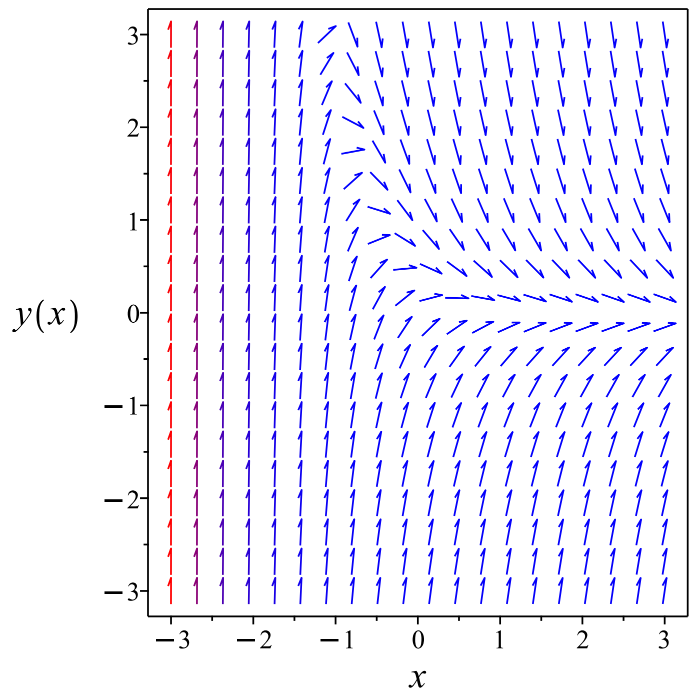


Figure 130: Slope field plot

Verification of solutions

$$y = \frac{e^{-2x}(3x + 4c_1)}{4}$$

Verified OK.

5.11.4 Maple step by step solution

Let's solve

$y' + 2y = \frac{3\,\mathrm{e}^{-2x}}{4}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2y + \frac{3\,\mathrm{e}^{-2x}}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2y = \frac{3\,\mathrm{e}^{-2x}}{4}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + 2y) = \frac{3\mu(x)\mathrm{e}^{-2x}}{4}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' + 2y) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = 2\mu(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{2x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \frac{3\mu(x)\mathrm{e}^{-2x}}{4}dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{3\mu(x)\mathrm{e}^{-2x}}{4}dx + c_1$

- Solve for y

  $y = \frac{\int \frac{3\mu(x)\mathrm{e}^{-2x}}{4}dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{2x}$

  $y = \frac{\int \frac{3\,\mathrm{e}^{-2x}\mathrm{e}^{2x}}{4}dx + c_1}{\mathrm{e}^{2x}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{3x}{4} + c_1}{\mathrm{e}^{2x}}$

- Simplify

$$y = \frac{e^{-2x}(3x+4c_1)}{4}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)+2*y(x)=3/4*exp(-2*x),y(x), singsol=all)
```

$$y(x) = \frac{(3x+4c_1)\,e^{-2x}}{4}$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 22

```
DSolve[y'[x]+2*y[x]==3/4*Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{4}e^{-2x}(3x+4c_1)$$

5.12 problem Exercise 11.11, page 97

Internal problem ID [4506]
Internal file name [`OUTPUT/3999_Sunday_June_05_2022_12_02_48_PM_1542658/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.11, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 2y = \sin(x)}$$

5.12.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2$$
$$q(x) = \sin(x)$$

Hence the ode is

$$y' + 2y = \sin(x)$$

The integrating factor μ is

$$\mu = e^{\int 2dx}$$
$$= e^{2x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(\sin(x))$$
$$\frac{d}{dx}(e^{2x}y) = (e^{2x})(\sin(x))$$
$$d(e^{2x}y) = (\sin(x)e^{2x})\, dx$$

Integrating gives

$$e^{2x}y = \int \sin(x)e^{2x}\, dx$$
$$e^{2x}y = -\frac{\cos(x)e^{2x}}{5} + \frac{2\sin(x)e^{2x}}{5} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = e^{-2x}\left(-\frac{\cos(x)e^{2x}}{5} + \frac{2\sin(x)e^{2x}}{5}\right) + c_1 e^{-2x}$$

which simplifies to

$$y = \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + c_1 e^{-2x}$$

Summary

The solution(s) found are the following

$$y = \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + c_1 e^{-2x} \tag{1}$$

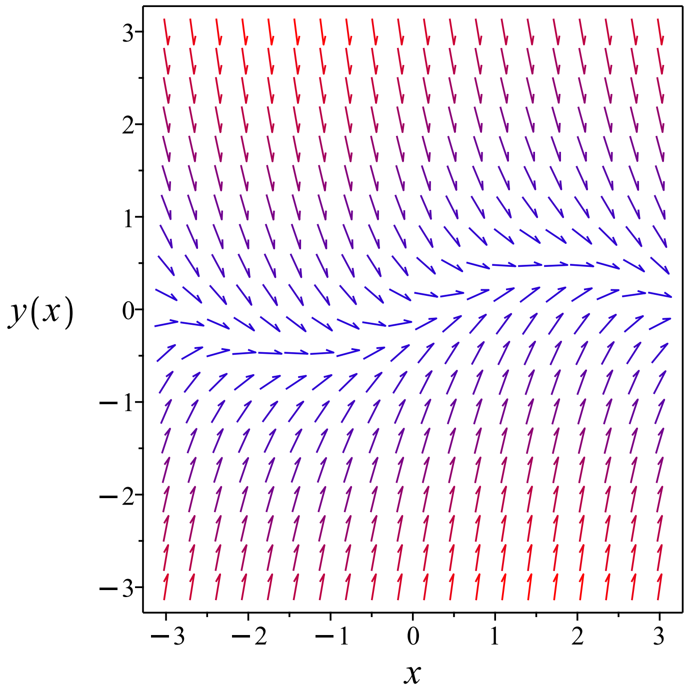


Figure 131: Slope field plot

Verification of solutions

$$y = \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + c_1 e^{-2x}$$

Verified OK.

5.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -2y + \sin(x) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 67: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= e^{-2x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-2x}} dy \end{aligned}$$

Which results in

$$S = e^{2x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2y + \sin(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2\,e^{2x} y \\ S_y &= e^{2x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(x)\, e^{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(R)\, e^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 - \frac{e^{2R}(\cos(R) - 2\sin(R))}{5} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{2x}y = c_1 - \frac{e^{2x}(\cos(x) - 2\sin(x))}{5}$$

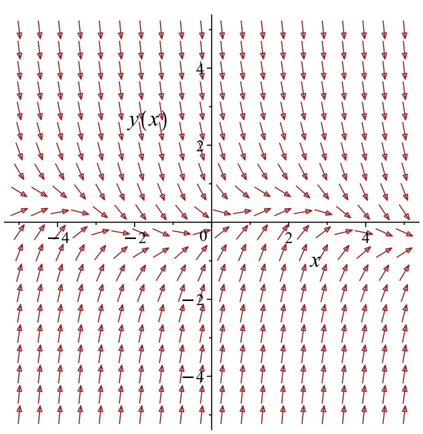
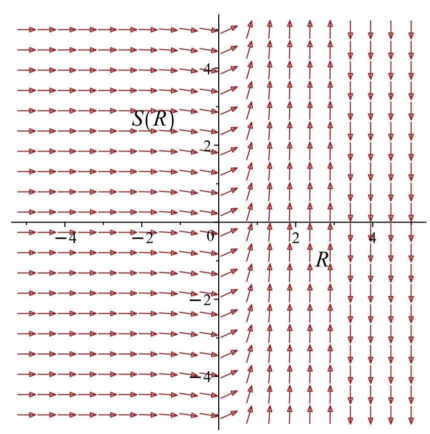
Which simplifies to

$$e^{2x}y = c_1 - \frac{e^{2x}(\cos(x) - 2\sin(x))}{5}$$

Which gives

$$y = \frac{e^{-2x}(2\sin(x)e^{2x} - \cos(x)e^{2x} + 5c_1)}{5}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2y + \sin(x)$ 	$R = x$ $S = e^{2x}y$	$\frac{dS}{dR} = \sin(R)e^{2R}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{-2x}(2\sin(x)e^{2x} - \cos(x)e^{2x} + 5c_1)}{5} \tag{1}$$

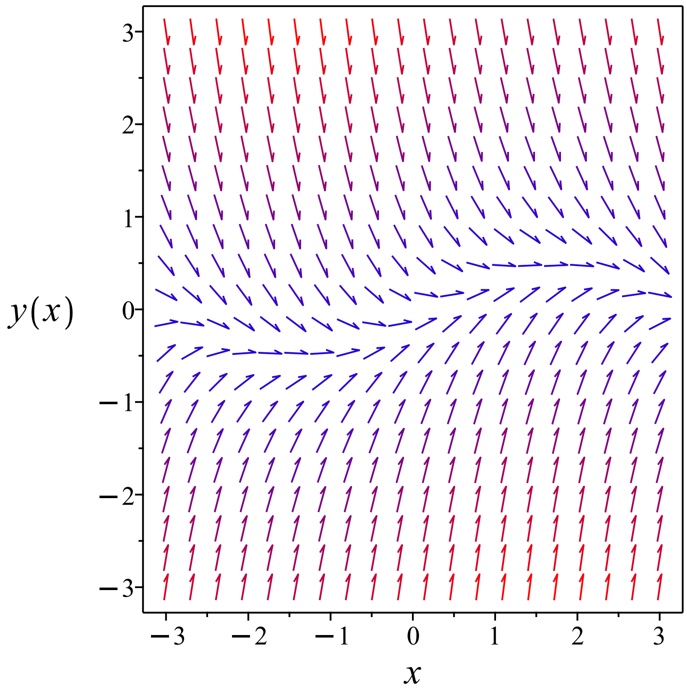


Figure 132: Slope field plot

Verification of solutions

$$y = \frac{e^{-2x}(2\sin(x)e^{2x} - \cos(x)e^{2x} + 5c_1)}{5}$$

Verified OK.

5.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-2y + \sin(x))\,\mathrm{d}x$$
$$(2y - \sin(x))\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 2y - \sin(x)$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2y - \sin(x)) \\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((2) - (0)) \\
&= 2
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 2 \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{2x} \\
&= \mathrm{e}^{2x}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{2x}(2y - \sin(x)) \\
&= (2y - \sin(x))\,\mathrm{e}^{2x}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{2x}(1) \\
&= \mathrm{e}^{2x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left((2y - \sin(x))\,\mathrm{e}^{2x}\right) + \left(\mathrm{e}^{2x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int (2y - \sin(x)) \mathrm{e}^{2x} \mathrm{d}x$$

$$\phi = \frac{(5y + \cos(x) - 2\sin(x)) \mathrm{e}^{2x}}{5} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x}$. Therefore equation (4) becomes

$$\mathrm{e}^{2x} = \mathrm{e}^{2x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(5y + \cos(x) - 2\sin(x)) \mathrm{e}^{2x}}{5} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(5y + \cos(x) - 2\sin(x)) \mathrm{e}^{2x}}{5}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-2x}(2\sin(x) \mathrm{e}^{2x} - \cos(x) \mathrm{e}^{2x} + 5c_1)}{5}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-2x}(2\sin(x)e^{2x} - \cos(x)e^{2x} + 5c_1)}{5} \tag{1}$$

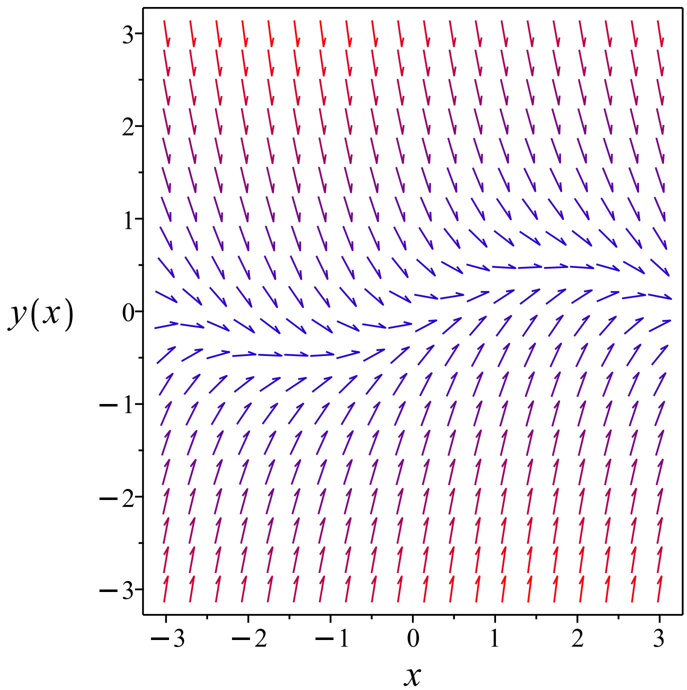


Figure 133: Slope field plot

Verification of solutions

$$y = \frac{e^{-2x}(2\sin(x)e^{2x} - \cos(x)e^{2x} + 5c_1)}{5}$$

Verified OK.

5.12.4 Maple step by step solution

Let's solve

$y' + 2y = \sin(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = -2y + \sin(x)$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE
$$y' + 2y = \sin(x)$$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
$$\mu(x)(y' + 2y) = \mu(x)\sin(x)$$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$
$$\mu(x)(y' + 2y) = \mu'(x)\, y + \mu(x)\, y'$$
- Isolate $\mu'(x)$
$$\mu'(x) = 2\mu(x)$$
- Solve to find the integrating factor
$$\mu(x) = \mathrm{e}^{2x}$$
- Integrate both sides with respect to x
$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\sin(x)\, dx + c_1$$
- Evaluate the integral on the lhs
$$\mu(x)\, y = \int \mu(x)\sin(x)\, dx + c_1$$
- Solve for y
$$y = \frac{\int \mu(x)\sin(x)dx + c_1}{\mu(x)}$$
- Substitute $\mu(x) = \mathrm{e}^{2x}$
$$y = \frac{\int \sin(x)\mathrm{e}^{2x}dx + c_1}{\mathrm{e}^{2x}}$$
- Evaluate the integrals on the rhs
$$y = \frac{\frac{2\sin(x)\mathrm{e}^{2x}}{5} - \frac{\cos(x)\mathrm{e}^{2x}}{5} + c_1}{\mathrm{e}^{2x}}$$
- Simplify
$$y = \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + c_1\mathrm{e}^{-2x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)+2*y(x)=sin(x),y(x), singsol=all)
```

$$y(x) = -\frac{\cos(x)}{5} + \frac{2\sin(x)}{5} + e^{-2x}c_1$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 26

```
DSolve[y'[x]+2*y[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + c_1e^{-2x}$$

5.13 problem Exercise 11.14, page 97

Internal problem ID [4507]
Internal file name [`OUTPUT/4000_Sunday_June_05_2022_12_02_58_PM_16891722/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.14, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cos(x) = e^{2x}}$$

5.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cos(x)$$
$$q(x) = e^{2x}$$

Hence the ode is

$$y' + y\cos(x) = e^{2x}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \cos(x)dx} \\
&= \mathrm{e}^{\sin(x)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\mathrm{e}^{2x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\sin(x)} y\right) &= \left(\mathrm{e}^{\sin(x)}\right)\left(\mathrm{e}^{2x}\right) \\
\mathrm{d}\left(\mathrm{e}^{\sin(x)} y\right) &= \mathrm{e}^{2x+\sin(x)}\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\sin(x)} y &= \int \mathrm{e}^{2x+\sin(x)}\,\mathrm{d}x \\
\mathrm{e}^{\sin(x)} y &= \int \mathrm{e}^{2x+\sin(x)} dx + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\sin(x)}$ results in

$$
y = \mathrm{e}^{-\sin(x)}\left(\int \mathrm{e}^{2x+\sin(x)} dx\right) + c_1 \mathrm{e}^{-\sin(x)}
$$

which simplifies to

$$
y = \mathrm{e}^{-\sin(x)}\left(\int \mathrm{e}^{2x+\sin(x)} dx + c_1\right)
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \mathrm{e}^{-\sin(x)}\left(\int \mathrm{e}^{2x+\sin(x)} dx + c_1\right) \tag{1}
$$

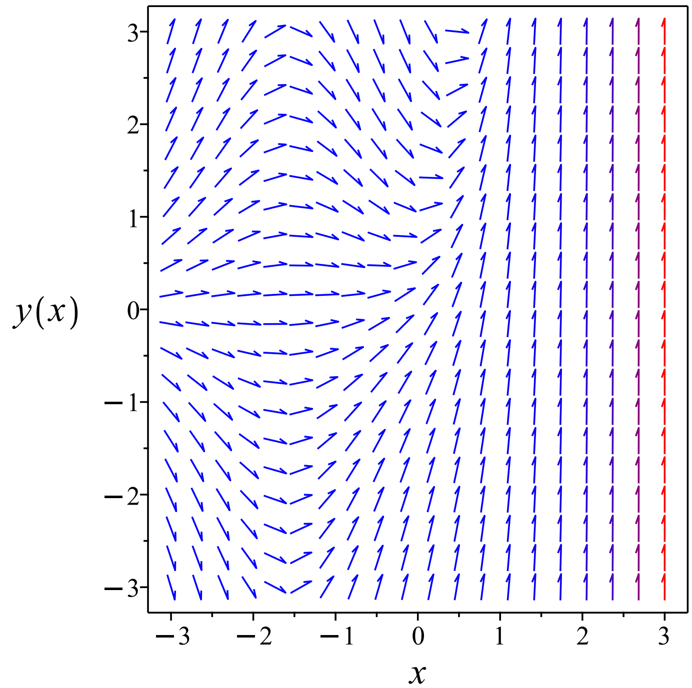


Figure 134: Slope field plot

Verification of solutions

$$y = e^{-\sin(x)} \left(\int e^{2x+\sin(x)} dx + c_1 \right)$$

Verified OK.

5.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y\cos(x) + e^{2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 70: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= e^{-\sin(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\sin(x)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\sin(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y\cos(x) + \mathrm{e}^{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x)\,\mathrm{e}^{\sin(x)} y \\ S_y &= \mathrm{e}^{\sin(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{2x+\sin(x)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{2R+\sin(R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int e^{2R+\sin(R)} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\sin(x)} y = \int e^{2x+\sin(x)} dx + c_1$$

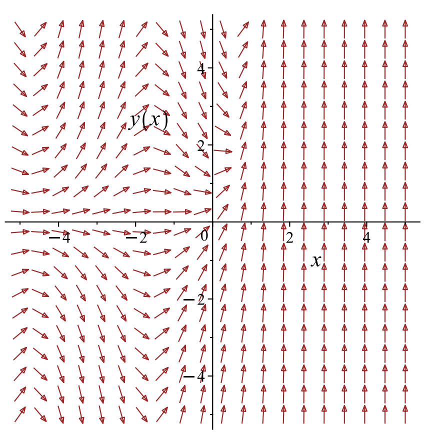
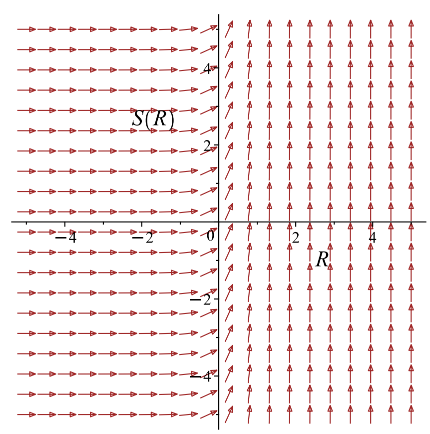
Which simplifies to

$$e^{\sin(x)} y = \int e^{2x+\sin(x)} dx + c_1$$

Which gives

$$y = \left(\int e^{2x+\sin(x)} dx + c_1\right) e^{-\sin(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y\cos(x) + e^{2x}$ 	$R = x$ $S = e^{\sin(x)} y$	$\frac{dS}{dR} = e^{2R+\sin(R)}$ 

Summary

The solution(s) found are the following

$$y = \left(\int e^{2x+\sin(x)} dx + c_1\right) e^{-\sin(x)} \tag{1}$$

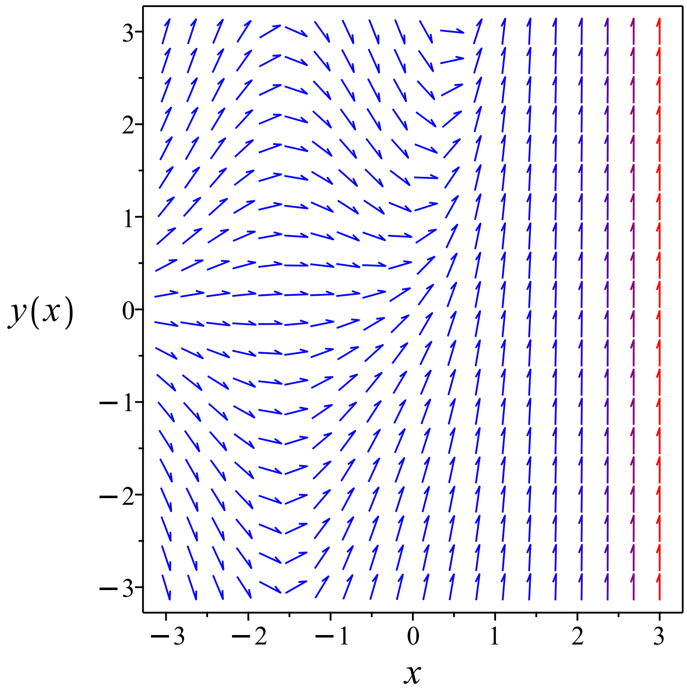


Figure 135: Slope field plot

Verification of solutions

$$y = \left(\int e^{2x+\sin(x)}dx + c_1\right) e^{-\sin(x)}$$

Verified OK.

5.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(-y \cos (x) + \mathrm{e}^{2x}\right) \mathrm{d}x\\ \left(y \cos (x) - \mathrm{e}^{2x}\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y \cos (x) - \mathrm{e}^{2x}\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y \cos (x) - \mathrm{e}^{2x}\right)\\ &= \cos (x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\cos(x)) - (0)) \\ &= \cos(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, dx} \\ &= e^{\int \cos(x) \, dx} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\sin(x)} \\ &= e^{\sin(x)} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= e^{\sin(x)}\left(y\cos(x) - e^{2x}\right) \\ &= \left(y\cos(x) - e^{2x}\right) e^{\sin(x)} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= e^{\sin(x)}(1) \\ &= e^{\sin(x)} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{dy}{dx} &= 0 \\ \left(\left(y\cos(x) - e^{2x}\right) e^{\sin(x)}\right) + \left(e^{\sin(x)}\right)\frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \left(y \cos(x) - \mathrm{e}^{2x}\right) \mathrm{e}^{\sin(x)} \mathrm{d}x$$

$$\phi = \int^{x} \left(y \cos(\_a) - \mathrm{e}^{2\_a}\right) \mathrm{e}^{\sin(\_a)} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{\sin(x)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\sin(x)}$. Therefore equation (4) becomes

$$\mathrm{e}^{\sin(x)} = \mathrm{e}^{\sin(x)} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^{x} \left(y \cos(\_a) - \mathrm{e}^{2\_a}\right) \mathrm{e}^{\sin(\_a)} d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^{x} \left(y \cos(\_a) - \mathrm{e}^{2\_a}\right) \mathrm{e}^{\sin(\_a)} d\_a$$

Summary

The solution(s) found are the following

$$\int^{x} \left(y \cos(\_a) - \mathrm{e}^{2\_a}\right) \mathrm{e}^{\sin(\_a)} d\_a = c_1 \tag{1}$$

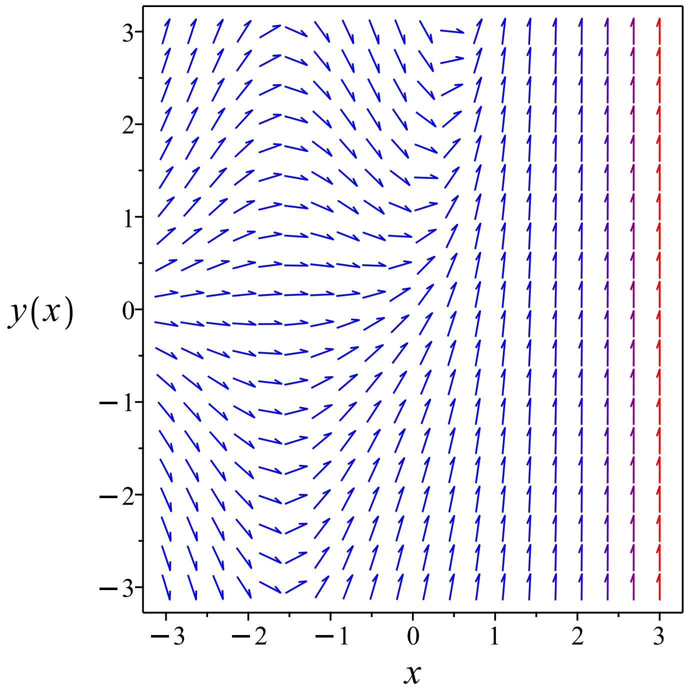


Figure 136: Slope field plot

Verification of solutions

$$\int^x \left(y\cos(\_a) - e^{2\_a}\right) e^{\sin(\_a)} d\_a = c_1$$

Verified OK.

5.13.4 Maple step by step solution

Let's solve

$y' + y\cos(x) = e^{2x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y\cos(x) + e^{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y\cos(x) = e^{2x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' + y\cos(x)\right) = \mu(x)\,\mathrm{e}^{2x}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y' + y\cos(x)\right) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \mu(x)\cos(x)$$

- Solve to find the integrating factor

$$\mu(x) = \mathrm{e}^{\sin(x)}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \mu(x)\,\mathrm{e}^{2x}dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \mu(x)\,\mathrm{e}^{2x}dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x)\mathrm{e}^{2x}dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \mathrm{e}^{\sin(x)}$

$$y = \frac{\int \mathrm{e}^{2x}\mathrm{e}^{\sin(x)}dx + c_1}{\mathrm{e}^{\sin(x)}}$$

- Simplify

$$y = \left(\int \mathrm{e}^{2x+\sin(x)}dx + c_1\right)\mathrm{e}^{-\sin(x)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(x),x)+y(x)*cos(x)=exp(2*x),y(x), singsol=all)
```

$$y(x) = \left(\int \mathrm{e}^{2x+\sin(x)}dx + c_1\right)\mathrm{e}^{-\sin(x)}$$

✓ Solution by Mathematica

Time used: 0.735 (sec). Leaf size: 32

```
DSolve[y'[x]+y[x]*Cos[x]==Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\sin(x)} \left(\int_1^x e^{2K[1]+\sin(K[1])} dK[1] + c_1 \right)$$

5.14 problem Exercise 11.15, page 97

Internal problem ID [4508]
Internal file name [`OUTPUT/4001_Sunday_June_05_2022_12_03_08_PM_81606101/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.15, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cos(x) = \frac{\sin(2x)}{2}}$$

5.14.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cos(x)$$
$$q(x) = \frac{\sin(2x)}{2}$$

Hence the ode is

$$y' + y\cos(x) = \frac{\sin(2x)}{2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \cos(x)dx} \\ &= e^{\sin(x)}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\sin(2x)}{2}\right) \\ \frac{d}{dx}\left(e^{\sin(x)}y\right) &= \left(e^{\sin(x)}\right)\left(\frac{\sin(2x)}{2}\right) \\ d\left(e^{\sin(x)}y\right) &= \left(\frac{\sin(2x)\,e^{\sin(x)}}{2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{\sin(x)}y &= \int \frac{\sin(2x)\,e^{\sin(x)}}{2}\,dx \\ e^{\sin(x)}y &= \sin(x)\,e^{\sin(x)} - e^{\sin(x)} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\sin(x)}$ results in

$$y = e^{-\sin(x)}\left(\sin(x)\,e^{\sin(x)} - e^{\sin(x)}\right) + c_1 e^{-\sin(x)}$$

which simplifies to

$$y = \sin(x) - 1 + c_1 e^{-\sin(x)}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \sin(x) - 1 + c_1 e^{-\sin(x)} \tag{1}$$

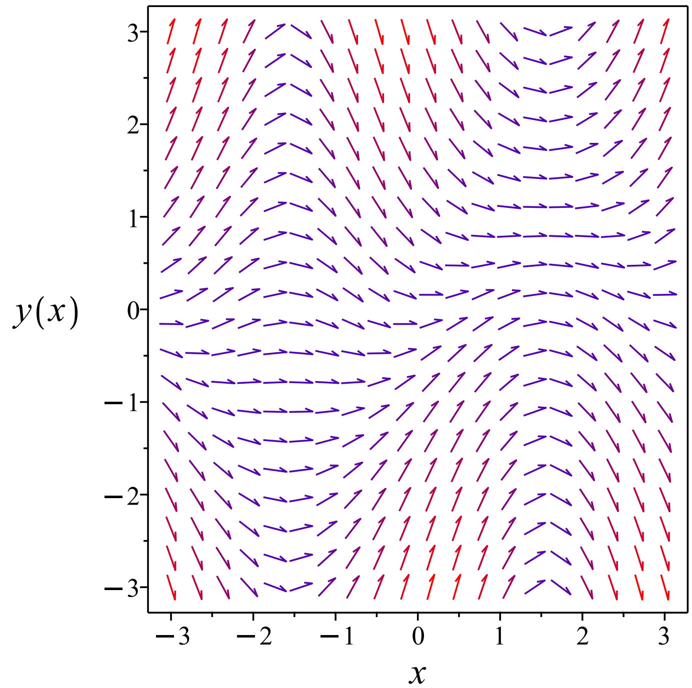


Figure 137: Slope field plot

Verification of solutions

$$y = \sin(x) - 1 + c_1 e^{-\sin(x)}$$

Verified OK.

5.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -y\cos(x) + \frac{\sin(2x)}{2} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 73: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{-\sin(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\sin(x)}} dy \end{aligned}$$

Which results in

$$S = e^{\sin(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -y \cos(x) + \frac{\sin(2x)}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x) e^{\sin(x)} y \\ S_y &= e^{\sin(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sin(2x) e^{\sin(x)}}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\sin(2R) e^{\sin(R)}}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 + e^{\sin(R)}(-1 + \sin(R)) \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\sin(x)} y = e^{\sin(x)}(-1 + \sin(x)) + c_1$$

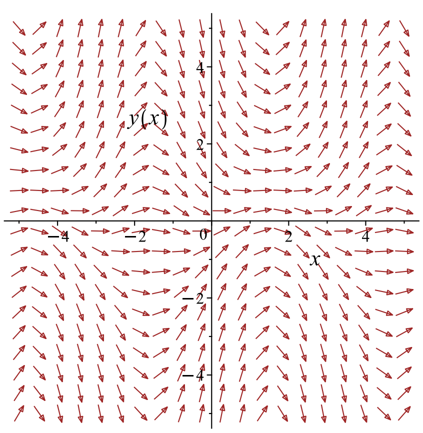
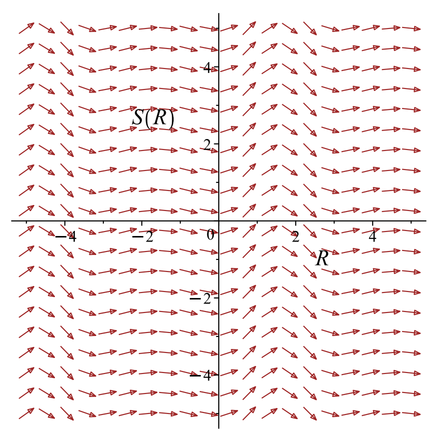
Which simplifies to

$$e^{\sin(x)} y = e^{\sin(x)}(-1 + \sin(x)) + c_1$$

Which gives

$$y = e^{-\sin(x)}\left(\sin(x) e^{\sin(x)} - e^{\sin(x)} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y\cos(x) + \frac{\sin(2x)}{2}$ 	$R = x$ $S = e^{\sin(x)} y$	$\frac{dS}{dR} = \frac{\sin(2R) e^{\sin(R)}}{2}$ 

Summary
The solution(s) found are the following

$$y = e^{-\sin(x)}\left(\sin(x) e^{\sin(x)} - e^{\sin(x)} + c_1\right) \tag{1}$$

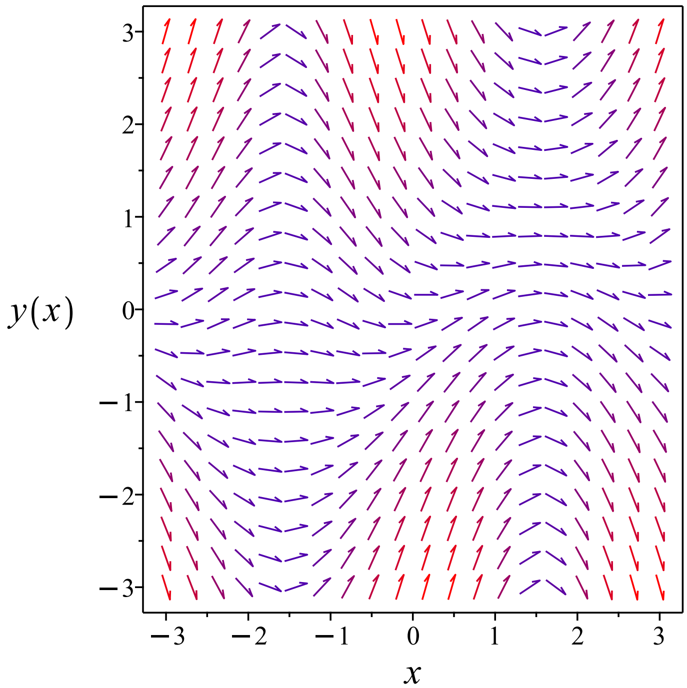


Figure 138: Slope field plot

Verification of solutions

$$y = e^{-\sin(x)}\left(\sin(x)\, e^{\sin(x)} - e^{\sin(x)} + c_1\right)$$

Verified OK.

5.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-y \cos(x) + \frac{\sin(2x)}{2}\right) \mathrm{d}x$$
$$\left(y \cos(x) - \frac{\sin(2x)}{2}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y \cos(x) - \frac{\sin(2x)}{2}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y \cos(x) - \frac{\sin(2x)}{2}\right)\\ &= \cos(x)\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((\cos(x)) - (0)) \\
&= \cos(x)
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int \cos(x) \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\sin(x)} \\
&= \mathrm{e}^{\sin(x)}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{\sin(x)}\left(y\cos(x) - \frac{\sin(2x)}{2}\right) \\
&= \cos(x)(-\sin(x) + y)\,\mathrm{e}^{\sin(x)}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\sin(x)}(1) \\
&= \mathrm{e}^{\sin(x)}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\cos(x)(-\sin(x) + y)\,\mathrm{e}^{\sin(x)}\right) + \left(\mathrm{e}^{\sin(x)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \cos(x)\,(-\sin(x) + y)\,\mathrm{e}^{\sin(x)} \, \mathrm{d}x$$
$$\phi = (y - \sin(x) + 1)\,\mathrm{e}^{\sin(x)} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{\sin(x)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\sin(x)}$. Therefore equation (4) becomes

$$\mathrm{e}^{\sin(x)} = \mathrm{e}^{\sin(x)} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (y - \sin(x) + 1)\,\mathrm{e}^{\sin(x)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y - \sin(x) + 1)\,\mathrm{e}^{\sin(x)}$$

The solution becomes

$$y = e^{-\sin(x)} \left(\sin(x) e^{\sin(x)} - e^{\sin(x)} + c_1 \right)$$

Summary

The solution(s) found are the following

$$y = e^{-\sin(x)} \left(\sin(x) e^{\sin(x)} - e^{\sin(x)} + c_1 \right) \tag{1}$$

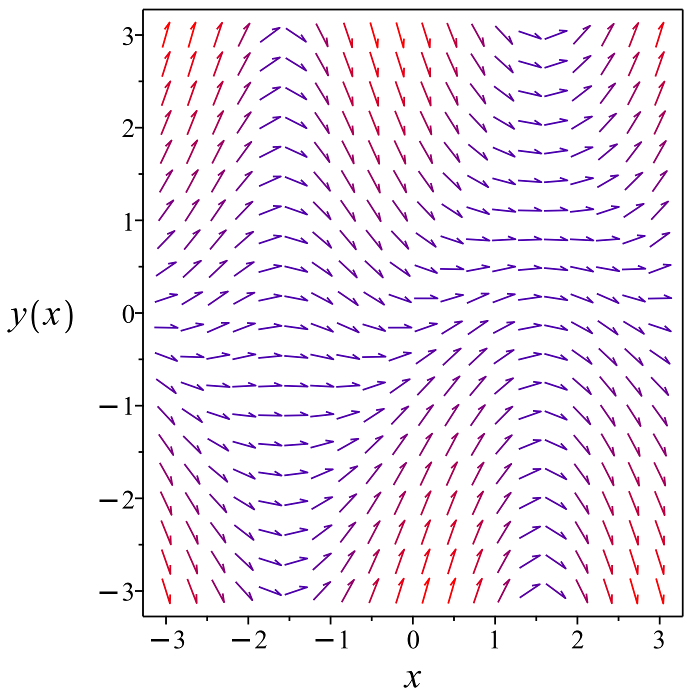


Figure 139: Slope field plot

Verification of solutions

$$y = e^{-\sin(x)} \left(\sin(x) e^{\sin(x)} - e^{\sin(x)} + c_1 \right)$$

Verified OK.

5.14.4 Maple step by step solution

Let's solve

$y' + y\cos(x) = \frac{\sin(2x)}{2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y\cos(x) + \frac{\sin(2x)}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y\cos(x) = \frac{\sin(2x)}{2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' + y\cos(x)) = \frac{\mu(x)\sin(2x)}{2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)(y' + y\cos(x)) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\cos(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\sin(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)\sin(2x)}{2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)\sin(2x)}{2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\sin(2x)}{2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{\sin(x)}$

  $y = \frac{\int \frac{\sin(2x)\mathrm{e}^{\sin(x)}}{2} dx + c_1}{\mathrm{e}^{\sin(x)}}$

- Evaluate the integrals on the rhs

  $y = \frac{\sin(x)\mathrm{e}^{\sin(x)} - \mathrm{e}^{\sin(x)} + c_1}{\mathrm{e}^{\sin(x)}}$

- Simplify

$$y = \sin(x) - 1 + c_1 e^{-\sin(x)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)+y(x)*cos(x)=1/2*sin(2*x),y(x), singsol=all)
```

$$y(x) = \sin(x) - 1 + e^{-\sin(x)} c_1$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 18

```
DSolve[y'[x]+y[x]*Cos[x]==1/2*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x) + c_1 e^{-\sin(x)} - 1$$

5.15 problem Exercise 11.16, page 97

Internal problem ID [4509]
Internal file name [`OUTPUT/4002_Sunday_June_05_2022_12_03_18_PM_22890332/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.16, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' + y = \sin(x)\, x}$$

5.15.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = \sin(x)$$

Hence the ode is

$$y' + \frac{y}{x} = \sin(x)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x}dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(\sin(x)) \\
\frac{d}{dx}(xy) &= (x)(\sin(x)) \\
d(xy) &= (\sin(x)\, x)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
xy &= \int \sin(x)\, x\, dx \\
xy &= -\cos(x)\, x + \sin(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
y = \frac{-\cos(x)\, x + \sin(x)}{x} + \frac{c_1}{x}
$$

which simplifies to

$$
y = \frac{-\cos(x)\, x + \sin(x) + c_1}{x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{-\cos(x)\, x + \sin(x) + c_1}{x} \tag{1}
$$

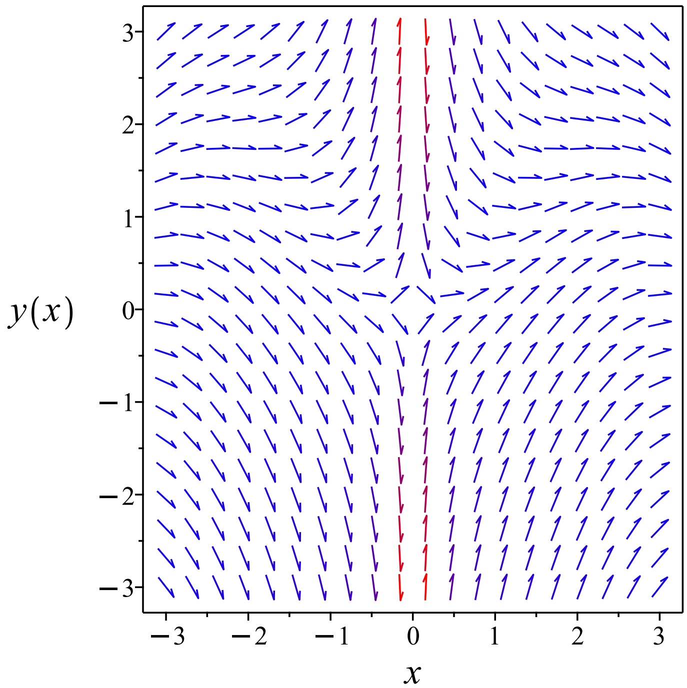


Figure 140: Slope field plot

Verification of solutions

$$y = \frac{-\cos(x)\,x + \sin(x) + c_1}{x}$$

Verified OK.

5.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-y + \sin(x)\,x}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 76: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = xy$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-y + \sin(x)\, x}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(x)\, x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(R)\, R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \sin(R) - \cos(R) R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$xy = -\cos(x) x + \sin(x) + c_1$$

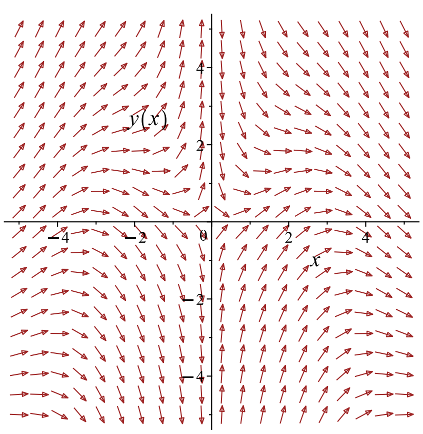
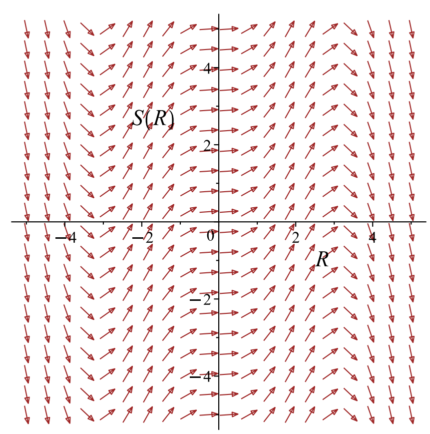
Which simplifies to

$$xy = -\cos(x) x + \sin(x) + c_1$$

Which gives

$$y = \frac{-\cos(x) x + \sin(x) + c_1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-y+\sin(x)x}{x}$ 	$R = x$ $S = xy$	$\frac{dS}{dR} = \sin(R) R$ 

Summary

The solution(s) found are the following

$$y = \frac{-\cos(x) x + \sin(x) + c_1}{x} \tag{1}$$

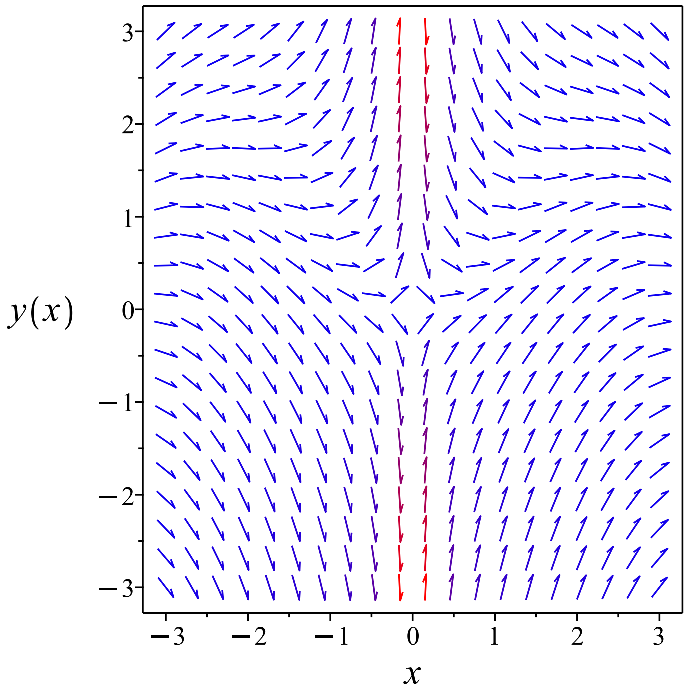


Figure 141: Slope field plot

Verification of solutions

$$y = \frac{-\cos(x)\,x + \sin(x) + c_1}{x}$$

Verified OK.

5.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = (-y + \sin(x)\,x)\,\mathrm{d}x$$
$$(y - \sin(x)\,x)\,\mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y - \sin(x)\,x$$
$$N(x, y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - \sin(x)\,x) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int y - \sin(x)\, x \mathrm{d}x$$

$$\phi = xy - \sin(x) + \cos(x)\, x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x$. Therefore equation (4) becomes

$$x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = xy - \sin(x) + \cos(x)\, x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = xy - \sin(x) + \cos(x)\, x$$

The solution becomes

$$y = \frac{-\cos(x)\, x + \sin(x) + c_1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{-\cos(x)\, x + \sin(x) + c_1}{x} \tag{1}$$

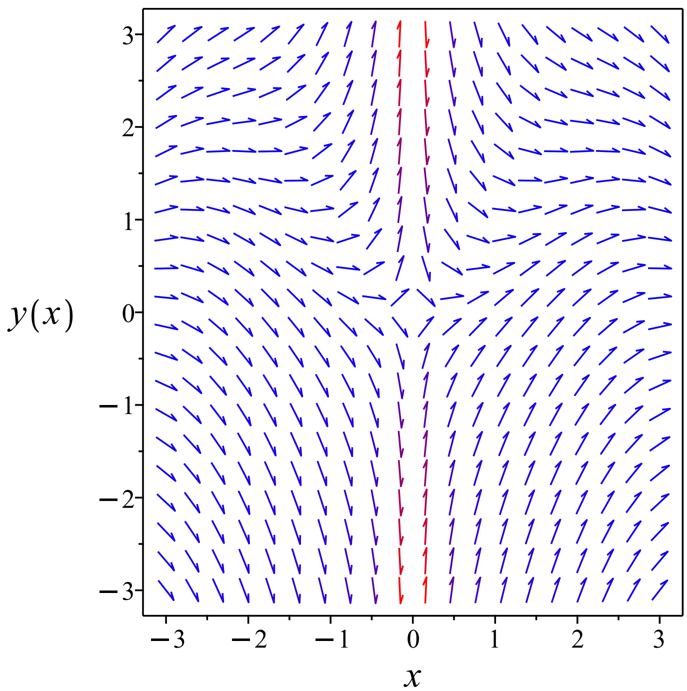


Figure 142: Slope field plot

Verification of solutions

$$y = \frac{-\cos(x)\, x + \sin(x) + c_1}{x}$$

Verified OK.

5.15.4 Maple step by step solution

Let's solve

$xy' + y = \sin(x)\, x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x} + \sin(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = \sin(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu(x)\sin(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\sin(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\sin(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\sin(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int \sin(x)x dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{-\cos(x)x + \sin(x) + c_1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x)+y(x)=x*sin(x),y(x), singsol=all)
```

$$y(x) = \frac{-x\cos(x) + \sin(x) + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 19

```
DSolve[x*y'[x]+y[x]==x*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sin(x) - x\cos(x) + c_1}{x}$$

5.16 problem Exercise 11.17, page 97

Internal problem ID [4510]
Internal file name [`OUTPUT/4003_Sunday_June_05_2022_12_03_28_PM_22281092/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.17, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-y + xy' = x^2 \sin(x)}$$

5.16.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \sin(x)\, x$$

Hence the ode is

$$y' - \frac{y}{x} = \sin(x)\, x$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(\sin(x)\, x)$$
$$\frac{d}{dx}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)(\sin(x)\, x)$$
$$d\left(\frac{y}{x}\right) = \sin(x)\; dx$$

Integrating gives

$$\frac{y}{x} = \int \sin(x)\; dx$$
$$\frac{y}{x} = -\cos(x) + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\cos(x)\, x + c_1 x$$

which simplifies to

$$y = x(-\cos(x) + c_1)$$

Summary
The solution(s) found are the following

$$y = x(-\cos(x) + c_1) \tag{1}$$

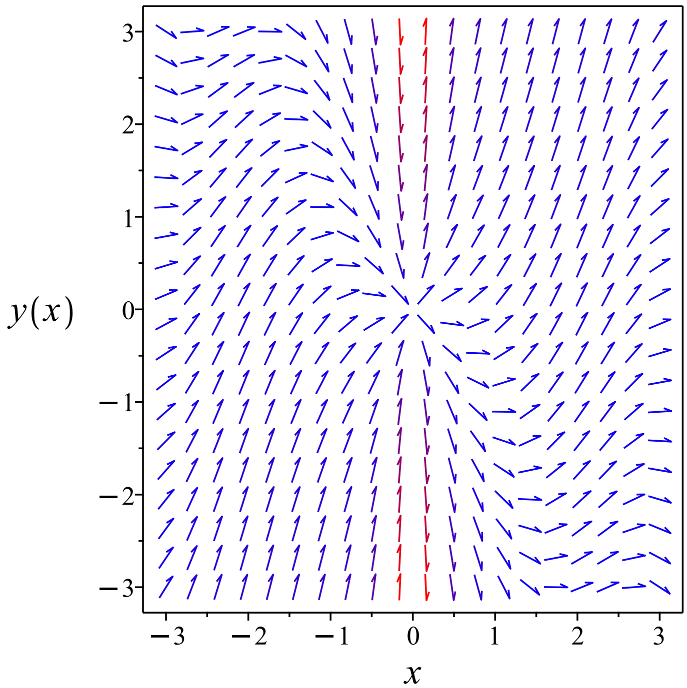


Figure 143: Slope field plot

Verification of solutions

$$y = x(-\cos(x) + c_1)$$

Verified OK.

5.16.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$-u(x)\, x + x(u'(x)\, x + u(x)) = x^2 \sin(x)$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \sin(x)\, dx \\ &= -\cos(x) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x(-\cos(x) + c_2) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x(-\cos(x) + c_2) \tag{1}$$

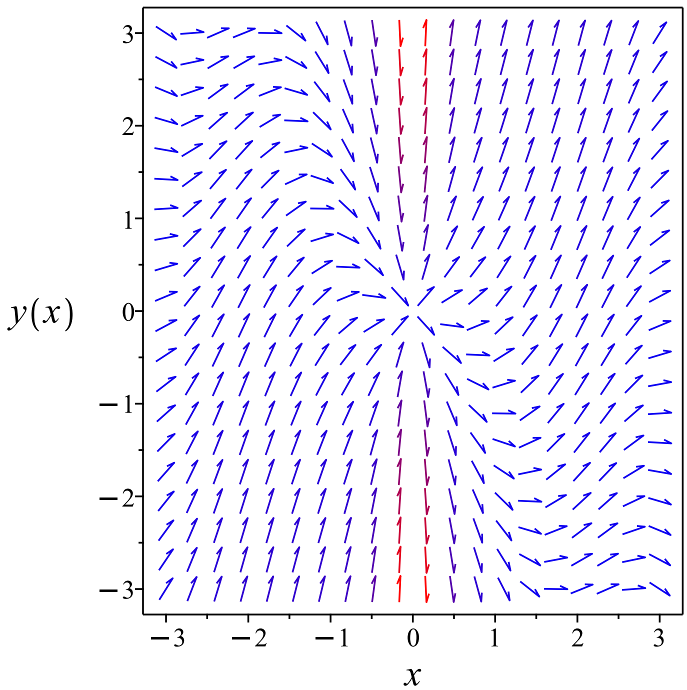


Figure 144: Slope field plot

Verification of solutions

$$y = x(-\cos(x) + c_2)$$

Verified OK.

5.16.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + x^2 \sin(x)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 79: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}S&=\int\frac{1}{\eta}dy\\&=\int\frac{1}{x}dy\end{aligned}$$

Which results in

$$S=\frac{y}{x}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{y+x^2\sin(x)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x&=1\\R_y&=0\\S_x&=-\frac{y}{x^2}\\S_y&=\frac{1}{x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=\sin(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=\sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\cos(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = -\cos(x) + c_1$$

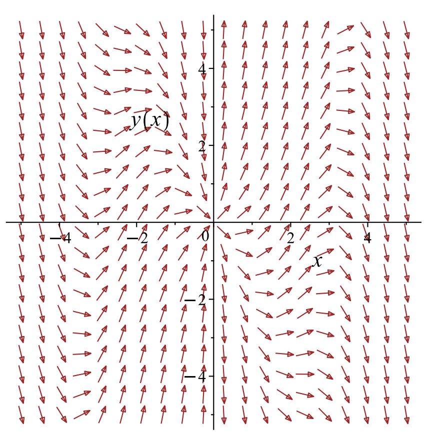
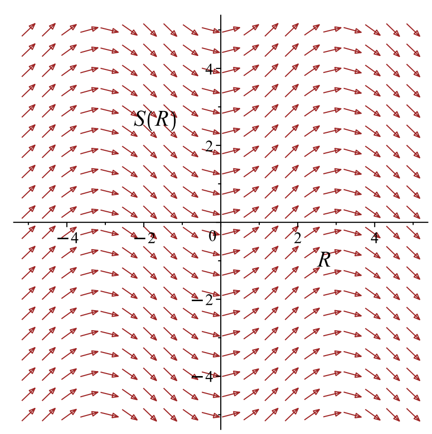
Which simplifies to

$$\frac{y}{x} = -\cos(x) + c_1$$

Which gives

$$y = -x(\cos(x) - c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+x^2\sin(x)}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \sin(R)$ 

Summary

The solution(s) found are the following

$$y = -x(\cos(x) - c_1) \tag{1}$$

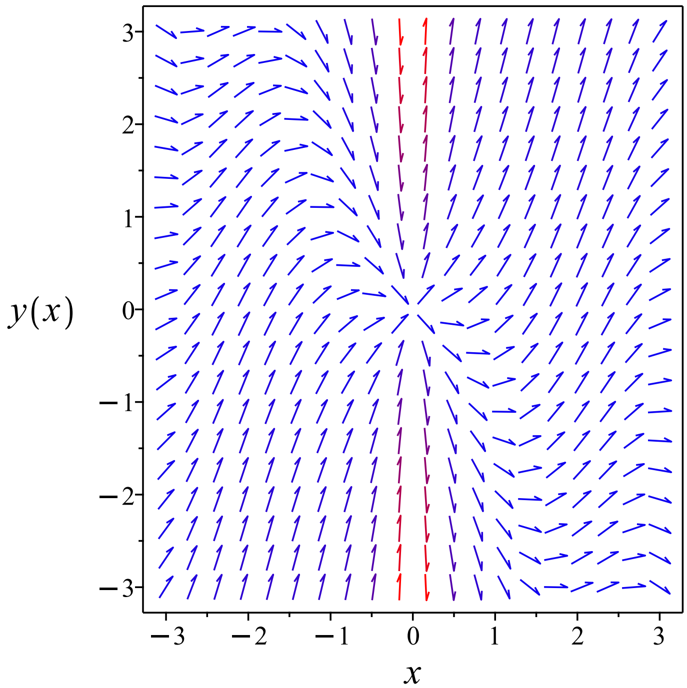


Figure 145: Slope field plot

Verification of solutions

$$y = -x(\cos(x) - c_1)$$

Verified OK.

5.16.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x)\,\mathrm{d}y &= \left(y + x^2 \sin(x)\right) \mathrm{d}x \\
\left(-y - x^2 \sin(x)\right) \mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -y - x^2 \sin(x) \\
N(x, y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y - x^2 \sin(x)\right) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((-1) - (1)) \\
&= -\frac{2}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}\left(-y - x^2 \sin(x)\right) \\
&= \frac{-y - x^2 \sin(x)}{x^2}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(x) \\
&= \frac{1}{x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-y - x^2 \sin(x)}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-y - x^2 \sin(x)}{x^2} \, \mathrm{d}x$$
$$\phi = \cos(x) + \frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \cos(x) + \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(x) + \frac{y}{x}$$

The solution becomes

$$y = -x(\cos(x) - c_1)$$

Summary

The solution(s) found are the following

$$y = -x(\cos(x) - c_1) \tag{1}$$

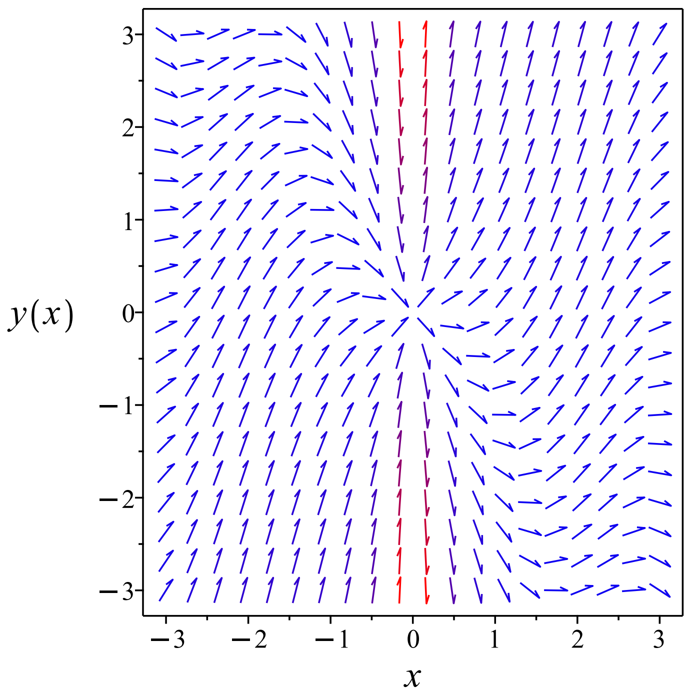


Figure 146: Slope field plot

Verification of solutions

$$y = -x(\cos(x) - c_1)$$

Verified OK.

5.16.5 Maple step by step solution

Let's solve

$-y + xy' = x^2 \sin(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + \sin(x)\, x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = \sin(x)\, x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)\sin(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\sin(x)\, x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\sin(x)\, x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\sin(x) x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \sin(x)\, dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(-\cos(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x)-y(x)=x^2*sin(x),y(x), singsol=all)
```

$$y(x) = (-\cos(x) + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 14

```
DSolve[x*y'[x]-y[x]==x^2*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(-\cos(x) + c_1)$$

5.17 problem Exercise 11.18, page 97

Internal problem ID [4511]
Internal file name [`OUTPUT/4004_Sunday_June_05_2022_12_03_37_PM_81568832/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.18, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Bernoulli]
```

$$\boxed{xy' + xy^2 - y = 0}$$

5.17.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)\,x + u(x)) + x^3u(x)^2 - u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -u^2x \end{aligned}$$

Where $f(x) = -x$ and $g(u) = u^2$. Integrating both sides gives

$$\frac{1}{u^2}\,du = -x\,dx$$
$$\int \frac{1}{u^2}\,du = \int -x\,dx$$
$$-\frac{1}{u} = -\frac{x^2}{2} + c_2$$

The solution is

$$-\frac{1}{u(x)} + \frac{x^2}{2} - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{x}{y} + \frac{x^2}{2} - c_2 = 0$$
$$-\frac{x}{y} + \frac{x^2}{2} - c_2 = 0$$

Summary

The solution(s) found are the following

$$-\frac{x}{y} + \frac{x^2}{2} - c_2 = 0 \tag{1}$$

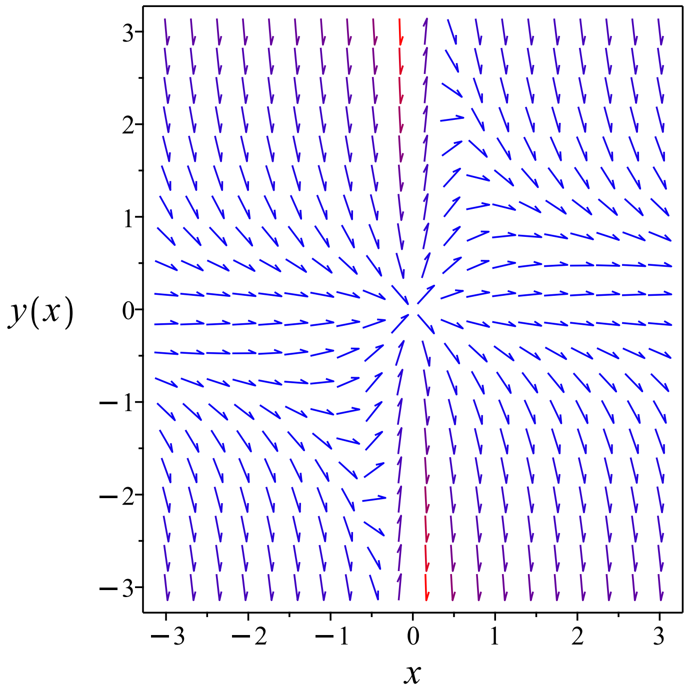


Figure 147: Slope field plot

Verification of solutions

$$-\frac{x}{y}+\frac{x^2}{2}-c_2=0$$

Verified OK.

5.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=-\frac{y(xy-1)}{x}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 82: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{y^2}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(xy - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{y} \\ S_y &= \frac{x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{R^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = -\frac{x^2}{2} + c_1$$

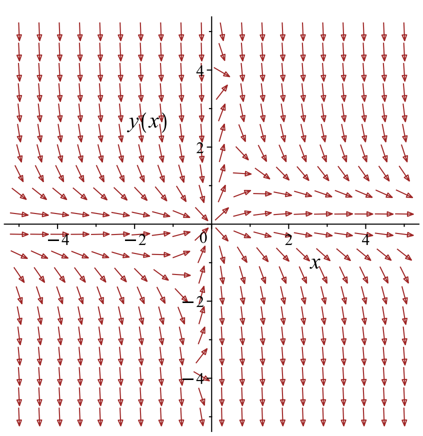
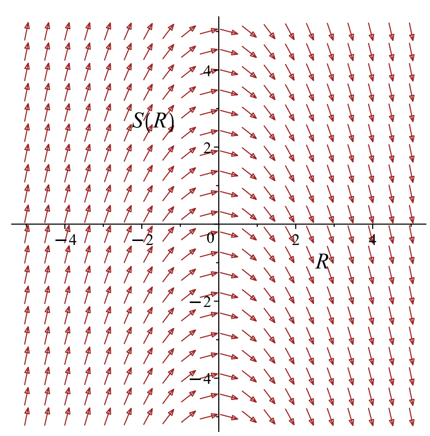
Which simplifies to

$$-\frac{x}{y} = -\frac{x^2}{2} + c_1$$

Which gives

$$y = -\frac{2x}{-x^2 + 2c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(xy-1)}{x}$ 	$R = x$ $S = -\frac{x}{y}$	$\frac{dS}{dR} = -R$ 

Summary

The solution(s) found are the following

$$y = -\frac{2x}{-x^2 + 2c_1} \tag{1}$$

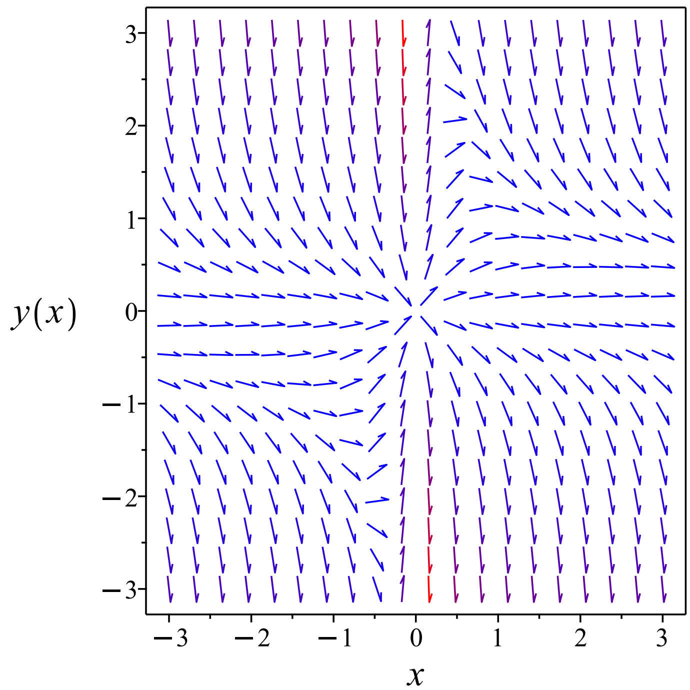


Figure 148: Slope field plot

Verification of solutions

$$y = -\frac{2x}{-x^2 + 2c_1}$$

Verified OK.

5.17.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(xy - 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y - y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= -1 \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y' \frac{1}{y^2} = \frac{1}{yx} - 1 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{w(x)}{x} - 1 \\ w' &= -\frac{w}{x} + 1 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= 1
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{w(x)}{x} = 1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x}dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= \mu \\
\frac{d}{dx}(xw) &= x \\
d(xw) &= x dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
xw &= \int x \, dx \\
xw &= \frac{x^2}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
w(x) = \frac{x}{2} + \frac{c_1}{x}
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = \frac{x}{2} + \frac{c_1}{x}
$$

Or

$$
y = \frac{1}{\frac{x}{2} + \frac{c_1}{x}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{1}{\frac{x}{2} + \frac{c_1}{x}} \tag{1}
$$

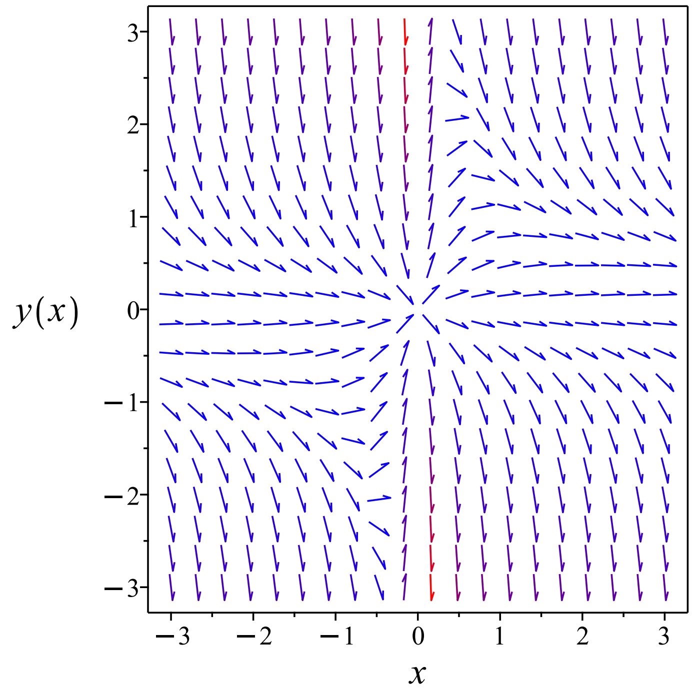


Figure 149: Slope field plot

Verification of solutions

$$y = \frac{1}{\frac{x}{2} + \frac{c_1}{x}}$$

Verified OK.

5.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = \left(-y^2 x + y\right)\mathrm{d}x$$
$$\left(y^2 x - y\right)\mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y^2 x - y$$
$$N(x, y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 x - y\right)\\ &= 2xy - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((2xy - 1) - (1)) \\
&= \frac{2xy - 2}{x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y\,(xy - 1)}((1) - (2xy - 1)) \\
&= -\frac{2}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int -\frac{2}{y} \,\mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(y)} \\
&= \frac{1}{y^2}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^2}\left(y^2 x - y\right) \\
&= \frac{xy - 1}{y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}(x) \\
&= \frac{x}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(\frac{xy-1}{y}\right) + \left(\frac{x}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{xy-1}{y}\,\mathrm{d}x$$

$$\phi = \frac{x(xy-2)}{2y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x^2}{2y} - \frac{x(xy-2)}{2y^2} + f'(y) \tag{4}$$

$$= \frac{x}{y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{y^2}$. Therefore equation (4) becomes

$$\frac{x}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(xy - 2)}{2y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(xy - 2)}{2y}$$

The solution becomes

$$y = -\frac{2x}{-x^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{2x}{-x^2 + 2c_1} \tag{1}$$

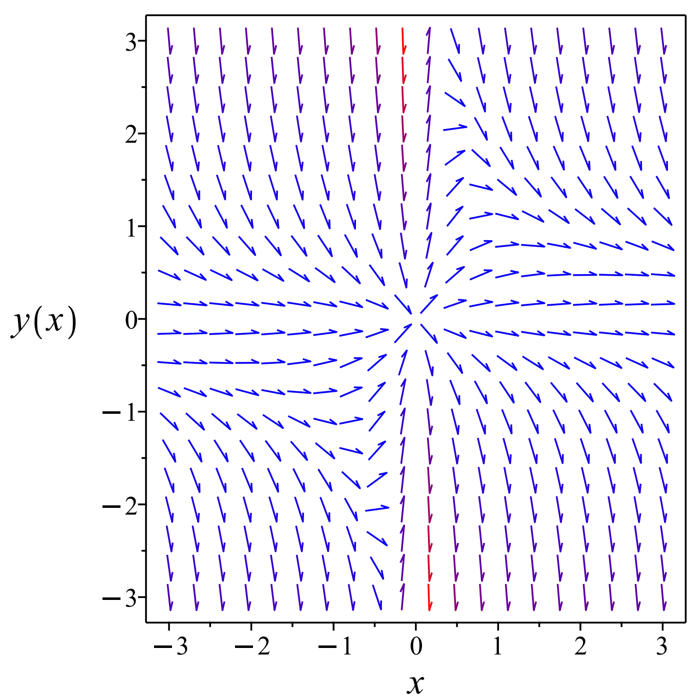


Figure 150: Slope field plot

Verification of solutions

$$y = -\frac{2x}{-x^2 + 2c_1}$$

Verified OK.

5.17.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= -\frac{y(xy - 1)}{x}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y^2 + \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = -1$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{-u}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}f_2' &= 0\\ f_1 f_2 &= -\frac{1}{x}\\ f_2^2 f_0 &= 0\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) + \frac{u'(x)}{x} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 x^2 + c_1$$

The above shows that

$$u'(x) = 2c_2 x$$

Using the above in (1) gives the solution

$$y = \frac{2c_2 x}{c_2 x^2 + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2x}{x^2 + c_3}$$

Summary
The solution(s) found are the following

$$y = \frac{2x}{x^2 + c_3} \tag{1}$$

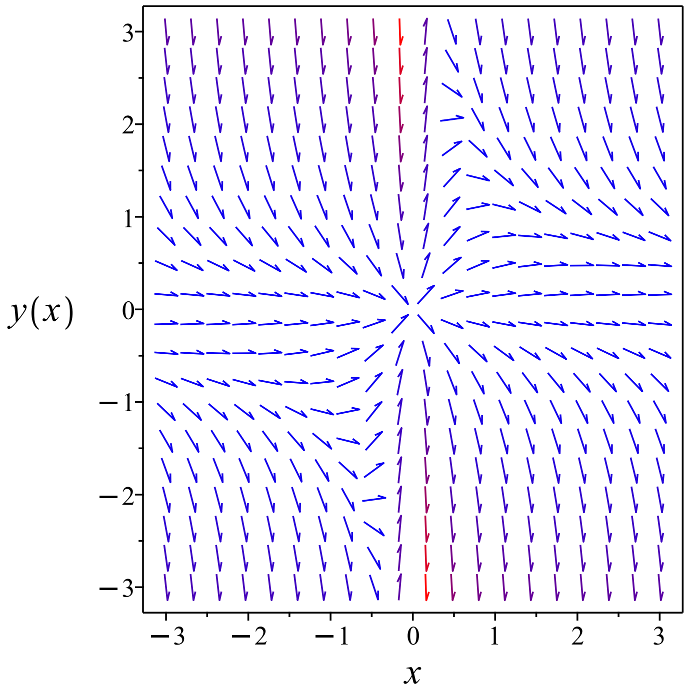


Figure 151: Slope field plot

Verification of solutions

$$y = \frac{2x}{x^2 + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x)+x*y(x)^2-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{2x}{x^2 + 2c_1}$$

✓ Solution by Mathematica

Time used: 0.149 (sec). Leaf size: 23

```
DSolve[x*y'[x]+x*y[x]^2-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2x}{x^2 + 2c_1}$$
$$y(x) \to 0$$

5.18 problem Exercise 11.19, page 97

Internal problem ID [4512]
Internal file name [`OUTPUT/4005_Sunday_June_05_2022_12_03_46_PM_83625075/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.19, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{xy' - y(2\ln(x)\, y - 1) = 0}$$

5.18.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(2\ln(x)\, y - 1)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 84: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= y^2x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^2 x} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{yx}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(2\ln(x)\, y - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{y\, x^2} \\ S_y &= \frac{1}{y^2 x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2\ln(x)}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2\ln(R)}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{2\ln(R)}{R} - \frac{2}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{yx} = -\frac{2\ln(x)}{x} - \frac{2}{x} + c_1$$

Which simplifies to

$$-\frac{1}{yx} = -\frac{2\ln(x)}{x} - \frac{2}{x} + c_1$$

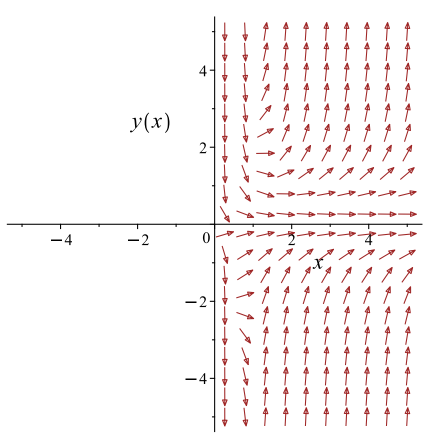
Which gives

$$y = \frac{1}{-c_1 x + 2\ln(x) + 2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{y(2\ln(x)y-1)}{x}$



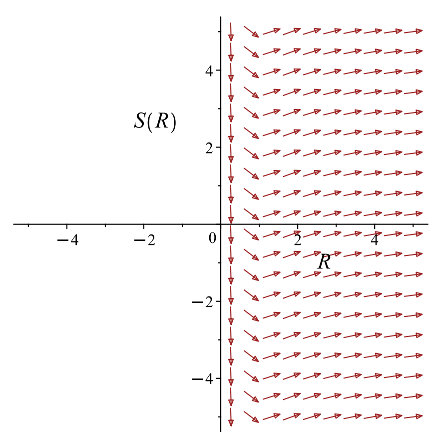
</td><td>

$R = x$

$S = -\frac{1}{yx}$

</td><td>

$\frac{dS}{dR} = \frac{2\ln(R)}{R^2}$



</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = \frac{1}{-c_1 x + 2\ln(x) + 2} \tag{1}$$

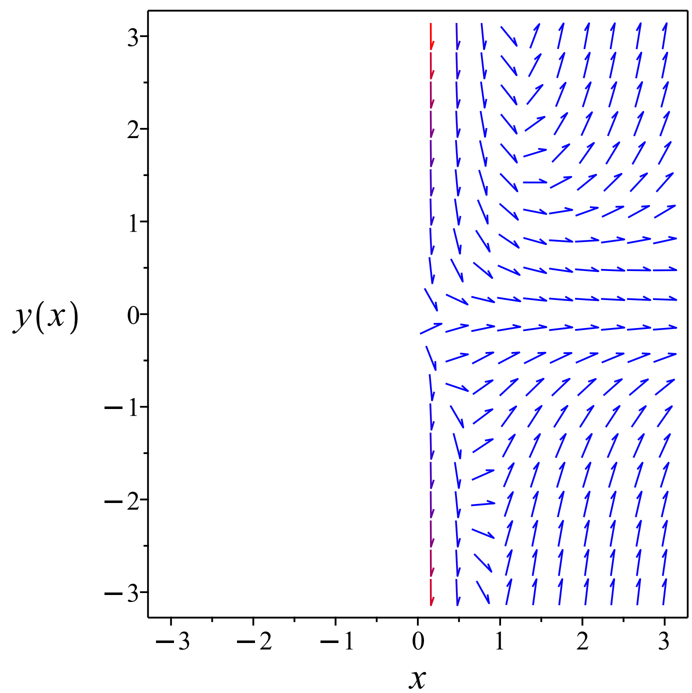


Figure 152: Slope field plot

Verification of solutions

$$y = \frac{1}{-c_1 x + 2\ln(x) + 2}$$

Verified OK.

5.18.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(2\ln(x)\,y - 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + \frac{2\ln(x)}{x}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= \frac{2\ln(x)}{x} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{yx} + \frac{2\ln(x)}{x} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{w(x)}{x} + \frac{2\ln(x)}{x} \\ w' &= \frac{w}{x} - \frac{2\ln(x)}{x} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -\frac{2\ln(x)}{x} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = -\frac{2\ln(x)}{x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-\frac{2\ln(x)}{x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(-\frac{2\ln(x)}{x}\right) \\ \mathrm{d}\left(\frac{w}{x}\right) &= \left(-\frac{2\ln(x)}{x^2}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{w}{x} &= \int -\frac{2\ln(x)}{x^2}\,\mathrm{d}x \\ \frac{w}{x} &= \frac{2\ln(x)}{x} + \frac{2}{x} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = x\left(\frac{2\ln(x)}{x} + \frac{2}{x}\right) + c_1 x$$

which simplifies to

$$w(x) = c_1 x + 2\ln(x) + 2$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = c_1 x + 2\ln(x) + 2$$

Or

$$y = \frac{1}{c_1 x + 2\ln(x) + 2}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{c_1 x + 2\ln(x) + 2} \tag{1}$$

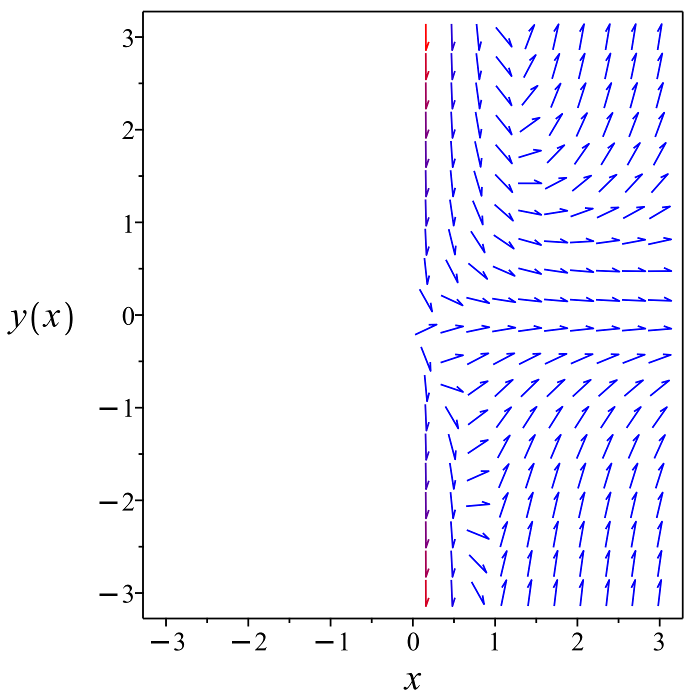


Figure 153: Slope field plot

Verification of solutions

$$y = \frac{1}{c_1 x + 2\ln(x) + 2}$$

Verified OK.

5.18.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=(y(2\ln(x)\,y-1))\,\mathrm{d}x\\ (-y(2\ln(x)\,y-1))\,\mathrm{d}x+(x)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y(2\ln(x)\,y-1)\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y(2\ln(x)\,y-1)) \\ &= -4\ln(x)\,y+1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((-4\ln(x)\,y+1)-(1)) \\ &= -\frac{4\ln(x)\,y}{x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{2\ln(x)\,y^2-y}((1)-(-4\ln(x)\,y+1)) \\ &= -\frac{4\ln(x)}{2\ln(x)\,y-1}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - (-4\ln(x)\,y + 1)}{x(-y(2\ln(x)\,y - 1)) - y(x)} \\
&= -\frac{2}{yx}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R\,\mathrm{d}t} \\
&= e^{\int \left(-\frac{2}{t}\right)\mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^2y^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2y^2}(-y(2\ln(x)\,y - 1)) \\
&= \frac{-2\ln(x)\,y + 1}{y\,x^2}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2y^2}(x) \\
&= \frac{1}{y^2x}
\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-2\ln(x)\,y+1}{y\,x^2}\right) + \left(\frac{1}{y^2 x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-2\ln(x)\,y+1}{y\,x^2}\,\mathrm{d}x$$
$$\phi = \frac{2\ln(x)\,y + 2y - 1}{xy} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2\ln(x)+2}{xy} - \frac{2\ln(x)\,y+2y-1}{x\,y^2} + f'(y) \tag{4}$$
$$= \frac{1}{y^2 x} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2 x}$. Therefore equation (4) becomes

$$\frac{1}{y^2 x} = \frac{1}{y^2 x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{2\ln(x)\,y + 2y - 1}{xy} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{2\ln(x)\,y + 2y - 1}{xy}$$

The solution becomes

$$y = \frac{1}{-c_1 x + 2\ln(x) + 2}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{-c_1 x + 2\ln(x) + 2} \tag{1}$$

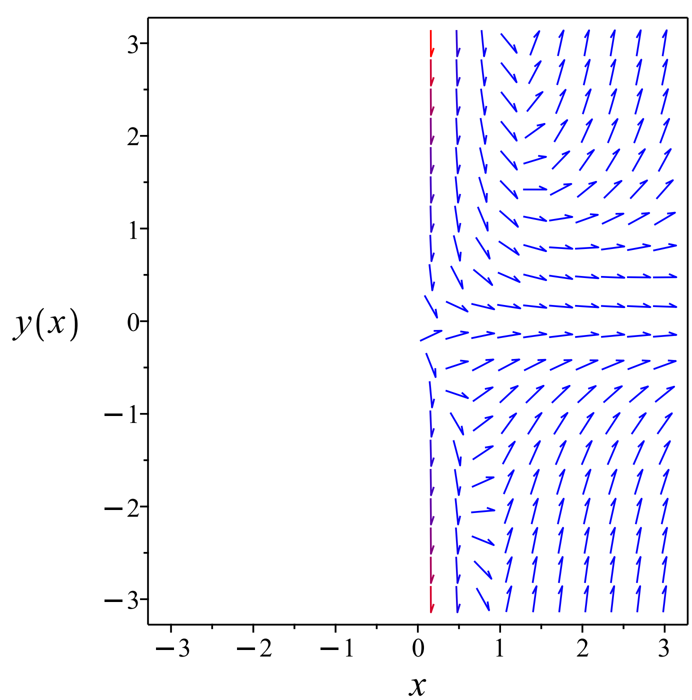


Figure 154: Slope field plot

Verification of solutions

$$y = \frac{1}{-c_1 x + 2\ln(x) + 2}$$

Verified OK.

5.18.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(2\ln(x)\,y - 1)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{2\ln(x)\,y^2}{x} - \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = \frac{2\ln(x)}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{2\ln(x)u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2\ln(x)}{x^2} + \frac{2}{x^2} \\ f_1 f_2 &= -\frac{2\ln(x)}{x^2} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{2\ln(x)\,u''(x)}{x} - \left(-\frac{4\ln(x)}{x^2} + \frac{2}{x^2}\right)u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{-c_2 \ln(x) + c_1 x - c_2}{x}$$

The above shows that

$$u'(x) = \frac{c_2 \ln(x)}{x^2}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{2(-c_2 \ln(x) + c_1 x - c_2)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{1}{-2c_3 x + 2\ln(x) + 2}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{-2c_3 x + 2\ln(x) + 2} \tag{1}$$

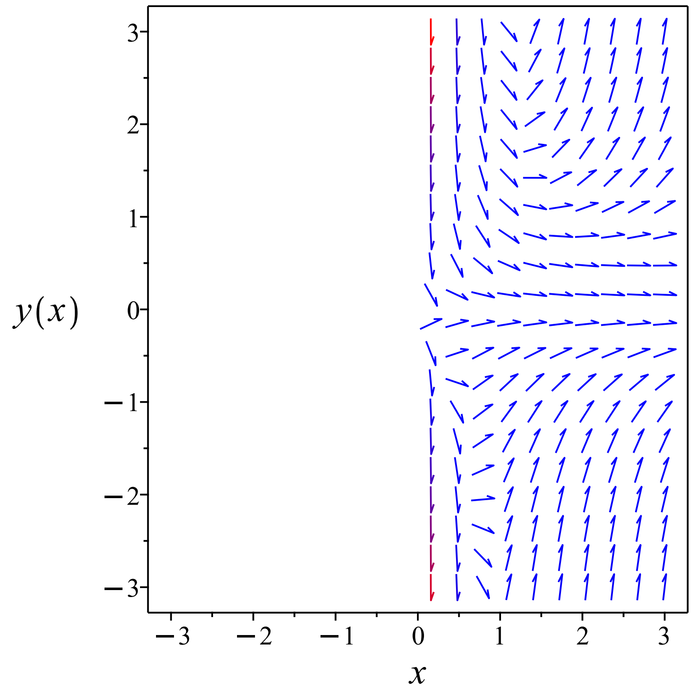


Figure 155: Slope field plot

Verification of solutions

$$y = \frac{1}{-2c_3 x + 2\ln(x) + 2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x*diff(y(x),x)-y(x)*(2*y(x)*ln(x)-1)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{2 + c_1 x + 2\ln(x)}$$

✓ Solution by Mathematica

Time used: 0.14 (sec). Leaf size: 22

```
DSolve[x*y'[x]-y[x]*(2*y[x]*Log[x]-1)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2\log(x) + c_1 x + 2}$$
$$y(x) \to 0$$

5.19 problem Exercise 11.20, page 97

Internal problem ID [4513]
Internal file name [`OUTPUT/4006_Sunday_June_05_2022_12_03_55_PM_33609402/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.20, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Bernoulli]
```

$$\boxed{x^2(x-1)\,y'-y^2-x(-2+x)\,y=0}$$

5.19.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^2(x-1)\,(u'(x)\,x+u(x))-u(x)^2\,x^2-x^2(-2+x)\,u(x)=0$$

In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=\frac{u(u-1)}{x\,(x-1)}\end{aligned}$$

Where $f(x) = \frac{1}{x(x-1)}$ and $g(u) = u(u-1)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u(u-1)}\,du &= \frac{1}{x(x-1)}\,dx \\
\int \frac{1}{u(u-1)}\,du &= \int \frac{1}{x(x-1)}\,dx \\
\ln(u-1) - \ln(u) &= \ln(x-1) - \ln(x) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{\ln(u-1)-\ln(u)} = e^{\ln(x-1)-\ln(x)+c_2}
$$

Which simplifies to

$$
\frac{u-1}{u} = c_3 e^{\ln(x-1)-\ln(x)}
$$

Which simplifies to

$$
u(x) = -\frac{1}{-1 + c_3\left(1 - \frac{1}{x}\right)}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= -\frac{x}{-1 + c_3\left(1 - \frac{1}{x}\right)}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{x}{-1 + c_3\left(1 - \frac{1}{x}\right)} \tag{1}
$$

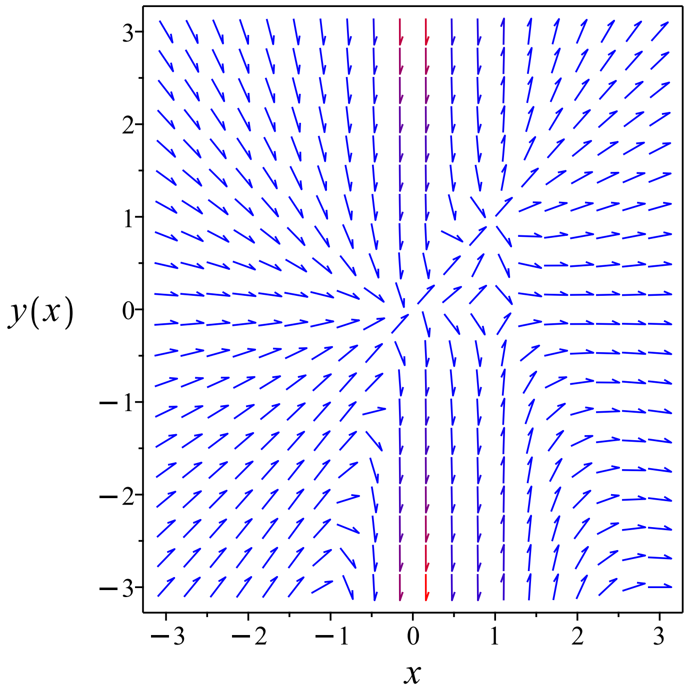


Figure 156: Slope field plot

Verification of solutions

$$y = -\frac{x}{-1 + c_3 \left(1 - \frac{1}{x}\right)}$$

Verified OK.

5.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(x^2 - 2x + y)}{x^2 (x - 1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 86: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= y^2 e^{\ln(x-1) - 2\ln(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^2 \mathrm{e}^{\ln(x-1)-2\ln(x)}} dy \end{aligned}$$

Which results in

$$S = -\frac{x^2}{(x-1)\, y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(x^2 - 2x + y)}{x^2\,(x-1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x(-2+x)}{(x-1)^2\, y} \\ S_y &= \frac{x^2}{(x-1)\, y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(x-1)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(R-1)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R-1} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{(x-1)\,y} = -\frac{1}{x-1} + c_1$$

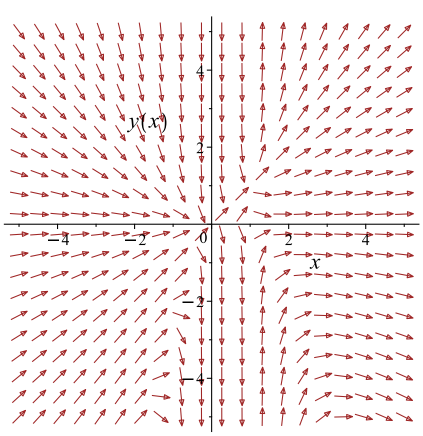
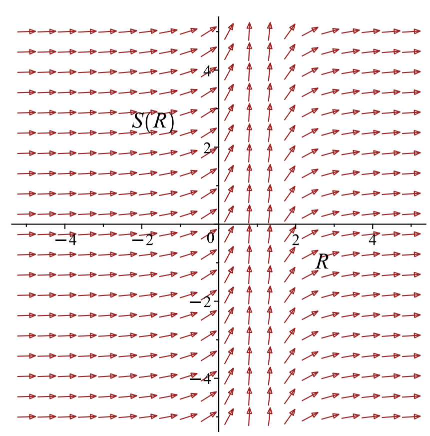
Which simplifies to

$$-\frac{x^2}{(x-1)\,y} = -\frac{1}{x-1} + c_1$$

Which gives

$$y = -\frac{x^2}{c_1 x - c_1 - 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(x^2-2x+y)}{x^2(x-1)}$ 	$R = x$ $S = -\frac{x^2}{(x-1)\,y}$	$\frac{dS}{dR} = \frac{1}{(R-1)^2}$ 

Summary
The solution(s) found are the following

$$y = -\frac{x^2}{c_1 x - c_1 - 1} \tag{1}$$

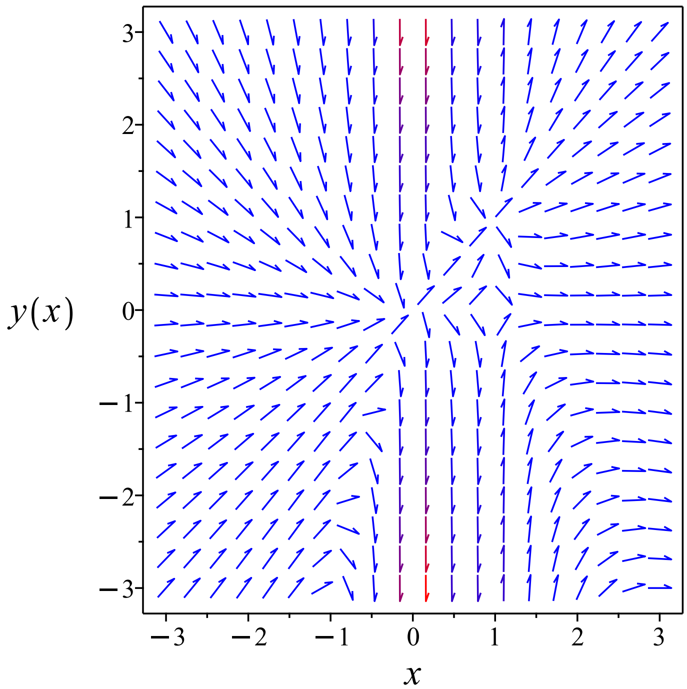


Figure 157: Slope field plot

Verification of solutions

$$y = -\frac{x^2}{c_1 x - c_1 - 1}$$

Verified OK.

5.19.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(x^2 - 2x + y)}{x^2 (x - 1)} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{x^2 - 2x}{x^2 (x - 1)} y + \frac{1}{x^2 (x - 1)} y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{x^2 - 2x}{x^2 (x-1)} \\ f_1(x) &= \frac{1}{x^2 (x-1)} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y' \frac{1}{y^2} = \frac{x^2 - 2x}{x^2 (x-1) y} + \frac{1}{x^2 (x-1)} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{(x^2 - 2x) w(x)}{x^2 (x-1)} + \frac{1}{x^2 (x-1)} \\ w' &= -\frac{(x^2 - 2x) w}{x^2 (x-1)} - \frac{1}{x^2 (x-1)} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2-x}{x(x-1)} \\ q(x) &= -\frac{1}{x^2(x-1)} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{(2-x)\,w(x)}{x(x-1)} = -\frac{1}{x^2(x-1)}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{2-x}{x(x-1)}dx} \\ &= e^{-\ln(x-1)+2\ln(x)} \end{aligned}$$

Which simplifies to

$$\mu = \frac{x^2}{x-1}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{1}{x^2(x-1)}\right) \\ \frac{d}{dx}\left(\frac{x^2 w}{x-1}\right) &= \left(\frac{x^2}{x-1}\right)\left(-\frac{1}{x^2(x-1)}\right) \\ d\left(\frac{x^2 w}{x-1}\right) &= \left(-\frac{1}{(x-1)^2}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{x^2 w}{x-1} &= \int -\frac{1}{(x-1)^2}\, dx \\ \frac{x^2 w}{x-1} &= \frac{1}{x-1} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{x^2}{x-1}$ results in

$$w(x) = \frac{1}{x^2} + \frac{c_1(x-1)}{x^2}$$

which simplifies to

$$w(x) = \frac{1 + (x-1)\, c_1}{x^2}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{1 + (x-1)\, c_1}{x^2}$$

Or

$$y = \frac{x^2}{1 + (x-1)\, c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{1 + (x-1)\, c_1} \tag{1}$$

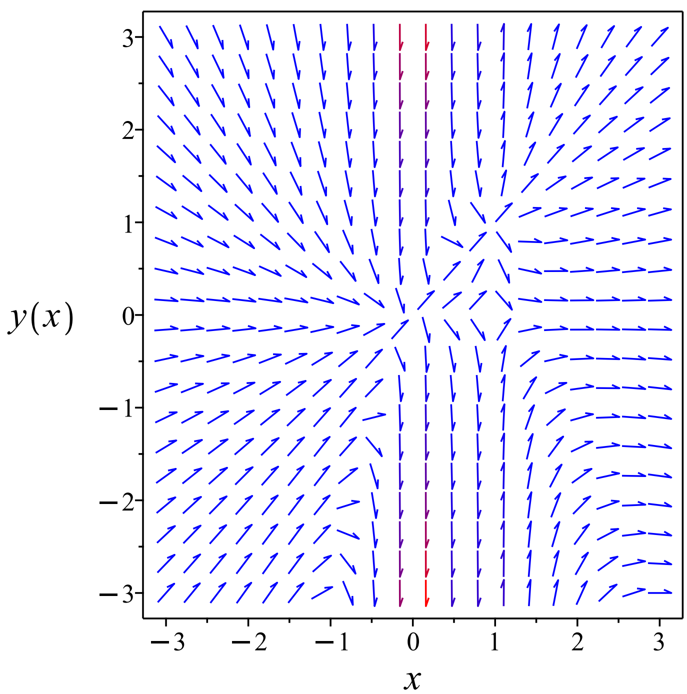


Figure 158: Slope field plot

Verification of solutions

$$y = \frac{x^2}{1 + (x-1)\, c_1}$$

Verified OK.

5.19.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= \frac{y(x^2-2x+y)}{x^2(x-1)}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{x-1} - \frac{2y}{x(x-1)} + \frac{y^2}{x^2(x-1)}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{x^2-2x}{x^2(x-1)}$ and $f_2(x) = \frac{1}{x^2(x-1)}$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{\frac{u}{x^2(x-1)}}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}f_2' &= -\frac{2}{x^3(x-1)} - \frac{1}{x^2(x-1)^2}\\ f_1 f_2 &= \frac{x^2-2x}{x^4(x-1)^2}\\ f_2^2 f_0 &= 0\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2(x-1)} - \left(-\frac{2}{x^3(x-1)} - \frac{1}{x^2(x-1)^2} + \frac{x^2-2x}{x^4(x-1)^2}\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{x-1}$$

The above shows that

$$u'(x) = -\frac{c_2}{(x-1)^2}$$

Using the above in (1) gives the solution

$$y = \frac{c_2 x^2}{(x-1)\left(c_1 + \frac{c_2}{x-1}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x^2}{c_3(x-1)+1}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{c_3(x-1)+1} \tag{1}$$

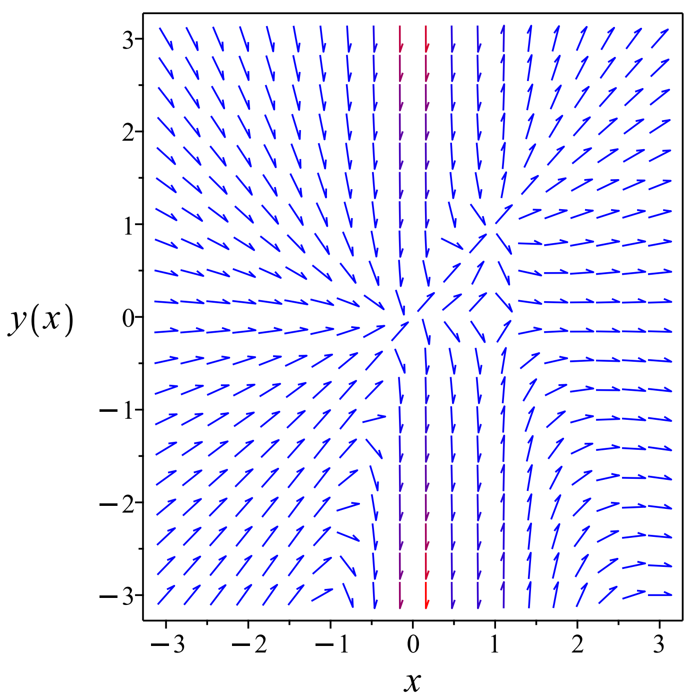


Figure 159: Slope field plot

Verification of solutions

$$y = \frac{x^2}{c_3 (x - 1) + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*(x-1)*diff(y(x),x)-y(x)^2-x*(x-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{1 + c_1 (x - 1)}$$

✓ Solution by Mathematica

Time used: 0.191 (sec). Leaf size: 25

```
DSolve[x^2*(x-1)*y'[x]-y[x]^2-x*(x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{c_1(-x) + 1 + c_1}$$
$$y(x) \to 0$$

5.20 problem Exercise 11.21, page 97

Internal problem ID [4514]
Internal file name [`OUTPUT/4007_Sunday_June_05_2022_12_04_04_PM_8179276/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.21, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = e^x}$$

With initial conditions

$$[y(0) = 1]$$

5.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = e^x$$

Hence the ode is

$$y' - y = e^{x}$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = e^{x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

5.20.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-1)dx} \\ &= e^{-x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(e^{x}\right) \\ \frac{d}{dx}\left(e^{-x} y\right) &= \left(e^{-x}\right)\left(e^{x}\right) \\ d\left(e^{-x} y\right) &= dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-x} y &= \int dx \\ e^{-x} y &= x + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = e^{x} x + c_1 e^{x}$$

which simplifies to

$$y = e^{x}(x + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

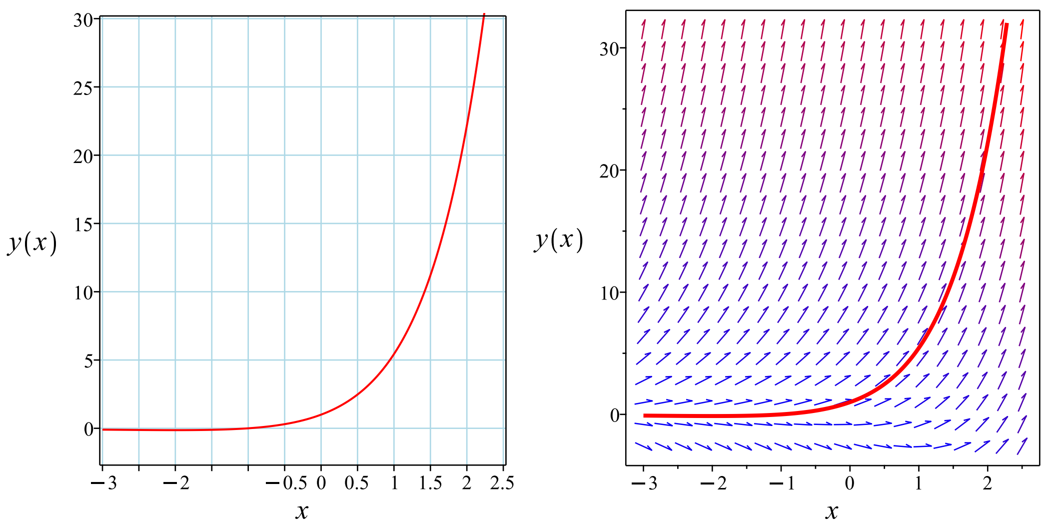
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = e^x(x + 1)$$

Summary
The solution(s) found are the following

$$y = e^x(x + 1) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = e^x(x + 1)$$

Verified OK.

5.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= y + e^{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 88: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{e^{x}} dy
\end{aligned}
$$

Which results in

$$
S = e^{-x} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = y + e^{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -e^{-x} y \\
S_y &= e^{-x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x} y = x + c_1$$

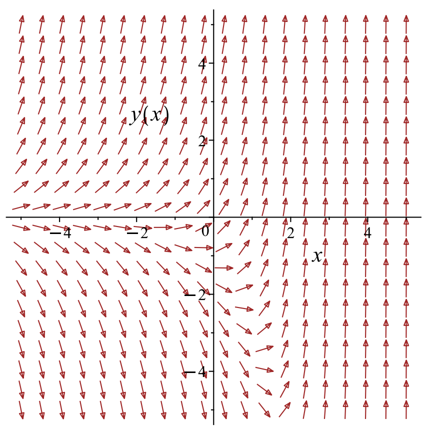
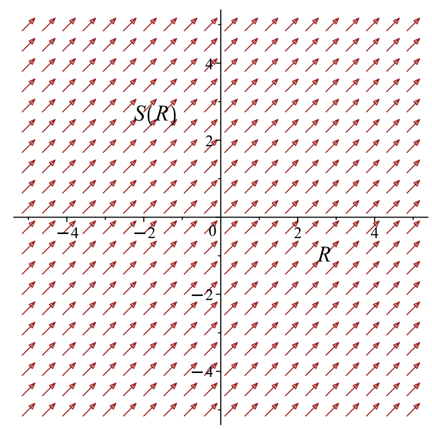
Which simplifies to

$$e^{-x} y = x + c_1$$

Which gives

$$y = e^{x}(x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y + e^{x}$ 	$R = x$ $S = e^{-x} y$	$\frac{dS}{dR} = 1$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

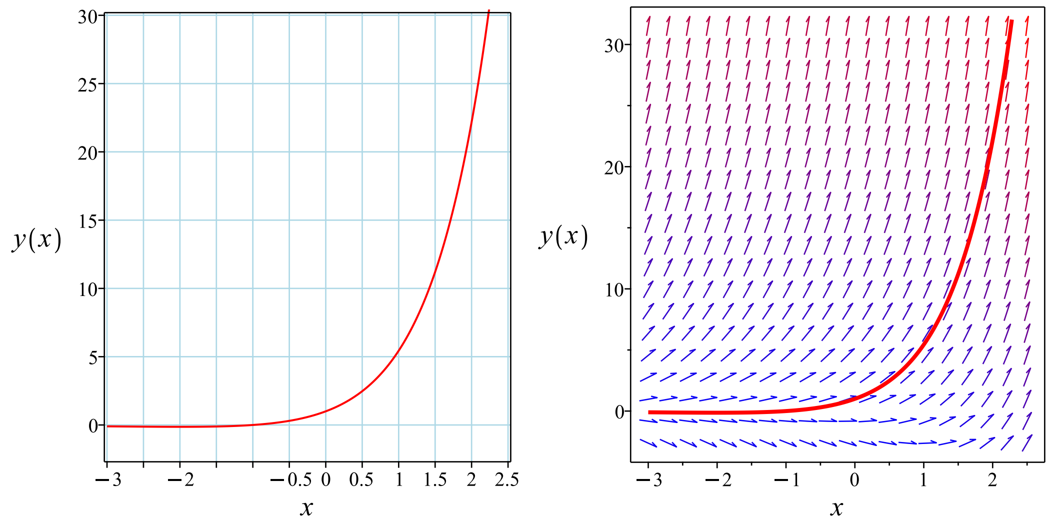
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = e^{x} x + e^{x}$$

Summary
The solution(s) found are the following

$$y = e^{x} x + e^{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^x x + e^x$$

Verified OK.

5.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (y + \mathrm{e}^x)\,\mathrm{d}x\\ (-y - \mathrm{e}^x)\,\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y - \mathrm{e}^x\\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y - \mathrm{e}^x)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((-1) - (0))\\ &= -1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -1 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-x} \\ &= \mathrm{e}^{-x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-x}(-y - \mathrm{e}^{x}) \\ &= -\mathrm{e}^{-x} y - 1\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{-x} y - 1\right) + \left(\mathrm{e}^{-x}\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\mathrm{e}^{-x} y - 1 \, \mathrm{d}x\end{aligned}$$

$$\phi = -x + \mathrm{e}^{-x} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \mathrm{e}^{-x}y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \mathrm{e}^{-x}y$$

The solution becomes

$$y = \mathrm{e}^{x}(x + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

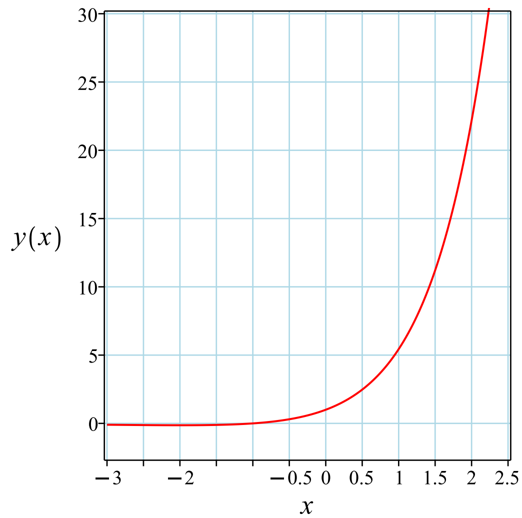
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

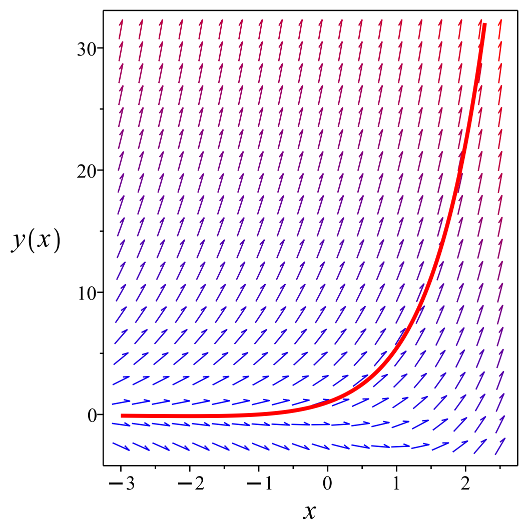
$$y = \mathrm{e}^{x}x + \mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = e^{x} x + e^{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{x} x + e^{x}$$

Verified OK.

5.20.5 Maple step by step solution

Let's solve

$[y' - y = e^{x}, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + e^{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = e^{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y'-y\right)=\mu(x)\,\mathrm{e}^{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'-y\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=-\mu(x)$

- Solve to find the integrating factor

  $\mu(x)=\mathrm{e}^{-x}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\,\mathrm{e}^{x}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\mu(x)\,\mathrm{e}^{x}dx+c_1$

- Solve for y

  $y=\frac{\int\mu(x)\mathrm{e}^{x}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\mathrm{e}^{-x}$

  $y=\frac{\int\mathrm{e}^{x}\mathrm{e}^{-x}dx+c_1}{\mathrm{e}^{-x}}$

- Evaluate the integrals on the rhs

  $y=\frac{x+c_1}{\mathrm{e}^{-x}}$

- Simplify

  $y=\mathrm{e}^{x}(x+c_1)$

- Use initial condition $y(0)=1$

  $1=c_1$

- Solve for c_1

  $c_1=1$

- Substitute $c_1=1$ into general solution and simplify

  $y=\mathrm{e}^{x}(x+1)$

- Solution to the IVP

  $y=\mathrm{e}^{x}(x+1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve([diff(y(x),x)-y(x)=exp(x),y(0) = 1],y(x), singsol=all)
```

$$y(x) = e^x(1 + x)$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 12

```
DSolve[{y'[x]-y[x]==Exp[x],{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(x + 1)$$

5.21 problem Exercise 11.22, page 97

Internal problem ID [4515]
Internal file name [`OUTPUT/4008_Sunday_June_05_2022_12_04_15_PM_9426759/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.22, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' + \frac{y}{x} - \frac{y^2}{x} = 0}$$

With initial conditions

$$[y(-1) = 1]$$

5.21.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y(y-1)}{x} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0=-1$ is inside this domain. The y domain of $f(x,y)$ when $x=-1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0=1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y(y-1)}{x}\right) \\ &= \frac{y-1}{x} + \frac{y}{x} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0=-1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=-1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0=1$ is inside this domain. Therefore solution exists and is unique.

5.21.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y(y-1)}{x} \end{aligned}$$

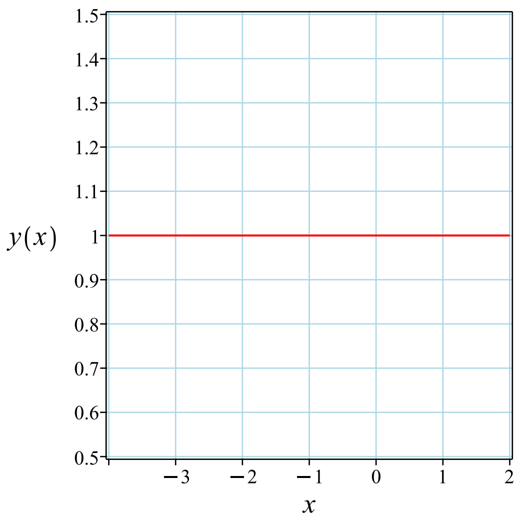
Where $f(x) = \frac{1}{x}$ and $g(y) = y(y-1)$. Since unique solution exists and $g(y)$ evaluated at $y_0 = 1$ is zero, then the solution is

$$\begin{aligned} y &= y_0 \\ &= 1 \end{aligned}$$

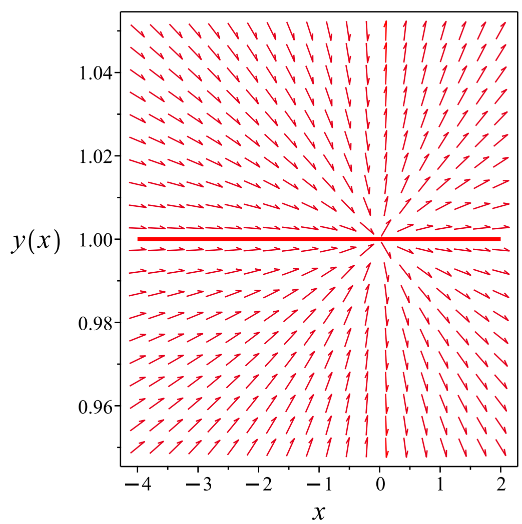
Summary

The solution(s) found are the following

$$y = 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1$$

Verified OK.

5.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y(y-1)}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 91: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x \\
\eta(x, y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta=0$ then in this special case

$$R=y$$

S is found from

$$\begin{aligned}S&=\int\frac{1}{\xi}dx\\&=\int\frac{1}{x}dx\end{aligned}$$

Which results in

$$S=\ln(x)$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{y(y-1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x&=0\\R_y&=1\\S_x&=\frac{1}{x}\\S_y&=0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=\frac{1}{y(y-1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=\frac{1}{R(R-1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R-1) - \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = \ln(y-1) - \ln(y) + c_1$$

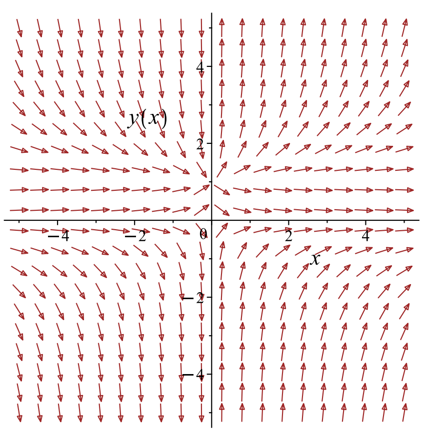
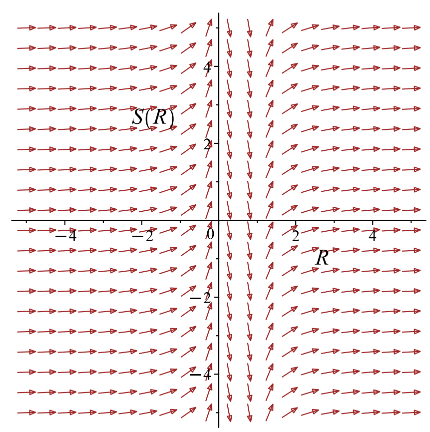
Which simplifies to

$$\ln(x) = \ln(y-1) - \ln(y) + c_1$$

Which gives

$$y = \frac{e^{c_1}}{-x + e^{c_1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(y-1)}{x}$ 	$R = y$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{1}{R(R-1)}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

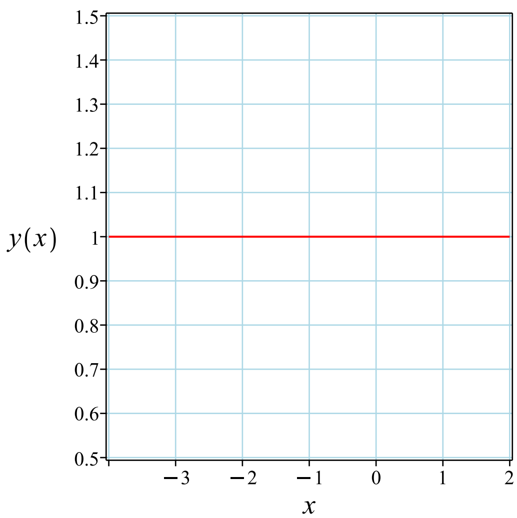
$$1 = \frac{e^{c_1}}{1 + e^{c_1}}$$

Unable to solve for constant of integration. Since $\lim_{c_1\to\infty}$ gives $y = \frac{e^{c_1}}{-x+e^{c_1}} = y = 1$ and this result satisfies the given initial condition.

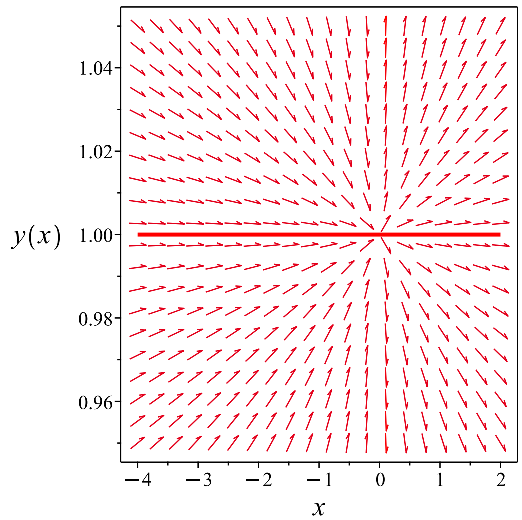
Summary

The solution(s) found are the following

$$y = 1$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1$$

Verified OK.

5.21.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(y-1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + \frac{1}{x}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= \frac{1}{x} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{yx} + \frac{1}{x} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{w(x)}{x} + \frac{1}{x} \\ w' &= \frac{w}{x} - \frac{1}{x} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= -\frac{1}{x}
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{w(x)}{x} = -\frac{1}{x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{1}{x}\right) \\
\frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(-\frac{1}{x}\right) \\
d\left(\frac{w}{x}\right) &= \left(-\frac{1}{x^2}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x} &= \int -\frac{1}{x^2} dx \\
\frac{w}{x} &= \frac{1}{x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
w(x) = c_1 x + 1
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = c_1 x + 1
$$

Or

$$
y = \frac{1}{c_1 x + 1}
$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{-1 + c_1}$$

$$c_1 = 0$$

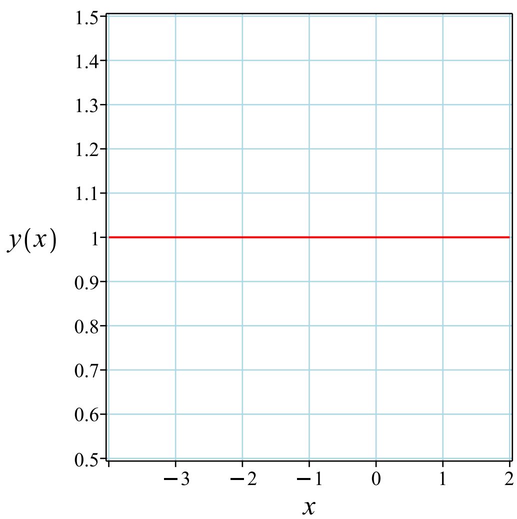
Substituting c_1 found above in the general solution gives

$$y = 1$$

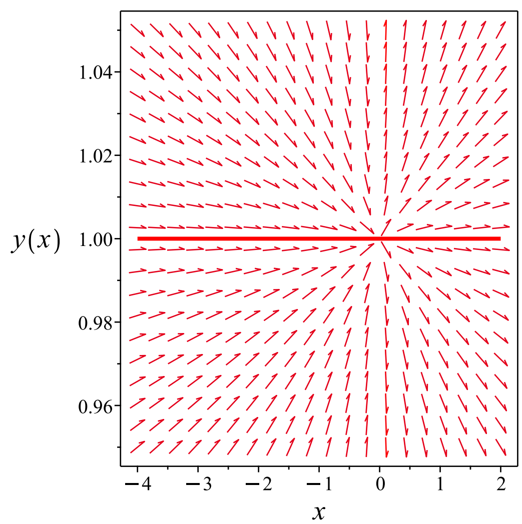
Summary

The solution(s) found are the following

$$y = 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1$$

Verified OK.

5.21.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y\,(y-1)}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{1}{y\,(y-1)}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x} \\
N(x,y) &= \frac{1}{y\left(y-1\right)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y\left(y-1\right)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln\left(x\right) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{y(y-1)}$. Therefore equation (4) becomes

$$\frac{1}{y(y-1)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y(y-1)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y(y-1)}\right) \mathrm{d}y$$
$$f(y) = \ln(y-1) - \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(y-1) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(y-1) - \ln(y)$$

The solution becomes

$$y = -\frac{1}{-1 + x\,\mathrm{e}^{c_1}}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{1 + \mathrm{e}^{c_1}}$$

Unable to solve for constant of integration. Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid.

Verification of solutions N/A

5.21.6 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(y-1)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y}{x} + \frac{y^2}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = \frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{x^2} \\ f_1 f_2 &= -\frac{1}{x^2} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x} + \frac{2u'(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{x}$$

The above shows that

$$u'(x) = -\frac{c_2}{x^2}$$

Using the above in (1) gives the solution

$$y = \frac{c_2}{x\left(c_1 + \frac{c_2}{x}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{1}{c_3 x + 1}$$

Initial conditions are used to solve for c_3. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{c_3 - 1}$$

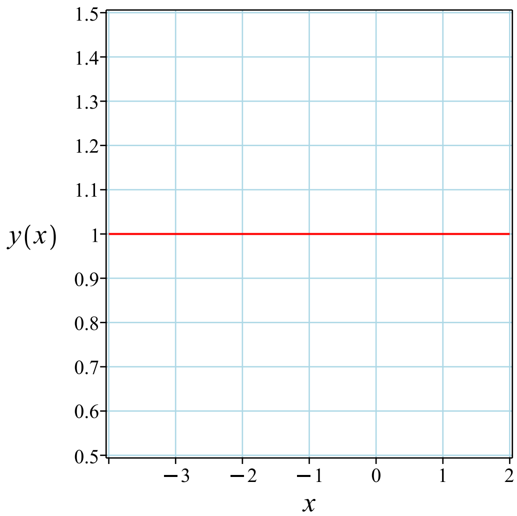
$$c_3 = 0$$

Substituting c_3 found above in the general solution gives

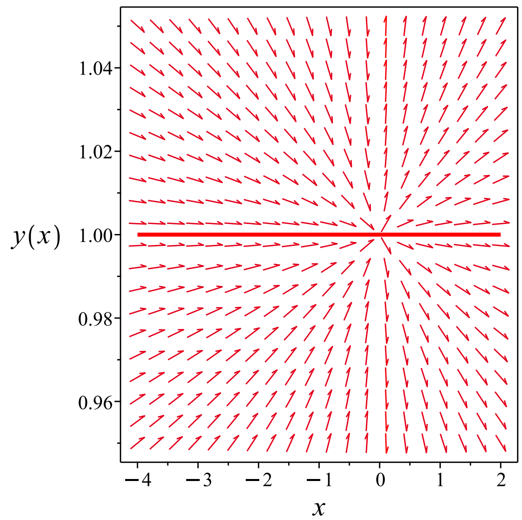
$$y = 1$$

Summary
The solution(s) found are the following

$$y = 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1$$

Verified OK.

5.21.7 Maple step by step solution

Let's solve

$\left[y' + \frac{y}{x} - \frac{y^2}{x} = 0, y(-1) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(y-1)} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(y-1)} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $\ln(y-1) - \ln(y) = \ln(x) + c_1$

- Solve for y

  $y = -\frac{1}{-1+x\,\mathrm{e}^{c_1}}$

- Use initial condition $y(-1) = 1$

  $1 = -\frac{1}{-1-e^{c_1}}$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)+y(x)/x=y(x)^2/x,y(-1) = 1],y(x), singsol=all)
```

$$y(x) = 1$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 6

```
DSolve[{y'[x]+y[x]/x==y[x]^2/x,{y[-1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1$$

5.22 problem Exercise 11.23, page 97

Internal problem ID [4516]
Internal file name [`OUTPUT/4009_Sunday_June_05_2022_12_04_24_PM_7975492/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.23, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{2\cos(x)\,y' - \sin(x)\,y + y^3 = 0}$$

With initial conditions

$$[y(0) = 1]$$

5.22.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y(-y^2+\sin(x))}{2\cos(x)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z35 \vee \frac{1}{2}\pi + \pi\_Z35 < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y(-y^2 + \sin(x))}{2\cos(x)}\right) \\ &= \frac{-y^2 + \sin(x)}{2\cos(x)} - \frac{y^2}{\cos(x)}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z35 \vee \frac{1}{2}\pi + \pi\_Z35 < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

5.22.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= \frac{y(-y^2 + \sin(x))}{2\cos(x)} \\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 94: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^3 \cos(x)
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^3 \cos(x)} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{2\cos(x)\, y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(-y^2 + \sin(x))}{2\cos(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{\sec(x)\tan(x)}{2y^2} \\ S_y &= \frac{\sec(x)}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\sec(x)^2}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{\sec(R)^2}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{\tan(R)}{2} + c_1 \tag{4}$$

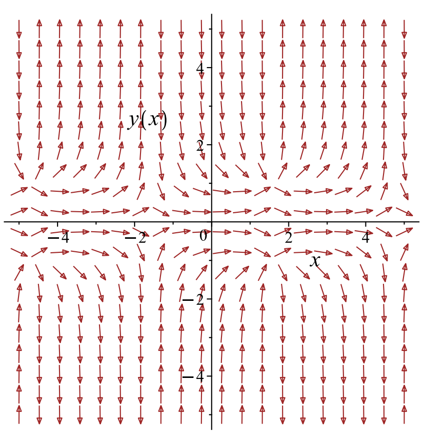
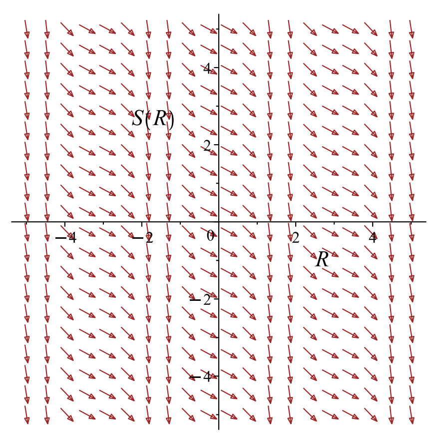
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\sec(x)}{2y^2} = -\frac{\tan(x)}{2} + c_1$$

Which simplifies to

$$-\frac{\sec(x)}{2y^2} = -\frac{\tan(x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(-y^2+\sin(x))}{2\cos(x)}$ 	$R = x$ $S = -\frac{\sec(x)}{2y^2}$	$\frac{dS}{dR} = -\frac{\sec(R)^2}{2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = c_1$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-\frac{\sec(x)}{2y^2} = -\frac{\tan(x)}{2} - \frac{1}{2}$$

The above simplifies to

$$y^2 \tan(x) + y^2 - \sec(x) = 0$$

Summary

The solution(s) found are the following

$$y^2 \tan(x) + y^2 - \sec(x) = 0 \tag{1}$$

Verification of solutions

$$y^2 \tan(x) + y^2 - \sec(x) = 0$$

Verified OK.

5.22.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(-y^2 + \sin(x))}{2\cos(x)} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{\sin(x)}{2\cos(x)} y - \frac{1}{2\cos(x)} y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{\sin(x)}{2\cos(x)} \\
f_1(x) &= -\frac{1}{2\cos(x)} \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y' \frac{1}{y^3} = \frac{\sin(x)}{2\cos(x)\, y^2} - \frac{1}{2\cos(x)} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{2}{y^3} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= \frac{\sin(x)\, w(x)}{2\cos(x)} - \frac{1}{2\cos(x)} \\
w' &= -\frac{\sin(x)\, w}{\cos(x)} + \frac{1}{\cos(x)}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \tan(x) \\
q(x) &= \sec(x)
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \tan(x)\, w(x) = \sec(x)
$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \tan(x)dx} \\ &= \frac{1}{\cos(x)}\end{aligned}$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(\sec(x)) \\ \frac{\mathrm{d}}{\mathrm{d}x}(\sec(x)\, w) &= (\sec(x))(\sec(x)) \\ \mathrm{d}(\sec(x)\, w) &= \sec(x)^2\, \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\sec(x)\, w &= \int \sec(x)^2\, \mathrm{d}x \\ \sec(x)\, w &= \tan(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$w(x) = \cos(x)\tan(x) + \cos(x)\, c_1$$

which simplifies to

$$w(x) = \cos(x)\, c_1 + \sin(x)$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = \cos(x)\, c_1 + \sin(x)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{y^2} = \cos(x) + \sin(x)$$

The above simplifies to

$$-\cos(x)\, y^2 - \sin(x)\, y^2 + 1 = 0$$

Summary

The solution(s) found are the following

$$-\cos(x)\, y^2 - \sin(x)\, y^2 + 1 = 0 \tag{1}$$

Verification of solutions

$$-\cos(x)\, y^2 - \sin(x)\, y^2 + 1 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.578 (sec). Leaf size: 33

```
dsolve([2*cos(x)*diff(y(x),x)=y(x)*sin(x)-y(x)^3,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{(2\cos(x)^2 - 1)(-\sin(x) + \cos(x))}}{2\cos(x)^2 - 1}$$

✓ Solution by Mathematica

Time used: 0.369 (sec). Leaf size: 14

```
DSolve[{2*Cos[x]*y'[x]==y[x]*Sin[x]-y[x]^3,{y[0]==1}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{\sqrt{\sin(x) + \cos(x)}}$$

5.23 problem Exercise 11.24, page 97

Internal problem ID [4517]
Internal file name [`OUTPUT/4010_Sunday_June_05_2022_12_04_39_PM_21440736/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.24, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{(x-\cos(y))\,y'+\tan(y)=0}$$

With initial conditions

$$\left[y(1)=\frac{\pi}{6}\right]$$

5.23.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{\tan(y)}{-x+\cos(y)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=\frac{\pi}{6}$ is

$$\left\{x<\frac{\sqrt{3}}{2}\vee\frac{\sqrt{3}}{2}<x\right\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\left\{-\infty \leq y < 2\pi\_Z37, 2\pi\_Z37 < y < \frac{1}{2}\pi + \pi\_Z36, \frac{1}{2}\pi + \pi\_Z36 < y \leq \infty\right\}$$

But the point $y_0 = \frac{\pi}{6}$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

5.23.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
(x-\cos(y))\,\mathrm{d}y &= (-\tan(y))\,\mathrm{d}x \\
(\tan(y))\,\mathrm{d}x + (x-\cos(y))\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= \tan(y) \\
N(x,y) &= x-\cos(y)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\tan(y)) \\
&= \sec(y)^2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x-\cos(y)) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x-\cos(y)}\left(\left(1+\tan(y)^2\right) - (1)\right) \\
&= \frac{\tan(y)^2}{x-\cos(y)}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \cot(y)\left((1) - \left(1+\tan(y)^2\right)\right) \\
&= -\tan(y)
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\tan(y) \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(\cos(y))} \\ &= \cos(y)\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \cos(y)(\tan(y)) \\ &= \sin(y)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \cos(y)(x - \cos(y)) \\ &= (x - \cos(y))\cos(y)\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (\sin(y)) + ((x - \cos(y))\cos(y))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \sin(y) \, \mathrm{d}x \\ \phi &= \sin(y)x + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cos(y)\, x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x - \cos(y))\cos(y)$. Therefore equation (4) becomes

$$(x - \cos(y))\cos(y) = \cos(y)\, x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\cos(y)^2$$

Integrating the above w.r.t y gives

$$\int f'(y)\, dy = \int \left(-\cos(y)^2\right) dy$$

$$f(y) = -\frac{\cos(y)\sin(y)}{2} - \frac{y}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin(y)\, x - \frac{\cos(y)\sin(y)}{2} - \frac{y}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin(y)\, x - \frac{\cos(y)\sin(y)}{2} - \frac{y}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \frac{\pi}{6}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} - \frac{\sqrt{3}}{8} - \frac{\pi}{12} = c_1$$

$$c_1 = \frac{1}{2} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}$$

Substituting c_1 found above in the general solution gives

$$\sin(y)\, x - \frac{\cos(y)\sin(y)}{2} - \frac{y}{2} = \frac{1}{2} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}$$

Summary

The solution(s) found are the following

$$\frac{\sin(y)\,(-\cos(y) + 2x)}{2} - \frac{y}{2} = \frac{1}{2} - \frac{\sqrt{3}}{8} - \frac{\pi}{12} \tag{1}$$

Verification of solutions

$$\frac{\sin(y)\,(-\cos(y) + 2x)}{2} - \frac{y}{2} = \frac{1}{2} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 1.172 (sec). Leaf size: 29

```
dsolve([(x-cos(y(x)))*diff(y(x),x)+tan(y(x))=0,y(1) = 1/6*Pi],y(x), singsol=all)
```

$$y(x) = \text{RootOf}\left(24\sin(\_Z)\, x - 6\sin(2\_Z) + 2\pi + 3\sqrt{3} - 12\_Z - 12\right)$$

✓ Solution by Mathematica

Time used: 0.216 (sec). Leaf size: 45

```
DSolve[{(x-Cos[y[x]])*y'[x]+Tan[y[x]]==0,{y[1]==Pi/6}},y[x],x,IncludeSingularSolutions -> Tr
```

$$\text{Solve}\left[x=\frac{1}{24}\left(12-3\sqrt{3}-2\pi\right)\csc(y(x))+\left(\frac{y(x)}{2}+\frac{1}{4}\sin(2y(x))\right)\csc(y(x)),y(x)\right]$$

5.24 problem Exercise 11.26, page 97

Internal problem ID [4518]
Internal file name [OUTPUT/4011_Sunday_June_05_2022_12_07_58_PM_94540826/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.26, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{y' - \frac{2y}{x} + \frac{y^2}{x} = x^3}$$

5.24.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{-x^4 + y^2 - 2y}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^3 + \frac{2y}{x} - \frac{y^2}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^3$, $f_1(x) = \frac{2}{x}$ and $f_2(x) = -\frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{1}{x^2} \\ f_1f_2 &= -\frac{2}{x^2} \\ f_2^2f_0 &= x \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x} + \frac{u'(x)}{x^2} + xu(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1\sinh\left(\frac{x^2}{2}\right) + c_2\cosh\left(\frac{x^2}{2}\right)$$

The above shows that

$$u'(x) = x\left(c_1\cosh\left(\frac{x^2}{2}\right) + c_2\sinh\left(\frac{x^2}{2}\right)\right)$$

Using the above in (1) gives the solution

$$y = \frac{x^2\left(c_1\cosh\left(\frac{x^2}{2}\right) + c_2\sinh\left(\frac{x^2}{2}\right)\right)}{c_1\sinh\left(\frac{x^2}{2}\right) + c_2\cosh\left(\frac{x^2}{2}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x^2\left(c_3\cosh\left(\frac{x^2}{2}\right) + \sinh\left(\frac{x^2}{2}\right)\right)}{c_3\sinh\left(\frac{x^2}{2}\right) + \cosh\left(\frac{x^2}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 \left(c_3 \cosh\left(\frac{x^2}{2}\right) + \sinh\left(\frac{x^2}{2}\right)\right)}{c_3 \sinh\left(\frac{x^2}{2}\right) + \cosh\left(\frac{x^2}{2}\right)} \tag{1}$$

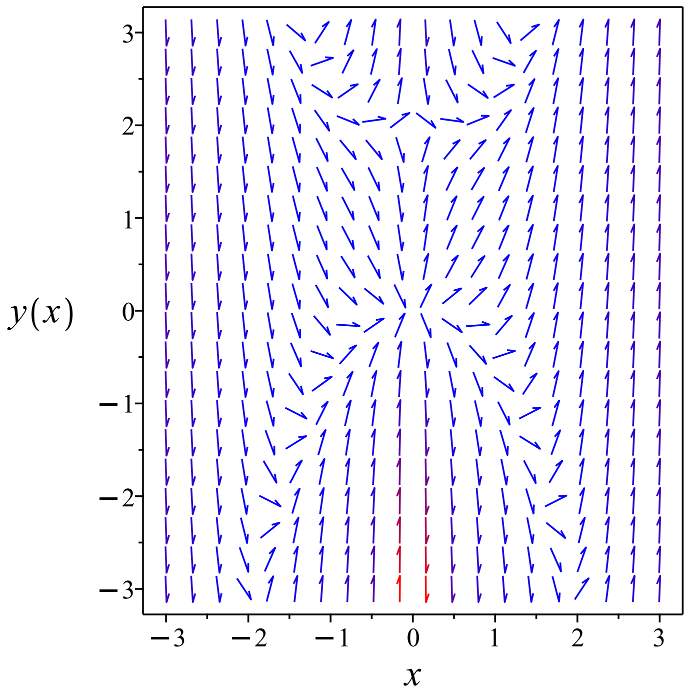


Figure 167: Slope field plot

Verification of solutions

$$y = \frac{x^2 \left(c_3 \cosh\left(\frac{x^2}{2}\right) + \sinh\left(\frac{x^2}{2}\right)\right)}{c_3 \sinh\left(\frac{x^2}{2}\right) + \cosh\left(\frac{x^2}{2}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)=x^3+2/x*y(x)-1/x*y(x)^2,y(x), singsol=all)
```

$$y(x) = i \tan\left(-\frac{i x^2}{2} + c_1\right) x^2$$

✓ Solution by Mathematica

Time used: 0.162 (sec). Leaf size: 75

```
DSolve[y'[x]==x^3+2/x*y[x]-1/x*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2\left(i \cosh\left(\frac{x^2}{2}\right) + c_1 \sinh\left(\frac{x^2}{2}\right)\right)}{i \sinh\left(\frac{x^2}{2}\right) + c_1 \cosh\left(\frac{x^2}{2}\right)}$$
$$y(x) \to x^2 \tanh\left(\frac{x^2}{2}\right)$$

5.25 problem Exercise 11.27, page 97

Internal problem ID [4519]
Internal file name [OUTPUT/4012_Sunday_June_05_2022_12_08_08_PM_29117838/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.27, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + \sin(x)\, y^2 = 2\sec(x)\tan(x)}$$

5.25.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= 2\sec(x)\tan(x) - \sin(x)\, y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 2\sec(x)\tan(x) - \sin(x)\, y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 2\sec(x)\tan(x)$, $f_1(x) = 0$ and $f_2(x) = -\sin(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\sin(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\cos(x) \\ f_1f_2 &= 0 \\ f_2^2f_0 &= 2\sin(x)^2\sec(x)\tan(x) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\sin(x)\,u''(x) + \cos(x)\,u'(x) + 2\sin(x)^2\sec(x)\tan(x)\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1\sec(x) + c_2\cos(x)^2$$

The above shows that

$$u'(x) = \left(-2c_2\cos(x)^3 + c_1\right)\sec(x)\tan(x)$$

Using the above in (1) gives the solution

$$y = \frac{\left(-2c_2\cos(x)^3 + c_1\right)\sec(x)\tan(x)}{\sin(x)\left(c_1\sec(x) + c_2\cos(x)^2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-2\cos(x)^2 + c_3\sec(x)}{\cos(x)^3 + c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{-2\cos(x)^2 + c_3\sec(x)}{\cos(x)^3 + c_3} \tag{1}$$

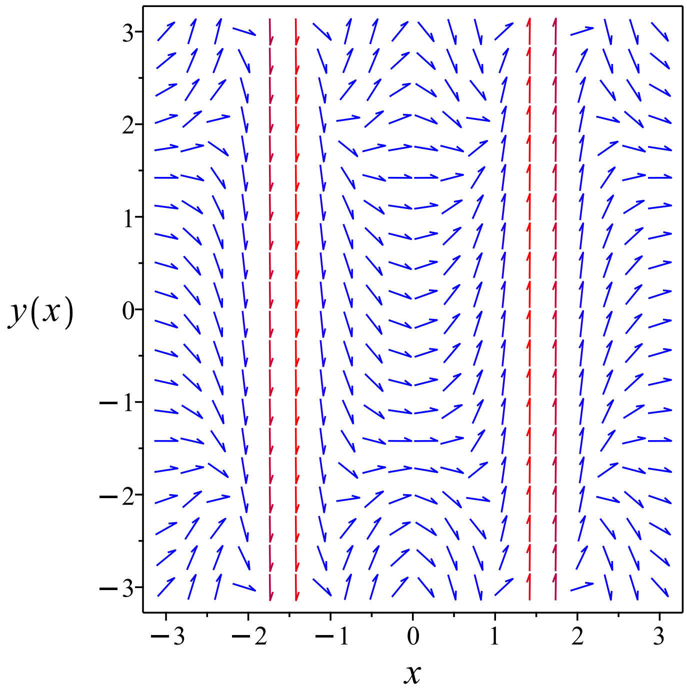


Figure 168: Slope field plot

Verification of solutions

$$y = \frac{-2\cos(x)^2 + c_3 \sec(x)}{\cos(x)^3 + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = cos(x)*(diff(y(x), x))/sin(x)+
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         <- LODE of Euler type successful
         Change of variables used:
            [x = arccos(t)]
         Linear ODE actually solved:
            (2*t^2-2)*u(t)+(-t^4+t^2)*diff(diff(u(t),t),t) = 0
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x)=2*tan(x)*sec(x)-y(x)^2*sin(x),y(x), singsol=all)
```

$$y(x) = \frac{-2c_1 \cos(x)^2 + \sec(x)}{c_1 \cos(x)^3 + 1}$$

✓ Solution by Mathematica

Time used: 0.88 (sec). Leaf size: 32

```
DSolve[y'[x]==2*Tan[x]*Sec[x]-y[x]^2*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sec(x)\left(-2\cos^3(x) + c_1\right)}{\cos^3(x) + c_1}$$

$$y(x) \to \sec(x)$$

5.26 problem Exercise 11.28, page 97

Internal problem ID [4520]
Internal file name [`OUTPUT/4013_Sunday_June_05_2022_12_08_20_PM_21730724/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.28, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_homogeneous, `class G`], _rational, _Riccati]`

$$\boxed{y' + \frac{y}{x} + y^2 = \frac{1}{x^2}}$$

5.26.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y^2x^2 + xy - 1}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(y^2x^2 + xy - 1)(b_3 - a_2)}{x^2} - \frac{(y^2x^2 + xy - 1)^2 a_3}{x^4} - \left(-\frac{2y^2x + y}{x^2} + \frac{2y^2x^2 + 2xy - 2}{x^3}\right)(xa_2 + ya_3 + a_1) + \frac{(2y\,x^2 + x)(xb_2 + yb_3 + b_1)}{x^2} = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{-x^4y^4a_3 + 2x^5yb_2 + x^4y^2a_2 + x^4y^2b_3 - 2x^3y^3a_3 + 2x^4yb_1 + 2b_2x^4 + x^3b_1 - x^2ya_1 + x^2a_2 + x^2b_3 + 4xya_3}{x^4} = 0$$

Setting the numerator to zero gives

$$-x^4y^4a_3 + 2x^5yb_2 + x^4y^2a_2 + x^4y^2b_3 - 2x^3y^3a_3 + 2x^4yb_1 + 2b_2x^4 + x^3b_1 - x^2ya_1 + x^2a_2 + x^2b_3 + 4xya_3 + 2xa_1 - a_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_3v_1^4v_2^4 + a_2v_1^4v_2^2 - 2a_3v_1^3v_2^3 + 2b_2v_1^5v_2 + b_3v_1^4v_2^2 + 2b_1v_1^4v_2 + 2b_2v_1^4 - a_1v_1^2v_2 + b_1v_1^3 + a_2v_1^2 + 4a_3v_1v_2 + b_3v_1^2 + 2a_1v_1 - a_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$2b_2v_1^5v_2 - a_3v_1^4v_2^4 + (a_2 + b_3)\,v_1^4v_2^2 + 2b_1v_1^4v_2 + 2b_2v_1^4 - 2a_3v_1^3v_2^3 + b_1v_1^3 - a_1v_1^2v_2 + (a_2 + b_3)\,v_1^2 + 4a_3v_1v_2 + 2a_1v_1 - a_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -a_1 &= 0 \\ 2a_1 &= 0 \\ -2a_3 &= 0 \\ -a_3 &= 0 \\ 4a_3 &= 0 \\ 2b_1 &= 0 \\ 2b_2 &= 0 \\ a_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y^2x^2+xy-1}{x^2}\right)(-x) \\
&= \frac{-y^2x^2+1}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-y^2x^2+1}{x}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln(xy+1)}{2} - \frac{\ln(xy-1)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y^2x^2+xy-1}{x^2}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y}{y^2x^2 - 1} \\
S_y &= -\frac{x}{y^2x^2 - 1}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(xy+1)}{2} - \frac{\ln(xy-1)}{2} = \ln(x) + c_1$$

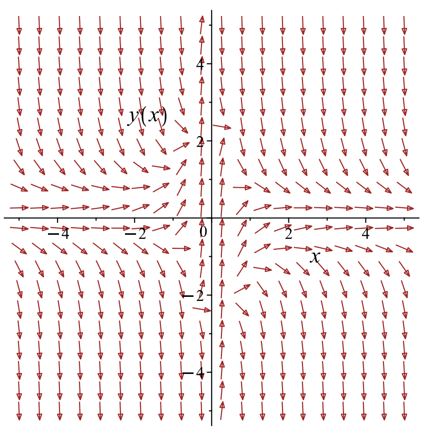
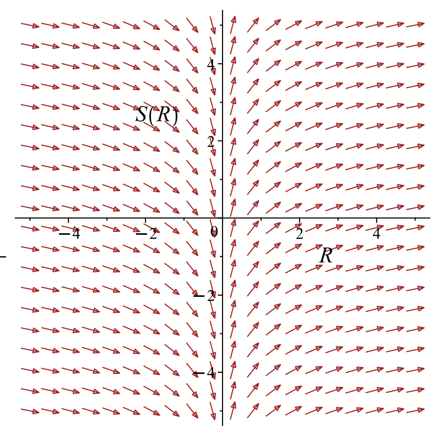
Which simplifies to

$$\frac{\ln(xy+1)}{2} - \frac{\ln(xy-1)}{2} = \ln(x) + c_1$$

Which gives

$$y = \frac{e^{2c_1}x^2 + 1}{x\left(e^{2c_1}x^2 - 1\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2x^2+xy-1}{x^2}$ 	$R = x$ $S = \frac{\ln(xy+1)}{2} - \frac{\ln(xy}{2}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{2c_1}x^2 + 1}{x\left(e^{2c_1}x^2 - 1\right)} \tag{1}$$

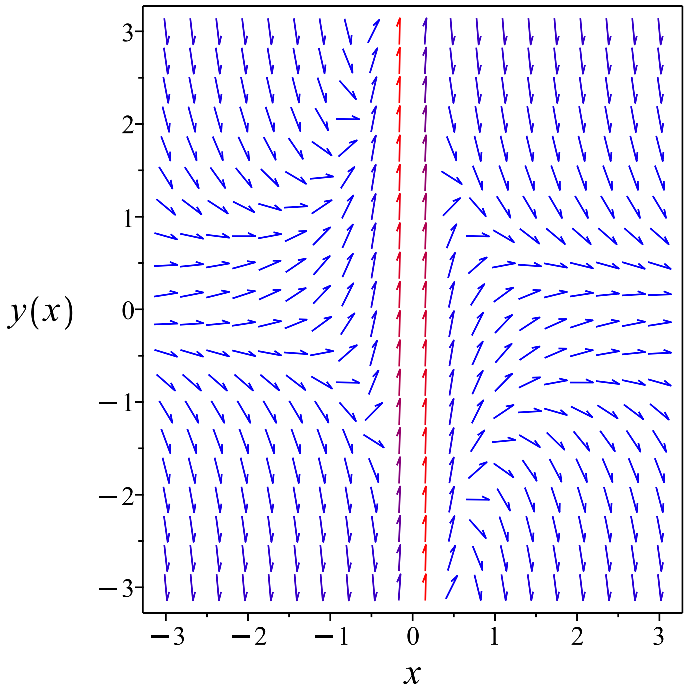


Figure 169: Slope field plot

Verification of solutions

$$y = \frac{e^{2c_1}x^2 + 1}{x\left(e^{2c_1}x^2 - 1\right)}$$

Verified OK.

5.26.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{1}{x^2} - \frac{y}{x} - y^2\right) \mathrm{d}x$$
$$\left(y^2 + \frac{y}{x} - \frac{1}{x^2}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y^2 + \frac{y}{x} - \frac{1}{x^2}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 + \frac{y}{x} - \frac{1}{x^2}\right) \\ &= 2y + \frac{1}{x}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(2y + \frac{1}{x}\right) - (0)\right) \\
&= 2y + \frac{1}{x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y^2 + \frac{y}{x} - \frac{1}{x^2}}\left((0) - \left(2y + \frac{1}{x}\right)\right) \\
&= \frac{-2y\,x^2 - x}{y^2x^2 + xy - 1}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(0) - \left(2y + \frac{1}{x}\right)}{x\left(y^2 + \frac{y}{x} - \frac{1}{x^2}\right) - y\,(1)} \\
&= \frac{-2xy - 1}{y^2x^2 - 1}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = \frac{-2t - 1}{t^2 - 1}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}\mu &= e^{\int R \, \mathrm{d}t} \\ &= e^{\int \left(\frac{-2t-1}{t^2-1}\right) \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{3\ln(t-1)}{2}-\frac{\ln(t+1)}{2}} \\ &= \frac{1}{(t-1)^{\frac{3}{2}}\sqrt{t+1}}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{(xy-1)^{\frac{3}{2}}\sqrt{xy+1}}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{(xy-1)^{\frac{3}{2}}\sqrt{xy+1}}\left(y^2+\frac{y}{x}-\frac{1}{x^2}\right) \\ &= \frac{y^2x^2+xy-1}{x^2(xy-1)^{\frac{3}{2}}\sqrt{xy+1}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{(xy-1)^{\frac{3}{2}}\sqrt{xy+1}}(1) \\ &= \frac{1}{(xy-1)^{\frac{3}{2}}\sqrt{xy+1}}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{y^2x^2+xy-1}{x^2(xy-1)^{\frac{3}{2}}\sqrt{xy+1}}\right)+\left(\frac{1}{(xy-1)^{\frac{3}{2}}\sqrt{xy+1}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{y^2x^2 + xy - 1}{x^2 (xy-1)^{\frac{3}{2}} \sqrt{xy+1}} \mathrm{d}x$$

$$\phi = -\frac{\sqrt{xy+1}}{x\sqrt{xy-1}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= -\frac{1}{2\sqrt{xy+1}\sqrt{xy-1}} + \frac{\sqrt{xy+1}}{2(xy-1)^{\frac{3}{2}}} + f'(y) \\ &= \frac{1}{(xy-1)^{\frac{3}{2}} \sqrt{xy+1}} + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{(xy-1)^{\frac{3}{2}}\sqrt{xy+1}}$. Therefore equation (4) becomes

$$\frac{1}{(xy-1)^{\frac{3}{2}} \sqrt{xy+1}} = \frac{1}{(xy-1)^{\frac{3}{2}} \sqrt{xy+1}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\sqrt{xy+1}}{x\sqrt{xy-1}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\sqrt{xy+1}}{x\sqrt{xy-1}}$$

The solution becomes

$$y = \frac{c_1^2 x^2 + 1}{(c_1^2 x^2 - 1)\, x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1^2 x^2 + 1}{(c_1^2 x^2 - 1)\, x} \tag{1}$$

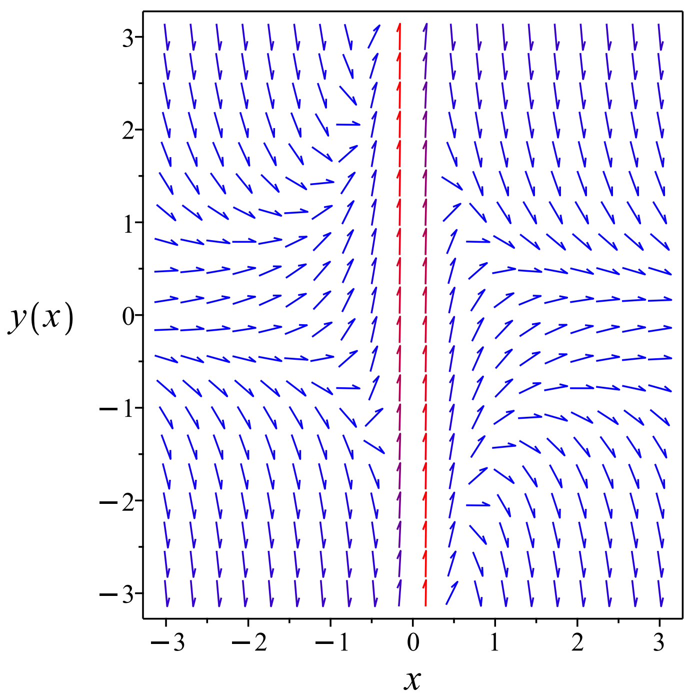


Figure 170: Slope field plot

Verification of solutions

$$y = \frac{c_1^2 x^2 + 1}{(c_1^2 x^2 - 1)\, x}$$

Verified OK.

5.26.3 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= -\frac{y^2x^2 + xy - 1}{x^2}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = \frac{1}{x^2} - \frac{y}{x} - y^2
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = \frac{1}{x^2}$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = -1$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-u} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= \frac{1}{x} \\
f_2^2 f_0 &= \frac{1}{x^2}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-u''(x) - \frac{u'(x)}{x} + \frac{u(x)}{x^2} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = \frac{c_1 x^2 + c_2}{x}
$$

The above shows that

$$u'(x) = \frac{c_1 x^2 - c_2}{x^2}$$

Using the above in (1) gives the solution

$$y = \frac{c_1 x^2 - c_2}{x\,(c_1 x^2 + c_2)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 x^2 - 1}{x\,(c_3 x^2 + 1)}$$

Summary

The solution(s) found are the following

$$y = \frac{c_3 x^2 - 1}{x\,(c_3 x^2 + 1)} \tag{1}$$

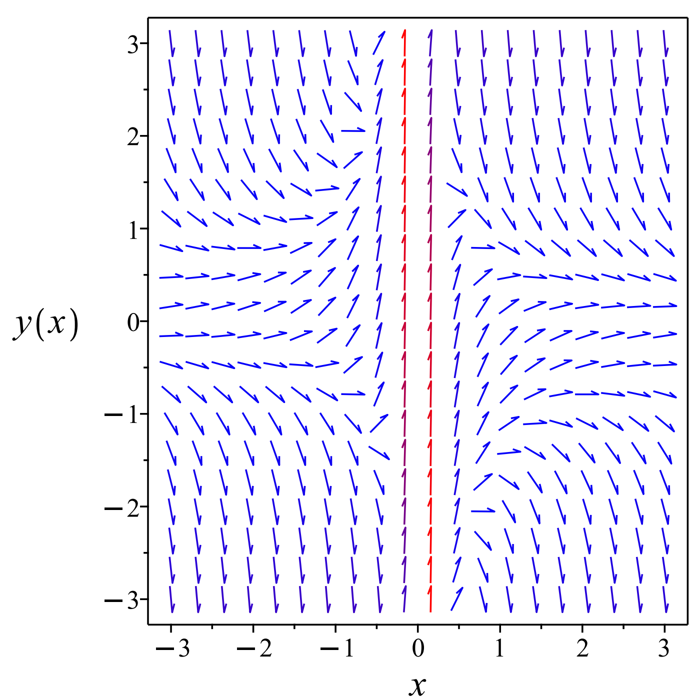


Figure 171: Slope field plot

Verification of solutions

$$y = \frac{c_3 x^2 - 1}{x (c_3 x^2 + 1)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)=1/x^2-y(x)/x-y(x)^2,y(x), singsol=all)
```

$$y(x) = -\frac{\tanh(-\ln(x) + c_1)}{x}$$

✓ Solution by Mathematica

Time used: 1.192 (sec). Leaf size: 62

```
DSolve[y'[x]==1/x^2-y[x]/x-y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{i \tan(c_1 - i \log(x))}{x}$$
$$y(x) \to -\frac{-x^2 + e^{2i\text{Interval}[\{0,\pi\}]}}{x^3 + x e^{2i\text{Interval}[\{0,\pi\}]}}$$

5.27 problem Exercise 11.29, page 97

Internal problem ID [4521]
Internal file name [`OUTPUT/4014_Sunday_June_05_2022_12_08_28_PM_62548136/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 11, Bernoulli Equations
Problem number: Exercise 11.29, page 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y' - \frac{y}{x} + \frac{y^2}{x^2} = 1}$$

5.27.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x)^2 = 1$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-u^2+1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -u^2 + 1$. Integrating both sides gives

$$\frac{1}{-u^2+1}\, du = \frac{1}{x}\, dx$$
$$\int \frac{1}{-u^2+1}\, du = \int \frac{1}{x}\, dx$$
$$\operatorname{arctanh}(u) = \ln(x) + c_2$$

The solution is

$$\operatorname{arctanh}(u(x)) - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\operatorname{arctanh}\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$
$$\operatorname{arctanh}\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\operatorname{arctanh}\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}$$

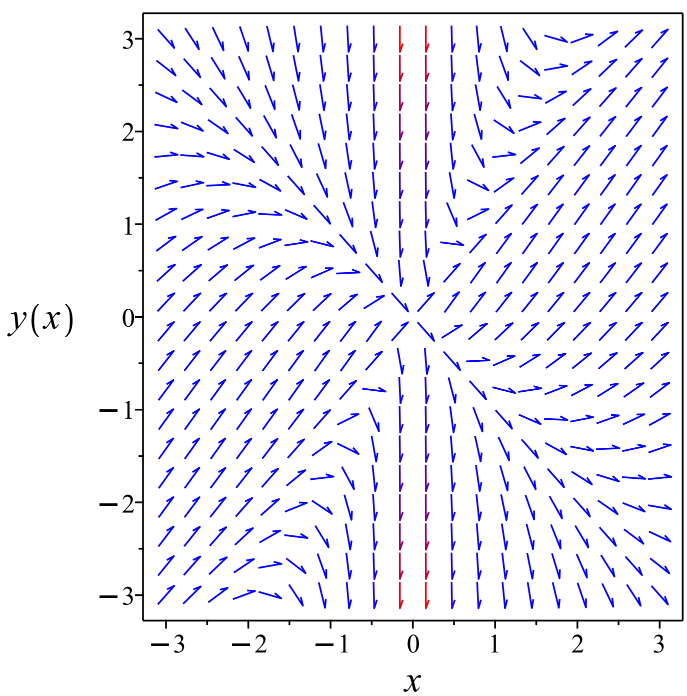


Figure 172: Slope field plot

Verification of solutions

$$\operatorname{arctanh}\left(\frac{y}{x}\right)-\ln\left(x\right)-c_2=0$$

Verified OK.

5.27.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{-x^2-xy+y^2}{x^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2-\frac{(-x^2-xy+y^2)(b_3-a_2)}{x^2}-\frac{(-x^2-xy+y^2)^2a_3}{x^4}-\left(-\frac{-2x-y}{x^2}+\frac{-2x^2-2xy+2y^2}{x^3}\right)(xa_2+ya_3+a_1)+\frac{(-x+2y)(xb_2+yb_3+b_1)}{x^2}=0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4a_2+x^4a_3-x^4b_3+2x^3ya_3-2x^3yb_2+x^2y^2a_2-2x^2y^2a_3-x^2y^2b_3+y^4a_3+x^3b_1-x^2ya_1-2x^2yb_1+}{x^4}=0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-x^4 a_2 - x^4 a_3 + x^4 b_3 - 2x^3 y a_3 + 2x^3 y b_2 - x^2 y^2 a_2 + 2x^2 y^2 a_3 \\
&+ x^2 y^2 b_3 - y^4 a_3 - x^3 b_1 + x^2 y a_1 + 2x^2 y b_1 - 2x\, y^2 a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-a_2 v_1^4 - a_2 v_1^2 v_2^2 - a_3 v_1^4 - 2a_3 v_1^3 v_2 + 2a_3 v_1^2 v_2^2 - a_3 v_2^4 + 2b_2 v_1^3 v_2 \\
&+ b_3 v_1^4 + b_3 v_1^2 v_2^2 + a_1 v_1^2 v_2 - 2a_1 v_1 v_2^2 - b_1 v_1^3 + 2b_1 v_1^2 v_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-a_2 - a_3 + b_3) v_1^4 + (-2a_3 + 2b_2) v_1^3 v_2 - b_1 v_1^3 \\
&+ (-a_2 + 2a_3 + b_3) v_1^2 v_2^2 + (a_1 + 2b_1) v_1^2 v_2 - 2a_1 v_1 v_2^2 - a_3 v_2^4 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_1 &= 0 \\
-a_3 &= 0 \\
-b_1 &= 0 \\
a_1 + 2b_1 &= 0 \\
-2a_3 + 2b_2 &= 0 \\
-a_2 - a_3 + b_3 &= 0 \\
-a_2 + 2a_3 + b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{-x^2 - xy + y^2}{x^2}\right)(x) \\
&= \frac{-x^2 + y^2}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2 + y^2}{x}} dy
\end{aligned}
$$

Which results in

$$S = -\frac{\ln(x+y)}{2} + \frac{\ln(-x+y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-x^2 - xy + y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{x^2 - y^2} \\ S_y &= -\frac{x}{x^2 - y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(x+y)}{2} + \frac{\ln(-x+y)}{2} = -\ln(x) + c_1$$

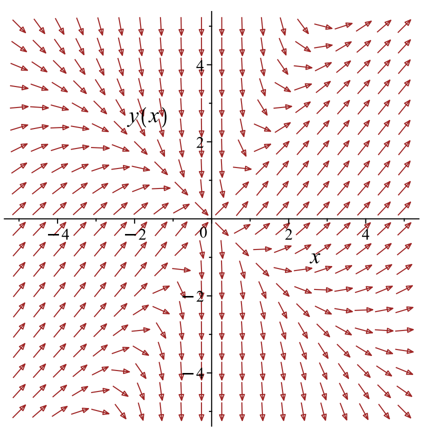
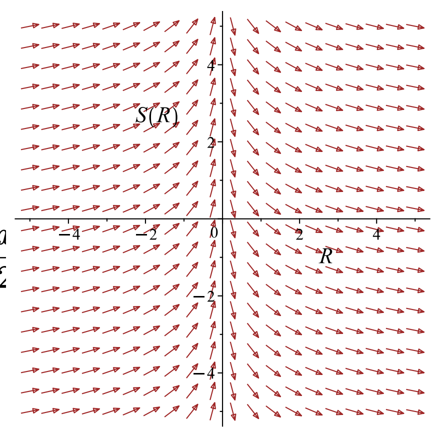
Which simplifies to

$$-\frac{\ln \left(x+y\right)}{2}+\frac{\ln \left(-x+y\right)}{2} = -\ln \left(x\right)+c_1$$

Which gives

$$y = -\frac{\left({\mathrm e}^{2 c_1}+x^{2}\right) x}{{\mathrm e}^{2 c_1}-x^{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{-x^{2}-x y+y^{2}}{x^{2}}$

</td><td>$R = x$
$S = -\frac{\ln \left(x+y\right)}{2}+\ln(-$</td><td>$\frac{dS}{dR} = -\frac{1}{R}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\frac{\left({\mathrm e}^{2 c_1}+x^{2}\right) x}{{\mathrm e}^{2 c_1}-x^{2}} \tag{1}$$

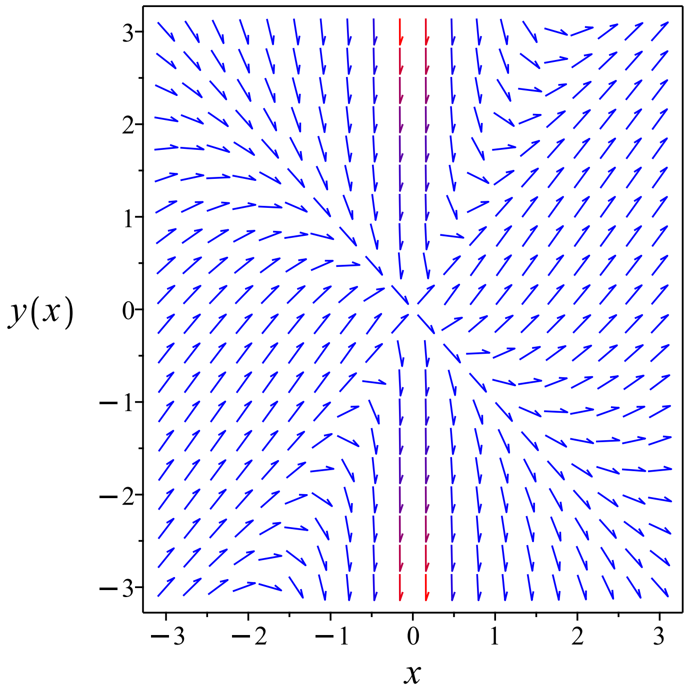


Figure 173: Slope field plot

Verification of solutions

$$y = -\frac{\left(e^{2c_1} + x^2\right) x}{e^{2c_1} - x^2}$$

Verified OK.

5.27.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{-x^2 - xy + y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 1 + \frac{y}{x} - \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 1$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = -\frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2}{x^3} \\ f_1 f_2 &= -\frac{1}{x^3} \\ f_2^2 f_0 &= \frac{1}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x^2} - \frac{u'(x)}{x^3} + \frac{u(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_1 x^2 + c_2}{x}$$

The above shows that

$$u'(x) = \frac{c_1 x^2 - c_2}{x^2}$$

Using the above in (1) gives the solution

$$y = \frac{(c_1 x^2 - c_2) x}{c_1 x^2 + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(c_3 x^2 - 1) x}{c_3 x^2 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{(c_3 x^2 - 1) x}{c_3 x^2 + 1} \tag{1}$$

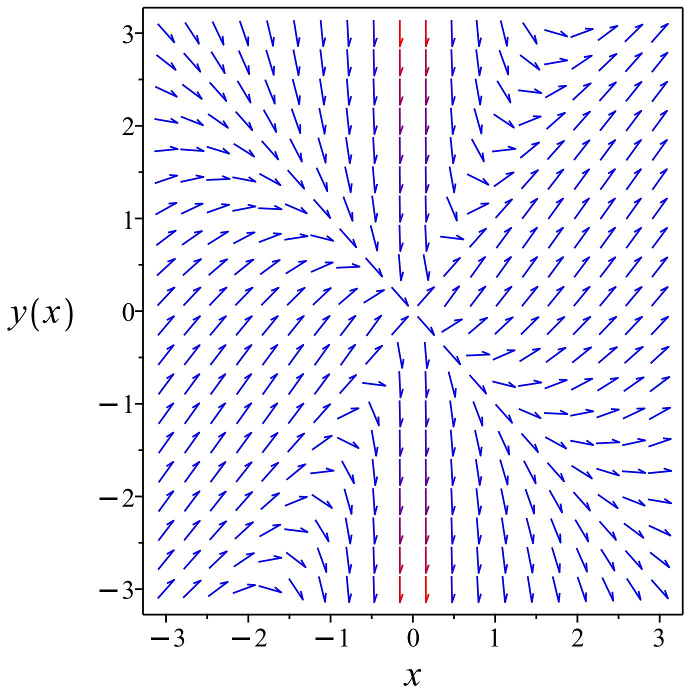


Figure 174: Slope field plot

Verification of solutions

$$y = \frac{(c_3 x^2 - 1) x}{c_3 x^2 + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)=1+y(x)/x-y(x)^2/x^2,y(x), singsol=all)
```

$$y(x) = \tanh\left(\ln\left(x\right) + c_1\right) x$$

✓ Solution by Mathematica

Time used: 0.539 (sec). Leaf size: 43

```
DSolve[y'[x]==1+y[x]/x-y[x]^2/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x(x^2 - e^{2c_1})}{x^2 + e^{2c_1}}$$
$$y(x) \to -x$$
$$y(x) \to x$$

6 Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods

6.1 problem Exercise 12.1, page 103

Internal problem ID [4522]
Internal file name [`OUTPUT/4015_Sunday_June_05_2022_12_08_38_PM_1734147/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.1, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{2xyy' + y^2(x+1) = \mathrm{e}^x}$$

6.1.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y^2x + y^2 - \mathrm{e}^x}{2xy} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 96: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{e^{-x-\ln(x)}}{y}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{e^{-x-\ln(x)}}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2 e^x x}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y^2 x + y^2 - e^x}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{e^x y^2 (x+1)}{2} \\ S_y &= y\, e^x x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{2x}}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^{2R}}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{2R}}{4} + c_1 \tag{4}$$

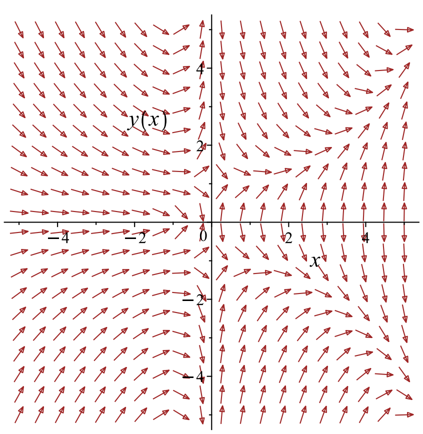
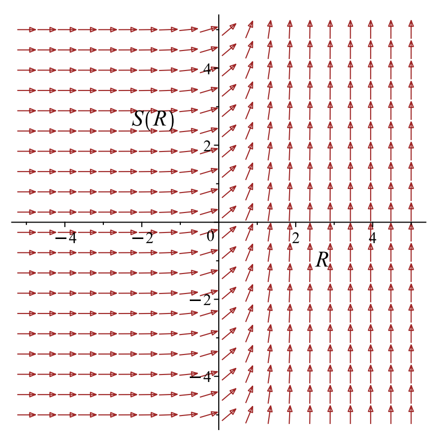
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2 e^x x}{2} = \frac{e^{2x}}{4} + c_1$$

Which simplifies to

$$\frac{y^2 e^x x}{2} = \frac{e^{2x}}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2x+y^2-e^x}{2xy}$ 	$R = x$ $S = \frac{y^2 e^x x}{2}$	$\frac{dS}{dR} = \frac{e^{2R}}{2}$ 

Summary

The solution(s) found are the following

$$\frac{y^2 e^x x}{2} = \frac{e^{2x}}{4} + c_1 \tag{1}$$

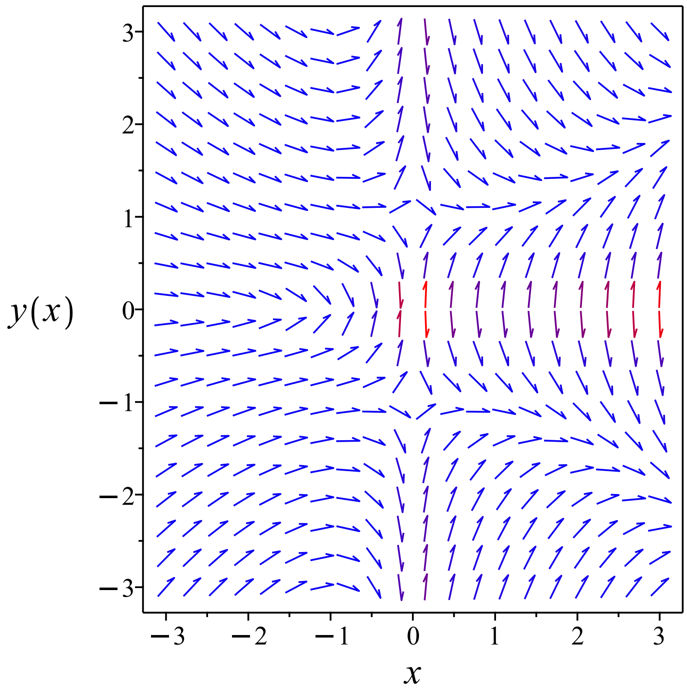


Figure 175: Slope field plot

Verification of solutions

$$\frac{y^2 \mathrm{e}^x x}{2} = \frac{\mathrm{e}^{2x}}{4} + c_1$$

Verified OK.

6.1.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y^2 x + y^2 - \mathrm{e}^x}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{x+1}{2x} y + \frac{\mathrm{e}^x}{2x} \frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x) y + f_1(x) y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x) y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{x+1}{2x} \\ f_1(x) &= \frac{e^x}{2x} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{(x+1)\,y^2}{2x} + \frac{e^x}{2x} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= -\frac{(x+1)\,w(x)}{2x} + \frac{e^x}{2x} \\ w' &= -\frac{(x+1)\,w}{x} + \frac{e^x}{x} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{-1-x}{x} \\ q(x) &= \frac{e^x}{x} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{(-1-x)\, w(x)}{x} = \frac{e^x}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{-1-x}{x} dx} \\ &= e^{x+\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = e^x x$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{e^x}{x}\right) \\ \frac{d}{dx}(e^x x w) &= (e^x x)\left(\frac{e^x}{x}\right) \\ d(e^x x w) &= e^{2x}\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^x x w &= \int e^{2x}\, dx \\ e^x x w &= \frac{e^{2x}}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x x$ results in

$$w(x) = \frac{e^{-x} e^{2x}}{2x} + \frac{c_1 e^{-x}}{x}$$

which simplifies to

$$w(x) = \frac{2c_1 e^{-x} + e^x}{2x}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{2c_1 e^{-x} + e^x}{2x}$$

Solving for y gives

$$y(x) = \frac{\sqrt{2}\,\sqrt{\mathrm{e}^{x} x\left(\mathrm{e}^{2x}+2c_1\right)}\,\mathrm{e}^{-x}}{2x}$$
$$y(x) = -\frac{\sqrt{2}\,\sqrt{\mathrm{e}^{x} x\left(\mathrm{e}^{2x}+2c_1\right)}\,\mathrm{e}^{-x}}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\,\sqrt{\mathrm{e}^{x} x\left(\mathrm{e}^{2x}+2c_1\right)}\,\mathrm{e}^{-x}}{2x} \tag{1}$$
$$y = -\frac{\sqrt{2}\,\sqrt{\mathrm{e}^{x} x\left(\mathrm{e}^{2x}+2c_1\right)}\,\mathrm{e}^{-x}}{2x} \tag{2}$$

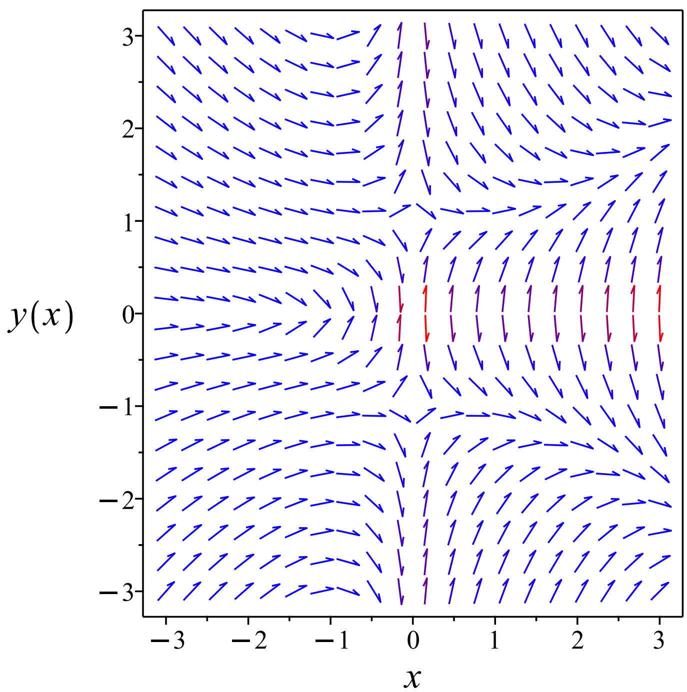


Figure 176: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{2}\,\sqrt{\mathrm{e}^{x} x\left(\mathrm{e}^{2x}+2c_1\right)}\,\mathrm{e}^{-x}}{2x}$$

Verified OK.

$$y = -\frac{\sqrt{2}\,\sqrt{\mathrm{e}^{x} x\left(\mathrm{e}^{2x}+2c_1\right)}\,\mathrm{e}^{-x}}{2x}$$

Verified OK.

6.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
(2xy)\,\mathrm{d}y &= \left(-y^2(x+1)+\mathrm{e}^x\right)\mathrm{d}x \\
\left(-\mathrm{e}^x + y^2(x+1)\right)\mathrm{d}x + (2xy)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\mathrm{e}^x + y^2(x+1) \\
N(x,y) &= 2xy
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\mathrm{e}^x + y^2(x+1)\right) \\
&= 2(x+1)\,y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy) \\
&= 2y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{2yx}\left(\left(2(x+1)\,y\right) - (2y)\right) \\
&= 1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A\,\mathrm{d}x} \\
&= e^{\int 1\,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x}\left(-\mathrm{e}^{x} + y^{2}(x+1)\right) \\
&= \left(-\mathrm{e}^{x} + y^{2}(x+1)\right)\mathrm{e}^{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x}(2xy) \\
&= 2y\,\mathrm{e}^{x} x
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(-\mathrm{e}^{x} + y^{2}(x+1)\right)\mathrm{e}^{x}\right) + \left(2y\,\mathrm{e}^{x} x\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \left(-\mathrm{e}^{x} + y^{2}(x+1)\right)\mathrm{e}^{x}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = y^{2}\mathrm{e}^{x} x - \frac{\mathrm{e}^{2x}}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2y\,\mathrm{e}^{x}x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y\,\mathrm{e}^{x}x$. Therefore equation (4) becomes

$$2y\,\mathrm{e}^{x}x = 2y\,\mathrm{e}^{x}x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y^2\mathrm{e}^{x}x - \frac{\mathrm{e}^{2x}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^2\mathrm{e}^{x}x - \frac{\mathrm{e}^{2x}}{2}$$

Summary

The solution(s) found are the following

$$y^2\mathrm{e}^{x}x - \frac{\mathrm{e}^{2x}}{2} = c_1 \tag{1}$$

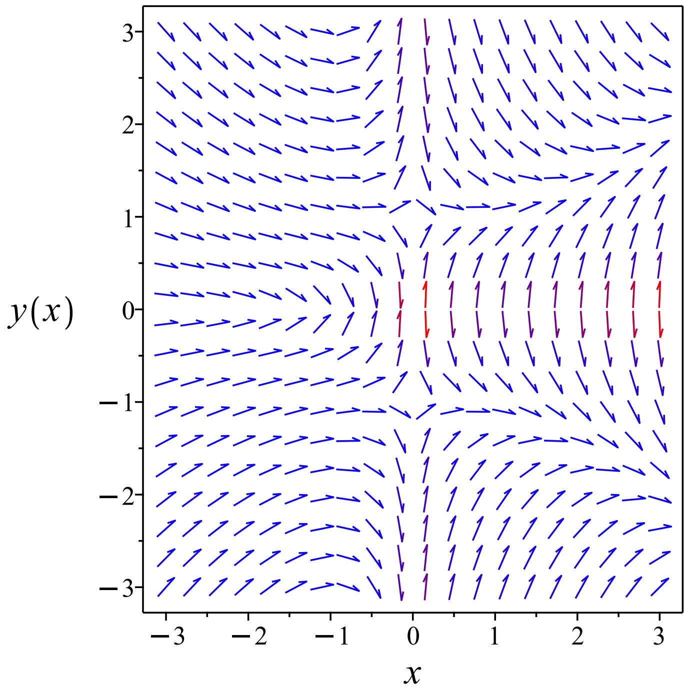


Figure 177: Slope field plot

Verification of solutions

$$y^2 e^x x - \frac{e^{2x}}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(2*x*y(x)*diff(y(x),x)+(1+x)*y(x)^2=exp(x),y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{2}\,\sqrt{x\,\mathrm{e}^{x}\left(\mathrm{e}^{2x}+2c_1\right)}\,\mathrm{e}^{-x}}{2x}$$
$$y(x) = \frac{\sqrt{2}\,\sqrt{x\,\mathrm{e}^{x}\left(\mathrm{e}^{2x}+2c_1\right)}\,\mathrm{e}^{-x}}{2x}$$

✓ Solution by Mathematica

Time used: 7.324 (sec). Leaf size: 66

```
DSolve[2*x*y[x]*y'[x]+(1+x)*y[x]^2==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{e^x + 2c_1e^{-x}}}{\sqrt{2}\sqrt{x}}$$
$$y(x) \to \frac{\sqrt{e^x + 2c_1e^{-x}}}{\sqrt{2}\sqrt{x}}$$

6.2 problem Exercise 12.2, page 103

Internal problem ID [4523]
Internal file name [OUTPUT/4016_Sunday_June_05_2022_12_08_50_PM_42057819/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.2, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{\cos(y)\, y' + \sin(y) = x^2}$$

6.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(y))\,\mathrm{d}y &= \left(-\sin(y) + x^2\right)\mathrm{d}x\\ \left(\sin(y) - x^2\right)\mathrm{d}x + (\cos(y))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \sin(y) - x^2\\ N(x,y) &= \cos(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\sin(y) - x^2\right)\\ &= \cos(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(y))\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \sec(y)\,((\cos(y)) - (0))\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x} \\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{x}\left(\sin\left(y\right) - x^{2}\right) \\ &= \left(\sin\left(y\right) - x^{2}\right)\mathrm{e}^{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{x}(\cos\left(y\right)) \\ &= \mathrm{e}^{x}\cos\left(y\right)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\left(\sin\left(y\right) - x^{2}\right)\mathrm{e}^{x}\right) + \left(\mathrm{e}^{x}\cos\left(y\right)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \left(\sin\left(y\right) - x^{2}\right)\mathrm{e}^{x} \, \mathrm{d}x\end{aligned}$$

$$\phi = \left(-x^{2} + 2x + \sin\left(y\right) - 2\right)\mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^x \cos(y) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^x \cos(y)$. Therefore equation (4) becomes

$$e^x \cos(y) = e^x \cos(y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \left(-x^2 + 2x + \sin(y) - 2\right) e^x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \left(-x^2 + 2x + \sin(y) - 2\right) e^x$$

<u>Summary</u>
The solution(s) found are the following

$$\left(-x^2 + 2x + \sin(y) - 2\right) e^x = c_1 \tag{1}$$

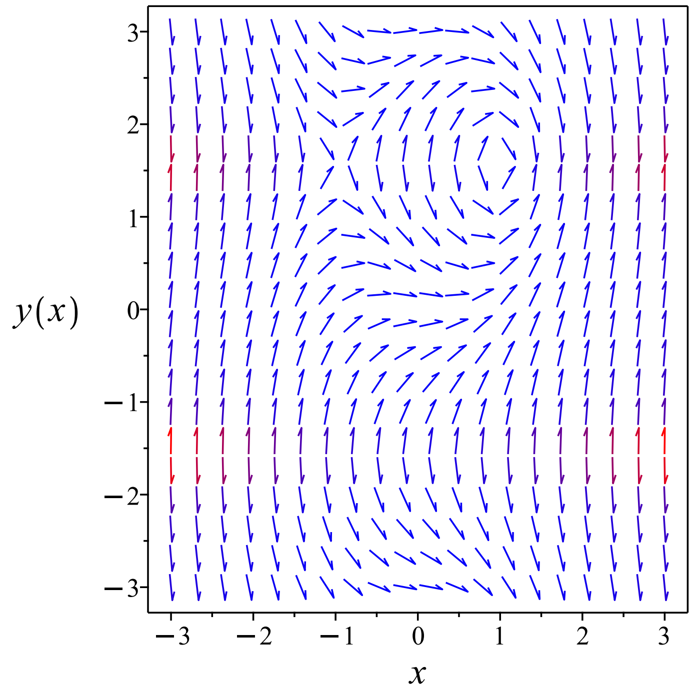


Figure 178: Slope field plot

Verification of solutions

$$\left(-x^2+2x+\sin\left(y\right)-2\right)\mathrm{e}^{x}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(cos(y(x))*diff(y(x),x)+sin(y(x))=x^2,y(x), singsol=all)
```

$$y(x) = -\arcsin\left(-x^2 + 2x - 2 + \mathrm{e}^{-x} c_1\right)$$

✓ Solution by Mathematica

Time used: 14.047 (sec). Leaf size: 23

```
DSolve[Cos[y[x]]*y'[x]+Sin[y[x]]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin\left(x^2 - 2x - 2c_1 e^{-x} + 2\right)$$

6.3 problem Exercise 12.3, page 103

Internal problem ID [4524]
Internal file name [`OUTPUT/4017_Sunday_June_05_2022_12_08_59_PM_84558491/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.3, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries]]`

$$\boxed{(x+1)\,y'-y-(x+1)\,\sqrt{1+y}=1}$$

6.3.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{\sqrt{1+y}\,x+\sqrt{1+y}+y+1}{x+1} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{\left(\sqrt{1+y}\,x + \sqrt{1+y} + y + 1\right)(b_3 - a_2)}{x+1} \\ &- \frac{\left(\sqrt{1+y}\,x + \sqrt{1+y} + y + 1\right)^2 a_3}{(x+1)^2} \\ &- \left(\frac{\sqrt{1+y}}{x+1} - \frac{\sqrt{1+y}\,x + \sqrt{1+y} + y + 1}{(x+1)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \frac{\left(\frac{x}{2\sqrt{1+y}} + \frac{1}{2\sqrt{1+y}} + 1\right)(xb_2 + yb_3 + b_1)}{x+1} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{8xya_3 + 4y^2a_3 + 2a_2y - x^2yb_3 - 2xyb_3 + 4xa_3 + 2a_2 + 4a_3 + b_1 - 2b_3 + 2\sqrt{1+y}\,xb_1 - 2\sqrt{1+y}\,xb_2 - }{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-8xya_3 - 4y^2a_3 - 2a_2y + x^2yb_3 + 2xyb_3 - 4xa_3 - 2a_2 - 4a_3 - b_1 \\ &+ 2b_3 - 2\sqrt{1+y}\,xb_1 + 2\sqrt{1+y}\,xb_2 + 2\sqrt{1+y}\,xb_3 + 2\sqrt{1+y}\,ya_1 \\ &- 2\sqrt{1+y}\,ya_2 - 2\sqrt{1+y}\,ya_3 + 4xb_3 - 4x\,y^2a_3 - 2x^2a_2 - 2x^2a_2y \\ &+ 2x^2b_3 - 4xa_2y - 2(1+y)^{\frac{3}{2}}\,x^2a_3 - 4(1+y)^{\frac{3}{2}}\,xa_3 - x^3b_2 - x^2b_1 - 2x^2b_2 \\ &- 2xb_1 - 2(1+y)^{\frac{3}{2}}\,a_3 + 2\sqrt{1+y}\,a_1 - 2\sqrt{1+y}\,a_2 - 2a_3\sqrt{1+y} \\ &- 2\sqrt{1+y}\,b_1 + 2b_2\sqrt{1+y} + 2\sqrt{1+y}\,b_3 - 4xa_2 - 8ya_3 - xb_2 + yb_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-4(1+y)\,xya_3 - x^2yb_3 - 2xyb_3 - b_1 - 2\sqrt{1+y}\,xb_1 + 2\sqrt{1+y}\,xb_2 \\ &+ 2\sqrt{1+y}\,xb_3 + 2\sqrt{1+y}\,ya_1 - 2\sqrt{1+y}\,ya_2 - 2\sqrt{1+y}\,ya_3 \\ &- 2(1+y)^{\frac{3}{2}}\,x^2a_3 - 4(1+y)^{\frac{3}{2}}\,xa_3 - 2(1+y)\,x^2a_2 + 2(1+y)\,x^2b_3 \\ &- 4(1+y)\,xa_2 - 4(1+y)\,xa_3 + 4(1+y)\,xb_3 - 4(1+y)\,ya_3 \\ &- x^3b_2 - x^2b_1 - 2x^2b_2 - 2xb_1 - 2(1+y)^{\frac{3}{2}}\,a_3 - 2(1+y)\,a_2 \\ &- 4(1+y)\,a_3 + 2(1+y)\,b_3 + 2\sqrt{1+y}\,a_1 - 2\sqrt{1+y}\,a_2 - 2a_3\sqrt{1+y} \\ &- 2\sqrt{1+y}\,b_1 + 2b_2\sqrt{1+y} + 2\sqrt{1+y}\,b_3 - xb_2 - yb_3 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-2\sqrt{1+y}\,x^2 a_3 y - 4\sqrt{1+y}\,x a_3 y - 8xya_3 - 4y^2 a_3 - 2a_2 y + x^2 y b_3 \\
&+ 2xyb_3 - 4xa_3 - 2a_2 - 4a_3 - b_1 + 2b_3 - 2\sqrt{1+y}\,x b_1 + 2\sqrt{1+y}\,x b_2 \\
&+ 2\sqrt{1+y}\,x b_3 + 2\sqrt{1+y}\,y a_1 - 2\sqrt{1+y}\,y a_2 - 4\sqrt{1+y}\,y a_3 + 4xb_3 \\
&- 4x\,y^2 a_3 - 2x^2 a_2 - 2x^2 a_2 y + 2x^2 b_3 - 4xa_2 y - x^3 b_2 - x^2 b_1 - 2x^2 b_2 - 2xb_1 \\
&- 4\sqrt{1+y}\,x a_3 + 2\sqrt{1+y}\,a_1 - 2\sqrt{1+y}\,a_2 - 4a_3\sqrt{1+y} - 2\sqrt{1+y}\,b_1 \\
&+ 2b_2\sqrt{1+y} + 2\sqrt{1+y}\,b_3 - 2\sqrt{1+y}\,x^2 a_3 - 4xa_2 - 8ya_3 - xb_2 + yb_3 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{1+y}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{1+y} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2v_3 v_1^2 a_3 v_2 - 2v_1^2 a_2 v_2 - 2v_3 v_1^2 a_3 - 4v_1 v_2^2 a_3 - 4v_3 v_1 a_3 v_2 - v_1^3 b_2 + v_1^2 v_2 b_3 \\
&+ 2v_3 v_2 a_1 - 2v_1^2 a_2 - 4v_1 a_2 v_2 - 2v_3 v_2 a_2 - 8v_1 v_2 a_3 - 4v_3 v_1 a_3 - 4v_2^2 a_3 \\
&- 4v_3 v_2 a_3 - v_1^2 b_1 - 2v_3 v_1 b_1 - 2v_1^2 b_2 + 2v_3 v_1 b_2 + 2v_1^2 b_3 + 2v_1 v_2 b_3 + 2v_3 v_1 b_3 \\
&+ 2v_3 a_1 - 4v_1 a_2 - 2a_2 v_2 - 2v_3 a_2 - 4v_1 a_3 - 8v_2 a_3 - 4a_3 v_3 - 2v_1 b_1 - 2v_3 b_1 \\
&- v_1 b_2 + 2b_2 v_3 + 4v_1 b_3 + v_2 b_3 + 2v_3 b_3 - 2a_2 - 4a_3 - b_1 + 2b_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-v_1^3 b_2 - 2v_3 v_1^2 a_3 v_2 + (-2a_2 + b_3)\,v_1^2 v_2 - 2v_3 v_1^2 a_3 + (-2a_2 - b_1 - 2b_2 + 2b_3)\,v_1^2 \\
&- 4v_1 v_2^2 a_3 - 4v_3 v_1 a_3 v_2 + (-4a_2 - 8a_3 + 2b_3)\,v_1 v_2 \\
&+ (-4a_3 - 2b_1 + 2b_2 + 2b_3)\,v_1 v_3 + (-4a_2 - 4a_3 - 2b_1 - b_2 + 4b_3)\,v_1 \\
&- 4v_2^2 a_3 + (2a_1 - 2a_2 - 4a_3)\,v_2 v_3 + (-2a_2 - 8a_3 + b_3)\,v_2 \\
&+ (2a_1 - 2a_2 - 4a_3 - 2b_1 + 2b_2 + 2b_3)\,v_3 - 2a_2 - 4a_3 - b_1 + 2b_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-4a_3 &= 0 \\
-2a_3 &= 0 \\
-b_2 &= 0 \\
-2a_2 + b_3 &= 0 \\
2a_1 - 2a_2 - 4a_3 &= 0 \\
-4a_2 - 8a_3 + 2b_3 &= 0 \\
-2a_2 - 8a_3 + b_3 &= 0 \\
-2a_2 - 4a_3 - b_1 + 2b_3 &= 0 \\
-2a_2 - b_1 - 2b_2 + 2b_3 &= 0 \\
-4a_3 - 2b_1 + 2b_2 + 2b_3 &= 0 \\
-4a_2 - 4a_3 - 2b_1 - b_2 + 4b_3 &= 0 \\
2a_1 - 2a_2 - 4a_3 - 2b_1 + 2b_2 + 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_2 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 2a_2 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x + 1 \\
\eta &= 2y + 2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= 2y + 2 - \left(\frac{\sqrt{1+y}\, x + \sqrt{1+y} + y + 1}{x+1}\right)(x+1) \\
&= -\sqrt{1+y}\, x - \sqrt{1+y} + y + 1 \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\sqrt{1+y}\, x - \sqrt{1+y} + y + 1} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(-x^2 - 2x + y)}{x^2 + 2x + 1} + \frac{x(x+2)\ln(-x^2 - 2x + y)}{x^2 + 2x + 1} + \frac{2\ln\left(\sqrt{1+y} - 1 - x\right)}{2 + 2x} - \frac{2\ln\left(\sqrt{1+y} + x + 1\right)}{2 + 2x} +$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{\sqrt{1+y}\, x + \sqrt{1+y} + y + 1}{x + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2}{\sqrt{1+y} - 1 - x} \\ S_y &= \frac{1}{\sqrt{1+y}\,\left(\sqrt{1+y} - 1 - x\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{(x+1)\sqrt{1+y}-1-y}{\sqrt{1+y}\,\left(-\sqrt{1+y}+1+x\right)(x+1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R+1) + c_1 \tag{4}$$

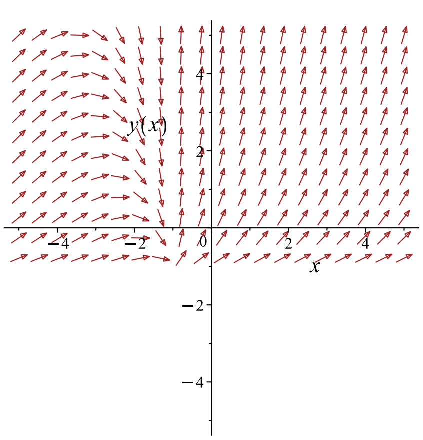
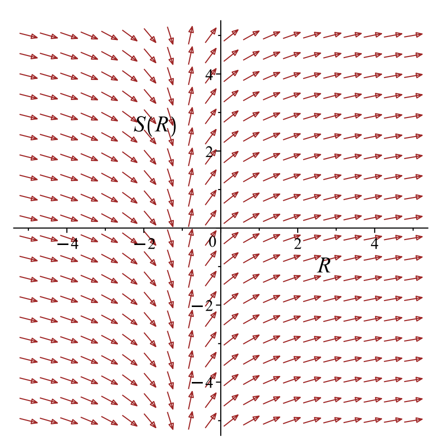
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(-x^2-2x+y\right) + \ln\left(\sqrt{1+y}-1-x\right) - \ln\left(\sqrt{1+y}+x+1\right) = \ln(x+1) + c_1$$

Which simplifies to

$$\ln\left(-x^2-2x+y\right) + \ln\left(\sqrt{1+y}-1-x\right) - \ln\left(\sqrt{1+y}+x+1\right) = \ln(x+1) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sqrt{1+y}\,x+\sqrt{1+y}+y+1}{x+1}$ 	$R = x$ $S = \ln\left(-x^2-2x+y\right) +$ $+ y + x$	$\frac{dS}{dR} = \frac{1}{R+1}$ 

Summary

The solution(s) found are the following

$$\ln \left(-x^{2}-2x+y\right)+\ln \left(\sqrt{1+y}-1-x\right)-\ln \left(\sqrt{1+y}+x+1\right)=\ln \left(x+1\right)+c_{1} \tag{1}$$

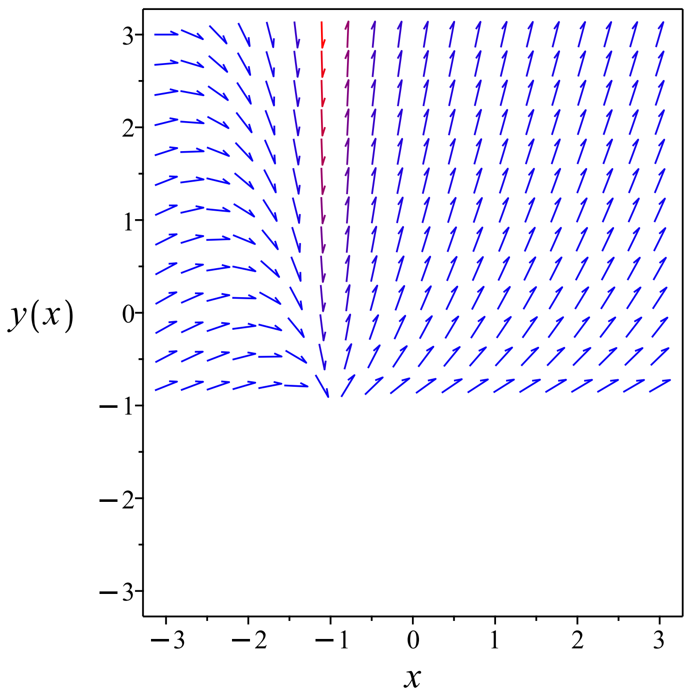


Figure 179: Slope field plot

Verification of solutions

$$\ln \left(-x^{2}-2x+y\right)+\ln \left(\sqrt{1+y}-1-x\right)-\ln \left(\sqrt{1+y}+x+1\right)=\ln \left(x+1\right)+c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = (2*y(x)+2)/(x+1), y(x)`      *** Sublev
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 81

```
dsolve((x+1)*diff(y(x),x)-(y(x)+1)=(x+1)*sqrt(y(x)+1),y(x), singsol=all)
```

$$\frac{(-c_1y(x)+1+c_1x^2+(2c_1+1)\,x)\,\sqrt{y\,(x)+1}-(1+x)\,(-c_1y(x)-1+c_1x^2+(2c_1-1)\,x)}{(x^2+2x-y\,(x))\left(-\sqrt{y\,(x)+1}+1+x\right)} = 0$$

✓ Solution by Mathematica

Time used: 0.244 (sec). Leaf size: 60

```
DSolve[(x+1)*y'[x]-(y[x]+1)==(x+1)*Sqrt[y[x]+1],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{2\sqrt{y(x)+1}\arctan\left(\frac{x+1}{\sqrt{-y(x)-1}}\right)}{\sqrt{-y(x)-1}}+\log\left(y(x)-(x+1)^2+1\right)-\log(x+1)=c_1,y(x)\right]$$

6.4 problem Exercise 12.4, page 103

Internal problem ID [4525]
Internal file name [OUTPUT/4018_Sunday_June_05_2022_12_09_06_PM_27291563/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.4, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{e^{y}(1+y')=e^{x}}$$

6.4.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -(e^{y}-e^{x})\,e^{-y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - (\mathrm{e}^y - \mathrm{e}^x)\,\mathrm{e}^{-y}(b_3 - a_2) - (\mathrm{e}^y - \mathrm{e}^x)^2\,\mathrm{e}^{-2y} a_3 \\ & \quad - \mathrm{e}^{-y}\mathrm{e}^x(x a_2 + y a_3 + a_1) - \left(-1 + (\mathrm{e}^y - \mathrm{e}^x)\,\mathrm{e}^{-y}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} & \left(-\mathrm{e}^y\mathrm{e}^x x a_2 + \mathrm{e}^y\mathrm{e}^x x b_2 - \mathrm{e}^y\mathrm{e}^x y a_3 + \mathrm{e}^y\mathrm{e}^x y b_3 + \mathrm{e}^{2y} a_2 - \mathrm{e}^{2y} a_3 + b_2\mathrm{e}^{2y} - \mathrm{e}^{2y} b_3 \right. \\ & \quad \left. - \mathrm{e}^y\mathrm{e}^x a_1 - \mathrm{e}^y\mathrm{e}^x a_2 + 2\,\mathrm{e}^y\mathrm{e}^x a_3 + \mathrm{e}^y\mathrm{e}^x b_1 + \mathrm{e}^y\mathrm{e}^x b_3 - \mathrm{e}^{2x} a_3\right)\mathrm{e}^{-2y} = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} & -\mathrm{e}^y\mathrm{e}^x x a_2 + \mathrm{e}^y\mathrm{e}^x x b_2 - \mathrm{e}^y\mathrm{e}^x y a_3 + \mathrm{e}^y\mathrm{e}^x y b_3 + \mathrm{e}^{2y} a_2 - \mathrm{e}^{2y} a_3 + b_2\mathrm{e}^{2y} \\ & \quad - \mathrm{e}^{2y} b_3 - \mathrm{e}^y\mathrm{e}^x a_1 - \mathrm{e}^y\mathrm{e}^x a_2 + 2\,\mathrm{e}^y\mathrm{e}^x a_3 + \mathrm{e}^y\mathrm{e}^x b_1 + \mathrm{e}^y\mathrm{e}^x b_3 - \mathrm{e}^{2x} a_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} & -x a_2\mathrm{e}^{x+y} + x b_2\mathrm{e}^{x+y} - y a_3\mathrm{e}^{x+y} + y b_3\mathrm{e}^{x+y} + \mathrm{e}^{2y} a_2 - \mathrm{e}^{2y} a_3 + b_2\mathrm{e}^{2y} \\ & \quad - \mathrm{e}^{2y} b_3 - a_1\mathrm{e}^{x+y} - a_2\mathrm{e}^{x+y} + 2a_3\mathrm{e}^{x+y} + b_1\mathrm{e}^{x+y} + b_3\mathrm{e}^{x+y} - \mathrm{e}^{2x} a_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \mathrm{e}^{2x}, \mathrm{e}^{2y}, \mathrm{e}^{x+y}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \mathrm{e}^{2x} = v_3, \mathrm{e}^{2y} = v_4, \mathrm{e}^{x+y} = v_5\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -v_1 a_2 v_5 - v_2 a_3 v_5 + v_1 b_2 v_5 + v_2 b_3 v_5 - a_1 v_5 + v_4 a_2 - a_2 v_5 \\ & \quad - v_3 a_3 - v_4 a_3 + 2a_3 v_5 + b_1 v_5 + b_2 v_4 - v_4 b_3 + b_3 v_5 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned}&(-a_2 + b_2)\, v_1 v_5 + (-a_3 + b_3)\, v_2 v_5 - v_3 a_3 \\ &+ (a_2 - a_3 + b_2 - b_3)\, v_4 + (-a_1 - a_2 + 2a_3 + b_1 + b_3)\, v_5 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-a_3 &= 0\\ -a_2 + b_2 &= 0\\ -a_3 + b_3 &= 0\\ a_2 - a_3 + b_2 - b_3 &= 0\\ -a_1 - a_2 + 2a_3 + b_1 + b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= b_1\\ a_2 &= 0\\ a_3 &= 0\\ b_1 &= b_1\\ b_2 &= 0\\ b_3 &= 0\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= 1\\ \eta &= 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi\\ &= 1 - \left(-(\mathrm{e}^{y} - \mathrm{e}^{x})\, \mathrm{e}^{-y}\right) (1)\\ &= (2\, \mathrm{e}^{y} - \mathrm{e}^{x})\, \mathrm{e}^{-y}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{(2\,e^{y} - e^{x})\,e^{-y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(2\,e^{y} - e^{x})}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -(e^{y} - e^{x})\,e^{-y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{e^{x}}{-4\,e^{y} + 2\,e^{x}} \\ S_y &= -\frac{e^{y}}{-2\,e^{y} + e^{x}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(2\,\mathrm{e}^{y}-\mathrm{e}^{x}\right)}{2} = -\frac{x}{2} + c_1$$

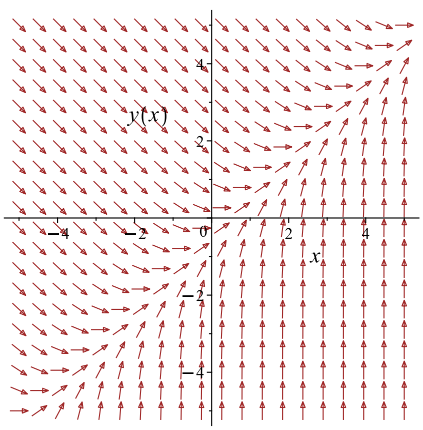
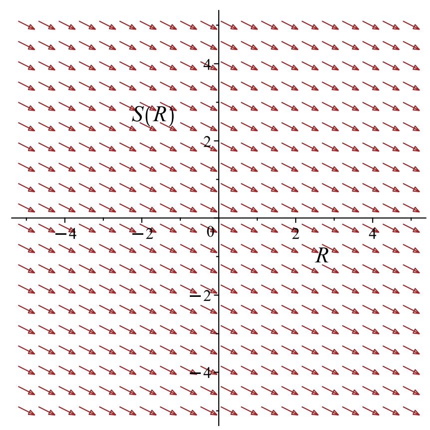
Which simplifies to

$$\frac{\ln\left(2\,\mathrm{e}^{y}-\mathrm{e}^{x}\right)}{2} = -\frac{x}{2} + c_1$$

Which gives

$$y = \ln\left(\frac{\mathrm{e}^{2c_1-x}}{2} + \frac{\mathrm{e}^{x}}{2}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -(\mathrm{e}^{y} - \mathrm{e}^{x})\,\mathrm{e}^{-y}$ 	$R = x$ $S = \frac{\ln(2\,\mathrm{e}^{y} - \mathrm{e}^{x})}{2}$	$\frac{dS}{dR} = -\frac{1}{2}$ 

Summary

The solution(s) found are the following

$$y = \ln\left(\frac{\mathrm{e}^{2c_1 - x}}{2} + \frac{\mathrm{e}^{x}}{2}\right) \tag{1}$$

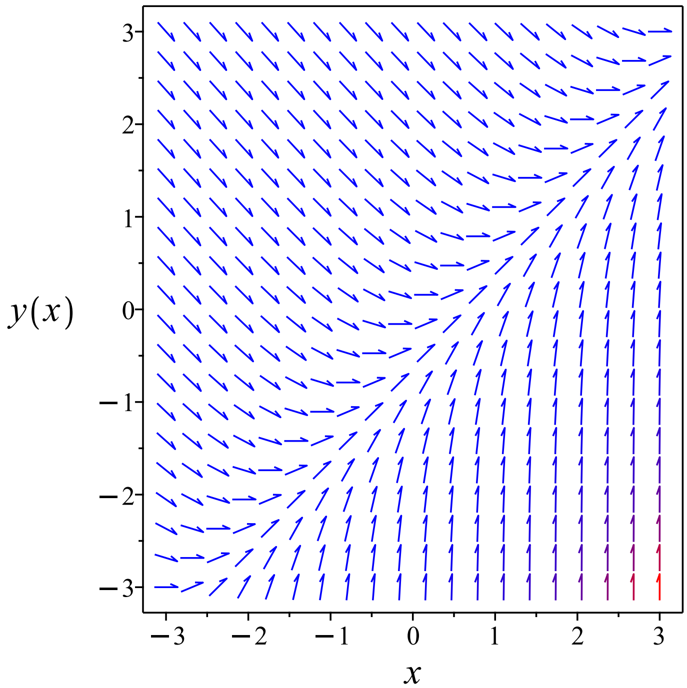


Figure 180: Slope field plot

<u>Verification of solutions</u>

$$y = \ln\left(\frac{e^{2c_1 - x}}{2} + \frac{e^x}{2}\right)$$

Verified OK.

6.4.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(\mathrm{e}^y)\,\mathrm{d}y &= (-\mathrm{e}^y + \mathrm{e}^x)\,\mathrm{d}x \\
(\mathrm{e}^y - \mathrm{e}^x)\,\mathrm{d}x + (\mathrm{e}^y)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= \mathrm{e}^y - \mathrm{e}^x \\
N(x,y) &= \mathrm{e}^y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\mathrm{e}^y - \mathrm{e}^x) \\
&= \mathrm{e}^y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^y) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \mathrm{e}^{-y}((\mathrm{e}^{y}) - (0)) \\
&= 1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 1 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x}(\mathrm{e}^{y} - \mathrm{e}^{x}) \\
&= (\mathrm{e}^{y} - \mathrm{e}^{x})\,\mathrm{e}^{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x}(\mathrm{e}^{y}) \\
&= \mathrm{e}^{x+y}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
((\mathrm{e}^{y} - \mathrm{e}^{x})\,\mathrm{e}^{x}) + \left(\mathrm{e}^{x+y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int (\mathrm{e}^y - \mathrm{e}^x) \mathrm{e}^x \mathrm{d}x$$

$$\phi = \mathrm{e}^{x+y} - \frac{\mathrm{e}^{2x}}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x+y} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x+y}$. Therefore equation (4) becomes

$$\mathrm{e}^{x+y} = \mathrm{e}^{x+y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{x+y} - \frac{\mathrm{e}^{2x}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{x+y} - \frac{\mathrm{e}^{2x}}{2}$$

The solution becomes

$$y = -x + \ln\left(\frac{\mathrm{e}^{2x}}{2} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = -x + \ln\left(\frac{e^{2x}}{2} + c_1\right) \tag{1}$$

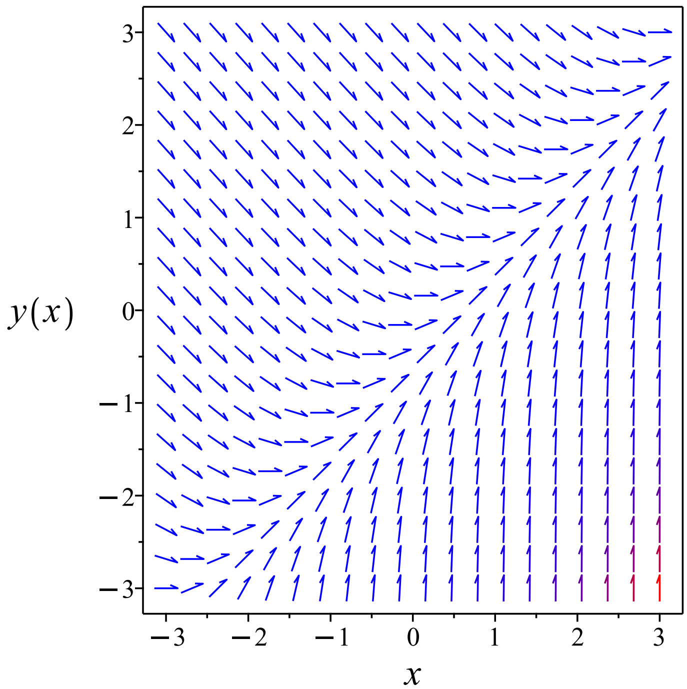


Figure 181: Slope field plot

Verification of solutions

$$y = -x + \ln\left(\frac{e^{2x}}{2} + c_1\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve(exp(y(x))*(diff(y(x),x)+1)=exp(x),y(x), singsol=all)
```

$$y(x) = x - \ln(2) + \ln\left(1 + e^{-2x} c_1\right)$$

✓ Solution by Mathematica

Time used: 1.32 (sec). Leaf size: 22

```
DSolve[Exp[y[x]]*(y'[x]+1)==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x + \log\left(\frac{e^{2x}}{2} + c_1\right)$$

6.5 problem Exercise 12.5, page 103

Internal problem ID [4526]
Internal file name [`OUTPUT/4019_Sunday_June_05_2022_12_09_15_PM_26704375/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.5, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' \sin(y) + \sin(x)\cos(y) = \sin(x)}$$

6.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\sin(x)\left(-\csc(y) + \cot(y)\right) \end{aligned}$$

Where $f(x) = -\sin(x)$ and $g(y) = -\csc(y) + \cot(y)$. Integrating both sides gives

$$\frac{1}{-\csc(y) + \cot(y)}\, dy = -\sin(x)\, dx$$

$$\int \frac{1}{-\csc(y) + \cot(y)}\, dy = \int -\sin(x)\, dx$$

$$-\ln(-1 + \cos(y)) = \cos(x) + c_1$$

Raising both side to exponential gives

$$\frac{1}{-1+\cos(y)} = \mathrm{e}^{\cos(x)+c_1}$$

Which simplifies to

$$\frac{1}{-1+\cos(y)} = c_2 \mathrm{e}^{\cos(x)}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \arccos\left(\frac{\left(c_2 \mathrm{e}^{\cos(x)+c_1}+1\right)\mathrm{e}^{-\cos(x)-c_1}}{c_2}\right) \tag{1}$$

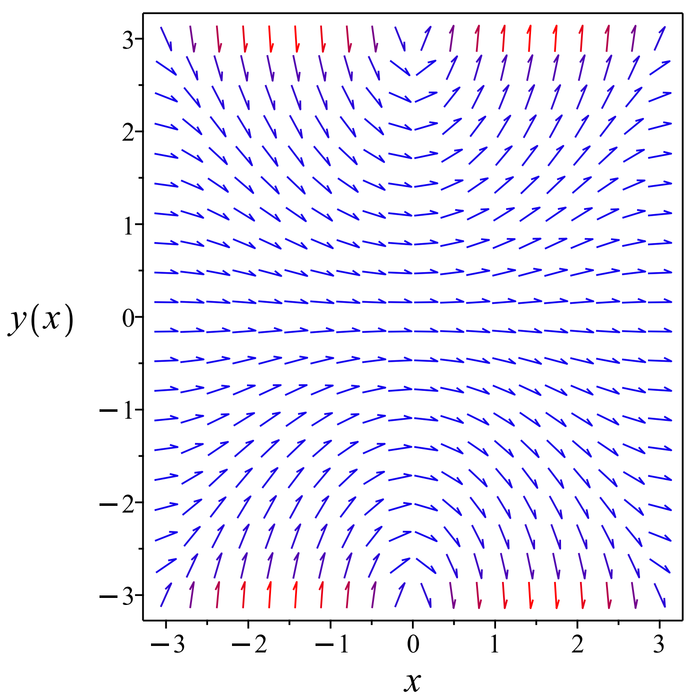


Figure 182: Slope field plot

<u>Verification of solutions</u>

$$y = \arccos\left(\frac{\left(c_2 \mathrm{e}^{\cos(x)+c_1}+1\right)\mathrm{e}^{-\cos(x)-c_1}}{c_2}\right)$$

Verified OK.

6.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\sin(x)(-1+\cos(y))}{\sin(y)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \qquad \text{(A)}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 98: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = -\frac{1}{\sin(x)}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{\sin(x)}} dx \end{aligned}$$

Which results in

$$S = \cos(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sin(x)(-1+\cos(y))}{\sin(y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\sin(x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{\sin(y)}{-1+\cos(y)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{\sin(R)}{-1+\cos(R)}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(-1+\cos(R)) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\cos(x) = -\ln(-1+\cos(y)) + c_1
$$

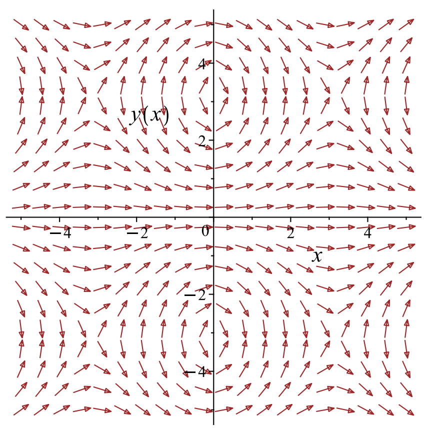
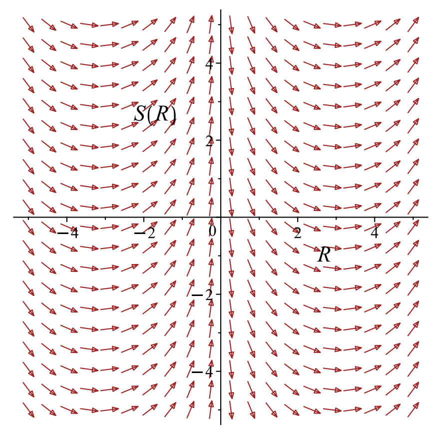
Which simplifies to

$$
\cos(x) = -\ln(-1+\cos(y)) + c_1
$$

Which gives

$$
y = \arccos\left(e^{-\cos(x)+c_1} + 1\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sin(x)(-1+\cos(y))}{\sin(y)}$ 	$R = y$ $S = \cos(x)$	$\frac{dS}{dR} = \frac{\sin(R)}{-1+\cos(R)}$ 

Summary

The solution(s) found are the following

$$y = \arccos\left(\mathrm{e}^{-\cos(x)+c_1} + 1\right) \tag{1}$$

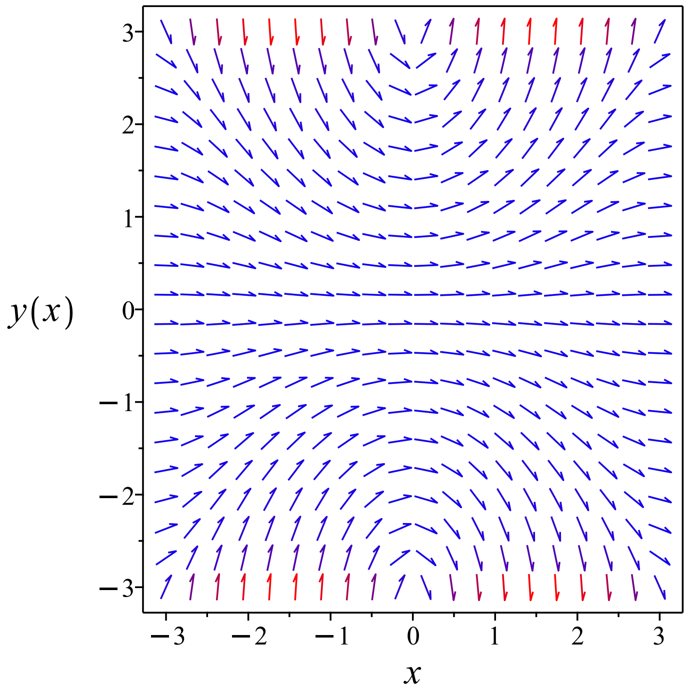


Figure 183: Slope field plot

Verification of solutions

$$y = \arccos\left(e^{-\cos(x)+c_1} + 1\right)$$

Verified OK.

6.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{\sin(y)}{-1 + \cos(y)}\right)\mathrm{d}y = (\sin(x))\,\mathrm{d}x$$
$$(-\sin(x))\,\mathrm{d}x + \left(-\frac{\sin(y)}{-1 + \cos(y)}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\sin(x)$$
$$N(x, y) = -\frac{\sin(y)}{-1 + \cos(y)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(x)) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\sin(y)}{-1+\cos(y)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x
$$

$$
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\sin(x) \, \mathrm{d}x
$$

$$
\phi = \cos(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\sin(y)}{-1+\cos(y)}$. Therefore equation (4) becomes

$$
-\frac{\sin(y)}{-1+\cos(y)} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{\sin(y)}{-1+\cos(y)}
$$

Integrating the above w.r.t y gives

$$
\int f'(y) \, \mathrm{d}y = \int \left(-\frac{\sin(y)}{-1+\cos(y)}\right) \mathrm{d}y
$$

$$
f(y) = \ln(-1+\cos(y)) + c_1
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \cos(x) + \ln(-1 + \cos(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(x) + \ln(-1 + \cos(y))$$

Summary
The solution(s) found are the following

$$\cos(x) + \ln(-1 + \cos(y)) = c_1 \tag{1}$$

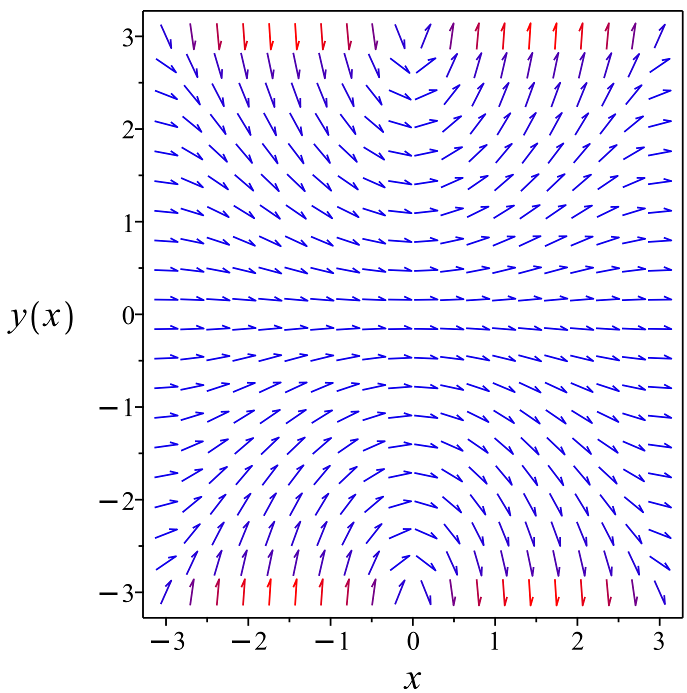


Figure 184: Slope field plot

Verification of solutions

$$\cos(x) + \ln(-1 + \cos(y)) = c_1$$

Verified OK.

6.5.4 Maple step by step solution

Let's solve

$y' \sin(y) + \sin(x) \cos(y) = \sin(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' \sin(y)}{-1+\cos(y)} = -\sin(x)$

- Integrate both sides with respect to x

  $\int \frac{y' \sin(y)}{-1+\cos(y)} dx = \int -\sin(x)\, dx + c_1$

- Evaluate integral

  $-\ln(-1+\cos(y)) = \cos(x) + c_1$

- Solve for y

  $y = \arccos\left(e^{-\cos(x)-c_1} + 1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)*sin(y(x))+sin(x)*cos(y(x))=sin(x),y(x), singsol=all)
```

$$y(x) = \arccos\left(e^{-\cos(x)} c_1 + 1\right)$$

✓ Solution by Mathematica

Time used: 0.792 (sec). Leaf size: 81

```
DSolve[y'[x]*Sin[y[x]]+Sin[x]*Cos[y[x]]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

$$\text{Solve}\left[2\cos(x)\tan\left(\frac{y(x)}{2}\right)e^{\operatorname{arctanh}(\cos(y(x)))} - \sqrt{\sin^2(y(x))}\csc\left(\frac{y(x)}{2}\right)\sec\left(\frac{y(x)}{2}\right)\left(\log\left(\sec^2\left(\frac{y(x)}{2}\right)\right) - 2\log\left(\tan\left(\frac{y(x)}{2}\right)\right)\right) = c_1, y(x)\right]$$

$$y(x) \to 0$$

6.6 problem Exercise 12.6, page 103

Internal problem ID [4527]
Internal file name [OUTPUT/4020_Sunday_June_05_2022_12_09_24_PM_67430551/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.6, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{(x-y)^2 y' = 4}$$

6.6.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{4}{x^2 - 2xy + y^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{4b_3 - 4a_2}{x^2 - 2xy + y^2} - \frac{16a_3}{(x^2 - 2xy + y^2)^2} \\ & + \frac{4(-2y + 2x)(xa_2 + ya_3 + a_1)}{(x^2 - 2xy + y^2)^2} + \frac{4(-2x + 2y)(xb_2 + yb_3 + b_1)}{(x^2 - 2xy + y^2)^2} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^4b_2 - 4x^3yb_2 + 6x^2y^2b_2 - 4x\,y^3b_2 + y^4b_2 + 4x^2a_2 - 8x^2b_2 + 4x^2b_3 + 8xya_3 + 8xyb_2 - 16xyb_3 - 4y^2a_2 -}{(x^2 - 2xy + y^2)^2}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & x^4b_2 - 4x^3yb_2 + 6x^2y^2b_2 - 4x\,y^3b_2 + y^4b_2 + 4x^2a_2 - 8x^2b_2 + 4x^2b_3 + 8xya_3 + 8xyb_2 \\ & - 16xyb_3 - 4y^2a_2 - 8y^2a_3 + 12y^2b_3 + 8xa_1 - 8xb_1 - 8ya_1 + 8yb_1 - 16a_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & b_2v_1^4 - 4b_2v_1^3v_2 + 6b_2v_1^2v_2^2 - 4b_2v_1v_2^3 + b_2v_2^4 + 4a_2v_1^2 - 4a_2v_2^2 \\ & + 8a_3v_1v_2 - 8a_3v_2^2 - 8b_2v_1^2 + 8b_2v_1v_2 + 4b_3v_1^2 - 16b_3v_1v_2 \\ & + 12b_3v_2^2 + 8a_1v_1 - 8a_1v_2 - 8b_1v_1 + 8b_1v_2 - 16a_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$b_2v_1^4-4b_2v_1^3v_2+6b_2v_1^2v_2^2+(4a_2-8b_2+4b_3)\,v_1^2-4b_2v_1v_2^3+(8a_3+8b_2-16b_3)\,v_1v_2 \\ +(8a_1-8b_1)\,v_1+b_2v_2^4+(-4a_2-8a_3+12b_3)\,v_2^2+(-8a_1+8b_1)\,v_2-16a_3=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ -16a_3 &= 0 \\ -4b_2 &= 0 \\ 6b_2 &= 0 \\ -8a_1+8b_1 &= 0 \\ 8a_1-8b_1 &= 0 \\ -4a_2-8a_3+12b_3 &= 0 \\ 4a_2-8b_2+4b_3 &= 0 \\ 8a_3+8b_2-16b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x,y)\,\xi \\ &= 1 - \left(\frac{4}{x^2-2xy+y^2}\right)(1) \\ &= \frac{x^2-2xy+y^2-4}{x^2-2xy+y^2} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2-2xy+y^2-4}{x^2-2xy+y^2}} dy \end{aligned}$$

Which results in

$$S = y - \ln(y - x + 2) + \ln(y - x - 2)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{4}{x^2 - 2xy + y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{4}{(x - y + 2)(x - y - 2)} \\ S_y &= \frac{(x - y)^2}{(x - y + 2)(x - y - 2)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

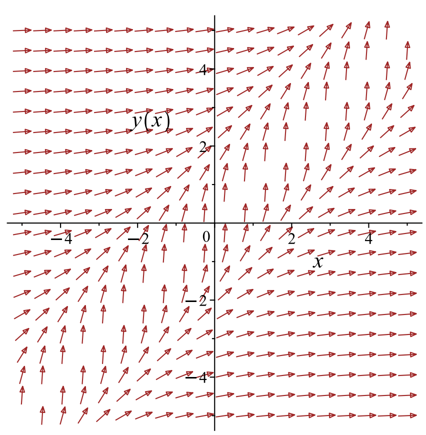
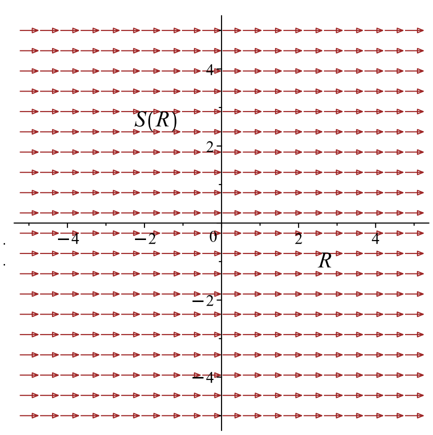
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y - \ln(y - x + 2) + \ln(y - x - 2) = c_1$$

Which simplifies to

$$y - \ln(y - x + 2) + \ln(y - x - 2) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{4}{x^2-2xy+y^2}$ 	$R = x$ $S = y - \ln(y - x + 2) +$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y - \ln(y - x + 2) + \ln(y - x - 2) = c_1 \tag{1}$$

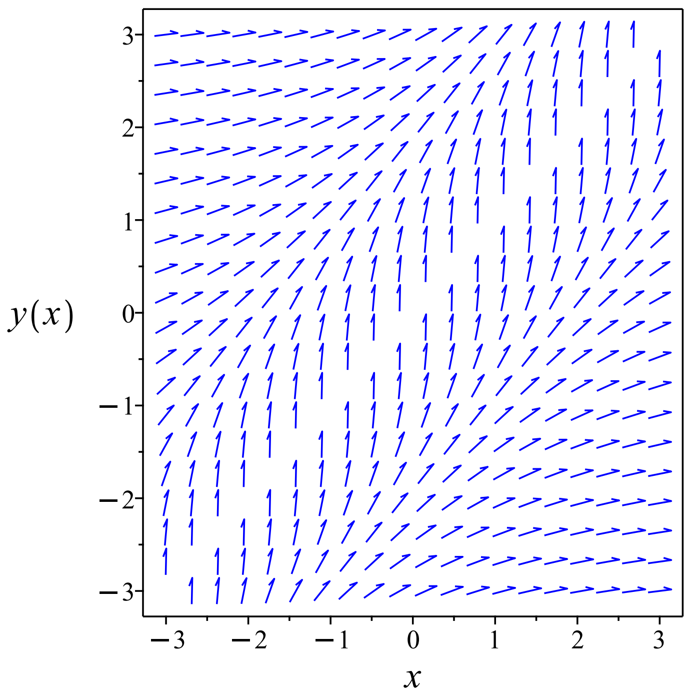


Figure 185: Slope field plot

Verification of solutions

$$y - \ln(y - x + 2) + \ln(y - x - 2) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 27

```
dsolve((x-y(x))^2*diff(y(x),x)=4,y(x), singsol=all)
```

$$y(x)+\ln(y(x)-x-2)-\ln(y(x)-x+2)-c_1=0$$

✓ Solution by Mathematica

Time used: 0.202 (sec). Leaf size: 36

```
DSolve[(x-y[x])^2*y'[x]==4,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[y(x)-4\left(\frac{1}{4}\log(y(x)-x+2)-\frac{1}{4}\log(-y(x)+x+2)\right)=c_1,y(x)\right]$$

6.7 problem Exercise 12.7, page 103

Internal problem ID [4528]
Internal file name [OUTPUT/4021_Sunday_June_05_2022_12_10_28_PM_83397190/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.7, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{-y + xy' - \sqrt{x^2 + y^2} = 0}$$

6.7.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y + \sqrt{x^2 + y^2}}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(y+\sqrt{x^2+y^2}\right)(b_3-a_2)}{x} - \frac{\left(y+\sqrt{x^2+y^2}\right)^2 a_3}{x^2} - \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{y+\sqrt{x^2+y^2}}{x^2}\right)(xa_2+ya_3+a_1) - \frac{\left(1+\frac{y}{\sqrt{x^2+y^2}}\right)(xb_2+yb_3+b_1)}{x} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(x^2+y^2)^{\frac{3}{2}} a_3 + x^3a_2 - x^3b_3 + 2x^2ya_3 + x^2yb_2 + y^3a_3 + \sqrt{x^2+y^2}\, xb_1 - \sqrt{x^2+y^2}\, ya_1 + xyb_1 - y^2a_1}{\sqrt{x^2+y^2}\, x^2} = 0$$

Setting the numerator to zero gives

$$-(x^2+y^2)^{\frac{3}{2}} a_3 - x^3a_2 + x^3b_3 - 2x^2ya_3 - x^2yb_2 - y^3a_3 - \sqrt{x^2+y^2}\, xb_1 + \sqrt{x^2+y^2}\, ya_1 - xyb_1 + y^2a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-(x^2+y^2)^{\frac{3}{2}} a_3 + (x^2+y^2)\, xb_3 - (x^2+y^2)\, ya_3 - x^3a_2 - x^2ya_3 - x^2yb_2 - x\,y^2b_3 + (x^2+y^2)\, a_1 - \sqrt{x^2+y^2}\, xb_1 + \sqrt{x^2+y^2}\, ya_1 - x^2a_1 - xyb_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-x^3a_2 + x^3b_3 - x^2\sqrt{x^2+y^2}\, a_3 - 2x^2ya_3 - x^2yb_2 - \sqrt{x^2+y^2}\, y^2a_3 - y^3a_3 - \sqrt{x^2+y^2}\, xb_1 - xyb_1 + \sqrt{x^2+y^2}\, ya_1 + y^2a_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2+y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2+y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$-v_1^3 a_2 - 2v_1^2 v_2 a_3 - v_1^2 v_3 a_3 - v_2^3 a_3 - v_3 v_2^2 a_3 - v_1^2 v_2 b_2 + v_1^3 b_3 + v_2^2 a_1 + v_3 v_2 a_1 - v_1 v_2 b_1 - v_3 v_1 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$(b_3 - a_2) v_1^3 + (-2a_3 - b_2) v_1^2 v_2 - v_1^2 v_3 a_3 - v_1 v_2 b_1 - v_3 v_1 b_1 - v_2^3 a_3 - v_3 v_2^2 a_3 + v_2^2 a_1 + v_3 v_2 a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -2a_3 - b_2 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y + \sqrt{x^2 + y^2}}{x}\right)(x) \\
&= -\sqrt{x^2 + y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{x^2 + y^2}} dy
\end{aligned}$$

Which results in

$$S = -\ln\left(y + \sqrt{x^2 + y^2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y + \sqrt{x^2+y^2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x}{\sqrt{x^2+y^2}\,\left(y+\sqrt{x^2+y^2}\right)} \\ S_y &= -\frac{1}{\sqrt{x^2+y^2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2\left(\sqrt{x^2+y^2}\,y + x^2 + y^2\right)}{x\sqrt{x^2+y^2}\,\left(y+\sqrt{x^2+y^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(y + \sqrt{x^2+y^2}\right) = -2\ln(x) + c_1$$

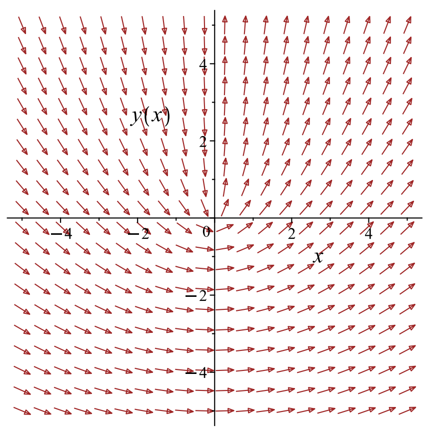
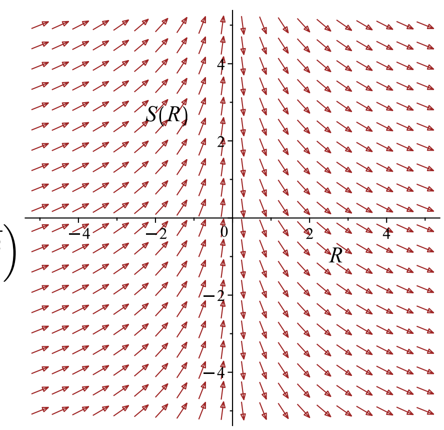
Which simplifies to

$$-\ln\left(y + \sqrt{x^2+y^2}\right) = -2\ln(x) + c_1$$

Which gives

$$y = -\frac{\mathrm{e}^{-c_1}\left(\mathrm{e}^{2c_1} - x^2\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$ 	$R = x$ $S = -\ln\left(y+\sqrt{x^2+y^2}\right)$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary
The solution(s) found are the following

$$y = -\frac{e^{-c_1}\left(e^{2c_1} - x^2\right)}{2} \tag{1}$$

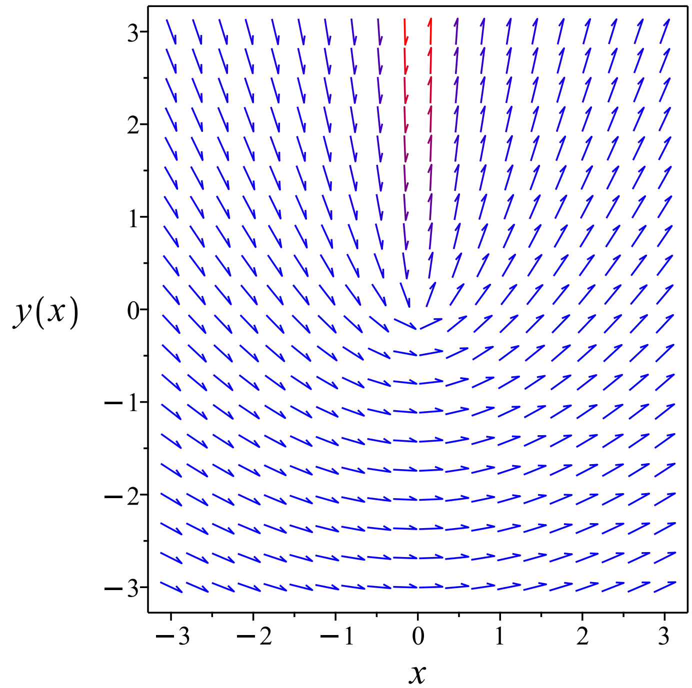


Figure 186: Slope field plot

Verification of solutions

$$y = -\frac{e^{-c_1}(e^{2c_1} - x^2)}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(x*diff(y(x),x)-y(x)=sqrt(x^2+y(x)^2),y(x), singsol=all)
```

$$\frac{-c_1 x^2 + \sqrt{x^2 + y(x)^2} + y(x)}{x^2} = 0$$

✓ Solution by Mathematica

Time used: 0.337 (sec). Leaf size: 27

```
DSolve[x*y'[x]-y[x]==Sqrt[x^2+y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-c_1} \left(-1 + e^{2c_1} x^2\right)$$

6.8 problem Exercise 12.8, page 103

Internal problem ID [4529]
Internal file name [`OUTPUT/4022_Sunday_June_05_2022_12_10_39_PM_27772936/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.8, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{(3x+2y+1)\,y'+3y=-4x-2}$$

6.8.1 Solving as differentialType ode

Writing the ode as

$$y'=\frac{-4x-3y-2}{3x+2y+1} \tag{1}$$

Which becomes

$$(1+2y)\,dy=(-3x)\,dy+(-4x-3y-2)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-3x)\,dy+(-4x-3y-2)\,dx=d\left(-2x^2-3xy-2x\right)$$

Hence (2) becomes

$$(1+2y)\,dy = d\left(-2x^2 - 3xy - 2x\right)$$

Integrating both sides gives gives these solutions

$$y = -\frac{3x}{2} - \frac{1}{2} + \frac{\sqrt{x^2 + 4c_1 - 2x + 1}}{2} + c_1$$
$$y = -\frac{3x}{2} - \frac{1}{2} - \frac{\sqrt{x^2 + 4c_1 - 2x + 1}}{2} + c_1$$

Summary

The solution(s) found are the following

$$y = -\frac{3x}{2} - \frac{1}{2} + \frac{\sqrt{x^2 + 4c_1 - 2x + 1}}{2} + c_1 \tag{1}$$
$$y = -\frac{3x}{2} - \frac{1}{2} - \frac{\sqrt{x^2 + 4c_1 - 2x + 1}}{2} + c_1 \tag{2}$$

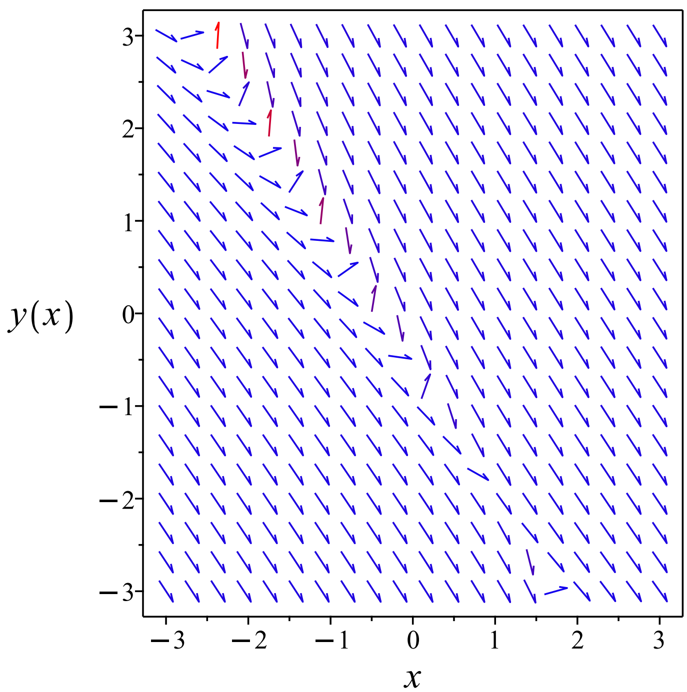


Figure 187: Slope field plot

Verification of solutions

$$
y=-\frac{3x}{2}-\frac{1}{2}+\frac{\sqrt{x^{2}+4c_{1}-2x+1}}{2}+c_{1}
$$

Verified OK.

$$
y=-\frac{3x}{2}-\frac{1}{2}-\frac{\sqrt{x^{2}+4c_{1}-2x+1}}{2}+c_{1}
$$

Verified OK.

6.8.2 Solving as homogeneousTypeMapleC ode

Let $Y=y+y_0$ and $X=x+x_0$ then the above is transformed to new ode in $Y(X)$

$$
\frac{d}{dX}Y(X)=-\frac{4X+4x_{0}+3Y(X)+3y_{0}+2}{3X+3x_{0}+2Y(X)+2y_{0}+1}
$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$
\begin{aligned}
x_{0}&=1\\
y_{0}&=-2
\end{aligned}
$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$
\frac{d}{dX}Y(X)=-\frac{4X+3Y(X)}{3X+2Y(X)}
$$

In canonical form, the ODE is

$$
\begin{aligned}
Y'&=F(X,Y)\\
&=-\frac{4X+3Y}{3X+2Y} \qquad (1)
\end{aligned}
$$

An ode of the form $Y'=\frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^{n}X,t^{n}Y)=t^{n}f(X,Y)
$$

In this case, it can be seen that both $M=-4X-3Y$ and $N=3X+2Y$ are both homogeneous and of the same order $n=1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}X}X + u = \frac{-3u - 4}{2u + 3}$$

$$\frac{\mathrm{d}u}{\mathrm{d}X} = \frac{\frac{-3u(X)-4}{2u(X)+3} - u(X)}{X}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-3u(X)-4}{2u(X)+3} - u(X)}{X} = 0$$

Or

$$2\left(\frac{d}{dX}u(X)\right)Xu(X) + 3\left(\frac{d}{dX}u(X)\right)X + 2u(X)^2 + 6u(X) + 4 = 0$$

Or

$$4 + X(2u(X) + 3)\left(\frac{d}{dX}u(X)\right) + 2u(X)^2 + 6u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{2(u^2 + 3u + 2)}{X\,(2u + 3)} \end{aligned}$$

Where $f(X) = -\frac{2}{X}$ and $g(u) = \frac{u^2+3u+2}{2u+3}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+3u+2}{2u+3}}\,du = -\frac{2}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+3u+2}{2u+3}}\,du = \int -\frac{2}{X}\,dX$$

$$\ln\left(u^2 + 3u + 2\right) = -2\ln(X) + c_2$$

Raising both side to exponential gives

$$u^2 + 3u + 2 = \mathrm{e}^{-2\ln(X)+c_2}$$

Which simplifies to

$$u^2 + 3u + 2 = \frac{c_3}{X^2}$$

Which simplifies to

$$u(X)^2 + 3u(X) + 2 = \frac{c_3 e^{c_2}}{X^2}$$

The solution is

$$u(X)^2 + 3u(X) + 2 = \frac{c_3 e^{c_2}}{X^2}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{Y(X)^2}{X^2} + \frac{3Y(X)}{X} + 2 = \frac{c_3 e^{c_2}}{X^2}$$

Which simplifies to

$$(X + Y(X))(2X + Y(X)) = c_3 e^{c_2}$$

Using the solution for $Y(X)$

$$(X + Y(X))(2X + Y(X)) = c_3 e^{c_2}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 2$$
$$X = x + 1$$

Then the solution in y becomes

$$(y + x + 1)(2x + y) = c_3 e^{c_2}$$

Summary
The solution(s) found are the following

$$(y + x + 1)(2x + y) = c_3 e^{c_2} \tag{1}$$

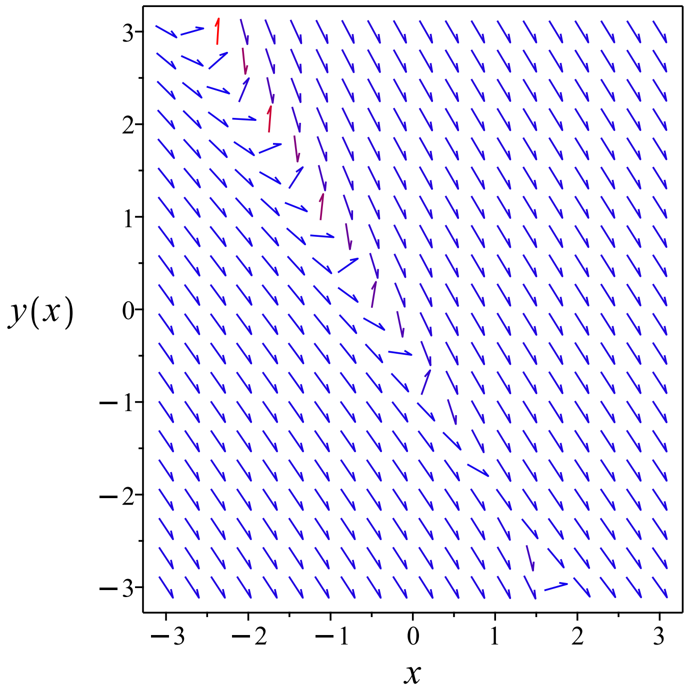


Figure 188: Slope field plot

Verification of solutions

$$(y+x+1)\,(2x+y) = c_3 e^{c_2}$$

Verified OK.

6.8.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{4x+3y+2}{3x+2y+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(4x+3y+2)(b_3-a_2)}{3x+2y+1} - \frac{(4x+3y+2)^2 a_3}{(3x+2y+1)^2} \\ & - \left(-\frac{4}{3x+2y+1} + \frac{12x+9y+6}{(3x+2y+1)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{3}{3x+2y+1} + \frac{8x+6y+4}{(3x+2y+1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{12x^2a_2 - 16x^2a_3 + 10x^2b_2 - 12x^2b_3 + 16xya_2 - 24xya_3 + 12xyb_2 - 16xyb_3 + 6y^2a_2 - 10y^2a_3 + 4y^2b_2 - 6}{(3x}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 12x^2a_2 - 16x^2a_3 + 10x^2b_2 - 12x^2b_3 + 16xya_2 - 24xya_3 + 12xyb_2 - 16xyb_3 \\ & + 6y^2a_2 - 10y^2a_3 + 4y^2b_2 - 6y^2b_3 + 8xa_2 - 16xa_3 + xb_1 + 5xb_2 - 10xb_3 \\ & - ya_1 + 7ya_2 - 14ya_3 + 4yb_2 - 8yb_3 - 2a_1 + 2a_2 - 4a_3 - b_1 + b_2 - 2b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 12a_2v_1^2 + 16a_2v_1v_2 + 6a_2v_2^2 - 16a_3v_1^2 - 24a_3v_1v_2 - 10a_3v_2^2 + 10b_2v_1^2 + 12b_2v_1v_2 \\ & + 4b_2v_2^2 - 12b_3v_1^2 - 16b_3v_1v_2 - 6b_3v_2^2 - a_1v_2 + 8a_2v_1 + 7a_2v_2 - 16a_3v_1 - 14a_3v_2 \\ & + b_1v_1 + 5b_2v_1 + 4b_2v_2 - 10b_3v_1 - 8b_3v_2 - 2a_1 + 2a_2 - 4a_3 - b_1 + b_2 - 2b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(12a_2 - 16a_3 + 10b_2 - 12b_3) v_1^2 + (16a_2 - 24a_3 + 12b_2 - 16b_3) v_1 v_2 \\ + (8a_2 - 16a_3 + b_1 + 5b_2 - 10b_3) v_1 + (6a_2 - 10a_3 + 4b_2 - 6b_3) v_2^2 \\ + (-a_1 + 7a_2 - 14a_3 + 4b_2 - 8b_3) v_2 - 2a_1 + 2a_2 - 4a_3 - b_1 + b_2 - 2b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}6a_2 - 10a_3 + 4b_2 - 6b_3 &= 0\\ 12a_2 - 16a_3 + 10b_2 - 12b_3 &= 0\\ 16a_2 - 24a_3 + 12b_2 - 16b_3 &= 0\\ -a_1 + 7a_2 - 14a_3 + 4b_2 - 8b_3 &= 0\\ 8a_2 - 16a_3 + b_1 + 5b_2 - 10b_3 &= 0\\ -2a_1 + 2a_2 - 4a_3 - b_1 + b_2 - 2b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -a_3 - b_3\\ a_2 &= 3a_3 + b_3\\ a_3 &= a_3\\ b_1 &= 2a_3 + 2b_3\\ b_2 &= -2a_3\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x - 1\\ \eta &= y + 2\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi\\ &= y + 2 - \left(-\frac{4x + 3y + 2}{3x + 2y + 1}\right)(x - 1)\\ &= \frac{4x^2 + 6xy + 2y^2 + 4x + 2y}{3x + 2y + 1}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4x^2+6xy+2y^2+4x+2y}{3x+2y+1}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln (2x^2 + 3xy + y^2 + 2x + y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{4x + 3y + 2}{3x + 2y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{2x + 2y + 2} + \frac{1}{2x + y} \\ S_y &= \frac{1}{2x + 2y + 2} + \frac{1}{4x + 2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

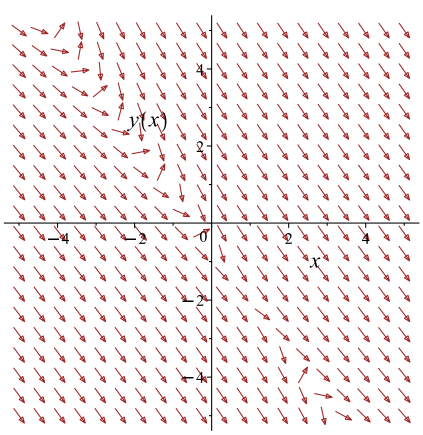
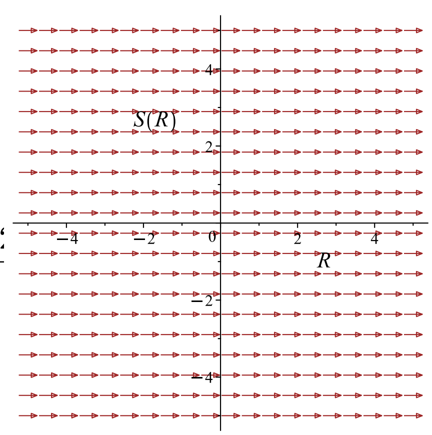
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y+x+1)}{2} + \frac{\ln(2x+y)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(y+x+1)}{2} + \frac{\ln(2x+y)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{4x+3y+2}{3x+2y+1}$ 	$R = x$ $S = \frac{\ln(x+y+1)}{2} + \ln($	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln(y+x+1)}{2}+\frac{\ln(2x+y)}{2}=c_1 \tag{1}$$

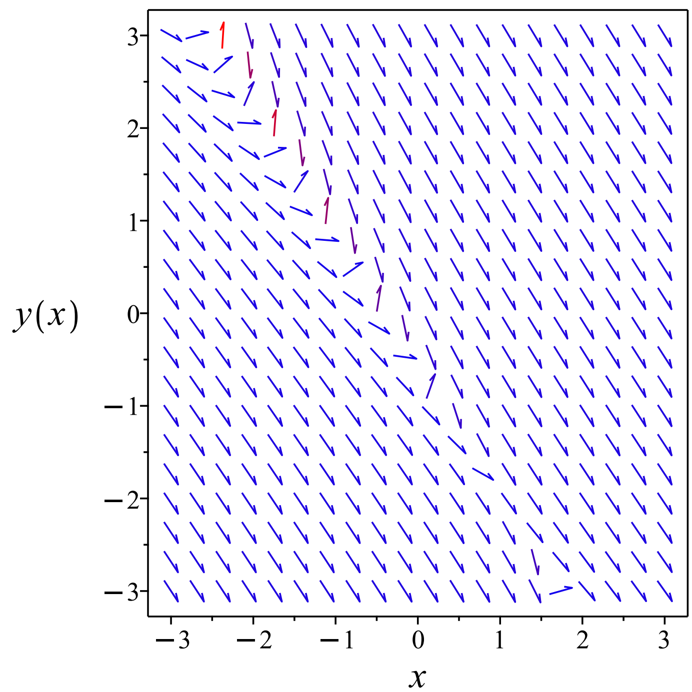


Figure 189: Slope field plot

Verification of solutions

$$\frac{\ln(y+x+1)}{2}+\frac{\ln(2x+y)}{2}=c_1$$

Verified OK.

6.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
(3 x+2 y+1) \mathrm{d} y &= (-4 x-3 y-2) \mathrm{d} x \\
(4 x+3 y+2) \mathrm{d} x+(3 x+2 y+1) \mathrm{d} y &= 0 \qquad (2\mathrm{A})
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= 4 x+3 y+2 \\
N(x, y) &= 3 x+2 y+1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(4 x+3 y+2) \\
&= 3
\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x + 2y + 1)\\ &= 3\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 4x + 3y + 2\,\mathrm{d}x\end{aligned}$$

$$\phi = x(2x + 3y + 2) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3x + 2y + 1$. Therefore equation (4) becomes

$$3x + 2y + 1 = 3x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1 + 2y$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int (1 + 2y)\,\mathrm{d}y\\ f(y) &= y^2 + y + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(2x + 3y + 2) + y^2 + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(2x + 3y + 2) + y^2 + y$$

Summary

The solution(s) found are the following

$$x(2x + 3y + 2) + y^2 + y = c_1 \tag{1}$$

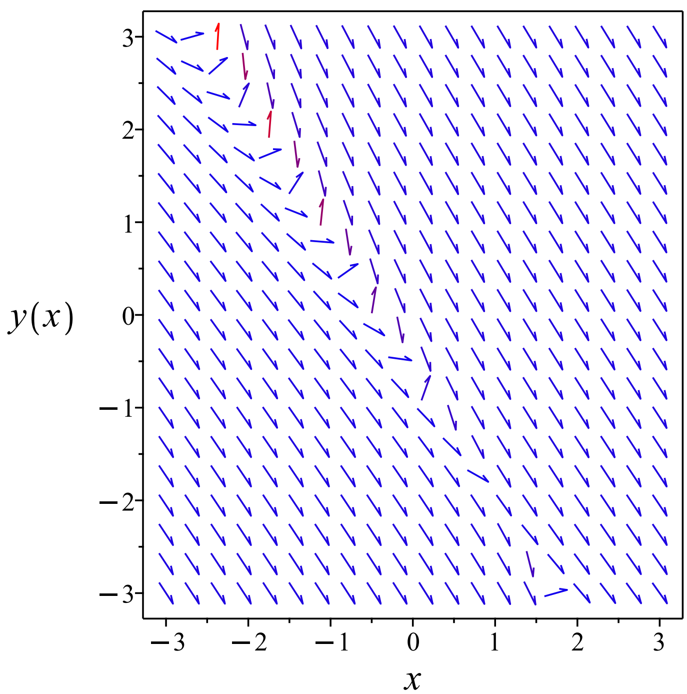


Figure 190: Slope field plot

Verification of solutions

$$x(2x + 3y + 2) + y^2 + y = c_1$$

Verified OK.

6.8.5 Maple step by step solution

Let's solve

$(3x + 2y + 1)\, y' + 3y = -4x - 2$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - ◦ Evaluate derivatives

    $3 = 3$

  - ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (4x + 3y + 2)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = 2x^2 + 3xy + 2x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $3x + 2y + 1 = 3x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 1 + 2y$

- Solve for $f_1(y)$

  $f_1(y) = y^2 + y$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = 2x^2 + 3xy + y^2 + 2x + y$$

- Substitute $F(x,y)$ into the solution of the ODE

$$2x^2 + 3xy + y^2 + 2x + y = c_1$$

- Solve for y

$$\left\{y = -\frac{3x}{2} - \frac{1}{2} - \frac{\sqrt{x^2+4c_1-2x+1}}{2}, y = -\frac{3x}{2} - \frac{1}{2} + \frac{\sqrt{x^2+4c_1-2x+1}}{2}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 32

```
dsolve((3*x+2*y(x)+1)*diff(y(x),x)+(4*x+3*y(x)+2)=0,y(x), singsol=all)
```

$$y(x) = \frac{-\sqrt{(x-1)^2 c_1^2 + 4} + (-3x-1)c_1}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.123 (sec). Leaf size: 61

```
DSolve[(3*x+2*y[x]+1)*y'[x]+(4*x+3*y[x]+2)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-\sqrt{x^2 - 2x + 1 + 4c_1} - 3x - 1\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{x^2 - 2x + 1 + 4c_1} - 3x - 1\right)$$

6.9 problem Exercise 12.9, page 103

Internal problem ID [4530]
Internal file name [`OUTPUT/4023_Sunday_June_05_2022_12_10_49_PM_4158373/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.9, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{\left(x^{2}-y^{2}\right) y^{\prime}-2 x y=0}$$

6.9.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$\left(x^{2}-u(x)^{2} x^{2}\right)\left(u^{\prime}(x) x+u(x)\right)-2 x^{2} u(x)=0$$

In canonical form the ODE is

$$\begin{aligned}
u^{\prime} &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{u(u^{2}+1)}{(u^{2}-1)\, x}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u^2+1)}{u^2-1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u^2+1)}{u^2-1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u(u^2+1)}{u^2-1}}\,du = \int -\frac{1}{x}\,dx$$

$$\ln\left(u^2+1\right) - \ln\left(u\right) = -\ln\left(x\right) + c_2$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(u^2+1)-\ln(u)} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{u^2+1}{u} = \frac{c_3}{x}$$

The solution is

$$\frac{u(x)^2+1}{u\left(x\right)} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\left(\frac{y^2}{x^2}+1\right)x}{y} = \frac{c_3}{x}$$

$$\frac{x^2+y^2}{xy} = \frac{c_3}{x}$$

Which simplifies to

$$\frac{x^2+y^2}{y} = c_3$$

Summary

The solution(s) found are the following

$$\frac{x^2+y^2}{y} = c_3 \tag{1}$$

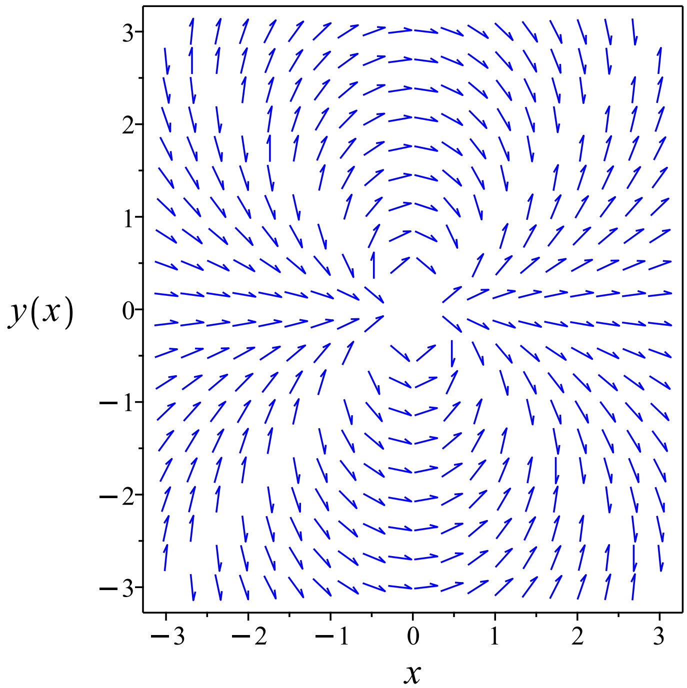


Figure 191: Slope field plot

Verification of solutions

$$\frac{x^2 + y^2}{y} = c_3$$

Verified OK.

6.9.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2xy}{-x^2 + y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{2xy(b_3 - a_2)}{-x^2 + y^2} - \frac{4x^2y^2a_3}{(-x^2 + y^2)^2} \\ & - \left(-\frac{2y}{-x^2 + y^2} - \frac{4x^2y}{(-x^2 + y^2)^2}\right)(xa_2 + ya_3 + a_1) \\ & - \left(-\frac{2x}{-x^2 + y^2} + \frac{4x\,y^2}{(-x^2 + y^2)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4b_2 + 2x^2y^2a_3 + 4x^2y^2b_2 - 4x\,y^3a_2 + 4x\,y^3b_3 - 2y^4a_3 - y^4b_2 + 2x^3b_1 - 2x^2ya_1 + 2x\,y^2b_1 - 2y^3a_1}{(x^2 - y^2)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -x^4b_2 - 2x^2y^2a_3 - 4x^2y^2b_2 + 4x\,y^3a_2 - 4x\,y^3b_3 + 2y^4a_3 \\ & + y^4b_2 - 2x^3b_1 + 2x^2ya_1 - 2x\,y^2b_1 + 2y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 4a_2v_1v_2^3 - 2a_3v_1^2v_2^2 + 2a_3v_2^4 - b_2v_1^4 - 4b_2v_1^2v_2^2 + b_2v_2^4 \\ & - 4b_3v_1v_2^3 + 2a_1v_1^2v_2 + 2a_1v_2^3 - 2b_1v_1^3 - 2b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_2 v_1^4 - 2b_1 v_1^3 + (-2a_3 - 4b_2) v_1^2 v_2^2 + 2a_1 v_1^2 v_2 + (4a_2 - 4b_3) v_1 v_2^3 - 2b_1 v_1 v_2^2 + (2a_3 + b_2) v_2^4 + 2a_1 v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
2a_1 &= 0 \\
-2b_1 &= 0 \\
-b_2 &= 0 \\
4a_2 - 4b_3 &= 0 \\
-2a_3 - 4b_2 &= 0 \\
2a_3 + b_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{2xy}{-x^2 + y^2}\right)(x) \\
&= \frac{-y\, x^2 - y^3}{x^2 - y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-y\,x^2 - y^3}{x^2 - y^2}} dy \end{aligned}$$

Which results in

$$S = \ln\left(x^2 + y^2\right) - \ln\left(y\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2xy}{-x^2 + y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x}{x^2 + y^2} \\ S_y &= \frac{2y}{x^2 + y^2} - \frac{1}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

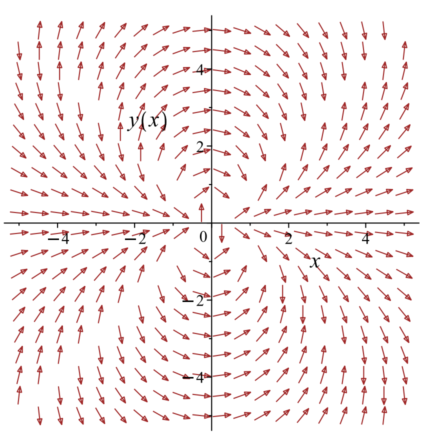
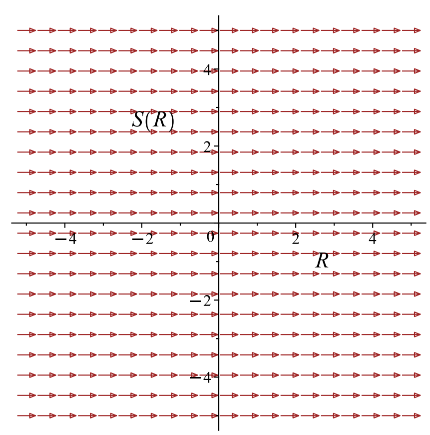
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(x^2+y^2\right) - \ln(y) = c_1$$

Which simplifies to

$$\ln\left(x^2+y^2\right) - \ln(y) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2xy}{-x^2+y^2}$ 	$R = x$ $S = \ln\left(x^2+y^2\right) - \ln(y)$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\ln\left(x^2+y^2\right)-\ln\left(y\right)=c_1 \tag{1}$$

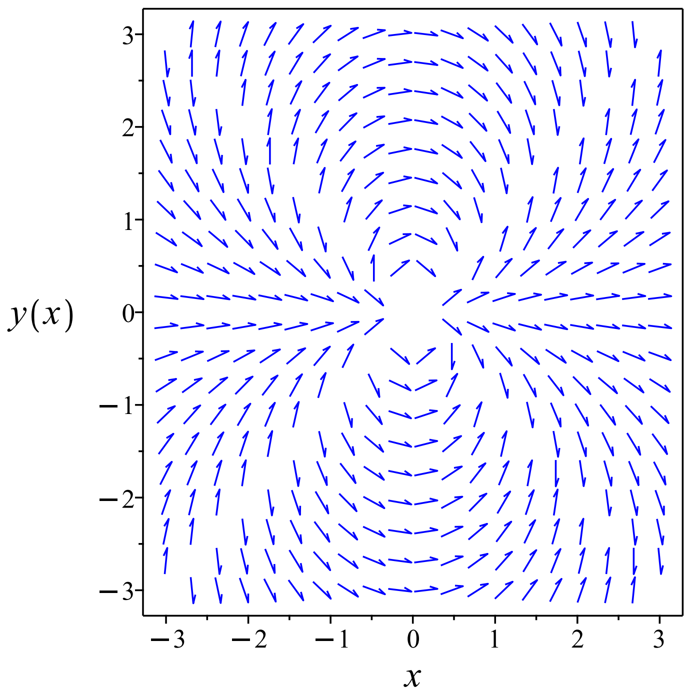


Figure 192: Slope field plot

Verification of solutions

$$\ln\left(x^2+y^2\right)-\ln\left(y\right)=c_1$$

Verified OK.

6.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2 - y^2\right)\mathrm{d}y &= (2xy)\,\mathrm{d}x \\ (-2xy)\,\mathrm{d}x + \left(x^2 - y^2\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -2xy \\ N(x, y) &= x^2 - y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2xy) \\ &= -2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 - y^2) \\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x^2 - y^2}((-2x) - (2x)) \\ &= -\frac{4x}{x^2 - y^2}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{2yx}((2x) - (-2x)) \\ &= -\frac{2}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{2}{y} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(y)} \\ &= \frac{1}{y^2}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{y^2}(-2xy) \\ &= -\frac{2x}{y}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}\left(x^2-y^2\right) \\
&= \frac{x^2-y^2}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{2x}{y}\right)+\left(\frac{x^2-y^2}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int-\frac{2x}{y}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{y}+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y} = \frac{x^2}{y^2}+f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\frac{x^2-y^2}{y^2}$. Therefore equation (4) becomes

$$
\frac{x^2-y^2}{y^2} = \frac{x^2}{y^2}+f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -1$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (-1) \,\mathrm{d}y$$
$$f(y) = -y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{y} - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{y} - y$$

Summary
The solution(s) found are the following

$$-\frac{x^2}{y} - y = c_1 \tag{1}$$

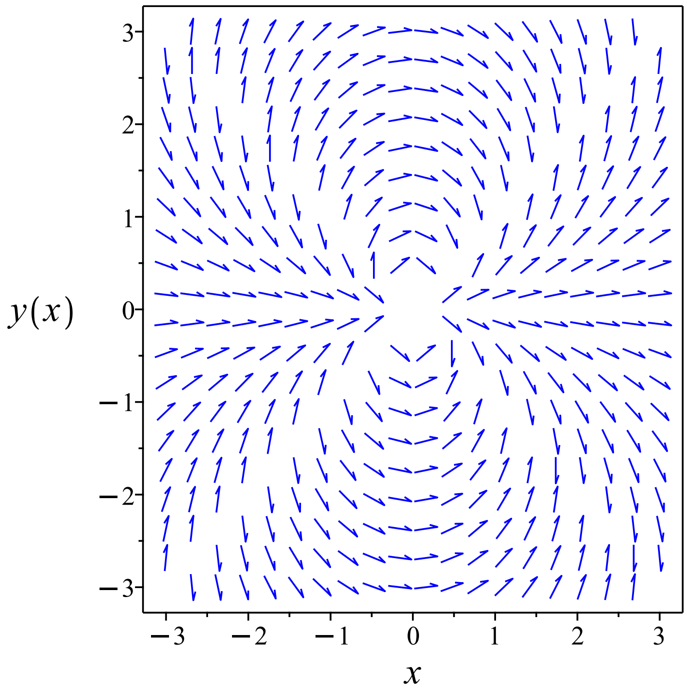


Figure 193: Slope field plot

Verification of solutions

$$-\frac{x^2}{y} - y = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
dsolve((x^2-y(x)^2)*diff(y(x),x)=2*x*y(x),y(x), singsol=all)
```

$$y(x) = \frac{1 - \sqrt{-4x^2c_1^2 + 1}}{2c_1}$$
$$y(x) = \frac{1 + \sqrt{-4x^2c_1^2 + 1}}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.982 (sec). Leaf size: 66

```
DSolve[(x^2-y[x]^2)*y'[x]==2*x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(e^{c_1} - \sqrt{-4x^2 + e^{2c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{-4x^2 + e^{2c_1}} + e^{c_1}\right)$$
$$y(x) \to 0$$

6.10 problem Exercise 12.10, page 103

Internal problem ID [4531]
Internal file name [`OUTPUT/4024_Sunday_June_05_2022_12_11_02_PM_20905807/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.10, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{y+\left(1+\mathrm{e}^{2x} y^{2}\right) y' = 0}$$

6.10.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}
y' &= -\frac{y}{1+\mathrm{e}^{2x} y^{2}} \\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(b_3 - a_2)}{1 + e^{2x}y^2} - \frac{y^2 a_3}{(1 + e^{2x}y^2)^2} - \frac{2y^3 e^{2x}(x a_2 + y a_3 + a_1)}{(1 + e^{2x}y^2)^2} - \left(-\frac{1}{1 + e^{2x}y^2} + \frac{2y^2 e^{2x}}{(1 + e^{2x}y^2)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{e^{4x}y^4 b_2 - 2\,e^{2x}x\,y^3 a_2 - 2\,e^{2x}y^4 a_3 - e^{2x}x\,y^2 b_2 - 2\,e^{2x}y^3 a_1 + e^{2x}y^3 a_2 - 2\,e^{2x}y^3 b_3 - e^{2x}y^2 b_1 + 2\,e^{2x}y^2 b_2 - y^2 a}{(1 + e^{2x}y^2)^2} = 0$$

Setting the numerator to zero gives

$$e^{4x}y^4 b_2 - 2\,e^{2x}x\,y^3 a_2 - 2\,e^{2x}y^4 a_3 - e^{2x}x\,y^2 b_2 - 2\,e^{2x}y^3 a_1 + e^{2x}y^3 a_2 - 2\,e^{2x}y^3 b_3 - e^{2x}y^2 b_1 + 2\,e^{2x}y^2 b_2 - y^2 a_3 + x b_2 + y a_2 + b_1 + b_2 = 0 \tag{6E}$$

Simplifying the above gives

$$e^{4x}y^4 b_2 - 2\,e^{2x}x\,y^3 a_2 - 2\,e^{2x}y^4 a_3 - e^{2x}x\,y^2 b_2 - 2\,e^{2x}y^3 a_1 + e^{2x}y^3 a_2 - 2\,e^{2x}y^3 b_3 - e^{2x}y^2 b_1 + 2\,e^{2x}y^2 b_2 - y^2 a_3 + x b_2 + y a_2 + b_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, e^{2x}, e^{4x}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, e^{2x} = v_3, e^{4x} = v_4\}$$

The above PDE (6E) now becomes

$$-2v_3v_1v_2^3a_2 - 2v_3v_2^4a_3 + v_4v_2^4b_2 - 2v_3v_2^3a_1 + v_3v_2^3a_2 - v_3v_1v_2^2b_2 - 2v_3v_2^3b_3 - v_3v_2^2b_1 + 2v_3v_2^2b_2 - v_2^2a_3 + v_2a_2 + v_1b_2 + b_1 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned}&-2v_3v_1v_2^3a_2 - v_3v_1v_2^2b_2 + v_1b_2 - 2v_3v_2^4a_3 + v_4v_2^4b_2 \\ &+ (-2a_1 + a_2 - 2b_3)\, v_2^3v_3 + (-b_1 + 2b_2)\, v_2^2v_3 - v_2^2a_3 + v_2a_2 + b_1 + b_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_2 &= 0 \\ b_2 &= 0 \\ -2a_2 &= 0 \\ -2a_3 &= 0 \\ -a_3 &= 0 \\ -b_2 &= 0 \\ -b_1 + 2b_2 &= 0 \\ b_1 + b_2 &= 0 \\ -2a_1 + a_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -b_3 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -1 \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y}{1+\mathrm{e}^{2x} y^2}\right)(-1) \\
&= \frac{y^3 \mathrm{e}^{2x}}{1+\mathrm{e}^{2x} y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y^3 \mathrm{e}^{2x}}{1+\mathrm{e}^{2x} y^2}} dy
\end{aligned}$$

Which results in

$$S = \mathrm{e}^{-2x} \left(\mathrm{e}^{2x} \ln(y) - \frac{1}{2y^2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y}{1+\mathrm{e}^{2x} y^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{e^{-2x}}{y^2} \\
S_y &= \frac{y^2 + e^{-2x}}{y^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{2 \ln(y) y^2 - e^{-2x}}{2y^2} = c_1
$$

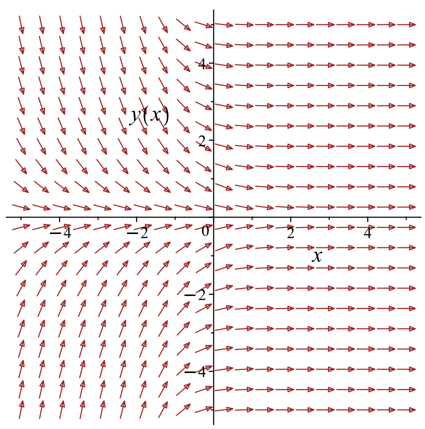
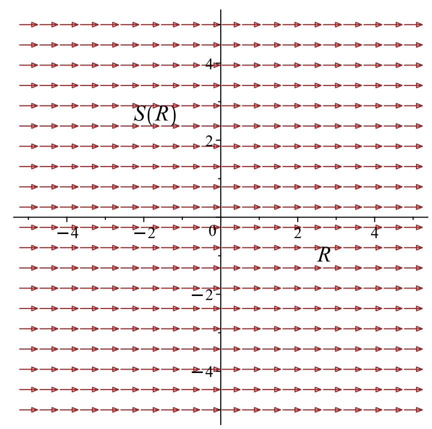
Which simplifies to

$$
\frac{2 \ln(y) y^2 - e^{-2x}}{2y^2} = c_1
$$

Which gives

$$
y = e^{\frac{\operatorname{LambertW}\left(e^{-2c_1 - 2x}\right)}{2} + c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{1+e^{2x}y^2}$ 	$R = x$ $S = \frac{2\ln(y)\,y^2 - e^{-2x}}{2y^2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = e^{\frac{\mathrm{LambertW}\left(e^{-2c_1-2x}\right)}{2}+c_1} \tag{1}$$

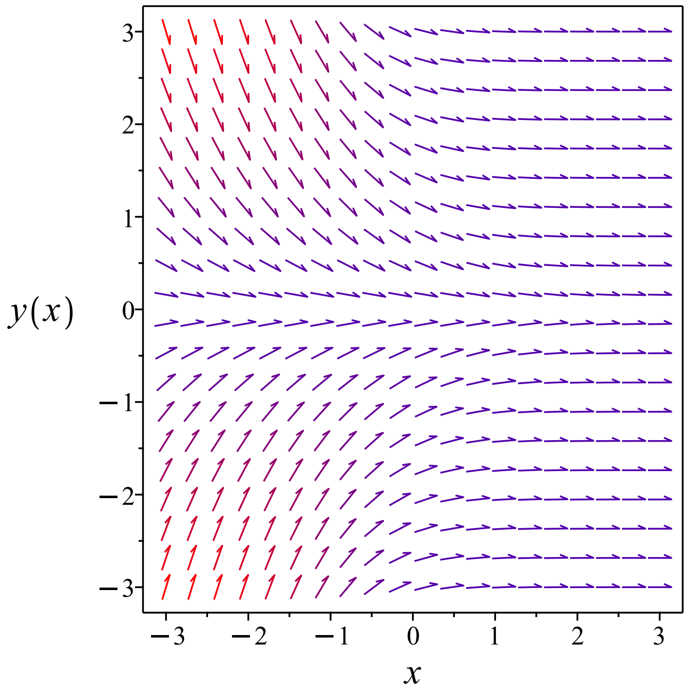


Figure 194: Slope field plot

Verification of solutions

$$y = e^{\frac{\mathrm{LambertW}\left(e^{-2c_1-2x}\right)}{2}+c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -y(x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve(y(x)+(1+y(x)^2*exp(2*x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{-x}}{\sqrt{\text{LambertW}\left(e^{-2x}c_1\right)}}$$

✓ Solution by Mathematica

Time used: 3.33 (sec). Leaf size: 57

```
DSolve[y[x]+(1+y[x]^2*Exp[2*x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{-x}}{\sqrt{W\left(e^{-2x+2c_1}\right)}}$$
$$y(x) \to \frac{e^{-x}}{\sqrt{W\left(e^{-2x+2c_1}\right)}}$$
$$y(x) \to 0$$

6.11 problem Exercise 12.11, page 103

Internal problem ID [4532]
Internal file name [`OUTPUT/4025_Sunday_June_05_2022_12_11_09_PM_93497014/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.11, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{yx^2 + y^2 + y'x^3 = 0}$$

6.11.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(x^2 + y)}{x^3} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 102: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^2 x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^2 x} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{yx}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(x^2 + y)}{x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{y\,x^2} \\ S_y &= \frac{1}{y^2 x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x^4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R^4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{3R^3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{xy} = \frac{1}{3x^3} + c_1$$

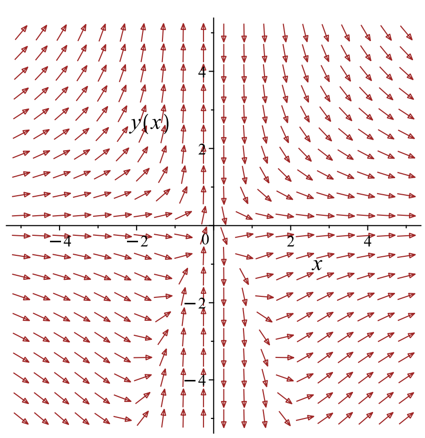
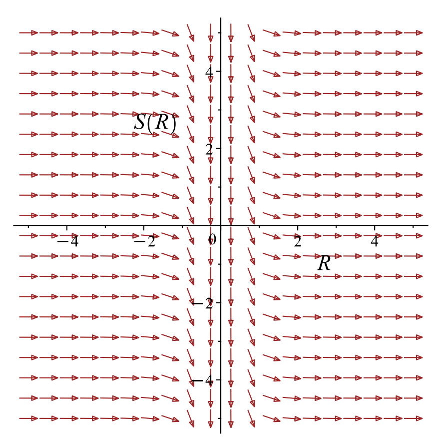
Which simplifies to

$$-\frac{1}{xy} = \frac{1}{3x^3} + c_1$$

Which gives

$$y = -\frac{3x^2}{3c_1x^3 + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(x^2+y)}{x^3}$ 	$R = x$ $S = -\frac{1}{yx}$	$\frac{dS}{dR} = -\frac{1}{R^4}$ 

Summary

The solution(s) found are the following

$$y = -\frac{3x^2}{3c_1x^3 + 1} \tag{1}$$

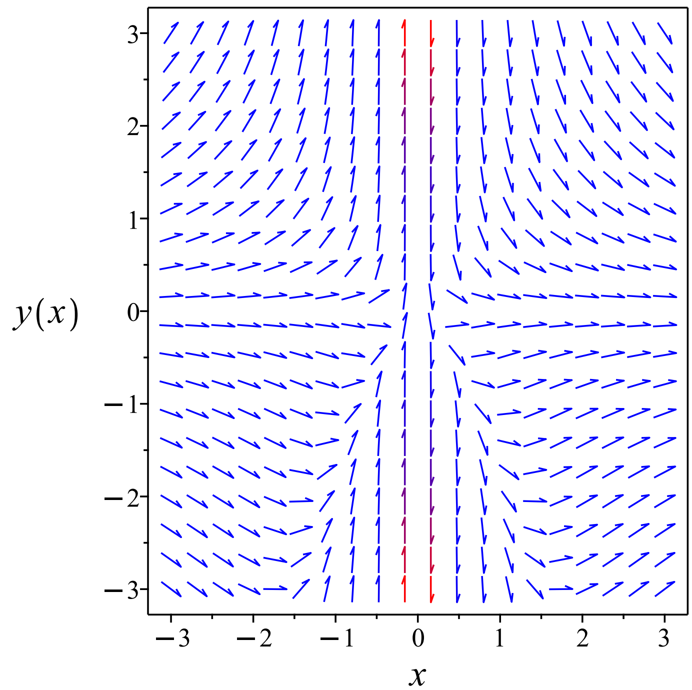


Figure 195: Slope field plot

Verification of solutions

$$y = -\frac{3x^2}{3c_1x^3 + 1}$$

Verified OK.

6.11.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(x^2 + y)}{x^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - \frac{1}{x^3}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= -\frac{1}{x^3} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{yx} - \frac{1}{x^3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{w(x)}{x} - \frac{1}{x^3} \\ w' &= \frac{w}{x} + \frac{1}{x^3} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{1}{x}\\ q(x) &= \frac{1}{x^3}\end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = \frac{1}{x^3}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx}\\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{1}{x^3}\right)\\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{1}{x^3}\right)\\ d\left(\frac{w}{x}\right) &= \frac{1}{x^4}\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int \frac{1}{x^4}\,dx\\ \frac{w}{x} &= -\frac{1}{3x^3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = -\frac{1}{3x^2} + c_1 x$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = -\frac{1}{3x^2} + c_1 x$$

Or

$$y = \frac{1}{-\frac{1}{3x^2} + c_1 x}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{-\frac{1}{3x^2} + c_1 x} \tag{1}$$

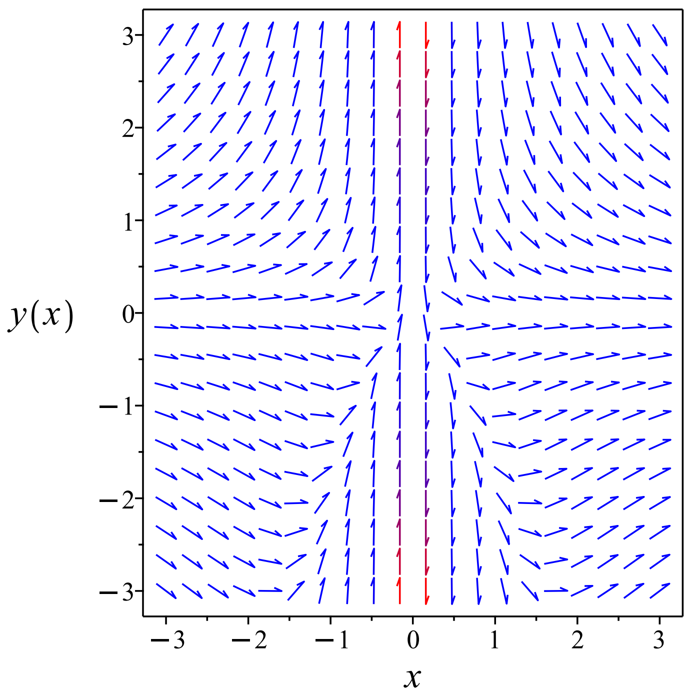


Figure 196: Slope field plot

Verification of solutions

$$y = \frac{1}{-\frac{1}{3x^2} + c_1 x}$$

Verified OK.

6.11.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(x^2 + y)}{x^3} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y}{x} - \frac{y^2}{x^3}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = -\frac{1}{x^3}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x^3}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{3}{x^4} \\ f_1 f_2 &= \frac{1}{x^4} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x^3} - \frac{4u'(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{x^3}$$

The above shows that

$$u'(x) = -\frac{3c_2}{x^4}$$

Using the above in (1) gives the solution

$$y = -\frac{3c_2}{x\left(c_1 + \frac{c_2}{x^3}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{3}{x\left(c_3 + \frac{1}{x^3}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{3}{x\left(c_3 + \frac{1}{x^3}\right)} \tag{1}$$

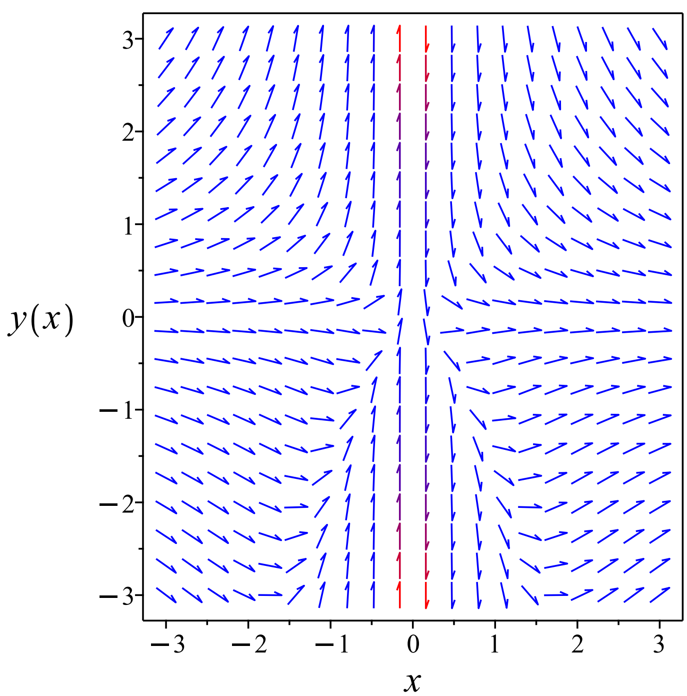


Figure 197: Slope field plot

Verification of solutions

$$y = -\frac{3}{x\left(c_3 + \frac{1}{x^3}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve((x^2*y(x)+y(x)^2)+x^3*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{3x^2}{3c_1x^3 - 1}$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 26

```
DSolve[(x^2*y[x]+y[x]^2)+x^3*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3x^2}{-1 + 3c_1x^3}$$
$$y(x) \to 0$$

6.12 problem Exercise 12.12, page 103

Internal problem ID [4533]
Internal file name [`OUTPUT/4026_Sunday_June_05_2022_12_11_19_PM_18044692/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.12, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{y^2 e^{x y^2}+\left(2 x y\, e^{x y^2}-3 y^2\right) y'=-4 x^3}$$

6.12.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x}=M$$
$$\frac{\partial \phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(2xy\,\mathrm{e}^{y^2x} - 3y^2\right)\mathrm{d}y = \left(-y^2\mathrm{e}^{y^2x} - 4x^3\right)\mathrm{d}x$$

$$\left(y^2\mathrm{e}^{y^2x} + 4x^3\right)\mathrm{d}x + \left(2xy\,\mathrm{e}^{y^2x} - 3y^2\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= y^2\mathrm{e}^{y^2x} + 4x^3 \\ N(x, y) &= 2xy\,\mathrm{e}^{y^2x} - 3y^2 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2\mathrm{e}^{y^2x} + 4x^3\right) \\ &= 2y\,\mathrm{e}^{y^2x}\left(y^2x + 1\right) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2xy\,\mathrm{e}^{y^2x} - 3y^2\right) \\ &= 2y\,\mathrm{e}^{y^2x}\left(y^2x + 1\right) \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int y^2 \mathrm{e}^{y^2 x} + 4x^3 \,\mathrm{d}x$$
$$\phi = \mathrm{e}^{y^2 x} + x^4 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2xy\,\mathrm{e}^{y^2 x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2xy\,\mathrm{e}^{y^2 x} - 3y^2$. Therefore equation (4) becomes

$$2xy\,\mathrm{e}^{y^2 x} - 3y^2 = 2xy\,\mathrm{e}^{y^2 x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -3y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (-3y^2) \,\mathrm{d}y$$
$$f(y) = -y^3 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{y^2 x} + x^4 - y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{y^2 x} + x^4 - y^3$$

Summary

The solution(s) found are the following

$$e^{x y^2}+x^4-y^3=c_1 \tag{1}$$

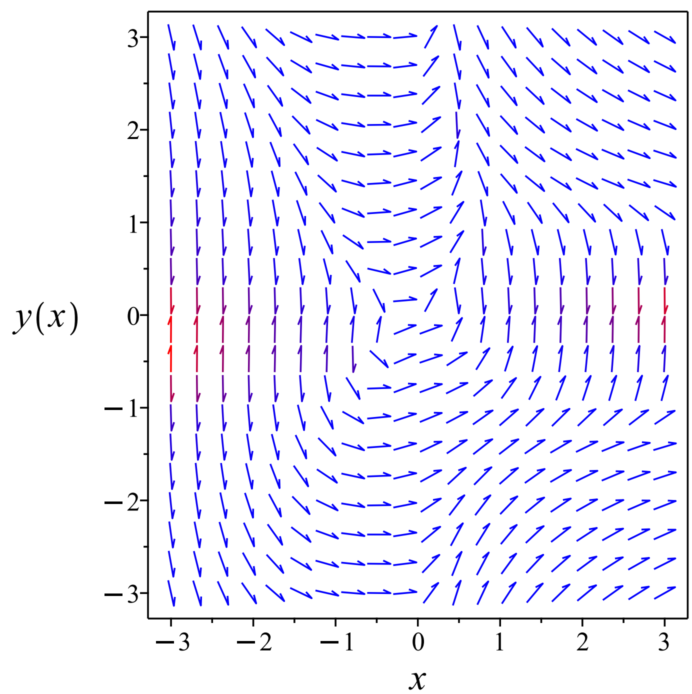


Figure 198: Slope field plot

Verification of solutions

$$e^{x y^2}+x^4-y^3=c_1$$

Verified OK.

6.12.2 Maple step by step solution

Let's solve

$y^2 e^{x y^2}+\left(2 x y\, e^{x y^2}-3 y^2\right) y'=-4 x^3$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $2y\,e^{y^2x} + 2y^3x\,e^{y^2x} = 2y\,e^{y^2x} + 2y^3x\,e^{y^2x}$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \left(y^2e^{y^2x} + 4x^3\right) dx + f_1(y)$

- Evaluate integral

  $F(x, y) = e^{y^2x} + x^4 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $2xy\,e^{y^2x} - 3y^2 = 2xy\,e^{y^2x} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = -3y^2$

- Solve for $f_1(y)$

  $f_1(y) = -y^3$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = e^{y^2x} + x^4 - y^3$

- Substitute $F(x, y)$ into the solution of the ODE

  $e^{y^2x} + x^4 - y^3 = c_1$

- Solve for y

  $y = RootOf\left(-e^{\_Z^2x} - x^4 + \_Z^3 + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve((y(x)^2*exp(x*y(x)^2)+4*x^3)+(2*x*y(x)*exp(x*y(x)^2)-3*y(x)^2)*diff(y(x),x)=0,y(x), s
```

$$e^{xy(x)^2} + x^4 - y(x)^3 + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.279 (sec). Leaf size: 24

```
DSolve[(y[x]^2*Exp[x*y[x]^2]+4*x^3)+(2*x*y[x]*Exp[x*y[x]^2]-3*y[x]^2)*y'[x]==0,y[x],x,Includ
```

$$\text{Solve}\left[x^4 + e^{xy(x)^2} - y(x)^3 = c_1, y(x)\right]$$

6.13 problem Exercise 12.13, page 103

Internal problem ID [4534]
Internal file name [OUTPUT/4027_Sunday_June_05_2022_12_11_27_PM_7134422/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.13, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{y' - \left(x^2+2y-1\right)^{\frac{2}{3}} = -x}$$

6.13.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}
y' &= \left(x^2+2y-1\right)^{\frac{2}{3}} - x \\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \left(\left(x^2+2y-1\right)^{\frac{2}{3}} - x\right)(b_3 - a_2) - \left(\left(x^2+2y-1\right)^{\frac{2}{3}} - x\right)^2 a_3 - \left(\frac{4x}{3\left(x^2+2y-1\right)^{\frac{1}{3}}} - 1\right)(xa_2 + ya_3 + a_1) - \frac{4(xb_2 + yb_3 + b_1)}{3\left(x^2+2y-1\right)^{\frac{1}{3}}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{3(x^2+2y-1)^{\frac{5}{3}} a_3 + 3(x^2+2y-1)^{\frac{1}{3}} x^2 a_3 - 6x^3 a_3 - 6(x^2+2y-1)^{\frac{1}{3}} x a_2 + 3(x^2+2y-1)^{\frac{1}{3}} x b_3 - 3(x^2}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}&-3\left(x^2+2y-1\right)^{\frac{5}{3}} a_3 - 3\left(x^2+2y-1\right)^{\frac{1}{3}} x^2 a_3 + 6x^3 a_3 \\ &+ 6\left(x^2+2y-1\right)^{\frac{1}{3}} x a_2 - 3\left(x^2+2y-1\right)^{\frac{1}{3}} x b_3 + 3\left(x^2+2y-1\right)^{\frac{1}{3}} y a_3 \\ &- 7x^2 a_2 + 3x^2 b_3 + 8xya_3 + 3\left(x^2+2y-1\right)^{\frac{1}{3}} a_1 + 3b_2\left(x^2+2y-1\right)^{\frac{1}{3}} \\ &- 4xa_1 - 6xa_3 - 4xb_2 - 6ya_2 + 2yb_3 + 3a_2 - 4b_1 - 3b_3 = 0\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}&-3\left(x^2+2y-1\right)^{\frac{5}{3}} a_3 + 6\left(x^2+2y-1\right) xa_3 - 3\left(x^2+2y-1\right) a_2 \\ &+ 3\left(x^2+2y-1\right) b_3 - 3\left(x^2+2y-1\right)^{\frac{1}{3}} x^2 a_3 + 6\left(x^2+2y-1\right)^{\frac{1}{3}} x a_2 \\ &- 3\left(x^2+2y-1\right)^{\frac{1}{3}} x b_3 + 3\left(x^2+2y-1\right)^{\frac{1}{3}} y a_3 - 4x^2 a_2 - 4xya_3 \\ &+ 3\left(x^2+2y-1\right)^{\frac{1}{3}} a_1 + 3b_2\left(x^2+2y-1\right)^{\frac{1}{3}} - 4xa_1 - 4xb_2 - 4yb_3 - 4b_1 = 0\end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}&-3x^2\left(x^2+2y-1\right)^{\frac{2}{3}} a_3 + 6x^3 a_3 - 3\left(x^2+2y-1\right)^{\frac{1}{3}} x^2 a_3 - 6\left(x^2+2y-1\right)^{\frac{2}{3}} y a_3 \\ &- 7x^2 a_2 + 3x^2 b_3 + 6\left(x^2+2y-1\right)^{\frac{1}{3}} x a_2 - 3\left(x^2+2y-1\right)^{\frac{1}{3}} x b_3 + 8xya_3 \\ &+ 3\left(x^2+2y-1\right)^{\frac{2}{3}} a_3 + 3\left(x^2+2y-1\right)^{\frac{1}{3}} y a_3 - 4xa_1 - 6xa_3 - 4xb_2 \\ &+ 3\left(x^2+2y-1\right)^{\frac{1}{3}} a_1 + 3b_2\left(x^2+2y-1\right)^{\frac{1}{3}} - 6ya_2 + 2yb_3 + 3a_2 - 4b_1 - 3b_3 = 0\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, (x^2+2y-1)^{\frac{1}{3}}, (x^2+2y-1)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, (x^2+2y-1)^{\frac{1}{3}} = v_3, (x^2+2y-1)^{\frac{2}{3}} = v_4\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&6v_1^3a_3 - 3v_3v_1^2a_3 - 3v_1^2v_4a_3 - 7v_1^2a_2 + 6v_3v_1a_2 + 8v_1v_2a_3 + 3v_3v_2a_3 \\
&- 6v_4v_2a_3 + 3v_1^2b_3 - 3v_3v_1b_3 - 4v_1a_1 + 3v_3a_1 - 6v_2a_2 - 6v_1a_3 \\
&+ 3v_4a_3 - 4v_1b_2 + 3b_2v_3 + 2v_2b_3 + 3a_2 - 4b_1 - 3b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned}
&6v_1^3a_3 - 3v_3v_1^2a_3 - 3v_1^2v_4a_3 + (-7a_2 + 3b_3)v_1^2 + 8v_1v_2a_3 \\
&+ (6a_2 - 3b_3)v_1v_3 + (-4a_1 - 6a_3 - 4b_2)v_1 + 3v_3v_2a_3 - 6v_4v_2a_3 \\
&+ (-6a_2 + 2b_3)v_2 + (3a_1 + 3b_2)v_3 + 3v_4a_3 + 3a_2 - 4b_1 - 3b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-6a_3 &= 0 \\
-3a_3 &= 0 \\
3a_3 &= 0 \\
6a_3 &= 0 \\
8a_3 &= 0 \\
3a_1 + 3b_2 &= 0 \\
-7a_2 + 3b_3 &= 0 \\
-6a_2 + 2b_3 &= 0 \\
6a_2 - 3b_3 &= 0 \\
-4a_1 - 6a_3 - 4b_2 &= 0 \\
3a_2 - 4b_1 - 3b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -b_2 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -1 \\ \eta &= x \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= x - \left(\left(x^2 + 2y - 1\right)^{\frac{2}{3}} - x\right)(-1) \\ &= \left(x^2 + 2y - 1\right)^{\frac{2}{3}} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\left(x^2 + 2y - 1\right)^{\frac{2}{3}}} dy \end{aligned}$$

Which results in

$$S = \frac{3(x^2 + 2y - 1)^{\frac{1}{3}}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \left(x^2 + 2y - 1\right)^{\frac{2}{3}} - x$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x}{(x^2 + 2y - 1)^{\frac{2}{3}}} \\
S_y &= \frac{1}{(x^2 + 2y - 1)^{\frac{2}{3}}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3(x^2 + 2y - 1)^{\frac{1}{3}}}{2} = x + c_1$$

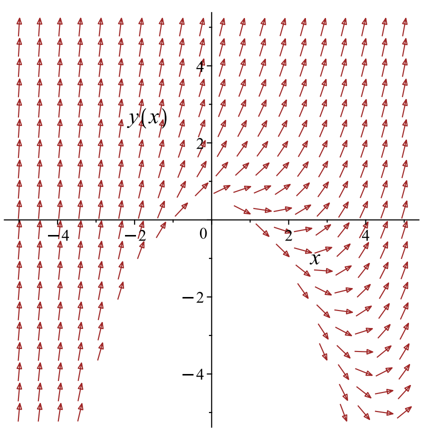
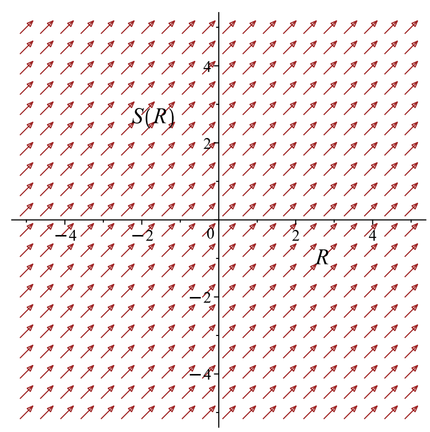
Which simplifies to

$$\frac{3(x^2+2y-1)^{\frac{1}{3}}}{2} = x + c_1$$

Which gives

$$y = \frac{4}{27}c_1^3 + \frac{4}{9}c_1^2 x + \frac{4}{9}c_1 x^2 + \frac{4}{27}x^3 - \frac{1}{2}x^2 + \frac{1}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (x^2+2y-1)^{\frac{2}{3}} - x$  	$R = x$ $S = \frac{3(x^2+2y-1)^{\frac{1}{3}}}{2}$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = \frac{4}{27}c_1^3 + \frac{4}{9}c_1^2 x + \frac{4}{9}c_1 x^2 + \frac{4}{27}x^3 - \frac{1}{2}x^2 + \frac{1}{2} \tag{1}$$

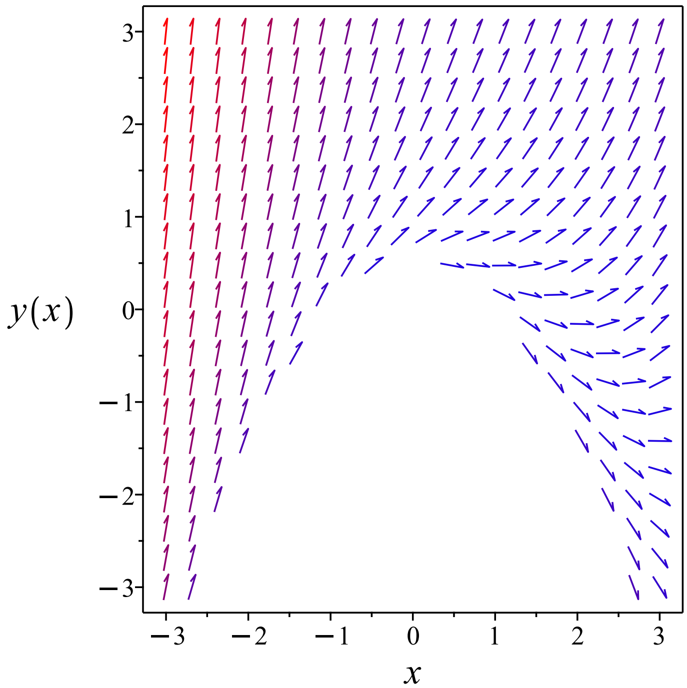


Figure 199: Slope field plot

Verification of solutions

$$y = \frac{4}{27}c_1^3 + \frac{4}{9}c_1^2 x + \frac{4}{9}c_1 x^2 + \frac{4}{27}x^3 - \frac{1}{2}x^2 + \frac{1}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -x, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      <- quadrature successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x)=(x^2+2*y(x)-1)^(2/3)-x,y(x), singsol=all)
```

$$x - \frac{3(x^2 + 2y(x) - 1)^{\frac{1}{3}}}{2} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.214 (sec). Leaf size: 40

```
DSolve[y'[x]==(x^2+2*y[x]-1)^(2/3)-x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{54}\left(8x^3 - 3(9 + 8c_1)x^2 + 24{c_1}^2x + 27 - 8{c_1}^3\right)$$

6.14 problem Exercise 12.14, page 103

Internal problem ID [4535]
Internal file name [`OUTPUT/4028_Sunday_June_05_2022_12_11_37_PM_3815086/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.14, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{xy' + y - x^2(1 + e^x)\, y^2 = 0}$$

6.14.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y(e^x x^2 y + y\, x^2 - 1)}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 105: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= y^2 x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^2 x} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{yx}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(\mathrm{e}^x x^2 y + y\,x^2 - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{y\,x^2} \\ S_y &= \frac{1}{y^2 x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 + \mathrm{e}^x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1 + \mathrm{e}^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = e^{R} + R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{xy} = e^{x} + x + c_1$$

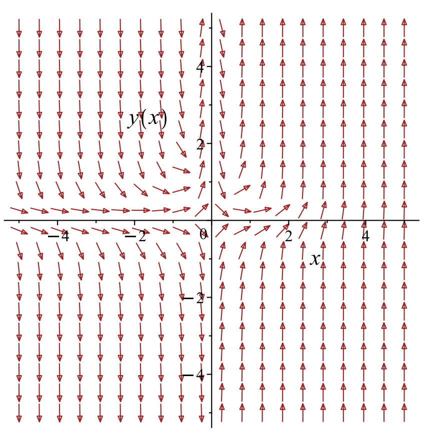
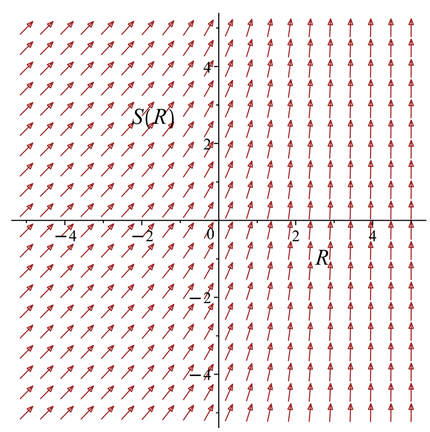
Which simplifies to

$$-\frac{1}{xy} = e^{x} + x + c_1$$

Which gives

$$y = -\frac{1}{x\left(e^{x} + x + c_1\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y\left(e^{x} x^{2} y + y\,x^{2} - 1\right)}{x}$ 	$R = x$ $S = -\frac{1}{yx}$	$\frac{dS}{dR} = 1 + e^{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{1}{x\left(\mathrm{e}^{x} + x + c_1\right)} \tag{1}$$

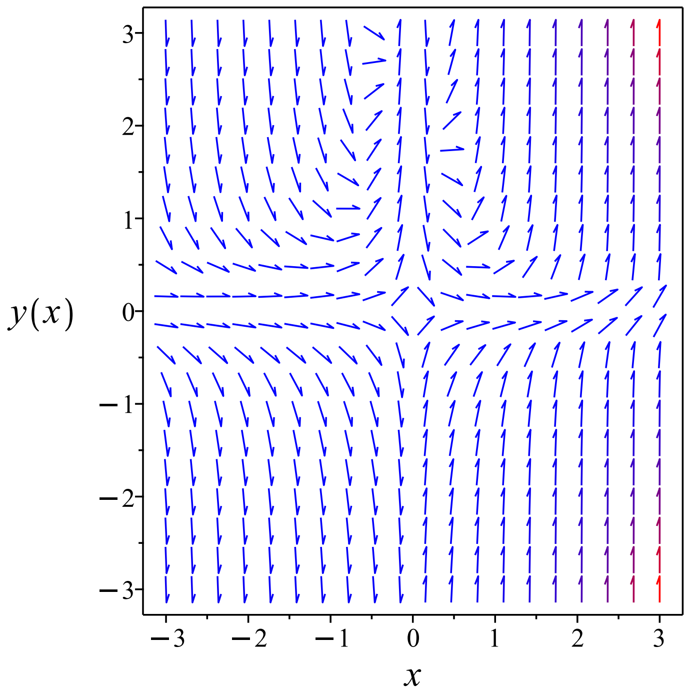


Figure 200: Slope field plot

Verification of solutions

$$y = -\frac{1}{x\left(\mathrm{e}^{x} + x + c_1\right)}$$

Verified OK.

6.14.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(\mathrm{e}^{x} x^{2} y + y\,x^{2} - 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x} y + \frac{\mathrm{e}^{x} x^{2} + x^{2}}{x} y^{2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= \frac{e^x x^2 + x^2}{x} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y' \frac{1}{y^2} = -\frac{1}{yx} + \frac{e^x x^2 + x^2}{x} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{w(x)}{x} + \frac{e^x x^2 + x^2}{x} \\ w' &= \frac{w}{x} - \frac{e^x x^2 + x^2}{x} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{1}{x}\\ q(x) &= -(1+\mathrm{e}^{x})\, x\end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = -(1+\mathrm{e}^{x})\, x$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{x}dx}\\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-(1+\mathrm{e}^{x})\, x\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(-(1+\mathrm{e}^{x})\, x\right)\\ \mathrm{d}\left(\frac{w}{x}\right) &= (-1-\mathrm{e}^{x})\, \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int -1-\mathrm{e}^{x}\, \mathrm{d}x\\ \frac{w}{x} &= -x-\mathrm{e}^{x}+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = x(-x-\mathrm{e}^{x}) + c_1 x$$

which simplifies to

$$w(x) = -x(\mathrm{e}^{x} - c_1 + x)$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = -x(\mathrm{e}^{x} - c_1 + x)$$

Or

$$y = -\frac{1}{x\left(e^{x} - c_1 + x\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{x\left(e^{x} - c_1 + x\right)} \tag{1}$$

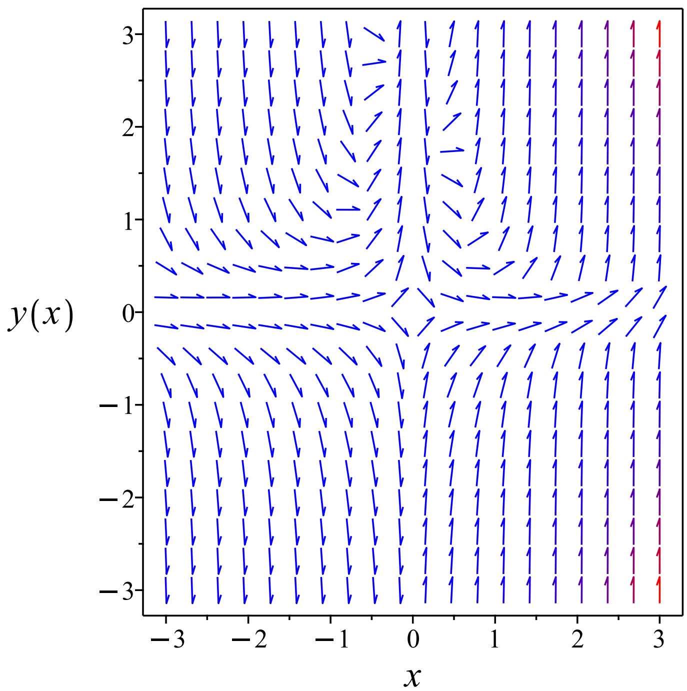


Figure 201: Slope field plot

Verification of solutions

$$y = -\frac{1}{x\left(e^{x} - c_1 + x\right)}$$

Verified OK.

6.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = \left(-y + x^2(1+\mathrm{e}^x)\,y^2\right)\mathrm{d}x$$
$$\left(y - x^2(1+\mathrm{e}^x)\,y^2\right)\mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = y - x^2(1+\mathrm{e}^x)\,y^2$$
$$N(x,y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y - x^2(1+\mathrm{e}^x)\, y^2\right)\\ &= -2\,\mathrm{e}^x x^2 y - 2y\, x^2 + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}\left(\left(1 - 2x^2(1+\mathrm{e}^x)\, y\right) - (1)\right)\\ &= -2x(1+\mathrm{e}^x)\, y\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -\frac{1}{y\,(\mathrm{e}^x x^2 y + y\, x^2 - 1)}\left((1) - \left(1 - 2x^2(1+\mathrm{e}^x)\, y\right)\right)\\ &= -\frac{2x^2(1+\mathrm{e}^x)}{\mathrm{e}^x x^2 y + y\, x^2 - 1}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - (1 - 2x^2(1+\mathrm{e}^x)\,y)}{x\,(y - x^2\,(1+\mathrm{e}^x)\,y^2) - y\,(x)} \\
&= -\frac{2}{yx}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{2}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{y^2x^2}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^2x^2}\left(y - x^2(1+\mathrm{e}^x)\,y^2\right) \\
&= \frac{-\mathrm{e}^x x^2 y - y\,x^2 + 1}{y\,x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2x^2}(x) \\
&= \frac{1}{y^2x}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-\mathrm{e}^x x^2 y - y\,x^2 + 1}{y\,x^2}\right) + \left(\frac{1}{y^2 x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-\mathrm{e}^x x^2 y - y\,x^2 + 1}{y\,x^2}\,\mathrm{d}x$$
$$\phi = \frac{-y\,\mathrm{e}^x x - y\,x^2 - 1}{xy} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{-\mathrm{e}^x x - x^2}{xy} - \frac{-y\,\mathrm{e}^x x - y\,x^2 - 1}{x\,y^2} + f'(y) \tag{4}$$
$$= \frac{1}{y^2 x} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2 x}$. Therefore equation (4) becomes

$$\frac{1}{y^2 x} = \frac{1}{y^2 x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-y\,\mathrm{e}^{x} x - y\,x^{2} - 1}{xy} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-y\,\mathrm{e}^{x} x - y\,x^{2} - 1}{xy}$$

The solution becomes

$$y = -\frac{1}{x\left(\mathrm{e}^{x} + x + c_1\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{x\left(\mathrm{e}^{x} + x + c_1\right)} \tag{1}$$

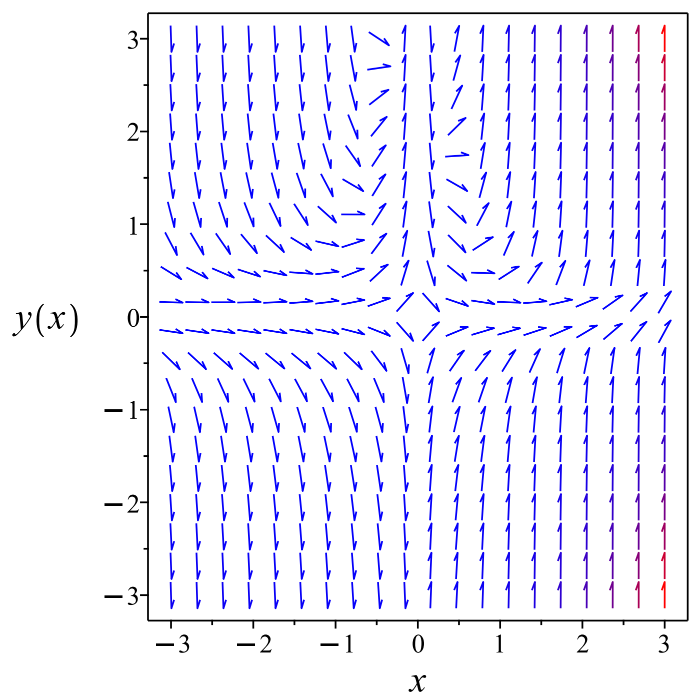


Figure 202: Slope field plot

Verification of solutions

$$y = -\frac{1}{x\left(e^{x}+x+c_1\right)}$$

Verified OK.

6.14.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= \frac{y(e^{x}x^{2}y+y\,x^{2}-1)}{x}
\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^{2}e^{x}x+y^{2}x-\frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x)+f_1(x)y+f_2(x)y^{2}$$

Shows that $f_0(x)=0$, $f_1(x)=-\frac{1}{x}$ and $f_2(x)=\frac{e^{x}x^{2}+x^{2}}{x}$. Let

$$\begin{aligned}
y &= \frac{-u'}{f_2 u}\\
&= \frac{-u'}{\frac{(e^{x}x^{2}+x^{2})u}{x}}
\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x)-\left(f_2'+f_1f_2\right)u'(x)+f_2^{2}f_0u(x)=0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{e^{x}x^{2}+2\,e^{x}x+2x}{x}-\frac{e^{x}x^{2}+x^{2}}{x^{2}}\\
f_1f_2 &= -\frac{e^{x}x^{2}+x^{2}}{x^{2}}\\
f_2^{2}f_0 &= 0
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\left(e^{x}x^{2}+x^{2}\right)u''(x)}{x}-\left(\frac{e^{x}x^{2}+2\,e^{x}x+2x}{x}-\frac{2(e^{x}x^{2}+x^{2})}{x^{2}}\right)u'(x)=0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 e^x + c_2 x + c_1$$

The above shows that

$$u'(x) = c_2(1 + e^x)$$

Using the above in (1) gives the solution

$$y = -\frac{c_2(1 + e^x)\, x}{(e^x x^2 + x^2)\,(c_2 e^x + c_2 x + c_1)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{x\,(e^x + x + c_3)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{x\,(e^x + x + c_3)} \tag{1}$$

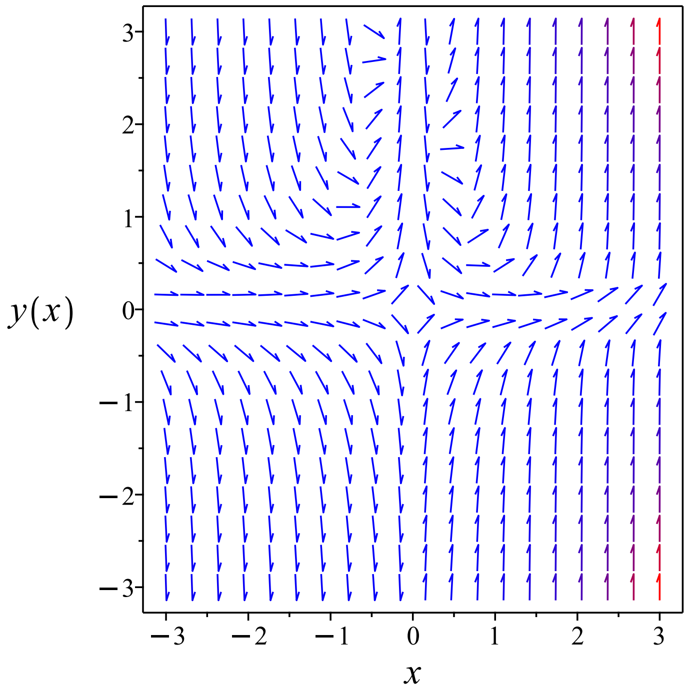


Figure 203: Slope field plot

Verification of solutions

$$y = -\frac{1}{x\left(e^{x} + x + c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(x*diff(y(x),x)+y(x)=x^2*(1+exp(x))*y(x)^2,y(x), singsol=all)
```

$$y(x) = -\frac{1}{(x + e^x - c_1)\, x}$$

✓ Solution by Mathematica

Time used: 0.249 (sec). Leaf size: 55

```
DSolve[x*y'[x]+y[x]==x^2*(1+exp[x])*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{-x \int_1^x (\exp(K[1]) + 1) dK[1] + c_1 x}$$
$$y(x) \to 0$$
$$y(x) \to -\frac{1}{x \int_1^x (\exp(K[1]) + 1) dK[1]}$$

6.15 problem Exercise 12.15, page 103

Internal problem ID [4536]
Internal file name [`OUTPUT/4029_Sunday_June_05_2022_12_11_47_PM_85802762/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.15, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2y - xy\ln(x) - 2x\ln(x)\, y' = 0}$$

6.15.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y(\ln(x)\, x - 2)}{2x\ln(x)} \end{aligned}$$

Where $f(x) = -\frac{\ln(x)x-2}{2x\ln(x)}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= -\frac{\ln(x)\,x-2}{2x\ln(x)}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{\ln(x)\,x-2}{2x\ln(x)}\,dx \\
\ln(y) &= -\frac{x}{2} + \ln(\ln(x)) + c_1 \\
y &= \mathrm{e}^{-\frac{x}{2}+\ln(\ln(x))+c_1} \\
&= c_1 \mathrm{e}^{-\frac{x}{2}+\ln(\ln(x))}
\end{aligned}
$$

Which simplifies to

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \ln(x)
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \ln(x) \tag{1}
$$

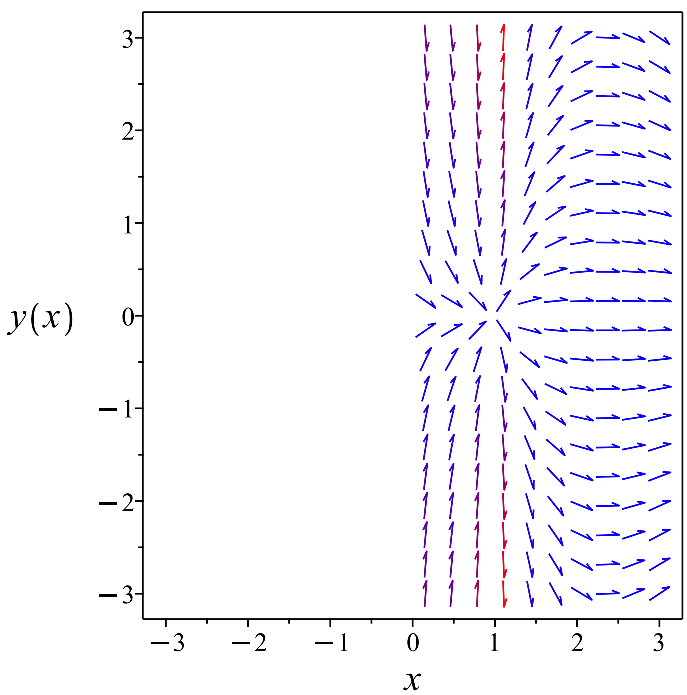


Figure 204: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x}{2}} \ln(x)$$

Verified OK.

6.15.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-\ln(x)x + 2}{2\ln(x)x}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{(-\ln(x)x + 2)y}{2\ln(x)x} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-\ln(x)x+2}{2\ln(x)x}dx}$$
$$= e^{\frac{x}{2} - \ln(\ln(x))}$$

Which simplifies to

$$\mu = \frac{e^{\frac{x}{2}}}{\ln(x)}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}\left(\frac{e^{\frac{x}{2}} y}{\ln(x)}\right) = 0$$

Integrating gives

$$\frac{e^{\frac{x}{2}} y}{\ln(x)} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{e^{\frac{x}{2}}}{\ln(x)}$ results in

$$y = c_1 e^{-\frac{x}{2}} \ln(x)$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-\frac{x}{2}} \ln(x) \tag{1}$$

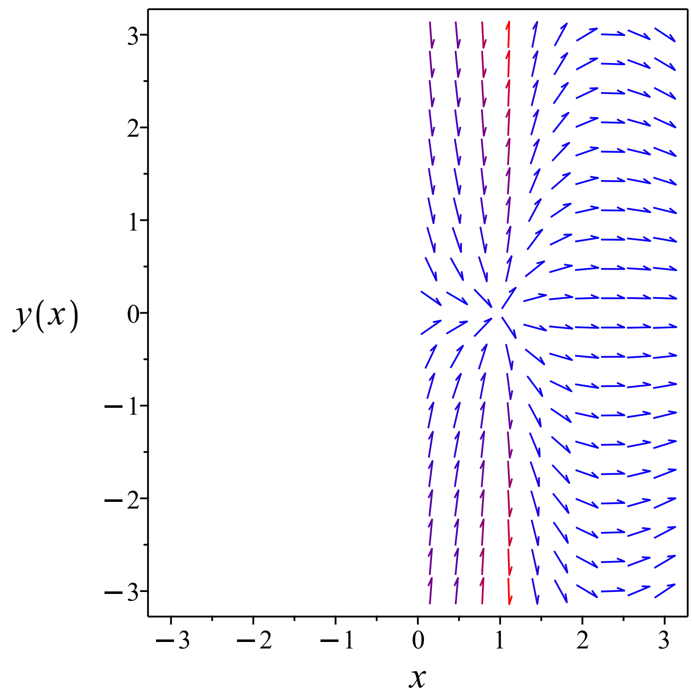


Figure 205: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x}{2}} \ln(x)$$

Verified OK.

6.15.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2u(x)\,x - x^2 u(x)\ln(x) - 2x\ln(x)\,(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(\ln(x)\,x + 2\ln(x) - 2)}{2\ln(x)\,x} \end{aligned}$$

Where $f(x) = -\frac{\ln(x)x+2\ln(x)-2}{2\ln(x)x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{\ln(x)\,x+2\ln(x)-2}{2\ln(x)\,x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{\ln(x)\,x+2\ln(x)-2}{2\ln(x)\,x}\,dx \\
\ln(u) &= -\frac{x}{2}-\ln(x)+\ln(\ln(x))+c_2 \\
u &= \mathrm{e}^{-\frac{x}{2}-\ln(x)+\ln(\ln(x))+c_2} \\
&= c_2\mathrm{e}^{-\frac{x}{2}-\ln(x)+\ln(\ln(x))}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2\mathrm{e}^{-\frac{x}{2}}\ln(x)}{x}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= c_2\mathrm{e}^{-\frac{x}{2}}\ln(x)
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_2\mathrm{e}^{-\frac{x}{2}}\ln(x) \tag{1}$$

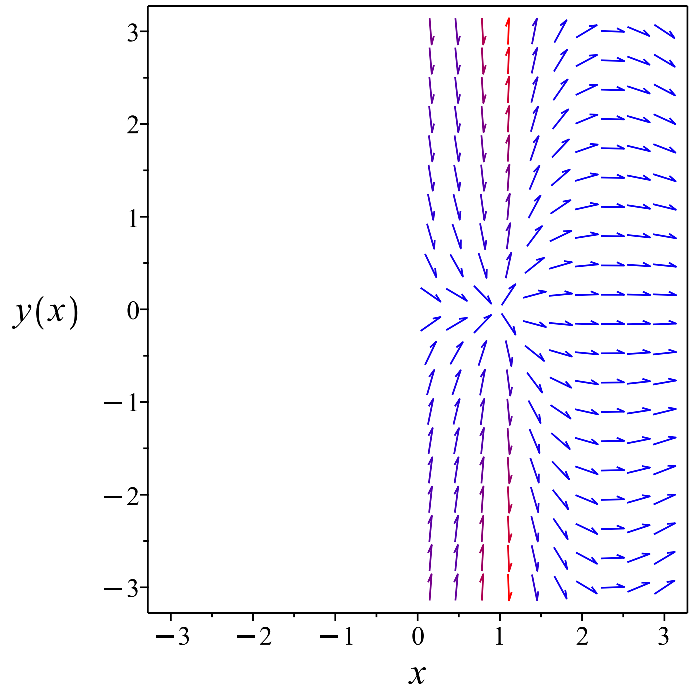


Figure 206: Slope field plot

Verification of solutions

$$y = c_2 \mathrm{e}^{-\frac{x}{2}} \ln (x)$$

Verified OK.

6.15.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(\ln (x) x - 2)}{2x \ln (x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 107: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{-\frac{x}{2}+\ln(\ln(x))} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{x}{2} + \ln(\ln(x))}} dy \end{aligned}$$

Which results in

$$S = \frac{e^{\frac{x}{2}} y}{\ln(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(\ln(x) x - 2)}{2x \ln(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{e^{\frac{x}{2}} y(\ln(x) x - 2)}{2x \ln(x)^2} \\ S_y &= \frac{e^{\frac{x}{2}}}{\ln(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{e^{\frac{x}{2}} y}{\ln(x)} = c_1$$

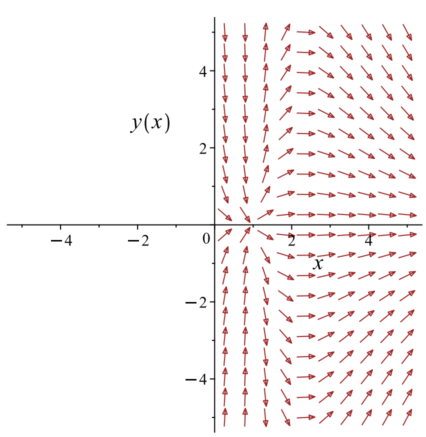
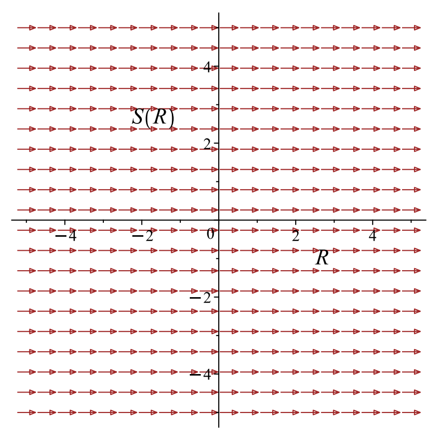
Which simplifies to

$$\frac{e^{\frac{x}{2}} y}{\ln(x)} = c_1$$

Which gives

$$y = c_1 e^{-\frac{x}{2}} \ln(x)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(\ln(x)x-2)}{2x\ln(x)}$ 	$R = x$ $S = \frac{e^{\frac{x}{2}} y}{\ln(x)}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 e^{-\frac{x}{2}} \ln(x) \tag{1}$$

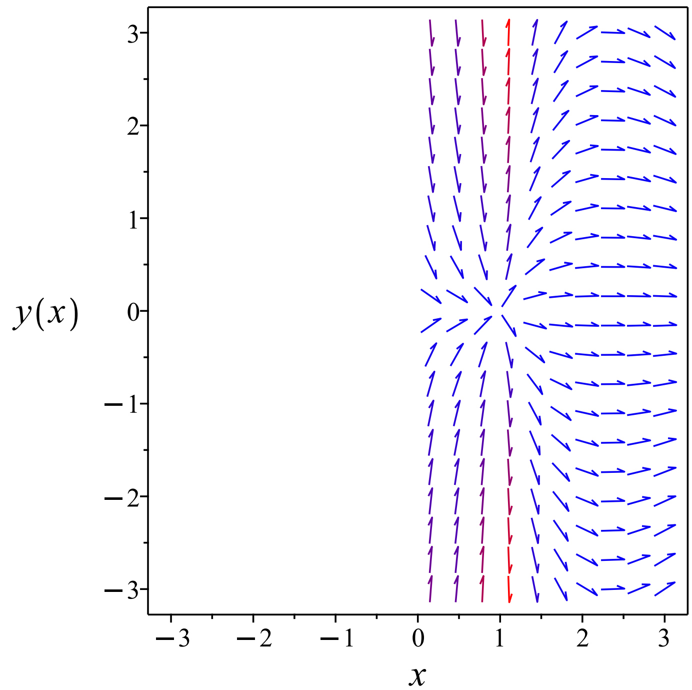


Figure 207: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x}{2}} \ln(x)$$

Verified OK.

6.15.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{2}{y}\right)\mathrm{d}y&=\left(\frac{\ln(x)\,x-2}{x\ln(x)}\right)\mathrm{d}x\\ \left(-\frac{\ln(x)\,x-2}{x\ln(x)}\right)\mathrm{d}x+\left(-\frac{2}{y}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{\ln(x)\,x-2}{x\ln(x)}\\ N(x,y)&=-\frac{2}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{\ln(x)\,x-2}{x\ln(x)}\right)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{2}{y}\right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{\ln(x)\,x - 2}{x\ln(x)}\,\mathrm{d}x$$

$$\phi = -x + 2\ln(\ln(x)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2}{y}$. Therefore equation (4) becomes

$$-\frac{2}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{2}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{2}{y}\right)\mathrm{d}y$$

$$f(y) = -2\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + 2\ln(\ln(x)) - 2\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + 2\ln(\ln(x)) - 2\ln(y)$$

The solution becomes

$$y = e^{-\frac{x}{2}-\frac{c_1}{2}} \ln(x)$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x}{2}-\frac{c_1}{2}} \ln(x) \tag{1}$$

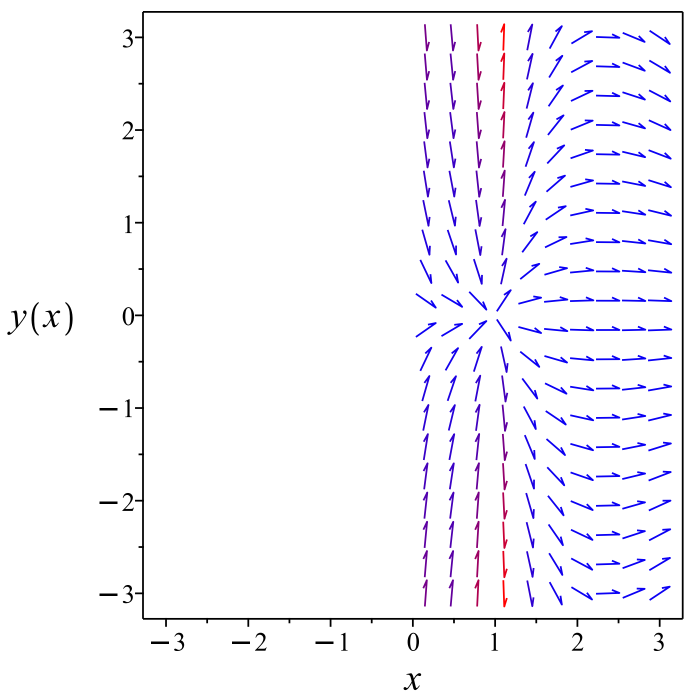


Figure 208: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}-\frac{c_1}{2}} \ln(x)$$

Verified OK.

6.15.6 Maple step by step solution

Let's solve

$2y - xy\ln(x) - 2x\ln(x)y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{\ln(x)x-2}{2x\ln(x)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -\frac{\ln(x)x-2}{2x\ln(x)}dx + c_1$

- Evaluate integral

  $\ln(y) = -\frac{x}{2} + \ln(\ln(x)) + c_1$

- Solve for y

  $y = e^{-\frac{x}{2}+c_1}\ln(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve((2*y(x)-x*y(x)*ln(x))-2*x*ln(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1e^{-\frac{x}{2}}\ln(x)$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 22

```
DSolve[(2*y[x]-x*y[x]*Log[x])-2*x*Log[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x/2} \log(x)$$
$$y(x) \to 0$$

6.16 problem Exercise 12.16, page 103

Internal problem ID [4537]
Internal file name [`OUTPUT/4030_Sunday_June_05_2022_12_11_56_PM_30189200/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.16, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + ya = k\,\mathrm{e}^{bx}}$$

6.16.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= a\\ q(x) &= k\,\mathrm{e}^{bx}\end{aligned}$$

Hence the ode is

$$y' + ya = k\,\mathrm{e}^{bx}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int a dx} \\
&= \mathrm{e}^{ax}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(k\,\mathrm{e}^{bx}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{ax} y) &= (\mathrm{e}^{ax})\left(k\,\mathrm{e}^{bx}\right) \\
\mathrm{d}(\mathrm{e}^{ax} y) &= \left(k\,\mathrm{e}^{x(a+b)}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{ax} y &= \int k\,\mathrm{e}^{x(a+b)}\,\mathrm{d}x \\
\mathrm{e}^{ax} y &= \frac{k\,\mathrm{e}^{x(a+b)}}{a+b} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{ax}$ results in

$$
y = \frac{\mathrm{e}^{-ax} k\,\mathrm{e}^{x(a+b)}}{a+b} + c_1 \mathrm{e}^{-ax}
$$

which simplifies to

$$
y = \frac{c_1(a+b)\,\mathrm{e}^{-ax} + k\,\mathrm{e}^{bx}}{a+b}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(a+b)\,\mathrm{e}^{-ax} + k\,\mathrm{e}^{bx}}{a+b} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(a+b)\,\mathrm{e}^{-ax} + k\,\mathrm{e}^{bx}}{a+b}
$$

Verified OK.

6.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -ya + k\,\mathrm{e}^{bx} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 110: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \mathrm{e}^{-ax} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-ax}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{ax} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -ya + k\,\mathrm{e}^{bx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= a\,\mathrm{e}^{ax} y \\ S_y &= \mathrm{e}^{ax} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = k\,\mathrm{e}^{x(a+b)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = k\,\mathrm{e}^{R(a+b)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{k\,\mathrm{e}^{R(a+b)}}{a+b} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{ax}y = \frac{k\,\mathrm{e}^{x(a+b)}}{a+b} + c_1$$

Which simplifies to

$$\mathrm{e}^{ax}y = \frac{k\,\mathrm{e}^{x(a+b)}}{a+b} + c_1$$

Which gives

$$y = \frac{\left(k\,\mathrm{e}^{x(a+b)} + ac_1 + c_1 b\right)\mathrm{e}^{-ax}}{a+b}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(k\,\mathrm{e}^{x(a+b)} + ac_1 + c_1 b\right)\mathrm{e}^{-ax}}{a+b} \tag{1}$$

Verification of solutions

$$y = \frac{\left(k\,\mathrm{e}^{x(a+b)} + ac_1 + c_1 b\right)\mathrm{e}^{-ax}}{a+b}$$

Verified OK.

6.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(-ya+k\,\mathrm{e}^{bx}\right)\mathrm{d}x\\ \left(ya-k\,\mathrm{e}^{bx}\right)\mathrm{d}x+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=ya-k\,\mathrm{e}^{bx}\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(ya - k\,\mathrm{e}^{bx}\right)\\ &= a\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((a) - (0))\\ &= a\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int a\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{ax}\\ &= \mathrm{e}^{ax}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{ax}\left(ya - k\,\mathrm{e}^{bx}\right)\\ &= \left(ya - k\,\mathrm{e}^{bx}\right)\mathrm{e}^{ax}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{ax}(1) \\
&= \mathrm{e}^{ax}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(ya - k\,\mathrm{e}^{bx}\right)\mathrm{e}^{ax}\right) + \left(\mathrm{e}^{ax}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \left(ya - k\,\mathrm{e}^{bx}\right)\mathrm{e}^{ax}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{-k\,\mathrm{e}^{x(a+b)} + y\,\mathrm{e}^{ax}(a+b)}{a+b} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{ax} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{ax}$. Therefore equation (4) becomes

$$
\mathrm{e}^{ax} = \mathrm{e}^{ax} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-k\,\mathrm{e}^{x(a+b)} + y\,\mathrm{e}^{ax}(a+b)}{a+b} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-k\,\mathrm{e}^{x(a+b)} + y\,\mathrm{e}^{ax}(a+b)}{a+b}$$

The solution becomes

$$y = \frac{\left(k\,\mathrm{e}^{x(a+b)} + ac_1 + c_1 b\right)\mathrm{e}^{-ax}}{a+b}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(k\,\mathrm{e}^{x(a+b)} + ac_1 + c_1 b\right)\mathrm{e}^{-ax}}{a+b} \tag{1}$$

Verification of solutions

$$y = \frac{\left(k\,\mathrm{e}^{x(a+b)} + ac_1 + c_1 b\right)\mathrm{e}^{-ax}}{a+b}$$

Verified OK.

6.16.4 Maple step by step solution

Let's solve

$y' + ya = k\,\mathrm{e}^{bx}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -ya + k\,\mathrm{e}^{bx}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + ya = k\,\mathrm{e}^{bx}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\,(y' + ya) = \mu(x)\,k\,\mathrm{e}^{bx}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\,(y' + ya) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \mu(x)\,a$$

- Solve to find the integrating factor

$$\mu(x) = \mathrm{e}^{ax}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,k\,\mathrm{e}^{bx} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \mu(x)\,k\,\mathrm{e}^{bx} dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x)k\,\mathrm{e}^{bx} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \mathrm{e}^{ax}$

$$y = \frac{\int k\,\mathrm{e}^{bx}\mathrm{e}^{ax} dx + c_1}{\mathrm{e}^{ax}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{k\,\mathrm{e}^{ax+bx}}{a+b} + c_1}{\mathrm{e}^{ax}}$$

- Simplify

$$y = \frac{\mathrm{e}^{-ax}\left(k\,\mathrm{e}^{x(a+b)} + (a+b)c_1\right)}{a+b}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x)+a*y(x)=k*exp(b*x),y(x), singsol=all)
```

$$y(x) = \frac{\left(k\,\mathrm{e}^{x(a+b)} + c_1(a+b)\right)\mathrm{e}^{-ax}}{a+b}$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 33

```
DSolve[y'[x]+a*y[x]==k*Exp[b*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ax}\left(ke^{x(a+b)} + c_1(a+b)\right)}{a+b}$$

6.17 problem Exercise 12.17, page 103

Internal problem ID [4538]
Internal file name [`OUTPUT/4031_Sunday_June_05_2022_12_12_05_PM_88804224/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.17, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _Riccati]
```

$$\boxed{y' - (x + y)^2 = 0}$$

6.17.1 Solving as homogeneousTypeC ode

Let

$$z = x + y \tag{1}$$

Then

$$z'(x) = 1 + y'$$

Therefore

$$y' = z'(x) - 1$$

Hence the given ode can now be written as

$$z'(x) - 1 = z^2$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{z^2+1} dz$$

$$x + c_1 = \arctan(z)$$

Replacing z back by its value from (1) then the above gives the solution as

$$y = -x + \tan(x + c_1)$$

$$y = -x + \tan(x + c_1)$$

Summary

The solution(s) found are the following

$$y = -x + \tan(x + c_1) \tag{1}$$

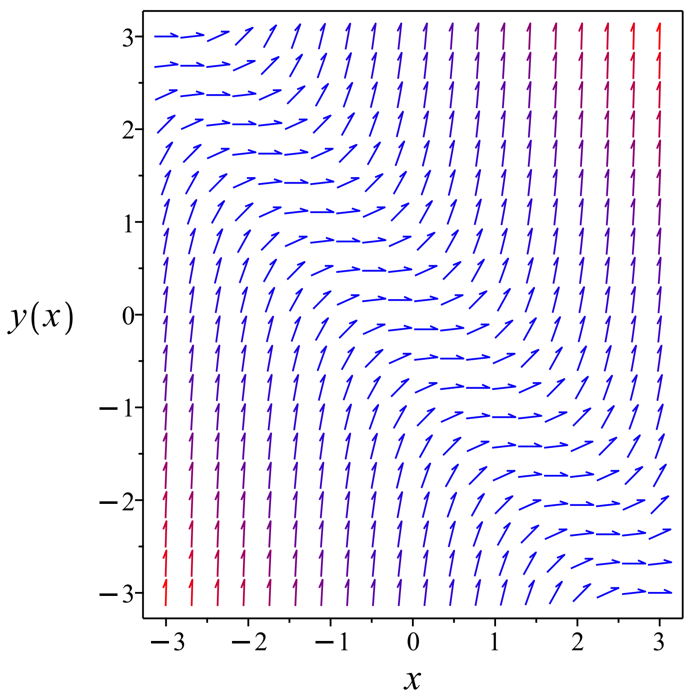


Figure 209: Slope field plot

Verification of solutions

$$y = -x + \tan(x + c_1)$$

Verified OK.

6.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = (x + y)^2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 113: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 1 \\ \eta(x, y) &= -1 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned} \frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{-1}{1} \\ &= -1 \end{aligned}$$

This is easily solved to give

$$y = -x + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = x + y$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{1} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= x \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (x+y)^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 1 \\ S_x &= 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{1+(x+y)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = \arctan(x+y) + c_1$$

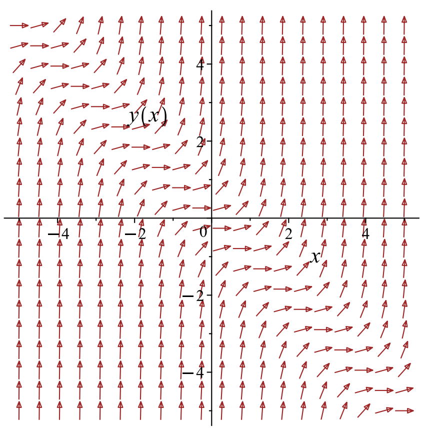
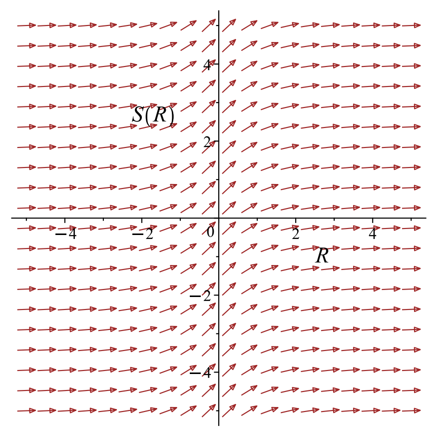
Which simplifies to

$$x = \arctan(x+y) + c_1$$

Which gives

$$y = -x - \tan(-x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (x+y)^2$ 	$R = x + y$ $S = x$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

Summary

The solution(s) found are the following

$$y = -x - \tan(-x + c_1) \tag{1}$$

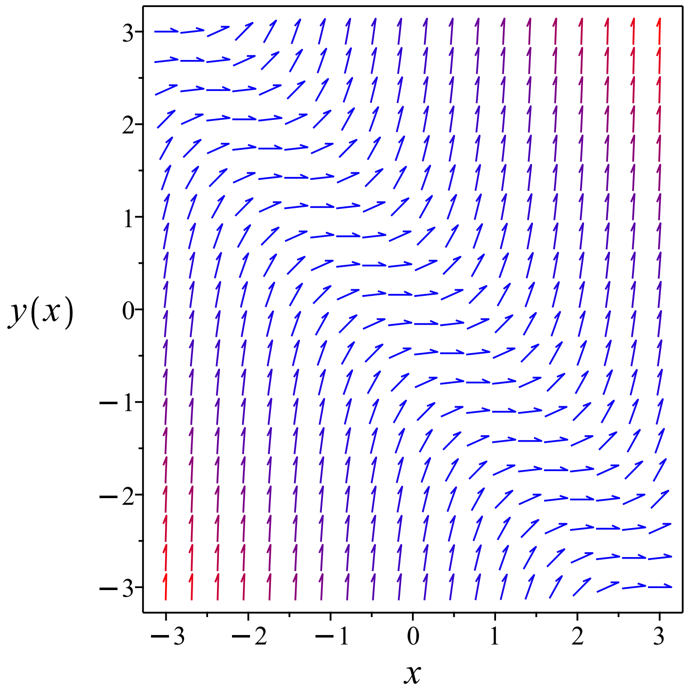


Figure 210: Slope field plot

Verification of solutions

$$y = -x - \tan(-x + c_1)$$

Verified OK.

6.17.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= (x + y)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 + 2xy + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2$, $f_1(x) = 2x$ and $f_2(x) = 1$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{u}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= 2x \\
f_2^2 f_0 &= x^2
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
u''(x) - 2xu'(x) + x^2 u(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = \mathrm{e}^{\frac{x^2}{2}} (\cos(x)\, c_1 + c_2 \sin(x))
$$

The above shows that

$$
u'(x) = ((c_1 x + c_2) \cos(x) + \sin(x)\, (c_2 x - c_1))\, \mathrm{e}^{\frac{x^2}{2}}
$$

Using the above in (1) gives the solution

$$
y = -\frac{(c_1 x + c_2) \cos(x) + \sin(x)\, (c_2 x - c_1)}{\cos(x)\, c_1 + c_2 \sin(x)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{(-c_3 x - 1) \cos(x) - \sin(x)\, (-c_3 + x)}{\cos(x)\, c_3 + \sin(x)}
$$

Summary
The solution(s) found are the following

$$y = \frac{(-c_3 x - 1)\cos(x) - \sin(x)(-c_3 + x)}{\cos(x) c_3 + \sin(x)} \tag{1}$$

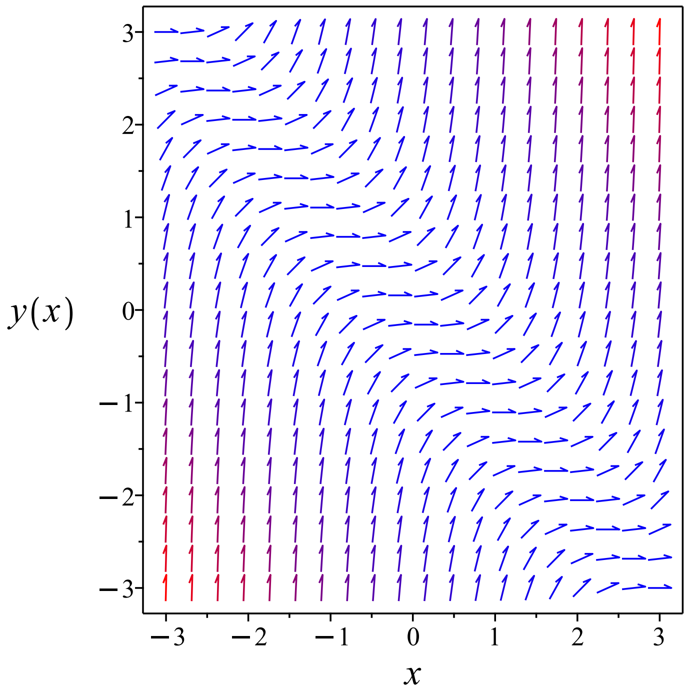


Figure 211: Slope field plot

Verification of solutions

$$y = \frac{(-c_3 x - 1)\cos(x) - \sin(x)(-c_3 + x)}{\cos(x) c_3 + \sin(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -1, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)=(x+y(x))^2,y(x), singsol=all)
```

$$y(x) = -x - \tan(-x + c_1)$$

✓ Solution by Mathematica

Time used: 0.472 (sec). Leaf size: 14

```
DSolve[y'[x]==(x+y[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x + \tan(x + c_1)$$

6.18 problem Exercise 12.18, page 103

Internal problem ID [4539]
Internal file name [`OUTPUT/4032_Sunday_June_05_2022_12_12_14_PM_84163132/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.18, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{y' + 8x^3y^3 + 2xy = 0}$$

6.18.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -8y^3x^3 - 2xy$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 115: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^3 \mathrm{e}^{2x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^3 e^{2x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{e^{-2x^2}}{2y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -8y^3x^3 - 2xy$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x\, e^{-2x^2}}{y^2} \\ S_y &= \frac{e^{-2x^2}}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -8x^3 e^{-2x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -8R^3 e^{-2R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = (2R^2 + 1)\,\mathrm{e}^{-2R^2} + c_1 \tag{4}$$

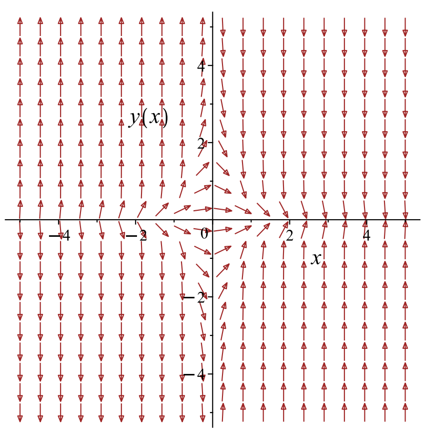
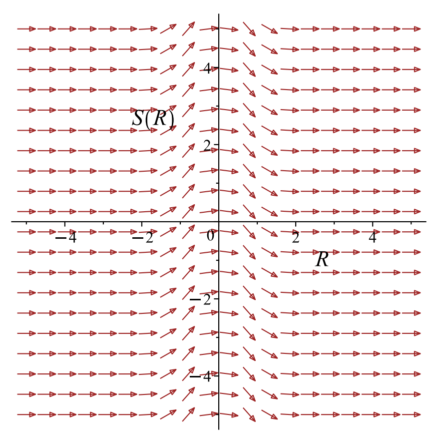
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\mathrm{e}^{-2x^2}}{2y^2} = (2x^2 + 1)\,\mathrm{e}^{-2x^2} + c_1$$

Which simplifies to

$$-\frac{\mathrm{e}^{-2x^2}}{2y^2} = (2x^2 + 1)\,\mathrm{e}^{-2x^2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -8y^3x^3 - 2xy$ 	$R = x$ $S = -\frac{\mathrm{e}^{-2x^2}}{2y^2}$	$\frac{dS}{dR} = -8R^3\mathrm{e}^{-2R^2}$ 

Summary

The solution(s) found are the following

$$-\frac{\mathrm{e}^{-2x^2}}{2y^2} = (2x^2 + 1)\,\mathrm{e}^{-2x^2} + c_1 \tag{1}$$

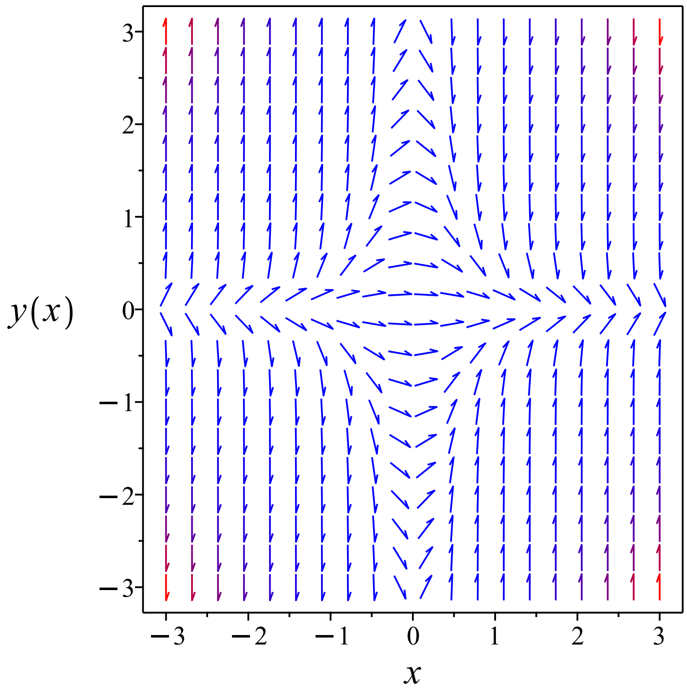


Figure 212: Slope field plot

Verification of solutions

$$-\frac{e^{-2x^2}}{2y^2} = \left(2x^2+1\right)e^{-2x^2} + c_1$$

Verified OK.

6.18.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -8y^3x^3 - 2xy \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -2xy - 8x^3y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -2x \\
f_1(x) &= -8x^3 \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y' \frac{1}{y^3} = -\frac{2x}{y^2} - 8x^3 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2} \qquad (5)
\end{aligned}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{2}{y^3} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= -2w(x)\, x - 8x^3 \\
w' &= 16x^3 + 4xw \qquad (7)
\end{aligned}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -4x \\
q(x) &= 16x^3
\end{aligned}
$$

Hence the ode is

$$
w'(x) - 4w(x)\, x = 16x^3
$$

The integrating factor μ is

$$\mu = e^{\int -4x dx}$$
$$= e^{-2x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)\left(16x^3\right)$$
$$\frac{d}{dx}\left(e^{-2x^2} w\right) = \left(e^{-2x^2}\right)\left(16x^3\right)$$
$$d\left(e^{-2x^2} w\right) = \left(16x^3 e^{-2x^2}\right) dx$$

Integrating gives

$$e^{-2x^2} w = \int 16x^3 e^{-2x^2} dx$$
$$e^{-2x^2} w = -2\left(2x^2+1\right) e^{-2x^2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-2x^2}$ results in

$$w(x) = -2 e^{2x^2}\left(2x^2+1\right) e^{-2x^2} + c_1 e^{2x^2}$$

which simplifies to

$$w(x) = -4x^2 - 2 + c_1 e^{2x^2}$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = -4x^2 - 2 + c_1 e^{2x^2}$$

Solving for y gives

$$y(x) = \frac{1}{\sqrt{-4x^2 - 2 + c_1 e^{2x^2}}}$$
$$y(x) = -\frac{1}{\sqrt{-4x^2 - 2 + c_1 e^{2x^2}}}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{-4x^2 - 2 + c_1 e^{2x^2}}} \tag{1}$$
$$y = -\frac{1}{\sqrt{-4x^2 - 2 + c_1 e^{2x^2}}} \tag{2}$$

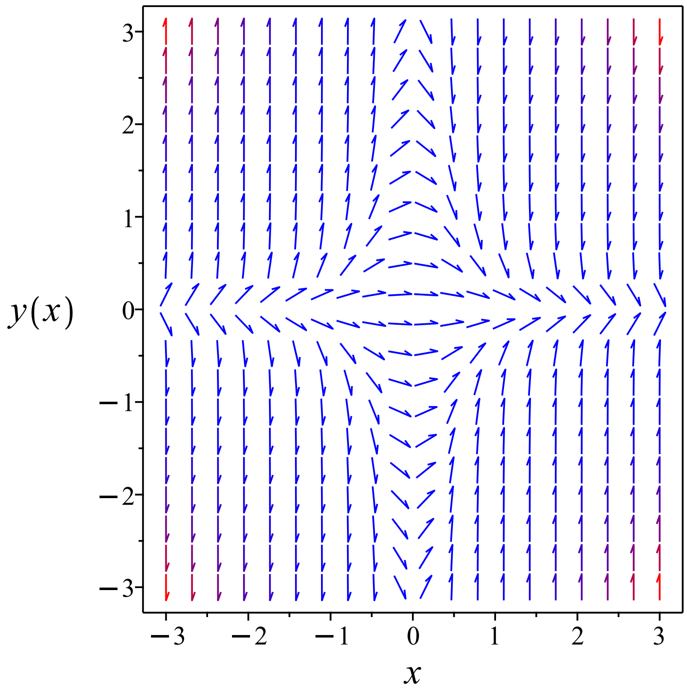


Figure 213: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{-4x^2 - 2 + c_1 e^{2x^2}}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{-4x^2 - 2 + c_1 e^{2x^2}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(diff(y(x),x)+8*x^3*y(x)^3+2*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{e^{2x^2}c_1 - 4x^2 - 2}}$$
$$y(x) = -\frac{1}{\sqrt{e^{2x^2}c_1 - 4x^2 - 2}}$$

✓ Solution by Mathematica

Time used: 7.034 (sec). Leaf size: 58

```
DSolve[y'[x]+8*x^3*y[x]^3+2*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{-4x^2 + c_1 e^{2x^2} - 2}}$$
$$y(x) \to \frac{1}{\sqrt{-4x^2 + c_1 e^{2x^2} - 2}}$$
$$y(x) \to 0$$

6.19 problem Exercise 12.19, page 103

Internal problem ID [4540]
Internal file name [`OUTPUT/4033_Sunday_June_05_2022_12_12_27_PM_21590560/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.19, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[NONE]`

Unable to solve or complete the solution.

$$\left(xy\sqrt{x^2-y^2}+x\right)y'-y+x^2\sqrt{x^2-y^2}=0$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5`[0, 1/((x^2-y^2)^(1/2)*y+1)*(x^2-y^2)^(1/2)]
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 34

```
dsolve((x*y(x)*sqrt(x^2-y(x)^2)+x)*diff(y(x),x)=y(x)-x^2*sqrt(x^2-y(x)^2),y(x), singsol=all)
```

$$\frac{y(x)^2}{2} + \arctan\left(\frac{y(x)}{\sqrt{x^2 - y(x)^2}}\right) + \frac{x^2}{2} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 1.772 (sec). Leaf size: 44

```
DSolve[(x*y[x]*Sqrt[x^2-y[x]^2]+x)*y'[x]==y[x]-x^2*Sqrt[x^2-y[x]^2],y[x],x,IncludeSingularSo
```

$$\text{Solve}\left[-\arctan\left(\frac{\sqrt{x^2 - y(x)^2}}{y(x)}\right) + \frac{x^2}{2} + \frac{y(x)^2}{2} = c_1, y(x)\right]$$

6.20 problem Exercise 12.20, page 103

Internal problem ID [4541]
Internal file name [`OUTPUT/4034_Sunday_June_05_2022_12_12_39_PM_31615192/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.20, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + ya = b\sin(kx)}$$

6.20.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= a \\ q(x) &= b\sin(kx) \end{aligned}$$

Hence the ode is

$$y' + ya = b\sin(kx)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int a dx} \\
&= e^{ax}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\,(b \sin(kx)) \\
\frac{d}{dx}(e^{ax} y) &= (e^{ax})\,(b \sin(kx)) \\
d(e^{ax} y) &= (b \sin(kx)\, e^{ax})\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{ax} y &= \int b \sin(kx)\, e^{ax}\, dx \\
e^{ax} y &= b\left(-\frac{k\, e^{ax} \cos(kx)}{a^2+k^2} + \frac{a\, e^{ax} \sin(kx)}{a^2+k^2}\right) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{ax}$ results in

$$
y = e^{-ax} b\left(-\frac{k\, e^{ax} \cos(kx)}{a^2+k^2} + \frac{a\, e^{ax} \sin(kx)}{a^2+k^2}\right) + c_1 e^{-ax}
$$

which simplifies to

$$
y = \frac{c_1(a^2+k^2)\, e^{-ax} + b(-k\cos(kx) + \sin(kx)\, a)}{a^2+k^2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(a^2+k^2)\, e^{-ax} + b(-k\cos(kx) + \sin(kx)\, a)}{a^2+k^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(a^2+k^2)\, e^{-ax} + b(-k\cos(kx) + \sin(kx)\, a)}{a^2+k^2}
$$

Verified OK.

6.20.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -ya + b\sin(kx) \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 117: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \mathrm{e}^{-ax}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\mathrm{e}^{-ax}} dy\end{aligned}$$

Which results in

$$S = \mathrm{e}^{ax} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -ya + b\sin(kx)$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= a\,\mathrm{e}^{ax} y\\ S_y &= \mathrm{e}^{ax}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = b \sin(kx) e^{ax} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = b \sin(kR) e^{aR}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{c_1(a^2 + k^2) + e^{aR} b(\sin(kR) a - k \cos(kR))}{a^2 + k^2} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{ax} y = \frac{c_1(a^2 + k^2) + e^{ax} b(-k \cos(kx) + \sin(kx) a)}{a^2 + k^2}$$

Which simplifies to

$$e^{ax} y = \frac{c_1(a^2 + k^2) + e^{ax} b(-k \cos(kx) + \sin(kx) a)}{a^2 + k^2}$$

Which gives

$$y = \frac{e^{-ax}(b \sin(kx) a e^{ax} - bk \cos(kx) e^{ax} + c_1 a^2 + c_1 k^2)}{a^2 + k^2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-ax}(b \sin(kx) a e^{ax} - bk \cos(kx) e^{ax} + c_1 a^2 + c_1 k^2)}{a^2 + k^2} \tag{1}$$

Verification of solutions

$$y = \frac{e^{-ax}(b \sin(kx) a e^{ax} - bk \cos(kx) e^{ax} + c_1 a^2 + c_1 k^2)}{a^2 + k^2}$$

Verified OK.

6.20.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=(-ya+b\sin(kx))\,\mathrm{d}x\\ (ya-b\sin(kx))\,\mathrm{d}x+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=ya-b\sin(kx)\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(ya - b\sin(kx)) \\ &= a\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((a) - (0)) \\ &= a\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int a \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{ax} \\ &= \mathrm{e}^{ax}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{ax}(ya - b\sin(kx)) \\ &= (ya - b\sin(kx))\,\mathrm{e}^{ax}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{ax}(1) \\
&= \mathrm{e}^{ax}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left((ya - b\sin(kx))\,\mathrm{e}^{ax}\right) + \left(\mathrm{e}^{ax}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int (ya - b\sin(kx))\,\mathrm{e}^{ax}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{(bk\cos(kx) - ab\sin(kx) + y(a^2+k^2))\,\mathrm{e}^{ax}}{a^2+k^2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{ax} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{ax}$. Therefore equation (4) becomes

$$
\mathrm{e}^{ax} = \mathrm{e}^{ax} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(bk\cos(kx) - ab\sin(kx) + y(a^2+k^2))\,\mathrm{e}^{ax}}{a^2+k^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(bk\cos(kx) - ab\sin(kx) + y(a^2+k^2))\,\mathrm{e}^{ax}}{a^2+k^2}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-ax}(b\sin(kx)\,a\,\mathrm{e}^{ax} - bk\cos(kx)\,\mathrm{e}^{ax} + c_1a^2 + c_1k^2)}{a^2+k^2}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-ax}(b\sin(kx)\,a\,\mathrm{e}^{ax} - bk\cos(kx)\,\mathrm{e}^{ax} + c_1a^2 + c_1k^2)}{a^2+k^2} \tag{1}$$

Verification of solutions

$$y = \frac{\mathrm{e}^{-ax}(b\sin(kx)\,a\,\mathrm{e}^{ax} - bk\cos(kx)\,\mathrm{e}^{ax} + c_1a^2 + c_1k^2)}{a^2+k^2}$$

Verified OK.

6.20.4 Maple step by step solution

Let's solve

$y' + ya = b\sin(kx)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -ya + b\sin(kx)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' + ya = b \sin(kx)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' + ya) = \mu(x) b \sin(kx)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x)(y' + ya) = \mu'(x) y + \mu(x) y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x) a$

- Solve to find the integrating factor

  $\mu(x) = e^{ax}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \mu(x) b \sin(kx) dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x) y = \int \mu(x) b \sin(kx) dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) b \sin(kx) dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{ax}$

  $y = \frac{\int b \sin(kx) e^{ax} dx + c_1}{e^{ax}}$

- Evaluate the integrals on the rhs

  $y = \frac{b\left(-\frac{k e^{ax} \cos(kx)}{a^2+k^2} + \frac{a e^{ax} \sin(kx)}{a^2+k^2}\right) + c_1}{e^{ax}}$

- Simplify

  $y = \frac{c_1(a^2+k^2) e^{-ax} + b(-k \cos(kx) + \sin(kx) a)}{a^2+k^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve(diff(y(x),x)+a*y(x)=b*sin(k*x),y(x), singsol=all)
```

$$y(x) = \frac{e^{-ax}c_1(a^2+k^2) + b(\sin(kx)\,a - k\cos(kx))}{a^2+k^2}$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 40

```
DSolve[y'[x]+a*y[x]==b*Sin[k*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{b(a\sin(kx) - k\cos(kx))}{a^2+k^2} + c_1e^{-ax}$$

6.21 problem Exercise 12.21, page 103

Internal problem ID [4542]
Internal file name [`OUTPUT/4035_Sunday_June_05_2022_12_12_48_PM_3010686/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.21, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{xy' - y^2 = -1}$$

6.21.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2-1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y^2 - 1$. Integrating both sides gives

$$\frac{1}{y^2-1}\,dy = \frac{1}{x}\,dx$$

$$\int \frac{1}{y^2-1}\,dy = \int \frac{1}{x}\,dx$$

$$-\operatorname{arctanh}(y) = \ln(x) + c_1$$

Which results in

$$y = -\tanh(\ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = -\tanh(\ln(x) + c_1) \tag{1}$$

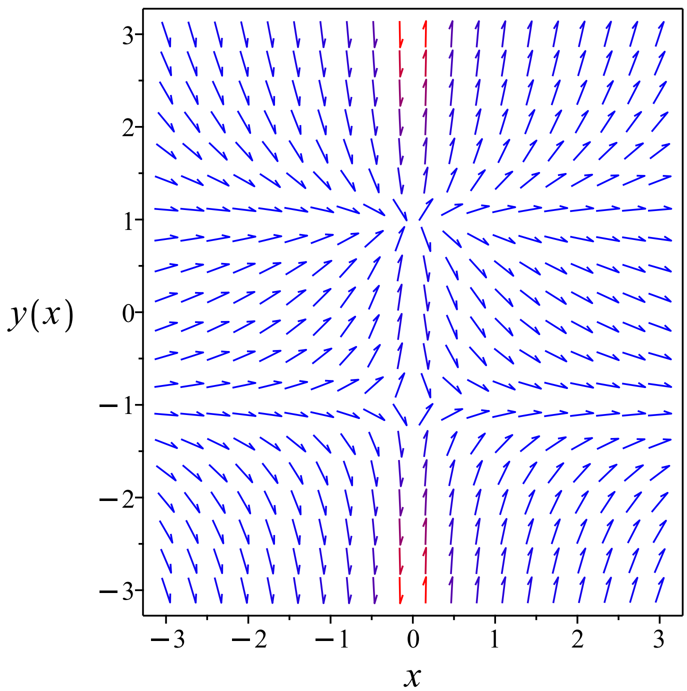


Figure 214: Slope field plot

Verification of solutions

$$y = -\tanh(\ln(x) + c_1)$$

Verified OK.

6.21.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2 - 1}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 120: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = x$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$S = \int \frac{1}{\xi} dx$$
$$= \int \frac{1}{x} dx$$

Which results in

$$S = \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2 - 1}{x}$$

Evaluating all the partial derivatives gives

$$R_x = 0$$
$$R_y = 1$$
$$S_x = \frac{1}{x}$$
$$S_y = 0$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2 - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\operatorname{arctanh}(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = -\operatorname{arctanh}(y) + c_1$$

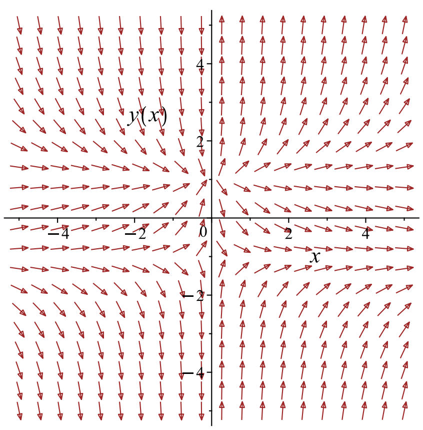
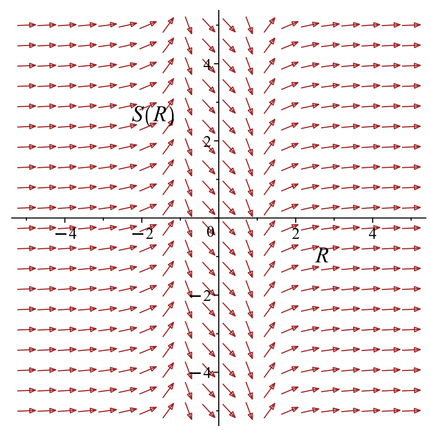
Which simplifies to

$$\ln(x) = -\operatorname{arctanh}(y) + c_1$$

Which gives

$$y = \tanh(-\ln(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2-1}{x}$ 	$R = y$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{1}{R^2-1}$ 

Summary

The solution(s) found are the following

$$y = \tanh(-\ln(x) + c_1) \quad (1)$$

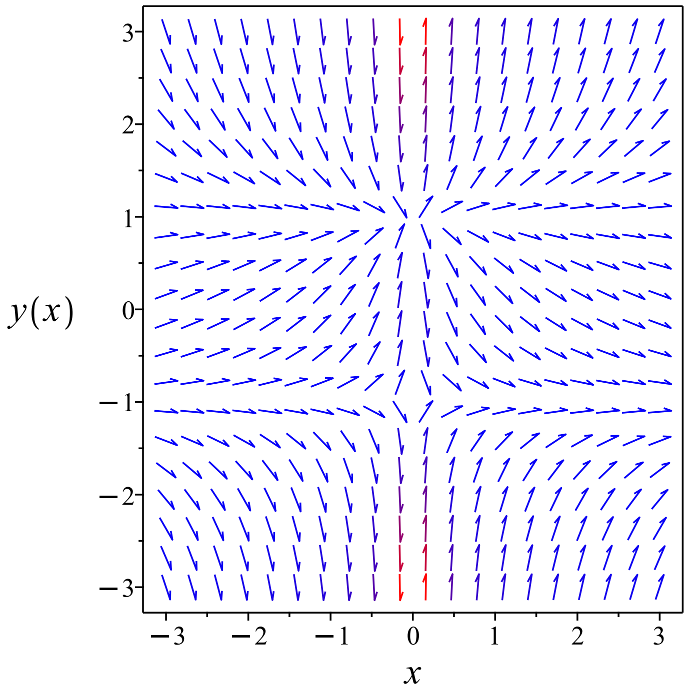


Figure 215: Slope field plot

Verification of solutions

$$y = \tanh(-\ln(x) + c_1)$$

Verified OK.

6.21.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^2 - 1}\right) \mathrm{d}y = \left(\frac{1}{x}\right) \mathrm{d}x$$
$$\left(-\frac{1}{x}\right) \mathrm{d}x + \left(\frac{1}{y^2 - 1}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = \frac{1}{y^2 - 1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y^2 - 1}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\frac{1}{x} \,\mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2-1}$. Therefore equation (4) becomes

$$\frac{1}{y^2 - 1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2 - 1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\frac{1}{y^2 - 1}\right) \mathrm{d}y$$
$$f(y) = -\operatorname{arctanh}(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \operatorname{arctanh}(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \operatorname{arctanh}(y)$$

The solution becomes

$$y = -\tanh(\ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = -\tanh(\ln(x) + c_1) \tag{1}$$

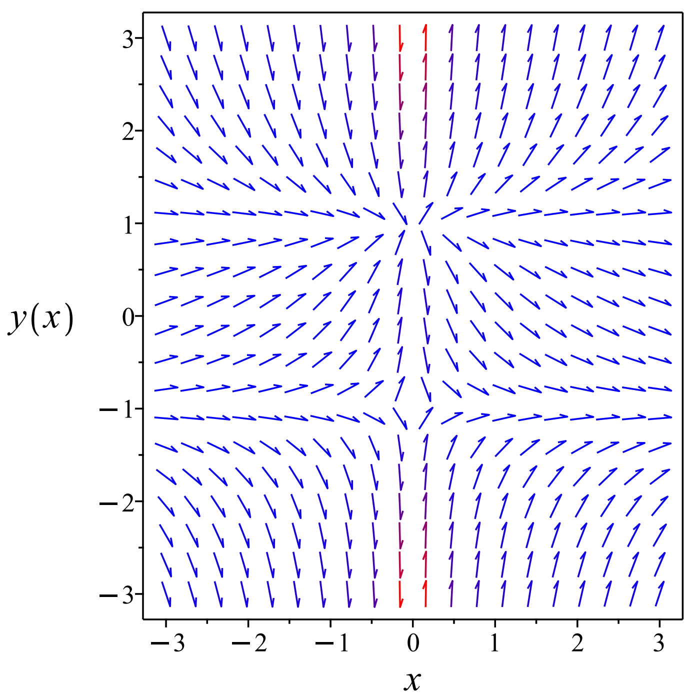


Figure 216: Slope field plot

Verification of solutions

$$y = -\tanh(\ln(x) + c_1)$$

Verified OK.

6.21.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2-1}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{x} - \frac{1}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{1}{x}$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{x^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -\frac{1}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x} + \frac{u'(x)}{x^2} - \frac{u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_1 x^2 + c_2}{x}$$

The above shows that

$$u'(x) = \frac{c_1 x^2 - c_2}{x^2}$$

Using the above in (1) gives the solution

$$y = -\frac{c_1 x^2 - c_2}{c_1 x^2 + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3 x^2 + 1}{c_3 x^2 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-c_3 x^2 + 1}{c_3 x^2 + 1} \tag{1}$$

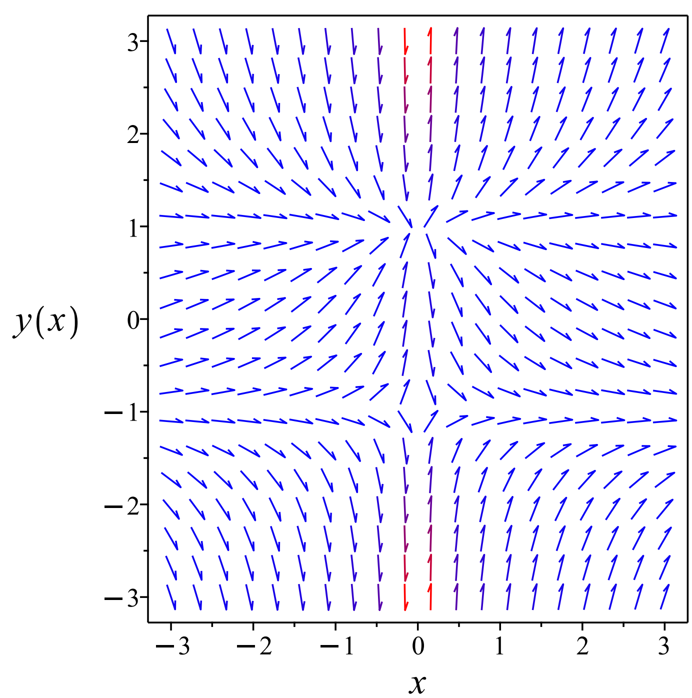


Figure 217: Slope field plot

Verification of solutions

$$y = \frac{-c_3 x^2 + 1}{c_3 x^2 + 1}$$

Verified OK.

6.21.5 Maple step by step solution

Let's solve

$x y' - y^2 = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-1} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2-1} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $-\mathrm{arctanh}(y) = \ln(x) + c_1$

- Solve for y

  $y = -\tanh(\ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(x*diff(y(x),x)-y(x)^2+1=0,y(x), singsol=all)
```

$$y(x) = -\tanh(\ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.486 (sec). Leaf size: 43

```
DSolve[x*y'[x]-y[x]^2+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} y(x) &\to \frac{1 - e^{2c_1}x^2}{1 + e^{2c_1}x^2} \\ y(x) &\to -1 \\ y(x) &\to 1 \end{aligned}$$

6.22 problem Exercise 12.22, page 103

Internal problem ID [4543]
Internal file name [OUTPUT/4036_Sunday_June_05_2022_12_12_57_PM_19149974/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.22, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{(y^2 + a\sin(x))\, y' = \cos(x)}$$

6.22.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d}x + N(x, y) \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(y^2 + a \sin(x)) \mathrm{d}y &= (\cos(x)) \mathrm{d}x \\
(-\cos(x)) \mathrm{d}x + (y^2 + a \sin(x)) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -\cos(x) \\
N(x, y) &= y^2 + a \sin(x)
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\cos(x)) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^2 + a \sin(x)) \\
&= a \cos(x)
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{y^2 + a\sin(x)}((0) - (a\cos(x))) \\
&= -\frac{a\cos(x)}{y^2 + a\sin(x)}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\sec(x)((a\cos(x)) - (0)) \\
&= -a
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B\,\mathrm{d}y} \\
&= e^{\int -a\,\mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-ya} \\
&= \mathrm{e}^{-ya}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-ya}(-\cos(x)) \\
&= -\cos(x)\,\mathrm{e}^{-ya}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-ya}\left(y^2 + a\sin(x)\right) \\
&= \left(y^2 + a\sin(x)\right)\mathrm{e}^{-ya}
\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-\cos\left(x\right)\mathrm{e}^{-ya}\right) + \left(\left(y^{2} + a\sin\left(x\right)\right)\mathrm{e}^{-ya}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\cos\left(x\right)\mathrm{e}^{-ya}\,\mathrm{d}x$$
$$\phi = -\sin\left(x\right)\mathrm{e}^{-ya} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin\left(x\right)a\,\mathrm{e}^{-ya} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \left(y^{2} + a\sin\left(x\right)\right)\mathrm{e}^{-ya}$. Therefore equation (4) becomes

$$\left(y^{2} + a\sin\left(x\right)\right)\mathrm{e}^{-ya} = \sin\left(x\right)a\,\mathrm{e}^{-ya} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \mathrm{e}^{-ya}y^{2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\mathrm{e}^{-ya}y^{2}\right)\mathrm{d}y$$
$$f(y) = -\frac{\left(y^{2}a^{2} + 2ya + 2\right)\mathrm{e}^{-ya}}{a^{3}} + c_{1}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(x)\,\mathrm{e}^{-ya} - \frac{(y^2a^2 + 2ya + 2)\,\mathrm{e}^{-ya}}{a^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(x)\,\mathrm{e}^{-ya} - \frac{(y^2a^2 + 2ya + 2)\,\mathrm{e}^{-ya}}{a^3}$$

Summary

The solution(s) found are the following

$$-\sin(x)\,\mathrm{e}^{-ya} - \frac{(y^2a^2 + 2ya + 2)\,\mathrm{e}^{-ya}}{a^3} = c_1 \tag{1}$$

Verification of solutions

$$-\sin(x)\,\mathrm{e}^{-ya} - \frac{(y^2a^2 + 2ya + 2)\,\mathrm{e}^{-ya}}{a^3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve((y(x)^2+a*sin(x))*diff(y(x),x)=cos(x),y(x), singsol=all)
```

$$\frac{\left(-\sin\left(x\right)a^{3}-y(x)^{2}a^{2}-2ay(x)-2\right)\mathrm{e}^{-ay(x)}+c_{1}a^{3}}{a^{3}}=0$$

✓ Solution by Mathematica

Time used: 0.194 (sec). Leaf size: 45

```
DSolve[(y[x]^2+a*Sin[x])*y'[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\sin(x)\left(-e^{-ay(x)}\right)-\frac{e^{-ay(x)}(a^{2}y(x)^{2}+2ay(x)+2)}{a^{3}}=c_{1},y(x)\right]$$

6.23 problem Exercise 12.23, page 103

Internal problem ID [4544]
Internal file name [`OUTPUT/4037_Sunday_June_05_2022_12_13_06_PM_62939991/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.23, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{xy' - x\,\mathrm{e}^{\frac{y}{x}} - y = x}$$

6.23.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)\,x + u(x)) - x\,\mathrm{e}^{u(x)} - u(x)\,x = x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\mathrm{e}^{u}+1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \mathrm{e}^{u} + 1$. Integrating both sides gives

$$\frac{1}{\mathrm{e}^{u}+1}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{\mathrm{e}^{u}+1}\,du = \int \frac{1}{x}\,dx$$

$$-\ln\left(\mathrm{e}^{u}+1\right) + \ln\left(\mathrm{e}^{u}\right) = \ln(x) + c_2$$

Raising both side to exponential gives

$$e^{-\ln(e^u+1)+\ln(e^u)} = e^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{e^u}{e^u+1} = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x\ln\left(-\frac{c_3 x}{c_3 x - 1}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\ln\left(-\frac{c_3 x}{c_3 x - 1}\right) \tag{1}$$

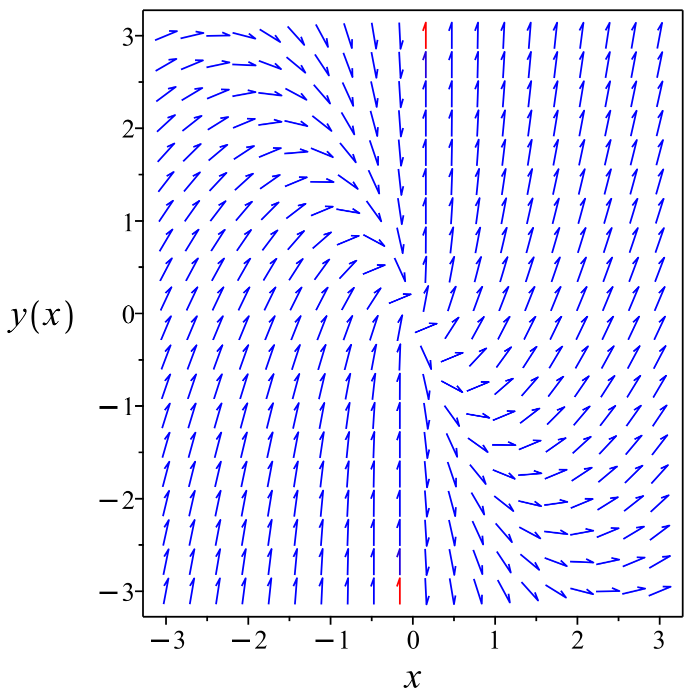


Figure 218: Slope field plot

Verification of solutions

$$y = x\ln\left(-\frac{c_3 x}{c_3 x - 1}\right)$$

Verified OK.

6.23.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x\,\mathrm{e}^{\frac{y}{x}} + x + y}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{\left(x\,\mathrm{e}^{\frac{y}{x}} + x + y\right)(b_3 - a_2)}{x} - \frac{\left(x\,\mathrm{e}^{\frac{y}{x}} + x + y\right)^2 a_3}{x^2} \\
&- \left(\frac{\mathrm{e}^{\frac{y}{x}} - \frac{y\,\mathrm{e}^{\frac{y}{x}}}{x} + 1}{x} - \frac{x\,\mathrm{e}^{\frac{y}{x}} + x + y}{x^2}\right)(x a_2 + y a_3 + a_1) \\
&- \frac{\left(\mathrm{e}^{\frac{y}{x}} + 1\right)(x b_2 + y b_3 + b_1)}{x} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{\mathrm{e}^{\frac{2y}{x}} x^2 a_3 + \mathrm{e}^{\frac{y}{x}} x^2 a_2 + 2\,\mathrm{e}^{\frac{y}{x}} x^2 a_3 + \mathrm{e}^{\frac{y}{x}} x^2 b_2 - \mathrm{e}^{\frac{y}{x}} x^2 b_3 - \mathrm{e}^{\frac{y}{x}} x y a_2 + 2\,\mathrm{e}^{\frac{y}{x}} x y a_3 + \mathrm{e}^{\frac{y}{x}} x y b_3 - \mathrm{e}^{\frac{y}{x}} y^2 a_3 + \mathrm{e}^{\frac{y}{x}} x b_1 - \mathrm{e}^{\frac{y}{x}}}{x^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-\mathrm{e}^{\frac{2y}{x}} x^2 a_3 - \mathrm{e}^{\frac{y}{x}} x^2 a_2 - 2\,\mathrm{e}^{\frac{y}{x}} x^2 a_3 - \mathrm{e}^{\frac{y}{x}} x^2 b_2 + \mathrm{e}^{\frac{y}{x}} x^2 b_3 + \mathrm{e}^{\frac{y}{x}} x y a_2 - 2\,\mathrm{e}^{\frac{y}{x}} x y a_3 - \mathrm{e}^{\frac{y}{x}} x y b_3 \\
&+ \mathrm{e}^{\frac{y}{x}} y^2 a_3 - \mathrm{e}^{\frac{y}{x}} x b_1 + \mathrm{e}^{\frac{y}{x}} y a_1 - x^2 a_2 - x^2 a_3 + x^2 b_3 - 2 x y a_3 - x b_1 + y a_1 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$-e^{\frac{2y}{x}}x^2a_3-e^{\frac{y}{x}}x^2a_2-2\,e^{\frac{y}{x}}x^2a_3-e^{\frac{y}{x}}x^2b_2+e^{\frac{y}{x}}x^2b_3+e^{\frac{y}{x}}xya_2-2\,e^{\frac{y}{x}}xya_3-e^{\frac{y}{x}}xyb_3 \\ +e^{\frac{y}{x}}y^2a_3-e^{\frac{y}{x}}xb_1+e^{\frac{y}{x}}ya_1-x^2a_2-x^2a_3+x^2b_3-2xya_3-xb_1+ya_1=0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, e^{\frac{y}{x}}, e^{\frac{2y}{x}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, e^{\frac{y}{x}} = v_3, e^{\frac{2y}{x}} = v_4\right\}$$

The above PDE (6E) now becomes

$$-v_3v_1^2a_2+v_3v_1v_2a_2-2v_3v_1^2a_3-v_4v_1^2a_3-2v_3v_1v_2a_3+v_3v_2^2a_3-v_3v_1^2b_2+v_3v_1^2b_3 \\ -v_3v_1v_2b_3+v_3v_2a_1-v_1^2a_2-v_1^2a_3-2v_1v_2a_3-v_3v_1b_1+v_1^2b_3+v_2a_1-v_1b_1=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$(-a_2-2a_3-b_2+b_3)\,v_1^2v_3-v_4v_1^2a_3+(-a_2-a_3+b_3)\,v_1^2 \\ +(a_2-2a_3-b_3)\,v_1v_2v_3-2v_1v_2a_3-v_3v_1b_1-v_1b_1+v_3v_2^2a_3+v_3v_2a_1+v_2a_1=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ a_3 &= 0 \\ -2a_3 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -a_2-a_3+b_3 &= 0 \\ a_2-2a_3-b_3 &= 0 \\ -a_2-2a_3-b_2+b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{x\,\mathrm{e}^{\frac{y}{x}} + x + y}{x}\right)(x) \\
&= -x\,\mathrm{e}^{\frac{y}{x}} - x \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-x\,\mathrm{e}^{\frac{y}{x}} - x} dy
\end{aligned}$$

Which results in

$$S = \ln\left(e^{\frac{y}{x}}+1\right)-\ln\left(e^{\frac{y}{x}}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x\,e^{\frac{y}{x}}+x+y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{y}{x^2\left(e^{\frac{y}{x}}+1\right)}\\
S_y &= -\frac{1}{x\left(e^{\frac{y}{x}}+1\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R)+c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(e^{\frac{y}{x}}+1\right)x-y}{x} = -\ln(x)+c_1$$

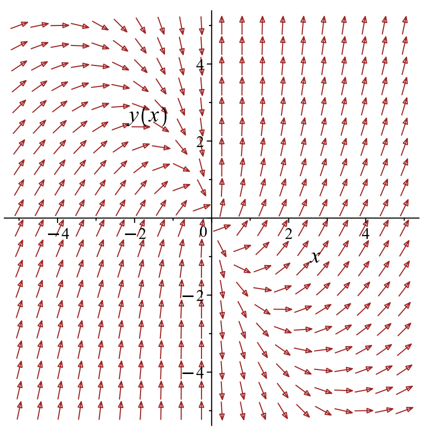
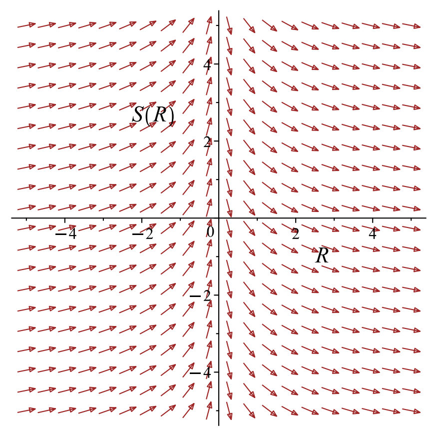
Which simplifies to

$$\frac{\ln\left(e^{\frac{y}{x}}+1\right)x - y}{x} = -\ln\left(x\right)+c_1$$

Which gives

$$y = \ln\left(x\right)x + x\ln\left(\frac{1}{-x+e^{c_1}}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{x\,e^{\frac{y}{x}}+x+y}{x}$
</td><td>$R = x$
$S = \frac{\ln\left(e^{\frac{y}{x}}+1\right)x - y}{x}$</td><td>$\frac{dS}{dR} = -\frac{1}{R}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \ln\left(x\right)x + x\ln\left(\frac{1}{-x+e^{c_1}}\right) \tag{1}$$

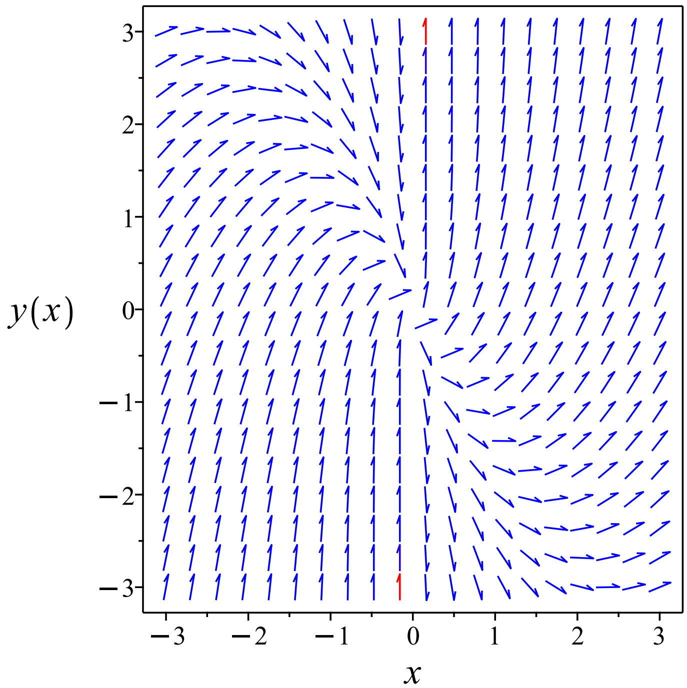


Figure 219: Slope field plot

Verification of solutions

$$y = \ln(x)\, x + x \ln\left(\frac{1}{-x + e^{c_1}}\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve(x*diff(y(x),x)=x*exp(y(x)/x)+x+y(x),y(x), singsol=all)
```

$$y(x) = \left(\ln\left(-\frac{x}{x\,e^{c_1}-1}\right)+c_1\right)x$$

✓ Solution by Mathematica

Time used: 4.512 (sec). Leaf size: 38

```
DSolve[x*y'[x]==x*Exp[y[x]/x]+x+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\log\left(\frac{1}{2}\left(-1+\tanh\left(\frac{1}{2}(-\log(x)-c_1)\right)\right)\right)$$
$$y(x) \to i\pi x$$

6.24 problem Exercise 12.24, page 103

Internal problem ID [4545]
Internal file name [`OUTPUT/4038_Sunday_June_05_2022_12_13_16_PM_43256292/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.24, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cos(x) = e^{-\sin(x)}}$$

6.24.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cos(x)$$
$$q(x) = e^{-\sin(x)}$$

Hence the ode is

$$y' + y\cos(x) = e^{-\sin(x)}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int \cos(x)dx}$$
$$= \mathrm{e}^{\sin(x)}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\mathrm{e}^{-\sin(x)}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\sin(x)} y\right) = \left(\mathrm{e}^{\sin(x)}\right)\left(\mathrm{e}^{-\sin(x)}\right)$$
$$\mathrm{d}\left(\mathrm{e}^{\sin(x)} y\right) = \mathrm{d}x$$

Integrating gives

$$\mathrm{e}^{\sin(x)} y = \int \mathrm{d}x$$
$$\mathrm{e}^{\sin(x)} y = x + c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\sin(x)}$ results in

$$y = \mathrm{e}^{-\sin(x)} x + c_1 \mathrm{e}^{-\sin(x)}$$

which simplifies to

$$y = \mathrm{e}^{-\sin(x)}(x + c_1)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\sin(x)}(x + c_1) \tag{1}$$

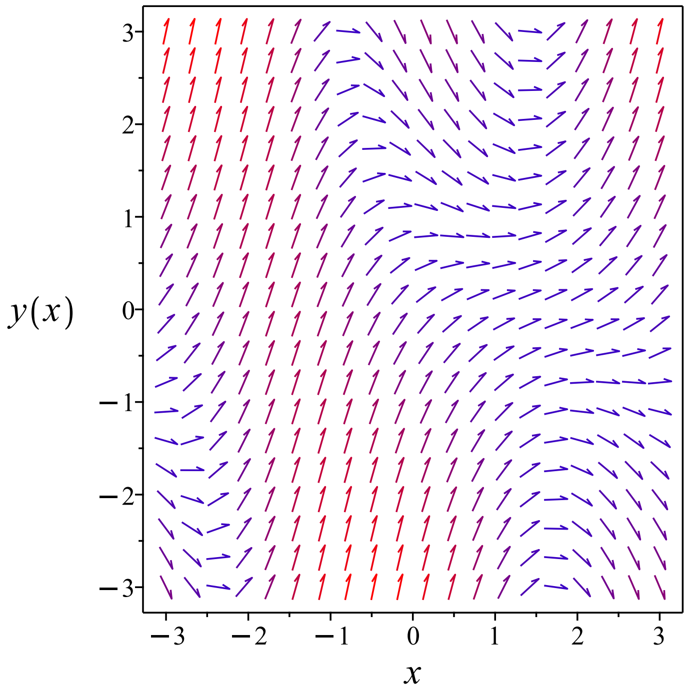


Figure 220: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\sin(x)}(x + c_1)$$

Verified OK.

6.24.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y\cos(x) + \mathrm{e}^{-\sin(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 123: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{-\sin(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\sin(x)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\sin(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -y \cos(x) + \mathrm{e}^{-\sin(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x) \mathrm{e}^{\sin(x)} y \\ S_y &= \mathrm{e}^{\sin(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\sin(x)} y = x + c_1$$

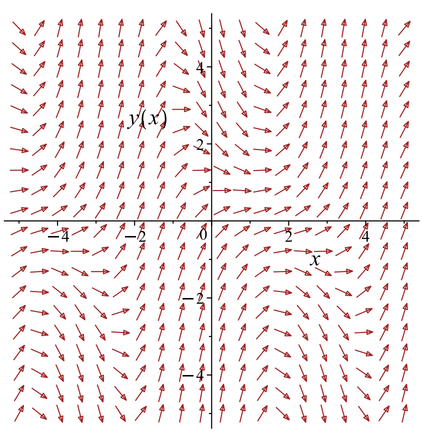
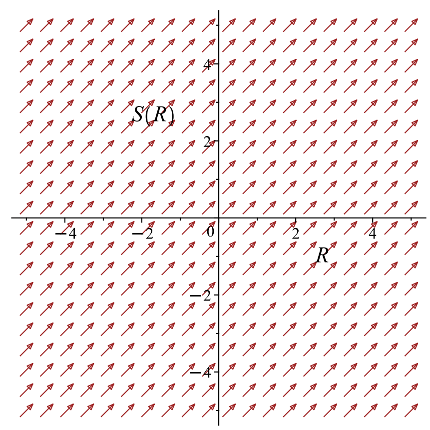
Which simplifies to

$$e^{\sin(x)} y = x + c_1$$

Which gives

$$y = e^{-\sin(x)} (x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y \cos(x) + e^{-\sin(x)}$ 	$R = x$ $S = e^{\sin(x)} y$	$\frac{dS}{dR} = 1$ 

Summary
The solution(s) found are the following

$$y = e^{-\sin(x)} (x + c_1) \tag{1}$$

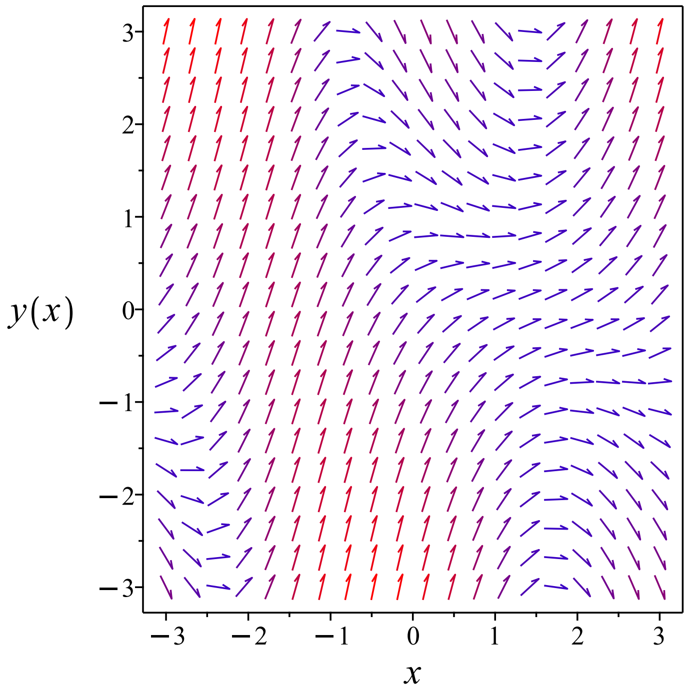


Figure 221: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\sin(x)}(x + c_1)$$

Verified OK.

6.24.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-y \cos(x) + \mathrm{e}^{-\sin(x)}\right) \mathrm{d}x$$
$$\left(y \cos(x) - \mathrm{e}^{-\sin(x)}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y \cos(x) - \mathrm{e}^{-\sin(x)}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(y \cos(x) - \mathrm{e}^{-\sin(x)}\right)$$
$$= \cos(x)$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\cos(x)) - (0)) \\ &= \cos(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \cos(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\sin(x)} \\ &= \mathrm{e}^{\sin(x)} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{\sin(x)}\left(y\cos(x) - \mathrm{e}^{-\sin(x)}\right) \\ &= \cos(x)\,\mathrm{e}^{\sin(x)} y - 1 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{\sin(x)}(1) \\ &= \mathrm{e}^{\sin(x)} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\cos(x)\,\mathrm{e}^{\sin(x)} y - 1\right) + \left(\mathrm{e}^{\sin(x)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \cos(x) \, \mathrm{e}^{\sin(x)} y - 1 \, \mathrm{d}x \\
\phi &= -x + \mathrm{e}^{\sin(x)} y + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{\sin(x)} + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\sin(x)}$. Therefore equation (4) becomes

$$
\mathrm{e}^{\sin(x)} = \mathrm{e}^{\sin(x)} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = -x + \mathrm{e}^{\sin(x)} y + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -x + \mathrm{e}^{\sin(x)} y
$$

The solution becomes

$$
y = \mathrm{e}^{-\sin(x)} (x + c_1)
$$

Summary

The solution(s) found are the following

$$y = e^{-\sin(x)}(x + c_1) \tag{1}$$

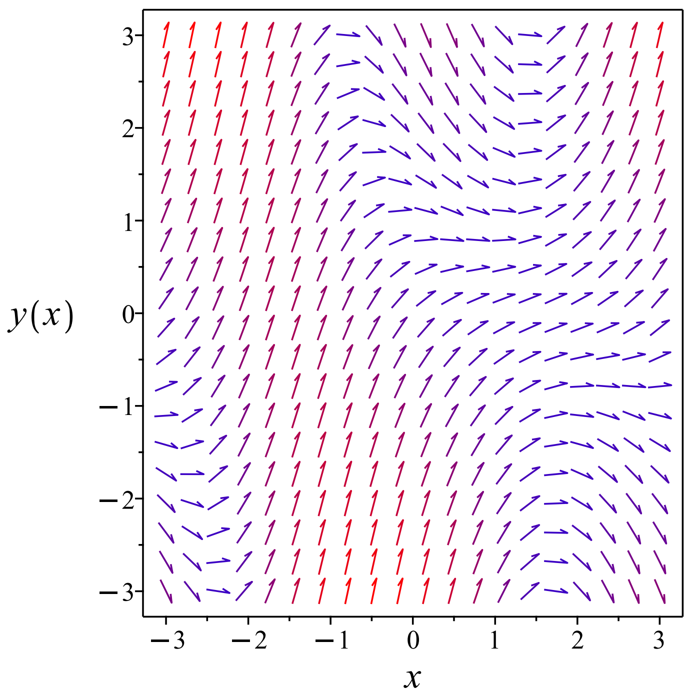


Figure 222: Slope field plot

Verification of solutions

$$y = e^{-\sin(x)}(x + c_1)$$

Verified OK.

6.24.4 Maple step by step solution

Let's solve

$y' + y\cos(x) = e^{-\sin(x)}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y\cos(x) + e^{-\sin(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y\cos(x) = e^{-\sin(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' + y\cos(x)) = \mu(x) e^{-\sin(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x)(y' + y\cos(x)) = \mu'(x) y + \mu(x) y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\cos(x)$

- Solve to find the integrating factor

  $\mu(x) = e^{\sin(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \mu(x) e^{-\sin(x)} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x) y = \int \mu(x) e^{-\sin(x)} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) e^{-\sin(x)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{\sin(x)}$

  $y = \frac{\int e^{-\sin(x)} e^{\sin(x)} dx + c_1}{e^{\sin(x)}}$

- Evaluate the integrals on the rhs

  $y = \frac{x + c_1}{e^{\sin(x)}}$

- Simplify

  $y = e^{-\sin(x)}(x + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)+y(x)*cos(x)=exp(-sin(x)),y(x), singsol=all)
```

$$y(x) = (x + c_1)\,\mathrm{e}^{-\sin(x)}$$

✓ Solution by Mathematica

Time used: 0.123 (sec). Leaf size: 16

```
DSolve[y'[x]+y[x]*Cos[x]==Exp[-Sin[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x + c_1)e^{-\sin(x)}$$

6.25 problem Exercise 12.25, page 103

Internal problem ID [4546]
Internal file name [`OUTPUT/4039_Sunday_June_05_2022_12_13_25_PM_95119008/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.25, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{xy' - y(\ln(xy) - 1) = 0}$$

6.25.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y(\ln(xy) - 1)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{y(\ln(xy) - 1)(b_3 - a_2)}{x} - \frac{y^2(\ln(xy) - 1)^2 a_3}{x^2} \\ &- \left(\frac{y}{x^2} - \frac{y(\ln(xy) - 1)}{x^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(\frac{\ln(xy) - 1}{x} + \frac{1}{x}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\ln(xy)^2 y^2 a_3 + \ln(xy) x^2 b_2 - 3\ln(xy) y^2 a_3 + \ln(xy) x b_1 - \ln(xy) y a_1 - b_2 x^2 + x y a_2 + x y b_3 + 3y^2 a_3 +}{x^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-\ln(xy)^2 y^2 a_3 - \ln(xy) x^2 b_2 + 3\ln(xy) y^2 a_3 - \ln(xy) x b_1 \\ &+ \ln(xy) y a_1 + b_2 x^2 - x y a_2 - x y b_3 - 3y^2 a_3 - 2y a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \ln(xy)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \ln(xy) = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-v_3^2 v_2^2 a_3 + 3 v_3 v_2^2 a_3 - v_3 v_1^2 b_2 + v_3 v_2 a_1 - v_1 v_2 a_2 \\ &- 3 v_2^2 a_3 - v_3 v_1 b_1 + b_2 v_1^2 - v_1 v_2 b_3 - 2 v_2 a_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}&-v_3v_1^2b_2 + b_2v_1^2 + (-a_2 - b_3)\,v_1v_2 - v_3v_1b_1 - v_3^2v_2^2a_3\\ &+ 3v_3v_2^2a_3 - 3v_2^2a_3 + v_3v_2a_1 - 2v_2a_1 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0\\ b_2 &= 0\\ -2a_1 &= 0\\ -3a_3 &= 0\\ -a_3 &= 0\\ 3a_3 &= 0\\ -b_1 &= 0\\ -b_2 &= 0\\ -a_2 - b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0\\ a_2 &= -b_3\\ a_3 &= 0\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y(\ln(xy)-1)}{x}\right)(-x) \\
&= y\ln(xy) \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{y\ln(xy)} dy
\end{aligned}
$$

Which results in

$$
S = \ln(\ln(xy))
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{y(\ln(xy)-1)}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{x\left(\ln\left(x\right)+\ln\left(y\right)\right)} \\
S_y &= \frac{1}{y\left(\ln\left(x\right)+\ln\left(y\right)\right)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{\ln\left(xy\right)}{x\left(\ln\left(x\right)+\ln\left(y\right)\right)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln\left(R\right)+c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln\left(\ln\left(x\right)+\ln\left(y\right)\right) = \ln\left(x\right)+c_1
$$

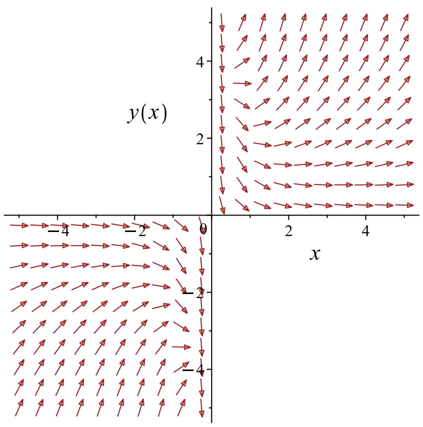
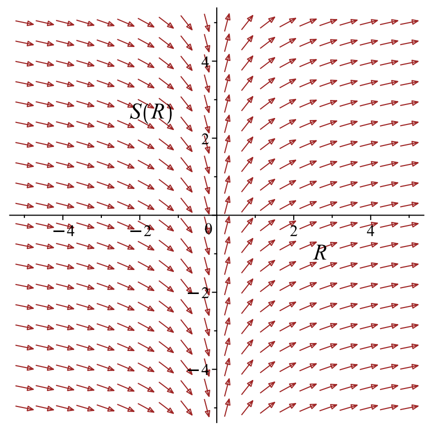
Which simplifies to

$$
\ln\left(\ln\left(x\right)+\ln\left(y\right)\right) = \ln\left(x\right)+c_1
$$

Which gives

$$
y = \frac{e^{x\,e^{c_1}}}{x}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(\ln(xy)-1)}{x}$ 	$R = x$ $S = \ln(\ln(x) + \ln(y))$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{x\,e^{c_1}}}{x} \tag{1}$$

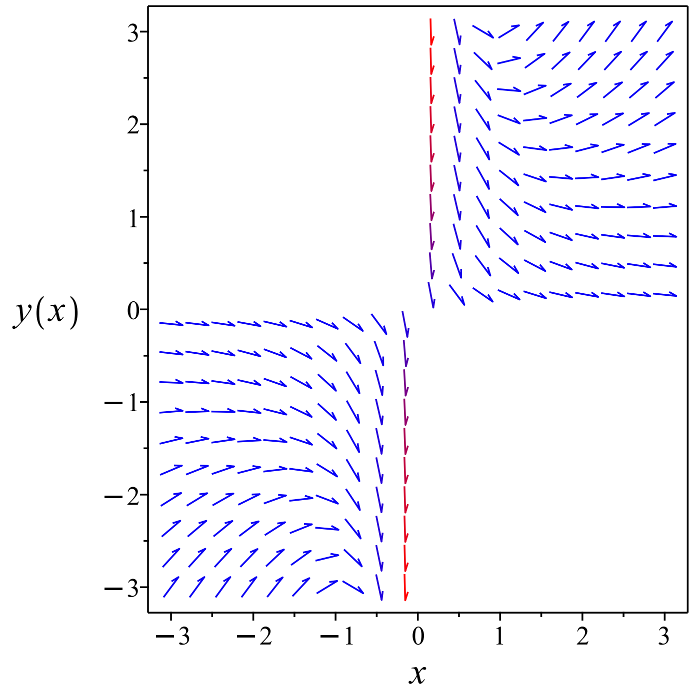


Figure 223: Slope field plot

Verification of solutions

$$y = \frac{e^{x e^{c_1}}}{x}$$

Verified OK.

6.25.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x) \,\mathrm{d}y &= (y(\ln(xy) - 1)) \,\mathrm{d}x \\
(-y(\ln(xy) - 1)) \,\mathrm{d}x + (x) \,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -y(\ln(xy) - 1) \\
N(x, y) &= x
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y(\ln(xy) - 1)) \\
&= -\ln(xy)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((-\ln(xy)) - (1)) \\
&= \frac{-\ln(xy) - 1}{x}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{y(\ln(xy) - 1)}((1) - (-\ln(xy))) \\
&= \frac{-\ln(xy) - 1}{y(\ln(xy) - 1)}
\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - (-\ln(xy))}{x(-y(\ln(xy) - 1)) - y(x)} \\
&= \frac{-\ln(xy) - 1}{xy\ln(xy)}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{-\ln(t) - 1}{t\ln(t)}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R\,\mathrm{d}t} \\
&= e^{\int \left(\frac{-\ln(t)-1}{t\ln(t)}\right)\mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\mu = e^{-\ln(t)-\ln(\ln(t))}$$
$$= \frac{1}{t\ln(t)}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{xy\ln(xy)}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\overline{M} = \mu M$$
$$= \frac{1}{xy\ln(xy)}(-y(\ln(xy)-1))$$
$$= \frac{-\ln(xy)+1}{x\ln(xy)}$$

And

$$\overline{N} = \mu N$$
$$= \frac{1}{xy\ln(xy)}(x)$$
$$= \frac{1}{y\ln(xy)}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{dy}{dx} = 0$$
$$\left(\frac{-\ln(xy)+1}{x\ln(xy)}\right) + \left(\frac{1}{y\ln(xy)}\right)\frac{dy}{dx} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-\ln(xy) + 1}{x \ln(xy)} \mathrm{d}x$$
$$\phi = -\ln(xy) + \ln(\ln(xy)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{y} + \frac{1}{y \ln(xy)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y \ln(xy)}$. Therefore equation (4) becomes

$$\frac{1}{y \ln(xy)} = -\frac{1}{y} + \frac{1}{y \ln(xy)} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(xy) + \ln(\ln(xy)) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(xy) + \ln(\ln(xy)) + \ln(y)$$

The solution becomes

$$y = \frac{e^{x e^{c_1}}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{x e^{c_1}}}{x} \tag{1}$$

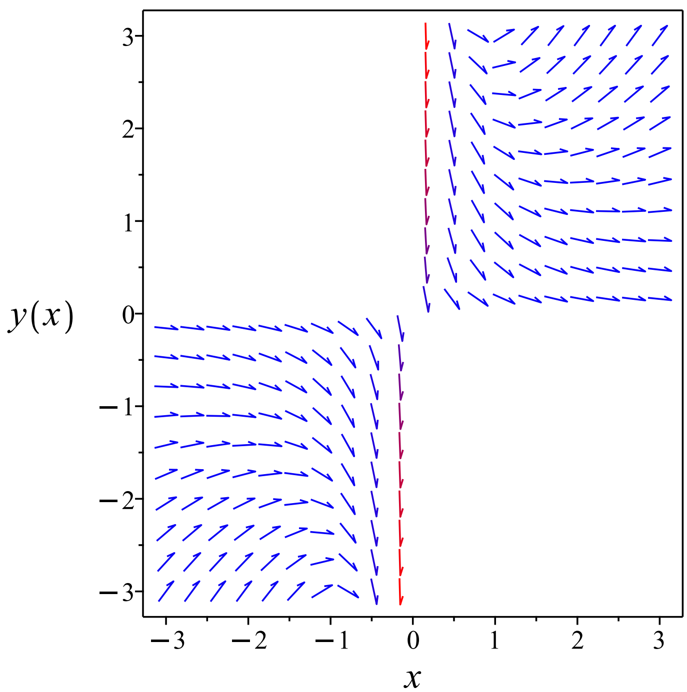


Figure 224: Slope field plot

Verification of solutions

$$y = \frac{e^{x e^{c_1}}}{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve(x*diff(y(x),x)-y(x)*(ln(x*y(x))-1)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{\frac{x}{c_1}}}{x}$$

✓ Solution by Mathematica

Time used: 0.186 (sec). Leaf size: 24

```
DSolve[x*y'[x]-y[x]*(Log[x*y[x]]-1)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{e^{c_1}x}}{x}$$
$$y(x) \to \frac{1}{x}$$

6.26 problem Exercise 12.26, page 103

Internal problem ID [4547]
Internal file name [`OUTPUT/4040_Sunday_June_05_2022_12_13_36_PM_14282714/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.26, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Bernoulli]
```

$$\boxed{y'x^3 - y^2 - yx^2 = 0}$$

6.26.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^3 - u(x)^2\,x^2 - u(x)\,x^3 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2}{x^2} \end{aligned}$$

Where $f(x) = \frac{1}{x^2}$ and $g(u) = u^2$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u^2}\, du &= \frac{1}{x^2}\, dx \\
\int \frac{1}{u^2}\, du &= \int \frac{1}{x^2}\, dx \\
-\frac{1}{u} &= -\frac{1}{x} + c_2
\end{aligned}
$$

The solution is

$$
-\frac{1}{u(x)} + \frac{1}{x} - c_2 = 0
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
-\frac{x}{y} + \frac{1}{x} - c_2 = 0 \\
-\frac{x}{y} + \frac{1}{x} - c_2 = 0
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
-\frac{x}{y} + \frac{1}{x} - c_2 = 0 \tag{1}
$$

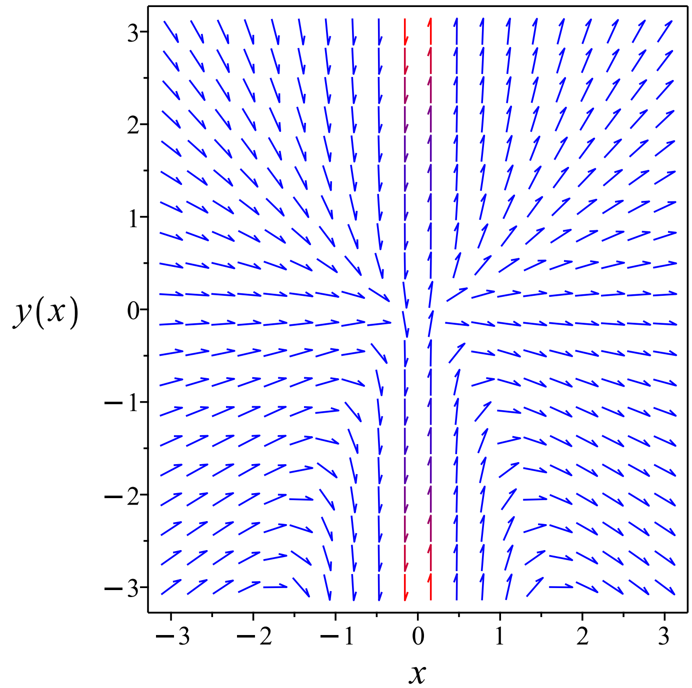


Figure 225: Slope field plot

Verification of solutions

$$-\frac{x}{y} + \frac{1}{x} - c_2 = 0$$

Verified OK.

6.26.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(x^2 + y)}{x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 126: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{y^2}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(x^2 + y)}{x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{y} \\ S_y &= \frac{x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = c_1 - \frac{1}{x}$$

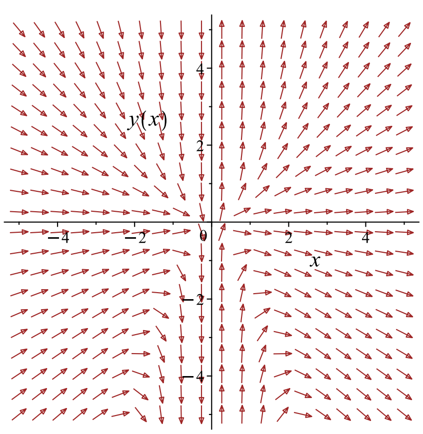
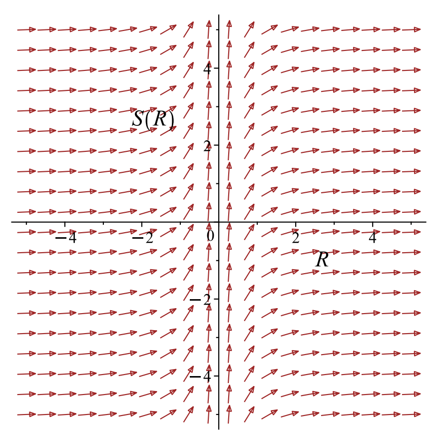
Which simplifies to

$$-\frac{x}{y} = c_1 - \frac{1}{x}$$

Which gives

$$y = -\frac{x^2}{c_1 x - 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(x^2+y)}{x^3}$ 	$R = x$ $S = -\frac{x}{y}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary
The solution(s) found are the following

$$y = -\frac{x^2}{c_1 x - 1} \tag{1}$$

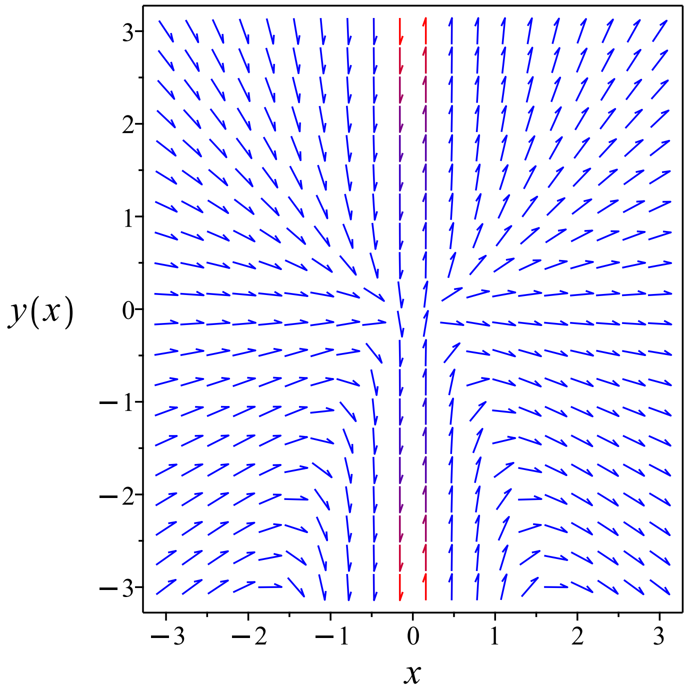


Figure 226: Slope field plot

Verification of solutions

$$y = -\frac{x^2}{c_1 x - 1}$$

Verified OK.

6.26.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(x^2 + y)}{x^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x} y + \frac{1}{x^3} y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= \frac{1}{x^3} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{yx} + \frac{1}{x^3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{w(x)}{x} + \frac{1}{x^3} \\ w' &= -\frac{w}{x} - \frac{1}{x^3} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^3} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = -\frac{1}{x^3}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x}dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{1}{x^3}\right) \\ \frac{d}{dx}(xw) &= (x)\left(-\frac{1}{x^3}\right) \\ d(xw) &= \left(-\frac{1}{x^2}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} xw &= \int -\frac{1}{x^2}\, dx \\ xw &= \frac{1}{x} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = \frac{1}{x^2} + \frac{c_1}{x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{1}{x^2} + \frac{c_1}{x}$$

Or

$$y = \frac{1}{\frac{1}{x^2} + \frac{c_1}{x}}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{\frac{1}{x^2} + \frac{c_1}{x}} \tag{1}$$

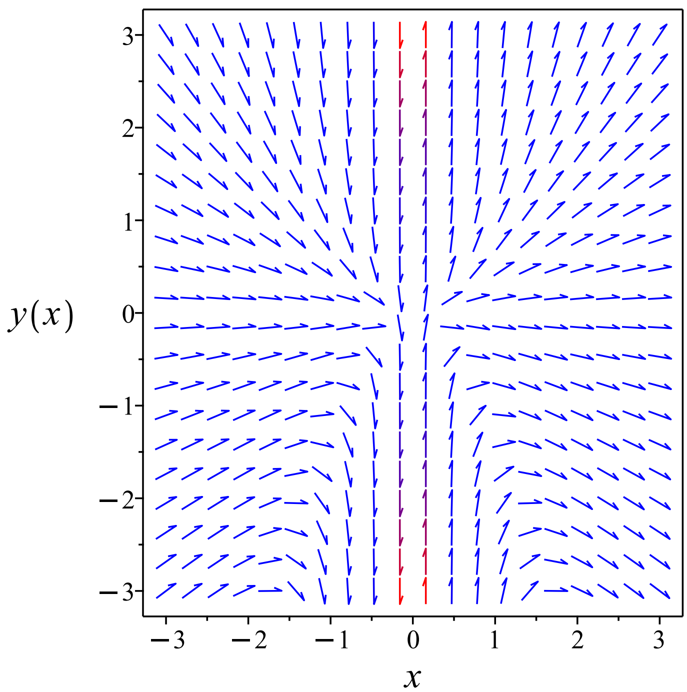


Figure 227: Slope field plot

Verification of solutions

$$y = \frac{1}{\frac{1}{x^2} + \frac{c_1}{x}}$$

Verified OK.

6.26.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^{3}\right)\mathrm{d}y&=\left(y\,x^{2}+y^{2}\right)\mathrm{d}x\\ \left(-y\,x^{2}-y^{2}\right)\mathrm{d}x+\left(x^{3}\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y\,x^{2}-y^{2}\\ N(x,y)&=x^{3}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y\,x^2 - y^2\right)\\ &= -x^2 - 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^3\right)\\ &= 3x^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x^3}\left(\left(-x^2 - 2y\right) - \left(3x^2\right)\right)\\ &= \frac{-4x^2 - 2y}{x^3}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -\frac{1}{y\,(x^2 + y)}\left(\left(3x^2\right) - \left(-x^2 - 2y\right)\right)\\ &= \frac{-4x^2 - 2y}{y\,(x^2 + y)}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(3x^2) - (-x^2 - 2y)}{x(-y\,x^2 - y^2) - y\,(x^3)} \\ &= -\frac{2}{yx} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \, \mathrm{d}t} \\ &= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(t)} \\ &= \frac{1}{t^2} \end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{y^2 x^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^2 x^2}\left(-y\,x^2 - y^2\right) \\ &= \frac{-x^2 - y}{y\,x^2} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{y^2 x^2}\left(x^3\right) \\ &= \frac{x}{y^2} \end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-x^2 - y}{y\,x^2}\right) + \left(\frac{x}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{-x^2 - y}{y\,x^2}\,\mathrm{d}x$$
$$\phi = \frac{-x^2 + y}{xy} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{1}{yx} - \frac{-x^2 + y}{x\,y^2} + f'(y) \tag{4}$$
$$= \frac{x}{y^2} + f'(y)$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{x}{y^2}$. Therefore equation (4) becomes

$$\frac{x}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x^2 + y}{xy} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x^2 + y}{xy}$$

The solution becomes

$$y = -\frac{x^2}{c_1 x - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^2}{c_1 x - 1} \tag{1}$$

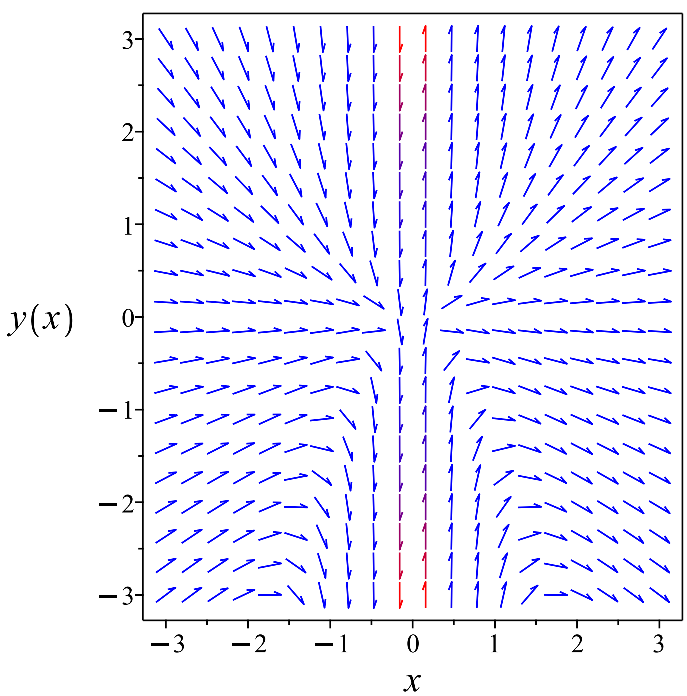


Figure 228: Slope field plot

Verification of solutions

$$y = -\frac{x^2}{c_1 x - 1}$$

Verified OK.

6.26.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(x^2 + y)}{x^3} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{x} + \frac{y^2}{x^3}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{x^3}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^3}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{3}{x^4} \\ f_1 f_2 &= \frac{1}{x^4} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^3} + \frac{2u'(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{x}$$

The above shows that

$$u'(x) = -\frac{c_2}{x^2}$$

Using the above in (1) gives the solution

$$y = \frac{c_2 x}{c_1 + \frac{c_2}{x}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x}{c_3 + \frac{1}{x}}$$

Summary
The solution(s) found are the following

$$y = \frac{x}{c_3 + \frac{1}{x}} \tag{1}$$

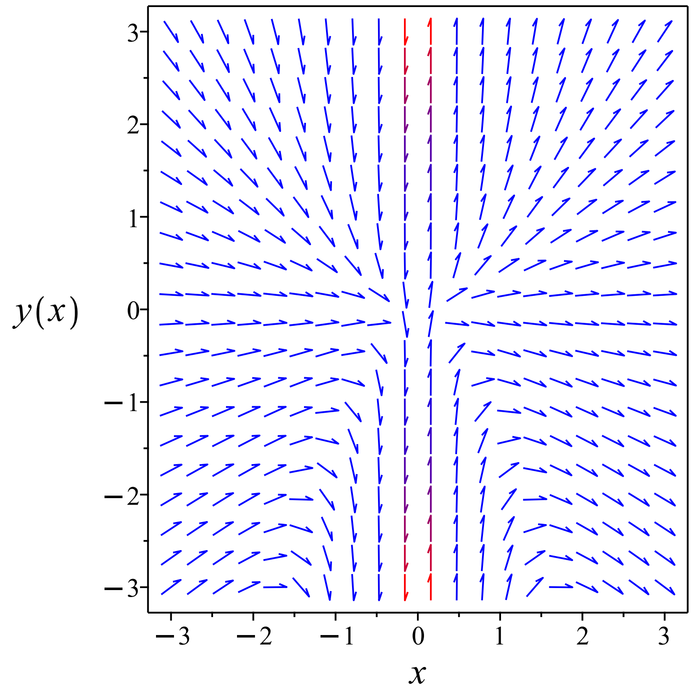


Figure 229: Slope field plot

Verification of solutions

$$y = \frac{x}{c_3 + \frac{1}{x}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^3*diff(y(x),x)-y(x)^2-x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{c_1x + 1}$$

✓ Solution by Mathematica

Time used: 0.129 (sec). Leaf size: 22

```
DSolve[x^3*y'[x]-y[x]^2-x^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{1 + c_1x}$$
$$y(x) \to 0$$

6.27 problem Exercise 12.27, page 103

Internal problem ID [4548]
Internal file name [`OUTPUT/4041_Sunday_June_05_2022_12_13_45_PM_12672693/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.27, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{xy' + ya = -b\,x^n}$$

6.27.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{a}{x} \\ q(x) &= -b\,x^{n-1} \end{aligned}$$

Hence the ode is

$$y' + \frac{ay}{x} = -b\,x^{n-1}$$

The integrating factor μ is

$$\mu = e^{\int \frac{a}{x}dx}$$
$$= e^{a\ln(x)}$$

Which simplifies to

$$\mu = x^a$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(-b\,x^{n-1}\right)$$
$$\frac{d}{dx}(x^a y) = (x^a)\left(-b\,x^{n-1}\right)$$
$$d(x^a y) = \left(-b\,x^{a+n-1}\right)\,dx$$

Integrating gives

$$x^a y = \int -b\,x^{a+n-1}\,dx$$
$$x^a y = -\frac{b\,x^{a+n}}{a+n} + c_1$$

Dividing both sides by the integrating factor $\mu = x^a$ results in

$$y = -\frac{x^{-a} b\,x^{a+n}}{a+n} + c_1 x^{-a}$$

which simplifies to

$$y = -\frac{b\,x^n}{a+n} + c_1 x^{-a}$$

Summary

The solution(s) found are the following

$$y = -\frac{b\,x^n}{a+n} + c_1 x^{-a} \tag{1}$$

Verification of solutions

$$y = -\frac{b\,x^n}{a+n} + c_1 x^{-a}$$

Verified OK.

6.27.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{ya + b\,x^n}{x}\\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 128: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = e^{-a\ln(x)} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-a\ln(x)}} dy \end{aligned}$$

Which results in

$$S = e^{a\ln(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{ya + b\,x^n}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= ay\,x^{a-1} \\ S_y &= x^a \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -b\,x^{a+n-1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -b\,R^{a+n-1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R^{a+n}b}{a+n} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^a y = -\frac{b\,x^{a+n}}{a+n} + c_1$$

Which simplifies to

$$x^a y = -\frac{b\,x^{a+n}}{a+n} + c_1$$

Which gives

$$y = -\frac{(b\,x^{a+n} - ac_1 - c_1 n)\,x^{-a}}{a+n}$$

Summary

The solution(s) found are the following

$$y = -\frac{(b\,x^{a+n} - ac_1 - c_1 n)\,x^{-a}}{a+n} \tag{1}$$

Verification of solutions

$$y = -\frac{(b\,x^{a+n} - ac_1 - c_1 n)\,x^{-a}}{a+n}$$

Verified OK.

6.27.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=(-ya-b\,x^n)\,\mathrm{d}x\\ (ya+b\,x^n)\,\mathrm{d}x+(x)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=ya+b\,x^n\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(ya + b\,x^n)\\ &= a\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((a) - (1))\\ &= \frac{a-1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{a-1}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{(a-1)\ln(x)}\\ &= x^{a-1}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= x^{a-1}(ya + b\,x^n)\\ &= \frac{x^a(ya + b\,x^n)}{x}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x^{a-1}(x) \\
&= x^{a}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^{a}(ya + b\,x^{n})}{x}\right) + (x^{a})\,\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x^{a}(ya + b\,x^{n})}{x}\,\mathrm{d}x \\
\phi &= x^{a}\left(y + \frac{b\,x^{n}}{a+n}\right) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x^{a} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^{a}$. Therefore equation (4) becomes

$$
x^{a} = x^{a} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x^a \left(y + \frac{b\,x^n}{a+n}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^a \left(y + \frac{b\,x^n}{a+n}\right)$$

The solution becomes

$$y = -\frac{(b\,x^a x^n - a c_1 - c_1 n)\,x^{-a}}{a+n}$$

Summary

The solution(s) found are the following

$$y = -\frac{(b\,x^a x^n - a c_1 - c_1 n)\,x^{-a}}{a+n} \tag{1}$$

Verification of solutions

$$y = -\frac{(b\,x^a x^n - a c_1 - c_1 n)\,x^{-a}}{a+n}$$

Verified OK.

6.27.4 Maple step by step solution

Let's solve

$xy' + ya = -b\,x^n$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{ay}{x} - \frac{b\,x^n}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + \frac{ay}{x} = -\frac{b\,x^n}{x}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' + \frac{ay}{x}\right) = -\frac{\mu(x) b\,x^n}{x}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y' + \frac{ay}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{\mu(x) a}{x}$$

- Solve to find the integrating factor

$$\mu(x) = x^a$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\frac{\mu(x) b\,x^n}{x} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int -\frac{\mu(x) b\,x^n}{x} dx + c_1$$

- Solve for y

$$y = \frac{\int -\frac{\mu(x) b\,x^n}{x} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = x^a$

$$y = \frac{\int -\frac{b\,x^a x^n}{x} dx + c_1}{x^a}$$

- Evaluate the integrals on the rhs

$$y = \frac{-\frac{b\,x^{a+n}}{a+n} + c_1}{x^a}$$

- Simplify

$$y = -\frac{b\,x^n}{a+n} + c_1 x^{-a}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x)+a*y(x)+b*x^n=0,y(x), singsol=all)
```

$$y(x) = -\frac{x^n b}{a+n} + x^{-a} c_1$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 25

```
DSolve[x*y'[x]+a*y[x]+b*x^n==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{b x^n}{a+n} + c_1 x^{-a}$$

6.28 problem Exercise 12.28, page 103

Internal problem ID [4549]
Internal file name [`OUTPUT/4042_Sunday_June_05_2022_12_13_57_PM_46057768/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.28, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{xy' - x\sin\left(\frac{y}{x}\right) - y = 0}$$

6.28.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \sin\left(\frac{y}{x}\right) + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$\begin{aligned}\frac{du}{dx}x+u&=u+g(x)\,f(bu)^{\frac{n}{m}}\\ u'&=\frac{1}{x}g(x)\,f(bu)^{\frac{n}{m}}\end{aligned}\tag{2}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y=ux$. Comapring the given ode (A) with the form (1) shows that

$$\begin{aligned}g(x)&=1\\ b&=1\\ f\left(\frac{bx}{y}\right)&=\sin\left(\frac{y}{x}\right)\end{aligned}$$

Substituting the above in (2) results in the $u(x)$ ode as

$$u'(x)=\frac{\sin(u(x))}{x}$$

Which is now solved as separable In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\ &=f(x)g(u)\\ &=\frac{\sin(u)}{x}\end{aligned}$$

Where $f(x)=\frac{1}{x}$ and $g(u)=\sin(u)$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\sin(u)}\,du&=\frac{1}{x}\,dx\\ \int\frac{1}{\sin(u)}\,du&=\int\frac{1}{x}\,dx\\ \ln(\csc(u)-\cot(u))&=\ln(x)+c_1\end{aligned}$$

Raising both side to exponential gives

$$\csc(u)-\cot(u)=e^{\ln(x)+c_1}$$

Which simplifies to

$$\csc(u)-\cot(u)=c_2x$$

Therefore the solution is

$$
\begin{aligned}
y &= ux \\
&= x \arctan\left(\frac{2c_2 x\, e^{c_1}}{c_2^2 x^2 e^{2c_1} + 1}, -\frac{c_2^2 x^2 e^{2c_1} - 1}{c_2^2 x^2 e^{2c_1} + 1}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x \arctan\left(\frac{2c_2 x\, e^{c_1}}{c_2^2 x^2 e^{2c_1} + 1}, -\frac{c_2^2 x^2 e^{2c_1} - 1}{c_2^2 x^2 e^{2c_1} + 1}\right) \tag{1}
$$

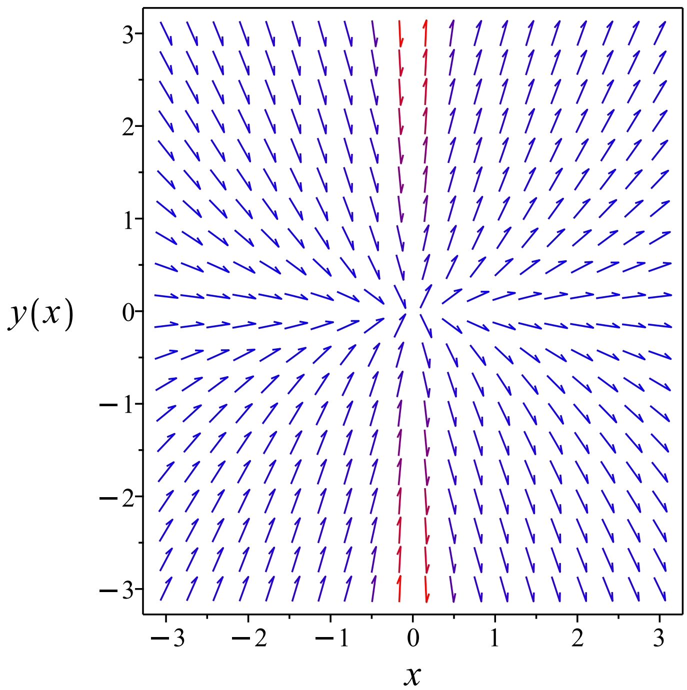


Figure 230: Slope field plot

Verification of solutions

$$
y = x \arctan\left(\frac{2c_2 x\, e^{c_1}}{c_2^2 x^2 e^{2c_1} + 1}, -\frac{c_2^2 x^2 e^{2c_1} - 1}{c_2^2 x^2 e^{2c_1} + 1}\right)
$$

Verified OK.

6.28.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)\,x + u(x)) - x\sin(u(x)) - u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\sin(u)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \sin(u)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\sin(u)}\,du &= \frac{1}{x}\,dx \\ \int \frac{1}{\sin(u)}\,du &= \int \frac{1}{x}\,dx \\ \ln(\csc(u) - \cot(u)) &= \ln(x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\csc(u) - \cot(u) = e^{\ln(x)+c_2}$$

Which simplifies to

$$\csc(u) - \cot(u) = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x\arctan\left(\frac{2c_3 x\,e^{c_2}}{e^{2c_2}c_3^2x^2+1}, -\frac{e^{2c_2}c_3^2x^2-1}{e^{2c_2}c_3^2x^2+1}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x\arctan\left(\frac{2c_3 x\,e^{c_2}}{e^{2c_2}c_3^2x^2+1}, -\frac{e^{2c_2}c_3^2x^2-1}{e^{2c_2}c_3^2x^2+1}\right) \tag{1}$$

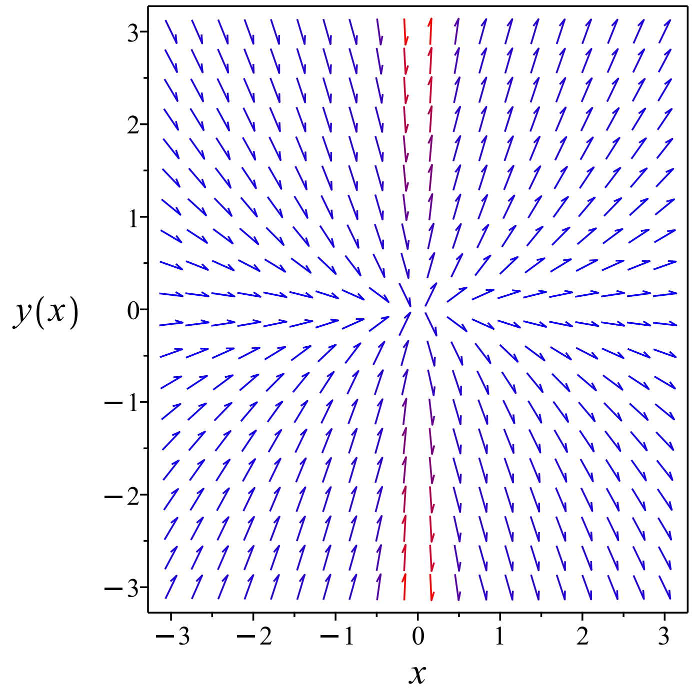


Figure 231: Slope field plot

<u>Verification of solutions</u>

$$y = x \arctan\left(\frac{2c_3 x\, \mathrm{e}^{c_2}}{\mathrm{e}^{2c_2} c_3^2 x^2 + 1}, -\frac{\mathrm{e}^{2c_2} c_3^2 x^2 - 1}{\mathrm{e}^{2c_2} c_3^2 x^2 + 1}\right)$$

Verified OK.

6.28.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + x \sin\left(\frac{y}{x}\right)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 131: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= x^2\\ \eta(x,y) &= xy\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{xy}{x^2}\\ &= \frac{y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{x^2}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= -\frac{1}{x}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y + x\sin\left(\frac{y}{x}\right)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{y}{x^2}\\ R_y &= \frac{1}{x}\\ S_x &= \frac{1}{x^2}\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\csc\left(\frac{y}{x}\right)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -S(R)\csc(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1(\csc(R) + \cot(R)) \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1\left(\csc\left(\frac{y}{x}\right) + \cot\left(\frac{y}{x}\right)\right)$$

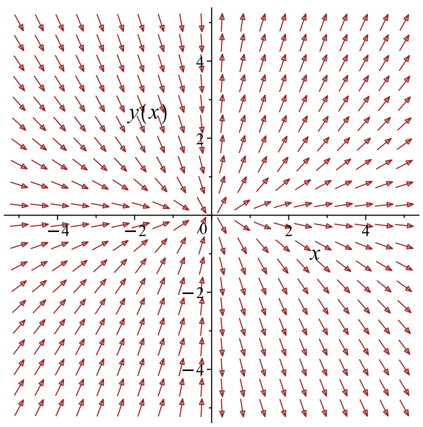
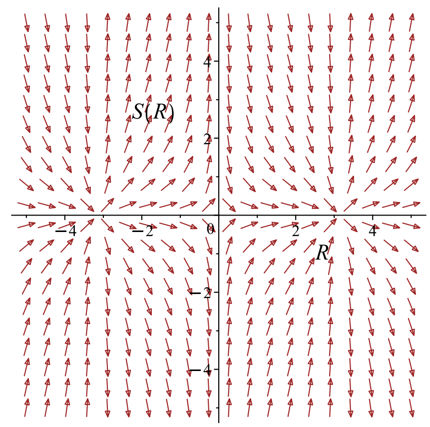
Which simplifies to

$$-\frac{1}{x} = c_1\left(\csc\left(\frac{y}{x}\right) + \cot\left(\frac{y}{x}\right)\right)$$

Which gives

$$y = \arctan\left(-\frac{2c_1x}{c_1^2x^2+1}, -\frac{c_1^2x^2-1}{c_1^2x^2+1}\right)x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y + x\sin\left(\frac{y}{x}\right)}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -S(R)\csc(R)$ 

Summary

The solution(s) found are the following

$$y = \arctan\left(-\frac{2c_1x}{c_1^2x^2+1}, -\frac{c_1^2x^2-1}{c_1^2x^2+1}\right)x \tag{1}$$

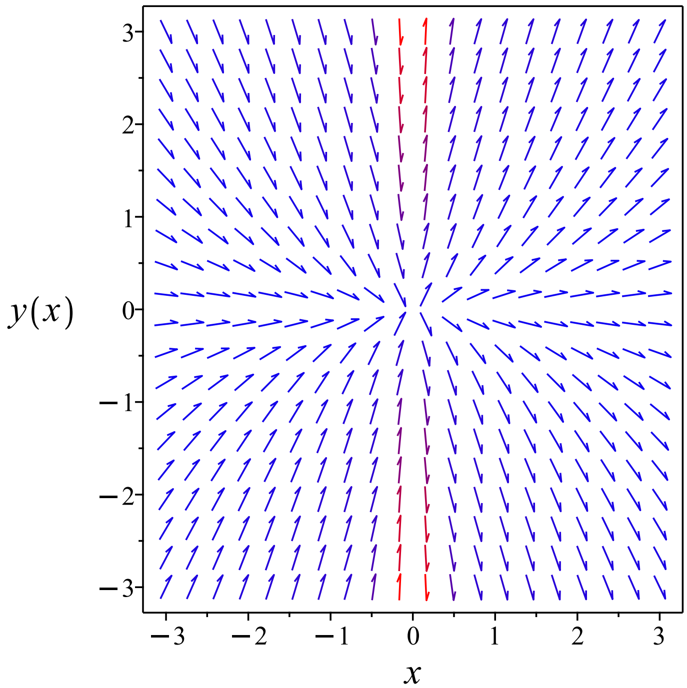


Figure 232: Slope field plot

Verification of solutions

$$y = \arctan\left(-\frac{2c_1x}{c_1^2x^2+1}, -\frac{c_1^2x^2-1}{c_1^2x^2+1}\right)x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
dsolve(x*diff(y(x),x)-x*sin(y(x)/x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \arctan\left(\frac{2xc_1}{x^2c_1^2+1}, \frac{-x^2c_1^2+1}{x^2c_1^2+1}\right)x$$

✓ Solution by Mathematica

Time used: 0.321 (sec). Leaf size: 52

```
DSolve[x*y'[x]-x*Sin[y[x]/x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\arccos(-\tanh(\log(x)+c_1))$$
$$y(x) \to x\arccos(-\tanh(\log(x)+c_1))$$
$$y(x) \to 0$$
$$y(x) \to -\pi x$$
$$y(x) \to \pi x$$

6.29 problem Exercise 12.29, page 103

Internal problem ID [4550]
Internal file name [`OUTPUT/4043_Sunday_June_05_2022_12_14_08_PM_48380251/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.29, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y^2 - 3xy + \left(xy - x^2\right) y' = 2x^2}$$

6.29.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)^2 x^2 - 3x^2 u(x) + \left(x^2 u(x) - x^2\right) \left(u'(x)\, x + u(x)\right) = 2x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2(u^2 - 2u - 1)}{x\,(u-1)} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u^2-2u-1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-2u-1}{u-1}}\,du = -\frac{2}{x}\,dx$$

$$\int \frac{1}{\frac{u^2-2u-1}{u-1}}\,du = \int -\frac{2}{x}\,dx$$

$$\frac{\ln\left(u^2-2u-1\right)}{2} = -2\ln\left(x\right) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2-2u-1} = \mathrm{e}^{-2\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u^2-2u-1} = \frac{c_3}{x^2}$$

Which simplifies to

$$\sqrt{u\left(x\right)^2 - 2u\left(x\right) - 1} = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$

The solution is

$$\sqrt{u\left(x\right)^2 - 2u\left(x\right) - 1} = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2} - \frac{2y}{x} - 1} = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$

$$\sqrt{\frac{-x^2-2xy+y^2}{x^2}} = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{-x^2-2xy+y^2}{x^2}} = \frac{c_3\mathrm{e}^{c_2}}{x^2} \tag{1}$$

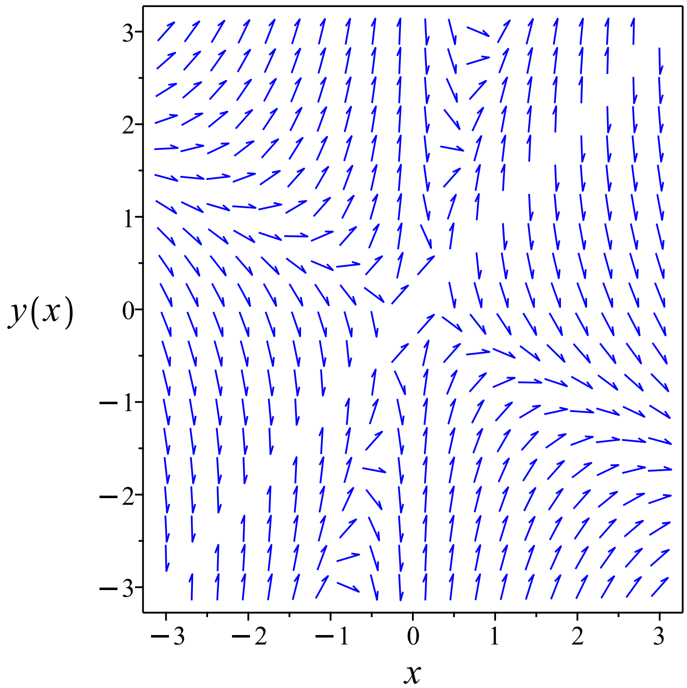


Figure 233: Slope field plot

Verification of solutions

$$\sqrt{\frac{-x^2-2xy+y^2}{x^2}} = \frac{c_3 e^{c_2}}{x^2}$$

Verified OK.

6.29.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{-2x^2-3xy+y^2}{x(-x+y)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{(-2x^2-3xy+y^2)(b_3-a_2)}{x(-x+y)} - \frac{(-2x^2-3xy+y^2)^2 a_3}{x^2(-x+y)^2} \\
&- \left(-\frac{-4x-3y}{x(-x+y)} + \frac{-2x^2-3xy+y^2}{x^2(-x+y)} - \frac{-2x^2-3xy+y^2}{x(-x+y)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{-3x+2y}{x(-x+y)} + \frac{-2x^2-3xy+y^2}{x(-x+y)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^4a_2 - 4x^4a_3 + 6x^4b_2 - 2x^4b_3 - 4x^3ya_2 - 12x^3ya_3 - 4x^3yb_2 + 4x^3yb_3 - 2x^2y^2a_2 - 10x^2y^2a_3 + 2x^2y^2b_2 -}{x^2(x-y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&2x^4a_2 - 4x^4a_3 + 6x^4b_2 - 2x^4b_3 - 4x^3ya_2 - 12x^3ya_3 - 4x^3yb_2 \\
&+ 4x^3yb_3 - 2x^2y^2a_2 - 10x^2y^2a_3 + 2x^2y^2b_2 + 2x^2y^2b_3 + 8x\,y^3a_3 \\
&- 2y^4a_3 + 5x^3b_1 - 5x^2ya_1 - 2x^2yb_1 + 2x\,y^2a_1 + x\,y^2b_1 - y^3a_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&2a_2v_1^4 - 4a_2v_1^3v_2 - 2a_2v_1^2v_2^2 - 4a_3v_1^4 - 12a_3v_1^3v_2 - 10a_3v_1^2v_2^2 + 8a_3v_1v_2^3 \\
&- 2a_3v_2^4 + 6b_2v_1^4 - 4b_2v_1^3v_2 + 2b_2v_1^2v_2^2 - 2b_3v_1^4 + 4b_3v_1^3v_2 + 2b_3v_1^2v_2^2 \\
&- 5a_1v_1^2v_2 + 2a_1v_1v_2^2 - a_1v_2^3 + 5b_1v_1^3 - 2b_1v_1^2v_2 + b_1v_1v_2^2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(2a_2 - 4a_3 + 6b_2 - 2b_3) v_1^4 + (-4a_2 - 12a_3 - 4b_2 + 4b_3) v_1^3 v_2 + 5b_1 v_1^3 \\ &+ (-2a_2 - 10a_3 + 2b_2 + 2b_3) v_1^2 v_2^2 + (-5a_1 - 2b_1) v_1^2 v_2 \\ &+ 8a_3 v_1 v_2^3 + (2a_1 + b_1) v_1 v_2^2 - 2a_3 v_2^4 - a_1 v_2^3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-a_1 &= 0\\ -2a_3 &= 0\\ 8a_3 &= 0\\ 5b_1 &= 0\\ -5a_1 - 2b_1 &= 0\\ 2a_1 + b_1 &= 0\\ -4a_2 - 12a_3 - 4b_2 + 4b_3 &= 0\\ -2a_2 - 10a_3 + 2b_2 + 2b_3 &= 0\\ 2a_2 - 4a_3 + 6b_2 - 2b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= b_3\\ a_3 &= 0\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{-2x^2 - 3xy + y^2}{x\,(-x+y)}\right)(x) \\ &= \frac{2x^2 + 4xy - 2y^2}{x - y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2x^2+4xy-2y^2}{x-y}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(-x^2 - 2xy + y^2)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-2x^2 - 3xy + y^2}{x\,(-x+y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x+y}{2x^2+4xy-2y^2} \\
S_y &= \frac{x-y}{2x^2+4xy-2y^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{2R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}
$$

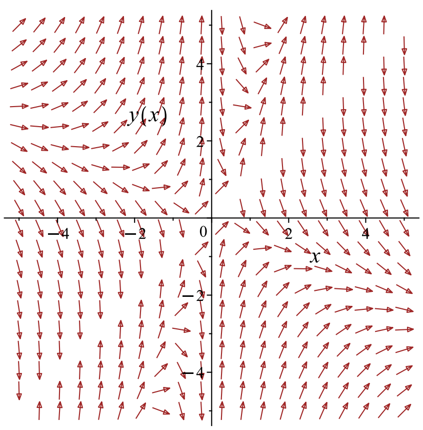
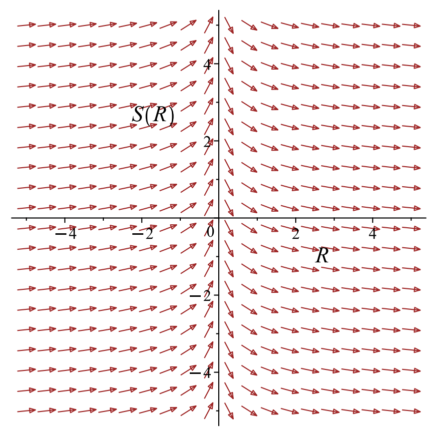
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(-x^2-2xy+y^2)}{4} = -\frac{\ln(x)}{2} + c_1
$$

Which simplifies to

$$
\frac{\ln(-x^2-2xy+y^2)}{4} = -\frac{\ln(x)}{2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-2x^2-3xy+y^2}{x(-x+y)}$ 	$R = x$ $S = \frac{\ln(-x^2-2xy+y^2)}{4}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$\frac{\ln(-x^2-2xy+y^2)}{4} = -\frac{\ln(x)}{2} + c_1 \tag{1}$$

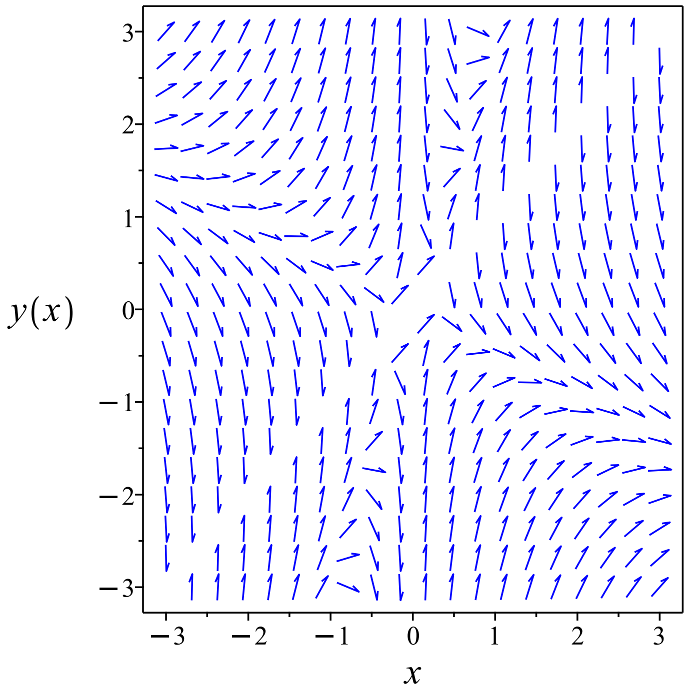


Figure 234: Slope field plot

Verification of solutions

$$\frac{\ln(-x^2-2xy+y^2)}{4} = -\frac{\ln(x)}{2} + c_1$$

Verified OK.

6.29.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-x^2 + xy\right)\mathrm{d}y &= \left(2x^2 + 3xy - y^2\right)\mathrm{d}x \\ \left(-2x^2 - 3xy + y^2\right)\mathrm{d}x + \left(-x^2 + xy\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -2x^2 - 3xy + y^2 \\ N(x, y) &= -x^2 + xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2x^2 - 3xy + y^2\right) \\ &= -3x + 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^2 + xy\right) \\ &= -2x + y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{x\left(x-y\right)}\left(\left(-3x+2y\right)-\left(-2x+y\right)\right) \\ &= \frac{1}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{1}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(x)} \\ &= x \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= x\left(-2x^2 - 3xy + y^2\right) \\ &= -2x^3 - 3y\,x^2 + y^2 x \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= x\left(-x^2 + xy\right) \\ &= -x^2(x-y) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-2x^3 - 3y\,x^2 + y^2 x\right) + \left(-x^2(x-y)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -2x^3 - 3y\,x^2 + y^2 x \,\mathrm{d}x \\
\phi &= -\frac{1}{2}x^4 - y\,x^3 + \frac{1}{2}y^2x^2 + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}
\frac{\partial \phi}{\partial y} &= -x^3 + y\,x^2 + f'(y) \qquad (4) \\
&= -x^2(x-y) + f'(y)
\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -x^2(x-y)$. Therefore equation (4) becomes

$$-x^2(x-y) = -x^2(x-y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{2}x^4 - y\,x^3 + \frac{1}{2}y^2x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{2}x^4 - y\,x^3 + \frac{1}{2}y^2x^2$$

Summary

The solution(s) found are the following

$$-\frac{x^4}{2} - yx^3 + \frac{y^2x^2}{2} = c_1 \tag{1}$$

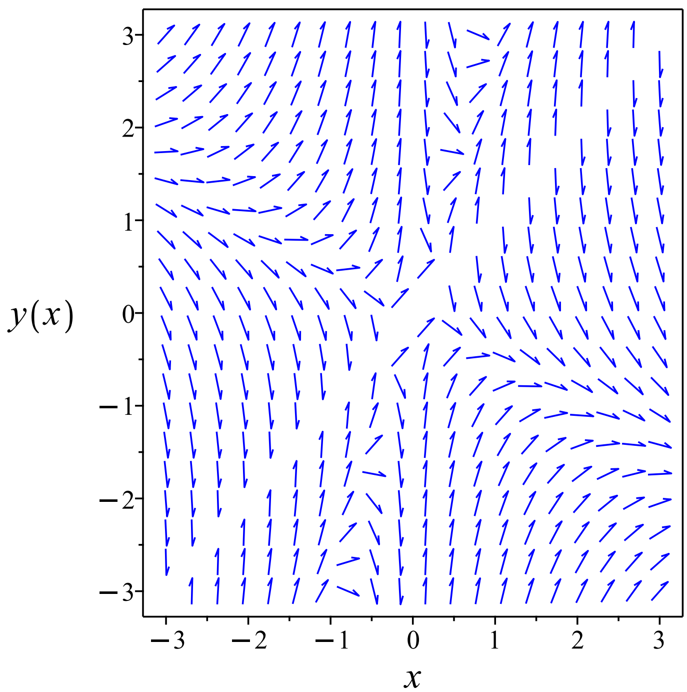


Figure 235: Slope field plot

Verification of solutions

$$-\frac{x^4}{2} - yx^3 + \frac{y^2x^2}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 59

```
dsolve((x*y(x)-x^2)*diff(y(x),x)+y(x)^2-3*x*y(x)-2*x^2=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2 - \sqrt{2c_1^2 x^4 + 1}}{c_1 x}$$
$$y(x) = \frac{c_1 x^2 + \sqrt{2c_1^2 x^4 + 1}}{c_1 x}$$

✓ Solution by Mathematica

Time used: 0.625 (sec). Leaf size: 99

```
DSolve[(x*y[x]-x^2)*y'[x]+y[x]^2-3*x*y[x]-2*x^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - \frac{\sqrt{2x^4 + e^{2c_1}}}{x}$$
$$y(x) \to x + \frac{\sqrt{2x^4 + e^{2c_1}}}{x}$$
$$y(x) \to x - \frac{\sqrt{2}\sqrt{x^4}}{x}$$
$$y(x) \to \frac{\sqrt{2}\sqrt{x^4}}{x} + x$$

6.30 problem Exercise 12.30, page 103

Internal problem ID [4551]
Internal file name [`OUTPUT/4044_Sunday_June_05_2022_12_14_16_PM_74322998/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.30, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{(6xy + x^2 + 3)\, y' + 3y^2 + 2xy = -2x}$$

6.30.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2 + 6xy + 3)\,\mathrm{d}y &= (-2xy - 3y^2 - 2x)\,\mathrm{d}x \\ (2xy + 3y^2 + 2x)\,\mathrm{d}x + (x^2 + 6xy + 3)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2xy + 3y^2 + 2x \\ N(x, y) &= x^2 + 6xy + 3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2xy + 3y^2 + 2x) \\ &= 2x + 6y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + 6xy + 3) \\ &= 2x + 6y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2xy + 3y^2 + 2x \, \mathrm{d}x$$
$$\phi = x\left(xy + 3y^2 + x\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x(x + 6y) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 + 6xy + 3$. Therefore equation (4) becomes

$$x^2 + 6xy + 3 = x(x + 6y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (3) \, \mathrm{d}y$$
$$f(y) = 3y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x\left(xy + 3y^2 + x\right) + 3y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\left(xy + 3y^2 + x\right) + 3y$$

Summary
The solution(s) found are the following

$$x(xy + 3y^2 + x) + 3y = c_1 \tag{1}$$

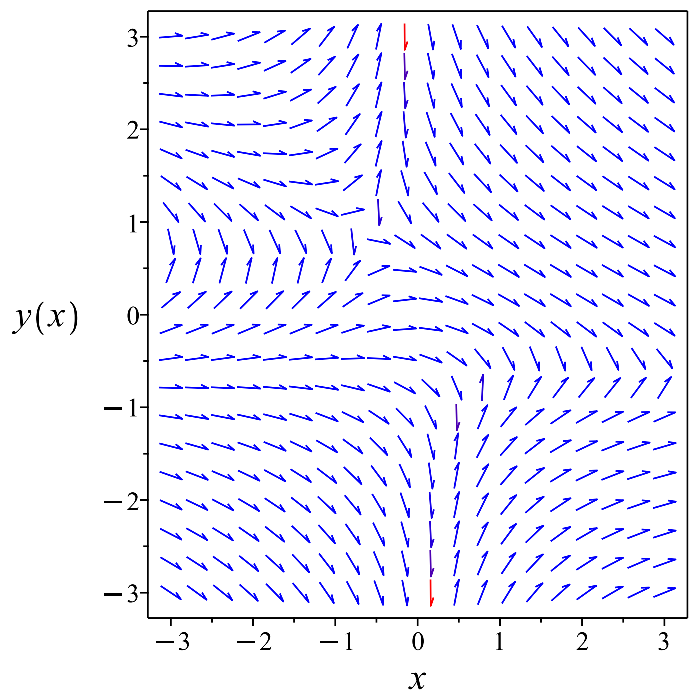


Figure 236: Slope field plot

Verification of solutions

$$x(xy + 3y^2 + x) + 3y = c_1$$

Verified OK.

6.30.2 Maple step by step solution

Let's solve

$(6xy + x^2 + 3) y' + 3y^2 + 2xy = -2x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $2x + 6y = 2x + 6y$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (2xy + 3y^2 + 2x)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = y\, x^2 + 3y^2 x + x^2 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $x^2 + 6xy + 3 = x^2 + 6xy + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 3$

- Solve for $f_1(y)$

  $f_1(y) = 3y$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = y\, x^2 + 3y^2 x + x^2 + 3y$

- Substitute $F(x, y)$ into the solution of the ODE

  $y\, x^2 + 3y^2 x + x^2 + 3y = c_1$

- Solve for y

  $\left\{y = \frac{-x^2 - 3 + \sqrt{x^4 - 12x^3 + 12c_1 x + 6x^2 + 9}}{6x}, y = -\frac{x^2 + \sqrt{x^4 - 12x^3 + 12c_1 x + 6x^2 + 9} + 3}{6x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 75

```
dsolve((6*x*y(x)+x^2+3)*diff(y(x),x)+3*y(x)^2+2*x*y(x)+2*x=0,y(x), singsol=all)
```

$$y(x) = \frac{-x^2 - 3 + \sqrt{x^4 - 12x^3 - 12c_1x + 6x^2 + 9}}{6x}$$
$$y(x) = \frac{-x^2 - 3 - \sqrt{x^4 - 12x^3 - 12c_1x + 6x^2 + 9}}{6x}$$

✓ Solution by Mathematica

Time used: 0.477 (sec). Leaf size: 83

```
DSolve[(6*x*y[x]+x^2+3)*y'[x]+3*y[x]^2+2*x*y[x]+2*x==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to -\frac{x^2 + \sqrt{x^4 - 12x^3 + 6x^2 + 36c_1x + 9} + 3}{6x}$$
$$y(x) \to -\frac{x^2 - \sqrt{x^4 - 12x^3 + 6x^2 + 36c_1x + 9} + 3}{6x}$$

6.31 problem Exercise 12.31, page 103

Internal problem ID [4552]
Internal file name [`OUTPUT/4045_Sunday_June_05_2022_12_14_24_PM_76345221/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.31, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{x^2y' + y^2 + xy = -x^2}$$

6.31.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^2(u'(x)\,x + u(x)) + u(x)^2\,x^2 + x^2u(x) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-u^2 - 2u - 1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -u^2 - 2u - 1$. Integrating both sides gives

$$\frac{1}{-u^2 - 2u - 1}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{-u^2 - 2u - 1}\,du = \int \frac{1}{x}\,dx$$

$$\frac{1}{u+1} = \ln(x) + c_2$$

The solution is

$$\frac{1}{u(x)+1} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{1}{\frac{y}{x}+1} - \ln(x) - c_2 = 0$$

$$\frac{(-c_2 - \ln(x))\,y - x(c_2 + \ln(x) - 1)}{x+y} = 0$$

Summary

The solution(s) found are the following

$$\frac{(-c_2 - \ln(x))\,y - x(c_2 + \ln(x) - 1)}{x+y} = 0 \tag{1}$$

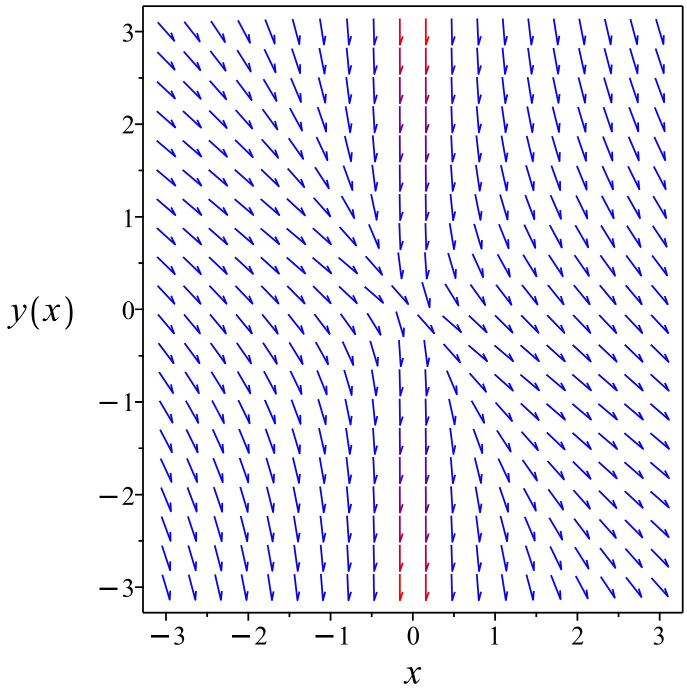


Figure 237: Slope field plot

Verification of solutions

$$\frac{(-c_2 - \ln(x))\, y - x(c_2 + \ln(x) - 1)}{x + y} = 0$$

Verified OK.

6.31.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x^2 + xy + y^2}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2-\frac{\left(x^2+x y+y^2\right)\left(b_3-a_2\right)}{x^2}-\frac{\left(x^2+x y+y^2\right)^2 a_3}{x^4} \\ & -\left(-\frac{2 x+y}{x^2}+\frac{2 x^2+2 x y+2 y^2}{x^3}\right)\left(x a_2+y a_3+a_1\right) \\ & +\frac{(2 y+x)\left(x b_2+y b_3+b_1\right)}{x^2}=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^4 a_2-x^4 a_3+2 b_2 x^4-x^4 b_3-2 x^3 y a_3+2 x^3 y b_2-x^2 y^2 a_2-4 x^2 y^2 a_3+x^2 y^2 b_3-4 x\, y^3 a_3-y^4 a_3+x^3 b_1-x}{x^4}$$
$$=0$$

Setting the numerator to zero gives

$$\begin{aligned} & x^4 a_2-x^4 a_3+2 b_2 x^4-x^4 b_3-2 x^3 y a_3+2 x^3 y b_2-x^2 y^2 a_2-4 x^2 y^2 a_3 \\ & +x^2 y^2 b_3-4 x\, y^3 a_3-y^4 a_3+x^3 b_1-x^2 y a_1+2 x^2 y b_1-2 x\, y^2 a_1=0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x=v_1, y=v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & a_2 v_1^4-a_2 v_1^2 v_2^2-a_3 v_1^4-2 a_3 v_1^3 v_2-4 a_3 v_1^2 v_2^2-4 a_3 v_1 v_2^3-a_3 v_2^4+2 b_2 v_1^4 \\ & +2 b_2 v_1^3 v_2-b_3 v_1^4+b_3 v_1^2 v_2^2-a_1 v_1^2 v_2-2 a_1 v_1 v_2^2+b_1 v_1^3+2 b_1 v_1^2 v_2=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(a_2 - a_3 + 2b_2 - b_3) v_1^4 + (-2a_3 + 2b_2) v_1^3 v_2 + b_1 v_1^3 + (-a_2 - 4a_3 + b_3) v_1^2 v_2^2 + (-a_1 + 2b_1) v_1^2 v_2 - 4a_3 v_1 v_2^3 - 2a_1 v_1 v_2^2 - a_3 v_2^4 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -2a_1 &= 0 \\ -4a_3 &= 0 \\ -a_3 &= 0 \\ -a_1 + 2b_1 &= 0 \\ -2a_3 + 2b_2 &= 0 \\ -a_2 - 4a_3 + b_3 &= 0 \\ a_2 - a_3 + 2b_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{x^2+xy+y^2}{x^2}\right)(x) \\ &= \frac{x^2+2xy+y^2}{x} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2+2xy+y^2}{x}} dy\end{aligned}$$

Which results in

$$S = -\frac{x}{x+y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x^2+xy+y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y}{\left(x+y\right)^2} \\
S_y &= \frac{x}{\left(x+y\right)^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln\left(R\right) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{x}{x+y} = -\ln\left(x\right) + c_1
$$

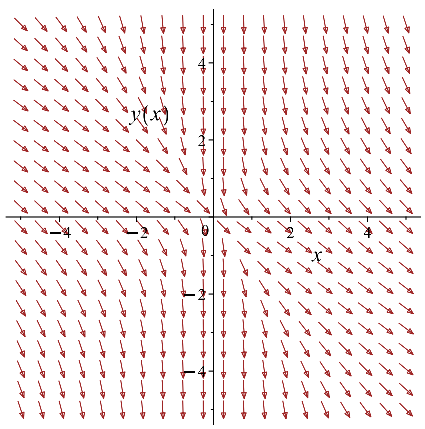
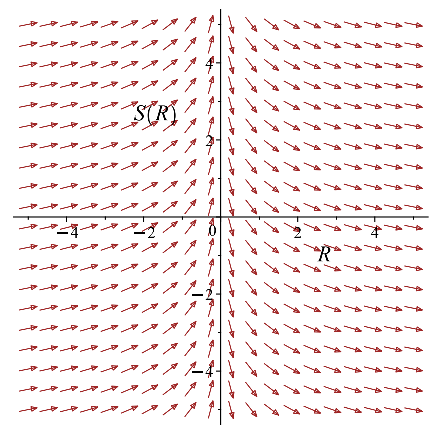
Which simplifies to

$$
-\frac{x}{x+y} = -\ln\left(x\right) + c_1
$$

Which gives

$$
y = -\frac{x(-1+\ln\left(x\right)-c_1)}{\ln\left(x\right)-c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2+xy+y^2}{x^2}$ 	$R = x$ $S = -\frac{x}{x+y}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{x(-1+\ln(x)-c_1)}{\ln(x)-c_1} \tag{1}$$

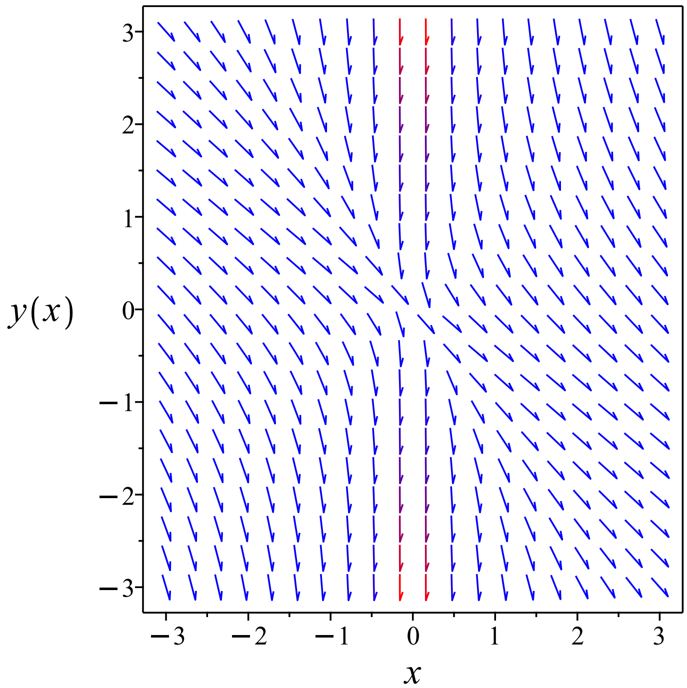


Figure 238: Slope field plot

Verification of solutions

$$y = -\frac{x(-1 + \ln(x) - c_1)}{\ln(x) - c_1}$$

Verified OK.

6.31.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{x^2 + xy + y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -1 - \frac{y}{x} - \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -1$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = -\frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2}{x^3} \\ f_1 f_2 &= \frac{1}{x^3} \\ f_2^2 f_0 &= -\frac{1}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x^2} - \frac{3u'(x)}{x^3} - \frac{u(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_2 \ln(x) + c_1}{x}$$

The above shows that

$$u'(x) = \frac{c_2 - c_2 \ln(x) - c_1}{x^2}$$

Using the above in (1) gives the solution

$$y = \frac{x(c_2 - c_2 \ln(x) - c_1)}{c_2 \ln(x) + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{(-1 + \ln(x) + c_3)\, x}{\ln(x) + c_3}$$

Summary

The solution(s) found are the following

$$y = -\frac{(-1+\ln (x)+c_3)\, x}{\ln (x)+c_3} \tag{1}$$

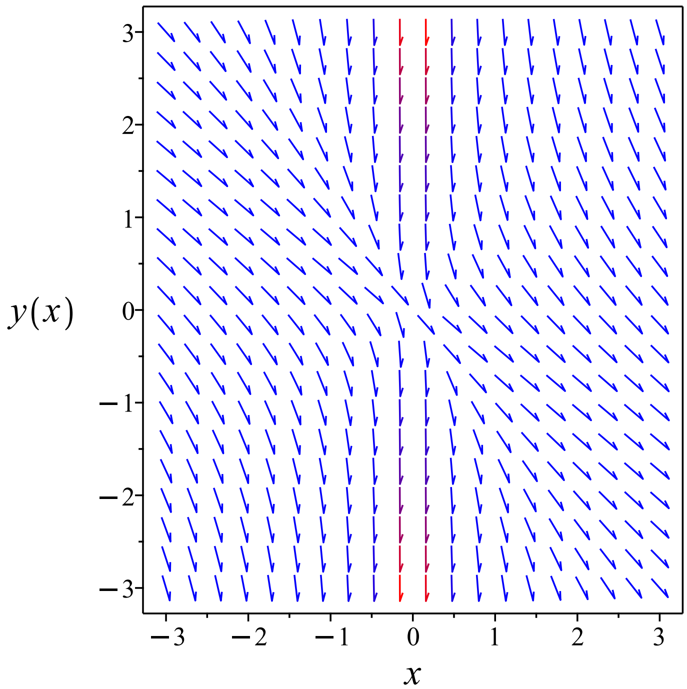


Figure 239: Slope field plot

Verification of solutions

$$y = -\frac{(-1+\ln (x)+c_3)\, x}{\ln (x)+c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(x^2*diff(y(x),x)+y(x)^2+x*y(x)+x^2=0,y(x), singsol=all)
```

$$y(x) = -\frac{x(\ln(x) + c_1 - 1)}{\ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.139 (sec). Leaf size: 31

```
DSolve[x^2*y'[x]+y[x]^2+x*y[x]+x^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x(\log(x) - 1 - c_1)}{-\log(x) + c_1}$$
$$y(x) \to -x$$

6.32 problem Exercise 12.32, page 103

Internal problem ID [4553]
Internal file name [`OUTPUT/4046_Sunday_June_05_2022_12_14_33_PM_43963202/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.32, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\left(x^2-1\right) y'+2xy=\cos\left(x\right)}$$

6.32.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{2x}{x^2-1}$$
$$q(x)=\frac{\cos\left(x\right)}{x^2-1}$$

Hence the ode is

$$y'+\frac{2xy}{x^2-1}=\frac{\cos\left(x\right)}{x^2-1}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{2x}{x^2-1}dx} \\
&= \mathrm{e}^{\ln(x-1)+\ln(x+1)}
\end{aligned}
$$

Which simplifies to

$$
\mu = x^2 - 1
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{\cos(x)}{x^2-1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2-1\right)y\right) &= \left(x^2-1\right)\left(\frac{\cos(x)}{x^2-1}\right) \\
\mathrm{d}\left(\left(x^2-1\right)y\right) &= \cos(x)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\left(x^2-1\right)y &= \int \cos(x)\,\mathrm{d}x \\
\left(x^2-1\right)y &= \sin(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2 - 1$ results in

$$
y = \frac{\sin(x)}{x^2-1} + \frac{c_1}{x^2-1}
$$

which simplifies to

$$
y = \frac{\sin(x) + c_1}{x^2-1}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\sin(x) + c_1}{x^2-1} \tag{1}
$$

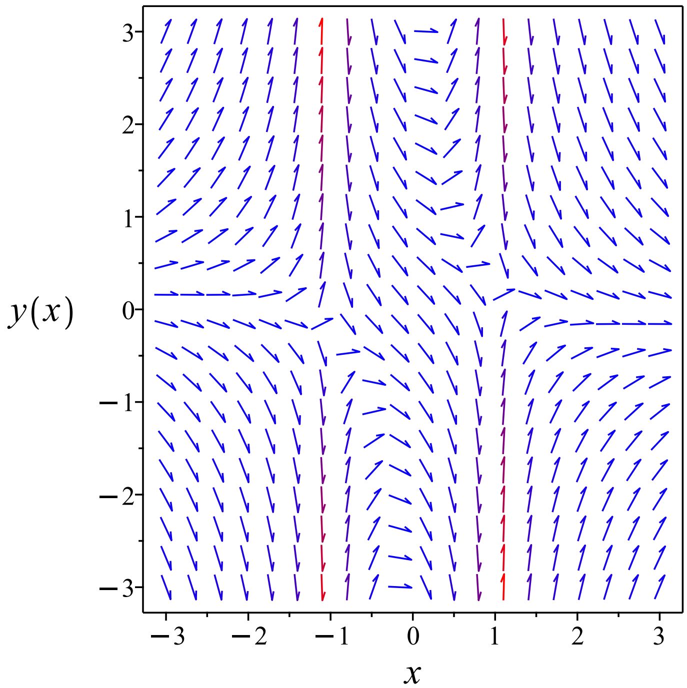


Figure 240: Slope field plot

Verification of solutions

$$y = \frac{\sin(x) + c_1}{x^2 - 1}$$

Verified OK.

6.32.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-2xy + \cos(x)}{x^2 - 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 134: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0\\ \eta(x, y) &= \mathrm{e}^{-\ln(x-1)-\ln(x+1)}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\ln(x-1)-\ln(x+1)}} dy \end{aligned}$$

Which results in

$$S = (x-1)(x+1)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-2xy + \cos(x)}{x^2 - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2xy \\ S_y &= x^2 - 1 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y x^2 - y = \sin(x) + c_1$$

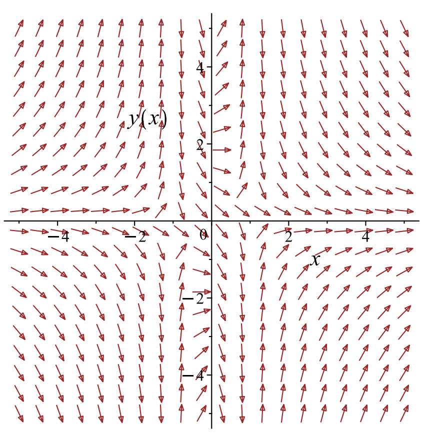
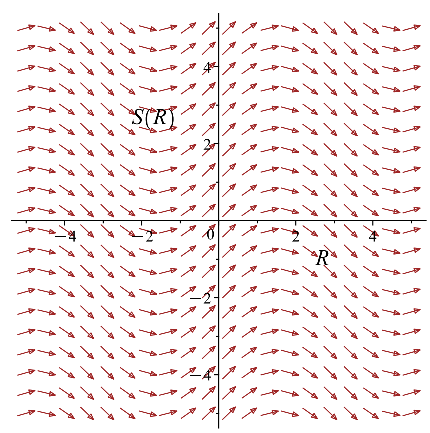
Which simplifies to

$$y x^2 - y = \sin(x) + c_1$$

Which gives

$$y = \frac{\sin(x) + c_1}{x^2 - 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-2xy+\cos(x)}{x^2-1}$ 	$R = x$ $S = y\,x^2 - y$	$\frac{dS}{dR} = \cos(R)$ 

Summary

The solution(s) found are the following

$$y = \frac{\sin(x) + c_1}{x^2 - 1} \tag{1}$$

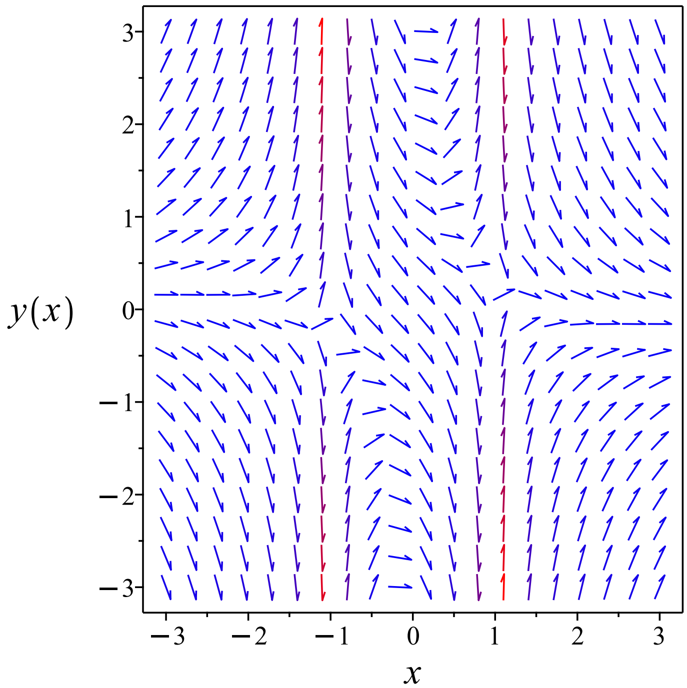


Figure 241: Slope field plot

Verification of solutions

$$y = \frac{\sin(x) + c_1}{x^2 - 1}$$

Verified OK.

6.32.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x^2 - 1\right)\mathrm{d}y = \left(-2xy + \cos\left(x\right)\right)\mathrm{d}x$$
$$\left(2xy - \cos\left(x\right)\right)\mathrm{d}x + \left(x^2 - 1\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 2xy - \cos\left(x\right)$$
$$N(x, y) = x^2 - 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(2xy - \cos\left(x\right)\right)$$
$$= 2x$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(x^2 - 1\right)$$
$$= 2x$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int 2xy-\cos(x)\,\mathrm{d}x$$
$$\phi=y\,x^2-\sin(x)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=x^2+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=x^2-1$. Therefore equation (4) becomes

$$x^2-1=x^2+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=-1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int(-1)\,\mathrm{d}y$$
$$f(y)=-y+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=y\,x^2-\sin(x)-y+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^2 - \sin(x) - y$$

The solution becomes

$$y = \frac{\sin(x) + c_1}{x^2 - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin(x) + c_1}{x^2 - 1} \tag{1}$$

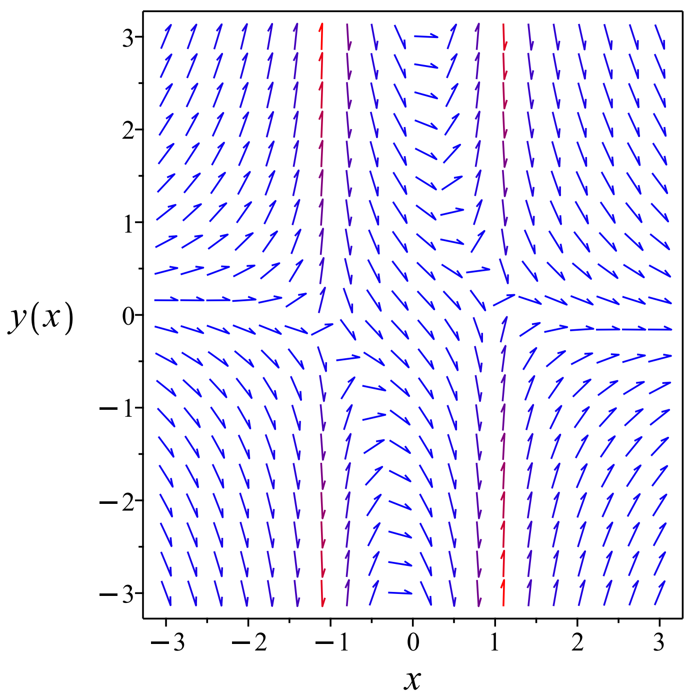


Figure 242: Slope field plot

Verification of solutions

$$y = \frac{\sin(x) + c_1}{x^2 - 1}$$

Verified OK.

6.32.4 Maple step by step solution

Let's solve

$(x^2 - 1)\, y' + 2xy = \cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2xy}{x^2-1} + \frac{\cos(x)}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2xy}{x^2-1} = \frac{\cos(x)}{x^2-1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2xy}{x^2-1}\right) = \frac{\mu(x)\cos(x)}{x^2-1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{2xy}{x^2-1}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)x}{x^2-1}$

- Solve to find the integrating factor

  $\mu(x) = (x-1)(x+1)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)\cos(x)}{x^2-1} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)\cos(x)}{x^2-1} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\cos(x)}{x^2-1} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = (x-1)(x+1)$

  $y = \frac{\int \frac{(x-1)(x+1)\cos(x)}{x^2-1} dx + c_1}{(x-1)(x+1)}$

- Evaluate the integrals on the rhs

  $y = \frac{\sin(x) + c_1}{(x-1)(x+1)}$

- Simplify

$y = \frac{\sin(x)+c_1}{x^2-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((x^2-1)*diff(y(x),x)+2*x*y(x)-cos(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sin(x) + c_1}{x^2 - 1}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 18

```
DSolve[(x^2-1)*y'[x]+2*x*y[x]-Cos[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sin(x) + c_1}{x^2 - 1}$$

6.33 problem Exercise 12.33, page 103

Internal problem ID [4554]
Internal file name [`OUTPUT/4047_Sunday_June_05_2022_12_14_43_PM_28867852/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.33, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{(y x^2-1)\, y'+x y^2 = 1}$$

6.33.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y\,x^2 - 1)\,\mathrm{d}y &= (-y^2x + 1)\,\mathrm{d}x\\ (y^2x - 1)\,\mathrm{d}x + (y\,x^2 - 1)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= y^2x - 1\\ N(x, y) &= y\,x^2 - 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y^2x - 1)\\ &= 2xy \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y\,x^2 - 1)\\ &= 2xy \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int y^2 x - 1 \, \mathrm{d}x$$
$$\phi = \frac{1}{2} y^2 x^2 - x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = y\,x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y\,x^2 - 1$. Therefore equation (4) becomes

$$y\,x^2 - 1 = y\,x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -1$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-1) \, \mathrm{d}y$$
$$f(y) = -y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{2} y^2 x^2 - x - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{2} y^2 x^2 - x - y$$

Summary
The solution(s) found are the following

$$\frac{y^2x^2}{2} - x - y = c_1 \tag{1}$$

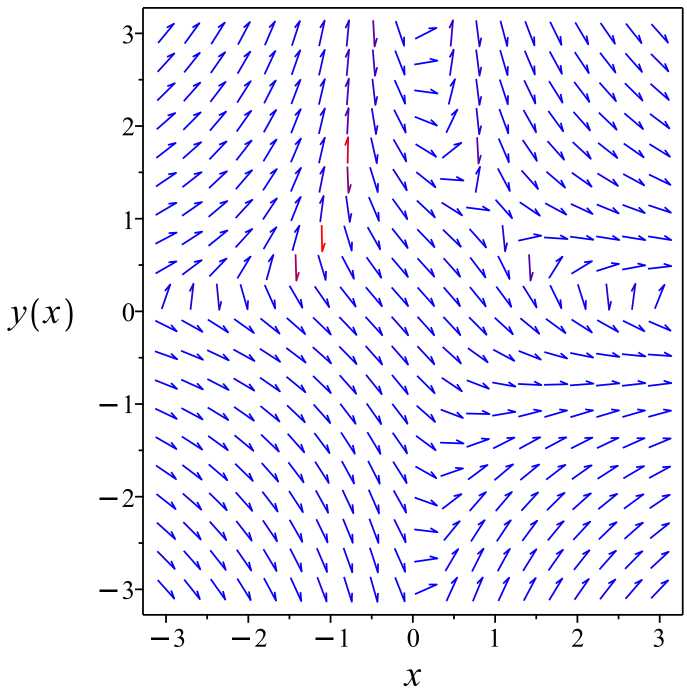


Figure 243: Slope field plot

Verification of solutions

$$\frac{y^2x^2}{2} - x - y = c_1$$

Verified OK.

6.33.2 Maple step by step solution

Let's solve

$(yx^2 - 1)\, y' + xy^2 = 1$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $2xy = 2xy$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (y^2x - 1)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{y^2x^2}{2} - x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $y\, x^2 - 1 = y\, x^2 + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = -1$

- Solve for $f_1(y)$

  $f_1(y) = -y$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = \frac{1}{2}y^2x^2 - x - y$

- Substitute $F(x, y)$ into the solution of the ODE

  $\frac{1}{2}y^2x^2 - x - y = c_1$

- Solve for y

  $\left\{y = \frac{1+\sqrt{2c_1x^2+2x^3+1}}{x^2}, y = -\frac{-1+\sqrt{2c_1x^2+2x^3+1}}{x^2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 51

```
dsolve((x^2*y(x)-1)*diff(y(x),x)+x*y(x)^2-1=0,y(x), singsol=all)
```

$$y(x) = \frac{1 + \sqrt{-2c_1x^2 + 2x^3 + 1}}{x^2}$$
$$y(x) = \frac{1 - \sqrt{-2c_1x^2 + 2x^3 + 1}}{x^2}$$

✓ Solution by Mathematica

Time used: 0.505 (sec). Leaf size: 57

```
DSolve[(x^2*y[x]-1)*y'[x]+x*y[x]^2-1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1 - \sqrt{2x^3 + c_1x^2 + 1}}{x^2}$$
$$y(x) \to \frac{1 + \sqrt{2x^3 + c_1x^2 + 1}}{x^2}$$

6.34 problem Exercise 12.34, page 103

Internal problem ID [4555]
Internal file name [`OUTPUT/4048_Sunday_June_05_2022_12_14_51_PM_83200267/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.34, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(x^2-1)\,y'+xy-3xy^2=0}$$

6.34.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(3y^2-y)}{x^2-1} \end{aligned}$$

Where $f(x)=\frac{x}{x^2-1}$ and $g(y)=3y^2-y$. Integrating both sides gives

$$\frac{1}{3y^2-y}\,dy=\frac{x}{x^2-1}\,dx$$

$$\int \frac{1}{3y^2 - y}\,dy = \int \frac{x}{x^2 - 1}\,dx$$

$$\ln\left(-1 + 3y\right) - \ln\left(y\right) = \frac{\ln\left(x - 1\right)}{2} + \frac{\ln\left(x + 1\right)}{2} + c_1$$

Raising both side to exponential gives

$$e^{\ln(-1+3y)-\ln(y)} = e^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}+c_1}$$

Which simplifies to

$$\frac{-1 + 3y}{y} = c_2 e^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}}$$

Which simplifies to

$$y = -\frac{1}{-3 + c_2\sqrt{x+1}\,\sqrt{x-1}}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{-3 + c_2\sqrt{x+1}\,\sqrt{x-1}} \tag{1}$$

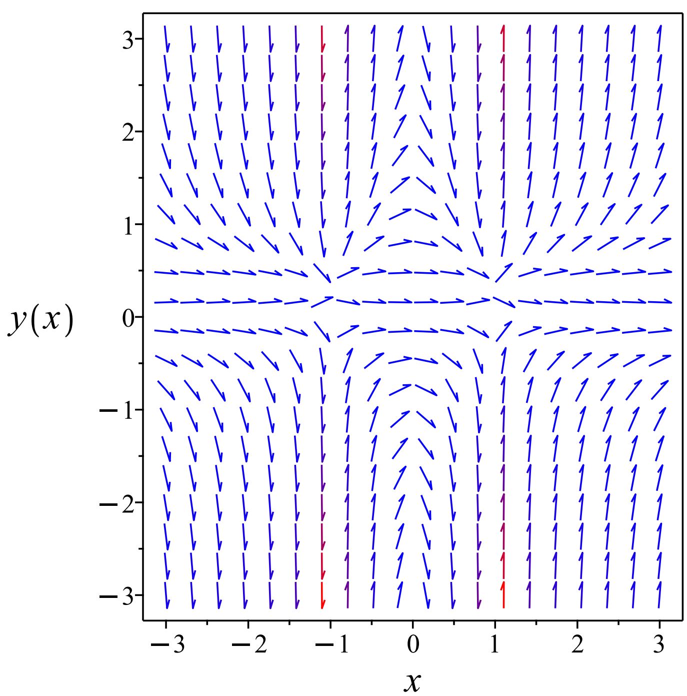


Figure 244: Slope field plot

Verification of solutions

$$y = -\frac{1}{-3 + c_2\sqrt{x+1}\,\sqrt{x-1}}$$

Verified OK.

6.34.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{xy(-1+3y)}{x^2-1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 138: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = \frac{x^2 - 1}{x}$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x^2-1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{xy(-1+3y)}{x^2-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{x}{x^2-1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y(-1+3y)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R(-1+3R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(-1+3R) - \ln(R) + c_1 \tag{4}$$

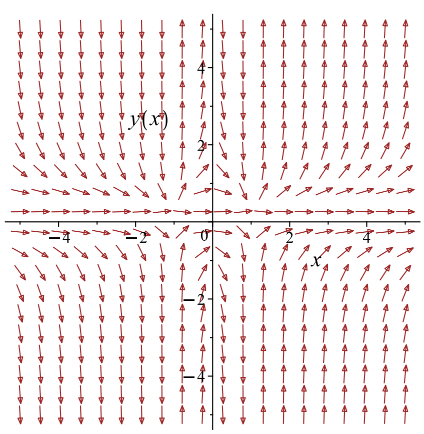
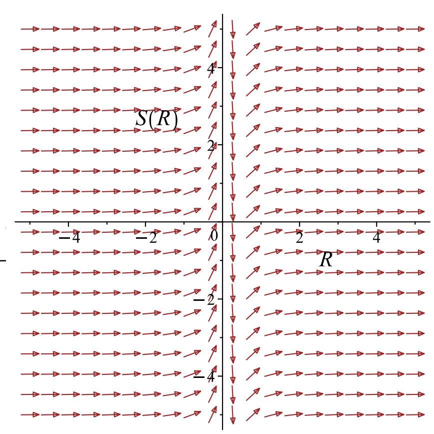
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2} = \ln(-1+3y) - \ln(y) + c_1$$

Which simplifies to

$$\frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2} = \ln(-1+3y) - \ln(y) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{xy(-1+3y)}{x^2-1}$ 	$R = y$ $S = \frac{\ln(x-1)}{2} + \frac{\ln(x+}{2}$	$\frac{dS}{dR} = \frac{1}{R(-1+3R)}$ 

Summary
The solution(s) found are the following

$$\frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2} = \ln(-1+3y) - \ln(y) + c_1 \tag{1}$$

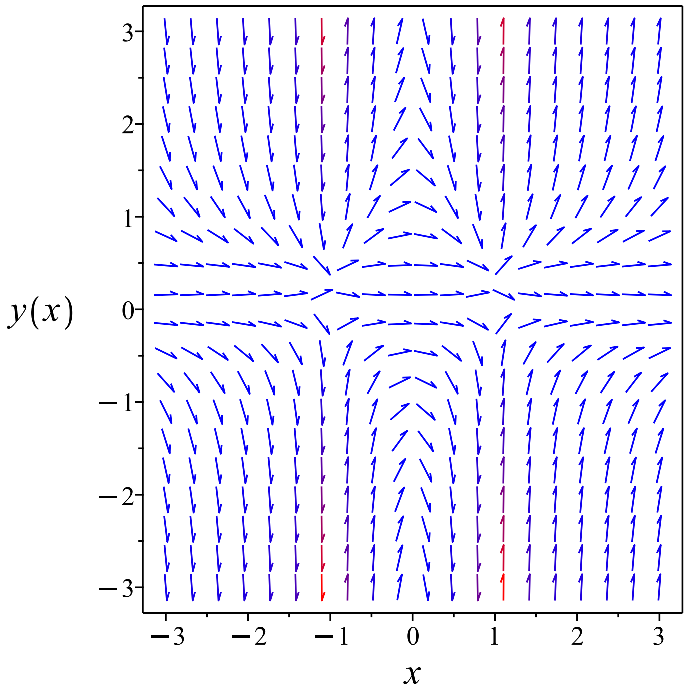


Figure 245: Slope field plot

Verification of solutions

$$\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}=\ln(-1+3y)-\ln(y)+c_1$$

Verified OK.

6.34.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{xy(-1+3y)}{x^2-1} \end{aligned}$$

This is a Bernoulli ODE.

$$y'=-\frac{x}{x^2-1}y+\frac{3x}{x^2-1}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y'=f_0(x)y+f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n}=f_0(x)y^{1-n}+f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{x}{x^2-1} \\ f_1(x) &= \frac{3x}{x^2-1} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y' \frac{1}{y^2} = -\frac{x}{(x^2-1)\, y} + \frac{3x}{x^2-1} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{x w(x)}{x^2-1} + \frac{3x}{x^2-1} \\ w' &= \frac{x w}{x^2-1} - \frac{3x}{x^2-1} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x) w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{x}{x^2-1} \\ q(x) &= -\frac{3x}{x^2-1} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{xw(x)}{x^2-1} = -\frac{3x}{x^2-1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{x}{x^2-1}dx} \\ &= e^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}}\end{aligned}$$

Which simplifies to

$$\mu = \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{3x}{x^2-1}\right) \\ \frac{d}{dx}\left(\frac{w}{\sqrt{x-1}\,\sqrt{x+1}}\right) &= \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)\left(-\frac{3x}{x^2-1}\right) \\ d\left(\frac{w}{\sqrt{x-1}\,\sqrt{x+1}}\right) &= \left(-\frac{3x}{(x^2-1)\sqrt{x-1}\,\sqrt{x+1}}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{\sqrt{x-1}\,\sqrt{x+1}} &= \int -\frac{3x}{(x^2-1)\sqrt{x-1}\,\sqrt{x+1}}\, dx \\ \frac{w}{\sqrt{x-1}\,\sqrt{x+1}} &= \frac{3\sqrt{x-1}\,\sqrt{x+1}}{x^2-1} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}$ results in

$$w(x) = \frac{3(x+1)(x-1)}{x^2-1} + c_1\sqrt{x+1}\,\sqrt{x-1}$$

which simplifies to

$$w(x) = 3 + c_1\sqrt{x+1}\,\sqrt{x-1}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = 3 + c_1\sqrt{x+1}\,\sqrt{x-1}$$

Or

$$y = \frac{1}{3 + c_1\sqrt{x+1}\,\sqrt{x-1}}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{3 + c_1\sqrt{x+1}\,\sqrt{x-1}} \tag{1}$$

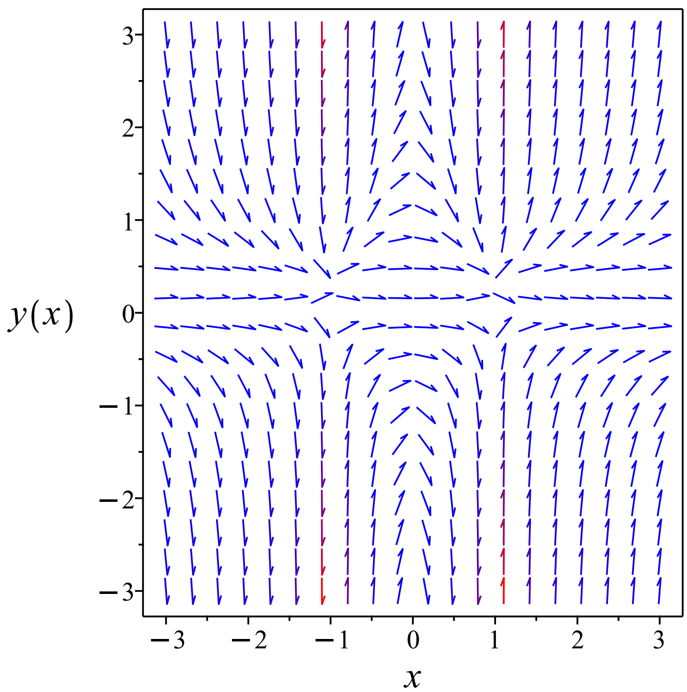


Figure 246: Slope field plot

Verification of solutions

$$y = \frac{1}{3 + c_1\sqrt{x+1}\,\sqrt{x-1}}$$

Verified OK.

6.34.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y(-1+3y)}\right)\mathrm{d}y&=\left(\frac{x}{x^2-1}\right)\mathrm{d}x\\ \left(-\frac{x}{x^2-1}\right)\mathrm{d}x+\left(\frac{1}{y(-1+3y)}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{x}{x^2-1} \\
N(x,y) &= \frac{1}{y\left(-1+3y\right)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2-1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y\left(-1+3y\right)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{x^2-1}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln\left(x-1\right)}{2} - \frac{\ln\left(x+1\right)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{y(-1+3y)}$. Therefore equation (4) becomes

$$\frac{1}{y(-1+3y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y(-1+3y)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y(-1+3y)}\right) \mathrm{d}y$$
$$f(y) = \ln(-1+3y) - \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \ln(-1+3y) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \ln(-1+3y) - \ln(y)$$

Summary
The solution(s) found are the following

$$-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \ln(-1+3y) - \ln(y) = c_1 \tag{1}$$

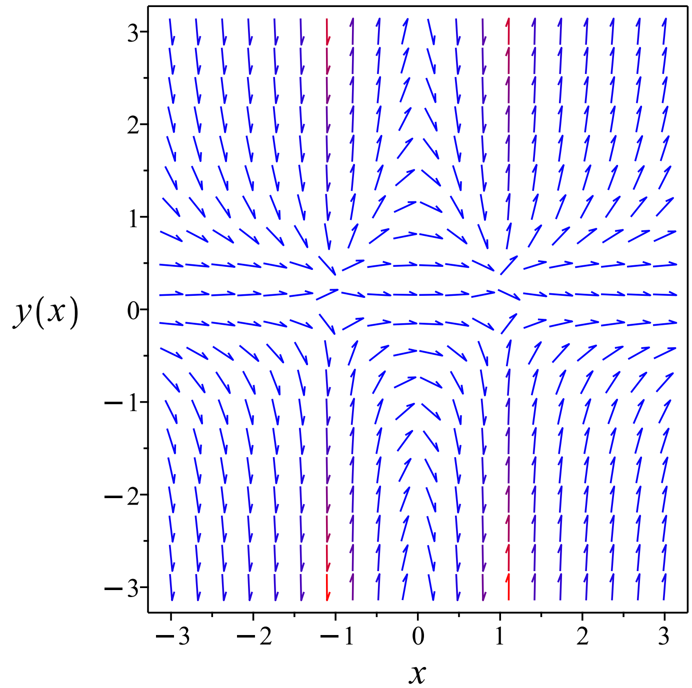


Figure 247: Slope field plot

Verification of solutions

$$-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}+\ln(-1+3y)-\ln(y)=c_1$$

Verified OK.

6.34.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= \frac{xy(-1+3y)}{x^2-1}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=\frac{3y^2x}{x^2-1}-\frac{xy}{x^2-1}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{x}{x^2-1}$ and $f_2(x) = \frac{3x}{x^2-1}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{3xu}{x^2-1}} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= \frac{3}{x^2-1} - \frac{6x^2}{(x^2-1)^2} \\
f_1 f_2 &= -\frac{3x^2}{(x^2-1)^2} \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{3xu''(x)}{x^2-1} - \left(\frac{3}{x^2-1} - \frac{9x^2}{(x^2-1)^2}\right) u'(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 + \frac{c_2}{\sqrt{x^2-1}}
$$

The above shows that

$$
u'(x) = -\frac{c_2 x}{(x^2-1)^{\frac{3}{2}}}
$$

Using the above in (1) gives the solution

$$
y = \frac{c_2}{3\sqrt{x^2-1}\left(c_1 + \frac{c_2}{\sqrt{x^2-1}}\right)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{1}{3c_3\sqrt{x^2-1}+3}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{3c_3\sqrt{x^2-1}+3} \tag{1}$$

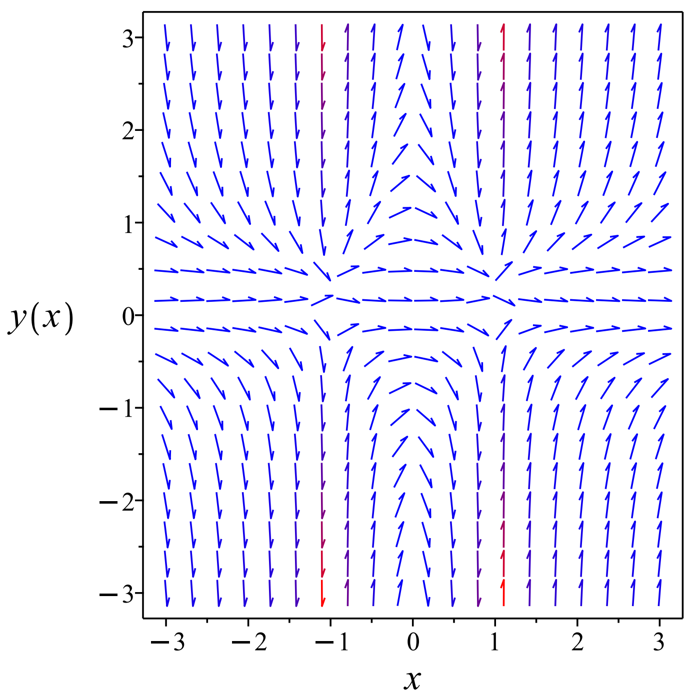


Figure 248: Slope field plot

Verification of solutions

$$y = \frac{1}{3c_3\sqrt{x^2-1}+3}$$

Verified OK.

6.34.6 Maple step by step solution

Let's solve

$\left(x^{2}-1\right) y'+x y-3 x y^{2}=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(-1+3y)}=\frac{x}{(x-1)(x+1)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(-1+3y)}dx=\int \frac{x}{(x-1)(x+1)}dx+c_1$

- Evaluate integral

  $\ln\left(-1+3y\right)-\ln\left(y\right)=\frac{\ln((x-1)(x+1))}{2}+c_1$

- Solve for y

  $\left\{y=\frac{-3+\sqrt{\mathrm{e}^{2c_1}x^{2}-\mathrm{e}^{2c_1}}}{\mathrm{e}^{2c_1}x^{2}-\mathrm{e}^{2c_1}-9}, y=-\frac{3+\sqrt{\mathrm{e}^{2c_1}x^{2}-\mathrm{e}^{2c_1}}}{\mathrm{e}^{2c_1}x^{2}-\mathrm{e}^{2c_1}-9}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve((x^2-1)*diff(y(x),x)+x*y(x)-3*x*y(x)^2=0,y(x), singsol=all)
```

$$y(x)=\frac{1}{3+\sqrt{x-1}\,\sqrt{1+x}\,c_1}$$

✓ Solution by Mathematica

Time used: 2.214 (sec). Leaf size: 35

```
DSolve[(x^2-1)*y'[x]+x*y[x]-3*x*y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3 + e^{c_1}\sqrt{x^2 - 1}}$$
$$y(x) \to 0$$
$$y(x) \to \frac{1}{3}$$

6.35 problem Exercise 12.35, page 103

Internal problem ID [4556]
Internal file name [`OUTPUT/4049_Sunday_June_05_2022_12_15_02_PM_96926832/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.35, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(x^2-1)\,y'-2xy\ln(y)=0}$$

6.35.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= \frac{2xy\ln(y)}{x^2-1}
\end{aligned}$$

Where $f(x)=\frac{2x}{x^2-1}$ and $g(y)=\ln(y)\,y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\ln(y)\,y}\,dy &= \frac{2x}{x^2-1}\,dx\\
\int \frac{1}{\ln(y)\,y}\,dy &= \int \frac{2x}{x^2-1}\,dx\\
\ln(\ln(y)) &= \ln(x-1)+\ln(x+1)+c_1
\end{aligned}$$

Raising both side to exponential gives

$$\ln(y) = e^{\ln(x-1)+\ln(x+1)+c_1}$$

Which simplifies to

$$\ln(y) = c_2 e^{\ln(x-1)+\ln(x+1)}$$

Summary

The solution(s) found are the following

$$y = e^{e^{c_1} c_2 x^2 - c_2 e^{c_1}} \tag{1}$$

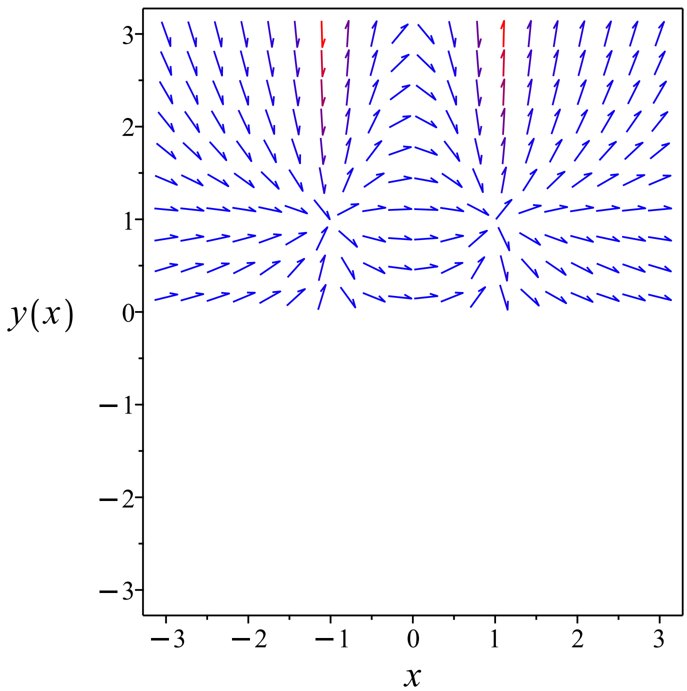


Figure 249: Slope field plot

Verification of solutions

$$y = e^{e^{c_1} c_2 x^2 - c_2 e^{c_1}}$$

Verified OK.

6.35.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2xy\ln(y)}{x^2-1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 141: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{x^2-1}{2x}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x^2-1}{2x}} dx \end{aligned}$$

Which results in

$$S = \ln(x-1) + \ln(x+1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2xy\ln(y)}{x^2-1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{2x}{x^2-1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y\ln(y)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R\ln(R)}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(\ln(R)) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln(x-1) + \ln(x+1) = \ln(\ln(y)) + c_1
$$

Which simplifies to

$$
\ln(x-1) + \ln(x+1) = \ln(\ln(y)) + c_1
$$

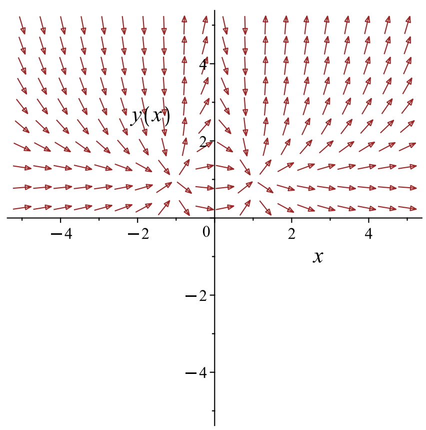
Which gives

$$
y = e^{e^{-c_1}(x+1)(x-1)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{2xy\ln(y)}{x^2-1}$



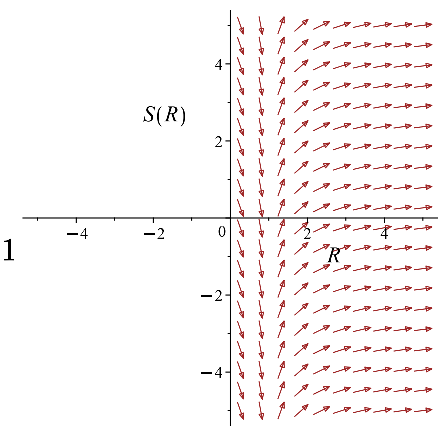
</td><td>

$R = y$

$S = \ln(x-1) + \ln(x+1$

</td><td>

$\frac{dS}{dR} = \frac{1}{R\ln(R)}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = e^{e^{-c_1}(x+1)(x-1)} \tag{1}$$

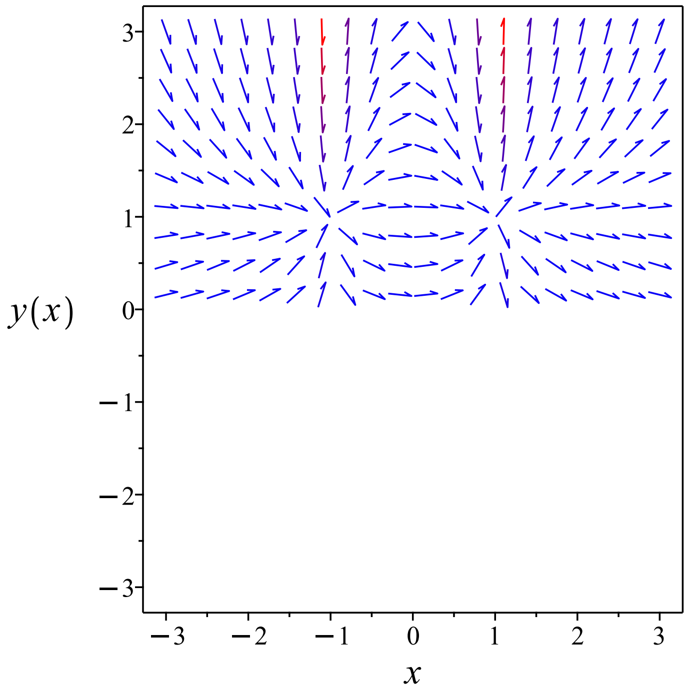


Figure 250: Slope field plot

<u>Verification of solutions</u>

$$y = e^{e^{-c_1}(x+1)(x-1)}$$

Verified OK.

6.35.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{2y \ln(y)}\right) \mathrm{d}y = \left(\frac{x}{x^2 - 1}\right) \mathrm{d}x$$
$$\left(-\frac{x}{x^2 - 1}\right) \mathrm{d}x + \left(\frac{1}{2y \ln(y)}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x}{x^2 - 1}$$
$$N(x, y) = \frac{1}{2y \ln(y)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{x}{x^2 - 1}\right)$$
$$= 0$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2y\ln\left(y\right)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{x^2-1}\,\mathrm{d}x\\ \phi &= -\frac{\ln\left(x-1\right)}{2} - \frac{\ln\left(x+1\right)}{2} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y\ln(y)}$. Therefore equation (4) becomes

$$\frac{1}{2y\ln\left(y\right)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{2y\ln\left(y\right)}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int \left(\frac{1}{2y \ln(y)}\right) dy$$
$$f(y) = \frac{\ln(\ln(y))}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \frac{\ln(\ln(y))}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \frac{\ln(\ln(y))}{2}$$

The solution becomes

$$y = e^{e^{2c_1}(x-1)(x+1)}$$

Summary

The solution(s) found are the following

$$y = e^{e^{2c_1}(x-1)(x+1)} \tag{1}$$

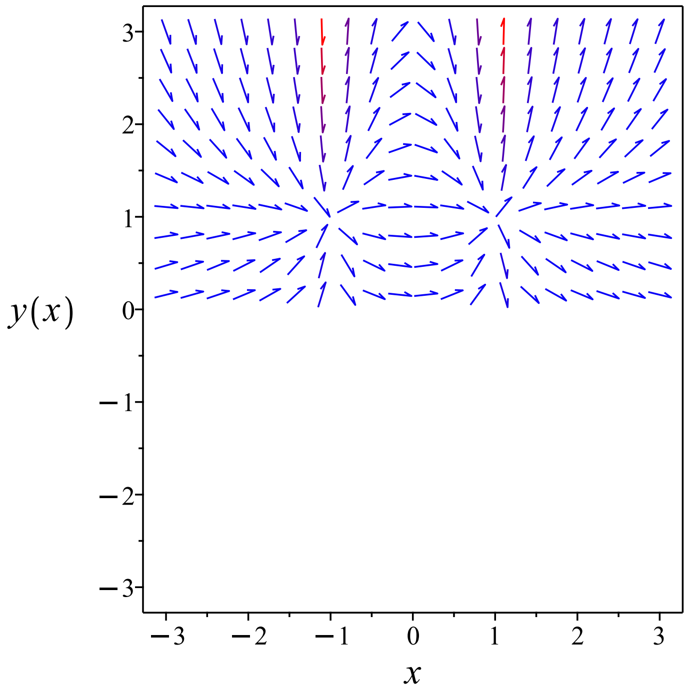


Figure 251: Slope field plot

Verification of solutions

$$y = e^{e^{2c_1}(x-1)(x+1)}$$

Verified OK.

6.35.4 Maple step by step solution

Let's solve

$(x^2-1)\,y' - 2xy\ln(y) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y\ln(y)} = \frac{2x}{x^2-1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y\ln(y)}dx = \int \frac{2x}{x^2-1}dx + c_1$

- Evaluate integral

$$\ln(\ln(y)) = \ln(x-1) + \ln(x+1) + c_1$$

- Solve for y

$$y = e^{e^{c_1}(x+1)(x-1)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve((x^2-1)*diff(y(x),x)-2*x*y(x)*ln(y(x))=0,y(x), singsol=all)
```

$$y(x) = e^{c_1(x-1)(1+x)}$$

✓ Solution by Mathematica

Time used: 0.223 (sec). Leaf size: 22

```
DSolve[(x^2-1)*y'[x]-2*x*y[x]*Log[y[x]]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{e^{c_1}(x^2-1)}$$
$$y(x) \to 1$$

6.36 problem Exercise 12.36, page 103

Internal problem ID [4557]
Internal file name [`OUTPUT/4050_Sunday_June_05_2022_12_15_12_PM_75214007/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.36, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{\left(1+x^{2}+y^{2}\right) y'+2xy = -x^{2}-3}$$

6.36.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-2xy-x^{2}-3}{1+x^{2}+y^{2}} \tag{1}$$

Which becomes

$$\left(y^{2}+1\right) dy = \left(-x^{2}\right) dy+\left(-x^{2}-2xy-3\right) dx \tag{2}$$

But the RHS is complete differential because

$$\left(-x^{2}\right) dy+\left(-x^{2}-2xy-3\right) dx = d\left(-\frac{1}{3}x^{3}-y\,x^{2}-3x\right)$$

Hence (2) becomes

$$\left(y^{2}+1\right) dy = d\left(-\frac{1}{3}x^{3}-y\,x^{2}-3x\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(-4x^3 + 12c_1 - 36x + 4\sqrt{5x^6 - 6c_1x^3 + 30x^4 + 9c_1^2 - 54c_1x + 93x^2 + 4}\right)^{\frac{1}{3}}}{2} - \frac{}{\left(-4x^3 + 12c_1 - 36x +\right.}$$

$$y = -\frac{\left(-4x^3 + 12c_1 - 36x + 4\sqrt{5x^6 - 6c_1x^3 + 30x^4 + 9c_1^2 - 54c_1x + 93x^2 + 4}\right)^{\frac{1}{3}}}{4} + \frac{}{\left(-4x^3 + 12c_1 - 36x\right.}$$

$$y = -\frac{\left(-4x^3 + 12c_1 - 36x + 4\sqrt{5x^6 - 6c_1x^3 + 30x^4 + 9c_1^2 - 54c_1x + 93x^2 + 4}\right)^{\frac{1}{3}}}{4} + \frac{}{\left(-4x^3 + 12c_1 - 36x\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}}{2} - \frac{2(x^2+1)}{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}} + c_1 \tag{1}$$

$$y = -\frac{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}}{4} + \frac{x^2+1}{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2x^2+2}{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+\cdots}\right.}\cdots\right)}{2} + c_1 \tag{2}$$

$$y = -\frac{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}}{4} + \frac{x^2+1}{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2x^2+2}{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+\cdots}\right.}\cdots\right)}{2} + c_1 \tag{3}$$

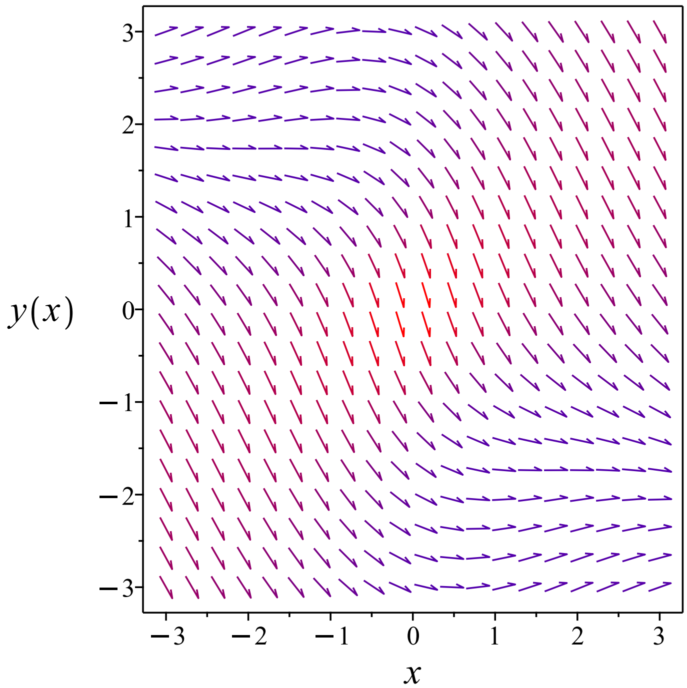


Figure 252: Slope field plot

Verification of solutions

$$y = \frac{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}}{2} - \frac{2(x^2+1)}{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}} + c_1$$

Verified OK.

$$y = -\frac{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}}{4} + \frac{x^2+1}{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2x^2+2}{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+\right.}}\right.}{2} + c_1$$

Verified OK.

$$y = -\frac{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}}{4} + \frac{x^2+1}{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2x^2+2}{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+\right.}}\right.}{2} + c_1$$

Verified OK.

6.36.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2+y^2+1\right)\mathrm{d}y&=\left(-x^2-2xy-3\right)\mathrm{d}x\\ \left(x^2+2xy+3\right)\mathrm{d}x+\left(x^2+y^2+1\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x^2+2xy+3\\ N(x,y)&=x^2+y^2+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + 2xy + 3\right) \\ &= 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 + y^2 + 1\right) \\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int x^2 + 2xy + 3 \, \mathrm{d}x \\ \phi &= \frac{1}{3}x^3 + y\, x^2 + 3x + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 + y^2 + 1$. Therefore equation (4) becomes

$$x^2 + y^2 + 1 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2 + 1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y^2 + 1)\,\mathrm{d}y$$
$$f(y) = \frac{1}{3}y^3 + y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{3}x^3 + y\,x^2 + 3x + \frac{1}{3}y^3 + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{3}x^3 + y\,x^2 + 3x + \frac{1}{3}y^3 + y$$

Summary

The solution(s) found are the following

$$\frac{x^3}{3} + yx^2 + 3x + \frac{y^3}{3} + y = c_1 \tag{1}$$

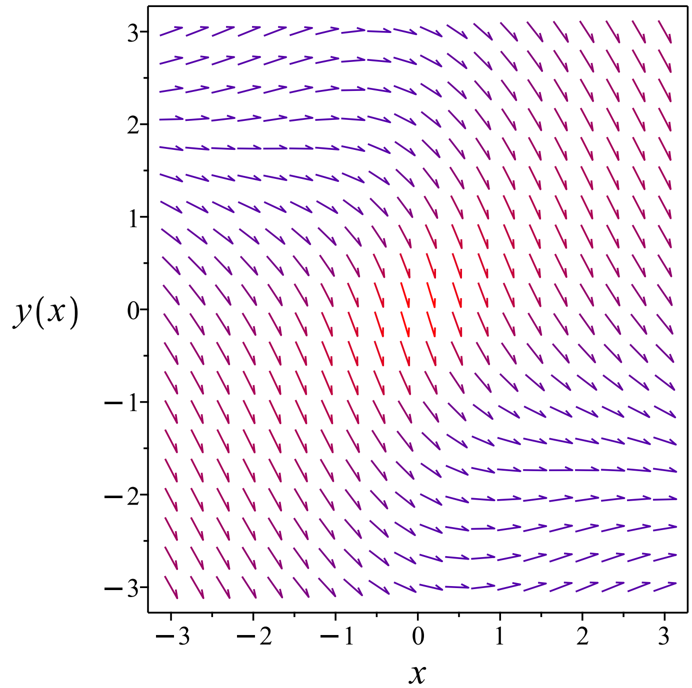


Figure 253: Slope field plot

Verification of solutions

$$\frac{x^3}{3} + yx^2 + 3x + \frac{y^3}{3} + y = c_1$$

Verified OK.

6.36.3 Maple step by step solution

Let's solve

$(1 + x^2 + y^2)\, y' + 2xy = -x^2 - 3$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $2x = 2x$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (x^2 + 2xy + 3)\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = \frac{x^3}{3} + y\,x^2 + 3x + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x^2 + y^2 + 1 = x^2 + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = y^2 + 1$

- Solve for $f_1(y)$

  $f_1(y) = \frac{1}{3}y^3 + y$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = \frac{1}{3}x^3 + y\,x^2 + 3x + \frac{1}{3}y^3 + y$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{1}{3}x^3 + y\,x^2 + 3x + \frac{1}{3}y^3 + y = c_1$

- Solve for y

$$\left\{ y = \frac{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-54c_1x+93x^2+4}\right)^{\frac{1}{3}}}{2} - \frac{2(x^2+1)}{\left(-4x^3+12c_1-36x+4\sqrt{5x^6-6c_1x^3+30x^4+9c_1^2-5\right.}}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 370

```
dsolve((x^2+y(x)^2+1)*diff(y(x),x)+2*x*y(x)+x^2+3=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(-4x^3-12c_1-36x+4\sqrt{5x^6+6c_1x^3+30x^4+9c_1^2+54c_1x+93x^2+4}\right)^{\frac{2}{3}}-4x^2-4}{2\left(-4x^3-12c_1-36x+4\sqrt{5x^6+6c_1x^3+30x^4+9c_1^2+54c_1x+93x^2+4}\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{\left(\frac{i\sqrt{3}}{4}+\frac{1}{4}\right)\left(-4x^3-12c_1-36x+4\sqrt{5x^6+6c_1x^3+30x^4+9c_1^2+54c_1x+93x^2+4}\right)^{\frac{2}{3}}+\left(i\sqrt{3}-1\right)(}{\left(-4x^3-12c_1-36x+4\sqrt{5x^6+6c_1x^3+30x^4+9c_1^2+54c_1x+93x^2+4}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left(-4x^3-12c_1-36x+4\sqrt{5x^6+6c_1x^3+30x^4+9c_1^2+54c_1x+93x^2+4}\right)^{\frac{1}{3}}}{4} + \frac{\left(1+i\sqrt{3}\right)\left(x^2+1\right)}{\left(-4x^3-12c_1-36x+4\sqrt{5x^6+6c_1x^3+30x^4+9c_1^2+54c_1x+93x^2+4}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 5.385 (sec). Leaf size: 411

```
DSolve[(x^2+y[x]^2+1)*y'[x]+2*x*y[x]+x^2+3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) \to & \frac{\sqrt[3]{-27x^3+\sqrt{4\left(9x^2+9\right)^3+729\left(x^3+9x-3c_1\right)^2}-243x+81c_1}}{3\sqrt[3]{2}} \\
& -\frac{3\sqrt[3]{2}(x^2+1)}{\sqrt[3]{-27x^3+\sqrt{4\left(9x^2+9\right)^3+729\left(x^3+9x-3c_1\right)^2}-243x+81c_1}} \\
y(x) \to & \frac{3\left(1+i\sqrt{3}\right)\left(x^2+1\right)}{2^{2/3}\sqrt[3]{-27x^3+\sqrt{4\left(9x^2+9\right)^3+729\left(x^3+9x-3c_1\right)^2}-243x+81c_1}} \\
& +\frac{\left(-1+i\sqrt{3}\right)\sqrt[3]{-27x^3+\sqrt{4\left(9x^2+9\right)^3+729\left(x^3+9x-3c_1\right)^2}-243x+81c_1}}{6\sqrt[3]{2}} \\
y(x) \to & \frac{3\left(1-i\sqrt{3}\right)\left(x^2+1\right)}{2^{2/3}\sqrt[3]{-27x^3+\sqrt{4\left(9x^2+9\right)^3+729\left(x^3+9x-3c_1\right)^2}-243x+81c_1}} \\
& -\frac{\left(1+i\sqrt{3}\right)\sqrt[3]{-27x^3+\sqrt{4\left(9x^2+9\right)^3+729\left(x^3+9x-3c_1\right)^2}-243x+81c_1}}{6\sqrt[3]{2}}
\end{aligned}
$$

6.37 problem Exercise 12.37, page 103

Internal problem ID [4558]
Internal file name [`OUTPUT/4051_Sunday_June_05_2022_12_15_21_PM_39830139/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.37, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\cos(x) y' + y = -(1 + \sin(x)) \cos(x)}$$

6.37.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \sec(x)$$
$$q(x) = -\sin(x) - 1$$

Hence the ode is

$$y' + y\sec(x) = -\sin(x) - 1$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \sec(x)dx} \\
&= \sec(x) + \tan(x)
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\sin(x) - 1\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(\sec(x) + \tan(x)\right) y\right) &= \left(\sec(x) + \tan(x)\right)\left(-\sin(x) - 1\right) \\
\mathrm{d}\left(\left(\sec(x) + \tan(x)\right) y\right) &= \left(\cos(x) - 2\tan(x) - 2\sec(x)\right) \,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\left(\sec(x) + \tan(x)\right) y &= \int \cos(x) - 2\tan(x) - 2\sec(x) \,\mathrm{d}x \\
\left(\sec(x) + \tan(x)\right) y &= -2\ln\left(\sec(x) + \tan(x)\right) + 2\ln\left(\cos(x)\right) + \sin(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sec(x) + \tan(x)$ results in

$$
y = \frac{-2\ln\left(\sec(x) + \tan(x)\right) + 2\ln\left(\cos(x)\right) + \sin(x)}{\sec(x) + \tan(x)} + \frac{c_1}{\sec(x) + \tan(x)}
$$

which simplifies to

$$
y = \frac{-2\ln\left(\sec(x) + \tan(x)\right) + 2\ln\left(\cos(x)\right) + \sin(x) + c_1}{\sec(x) + \tan(x)}
$$

Summary

The solution(s) found are the following

$$
y = \frac{-2\ln\left(\sec(x) + \tan(x)\right) + 2\ln\left(\cos(x)\right) + \sin(x) + c_1}{\sec(x) + \tan(x)} \tag{1}
$$

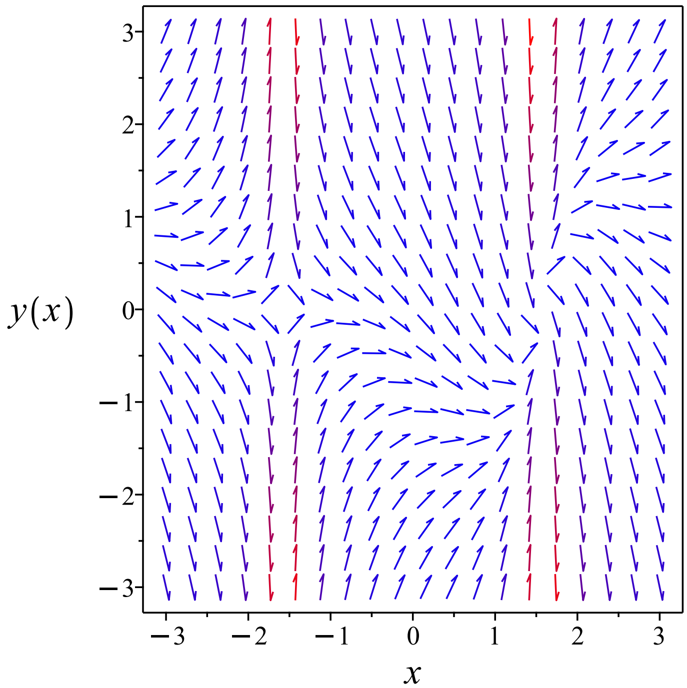


Figure 254: Slope field plot

Verification of solutions

$$y = \frac{-2\ln(\sec(x) + \tan(x)) + 2\ln(\cos(x)) + \sin(x) + c_1}{\sec(x) + \tan(x)}$$

Verified OK.

6.37.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\cos(x)\sin(x) + \cos(x) + y}{\cos(x)}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 145: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = 0$$
$$\eta(x, y) = \frac{1}{\sec(x) + \tan(x)} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{\sec(x)+\tan(x)}} dy
\end{aligned}$$

Which results in

$$S = (\sec(x) + \tan(x))\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\cos(x)\sin(x) + \cos(x) + y}{\cos(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y}{-1 + \sin(x)} \\
S_y &= \sec(x) + \tan(x)
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(x) - 2\tan(x) - 2\sec(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R) - 2\tan(R) - 2\sec(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -2\ln(\tan(R) + \sec(R)) + 2\ln(\cos(R)) + \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(\sec(x) + \tan(x))\, y = -2\ln(\sec(x) + \tan(x)) + 2\ln(\cos(x)) + \sin(x) + c_1$$

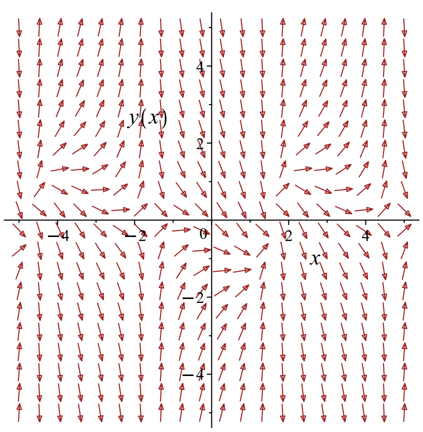
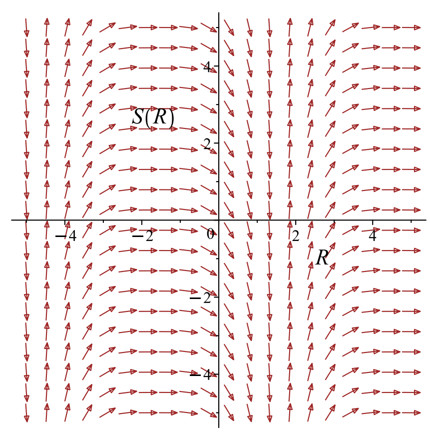
Which simplifies to

$$(\sec(x) + \tan(x))\, y = -2\ln(\sec(x) + \tan(x)) + 2\ln(\cos(x)) + \sin(x) + c_1$$

Which gives

$$y = \frac{-2\ln(\sec(x) + \tan(x)) + 2\ln(\cos(x)) + \sin(x) + c_1}{\sec(x) + \tan(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{\cos(x)\sin(x)+\cos(x)+y}{\cos(x)}$

</td>
<td>$R = x$
$S = (\sec(x) + \tan(x))\, y$</td>
<td>$\frac{dS}{dR} = \cos(R) - 2\tan(R) - 2\sec(R)$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{-2\ln(\sec(x) + \tan(x)) + 2\ln(\cos(x)) + \sin(x) + c_1}{\sec(x) + \tan(x)} \tag{1}$$

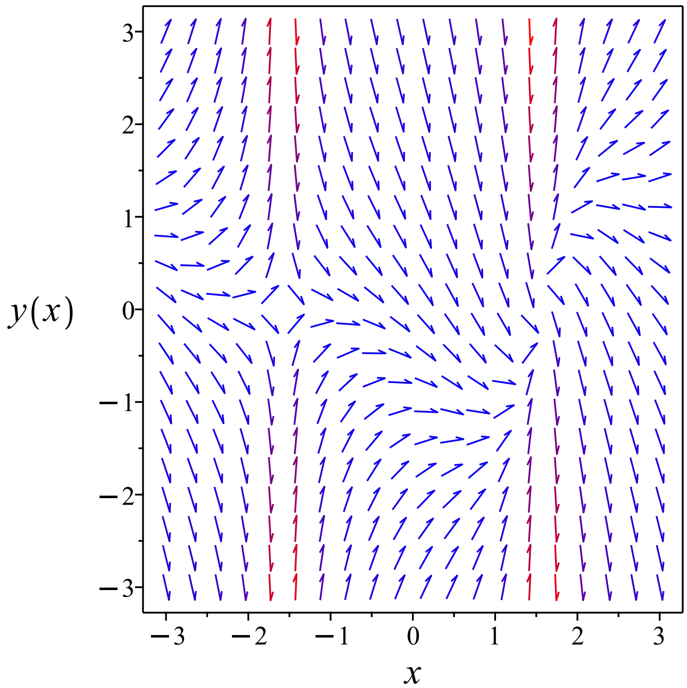


Figure 255: Slope field plot

Verification of solutions

$$y = \frac{-2\ln\left(\sec\left(x\right)+\tan\left(x\right)\right)+2\ln\left(\cos\left(x\right)\right)+\sin\left(x\right)+c_1}{\sec\left(x\right)+\tan\left(x\right)}$$

Verified OK.

6.37.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(x))\,\mathrm{d}y &= (-y - (1 + \sin(x))\cos(x))\,\mathrm{d}x\\ (y + (1 + \sin(x))\cos(x))\,\mathrm{d}x + (\cos(x))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y + (1 + \sin(x))\cos(x)\\ N(x, y) &= \cos(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y + (1 + \sin(x))\cos(x))\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(x))\\ &= -\sin(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \sec(x)\left((1) - (-\sin(x))\right) \\ &= \sec(x) + \tan(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \sec(x)+\tan(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(\sec(x)+\tan(x))-\ln(\cos(x))} \\ &= \frac{\sec(x) + \tan(x)}{\cos(x)} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{\sec(x) + \tan(x)}{\cos(x)}(y + (1 + \sin(x))\cos(x)) \\ &= \frac{(-\sin(x) - 1)\cos(x) - y}{-1 + \sin(x)} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{\sec(x) + \tan(x)}{\cos(x)}(\cos(x)) \\ &= \sec(x) + \tan(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{(-\sin(x) - 1)\cos(x) - y}{-1 + \sin(x)}\right) + (\sec(x) + \tan(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{(-\sin(x) - 1)\cos(x) - y}{-1 + \sin(x)} \, \mathrm{d}x$$
$$\phi = -\frac{2y}{\tan\left(\frac{x}{2}\right) - 1} - 4\ln\left(\tan\left(\frac{x}{2}\right) - 1\right) - \frac{2\tan\left(\frac{x}{2}\right)}{\tan\left(\frac{x}{2}\right)^2 + 1} + 2\ln\left(\sec\left(\frac{x}{2}\right)^2\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{2}{\tan\left(\frac{x}{2}\right) - 1} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sec(x) + \tan(x)$. Therefore equation (4) becomes

$$\sec(x) + \tan(x) = -\frac{2}{\tan\left(\frac{x}{2}\right) - 1} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{\tan(x)\tan\left(\frac{x}{2}\right) + \sec(x)\tan\left(\frac{x}{2}\right) - \tan(x) - \sec(x) + 2}{\tan\left(\frac{x}{2}\right) - 1}$$
$$= -1$$

Integrating the above w.r.t y results in

$$\int f'(y) \, \mathrm{d}y = \int (-1) \, \mathrm{d}y$$
$$f(y) = -y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{2y}{\tan\left(\frac{x}{2}\right)-1} - 4\ln\left(\tan\left(\frac{x}{2}\right)-1\right) - \frac{2\tan\left(\frac{x}{2}\right)}{\tan\left(\frac{x}{2}\right)^2+1} + 2\ln\left(\sec\left(\frac{x}{2}\right)^2\right) - y + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{2y}{\tan\left(\frac{x}{2}\right)-1} - 4\ln\left(\tan\left(\frac{x}{2}\right)-1\right) - \frac{2\tan\left(\frac{x}{2}\right)}{\tan\left(\frac{x}{2}\right)^2+1} + 2\ln\left(\sec\left(\frac{x}{2}\right)^2\right) - y
$$

The solution becomes

$$
y = \frac{2\tan\left(\frac{x}{2}\right)^3\ln\left(\sec\left(\frac{x}{2}\right)^2\right) - 4\tan\left(\frac{x}{2}\right)^3\ln\left(\tan\left(\frac{x}{2}\right)-1\right) - \tan\left(\frac{x}{2}\right)^3 c_1 - 2\tan\left(\frac{x}{2}\right)^2\ln\left(\sec\left(\frac{x}{2}\right)^2\right) + 4\tan\left(\right.}{}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2\tan\left(\frac{x}{2}\right)^3\ln\left(\sec\left(\frac{x}{2}\right)^2\right) - 4\tan\left(\frac{x}{2}\right)^3\ln\left(\tan\left(\frac{x}{2}\right)-1\right) - \tan\left(\frac{x}{2}\right)^3 c_1 - 2\tan\left(\frac{x}{2}\right)^2\ln\left(\sec\left(\frac{x}{2}\right)^2\right) + 4\tan\left(\right.}{} \tag{1}
$$

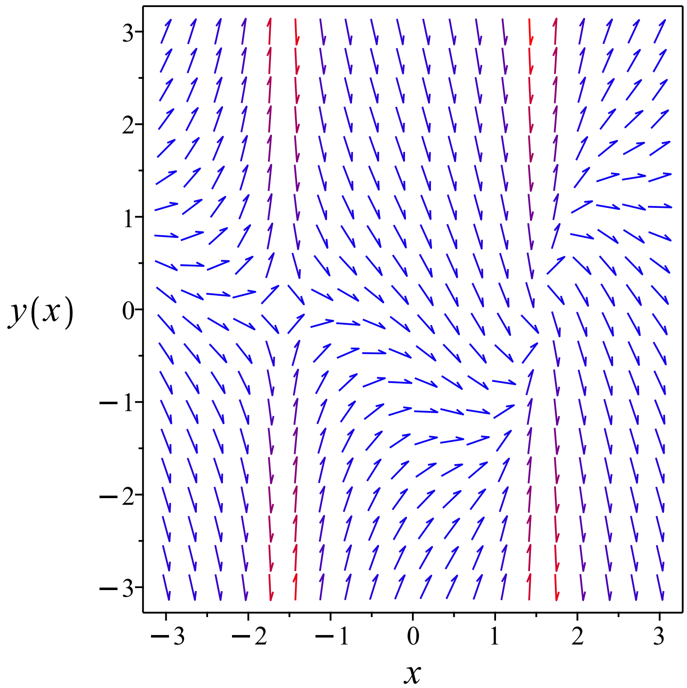


Figure 256: Slope field plot

Verification of solutions

$$y = \frac{2\tan\left(\frac{x}{2}\right)^3 \ln\left(\sec\left(\frac{x}{2}\right)^2\right) - 4\tan\left(\frac{x}{2}\right)^3 \ln\left(\tan\left(\frac{x}{2}\right) - 1\right) - \tan\left(\frac{x}{2}\right)^3 c_1 - 2\tan\left(\frac{x}{2}\right)^2 \ln\left(\sec\left(\frac{x}{2}\right)^2\right) + 4\tan\left(\frac{x}{2}\right. \cdots}{\cdots}$$

Verified OK.

6.37.4 Maple step by step solution

Let's solve

$\cos(x)\, y' + y = -(1 + \sin(x)) \cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\sin(x) - 1 - \frac{y}{\cos(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{\cos(x)} = -\sin(x) - 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{y}{\cos(x)}\right)=\mu(x)\left(-\sin\left(x\right)-1\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{y}{\cos(x)}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)}{\cos(x)}$

- Solve to find the integrating factor

  $\mu(x)=\sec\left(x\right)+\tan\left(x\right)$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\left(-\sin\left(x\right)-1\right)dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\mu(x)\left(-\sin\left(x\right)-1\right)dx+c_1$

- Solve for y

  $y=\frac{\int\mu(x)(-\sin(x)-1)dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\sec\left(x\right)+\tan\left(x\right)$

  $y=\frac{\int(-\sin(x)-1)(\sec(x)+\tan(x))dx+c_1}{\sec(x)+\tan(x)}$

- Evaluate the integrals on the rhs

  $y=\frac{-2\ln(\sec(x)+\tan(x))+2\ln(\cos(x))+\sin(x)+c_1}{\sec(x)+\tan(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x)*cos(x)+y(x)+(1+sin(x))*cos(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-2\ln\left(\sec\left(x\right)+\tan\left(x\right)\right)+2\ln\left(\cos\left(x\right)\right)+\sin\left(x\right)+c_1}{\sec\left(x\right)+\tan\left(x\right)}$$

✓ Solution by Mathematica

Time used: 0.671 (sec). Leaf size: 40

```
DSolve[y'[x]*Cos[x]+y[x]+(1+Sin[x])*Cos[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2\text{arctanh}\left(\tan\left(\frac{x}{2}\right)\right)}\left(\sin(x)+4\log\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)+c_1\right)$$

6.38 problem Exercise 12.38, page 103

Internal problem ID [4559]
Internal file name [`OUTPUT/4052_Sunday_June_05_2022_12_15_32_PM_17307676/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.38, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _exact, _rational, [_Abel, `2nd
   type`, `class B`]]
```

$$\boxed{(2xy + 4x^3)\, y' + y^2 + 12yx^2 = 0}$$

6.38.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(12x^2 + y)}{2x\,(2x^2 + y)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 - \frac{y(12x^2+y)\,(b_3-a_2)}{2x\,(2x^2+y)} - \frac{y^2(12x^2+y)^2\,a_3}{4x^2\,(2x^2+y)^2} \\
& - \left(-\frac{12y}{2x^2+y} + \frac{y(12x^2+y)}{2x^2\,(2x^2+y)} + \frac{2y(12x^2+y)}{(2x^2+y)^2}\right)(xa_2+ya_3+a_1) \\
& - \left(-\frac{12x^2+y}{2x\,(2x^2+y)} - \frac{y}{2\,(2x^2+y)\,x} + \frac{y(12x^2+y)}{2x\,(2x^2+y)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$\frac{64x^6b_2 - 192x^4y^2a_3 + 48x^5b_1 - 48x^4ya_1 + 24x^4yb_2 + 40x^3y^2a_2 - 20x^3y^2b_3 - 12x^2y^3a_3 + 8x^3yb_1 + 12x^2y^2}{4\,(2x^2+y)^2\,x^2} = 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
& 64x^6b_2 - 192x^4y^2a_3 + 48x^5b_1 - 48x^4ya_1 + 24x^4yb_2 + 40x^3y^2a_2 - 20x^3y^2b_3 \\
& - 12x^2y^3a_3 + 8x^3yb_1 + 12x^2y^2a_1 + 6x^2y^2b_2 - 3y^4a_3 + 2x\,y^2b_1 - 2y^3a_1 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& -192a_3v_1^4v_2^2 + 64b_2v_1^6 - 48a_1v_1^4v_2 + 40a_2v_1^3v_2^2 - 12a_3v_1^2v_2^3 + 48b_1v_1^5 + 24b_2v_1^4v_2 \\
& - 20b_3v_1^3v_2^2 + 12a_1v_1^2v_2^2 - 3a_3v_2^4 + 8b_1v_1^3v_2 + 6b_2v_1^2v_2^2 - 2a_1v_2^3 + 2b_1v_1v_2^2 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$64b_2v_1^6 + 48b_1v_1^5 - 192a_3v_1^4v_2^2 + (-48a_1 + 24b_2)\, v_1^4v_2 + (40a_2 - 20b_3)\, v_1^3v_2^2 + 8b_1v_1^3v_2 - 12a_3v_1^2v_2^3 + (12a_1 + 6b_2)\, v_1^2v_2^2 + 2b_1v_1v_2^2 - 3a_3v_2^4 - 2a_1v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-2a_1 &= 0 \\
-192a_3 &= 0 \\
-12a_3 &= 0 \\
-3a_3 &= 0 \\
2b_1 &= 0 \\
8b_1 &= 0 \\
48b_1 &= 0 \\
64b_2 &= 0 \\
-48a_1 + 24b_2 &= 0 \\
12a_1 + 6b_2 &= 0 \\
40a_2 - 20b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= 2y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 2y - \left(-\frac{y(12x^2 + y)}{2x\,(2x^2 + y)}\right)(x) \\
&= \frac{20y\,x^2 + 5y^2}{4x^2 + 2y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{20y\,x^2 + 5y^2}{4x^2 + 2y}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln\left(y(4x^2 + y)\right)}{5}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{y(12x^2 + y)}{2x\,(2x^2 + y)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{8x}{20x^2 + 5y} \\
S_y &= \frac{1}{5y} + \frac{1}{20x^2 + 5y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{5x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{5R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\ln (R)}{5} + c_1 \tag{4}
$$

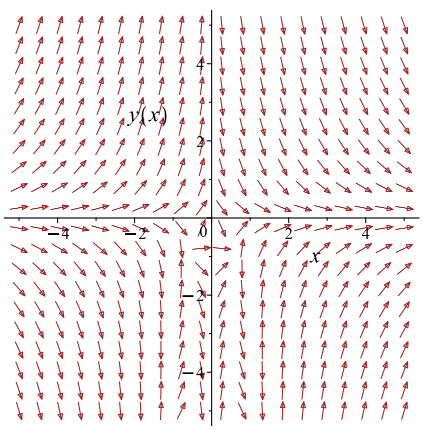
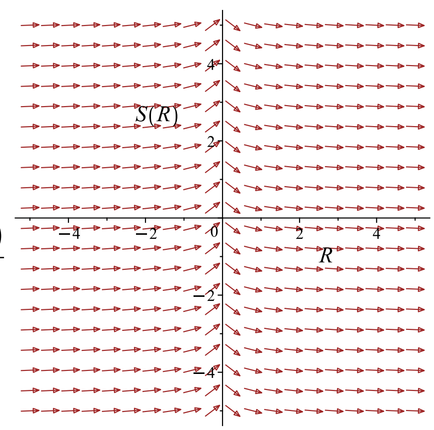
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln (y)}{5} + \frac{\ln (4x^2 + y)}{5} = -\frac{\ln (x)}{5} + c_1
$$

Which simplifies to

$$
\frac{\ln (y)}{5} + \frac{\ln (4x^2 + y)}{5} = -\frac{\ln (x)}{5} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(12x^2+y)}{2x(2x^2+y)}$ 	$R = x$ $S = \frac{\ln(y)}{5} + \frac{\ln(4x^2+y)}{5}$	$\frac{dS}{dR} = -\frac{1}{5R}$ 

Summary
The solution(s) found are the following

$$\frac{\ln(y)}{5} + \frac{\ln(4x^2+y)}{5} = -\frac{\ln(x)}{5} + c_1 \tag{1}$$

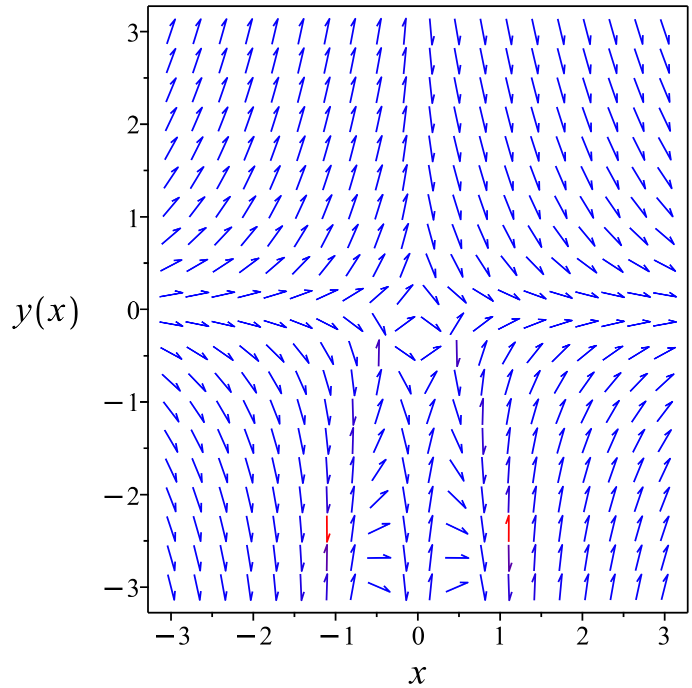


Figure 257: Slope field plot

Verification of solutions

$$\frac{\ln(y)}{5}+\frac{\ln(4x^2+y)}{5}=-\frac{\ln(x)}{5}+c_1$$

Verified OK.

6.38.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(4x^3 + 2xy\right) \mathrm{d}y &= \left(-12y\, x^2 - y^2\right) \mathrm{d}x \\
\left(12y\, x^2 + y^2\right) \mathrm{d}x + \left(4x^3 + 2xy\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= 12y\, x^2 + y^2 \\
N(x, y) &= 4x^3 + 2xy
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(12y\, x^2 + y^2\right) \\
&= 12x^2 + 2y
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(4x^3 + 2xy\right) \\
&= 12x^2 + 2y
\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int 12y\,x^2 + y^2\,\mathrm{d}x$$
$$\phi = 4y\,x^3 + y^2x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 4x^3 + 2xy + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 4x^3 + 2xy$. Therefore equation (4) becomes

$$4x^3 + 2xy = 4x^3 + 2xy + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 4y\,x^3 + y^2x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 4y\,x^3 + y^2x$$

Summary

The solution(s) found are the following

$$4yx^3 + xy^2 = c_1 \tag{1}$$

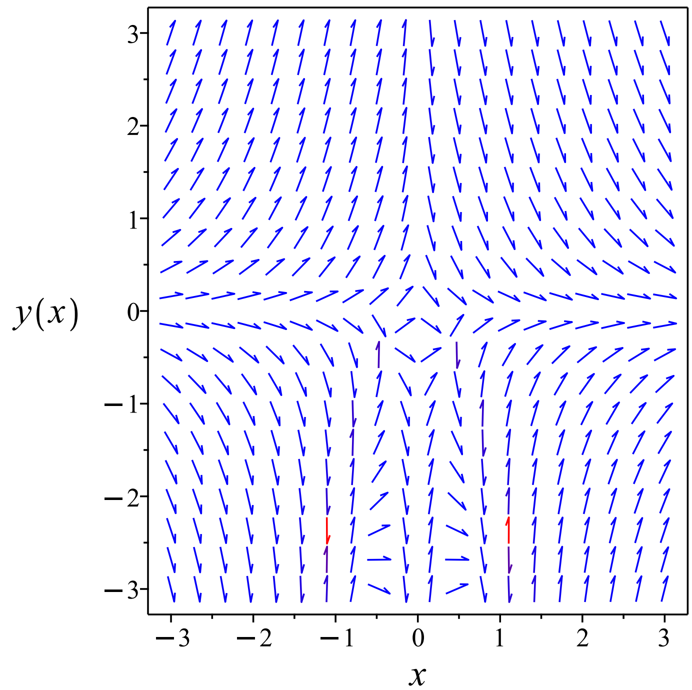


Figure 258: Slope field plot

Verification of solutions

$$4yx^3 + xy^2 = c_1$$

Verified OK.

6.38.3 Maple step by step solution

Let's solve

$(2xy + 4x^3)\, y' + y^2 + 12yx^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ∘ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $12x^2 + 2y = 12x^2 + 2y$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (12y\, x^2 + y^2)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = y(4x^3 + xy) + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $4x^3 + 2xy = 4x^3 + 2xy + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = y(4x^3 + xy)$

- Substitute $F(x, y)$ into the solution of the ODE

  $y(4x^3 + xy) = c_1$

- Solve for y

  $\left\{y = \frac{-2x^3 + \sqrt{4x^6 + c_1 x}}{x}, y = -\frac{2x^3 + \sqrt{4x^6 + c_1 x}}{x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 51

```
dsolve((2*x*y(x)+4*x^3)*diff(y(x),x)+y(x)^2+12*x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-2x^3 + \sqrt{4x^6 + c_1 x}}{x}$$
$$y(x) = \frac{-2x^3 - \sqrt{4x^6 + c_1 x}}{x}$$

✓ Solution by Mathematica

Time used: 0.441 (sec). Leaf size: 58

```
DSolve[(2*x*y[x]+4*x^3)*y'[x]+y[x]^2+12*x^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{2x^3 + \sqrt{x\left(4x^5 + c_1\right)}}{x}$$
$$y(x) \rightarrow \frac{-2x^3 + \sqrt{x\left(4x^5 + c_1\right)}}{x}$$

6.39 problem Exercise 12.39, page 103

Internal problem ID [4560]
Internal file name [`OUTPUT/4053_Sunday_June_05_2022_12_15_39_PM_40634094/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.39, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`], [
   _Abel, `2nd type`, `class C`]]
```

$$\boxed{(x^2 - y)\, y' = -x}$$

6.39.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x^2 - y)\,\mathrm{d}y &= (-x)\,\mathrm{d}x \\
(x)\,\mathrm{d}x + (x^2 - y)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= x \\
N(x, y) &= x^2 - y
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 - y) \\
&= 2x
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^2 - y}((0) - (2x)) \\
&= -\frac{2x}{x^2 - y}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{x}((2x) - (0)) \\ &= 2 \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int 2 \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{2y} \\ &= \mathrm{e}^{2y} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{2y}(x) \\ &= x\,\mathrm{e}^{2y} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{2y}\left(x^2 - y\right) \\ &= \left(x^2 - y\right)\mathrm{e}^{2y} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(x\,\mathrm{e}^{2y}\right) + \left(\left(x^2 - y\right)\mathrm{e}^{2y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int x \, \mathrm{e}^{2y} \, \mathrm{d}x$$

$$\phi = \frac{x^2 \mathrm{e}^{2y}}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 \mathrm{e}^{2y} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x^2 - y) \, \mathrm{e}^{2y}$. Therefore equation (4) becomes

$$\left(x^2 - y\right) \mathrm{e}^{2y} = x^2 \mathrm{e}^{2y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\mathrm{e}^{2y} y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\mathrm{e}^{2y} y\right) \mathrm{d}y$$

$$f(y) = -\frac{(2y-1) \, \mathrm{e}^{2y}}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2 \mathrm{e}^{2y}}{2} - \frac{(2y-1) \, \mathrm{e}^{2y}}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2 e^{2y}}{2} - \frac{(2y-1)\,e^{2y}}{4}$$

The solution becomes

$$y = x^2 + \frac{\mathrm{LambertW}\left(-4c_1 e^{-2x^2-1}\right)}{2} + \frac{1}{2}$$

<u>Summary</u>

The solution(s) found are the following

$$y = x^2 + \frac{\mathrm{LambertW}\left(-4c_1 e^{-2x^2-1}\right)}{2} + \frac{1}{2} \tag{1}$$

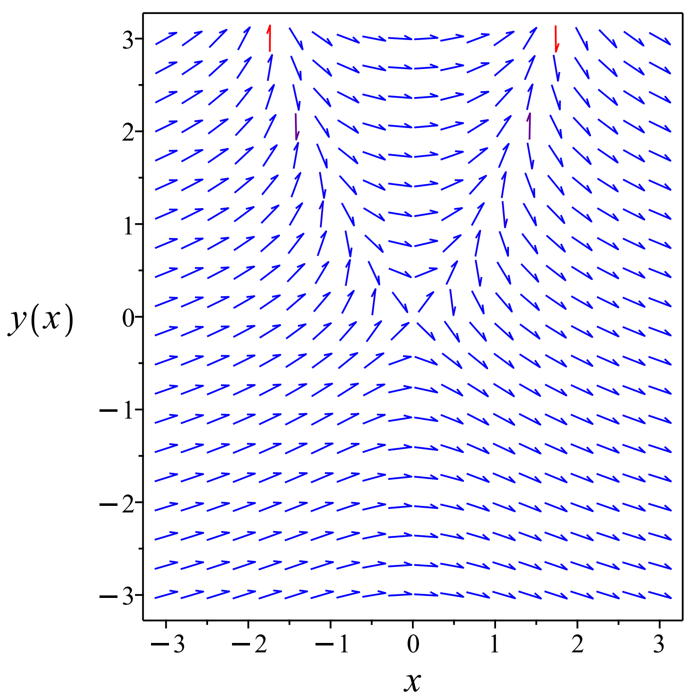


Figure 259: Slope field plot

<u>Verification of solutions</u>

$$y = x^2 + \frac{\mathrm{LambertW}\left(-4c_1 e^{-2x^2-1}\right)}{2} + \frac{1}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve((x^2-y(x))*diff(y(x),x)+x=0,y(x), singsol=all)
```

$$y(x) = x^2 + \frac{\text{LambertW}\left(4c_1 \mathrm{e}^{-2x^2-1}\right)}{2} + \frac{1}{2}$$

✓ Solution by Mathematica

Time used: 5.105 (sec). Leaf size: 40

```
DSolve[(x^2-y[x])*y'[x]+x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 + \frac{1}{2}\left(1 + W\left(-e^{-2x^2-1+c_1}\right)\right)$$
$$y(x) \to x^2 + \frac{1}{2}$$

6.40 problem Exercise 12.40, page 103

Internal problem ID [4561]
Internal file name [OUTPUT/4054_Sunday_June_05_2022_12_15_48_PM_34620777/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.40, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(x^2 - y)\, y' - 4xy = 0}$$

6.40.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{4xy}{-x^2 + y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{4xy(b_3 - a_2)}{-x^2 + y} - \frac{16x^2y^2a_3}{(-x^2 + y)^2} - \left(-\frac{4y}{-x^2 + y} - \frac{8x^2y}{(-x^2 + y)^2}\right)(xa_2 + ya_3 + a_1) - \left(-\frac{4x}{-x^2 + y} + \frac{4xy}{(-x^2 + y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{3x^4b_2 + 12x^2y^2a_3 + 4x^3b_1 - 4x^2ya_1 + 2x^2yb_2 - 8x\,y^2a_2 + 4x\,y^2b_3 - 4y^3a_3 - 4y^2a_1 - y^2b_2}{(x^2 - y)^2} = 0$$

Setting the numerator to zero gives

$$-3x^4b_2 - 12x^2y^2a_3 - 4x^3b_1 + 4x^2ya_1 - 2x^2yb_2 + 8x\,y^2a_2 - 4x\,y^2b_3 + 4y^3a_3 + 4y^2a_1 + y^2b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-12a_3v_1^2v_2^2 - 3b_2v_1^4 + 4a_1v_1^2v_2 + 8a_2v_1v_2^2 + 4a_3v_2^3 - 4b_1v_1^3 - 2b_2v_1^2v_2 - 4b_3v_1v_2^2 + 4a_1v_2^2 + b_2v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-3b_2v_1^4 - 4b_1v_1^3 - 12a_3v_1^2v_2^2 + (4a_1 - 2b_2)\, v_1^2v_2 \\ + (8a_2 - 4b_3)\, v_1v_2^2 + 4a_3v_2^3 + (4a_1 + b_2)\, v_2^2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -12a_3 &= 0 \\ 4a_3 &= 0 \\ -4b_1 &= 0 \\ -3b_2 &= 0 \\ 4a_1 - 2b_2 &= 0 \\ 4a_1 + b_2 &= 0 \\ 8a_2 - 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 2a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= 2y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= 2y - \left(-\frac{4xy}{-x^2 + y}\right)(x) \\ &= \frac{-2y\, x^2 - 2y^2}{x^2 - y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2y\,x^2 - 2y^2}{x^2 - y}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{2} + \ln\left(x^2 + y\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{4xy}{-x^2 + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x}{x^2 + y} \\ S_y &= -\frac{1}{2y} + \frac{1}{x^2 + y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

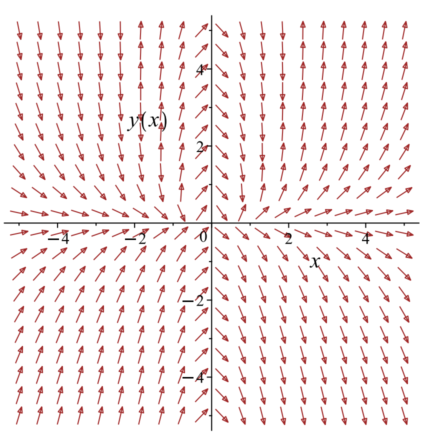
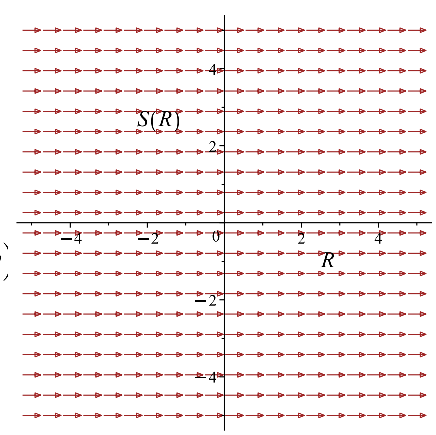
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{2} + \ln(x^2 + y) = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{2} + \ln(x^2 + y) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{4xy}{-x^2+y}$ 	$R = x$ $S = -\frac{\ln(y)}{2} + \ln(x^2 + y)$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$-\frac{\ln(y)}{2}+\ln(x^2+y)=c_1 \tag{1}$$

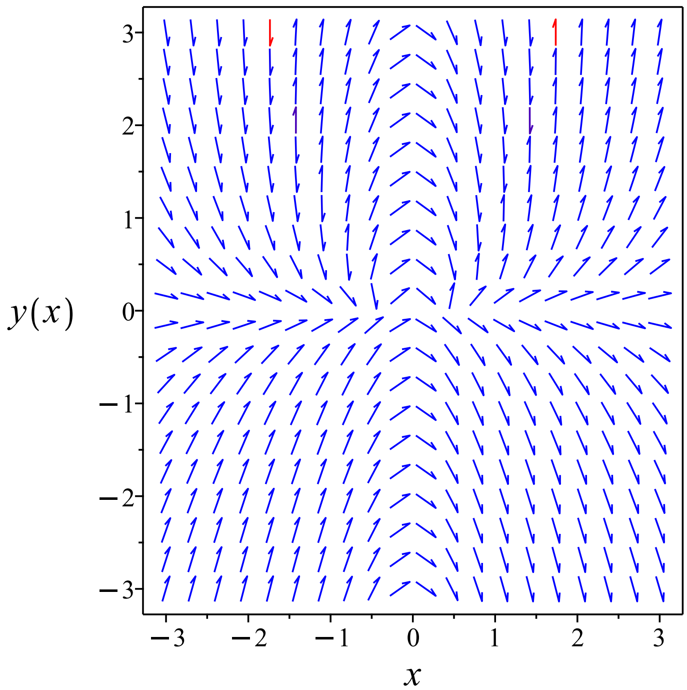


Figure 260: Slope field plot

Verification of solutions

$$-\frac{\ln(y)}{2}+\ln(x^2+y)=c_1$$

Verified OK.

6.40.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2 - y)\,\mathrm{d}y &= (4xy)\,\mathrm{d}x \\ (-4xy)\,\mathrm{d}x + (x^2 - y)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -4xy \\ N(x,y) &= x^2 - y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-4xy) \\ &= -4x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 - y)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x^2 - y}((-4x) - (2x))\\ &= -\frac{6x}{x^2 - y}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -\frac{1}{4yx}((2x) - (-4x))\\ &= -\frac{3}{2y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y}\\ &= e^{\int -\frac{3}{2y} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{3\ln(y)}{2}}\\ &= \frac{1}{y^{\frac{3}{2}}}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{y^{\frac{3}{2}}}(-4xy)\\ &= -\frac{4x}{\sqrt{y}}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^{\frac{3}{2}}}(x^2 - y) \\
&= \frac{x^2 - y}{y^{\frac{3}{2}}}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{4x}{\sqrt{y}}\right) + \left(\frac{x^2 - y}{y^{\frac{3}{2}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{4x}{\sqrt{y}}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{2x^2}{\sqrt{y}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{x^2}{y^{\frac{3}{2}}} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2-y}{y^{\frac{3}{2}}}$. Therefore equation (4) becomes

$$
\frac{x^2 - y}{y^{\frac{3}{2}}} = \frac{x^2}{y^{\frac{3}{2}}} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{\sqrt{y}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{\sqrt{y}}\right)\mathrm{d}y$$
$$f(y) = -2\sqrt{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{2x^2}{\sqrt{y}} - 2\sqrt{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{2x^2}{\sqrt{y}} - 2\sqrt{y}$$

Summary

The solution(s) found are the following

$$-\frac{2x^2}{\sqrt{y}} - 2\sqrt{y} = c_1 \tag{1}$$

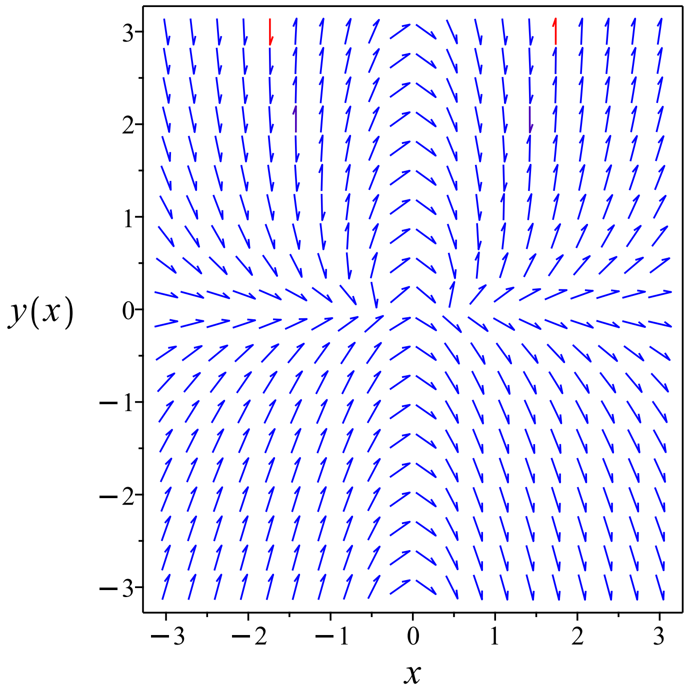


Figure 261: Slope field plot

Verification of solutions

$$-\frac{2x^2}{\sqrt{y}} - 2\sqrt{y} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 57

```
dsolve((x^2-y(x))*diff(y(x),x)-4*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{c_1\sqrt{c_1^2-4x^2}}{2}+\frac{c_1^2}{2}-x^2$$
$$y(x) = \frac{c_1\sqrt{c_1^2-4x^2}}{2}+\frac{c_1^2}{2}-x^2$$

✓ Solution by Mathematica

Time used: 2.441 (sec). Leaf size: 246

```
DSolve[(x^2-y[x])*y'[x]-4*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2\left(1+\frac{2-2i}{\frac{i\sqrt{2}}{\sqrt{x^2\cosh\left(\frac{2c_1}{9}\right)+x^2\sinh\left(\frac{2c_1}{9}\right)-i}}-(1-i)}\right)$$
$$y(x) \to x^2\left(1+\frac{2-2i}{(-1+i)-\frac{i\sqrt{2}}{\sqrt{x^2\cosh\left(\frac{2c_1}{9}\right)+x^2\sinh\left(\frac{2c_1}{9}\right)-i}}}\right)$$
$$y(x) \to x^2\left(1+\frac{2-2i}{(-1+i)-\frac{\sqrt{2}}{\sqrt{x^2\cosh\left(\frac{2c_1}{9}\right)+x^2\sinh\left(\frac{2c_1}{9}\right)+i}}}\right)$$
$$y(x) \to x^2\left(1+\frac{2-2i}{\frac{\sqrt{2}}{\sqrt{x^2\cosh\left(\frac{2c_1}{9}\right)+x^2\sinh\left(\frac{2c_1}{9}\right)+i}}-(1-i)}\right)$$
$$y(x) \to 0$$
$$y(x) \to -x^2$$

6.41 problem Exercise 12.41, page 103

Internal problem ID [4562]
Internal file name [OUTPUT/4055_Sunday_June_05_2022_12_15_56_PM_9821976/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.41, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{xyy' + y^2 = -x^2}$$

6.41.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^2 u(x)\,(u'(x)\,x + u(x)) + u(x)^2\,x^2 = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u^2+1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{2u^2+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2+1}{u}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{2u^2+1}{u}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(2u^2+1)}{4} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$(2u^2+1)^{\frac{1}{4}} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$(2u^2+1)^{\frac{1}{4}} = \frac{c_3}{x}$$

Which simplifies to

$$(2u(x)^2+1)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

The solution is

$$(2u(x)^2+1)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{2y^2}{x^2}+1\right)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

$$\left(\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\left(\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x} \tag{1}$$

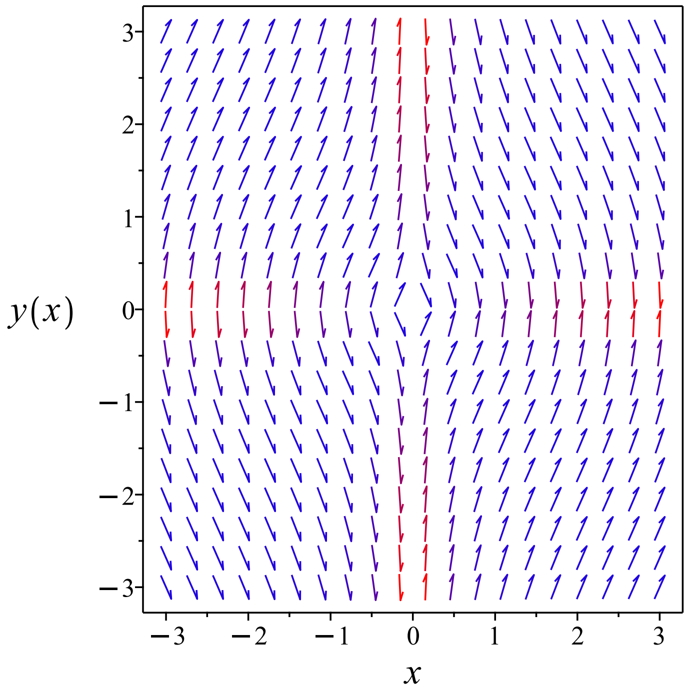


Figure 262: Slope field plot

Verification of solutions

$$\left(\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

6.41.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^2+y^2}{xy}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 149: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \frac{1}{y\, x^2}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y\,x^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2 x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x^2 + y^2}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y^2 x \\ S_y &= y\,x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{R^4}{4} + c_1 \tag{4}$$

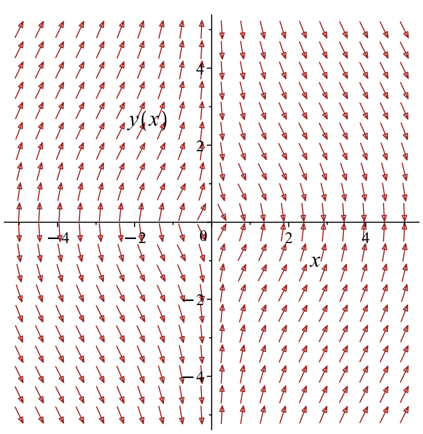
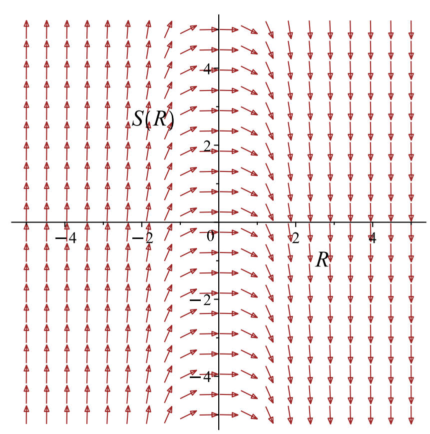
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2x^2}{2} = -\frac{x^4}{4} + c_1$$

Which simplifies to

$$\frac{y^2x^2}{2} = -\frac{x^4}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2+y^2}{xy}$ 	$R = x$ $S = \frac{y^2x^2}{2}$	$\frac{dS}{dR} = -R^3$ 

Summary

The solution(s) found are the following

$$\frac{y^2x^2}{2} = -\frac{x^4}{4} + c_1 \tag{1}$$

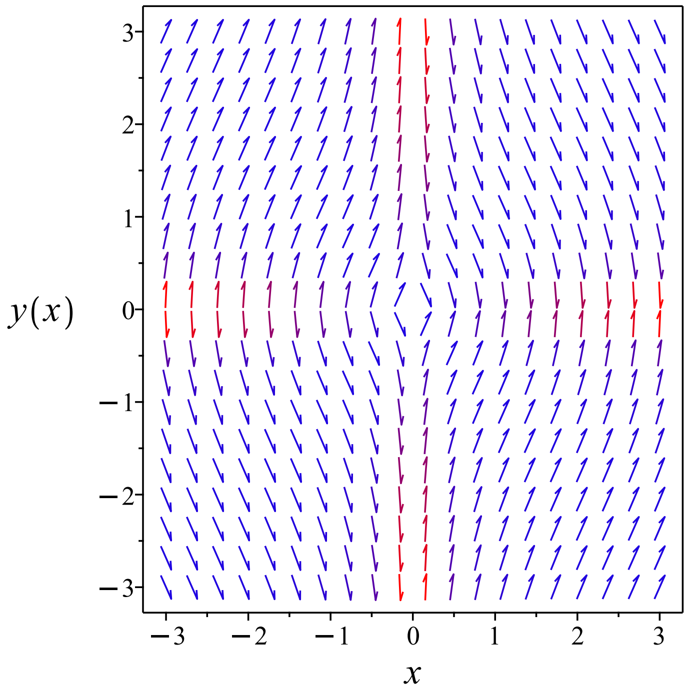


Figure 263: Slope field plot

Verification of solutions

$$\frac{y^2x^2}{2} = -\frac{x^4}{4} + c_1$$

Verified OK.

6.41.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{x^2 + y^2}{xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= -x \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = -\frac{y^2}{x} - x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2 \qquad (5)
\end{aligned}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= -\frac{w(x)}{x} - x \\
w' &= -\frac{2w}{x} - 2x \qquad (7)
\end{aligned}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= -2x
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x} = -2x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x} dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-2x) \\ \frac{d}{dx}(x^2 w) &= (x^2)(-2x) \\ d(x^2 w) &= (-2x^3)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^2 w &= \int -2x^3\, dx \\ x^2 w &= -\frac{x^4}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = -\frac{x^2}{2} + \frac{c_1}{x^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -\frac{x^2}{2} + \frac{c_1}{x^2}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{-2x^4 + 4c_1}}{2x} \\ y(x) &= -\frac{\sqrt{-2x^4 + 4c_1}}{2x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-2x^4 + 4c_1}}{2x} \tag{1}$$

$$y = -\frac{\sqrt{-2x^4 + 4c_1}}{2x} \tag{2}$$

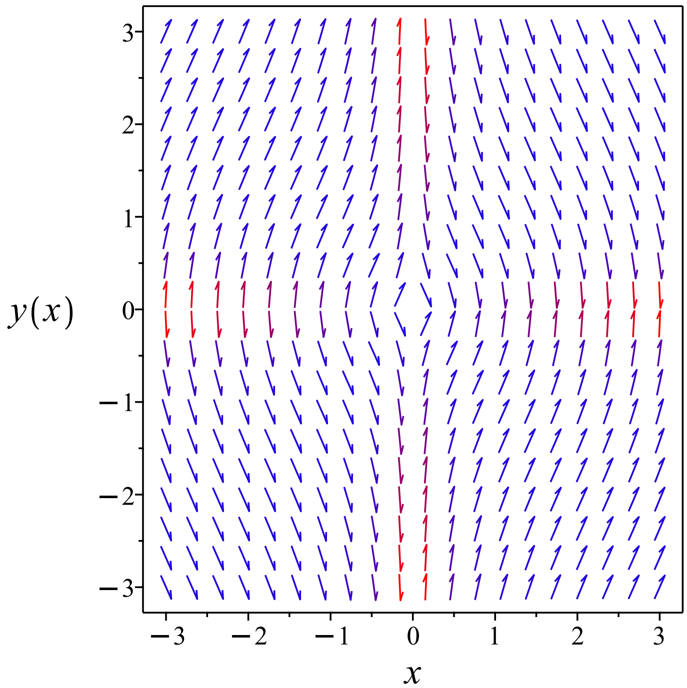


Figure 264: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-2x^4 + 4c_1}}{2x}$$

Verified OK.

$$y = -\frac{\sqrt{-2x^4 + 4c_1}}{2x}$$

Verified OK.

6.41.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(xy)\,\mathrm{d}y &= \left(-x^2 - y^2\right)\mathrm{d}x \\ \left(x^2 + y^2\right)\mathrm{d}x + (xy)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x^2 + y^2 \\ N(x, y) &= xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + y^2\right) \\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xy) \\ &= y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{yx}((2y) - (y)) \\ &= \frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{1}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)} \\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x\left(x^2 + y^2\right) \\ &= x\left(x^2 + y^2\right)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x(xy) \\ &= y\,x^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(x\left(x^2 + y^2\right)\right) + \left(y\,x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int x(x^2+y^2) \,\mathrm{d}x$$

$$\phi = \frac{(x^2+y^2)^2}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = (x^2+y^2)\,y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y\,x^2$. Therefore equation (4) becomes

$$y\,x^2 = (x^2+y^2)\,y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y^3$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (-y^3) \,\mathrm{d}y$$

$$f(y) = -\frac{y^4}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(x^2+y^2)^2}{4} - \frac{y^4}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4}$$

Summary

The solution(s) found are the following

$$\frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4} = c_1 \tag{1}$$

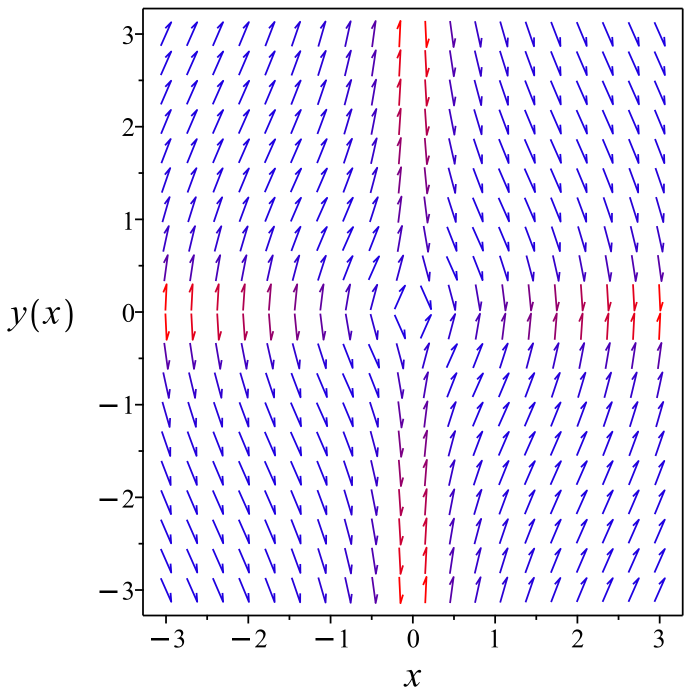


Figure 265: Slope field plot

Verification of solutions

$$\frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(x*y(x)*diff(y(x),x)+x^2+y(x)^2=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-2x^4 + 4c_1}}{2x}$$
$$y(x) = \frac{\sqrt{-2x^4 + 4c_1}}{2x}$$

✓ Solution by Mathematica

Time used: 0.211 (sec). Leaf size: 46

```
DSolve[x*y[x]*y'[x]+x^2+y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-\frac{x^4}{2} + c_1}}{x}$$
$$y(x) \to \frac{\sqrt{-\frac{x^4}{2} + c_1}}{x}$$

6.42 problem Exercise 12.42, page 103

Internal problem ID [4563]
Internal file name [`OUTPUT/4056_Sunday_June_05_2022_12_16_09_PM_51372307/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.42, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{2xyy' - y^2 = -3x^2}$$

6.42.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2x^2 u(x)\,(u'(x)\,x + u(x)) - u(x)^2\,x^2 = -3x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2+3}{2ux} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{u^2+3}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+3}{u}}\,du = -\frac{1}{2x}\,dx$$
$$\int \frac{1}{\frac{u^2+3}{u}}\,du = \int -\frac{1}{2x}\,dx$$
$$\frac{\ln\left(u^2+3\right)}{2} = -\frac{\ln\left(x\right)}{2} + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+3} = \mathrm{e}^{-\frac{\ln(x)}{2}+c_2}$$

Which simplifies to

$$\sqrt{u^2+3} = \frac{c_3}{\sqrt{x}}$$

Which simplifies to

$$\sqrt{u\left(x\right)^2+3} = \frac{c_3\,\mathrm{e}^{c_2}}{\sqrt{x}}$$

The solution is

$$\sqrt{u\left(x\right)^2+3} = \frac{c_3\,\mathrm{e}^{c_2}}{\sqrt{x}}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}+3} = \frac{c_3\,\mathrm{e}^{c_2}}{\sqrt{x}}$$
$$\sqrt{\frac{y^2+3x^2}{x^2}} = \frac{c_3\,\mathrm{e}^{c_2}}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{y^2+3x^2}{x^2}} = \frac{c_3\,\mathrm{e}^{c_2}}{\sqrt{x}} \tag{1}$$

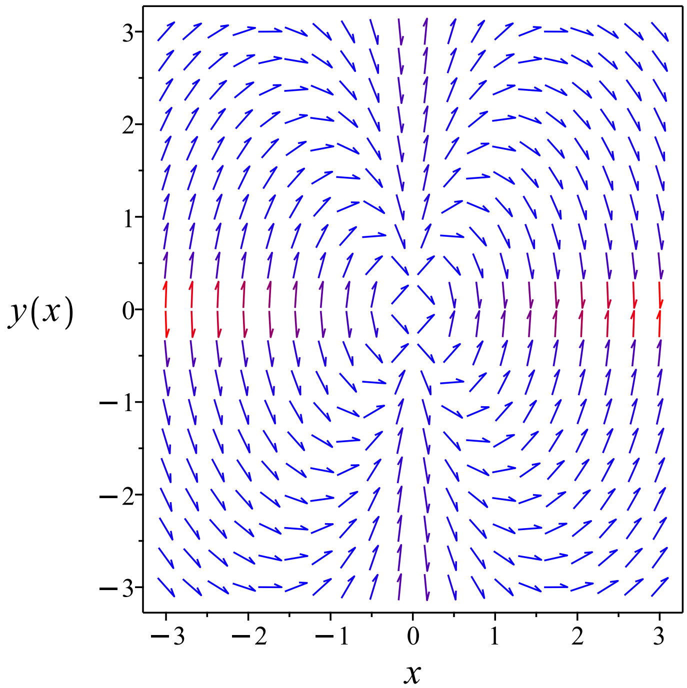


Figure 266: Slope field plot

Verification of solutions

$$\sqrt{\frac{y^2 + 3x^2}{x^2}} = \frac{c_3 e^{c_2}}{\sqrt{x}}$$

Verified OK.

6.42.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-3x^2 + y^2}{2xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 151: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-3x^2 + y^2}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^2}{2x^2} \\ S_y &= \frac{y}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{3}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{3}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{3R}{2} + c_1 \tag{4}$$

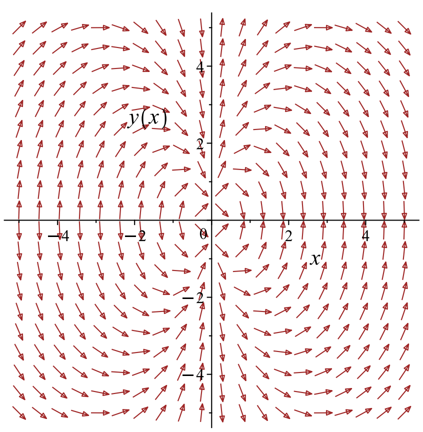
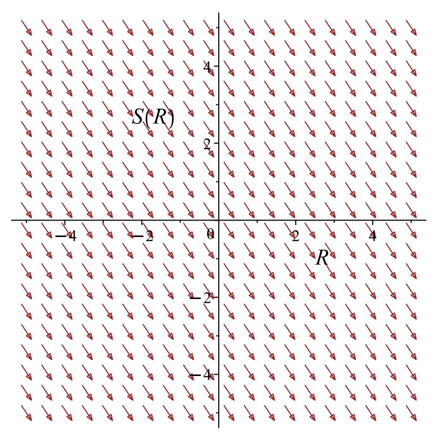
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x} = -\frac{3x}{2} + c_1$$

Which simplifies to

$$\frac{y^2}{2x} = -\frac{3x}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-3x^2+y^2}{2xy}$ 	$R = x$ $S = \frac{y^2}{2x}$	$\frac{dS}{dR} = -\frac{3}{2}$ 

Summary
The solution(s) found are the following

$$\frac{y^2}{2x} = -\frac{3x}{2} + c_1 \tag{1}$$

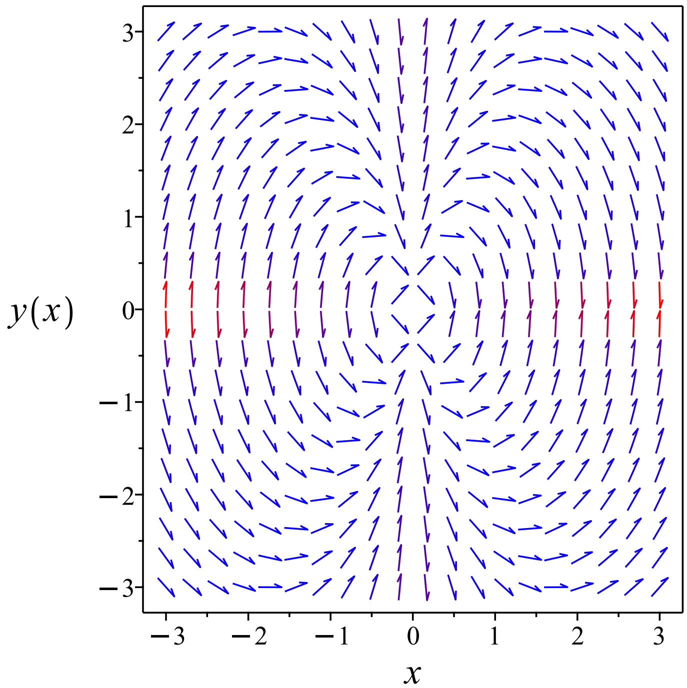


Figure 267: Slope field plot

Verification of solutions

$$\frac{y^2}{2x} = -\frac{3x}{2} + c_1$$

Verified OK.

6.42.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{-3x^2 + y^2}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x} y - \frac{3x}{2} \frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x) y + f_1(x) y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x) y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{2x} \\
f_1(x) &= -\frac{3x}{2} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = \frac{y^2}{2x} - \frac{3x}{2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{w(x)}{2x} - \frac{3x}{2} \\
w' &= \frac{w}{x} - 3x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= -3x
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = -3x$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)(-3x)$$
$$\frac{d}{dx}\left(\frac{w}{x}\right) = \left(\frac{1}{x}\right)(-3x)$$
$$d\left(\frac{w}{x}\right) = -3\,dx$$

Integrating gives

$$\frac{w}{x} = \int -3\,dx$$
$$\frac{w}{x} = -3x + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = c_1 x - 3x^2$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = c_1 x - 3x^2$$

Solving for y gives

$$y(x) = \sqrt{x(c_1 - 3x)}$$
$$y(x) = -\sqrt{c_1 x - 3x^2}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x(c_1 - 3x)} \tag{1}$$
$$y = -\sqrt{c_1 x - 3x^2} \tag{2}$$

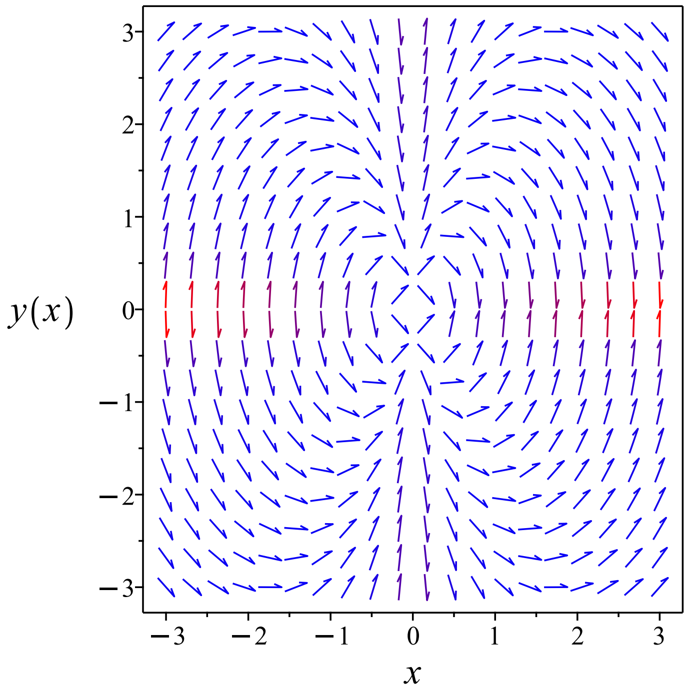


Figure 268: Slope field plot

Verification of solutions

$$y = \sqrt{x (c_1 - 3x)}$$

Verified OK.

$$y = -\sqrt{c_1 x - 3x^2}$$

Verified OK.

6.42.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2xy)\,\mathrm{d}y &= \left(-3x^2 + y^2\right)\mathrm{d}x\\ \left(3x^2 - y^2\right)\mathrm{d}x + (2xy)\,\mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 3x^2 - y^2\\ N(x, y) &= 2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3x^2 - y^2\right)\\ &= -2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy) \\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{2yx}((-2y) - (2y)) \\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{2}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^2}(3x^2 - y^2) \\ &= \frac{3x^2 - y^2}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^2}(2xy) \\ &= \frac{2y}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{3x^2 - y^2}{x^2}\right) + \left(\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{3x^2 - y^2}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = 3x + \frac{y^2}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{2y}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y}{x}$. Therefore equation (4) becomes

$$
\frac{2y}{x} = \frac{2y}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 3x + \frac{y^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 3x + \frac{y^2}{x}$$

Summary

The solution(s) found are the following

$$3x + \frac{y^2}{x} = c_1 \tag{1}$$

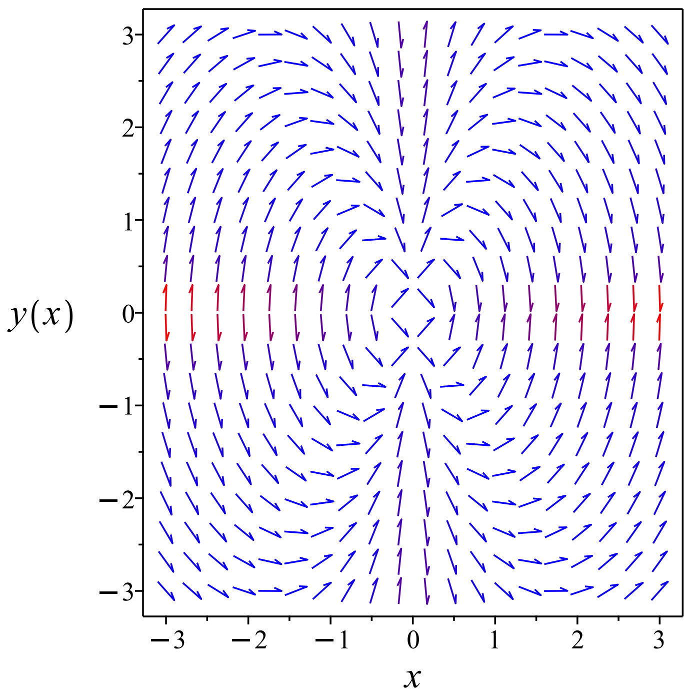


Figure 269: Slope field plot

Verification of solutions

$$3x + \frac{y^2}{x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve(2*x*y(x)*diff(y(x),x)+3*x^2-y(x)^2=0,y(x), singsol=all)
```

$$y(x) = \sqrt{(-3x + c_1)\, x}$$
$$y(x) = -\sqrt{c_1 x - 3x^2}$$

✓ Solution by Mathematica

Time used: 0.306 (sec). Leaf size: 35

```
DSolve[2*x*y[x]*y'[x]+3*x^2-y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x(-3x + c_1)}$$
$$y(x) \to \sqrt{x(-3x + c_1)}$$

6.43 problem Exercise 12.43, page 103

Internal problem ID [4564]
Internal file name [`OUTPUT/4057_Sunday_June_05_2022_12_16_21_PM_33963815/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.43, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{\left(2y^3x-x^4\right)y'+2yx^3-y^4=0}$$

6.43.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\left(2u(x)^3\,x^4-x^4\right)\left(u'(x)\,x+u(x)\right)+2u(x)\,x^4-u(x)^4\,x^4=0$$

In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=-\frac{u^4+u}{x\,(2u^3-1)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^4+u}{2u^3-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^4+u}{2u^3-1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u^4+u}{2u^3-1}}\, du = \int -\frac{1}{x}\, dx$$

$$\ln\left(u^2-u+1\right) + \ln\left(u+1\right) - \ln\left(u\right) = -\ln\left(x\right) + c_2$$

Raising both side to exponential gives

$$e^{\ln\left(u^2-u+1\right)+\ln(u+1)-\ln(u)} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{(u^2-u+1)(u+1)}{u} = \frac{c_3}{x}$$

The solution is

$$\frac{\left(u(x)^2-u(x)+1\right)\left(u(x)+1\right)}{u\left(x\right)} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\left(\frac{y^2}{x^2}-\frac{y}{x}+1\right)\left(\frac{y}{x}+1\right)x}{y} = \frac{c_3}{x}$$

$$\frac{(x^2-xy+y^2)(x+y)}{x^2y} = \frac{c_3}{x}$$

Which simplifies to

$$\frac{(x+y)(x^2-xy+y^2)}{xy} = c_3$$

<u>Summary</u>

The solution(s) found are the following

$$\frac{(x+y)(x^2-xy+y^2)}{xy} = c_3 \tag{1}$$

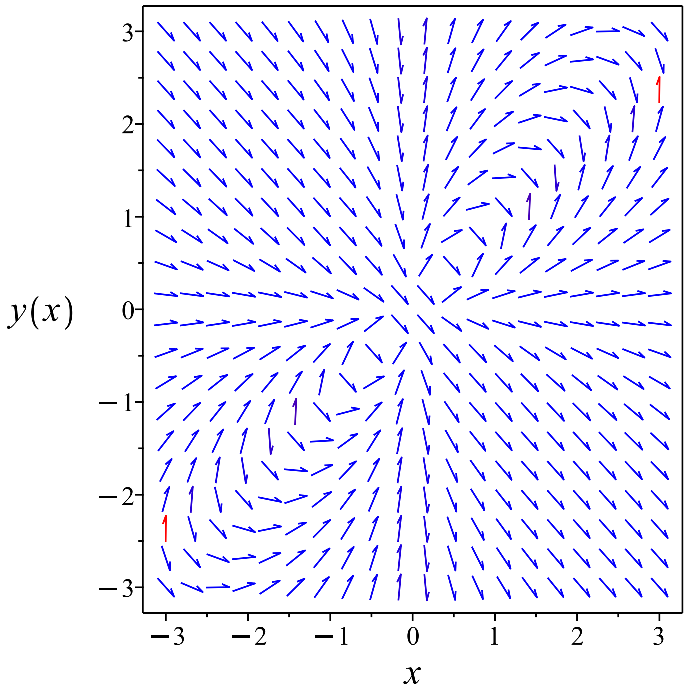


Figure 270: Slope field plot

Verification of solutions

$$\frac{(x+y)\,(x^2-xy+y^2)}{xy} = c_3$$

Verified OK.

6.43.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y(-2x^3+y^3)}{x\,(-x^3+2y^3)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{y(-2x^3+y^3)\,(b_3-a_2)}{x\,(-x^3+2y^3)} - \frac{y^2(-2x^3+y^3)^2\,a_3}{x^2\,(-x^3+2y^3)^2} \\
&- \left(-\frac{6yx}{-x^3+2y^3} - \frac{y(-2x^3+y^3)}{x^2\,(-x^3+2y^3)} + \frac{3y(-2x^3+y^3)\,x}{(-x^3+2y^3)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(\frac{-2x^3+y^3}{x\,(-x^3+2y^3)} + \frac{3y^3}{x\,(-x^3+2y^3)} - \frac{6y^3(-2x^3+y^3)}{x\,(-x^3+2y^3)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^8b_2 + 2x^6y^2a_3 + 8x^5y^3b_2 - 9x^4y^4a_2 + 9x^4y^4b_3 - 8x^3y^5a_3 - 2x^2y^6b_2 - y^8a_3 + 2x^7b_1 - 2x^6ya_1 + 4x^4y^3b_1}{(x^3-2y^3)^2\,x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-x^8b_2 - 2x^6y^2a_3 - 8x^5y^3b_2 + 9x^4y^4a_2 - 9x^4y^4b_3 + 8x^3y^5a_3 + 2x^2y^6b_2 \\
&+ y^8a_3 - 2x^7b_1 + 2x^6ya_1 - 4x^4y^3b_1 + 4x^3y^4a_1 - 2x\,y^6b_1 + 2y^7a_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&9a_2v_1^4v_2^4 - 2a_3v_1^6v_2^2 + 8a_3v_1^3v_2^5 + a_3v_2^8 - b_2v_1^8 - 8b_2v_1^5v_2^3 + 2b_2v_1^2v_2^6 \\
&- 9b_3v_1^4v_2^4 + 2a_1v_1^6v_2 + 4a_1v_1^3v_2^4 + 2a_1v_2^7 - 2b_1v_1^7 - 4b_1v_1^4v_2^3 - 2b_1v_1v_2^6 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_2 v_1^8 - 2b_1 v_1^7 - 2a_3 v_1^6 v_2^2 + 2a_1 v_1^6 v_2 - 8b_2 v_1^5 v_2^3 + (9a_2 - 9b_3) v_1^4 v_2^4 \\ - 4b_1 v_1^4 v_2^3 + 8a_3 v_1^3 v_2^5 + 4a_1 v_1^3 v_2^4 + 2b_2 v_1^2 v_2^6 - 2b_1 v_1 v_2^6 + a_3 v_2^8 + 2a_1 v_2^7 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_3 &= 0 \\ 2a_1 &= 0 \\ 4a_1 &= 0 \\ -2a_3 &= 0 \\ 8a_3 &= 0 \\ -4b_1 &= 0 \\ -2b_1 &= 0 \\ -8b_2 &= 0 \\ -b_2 &= 0 \\ 2b_2 &= 0 \\ 9a_2 - 9b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y(-2x^3+y^3)}{x\,(-x^3+2y^3)}\right)(x) \\
&= \frac{-y\,x^3-y^4}{x^3-2y^3} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-y\,x^3-y^4}{x^3-2y^3}} dy
\end{aligned}$$

Which results in

$$S = \ln(x+y) - \ln(y) + \ln\left(x^2-xy+y^2\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(-2x^3+y^3)}{x\,(-x^3+2y^3)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{3x^2}{(x+y)(x^2-xy+y^2)} \\
S_y &= \frac{-x^3+2y^3}{y(x+y)(x^2-xy+y^2)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(R) + c_1 \tag{4}
$$

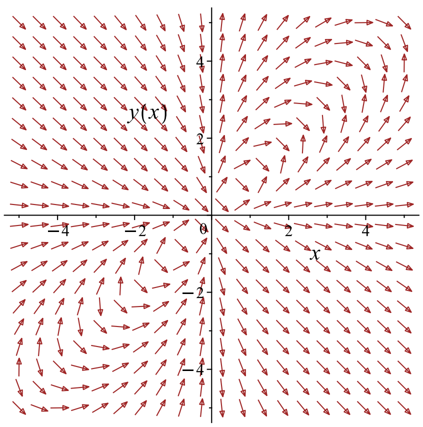
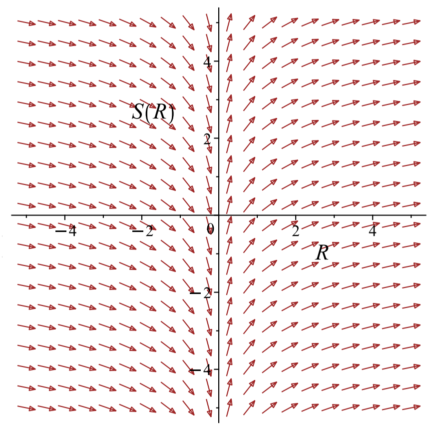
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln(x+y) - \ln(y) + \ln(x^2-xy+y^2) = \ln(x) + c_1
$$

Which simplifies to

$$
\ln(x+y) - \ln(y) + \ln(x^2-xy+y^2) = \ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(-2x^3+y^3)}{x(-x^3+2y^3)}$ 	$R = x$ $S = \ln(x+y) - \ln(y) +$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary
The solution(s) found are the following

$$\ln(x+y) - \ln(y) + \ln(x^2 - xy + y^2) = \ln(x) + c_1 \tag{1}$$

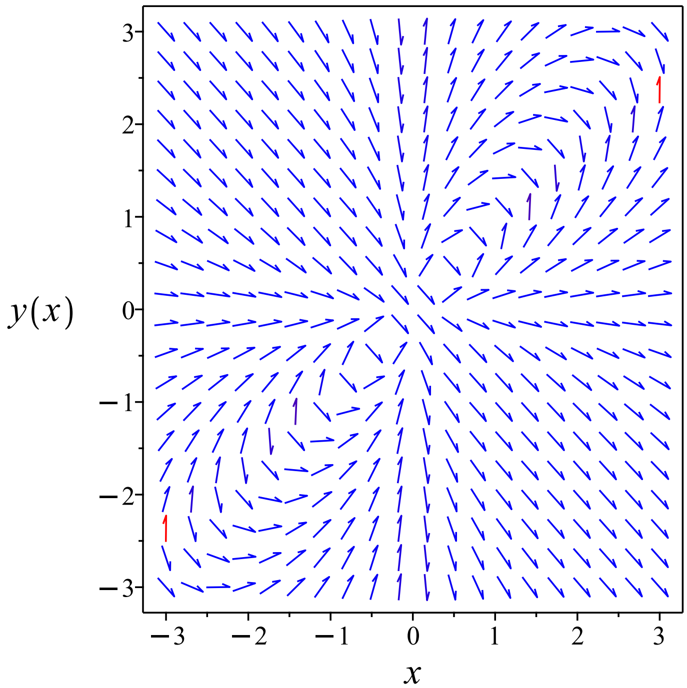


Figure 271: Slope field plot

Verification of solutions

$$\ln(x+y)-\ln(y)+\ln\left(x^2-xy+y^2\right)=\ln(x)+c_1$$

Verified OK.

6.43.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-x^4 + 2x\, y^3\right) \mathrm{d}y = \left(-2y\, x^3 + y^4\right) \mathrm{d}x$$
$$\left(2y\, x^3 - y^4\right) \mathrm{d}x + \left(-x^4 + 2x\, y^3\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 2y\, x^3 - y^4$$
$$N(x, y) = -x^4 + 2x\, y^3$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y\, x^3 - y^4\right) \\ &= 2x^3 - 4y^3 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^4 + 2x\, y^3\right) \\ &= -4x^3 + 2y^3 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{-x^4 + 2x\,y^3}\left(\left(2x^3 - 4y^3\right) - \left(-4x^3 + 2y^3\right)\right) \\ &= \frac{-6x^3 + 6y^3}{x\,(x^3 - 2y^3)} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{2y\,x^3 - y^4}\left(\left(-4x^3 + 2y^3\right) - \left(2x^3 - 4y^3\right)\right) \\ &= \frac{-6x^3 + 6y^3}{2y\,x^3 - y^4} \end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(-4x^3 + 2y^3) - (2x^3 - 4y^3)}{x\,(2y\,x^3 - y^4) - y\,(-x^4 + 2x\,y^3)} \\ &= -\frac{2}{yx} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \,\mathrm{d}t} \\ &= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\mu = e^{-2\ln(t)}$$
$$= \frac{1}{t^2}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{y^2 x^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{y^2 x^2}\left(2y\,x^3 - y^4\right) \\ &= \frac{2x^3 - y^3}{y\,x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{y^2 x^2}\left(-x^4 + 2x\,y^3\right) \\ &= \frac{-x^3 + 2y^3}{y^2 x}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{2x^3 - y^3}{y\,x^2}\right) + \left(\frac{-x^3 + 2y^3}{y^2 x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int \overline{M}\, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int \frac{2x^3 - y^3}{y\, x^2}\, \mathrm{d}x$$
$$\phi = \frac{x^3 + y^3}{yx} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x^3 + y^3}{y^2 x} + \frac{3y}{x} + f'(y) \tag{4}$$
$$= \frac{-x^3 + 2y^3}{y^2 x} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x^3+2y^3}{y^2 x}$. Therefore equation (4) becomes

$$\frac{-x^3 + 2y^3}{y^2 x} = \frac{-x^3 + 2y^3}{y^2 x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^3 + y^3}{yx} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^3 + y^3}{yx}$$

Summary

The solution(s) found are the following

$$\frac{x^3 + y^3}{yx} = c_1 \tag{1}$$

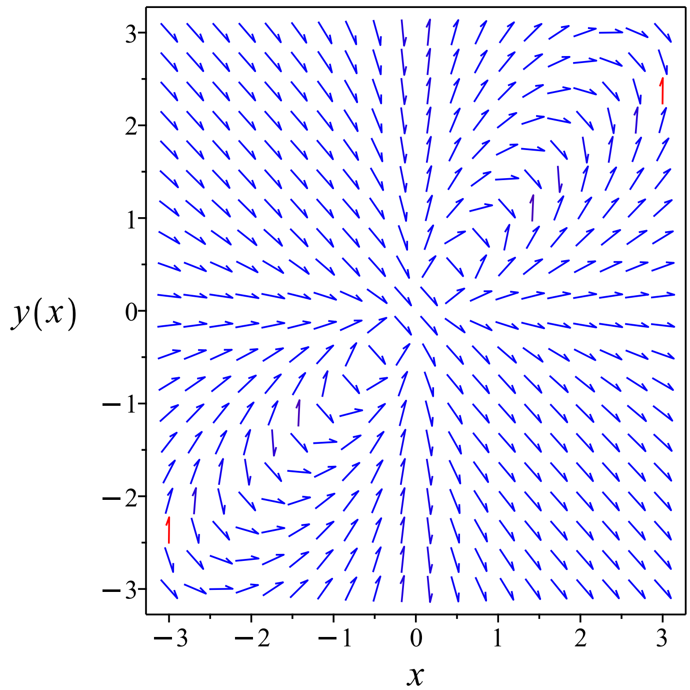


Figure 272: Slope field plot

Verification of solutions

$$\frac{x^3 + y^3}{yx} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 317

```
dsolve((2*x*y(x)^3-x^4)*diff(y(x),x)+2*x^3*y(x)-y(x)^4=0,y(x), singsol=all)
```

$$y(x) = \frac{12^{\frac{1}{3}}\left(x12^{\frac{1}{3}}c_1 + \left(x\left(-9c_1x^2+\sqrt{3}\,\sqrt{\frac{27c_1^3x^4-4x}{c_1}}\right)c_1^2\right)^{\frac{2}{3}}\right)}{6c_1\left(x\left(-9c_1x^2+\sqrt{3}\,\sqrt{\frac{27c_1^3x^4-4x}{c_1}}\right)c_1^2\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{3^{\frac{1}{3}}\left(\left(-i\sqrt{3}-1\right)\left(x\left(-9c_1x^2+\sqrt{3}\,\sqrt{\frac{27c_1^3x^4-4x}{c_1}}\right)c_1^2\right)^{\frac{2}{3}} + \left(i3^{\frac{5}{6}}-3^{\frac{1}{3}}\right)c_1 2^{\frac{2}{3}}x\right)2^{\frac{2}{3}}}{12\left(x\left(-9c_1x^2+\sqrt{3}\,\sqrt{\frac{27c_1^3x^4-4x}{c_1}}\right)c_1^2\right)^{\frac{1}{3}}c_1}$$

$$y(x) = -\frac{3^{\frac{1}{3}}\left(\left(1-i\sqrt{3}\right)\left(x\left(-9c_1x^2+\sqrt{3}\,\sqrt{\frac{27c_1^3x^4-4x}{c_1}}\right)c_1^2\right)^{\frac{2}{3}} + \left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}\right)c_1 2^{\frac{2}{3}}x\right)2^{\frac{2}{3}}}{12\left(x\left(-9c_1x^2+\sqrt{3}\,\sqrt{\frac{27c_1^3x^4-4x}{c_1}}\right)c_1^2\right)^{\frac{1}{3}}c_1}$$

✓ Solution by Mathematica

Time used: 60.224 (sec). Leaf size: 331

```
DSolve[(2*x*y[x]^3-x^4)*y'[x]+2*x^3*y[x]-y[x]^4==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{2}\left(-9x^3+\sqrt{81x^6-12e^{3c_1}x^3}\right)^{2/3}+2\sqrt[3]{3}e^{c_1}x}{6^{2/3}\sqrt[3]{-9x^3+\sqrt{81x^6-12e^{3c_1}x^3}}}$$

$$y(x) \to \frac{i\sqrt[3]{2}\sqrt[6]{3}\left(\sqrt{3}+i\right)\left(-9x^3+\sqrt{81x^6-12e^{3c_1}x^3}\right)^{2/3}-2\left(\sqrt{3}+3i\right)e^{c_1}x}{2\ 2^{2/3}3^{5/6}\sqrt[3]{-9x^3+\sqrt{81x^6-12e^{3c_1}x^3}}}$$

$$y(x) \to \frac{\sqrt[3]{2}\sqrt[6]{3}\left(-1-i\sqrt{3}\right)\left(-9x^3+\sqrt{81x^6-12e^{3c_1}x^3}\right)^{2/3}-2\left(\sqrt{3}-3i\right)e^{c_1}x}{2\ 2^{2/3}3^{5/6}\sqrt[3]{-9x^3+\sqrt{81x^6-12e^{3c_1}x^3}}}$$

6.44 problem Exercise 12.44, page 103

Internal problem ID [4565]
Internal file name [OUTPUT/4058_Sunday_June_05_2022_12_16_31_PM_15662884/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.44, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{(xy-1)^2\,xy'+\left(y^2x^2+1\right)y=0}$$

6.44.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(y^2x^2+1)}{x\,(y^2x^2-2xy+1)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{y(y^2x^2+1)(b_3-a_2)}{x(y^2x^2-2xy+1)} - \frac{y^2(y^2x^2+1)^2 a_3}{x^2(y^2x^2-2xy+1)^2} \\ &- \left(-\frac{2y^3}{y^2x^2-2xy+1} + \frac{y(y^2x^2+1)}{x^2(y^2x^2-2xy+1)} \right. \\ &\left. + \frac{y(y^2x^2+1)(2y^2x-2y)}{x(y^2x^2-2xy+1)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{y^2x^2+1}{x(y^2x^2-2xy+1)} \right. \\ &\left. - \frac{2y^2x}{y^2x^2-2xy+1} + \frac{y(y^2x^2+1)(2y\,x^2-2x)}{x(y^2x^2-2xy+1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^6y^4b_2 - 2x^4y^6a_3 + x^5y^4b_1 - x^4y^5a_1 - 8x^5y^3b_2 - 2x^4y^4a_2 - 2x^4y^4b_3 - 4x^4y^3b_1 + 8x^4y^2b_2 - 4x^2y^4a_3 + 2}{(y^2x^2 - 2xy +}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &2x^6y^4b_2 - 2x^4y^6a_3 + x^5y^4b_1 - x^4y^5a_1 - 8x^5y^3b_2 - 2x^4y^4a_2 - 2x^4y^4b_3 \\ &- 4x^4y^3b_1 + 8x^4y^2b_2 - 4x^2y^4a_3 + 2x^3y^2b_1 - 2x^2y^3a_1 - 4x^3yb_2 + 2x^2y^2a_2 \\ &+ 2x^2y^2b_3 + 4x\,y^3a_3 + 4x\,y^2a_1 + 2b_2x^2 - 2y^2a_3 + xb_1 - ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-2a_3v_1^4v_2^6 + 2b_2v_1^6v_2^4 - a_1v_1^4v_2^5 + b_1v_1^5v_2^4 - 2a_2v_1^4v_2^4 - 8b_2v_1^5v_2^3 - 2b_3v_1^4v_2^4 \\ &- 4b_1v_1^4v_2^3 - 4a_3v_1^2v_2^4 + 8b_2v_1^4v_2^2 - 2a_1v_1^2v_2^3 + 2b_1v_1^3v_2^2 + 2a_2v_1^2v_2^2 + 4a_3v_1v_2^3 \\ &- 4b_2v_1^3v_2 + 2b_3v_1^2v_2^2 + 4a_1v_1v_2^2 - 2a_3v_2^2 + 2b_2v_1^2 - a_1v_2 + b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &2b_2v_1^6v_2^4 + b_1v_1^5v_2^4 - 8b_2v_1^5v_2^3 - 2a_3v_1^4v_2^6 - a_1v_1^4v_2^5 + (-2a_2 - 2b_3)\, v_1^4v_2^4 \\ &- 4b_1v_1^4v_2^3 + 8b_2v_1^4v_2^2 + 2b_1v_1^3v_2^2 - 4b_2v_1^3v_2 - 4a_3v_1^2v_2^4 - 2a_1v_1^2v_2^3 \\ &+ (2a_2 + 2b_3)\, v_1^2v_2^2 + 2b_2v_1^2 + 4a_3v_1v_2^3 + 4a_1v_1v_2^2 + b_1v_1 - 2a_3v_2^2 - a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -2a_1 &= 0 \\ -a_1 &= 0 \\ 4a_1 &= 0 \\ -4a_3 &= 0 \\ -2a_3 &= 0 \\ 4a_3 &= 0 \\ -4b_1 &= 0 \\ 2b_1 &= 0 \\ -8b_2 &= 0 \\ -4b_2 &= 0 \\ 2b_2 &= 0 \\ 8b_2 &= 0 \\ -2a_2 - 2b_3 &= 0 \\ 2a_2 + 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = -x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{y(y^2x^2+1)}{x\,(y^2x^2-2xy+1)}\right)(-x) \\ &= -\frac{2y^2x}{y^2x^2-2xy+1} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{2y^2x}{y^2x^2-2xy+1}} dy\end{aligned}$$

Which results in

$$S = -\frac{xy}{2} + \ln(y) + \frac{1}{2yx}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(y^2x^2 + 1)}{x\,(y^2x^2 - 2xy + 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-y^2x^2 - 1}{2y\,x^2} \\ S_y &= -\frac{(xy - 1)^2}{2x\,y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

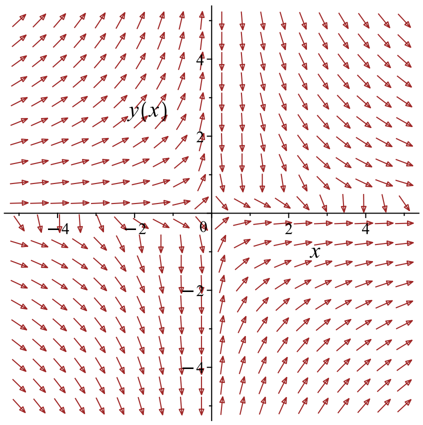
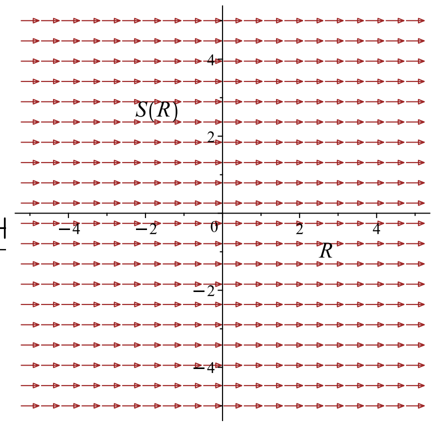
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{-y^2x^2 + 2xy\ln(y) + 1}{2xy} = c_1$$

Which simplifies to

$$\frac{-y^2x^2 + 2xy\ln(y) + 1}{2xy} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(y^2x^2+1)}{x(y^2x^2-2xy+1)}$ 	$R = x$ $S = \frac{-y^2x^2 + 2\ln(y)\,xy + 1}{2xy}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{-y^2x^2 + 2xy\ln(y) + 1}{2xy} = c_1 \tag{1}$$

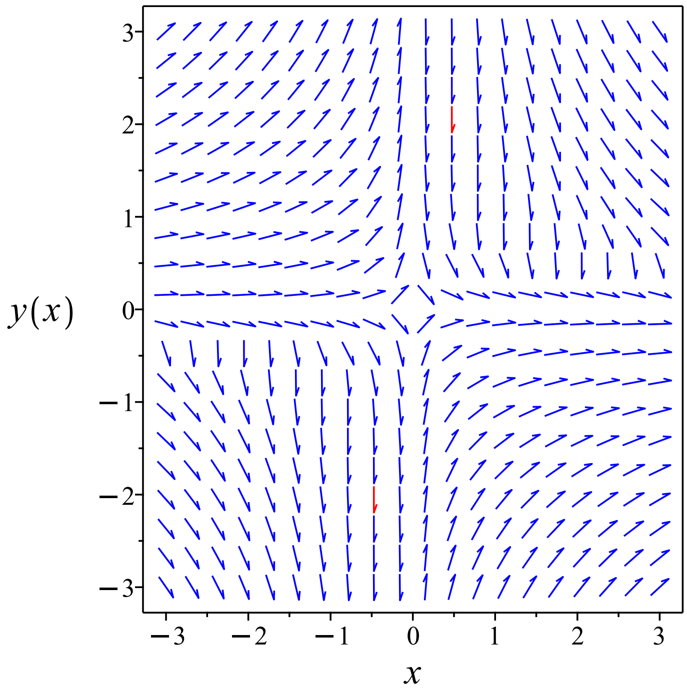


Figure 273: Slope field plot

Verification of solutions

$$\frac{-y^2x^2 + 2xy\ln(y) + 1}{2xy} = c_1$$

Verified OK.

6.44.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left((xy-1)^2 x\right) \mathrm{d}y = \left(-y\left(y^2x^2+1\right)\right) \mathrm{d}x$$
$$\left(y\left(y^2x^2+1\right)\right) \mathrm{d}x + \left((xy-1)^2 x\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = y\left(y^2x^2+1\right)$$
$$N(x,y) = (xy-1)^2 x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\left(y^2x^2+1\right)\right)\\ &= 3y^2x^2+1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left((xy-1)^2 x\right)\\ &= 3y^2x^2-4xy+1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x\left(xy-1\right)^2}\left(\left(3y^2x^2+1\right) - \left(2(xy-1)\,xy + \left(xy-1\right)^2\right)\right) \\
&= \frac{4y}{\left(xy-1\right)^2}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y\left(y^2x^2+1\right)}\left(\left(2(xy-1)\,xy + \left(xy-1\right)^2\right) - \left(3y^2x^2+1\right)\right) \\
&= -\frac{4x}{y^2x^2+1}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{\left(2(xy-1)\,xy + \left(xy-1\right)^2\right) - \left(3y^2x^2+1\right)}{x\left(y\left(y^2x^2+1\right)\right) - y\left(\left(xy-1\right)^2 x\right)} \\
&= -\frac{2}{yx}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{2}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(t)}\\ &= \frac{1}{t^2}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{y^2x^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{y^2x^2}\left(y\left(y^2x^2+1\right)\right)\\ &= \frac{y^2x^2+1}{y\,x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{y^2x^2}\left(\left(xy-1\right)^2 x\right)\\ &= \frac{\left(xy-1\right)^2}{x\,y^2}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{y^2x^2+1}{y\,x^2}\right)+\left(\frac{\left(xy-1\right)^2}{x\,y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{y^2x^2+1}{y\,x^2} \mathrm{d}x \\
\phi &= \frac{y^2x^2-1}{xy} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= 2x - \frac{y^2x^2-1}{x\,y^2} + f'(y) \qquad (4) \\
&= \frac{y^2x^2+1}{y^2x} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{(xy-1)^2}{x\,y^2}$. Therefore equation (4) becomes

$$
\frac{(xy-1)^2}{x\,y^2} = \frac{y^2x^2+1}{y^2x} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{2}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-\frac{2}{y}\right) \mathrm{d}y \\
f(y) &= -2\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{y^2x^2-1}{xy} - 2\ln(y) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y^2x^2 - 1}{xy} - 2\ln(y)$$

Summary

The solution(s) found are the following

$$\frac{y^2x^2 - 1}{xy} - 2\ln(y) = c_1 \tag{1}$$

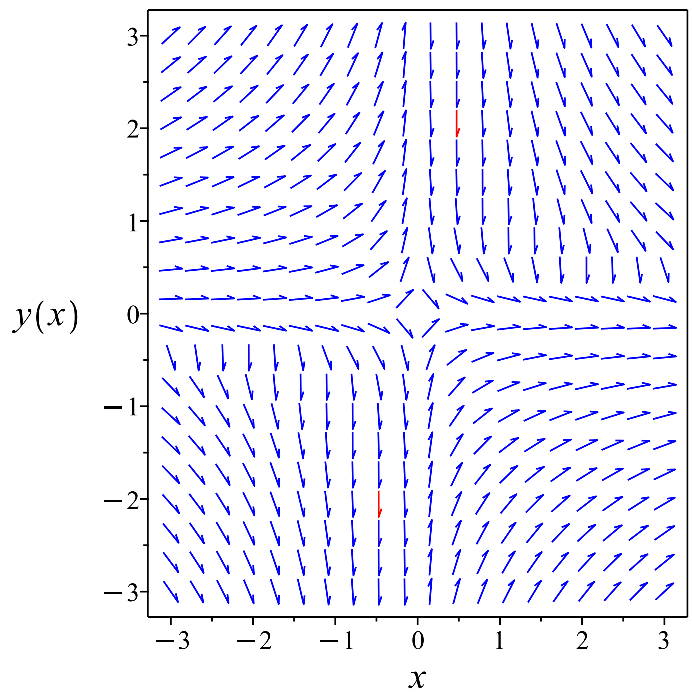


Figure 274: Slope field plot

Verification of solutions

$$\frac{y^2x^2 - 1}{xy} - 2\ln(y) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 34

```
dsolve((x*y(x)-1)^2*x*diff(y(x),x)+(x^2*y(x)^2+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{\mathrm{RootOf}\left(-e^{2\_Z}-2\ln(x)e^{\_Z}+2c_1e^{\_Z}+2\_Ze^{\_Z}+1\right)}}{x}$$

✓ Solution by Mathematica

Time used: 0.108 (sec). Leaf size: 25

```
DSolve[(x*y[x]-1)^2*x*y'[x]+(x^2*y[x]^2+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[xy(x) - \frac{1}{xy(x)} - 2\log(y(x)) = c_1, y(x)\right]$$

6.45 problem Exercise 12.45, page 103

Internal problem ID [4566]
Internal file name [`OUTPUT/4059_Sunday_June_05_2022_12_16_43_PM_99162800/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.45, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{\left(x^2+y^2\right) y'+2x(2x+y)=0}$$

6.45.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\left(x^2+u(x)^2 x^2\right) \left(u'(x)\,x+u(x)\right)+2x(2x+u(x)\,x)=0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^3+3u+4}{x\left(u^2+1\right)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^3+3u+4}{u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^3+3u+4}{u^2+1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^3+3u+4}{u^2+1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(u^3+3u+4)}{3} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$\left(u^3+3u+4\right)^{\frac{1}{3}} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\left(u^3+3u+4\right)^{\frac{1}{3}} = \frac{c_3}{x}$$

Which simplifies to

$$\left(u(x)^3+3u(x)+4\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\left(u(x)^3+3u(x)+4\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{y^3}{x^3}+\frac{3y}{x}+4\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

$$\left(\frac{y^3+3yx^2+4x^3}{x^3}\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\left(\frac{y^3+3yx^2+4x^3}{x^3}\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{x} \tag{1}$$

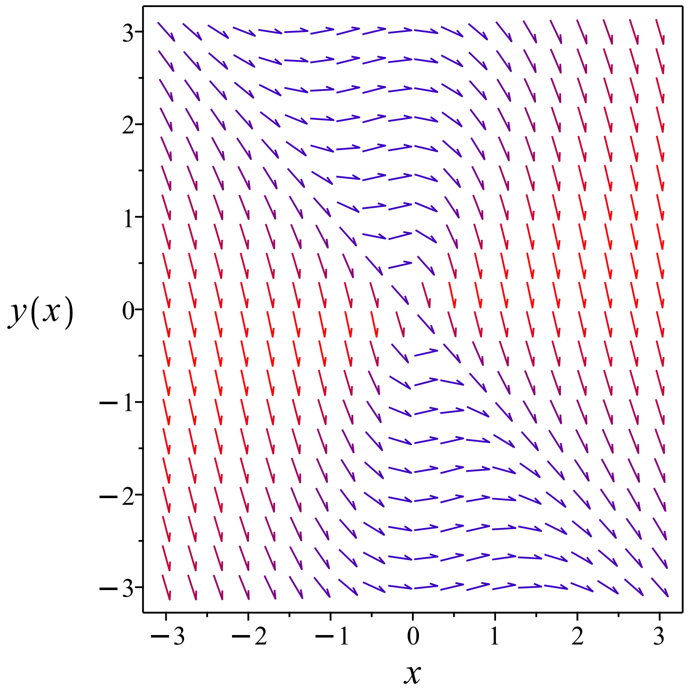


Figure 275: Slope field plot

Verification of solutions

$$\left(\frac{y^3+3yx^2+4x^3}{x^3}\right)^{\frac{1}{3}}=\frac{c_3 e^{c_2}}{x}$$

Verified OK.

6.45.2 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{2x(2x+y)}{x^2+y^2} \tag{1}$$

Which becomes

$$(y^2)\,dy = (-x^2)\,dy + (-2x(2x+y))\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x^2)\,dy + (-2x(2x+y))\,dx = d\left(-y\,x^2 - \frac{4}{3}x^3\right)$$

Hence (2) becomes

$$\left(y^{2}\right) dy = d\left(-y\,x^{2}-\frac{4}{3}x^{3}\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(-16x^{3}+12c_{1}+4\sqrt{20x^{6}-24c_{1}x^{3}+9c_{1}^{2}}\right)^{\frac{1}{3}}}{2}-\frac{2x^{2}}{\left(-16x^{3}+12c_{1}+4\sqrt{20x^{6}-24c_{1}x^{3}+9c_{1}^{2}}\right)^{\frac{1}{3}}}+c_{1}$$

$$y = -\frac{\left(-16x^{3}+12c_{1}+4\sqrt{20x^{6}-24c_{1}x^{3}+9c_{1}^{2}}\right)^{\frac{1}{3}}}{4}+\frac{x^{2}}{\left(-16x^{3}+12c_{1}+4\sqrt{20x^{6}-24c_{1}x^{3}+9c_{1}^{2}}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\cdots}{\cdots}$$

$$y = -\frac{\left(-16x^{3}+12c_{1}+4\sqrt{20x^{6}-24c_{1}x^{3}+9c_{1}^{2}}\right)^{\frac{1}{3}}}{4}+\frac{x^{2}}{\left(-16x^{3}+12c_{1}+4\sqrt{20x^{6}-24c_{1}x^{3}+9c_{1}^{2}}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\cdots}{\cdots}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{2x^2}{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}} + c_1 \tag{1}$$

$$y = -\frac{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}{4} + \frac{x^2}{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}{2} + \frac{2x^2}{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1 \tag{2}$$

$$y = -\frac{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}{4} + \frac{x^2}{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}{2} + \frac{2x^2}{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1 \tag{3}$$

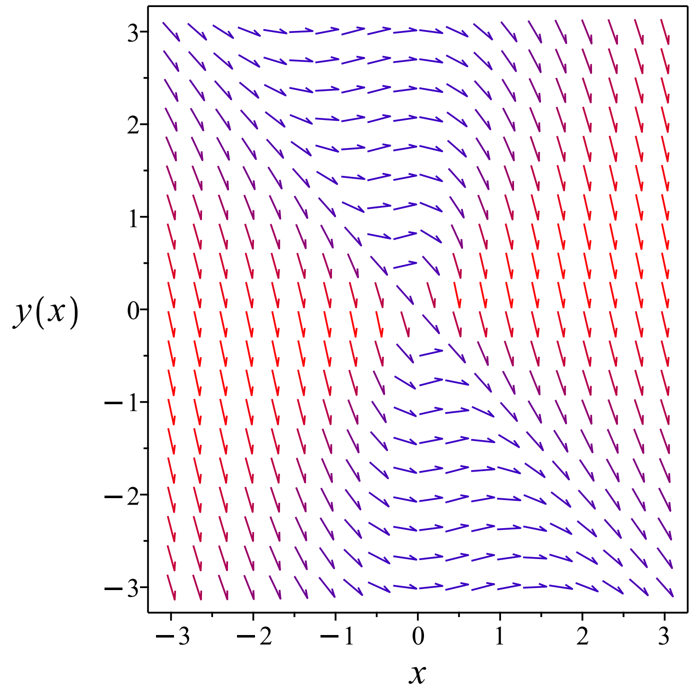


Figure 276: Slope field plot

Verification of solutions

$$y = \frac{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{2x^2}{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}} + c_1$$

Verified OK.

$$y = -\frac{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}{4} + \frac{x^2}{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}{2} + \frac{2x^2}{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1$$

Verified OK.

$$y = -\frac{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}{4} + \frac{x^2}{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}{2} + \frac{2x^2}{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1$$

Verified OK.

6.45.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2x(2x+y)}{x^2+y^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &- \frac{2x(2x+y)(b_3 - a_2)}{x^2+y^2} - \frac{4x^2(2x+y)^2 a_3}{(x^2+y^2)^2} \\ &- \left(-\frac{2(2x+y)}{x^2+y^2} - \frac{4x}{x^2+y^2} + \frac{4x^2(2x+y)}{(x^2+y^2)^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(-\frac{2x}{x^2+y^2} + \frac{4x(2x+y)\,y}{(x^2+y^2)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{4x^4 a_2 - 16x^4 a_3 + 3x^4 b_2 - 4x^4 b_3 - 16x^3 y a_3 - 8x^3 y b_2 + 12x^2 y^2 a_2 - 6x^2 y^2 a_3 - 12x^2 y^2 b_3 + 4x\,y^3 a_2 + 8x\,y^3}{(x^2+y^2)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &4x^4 a_2 - 16x^4 a_3 + 3x^4 b_2 - 4x^4 b_3 - 16x^3 y a_3 - 8x^3 y b_2 + 12x^2 y^2 a_2 \\ &- 6x^2 y^2 a_3 - 12x^2 y^2 b_3 + 4x\,y^3 a_2 + 8x\,y^3 a_3 - 4x\,y^3 b_3 + 2y^4 a_3 \\ &+ y^4 b_2 + 2x^3 b_1 - 2x^2 y a_1 - 8x^2 y b_1 + 8x\,y^2 a_1 - 2x\,y^2 b_1 + 2y^3 a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &4a_2v_1^4 + 12a_2v_1^2v_2^2 + 4a_2v_1v_2^3 - 16a_3v_1^4 - 16a_3v_1^3v_2 - 6a_3v_1^2v_2^2 + 8a_3v_1v_2^3 \\ &+ 2a_3v_2^4 + 3b_2v_1^4 - 8b_2v_1^3v_2 + b_2v_2^4 - 4b_3v_1^4 - 12b_3v_1^2v_2^2 - 4b_3v_1v_2^3 \\ &- 2a_1v_1^2v_2 + 8a_1v_1v_2^2 + 2a_1v_2^3 + 2b_1v_1^3 - 8b_1v_1^2v_2 - 2b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(4a_2 - 16a_3 + 3b_2 - 4b_3)\, v_1^4 + (-16a_3 - 8b_2)\, v_1^3v_2 \\ &+ 2b_1v_1^3 + (12a_2 - 6a_3 - 12b_3)\, v_1^2v_2^2 + (-2a_1 - 8b_1)\, v_1^2v_2 \\ &+ (4a_2 + 8a_3 - 4b_3)\, v_1v_2^3 + (8a_1 - 2b_1)\, v_1v_2^2 + (2a_3 + b_2)\, v_2^4 + 2a_1v_2^3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 &= 0 \\ 2b_1 &= 0 \\ -2a_1 - 8b_1 &= 0 \\ 8a_1 - 2b_1 &= 0 \\ -16a_3 - 8b_2 &= 0 \\ 2a_3 + b_2 &= 0 \\ 4a_2 + 8a_3 - 4b_3 &= 0 \\ 12a_2 - 6a_3 - 12b_3 &= 0 \\ 4a_2 - 16a_3 + 3b_2 - 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{2x(2x+y)}{x^2+y^2}\right)(x) \\
&= \frac{4x^3 + 3y\,x^2 + y^3}{x^2+y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{4x^3+3y\,x^2+y^3}{x^2+y^2}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\ln \left(4x^3 + 3y\,x^2 + y^3\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2x(2x+y)}{x^2+y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2x(2x+y)}{4x^3 + 3y\,x^2 + y^3} \\
S_y &= \frac{x^2+y^2}{4x^3 + 3y\,x^2 + y^3}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

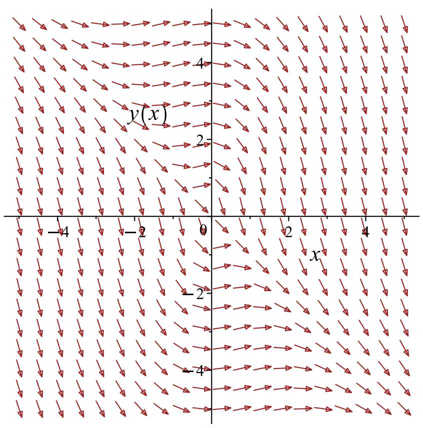
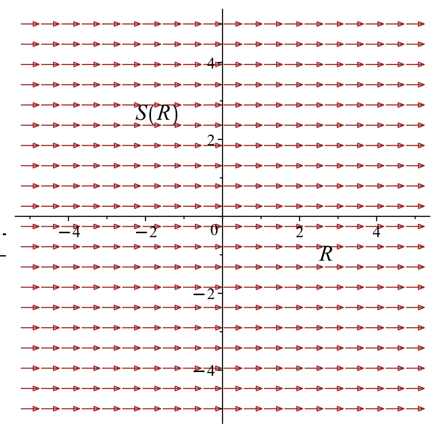
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln (x+y)}{3} + \frac{\ln \left(y^2 - xy + 4x^2\right)}{3} = c_1$$

Which simplifies to

$$\frac{\ln (x+y)}{3} + \frac{\ln (y^2 - xy + 4x^2)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x(2x+y)}{x^2+y^2}$ 	$R = x$ $S = \frac{\ln (x+y)}{3} + \frac{\ln (4x^2 \cdot}{}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln (x+y)}{3} + \frac{\ln (y^2 - xy + 4x^2)}{3} = c_1 \tag{1}$$

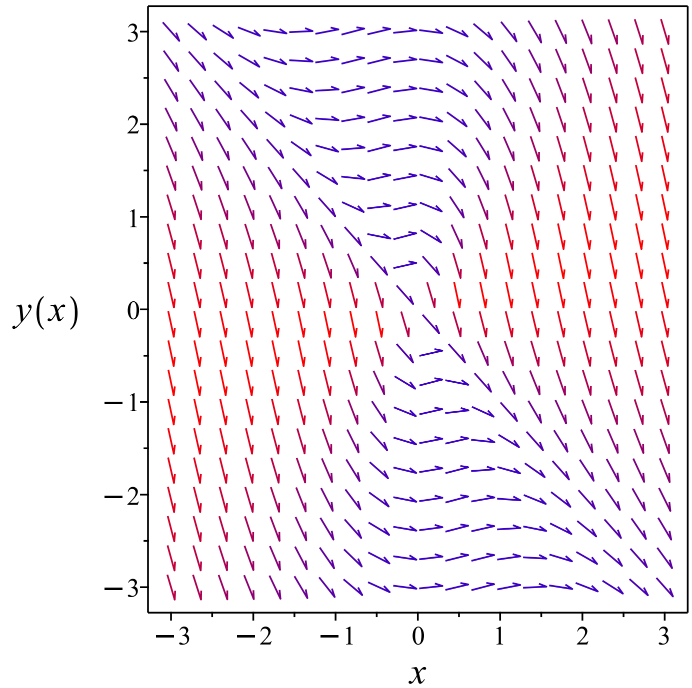


Figure 277: Slope field plot

Verification of solutions

$$\frac{\ln(x+y)}{3}+\frac{\ln(y^2-xy+4x^2)}{3}=c_1$$

Verified OK.

6.45.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2 + y^2\right) \mathrm{d}y &= (-2x(2x + y))\,\mathrm{d}x \\ (2x(2x + y))\,\mathrm{d}x + \left(x^2 + y^2\right) \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2x(2x + y) \\ N(x, y) &= x^2 + y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x(2x + y)) \\ &= 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 + y^2\right) \\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int 2x(2x + y) \, dx$$
$$\phi = \frac{x^2(4x + 3y)}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 + y^2$. Therefore equation (4) becomes

$$x^2 + y^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int (y^2) \, dy$$
$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2(4x+3y)}{3} + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2(4x+3y)}{3} + \frac{y^3}{3}$$

Summary

The solution(s) found are the following

$$\frac{x^2(4x+3y)}{3} + \frac{y^3}{3} = c_1 \tag{1}$$

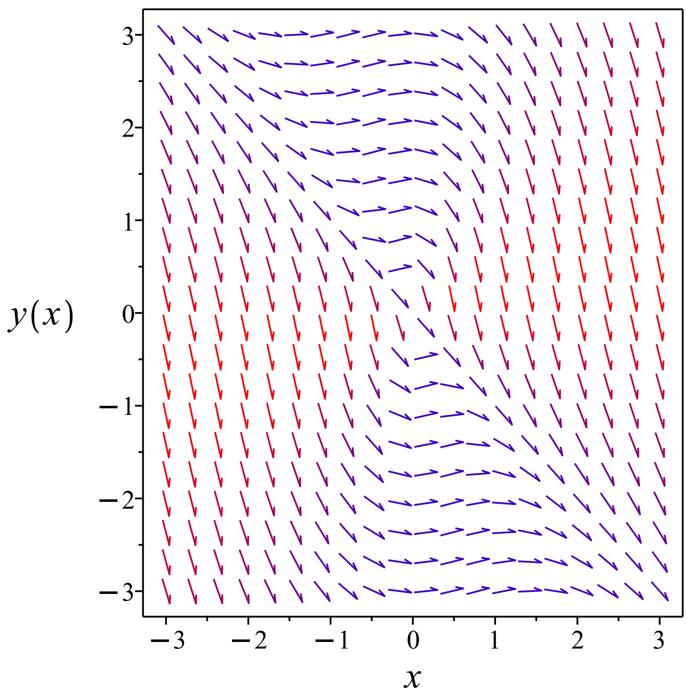


Figure 278: Slope field plot

Verification of solutions

$$\frac{x^2(4x+3y)}{3} + \frac{y^3}{3} = c_1$$

Verified OK.

6.45.5 Maple step by step solution

Let's solve

$(x^2+y^2)\,y'+2x(2x+y)=0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $2x=2x$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int 2x(2x+y)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{4x^3}{3}+y\,x^2+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x^2+y^2=x^2+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=y^2$

- Solve for $f_1(y)$

  $f_1(y)=\frac{y^3}{3}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = \frac{4}{3}x^3 + y\,x^2 + \frac{1}{3}y^3$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\frac{4}{3}x^3 + y\,x^2 + \frac{1}{3}y^3 = c_1$$

- Solve for y

$$\left\{y = \frac{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{2x^2}{\left(-16x^3+12c_1+4\sqrt{20x^6-24c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}, y = -\frac{\left(-16x^3+12c_1+4\sqrt{20x\right.}}{4}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 321

```
dsolve((x^2+y(x)^2)*diff(y(x),x)+2*x*(2*x+y(x))=0,y(x), singsol=all)
```

$$y(x) = -\frac{2\left(c_1x^2 - \frac{\left(4-16x^3c_1^{\frac{3}{2}}+4\sqrt{20x^6c_1^3-8x^3c_1^{\frac{3}{2}}+1}\right)^{\frac{2}{3}}}{4}\right)}{\sqrt{c_1}\left(4-16x^3c_1^{\frac{3}{2}}+4\sqrt{20x^6c_1^3-8x^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left(4-16x^3c_1^{\frac{3}{2}}+4\sqrt{20x^6c_1^3-8x^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}}{4\sqrt{c_1}} - \frac{\sqrt{c_1}\left(i\sqrt{3}-1\right)x^2}{\left(4-16x^3c_1^{\frac{3}{2}}+4\sqrt{20x^6c_1^3-8x^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{4i\sqrt{3}\,c_1x^2 + i\sqrt{3}\left(4-16x^3c_1^{\frac{3}{2}}+4\sqrt{20x^6c_1^3-8x^3c_1^{\frac{3}{2}}+1}\right)^{\frac{2}{3}} + 4c_1x^2 - \left(4-16x^3c_1^{\frac{3}{2}}+4\sqrt{20x^6c_1^3-8x^3c\right.}{4\left(4-16x^3c_1^{\frac{3}{2}}+4\sqrt{20x^6c_1^3-8x^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}\sqrt{c_1}}$$

✓ Solution by Mathematica

Time used: 18.874 (sec). Leaf size: 593

```
DSolve[(x^2+y[x]^2)*y'[x]+2*x*(2*x+y[x])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{\sqrt[3]{-4x^3+\sqrt{20x^6-8e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}}{\sqrt[3]{2}} - \frac{\sqrt[3]{2}x^2}{\sqrt[3]{-4x^3+\sqrt{20x^6-8e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}}$$

$$y(x) \rightarrow \frac{\sqrt[3]{2}\left(2+2i\sqrt{3}\right)x^2+i2^{2/3}\left(\sqrt{3}+i\right)\left(-4x^3+\sqrt{20x^6-8e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}\right)^{2/3}}{4\sqrt[3]{-4x^3+\sqrt{20x^6-8e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}}$$

$$y(x) \rightarrow \frac{\left(1-i\sqrt{3}\right)x^2}{2^{2/3}\sqrt[3]{-4x^3+\sqrt{20x^6-8e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{-4x^3+\sqrt{20x^6-8e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}}{2\sqrt[3]{2}}$$

$$y(x) \rightarrow \sqrt[3]{\sqrt{5}\sqrt{x^6}-2x^3} - \frac{x^2}{\sqrt[3]{\sqrt{5}\sqrt{x^6}-2x^3}}$$

$$y(x) \rightarrow \frac{\left(1-i\sqrt{3}\right)x^2+\left(-1-i\sqrt{3}\right)\left(\sqrt{5}\sqrt{x^6}-2x^3\right)^{2/3}}{2\sqrt[3]{\sqrt{5}\sqrt{x^6}-2x^3}}$$

$$y(x) \rightarrow \frac{\left(1+i\sqrt{3}\right)x^2+i\left(\sqrt{3}+i\right)\left(\sqrt{5}\sqrt{x^6}-2x^3\right)^{2/3}}{2\sqrt[3]{\sqrt{5}\sqrt{x^6}-2x^3}}$$

6.46 problem Exercise 12.46, page 103

Internal problem ID [4567]
Internal file name [`OUTPUT/4060_Sunday_June_05_2022_12_16_52_PM_45955166/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.46, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _exact, _rational, _Bernoulli]
```

$$\boxed{3xy^2y' + y^3 = 2x}$$

6.46.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y^3 - 2x}{3x\,y^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 154: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{y^2 x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y^2 x}} dy \end{aligned}$$

Which results in

$$S = \frac{x\,y^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y^3 - 2x}{3x\,y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y^3}{3} \\ S_y &= y^2 x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2x}{3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2R}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{3} + c_1 \tag{4}$$

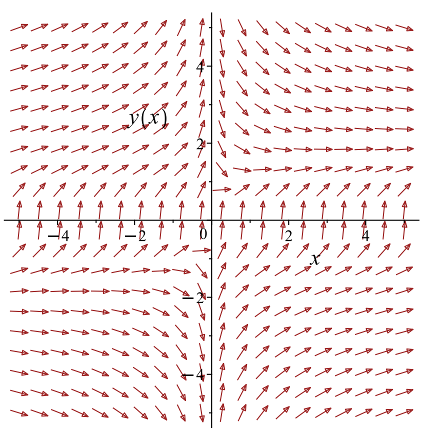
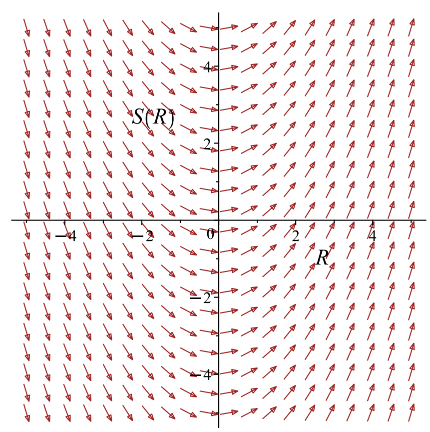
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3 x}{3} = \frac{x^2}{3} + c_1$$

Which simplifies to

$$\frac{y^3 x}{3} = \frac{x^2}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^3-2x}{3x\,y^2}$ 	$R = x$ $S = \frac{x\,y^3}{3}$	$\frac{dS}{dR} = \frac{2R}{3}$ 

Summary

The solution(s) found are the following

$$\frac{y^3 x}{3} = \frac{x^2}{3} + c_1 \tag{1}$$

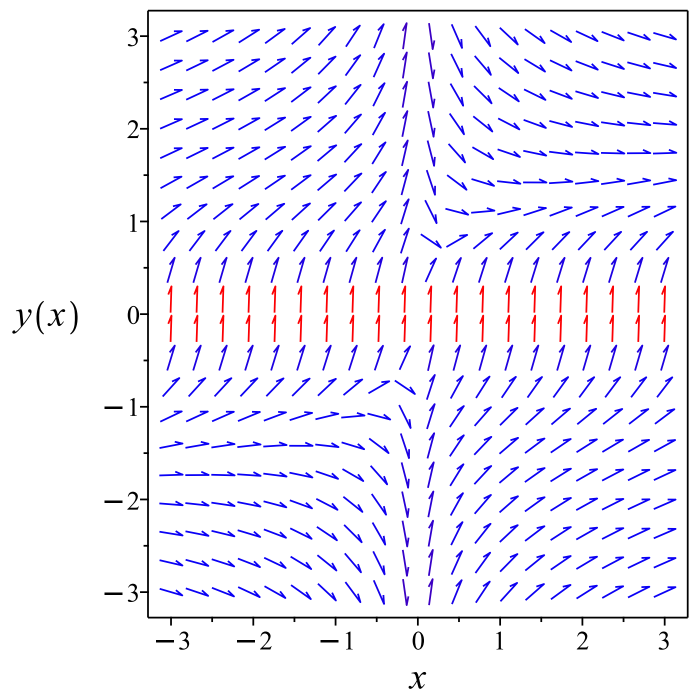


Figure 279: Slope field plot

Verification of solutions

$$\frac{y^3 x}{3} = \frac{x^2}{3} + c_1$$

Verified OK.

6.46.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y^3 - 2x}{3x\, y^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{3x}y + \frac{2}{3}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{3x} \\ f_1(x) &= \frac{2}{3} \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = -\frac{y^3}{3x} + \frac{2}{3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= -\frac{w(x)}{3x} + \frac{2}{3} \\ w' &= -\frac{w}{x} + 2 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= 2 \end{aligned}$$

Hence the ode is

$$w'(x)+\frac{w(x)}{x}=2$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{x}dx}$$
$$= x$$

The ode becomes

$$\frac{d}{dx}(\mu w)=(\mu)(2)$$
$$\frac{d}{dx}(xw)=(x)(2)$$
$$d(xw)=(2x)\,dx$$

Integrating gives

$$xw=\int 2x\,dx$$
$$xw=x^2+c_1$$

Dividing both sides by the integrating factor $\mu=x$ results in

$$w(x)=x+\frac{c_1}{x}$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3=x+\frac{c_1}{x}$$

Solving for y gives

$$y(x)=\frac{\left(\left(x^2+c_1\right)x^2\right)^{\frac{1}{3}}}{x}$$
$$y(x)=\frac{\left(\left(x^2+c_1\right)x^2\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{2x}$$
$$y(x)=-\frac{\left(\left(x^2+c_1\right)x^2\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\left(x^{2}+c_{1}\right) x^{2}\right)^{\frac{1}{3}}}{x} \tag{1}$$

$$y = \frac{\left(\left(x^{2}+c_{1}\right) x^{2}\right)^{\frac{1}{3}} \left(i\sqrt{3}-1\right)}{2x} \tag{2}$$

$$y = -\frac{\left(\left(x^{2}+c_{1}\right) x^{2}\right)^{\frac{1}{3}} \left(1+i\sqrt{3}\right)}{2x} \tag{3}$$

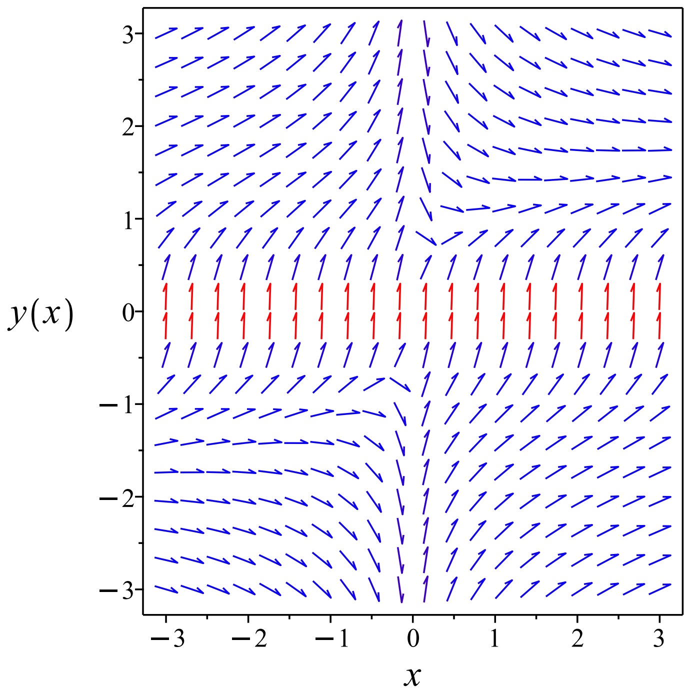


Figure 280: Slope field plot

Verification of solutions

$$y = \frac{\left(\left(x^{2}+c_{1}\right) x^{2}\right)^{\frac{1}{3}}}{x}$$

Verified OK.

$$y = \frac{\left(\left(x^{2}+c_{1}\right) x^{2}\right)^{\frac{1}{3}} \left(i\sqrt{3}-1\right)}{2x}$$

Verified OK.

$$y = -\frac{\left(\left(x^{2}+c_{1}\right) x^{2}\right)^{\frac{1}{3}} \left(1+i\sqrt{3}\right)}{2x}$$

Verified OK.

6.46.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(3y^2x)\,\mathrm{d}y &= (-y^3 + 2x)\,\mathrm{d}x \\
(y^3 - 2x)\,\mathrm{d}x + (3y^2x)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= y^3 - 2x \\
N(x,y) &= 3y^2x
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y^3 - 2x) \\
&= 3y^2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3y^2x) \\
&= 3y^2
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int y^3 - 2x\,\mathrm{d}x
\end{aligned}$$

$$\phi = x\,y^3 - x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3y^2 x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3y^2 x$. Therefore equation (4) becomes

$$3y^2 x = 3y^2 x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x\,y^3 - x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\,y^3 - x^2$$

Summary

The solution(s) found are the following

$$y^3 x - x^2 = c_1 \tag{1}$$

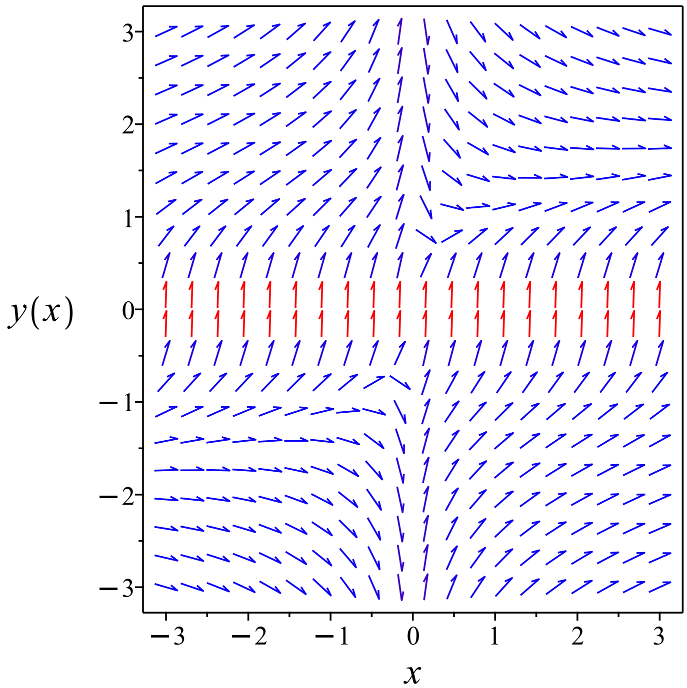


Figure 281: Slope field plot

Verification of solutions

$$y^3x - x^2 = c_1$$

Verified OK.

6.46.4 Maple step by step solution

Let's solve

$3xy^2y' + y^3 = 2x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

  ◦ Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right)y' = 0$

- Evaluate derivatives

  $3y^2 = 3y^2$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (y^3 - 2x)\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = x\,y^3 - x^2 + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $3y^2x = 3y^2x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = x\,y^3 - x^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $x\,y^3 - x^2 = c_1$

- Solve for y

  $\left\{y = \frac{((x^2+c_1)x^2)^{\frac{1}{3}}}{x}, y = -\frac{((x^2+c_1)x^2)^{\frac{1}{3}}}{2x} - \frac{\mathrm{I}\sqrt{3}\,((x^2+c_1)x^2)^{\frac{1}{3}}}{2x}, y = -\frac{((x^2+c_1)x^2)^{\frac{1}{3}}}{2x} + \frac{\mathrm{I}\sqrt{3}\,((x^2+c_1)x^2)^{\frac{1}{3}}}{2x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 73

```
dsolve(3*x*y(x)^2*diff(y(x),x)+y(x)^3-2*x=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(\left(x^2+c_1\right)x^2\right)^{\frac{1}{3}}}{x}$$
$$y(x) = -\frac{\left(\left(x^2+c_1\right)x^2\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{2x}$$
$$y(x) = \frac{\left(\left(x^2+c_1\right)x^2\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{2x}$$

✓ Solution by Mathematica

Time used: 0.224 (sec). Leaf size: 72

```
DSolve[3*x*y[x]^2*y'[x]+y[x]^3-2*x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{x^2+c_1}}{\sqrt[3]{x}}$$
$$y(x) \to -\frac{\sqrt[3]{-1}\sqrt[3]{x^2+c_1}}{\sqrt[3]{x}}$$
$$y(x) \to \frac{(-1)^{2/3}\sqrt[3]{x^2+c_1}}{\sqrt[3]{x}}$$

6.47 problem Exercise 12.47, page 103

Internal problem ID [4568]
Internal file name [`OUTPUT/4061_Sunday_June_05_2022_12_17_07_PM_35160420/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.47, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{2y^3y' + xy^2 = x^3}$$

6.47.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2u(x)^3\,x^3(u'(x)\,x + u(x)) + x^3u(x)^2 = x^3$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{2u^4 + u^2 - 1}{2u^3x}
\end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{2u^4+u^2-1}{u^3}$. Integrating both sides gives

$$\frac{1}{\frac{2u^4+u^2-1}{u^3}}\,du = -\frac{1}{2x}\,dx$$

$$\int \frac{1}{\frac{2u^4+u^2-1}{u^3}} du = \int -\frac{1}{2x} dx$$

$$\frac{\ln(2u^2-1)}{12} + \frac{\ln(u^2+1)}{6} = -\frac{\ln(x)}{2} + c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln(2u^2-1)}{12}+\frac{\ln(u^2+1)}{6}} = e^{-\frac{\ln(x)}{2}+c_2}$$

Which simplifies to

$$(2u^2-1)^{\frac{1}{12}} (u^2+1)^{\frac{1}{6}} = \frac{c_3}{\sqrt{x}}$$

The solution is

$$(2u(x)^2-1)^{\frac{1}{12}} (u(x)^2+1)^{\frac{1}{6}} = \frac{c_3}{\sqrt{x}}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{2y^2}{x^2}-1\right)^{\frac{1}{12}} \left(\frac{y^2}{x^2}+1\right)^{\frac{1}{6}} = \frac{c_3}{\sqrt{x}}$$

$$\left(\frac{2y^2-x^2}{x^2}\right)^{\frac{1}{12}} \left(\frac{x^2+y^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$\left(\frac{2y^2-x^2}{x^2}\right)^{\frac{1}{12}} \left(\frac{x^2+y^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3}{\sqrt{x}} \tag{1}$$

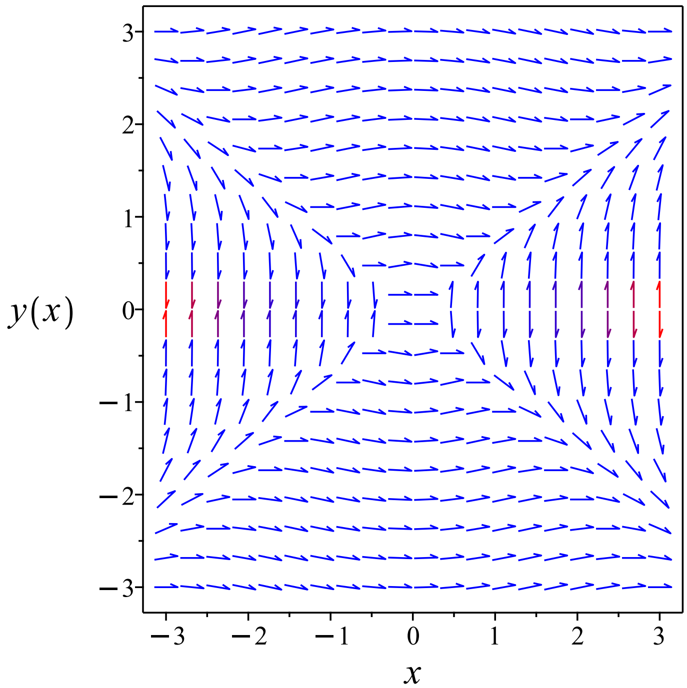


Figure 282: Slope field plot

Verification of solutions

$$\left(\frac{2y^2 - x^2}{x^2}\right)^{\frac{1}{12}} \left(\frac{x^2 + y^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3}{\sqrt{x}}$$

Verified OK.

6.47.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x(-x^2 + y^2)}{2y^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{x(-x^2+y^2)(b_3-a_2)}{2y^3} - \frac{x^2(-x^2+y^2)^2 a_3}{4y^6} \\
&- \left(-\frac{-x^2+y^2}{2y^3} + \frac{x^2}{y^3}\right)(xa_2 + ya_3 + a_1) \\
&- \left(-\frac{x}{y^2} + \frac{3x(-x^2+y^2)}{2y^4}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$-\frac{x^6a_3 - 2x^4y^2a_3 - 6x^4y^2b_2 + 8x^3y^3a_2 - 8x^3y^3b_3 + 7x^2y^4a_3 + 2x^2y^4b_2 - 4x\,y^5a_2 + 4x\,y^5b_3 - 2y^6a_3 - 4b_2y}{4y^6}$$
$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-x^6a_3 + 2x^4y^2a_3 + 6x^4y^2b_2 - 8x^3y^3a_2 + 8x^3y^3b_3 - 7x^2y^4a_3 - 2x^2y^4b_2 \\
&+ 4x\,y^5a_2 - 4x\,y^5b_3 + 2y^6a_3 + 4b_2y^6 + 6x^3y^2b_1 - 6x^2y^3a_1 - 2x\,y^4b_1 + 2y^5a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-8a_2v_1^3v_2^3 + 4a_2v_1v_2^5 - a_3v_1^6 + 2a_3v_1^4v_2^2 - 7a_3v_1^2v_2^4 + 2a_3v_2^6 + 6b_2v_1^4v_2^2 - 2b_2v_1^2v_2^4 \\
&+ 4b_2v_2^6 + 8b_3v_1^3v_2^3 - 4b_3v_1v_2^5 - 6a_1v_1^2v_2^3 + 2a_1v_2^5 + 6b_1v_1^3v_2^2 - 2b_1v_1v_2^4 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-a_3 v_1^6 + (2a_3 + 6b_2) v_1^4 v_2^2 + (-8a_2 + 8b_3) v_1^3 v_2^3 + 6b_1 v_1^3 v_2^2 + (-7a_3 - 2b_2) v_1^2 v_2^4 - 6a_1 v_1^2 v_2^3 + (4a_2 - 4b_3) v_1 v_2^5 - 2b_1 v_1 v_2^4 + (2a_3 + 4b_2) v_2^6 + 2a_1 v_2^5 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-6a_1 &= 0 \\
2a_1 &= 0 \\
-a_3 &= 0 \\
-2b_1 &= 0 \\
6b_1 &= 0 \\
-8a_2 + 8b_3 &= 0 \\
4a_2 - 4b_3 &= 0 \\
-7a_3 - 2b_2 &= 0 \\
2a_3 + 4b_2 &= 0 \\
2a_3 + 6b_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{x(-x^2+y^2)}{2y^3}\right)(x) \\
&= \frac{-x^4+y^2x^2+2y^4}{2y^3} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^4+y^2x^2+2y^4}{2y^3}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(x^2+y^2)}{3} + \frac{\ln(-x^2+2y^2)}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x(-x^2+y^2)}{2y^3}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x(x^2 - y^2)}{(x^2 - 2y^2)(x^2 + y^2)} \\
S_y &= -\frac{2y^3}{(x^2 - 2y^2)(x^2 + y^2)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

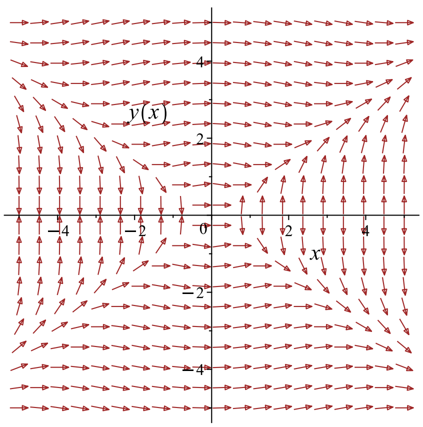
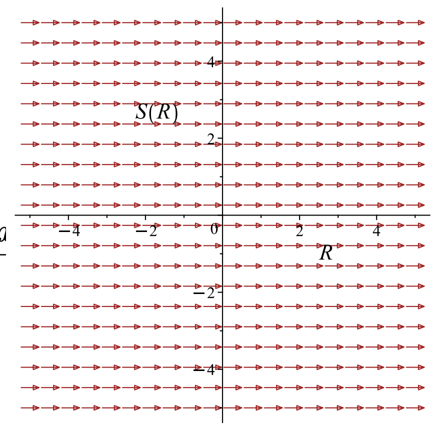
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(x^2 + y^2)}{3} + \frac{\ln(2y^2 - x^2)}{6} = c_1
$$

Which simplifies to

$$
\frac{\ln(x^2 + y^2)}{3} + \frac{\ln(2y^2 - x^2)}{6} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x(-x^2+y^2)}{2y^3}$ 	$R = x$ $S = \frac{\ln(x^2+y^2)}{3} + \ln(-x$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(x^2+y^2)}{3} + \frac{\ln(2y^2-x^2)}{6} = c_1 \tag{1}$$

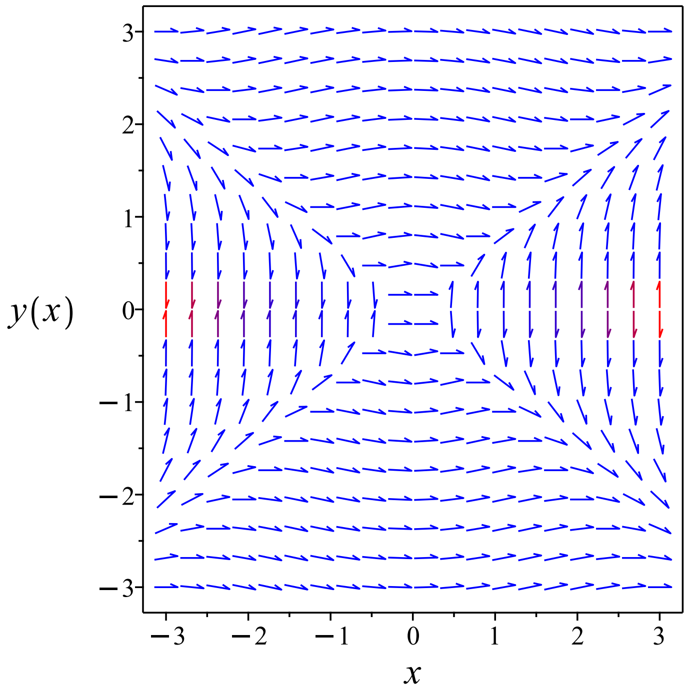


Figure 283: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+y^2)}{3}+\frac{\ln(2y^2-x^2)}{6}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.422 (sec). Leaf size: 649

```
dsolve(2*y(x)^3*diff(y(x),x)+x*y(x)^2-x^3=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{2}\,\sqrt{\frac{x^4c_1^2-c_1x^2\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}+\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{2}{3}}}{\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}}}}{2\sqrt{c_1}}$$

$$y(x) = \frac{\sqrt{2}\,\sqrt{\frac{x^4c_1^2-c_1x^2\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}+\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{2}{3}}}{\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}}}}{2\sqrt{c_1}}$$

$$y(x) = -\frac{\sqrt{\frac{\left(\left(-i\sqrt{3}-1\right)\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}+\left(i\sqrt{3}-1\right)x^2c_1\right)\left(c_1x^2+\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}\right)}{\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}}}}{2\sqrt{c_1}}$$

$$y(x) = \frac{\sqrt{\frac{\left(\left(-i\sqrt{3}-1\right)\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}+\left(i\sqrt{3}-1\right)x^2c_1\right)\left(c_1x^2+\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}\right)}{\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}}}}{2\sqrt{c_1}}$$

$$y(x) = -\frac{\sqrt{\frac{\left(\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)+\left(-i\sqrt{3}-1\right)x^2c_1\right)\left(c_1x^2+\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}\right)}{\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}}}}{2\sqrt{c_1}}$$

$$y(x) = \frac{\sqrt{\frac{\left(\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)+\left(-i\sqrt{3}-1\right)x^2c_1\right)\left(c_1x^2+\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}\right)}{\left(2+x^6c_1^3+2\sqrt{x^6c_1^3+1}\right)^{\frac{1}{3}}}}}{2\sqrt{c_1}}$$

✓ Solution by Mathematica

Time used: 60.13 (sec). Leaf size: 714

```
DSolve[2*y[x]^3*y'[x]+x*y[x]^2-x^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\sqrt[3]{x^6+2\sqrt{e^{24c_1}-e^{12c_1}x^6}-2e^{12c_1}}-x^2+\frac{x^4}{\sqrt[3]{x^6+2\sqrt{e^{24c_1}-e^{12c_1}x^6}-2e^{12c_1}}}}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{\sqrt[3]{x^6+2\sqrt{e^{24c_1}-e^{12c_1}x^6}-2e^{12c_1}}-x^2+\frac{x^4}{\sqrt[3]{x^6+2\sqrt{e^{24c_1}-e^{12c_1}x^6}-2e^{12c_1}}}}}{\sqrt{2}}$$

$$y(x) \to -\frac{1}{2}\sqrt{\left(-1-i\sqrt{3}\right)\sqrt[3]{x^6+2\sqrt{e^{24c_1}-e^{12c_1}x^6}-2e^{12c_1}}-2x^2+\frac{i\left(\sqrt{3}+i\right)x^4}{\sqrt[3]{x^6+2\sqrt{e^{24c_1}-e^{12c_1}x^6}-2e^{12c_1}}}}$$

$$y(x) \to \frac{1}{2}\sqrt{\left(-1-i\sqrt{3}\right)\sqrt[3]{x^6+2\sqrt{e^{24c_1}-e^{12c_1}x^6}-2e^{12c_1}}-2x^2+\frac{i\left(\sqrt{3}+i\right)x^4}{\sqrt[3]{x^6+2\sqrt{e^{24c_1}-e^{12c_1}x^6}-2e^{12c_1}}}}$$

$$y(x) \to -\frac{1}{2}\sqrt{i\left(\sqrt{3}+i\right)\sqrt[3]{x^6+2\sqrt{e^{24c_1}-e^{12c_1}x^6}-2e^{12c_1}}-2x^2+\frac{\left(-1-i\sqrt{3}\right)x^4}{\sqrt[3]{x^6+2\sqrt{e^{24c_1}-e^{12c_1}x^6}-2e^{12c_1}}}}$$

$$y(x) \to \frac{1}{2}\sqrt{i\left(\sqrt{3}+i\right)\sqrt[3]{x^6+2\sqrt{e^{24c_1}-e^{12c_1}x^6}-2e^{12c_1}}-2x^2+\frac{\left(-1-i\sqrt{3}\right)x^4}{\sqrt[3]{x^6+2\sqrt{e^{24c_1}-e^{12c_1}x^6}-2e^{12c_1}}}}$$

6.48 problem Exercise 12.48, page 103

Internal problem ID [4569]
Internal file name [OUTPUT/4062_Sunday_June_05_2022_12_17_16_PM_56735908/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.48, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{(2y^3x + xy + x^2)\, y' - xy + y^2 = 0}$$

6.48.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x\,y^3 + x^2 + xy)\,\mathrm{d}y &= (xy - y^2)\,\mathrm{d}x \\ (-xy + y^2)\,\mathrm{d}x + (2x\,y^3 + x^2 + xy)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -xy + y^2 \\ N(x, y) &= 2x\,y^3 + x^2 + xy \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-xy + y^2) \\ &= -x + 2y \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x\,y^3 + x^2 + xy) \\ &= 2y^3 + 2x + y \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = y^2 - xy$ and $N = 2y^3x + xy + x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{y^2 - xy}{xy^2}$$
$$N = \frac{2y^3x + xy + x^2}{xy^2}$$

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{2x\,y^3 + x^2 + xy}{x\,y^2}\right)\mathrm{d}y = \left(-\frac{-xy + y^2}{x\,y^2}\right)\mathrm{d}x$$

$$\left(\frac{-xy + y^2}{x\,y^2}\right)\mathrm{d}x + \left(\frac{2x\,y^3 + x^2 + xy}{x\,y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{-xy + y^2}{x\,y^2}$$
$$N(x,y) = \frac{2x\,y^3 + x^2 + xy}{x\,y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-xy + y^2}{x\,y^2}\right)\\ &= \frac{1}{y^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2x\,y^3 + x^2 + xy}{x\,y^2}\right)\\ &= \frac{1}{y^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-xy + y^2}{x\,y^2}\,\mathrm{d}x$$
$$\phi = \ln(x) - \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2x\,y^3+x^2+xy}{x\,y^2}$. Therefore equation (4) becomes

$$\frac{2x\,y^3 + x^2 + xy}{x\,y^2} = \frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{2y^2+1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{2y^2+1}{y}\right) \mathrm{d}y$$
$$f(y) = y^2 + \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) - \frac{x}{y} + y^2 + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) - \frac{x}{y} + y^2 + \ln(y)$$

Summary

The solution(s) found are the following

$$\ln(x) - \frac{x}{y} + y^2 + \ln(y) = c_1 \tag{1}$$

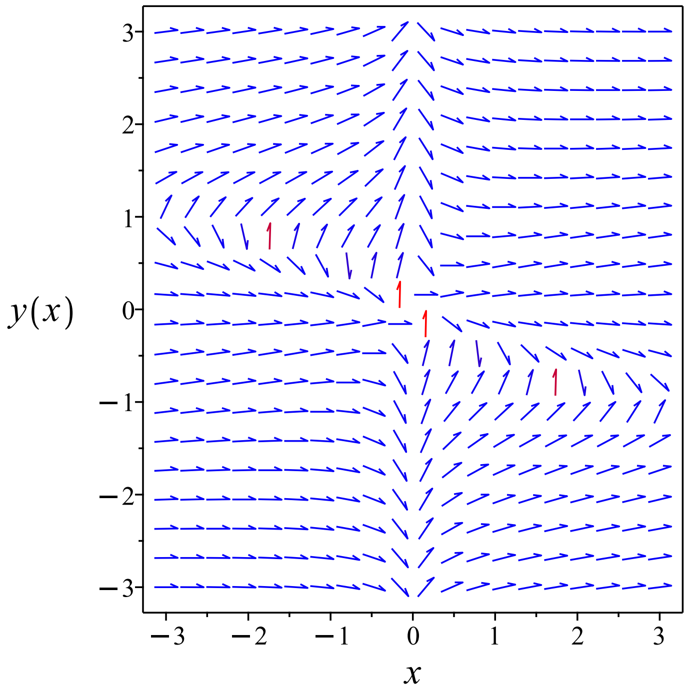


Figure 284: Slope field plot

Verification of solutions

$$\ln(x) - \frac{x}{y} + y^2 + \ln(y) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2`[0, y^2/(2*y^3+x+y)]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve((2*x*y(x)^3+x*y(x)+x^2)*diff(y(x),x)-x*y(x)+y(x)^2=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\mathrm{RootOf}\left(-\mathrm{e}^{3\_Z}-\ln(x)\mathrm{e}^{\_Z}+c_1\mathrm{e}^{\_Z}-\_Z\mathrm{e}^{\_Z}+x\right)}$$

✓ Solution by Mathematica

Time used: 0.225 (sec). Leaf size: 23

```
DSolve[(2*x*y[x]^3+x*y[x]+x^2)*y'[x]-x*y[x]+y[x]^2==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$\text{Solve}\left[y(x)^2 - \frac{x}{y(x)} + \log(y(x)) + \log(x) = c_1, y(x)\right]$$

6.49 problem Exercise 12.49, page 103

Internal problem ID [4570]
Internal file name [`OUTPUT/4063_Sunday_June_05_2022_12_17_27_PM_65044187/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.49, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(2y^3+y)\,y' = 2x^3+x}$$

6.49.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(2x^2+1)}{2y^3+y} \end{aligned}$$

Where $f(x) = x(2x^2+1)$ and $g(y) = \frac{1}{2y^3+y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{2y^3+y}}\,dy = x\left(2x^2+1\right)\,dx$$

$$\int \frac{1}{\frac{1}{2y^3+y}} dy = \int x(2x^2+1) \, dx$$

$$\frac{(2y^2+1)^2}{8} = \frac{(2x^2+1)^2}{8} + c_1$$

The solution is

$$\frac{(2y^2+1)^2}{8} - \frac{(2x^2+1)^2}{8} - c_1 = 0$$

Summary

The solution(s) found are the following

$$\frac{(2y^2+1)^2}{8} - \frac{(2x^2+1)^2}{8} - c_1 = 0 \tag{1}$$

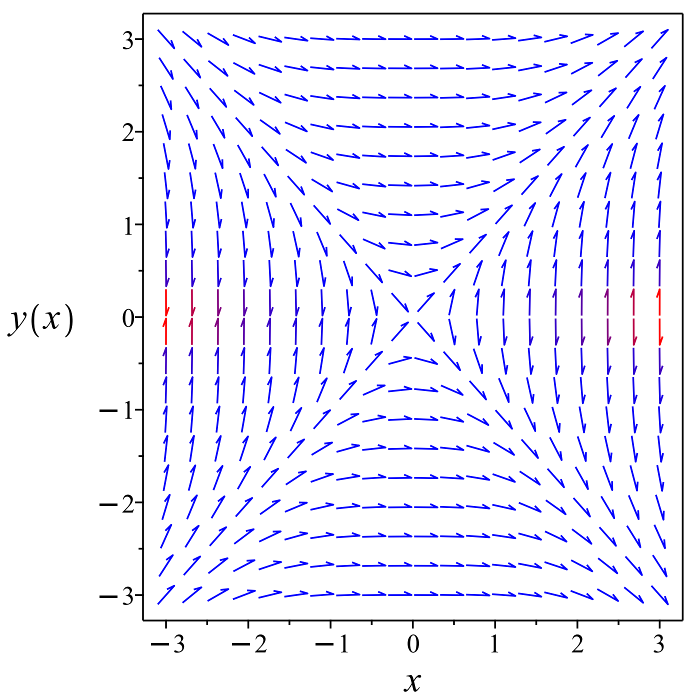


Figure 285: Slope field plot

Verification of solutions

$$\frac{(2y^2+1)^2}{8} - \frac{(2x^2+1)^2}{8} - c_1 = 0$$

Verified OK.

6.49.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{2x^3 + x}{2y^3 + y} \tag{1}$$

Which becomes

$$\left(2y^3 + y\right) dy = \left(x\left(2x^2 + 1\right)\right) dx \tag{2}$$

But the RHS is complete differential because

$$\left(x\left(2x^2 + 1\right)\right) dx = d\left(\frac{1}{2}x^2 + \frac{1}{2}x^4\right)$$

Hence (2) becomes

$$\left(2y^3 + y\right) dy = d\left(\frac{1}{2}x^2 + \frac{1}{2}x^4\right)$$

Integrating both sides gives gives these solutions

$$\begin{aligned}
y &= \frac{\sqrt{-2 + 4\sqrt{x^4 + x^2 + 2c_1}}}{2} + c_1 \\
y &= -\frac{\sqrt{-2 + 4\sqrt{x^4 + x^2 + 2c_1}}}{2} + c_1 \\
y &= \frac{\sqrt{-2 - 4\sqrt{x^4 + x^2 + 2c_1}}}{2} + c_1 \\
y &= -\frac{\sqrt{-2 - 4\sqrt{x^4 + x^2 + 2c_1}}}{2} + c_1
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-2 + 4\sqrt{x^4 + x^2 + 2c_1}}}{2} + c_1 \tag{1}$$

$$y = -\frac{\sqrt{-2 + 4\sqrt{x^4 + x^2 + 2c_1}}}{2} + c_1 \tag{2}$$

$$y = \frac{\sqrt{-2 - 4\sqrt{x^4 + x^2 + 2c_1}}}{2} + c_1 \tag{3}$$

$$y = -\frac{\sqrt{-2 - 4\sqrt{x^4 + x^2 + 2c_1}}}{2} + c_1 \tag{4}$$

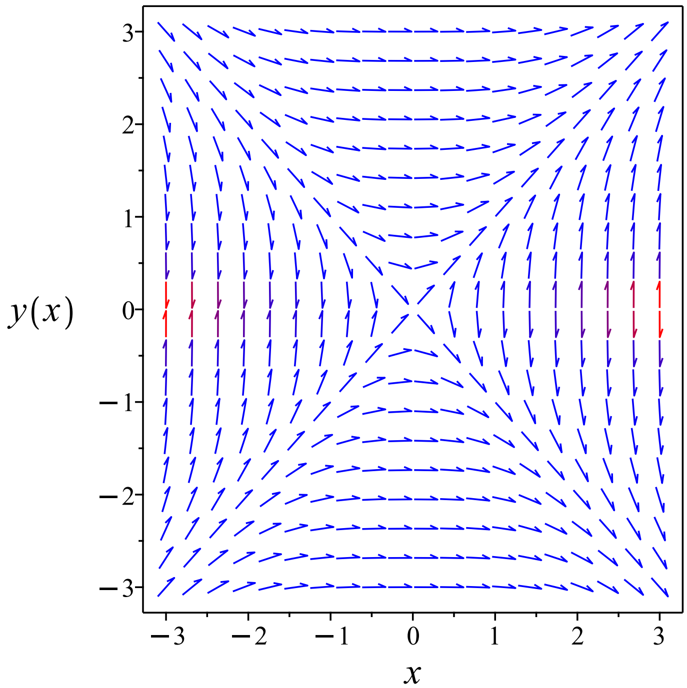


Figure 286: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\sqrt{-2+4\sqrt{x^4+x^2+2c_1}}}{2}+c_1$$

Verified OK.

$$y = -\frac{\sqrt{-2+4\sqrt{x^4+x^2+2c_1}}}{2}+c_1$$

Verified OK.

$$y = \frac{\sqrt{-2-4\sqrt{x^4+x^2+2c_1}}}{2}+c_1$$

Verified OK.

$$y = -\frac{\sqrt{-2-4\sqrt{x^4+x^2+2c_1}}}{2}+c_1$$

Verified OK.

6.49.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x(2x^2+1)}{y\,(2y^2+1)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 157: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{1}{x\left(2x^2+1\right)}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x(2x^2+1)}} dx \end{aligned}$$

Which results in

$$S = \frac{\left(2x^2+1\right)^2}{8}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x(2x^2+1)}{y\left(2y^2+1\right)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= 2x^3 + x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 2y^3 + y \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 2R^3 + R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{(2R^2 + 1)^2}{8} + c_1 \tag{4}
$$

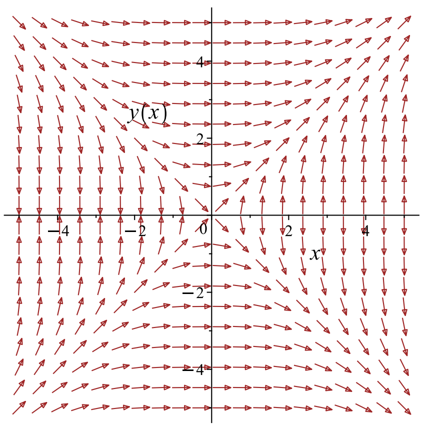
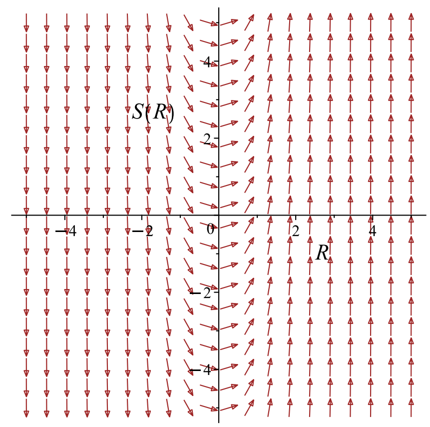
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{(2x^2 + 1)^2}{8} = \frac{(2y^2 + 1)^2}{8} + c_1
$$

Which simplifies to

$$
\frac{(2x^2 + 1)^2}{8} = \frac{(2y^2 + 1)^2}{8} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x(2x^2+1)}{y(2y^2+1)}$ 	$R = y$ $S = \frac{(2x^2+1)^2}{8}$	$\frac{dS}{dR} = 2R^3 + R$ 

Summary

The solution(s) found are the following

$$\frac{(2x^2+1)^2}{8} = \frac{(2y^2+1)^2}{8} + c_1 \tag{1}$$

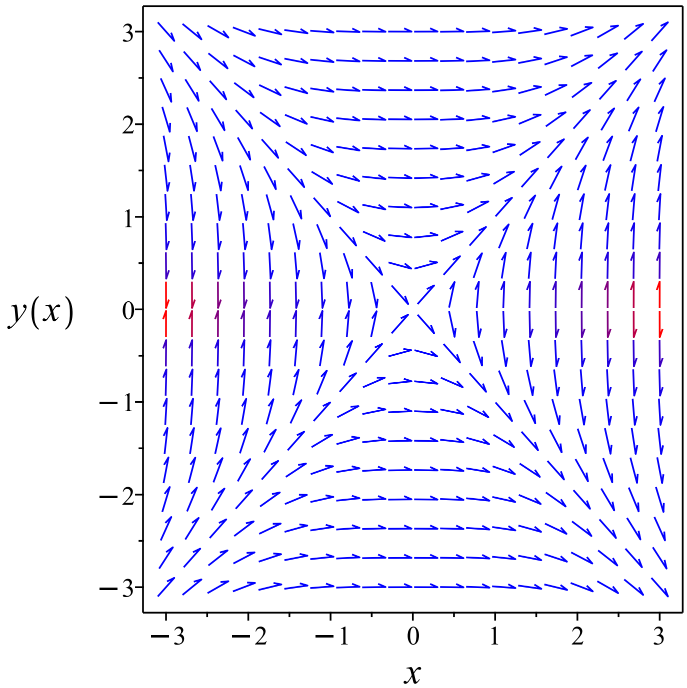


Figure 287: Slope field plot

Verification of solutions

$$\frac{(2x^2+1)^2}{8} = \frac{(2y^2+1)^2}{8} + c_1$$

Verified OK.

6.49.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, dx + N(x, y) \, dy = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(y\left(2y^2 + 1\right)\right) dy &= \left(x\left(2x^2 + 1\right)\right) dx \\
\left(-x\left(2x^2 + 1\right)\right) dx + \left(y\left(2y^2 + 1\right)\right) dy &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x\left(2x^2 + 1\right) \\
N(x, y) &= y\left(2y^2 + 1\right)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x\left(2x^2 + 1\right)\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y\left(2y^2 + 1\right)\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$

$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int -x(2x^2+1)\,\mathrm{d}x$$

$$\phi=-\frac{(2x^2+1)^2}{8}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=y(2y^2+1)$. Therefore equation (4) becomes

$$y(2y^2+1)=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y)&=y(2y^2+1)\\ &=2y^3+y \end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y=\int (2y^3+y)\,\mathrm{d}y$$

$$f(y)=\frac{(2y^2+1)^2}{8}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(2x^2+1)^2}{8} + \frac{(2y^2+1)^2}{8} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(2x^2+1)^2}{8} + \frac{(2y^2+1)^2}{8}$$

<u>Summary</u>

The solution(s) found are the following

$$\frac{(2y^2+1)^2}{8} - \frac{(2x^2+1)^2}{8} = c_1 \tag{1}$$

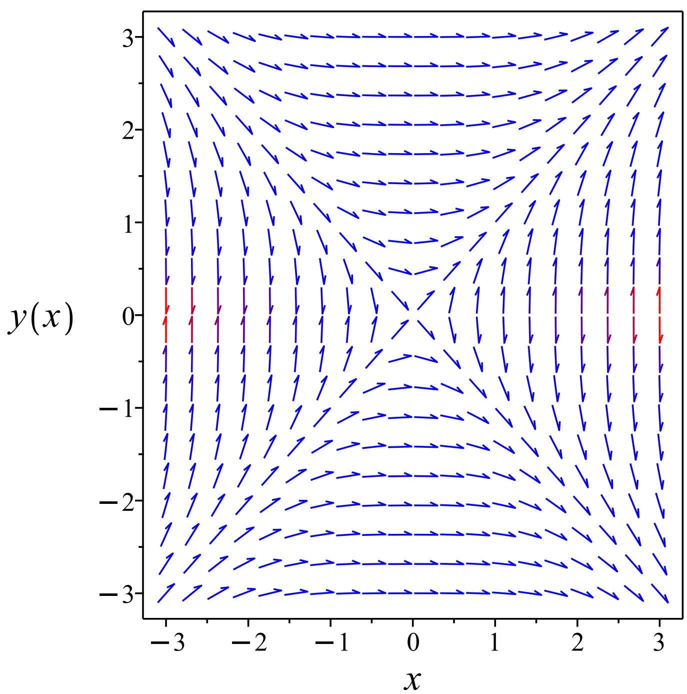


Figure 288: Slope field plot

<u>Verification of solutions</u>

$$\frac{(2y^2+1)^2}{8} - \frac{(2x^2+1)^2}{8} = c_1$$

Verified OK.

6.49.5 Maple step by step solution

Let's solve

$(2y^3 + y)\, y' = 2x^3 + x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (2y^3 + y)\, y' dx = \int (2x^3 + x)\, dx + c_1$

- Evaluate integral

  $\frac{(2y^2+1)^2}{8} = \frac{(2x^2+1)^2}{8} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 113

```
dsolve((2*y(x)^3+y(x))*diff(y(x),x)-2*x^3-x=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-2 - 2\sqrt{4x^4 + 4x^2 + 8c_1 + 1}}}{2}$$
$$y(x) = \frac{\sqrt{-2 - 2\sqrt{4x^4 + 4x^2 + 8c_1 + 1}}}{2}$$
$$y(x) = -\frac{\sqrt{-2 + 2\sqrt{4x^4 + 4x^2 + 8c_1 + 1}}}{2}$$
$$y(x) = \frac{\sqrt{-2 + 2\sqrt{4x^4 + 4x^2 + 8c_1 + 1}}}{2}$$

✓ Solution by Mathematica

Time used: 2.313 (sec). Leaf size: 151

```
DSolve[(2*y[x]^3+y[x])*y'[x]-2*x^3-x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-1-\sqrt{4x^4+4x^2+1+8c_1}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-1-\sqrt{4x^4+4x^2+1+8c_1}}}{\sqrt{2}}$$
$$y(x) \to -\frac{\sqrt{-1+\sqrt{4x^4+4x^2+1+8c_1}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-1+\sqrt{4x^4+4x^2+1+8c_1}}}{\sqrt{2}}$$

6.50 problem Exercise 12.50, page 103

Internal problem ID [4571]
Internal file name [`OUTPUT/4064_Sunday_June_05_2022_12_17_43_PM_91064341/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 2. Special types of differential equations of the first kind. Lesson 12, Miscellaneous Methods
Problem number: Exercise 12.50, page 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - e^{x-y} = -e^x}$$

6.50.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= e^x\left(e^{-y}-1\right) \end{aligned}$$

Where $f(x) = e^x$ and $g(y) = e^{-y} - 1$. Integrating both sides gives

$$\frac{1}{e^{-y}-1}\,dy = e^x\,dx$$

$$\int \frac{1}{e^{-y}-1}\,dy = \int e^x\,dx$$

$$-\ln\left(e^{-y}-1\right) + \ln\left(e^{-y}\right) = e^x + c_1$$

Raising both side to exponential gives

$$e^{-\ln(e^{-y}-1)+\ln(e^{-y})} = e^{e^x+c_1}$$

Which simplifies to

$$\frac{e^{-y}}{e^{-y}-1} = c_2 e^{e^x}$$

Summary

The solution(s) found are the following

$$y = -\ln\left(\frac{c_2}{-1+c_2 e^{e^x}}\right) - e^x \tag{1}$$

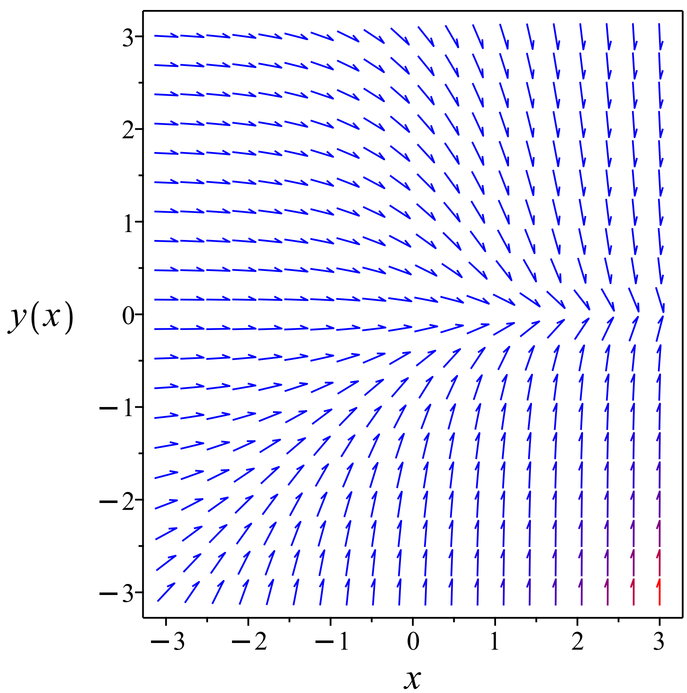


Figure 289: Slope field plot

Verification of solutions

$$y = -\ln\left(\frac{c_2}{-1+c_2 e^{e^x}}\right) - e^x$$

Verified OK.

6.50.2 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = e^{x-y} - e^{x} \tag{1}$$

And using the substitution $u = e^{y}$ then

$$u' = y' e^{y}$$

The above shows that

$$\begin{aligned} y' &= u'(x)\, e^{-y} \\ &= \frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$\frac{u'(x)}{u} = \frac{e^{x}}{u} - e^{x}$$

The above simplifies to

$$\begin{aligned} u'(x) &= e^{x} - e^{x} u(x) \\ u'(x) + e^{x} u(x) &= e^{x} \end{aligned} \tag{2}$$

Now ode (2) is solved for $u(x)$ In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= e^{x}(-u+1) \end{aligned}$$

Where $f(x) = e^{x}$ and $g(u) = -u + 1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-u+1}\, du &= e^{x}\, dx \\ \int \frac{1}{-u+1}\, du &= \int e^{x}\, dx \\ -\ln(u-1) &= e^{x} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{u-1} = e^{e^{x}+c_1}$$

Which simplifies to

$$\frac{1}{u-1} = c_2 e^{e^x}$$

Substituting the solution found for $u(x)$ in $u = e^y$ gives

$$\begin{aligned} y &= \ln(u(x)) \\ &= \ln\left(\frac{\left(c_2 e^{e^x + c_1} + 1\right) e^{-e^x - c_1}}{c_2}\right) \\ &= \ln\left(\frac{\left(c_2 e^{e^x + c_1} + 1\right) e^{-e^x - c_1}}{c_2}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \ln\left(\frac{\left(c_2 e^{e^x + c_1} + 1\right) e^{-e^x - c_1}}{c_2}\right) \tag{1}$$

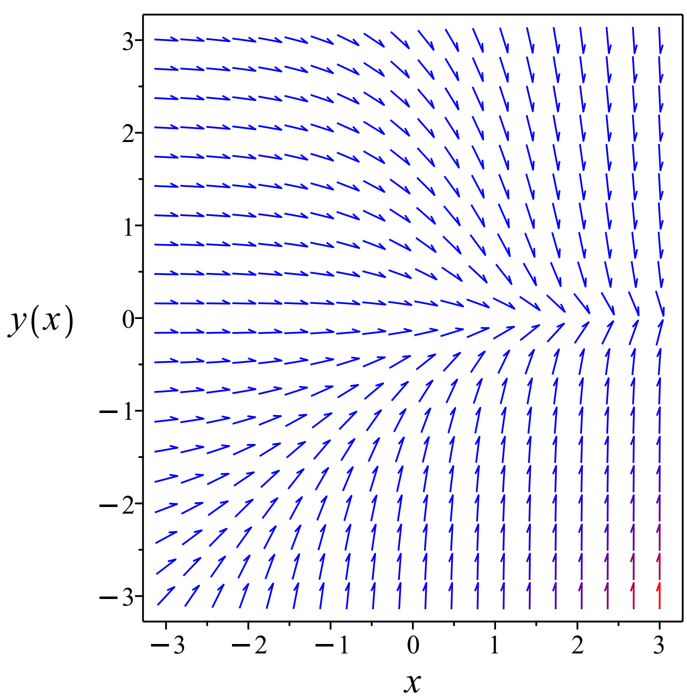


Figure 290: Slope field plot

Verification of solutions

$$y = \ln\left(\frac{\left(c_2 \mathrm{e}^{\mathrm{e}^x + c_1} + 1\right) \mathrm{e}^{-\mathrm{e}^x - c_1}}{c_2}\right)$$

Verified OK.

6.50.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \mathrm{e}^{x-y} - \mathrm{e}^x$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 160: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \mathrm{e}^{-x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{e^{-x}} dx \end{aligned}$$

Which results in

$$S = e^{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = e^{x-y} - e^{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= e^{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{x}}{e^{x-y} - e^{x}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{e^{-R} - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln\left(e^{-R} - 1\right) + \ln\left(e^{-R}\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^x = -\ln\left(e^{-y} - 1\right) + \ln\left(e^{-y}\right) + c_1$$

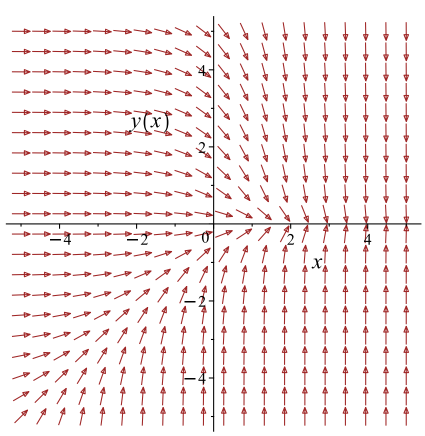
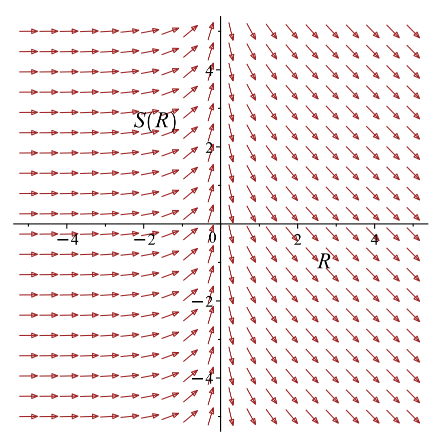
Which simplifies to

$$e^x + \ln\left(e^{-y} - 1\right) + y - c_1 = 0$$

Which gives

$$y = \ln\left(-e^{-e^x + c_1} + 1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = e^{x-y} - e^x$ 	$R = y$ $S = e^x$	$\frac{dS}{dR} = \frac{1}{e^{-R}-1}$ 

Summary
The solution(s) found are the following

$$y = \ln\left(-e^{-e^x + c_1} + 1\right) \tag{1}$$

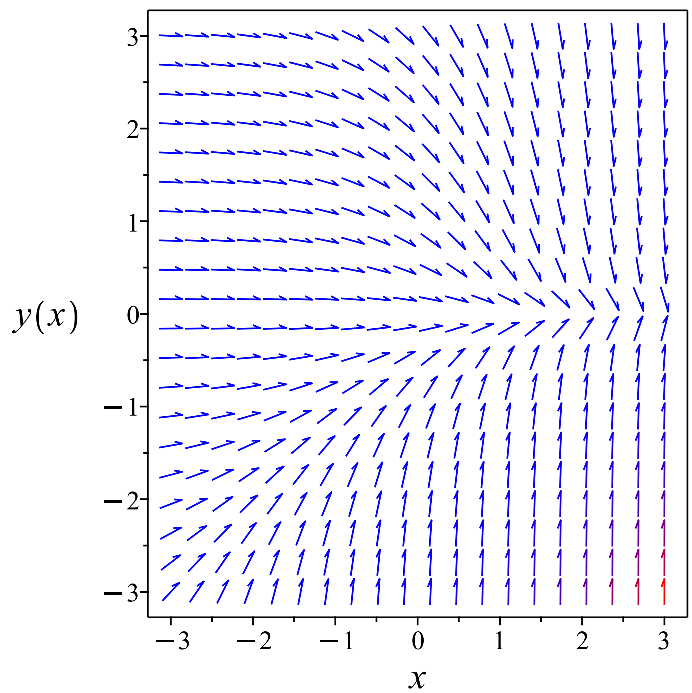


Figure 291: Slope field plot

Verification of solutions

$$y = \ln\left(-\mathrm{e}^{-\mathrm{e}^{x}+c_1} + 1\right)$$

Verified OK.

6.50.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{\mathrm{e}^{-y} - 1}\right) \mathrm{d}y = (\mathrm{e}^{x})\,\mathrm{d}x$$
$$(-\mathrm{e}^{x})\,\mathrm{d}x + \left(\frac{1}{\mathrm{e}^{-y} - 1}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\mathrm{e}^{x}$$
$$N(x, y) = \frac{1}{\mathrm{e}^{-y} - 1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\mathrm{e}^{x}) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{e^{-y}-1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -e^{x}\,\mathrm{d}x\\ \phi &= -e^{x} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{e^{-y}-1}$. Therefore equation (4) becomes

$$\frac{1}{e^{-y}-1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{e^{-y}-1}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{e^{-y}-1}\right)\mathrm{d}y\\ f(y) &= -\ln\left(e^{-y}-1\right) + \ln\left(e^{-y}\right) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -e^{x} - \ln\left(e^{-y} - 1\right) + \ln\left(e^{-y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -e^{x} - \ln\left(e^{-y} - 1\right) + \ln\left(e^{-y}\right)$$

The solution becomes

$$y = -e^{x} + \ln\left(-1 + e^{e^{x} + c_1}\right) - c_1$$

Summary

The solution(s) found are the following

$$y = -e^{x} + \ln\left(-1 + e^{e^{x} + c_1}\right) - c_1 \tag{1}$$

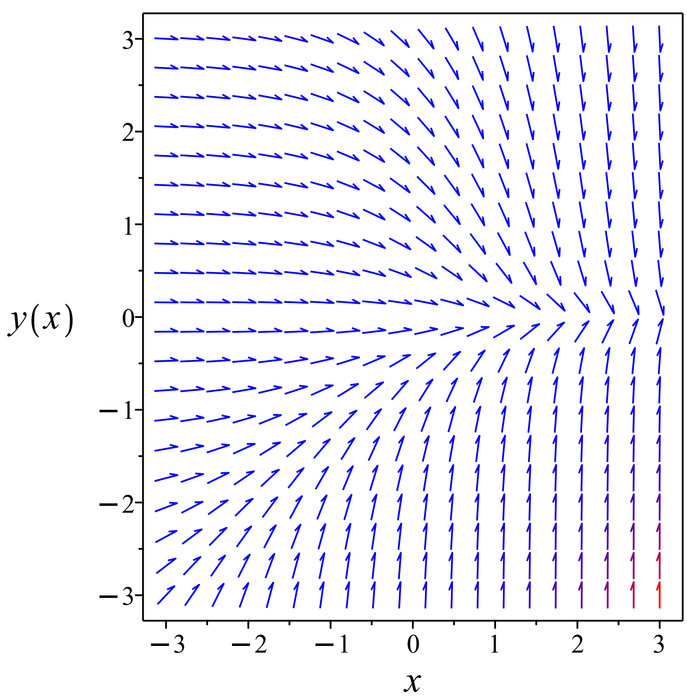


Figure 292: Slope field plot

Verification of solutions

$$y = -e^{x} + \ln\left(-1 + e^{e^{x} + c_1}\right) - c_1$$

Verified OK.

6.50.5 Maple step by step solution

Let's solve

$y' - e^{x-y} = -e^{x}$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' e^{y}}{e^{y}-1} = -e^{x}$

- Integrate both sides with respect to x

  $\int \frac{y' e^{y}}{e^{y}-1} dx = \int -e^{x} dx + c_1$

- Evaluate integral

  $\ln\left(e^{y} - 1\right) = -e^{x} + c_1$

- Solve for y

  $y = \ln\left(e^{-e^{x}+c_1} + 1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 20

```
dsolve(diff(y(x),x)-exp(x-y(x))+exp(x)=0,y(x), singsol=all)
```

$$y(x) = -e^{x} + \ln\left(-1 + e^{e^{x}+c_1}\right) - c_1$$

✓ Solution by Mathematica

Time used: 2.135 (sec). Leaf size: 23

```
DSolve[y'[x]-Exp[x-y[x]]+Exp[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log \left(1 + e^{-e^x + c_1}\right)$$
$$y(x) \to 0$$

7 Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients

7.1 problem Exercise 20.1, page 220

Internal problem ID [4572]
Internal file name [`OUTPUT/4065_Sunday_June_05_2022_12_17_53_PM_88199458/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.1, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' = 0}$$

7.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(0)}\\ &= -1 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 1\\ \lambda_2 &= -1 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(0)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 + c_2 \mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 \mathrm{e}^{-2x} \tag{1}$$

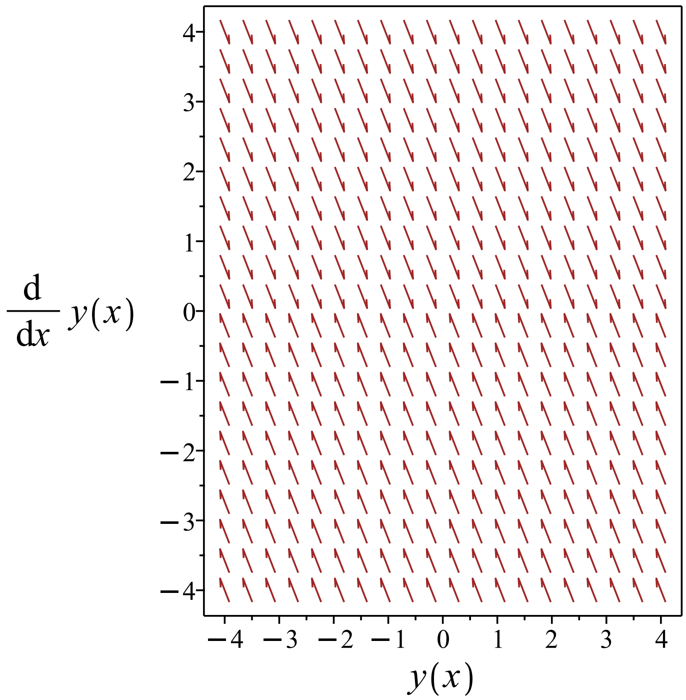


Figure 293: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-2x}$$

Verified OK.

7.1.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 2y')\,dx = 0$$
$$y' + 2y = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-2y + c_1} dy = \int dx$$
$$-\frac{\ln(-2y + c_1)}{2} = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{-2y + c_1}} = e^{x + c_2}$$

Which simplifies to

$$\frac{1}{\sqrt{-2y+c_1}} = c_3 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-2x}}{2c_3^2} + \frac{c_1}{2} \tag{1}$$

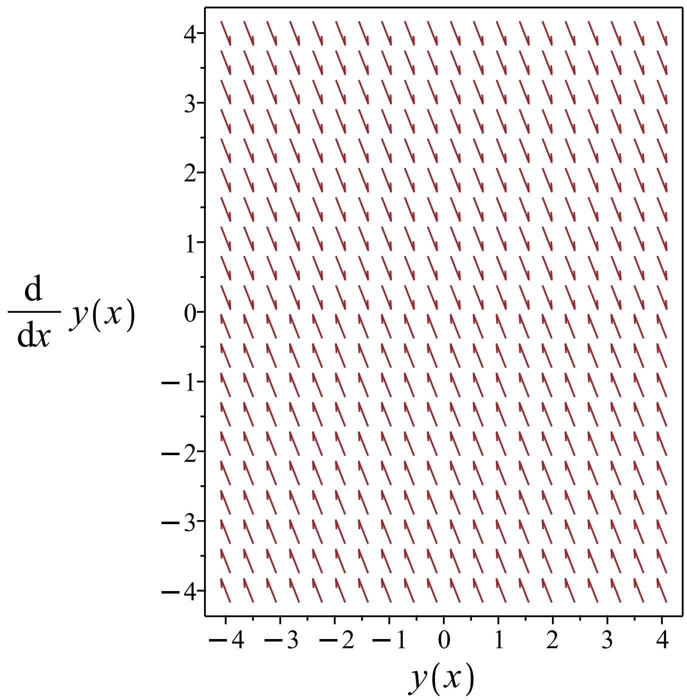


Figure 294: Slope field plot

Verification of solutions

$$y = -\frac{\mathrm{e}^{-2x}}{2c_3^2} + \frac{c_1}{2}$$

Verified OK.

7.1.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 2p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int -\frac{1}{2p} dp = \int dx$$
$$-\frac{\ln(p)}{2} = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{p}} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{p}} = c_2 e^x$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{e^{-2x}}{c_2^2}$$

Integrating both sides gives

$$y = \int \frac{e^{-2x}}{c_2^2} dx$$
$$= -\frac{e^{-2x}}{2c_2^2} + c_3$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{-2x}}{2c_2^2} + c_3 \tag{1}$$

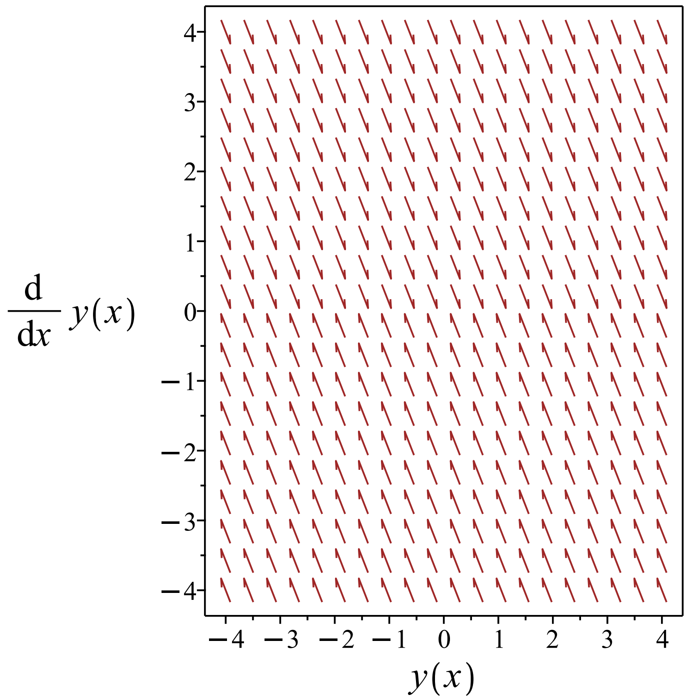


Figure 295: Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{e^{-2x}}{2c_2^2} + c_3$$

Verified OK.

7.1.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 2y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 2y')\, dx = 0$$
$$y' + 2y = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-2y+c_1}dy = \int dx$$
$$-\frac{\ln(-2y+c_1)}{2} = x+c_2$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{-2y+c_1}} = \mathrm{e}^{x+c_2}$$

Which simplifies to

$$\frac{1}{\sqrt{-2y+c_1}} = c_3\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-2x}}{2c_3^2} + \frac{c_1}{2} \tag{1}$$

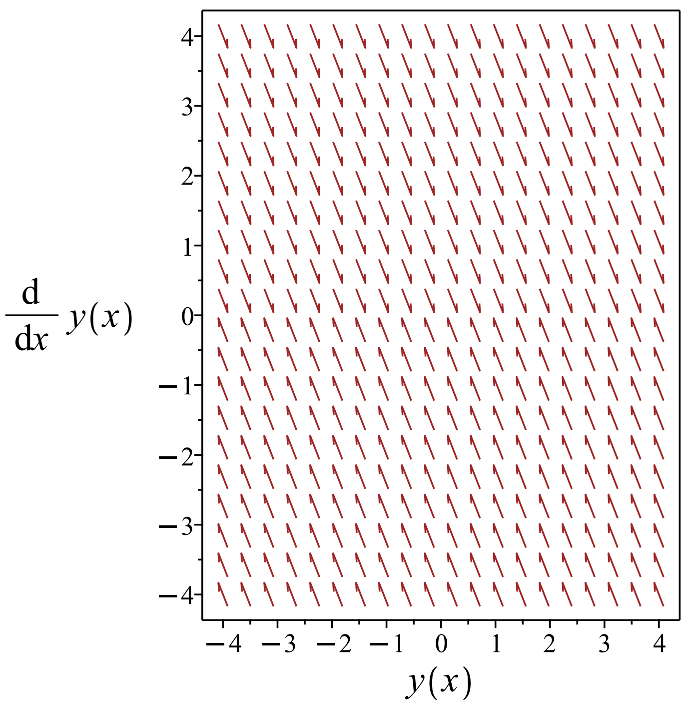


Figure 296: Slope field plot

Verification of solutions

$$y = -\frac{e^{-2x}}{2c_3^2} + \frac{c_1}{2}$$

Verified OK.

7.1.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 163: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-2x} + \frac{c_2}{2} \tag{1}
$$

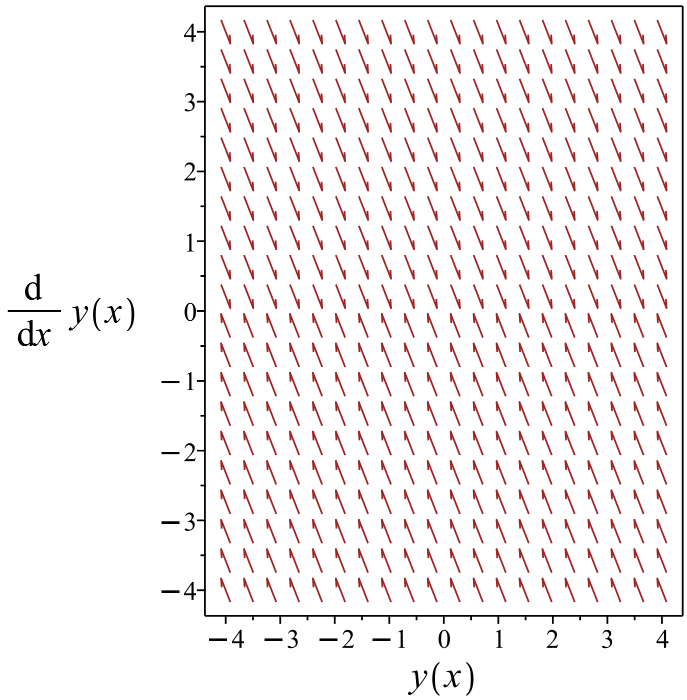


Figure 297: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2}{2}$$

Verified OK.

7.1.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x) y'' + q(x) y' + r(x) y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x) &= 1\\ q(x) &= 2\\ r(x) &= 0\\ s(x) &= 0\end{aligned}$$

Hence

$$p''(x) = 0$$
$$q'(x) = 0$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y' + 2y = c_1$$

We now have a first order ode to solve which is

$$y' + 2y = c_1$$

Integrating both sides gives

$$\int \frac{1}{-2y + c_1} dy = \int dx$$
$$-\frac{\ln(-2y + c_1)}{2} = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{-2y + c_1}} = e^{x + c_2}$$

Which simplifies to

$$\frac{1}{\sqrt{-2y + c_1}} = c_3 e^{x}$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{-2x}}{2c_3^2} + \frac{c_1}{2} \tag{1}$$

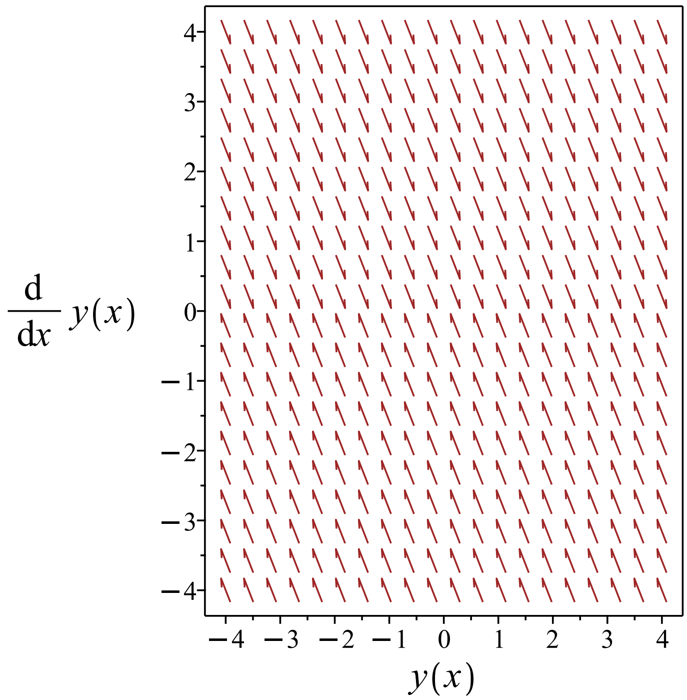


Figure 298: Slope field plot

Verification of solutions

$$y = -\frac{e^{-2x}}{2c_3^2} + \frac{c_1}{2}$$

Verified OK.

7.1.7 Maple step by step solution

Let's solve

$y'' + 2y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r = 0$

- Factor the characteristic polynomial

  $r(r + 2) = 0$

- Roots of the characteristic polynomial

$r = (-2, 0)$

- 1st solution of the ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-2x} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 19

```
DSolve[y''[x]+2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{1}{2} c_1 e^{-2x}$$

7.2 problem Exercise 20.2, page 220

Internal problem ID [4573]
Internal file name [`OUTPUT/4066_Sunday_June_05_2022_12_18_00_PM_27713857/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.2, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 3y' + 2y = 0}$$

7.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\lambda_{1,2} = \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-3^2 - (4)(1)(2)}$$
$$= \frac{3}{2} \pm \frac{1}{2}$$

Hence

$$\lambda_1 = \frac{3}{2} + \frac{1}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(1)x}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x} \tag{1}$$

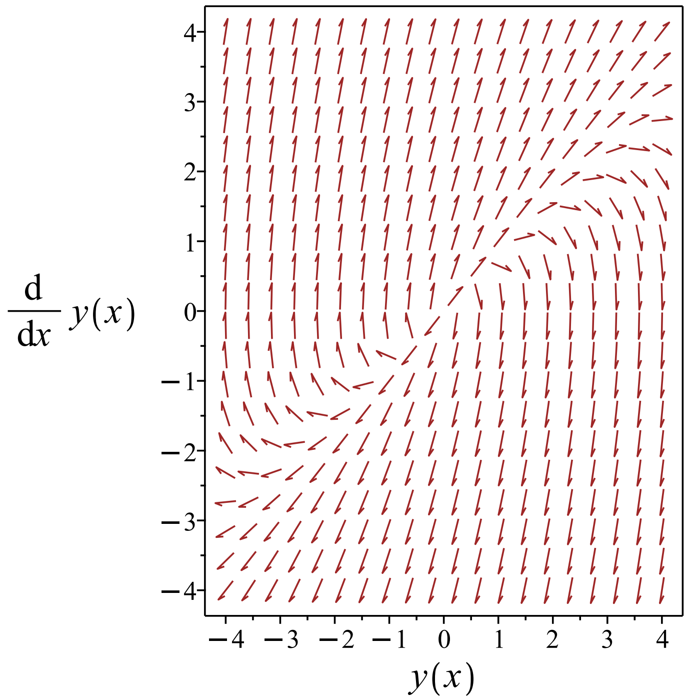


Figure 299: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{x}$$

Verified OK.

7.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 165: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3}{1} \, dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2} \, dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\mathrm{e}^x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{2x} \tag{1}
$$

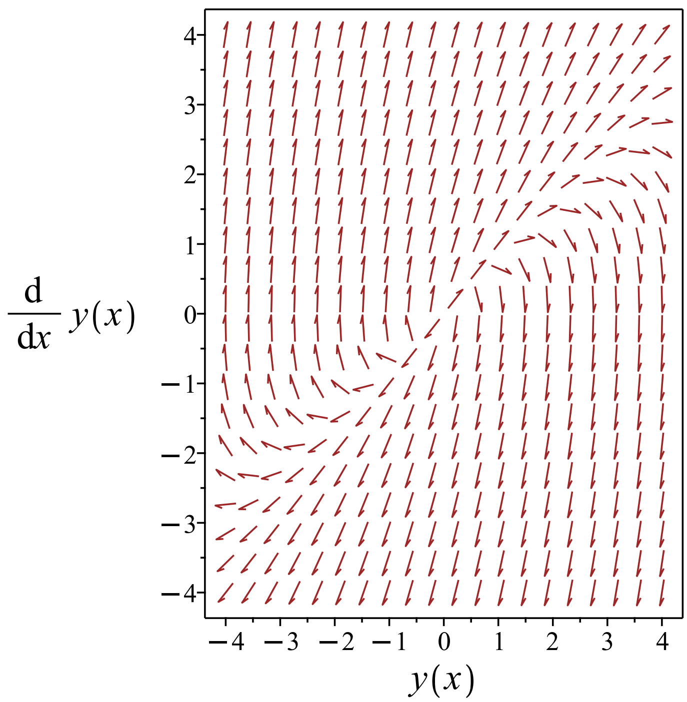


Figure 300: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{2x}
$$

Verified OK.

7.2.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the ODE

  $y_1(x) = e^x$

- 2nd solution of the ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^x + c_2 e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{2x}c_1 + c_2e^x$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 18

```
DSolve[y''[x]-3*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(c_2e^x + c_1)$$

7.3 problem Exercise 20.3, page 220

Internal problem ID [4574]
Internal file name [`OUTPUT/4067_Sunday_June_05_2022_12_18_07_PM_31973577/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.3, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

7.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\lambda_{1,2} = \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}$$
$$= \pm 1$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^{x} + c_2 e^{-x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x} + c_2 e^{-x} \tag{1}$$

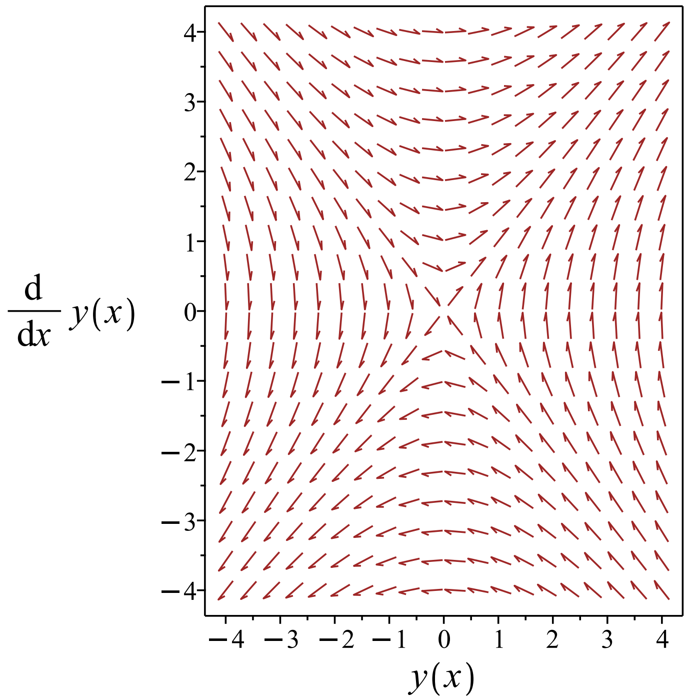


Figure 301: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{-x}$$

Verified OK.

7.3.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - y'y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - y'y)\, dx = 0$$

$$\frac{y'^2}{2} - \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2 + 2c_1} \tag{1}$$

$$y' = -\sqrt{y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{y^2+2c_1}}dy = \int dx$$
$$\ln\left(y+\sqrt{y^2+2c_1}\right) = x + c_2$$

Raising both side to exponential gives

$$y+\sqrt{y^2+2c_1} = \mathrm{e}^{x+c_2}$$

Which simplifies to

$$y+\sqrt{y^2+2c_1} = c_3\mathrm{e}^{x}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{y^2+2c_1}}dy = \int dx$$
$$-\ln\left(y+\sqrt{y^2+2c_1}\right) = x + c_4$$

Raising both side to exponential gives

$$\frac{1}{y+\sqrt{y^2+2c_1}} = \mathrm{e}^{x+c_4}$$

Which simplifies to

$$\frac{1}{y+\sqrt{y^2+2c_1}} = c_5\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\mathrm{e}^{2x}c_3^2-2c_1\right)\mathrm{e}^{-x}}{2c_3} \tag{1}$$
$$y = -\frac{\left(2c_1c_5^2\mathrm{e}^{2x}-1\right)\mathrm{e}^{-x}}{2c_5} \tag{2}$$

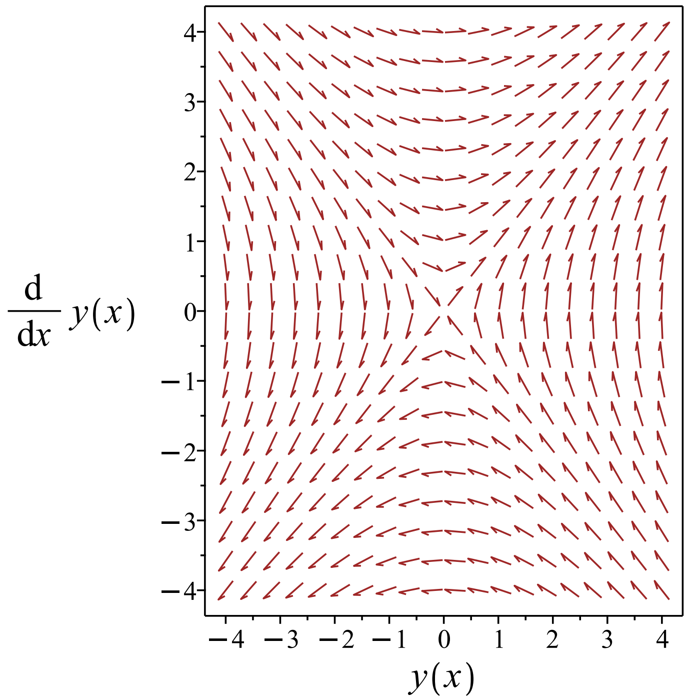


Figure 302: Slope field plot

Verification of solutions

$$y = \frac{(e^{2x}c_3^2 - 2c_1)\,e^{-x}}{2c_3}$$

Verified OK.

$$y = -\frac{(2c_1c_5^2e^{2x} - 1)\,e^{-x}}{2c_5}$$

Verified OK.

7.3.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 167: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-x}
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\,dx \\
&= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} \tag{1}$$

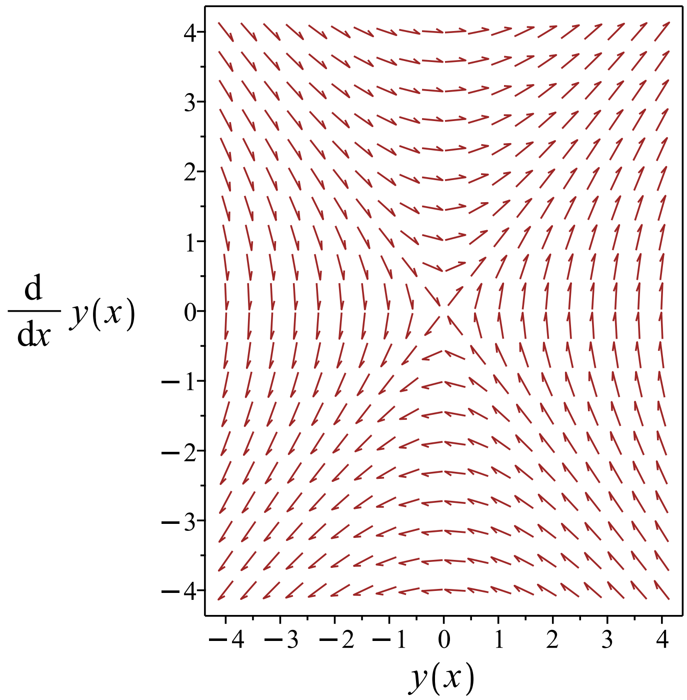


Figure 303: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}$$

Verified OK.

7.3.4 Maple step by step solution

Let's solve

$y'' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the ODE

$$y_2(x) = e^{x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{-x} + c_2 e^{x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{x} + c_2 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 20

```
DSolve[y''[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x} + c_2 e^{-x}$$

7.4 problem Exercise 20.5, page 220

Internal problem ID [4575]
Internal file name [`OUTPUT/4068_Sunday_June_05_2022_12_18_14_PM_11685603/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.5, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{6y'' - 11y' + 4y = 0}$$

7.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 6, B = -11, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$6\lambda^2 e^{\lambda x} - 11\lambda e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$6\lambda^2 - 11\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 6, B = -11, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{11}{(2)(6)} \pm \frac{1}{(2)(6)}\sqrt{-11^2 - (4)(6)(4)} \\ &= \frac{11}{12} \pm \frac{5}{12}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{11}{12} + \frac{5}{12} \\ \lambda_2 &= \frac{11}{12} - \frac{5}{12}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{4}{3} \\ \lambda_2 &= \frac{1}{2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(\frac{4}{3}\right)x} + c_2 e^{\left(\frac{1}{2}\right)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\frac{4x}{3}} + c_2 \mathrm{e}^{\frac{x}{2}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{4x}{3}} + c_2 \mathrm{e}^{\frac{x}{2}} \tag{1}$$

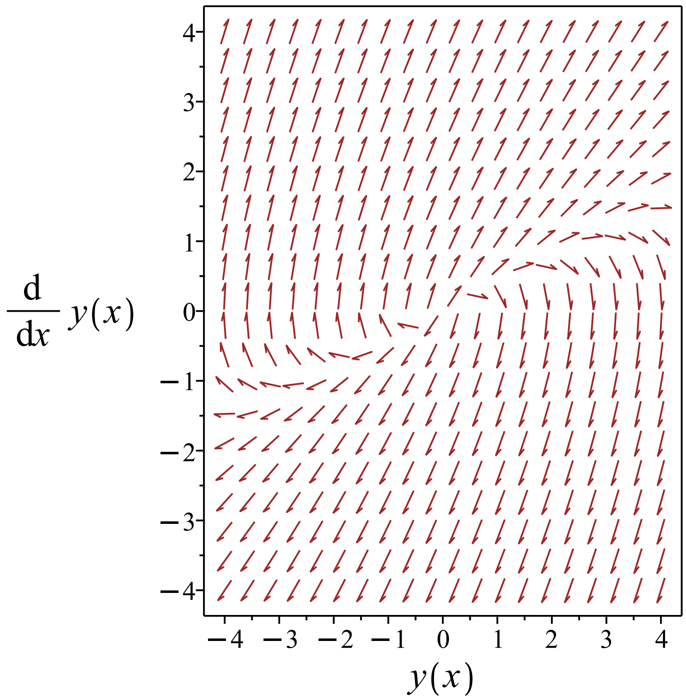


Figure 304: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{4x}{3}} + c_2 e^{\frac{x}{2}}$$

Verified OK.

7.4.2 Solving using Kovacic algorithm

Writing the ode as

$$6y'' - 11y' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6 \\ B &= -11 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{144} \tag{6}$$

Comparing the above to (5) shows that

$$s = 25$$
$$t = 144$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{144} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 169: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{144}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{12}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-11}{6}\,dx} \\ &= z_1 e^{\frac{11x}{12}} \\ &= z_1 \left(\mathrm{e}^{\frac{11x}{12}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-11}{6}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{11x}{6}}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{6\,\mathrm{e}^{\frac{5x}{6}}}{5}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{\frac{x}{2}}\left(\frac{6\,\mathrm{e}^{\frac{5x}{6}}}{5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\frac{x}{2}} + \frac{6 c_2 \mathrm{e}^{\frac{4x}{3}}}{5} \tag{1}
$$

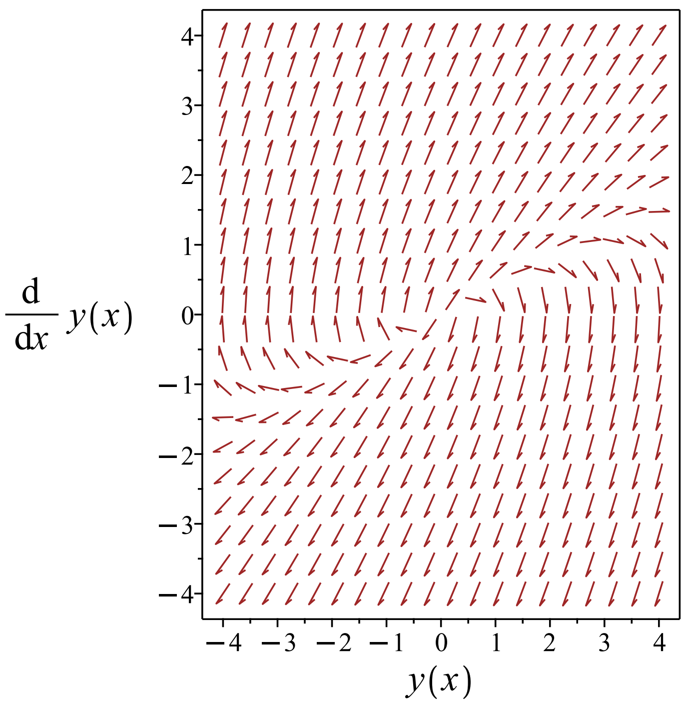


Figure 305: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{\frac{x}{2}} + \frac{6 c_2 \mathrm{e}^{\frac{4x}{3}}}{5}
$$

Verified OK.

7.4.3 Maple step by step solution

Let's solve

$6y'' - 11y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{11y'}{6} - \frac{2y}{3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{11y'}{6} + \frac{2y}{3} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{11}{6}r + \frac{2}{3} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)(3r-4)}{6} = 0$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2}, \frac{4}{3}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{\frac{x}{2}}$

- 2nd solution of the ODE

  $y_2(x) = e^{\frac{4x}{3}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{4x}{3}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(6*diff(y(x),x$2)-11*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{4x}{3}} + c_2 e^{\frac{x}{2}}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 35

```
DSolve[y''[x]-11*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{1}{2}\left(\sqrt{105}-11\right)x}\left(c_2 e^{\sqrt{105}x} + c_1\right)$$

7.5 problem Exercise 20.6, page 220

Internal problem ID [4576]
Internal file name [`OUTPUT/4069_Sunday_June_05_2022_12_18_21_PM_76263437/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.6, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' - y = 0}$$

7.5.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = -1$ into the above gives

$$\lambda_{1,2} = \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(-1)}$$
$$= -1 \pm \sqrt{2}$$

Hence

$$\lambda_1 = -1 + \sqrt{2}$$
$$\lambda_2 = -1 - \sqrt{2}$$

Which simplifies to

$$\lambda_1 = \sqrt{2} - 1$$
$$\lambda_2 = -1 - \sqrt{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\sqrt{2}-1\right)x} + c_2 e^{\left(-1-\sqrt{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\left(\sqrt{2}-1\right)x} + c_2 \mathrm{e}^{\left(-1-\sqrt{2}\right)x}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\left(\sqrt{2}-1\right)x} + c_2 \mathrm{e}^{\left(-1-\sqrt{2}\right)x} \tag{1}$$

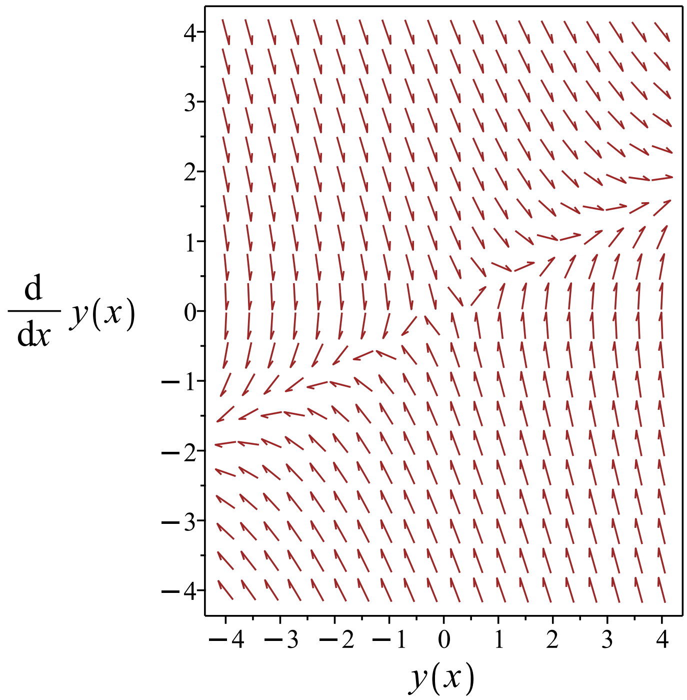


Figure 306: Slope field plot

Verification of solutions

$$y = c_1 e^{\left(\sqrt{2}-1\right)x} + c_2 e^{\left(-1-\sqrt{2}\right)x}$$

Verified OK.

7.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 171: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x\sqrt{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1 \left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\left(1+\sqrt{2}\right)x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\sqrt{2}\,\mathrm{e}^{2x\sqrt{2}}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\left(1+\sqrt{2}\right)x}\right) + c_2 \left(\mathrm{e}^{-\left(1+\sqrt{2}\right)x} \left(\frac{\sqrt{2}\,\mathrm{e}^{2x\sqrt{2}}}{4}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\left(1+\sqrt{2}\right)x} + \frac{c_2 \sqrt{2}\,\mathrm{e}^{\left(\sqrt{2}-1\right)x}}{4} \tag{1}$$

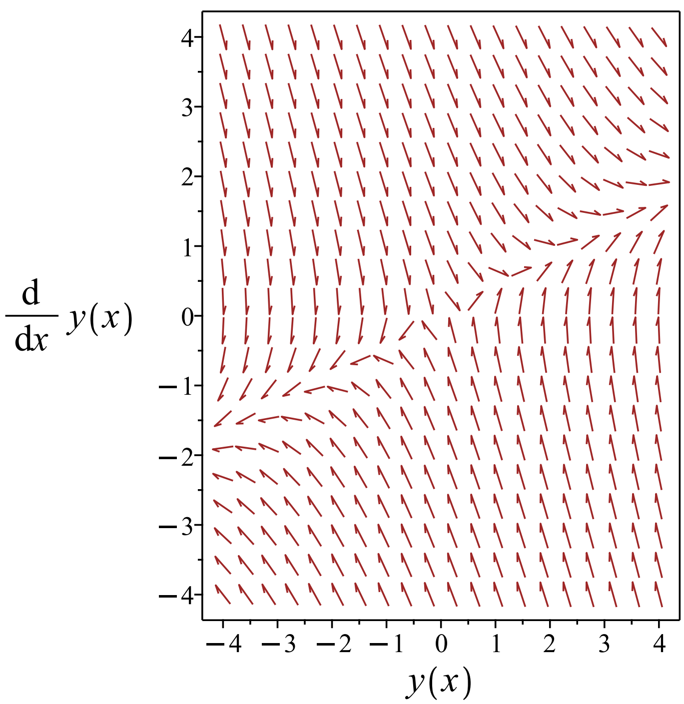


Figure 307: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-\left(1+\sqrt{2}\right)x} + \frac{c_2 \sqrt{2}\,\mathrm{e}^{\left(\sqrt{2}-1\right)x}}{4}$$

Verified OK.

7.5.3 Maple step by step solution

Let's solve

$y'' + 2y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r - 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm\left(\sqrt{8}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-1 - \sqrt{2}, \sqrt{2} - 1\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{\left(-1-\sqrt{2}\right)x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\left(\sqrt{2}-1\right)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{\left(-1-\sqrt{2}\right)x} + c_2 \mathrm{e}^{\left(\sqrt{2}-1\right)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\left(\sqrt{2}-1\right)x} + c_2 e^{-\left(1+\sqrt{2}\right)x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 34

```
DSolve[y''[x]+2*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\left(\left(1+\sqrt{2}\right)x\right)}\left(c_2 e^{2\sqrt{2}x} + c_1\right)$$

7.6 problem Exercise 20.7, page 220

Internal problem ID [4577]
Internal file name [`OUTPUT/4070_Sunday_June_05_2022_12_18_28_PM_73748754/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.7, page 220.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y'' - 10y' - 6y = 0}$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - 10\lambda - 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 3$$
$$\lambda_2 = -2 - \sqrt{2}$$
$$\lambda_3 = -2 + \sqrt{2}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{3x}c_1 + e^{\left(-2-\sqrt{2}\right)x}c_2 + e^{\left(-2+\sqrt{2}\right)x}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{3x}$$
$$y_2 = e^{\left(-2-\sqrt{2}\right)x}$$
$$y_3 = e^{\left(-2+\sqrt{2}\right)x}$$

Summary
The solution(s) found are the following

$$y = e^{3x}c_1 + e^{(-2-\sqrt{2})x}c_2 + e^{(-2+\sqrt{2})x}c_3 \tag{1}$$

Verification of solutions

$$y = e^{3x}c_1 + e^{(-2-\sqrt{2})x}c_2 + e^{(-2+\sqrt{2})x}c_3$$

Verified OK.

7.6.1 Maple step by step solution

Let's solve

$y''' + y'' - 10y' - 6y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -y_3(x) + 10y_2(x) + 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -y_3(x) + 10y_2(x) + 6y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 10 & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 10 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-2-\sqrt{2}, \begin{bmatrix} \frac{1}{\left(-2-\sqrt{2}\right)^2} \\ \frac{1}{-2-\sqrt{2}} \\ 1 \end{bmatrix}\right], \left[-2+\sqrt{2}, \begin{bmatrix} \frac{1}{\left(-2+\sqrt{2}\right)^2} \\ \frac{1}{-2+\sqrt{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-2-\sqrt{2}, \begin{bmatrix} \frac{1}{\left(-2-\sqrt{2}\right)^2} \\ \frac{1}{-2-\sqrt{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{\left(-2-\sqrt{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-2-\sqrt{2}\right)^2} \\ \frac{1}{-2-\sqrt{2}} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-2+\sqrt{2}, \left[\begin{array}{c} \frac{1}{\left(-2+\sqrt{2}\right)^2} \\ \frac{1}{-2+\sqrt{2}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{\left(-2+\sqrt{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-2+\sqrt{2}\right)^2} \\ \frac{1}{-2+\sqrt{2}} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = e^{3x}c_1 \cdot \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right] + e^{\left(-2-\sqrt{2}\right)x}c_2 \cdot \left[\begin{array}{c} \frac{1}{\left(-2-\sqrt{2}\right)^2} \\ \frac{1}{-2-\sqrt{2}} \\ 1 \end{array}\right] + e^{\left(-2+\sqrt{2}\right)x}c_3 \cdot \left[\begin{array}{c} \frac{1}{\left(-2+\sqrt{2}\right)^2} \\ \frac{1}{-2+\sqrt{2}} \\ 1 \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{c_2\left(3-2\sqrt{2}\right)e^{-\left(2+\sqrt{2}\right)x}}{2} + \frac{c_3\left(2\sqrt{2}+3\right)e^{\left(-2+\sqrt{2}\right)x}}{2} + \frac{e^{3x}c_1}{9}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)-10*diff(y(x),x)-6*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{\left(-2+\sqrt{2}\right)x} + c_3 \mathrm{e}^{-\left(2+\sqrt{2}\right)x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 43

```
DSolve[y'''[x]+y''[x]-10*y'[x]-6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\left(\left(2+\sqrt{2}\right)x\right)} + c_2 e^{\left(\sqrt{2}-2\right)x} + c_3 e^{3x}$$

7.7 problem Exercise 20.8, page 220

Internal problem ID [4578]
Internal file name [`OUTPUT/4071_Sunday_June_05_2022_12_18_35_PM_29085956/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.8, page 220.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - y''' - 4y'' + 4y' = 0}$$

The characteristic equation is

$$\lambda^4 - \lambda^3 - 4\lambda^2 + 4\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = 2$$
$$\lambda_4 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 \mathrm{e}^{-2x} + c_3 \mathrm{e}^{x} + \mathrm{e}^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = \mathrm{e}^{-2x}$$
$$y_3 = \mathrm{e}^{x}$$
$$y_4 = \mathrm{e}^{2x}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-2x} + c_3 e^{x} + e^{2x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^{-2x} + c_3 e^{x} + e^{2x} c_4$$

Verified OK.

7.7.1 Maple step by step solution

Let's solve

$y'''' - y''' - 4y'' + 4y' = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = y_4(x) + 4y_3(x) - 4y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = y_4(x) + 4y_3(x) - 4y_2(x)]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4 & 4 & 1 \end{array}\right] \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4 & 4 & 1 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right], \left[2, \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + c_4\vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + e^{2x} c_4 \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{(-e^{4x}c_4 - 8c_3 e^{3x} - 8c_2 e^{2x} + c_1) e^{-2x}}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$4)-diff(y(x),x$3)-4*diff(y(x),x$2)+4*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2 \mathrm{e}^{4x} + c_3 \mathrm{e}^{3x} + \mathrm{e}^{2x} c_1 + c_4\right) \mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 36

```
DSolve[y''''[x]-y'''[x]-4*y''[x]+4*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2} c_1 e^{-2x} + c_2 e^{x} + \frac{1}{2} c_3 e^{2x} + c_4$$

7.8 problem Exercise 20.9, page 220

Internal problem ID [4579]
Internal file name [`OUTPUT/4072_Sunday_June_05_2022_12_18_43_PM_61299716/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.9, page 220.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 4y''' + y'' - 4y' - 2y = 0}$$

The characteristic equation is

$$\lambda^4 + 4\lambda^3 + \lambda^2 - 4\lambda - 2 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -1 \\
\lambda_3 &= -2 - \sqrt{2} \\
\lambda_4 &= -2 + \sqrt{2}
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + e^{\left(-2-\sqrt{2}\right)x} c_3 + e^{\left(-2+\sqrt{2}\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{-x} \\
y_2 &= e^{x} \\
y_3 &= e^{\left(-2-\sqrt{2}\right)x} \\
y_4 &= e^{\left(-2+\sqrt{2}\right)x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-x} + c_2 e^{x} + e^{\left(-2-\sqrt{2}\right)x} c_3 + e^{\left(-2+\sqrt{2}\right)x} c_4 \tag{1}
$$

Verification of solutions

$$
y = c_1 e^{-x} + c_2 e^{x} + e^{\left(-2-\sqrt{2}\right)x} c_3 + e^{\left(-2+\sqrt{2}\right)x} c_4
$$

Verified OK.

7.8.1 Maple step by step solution

Let's solve

$y'''' + 4y''' + y'' - 4y' - 2y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -4y_4(x) - y_3(x) + 4y_2(x) + 2y_1(x)$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -4y_4(x) - y_3(x) + 4y_2(x) + 2y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & -1 & -4 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & -1 & -4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-2-\sqrt{2}, \begin{bmatrix} \frac{1}{\left(-2-\sqrt{2}\right)^3} \\ \frac{1}{\left(-2-\sqrt{2}\right)^2} \\ \frac{1}{-2-\sqrt{2}} \\ 1 \end{bmatrix}\right], \left[-2+\sqrt{2}, \begin{bmatrix} \frac{1}{\left(-2+\sqrt{2}\right)^3} \\ \frac{1}{\left(-2+\sqrt{2}\right)^2} \\ \frac{1}{-2+\sqrt{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-2-\sqrt{2}, \left[\begin{array}{c} \frac{1}{\left(-2-\sqrt{2}\right)^3} \\ \frac{1}{\left(-2-\sqrt{2}\right)^2} \\ \frac{1}{-2-\sqrt{2}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{\left(-2-\sqrt{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-2-\sqrt{2}\right)^{3}} \\ \frac{1}{\left(-2-\sqrt{2}\right)^{2}} \\ \frac{1}{-2-\sqrt{2}} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-2+\sqrt{2}, \left[\begin{array}{c} \frac{1}{\left(-2+\sqrt{2}\right)^{3}} \\ \frac{1}{\left(-2+\sqrt{2}\right)^{2}} \\ \frac{1}{-2+\sqrt{2}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = \mathrm{e}^{\left(-2+\sqrt{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-2+\sqrt{2}\right)^{3}} \\ \frac{1}{\left(-2+\sqrt{2}\right)^{2}} \\ \frac{1}{-2+\sqrt{2}} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + c_4\vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1\mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right] + c_2\mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right] + \mathrm{e}^{\left(-2-\sqrt{2}\right)x} c_3 \cdot \left[\begin{array}{c} \frac{1}{\left(-2-\sqrt{2}\right)^{3}} \\ \frac{1}{\left(-2-\sqrt{2}\right)^{2}} \\ \frac{1}{-2-\sqrt{2}} \\ 1 \end{array}\right] + \mathrm{e}^{\left(-2+\sqrt{2}\right)x} c_4 \cdot \left[\begin{array}{c} \frac{1}{\left(-2+\sqrt{2}\right)^{3}} \\ \frac{1}{\left(-2+\sqrt{2}\right)^{2}} \\ \frac{1}{-2+\sqrt{2}} \\ 1 \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{c_3\left(7\sqrt{2}-10\right)\mathrm{e}^{-\left(2+\sqrt{2}\right)x}}{4} + \frac{c_4\left(-7\sqrt{2}-10\right)\mathrm{e}^{\left(-2+\sqrt{2}\right)x}}{4} - c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{x}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$4)+4*diff(y(x),x$3)+diff(y(x),x$2)-4*diff(y(x),x)-2*y(x)=0,y(x), singsol=
```

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{\left(-2+\sqrt{2}\right)x} + c_4 e^{-\left(2+\sqrt{2}\right)x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 49

```
DSolve[y''''[x]+4*y'''[x]+y''[x]-4*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\left(\left(2+\sqrt{2}\right)x\right)} + c_2 e^{\left(\sqrt{2}-2\right)x} + c_3 e^{-x} + c_4 e^x$$

7.9 problem Exercise 20.10, page 220

Internal problem ID [4580]
Internal file name [`OUTPUT/4073_Sunday_June_05_2022_12_18_51_PM_53894104/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.10, page 220.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - ya^2 = 0}$$

The characteristic equation is

$$\lambda^4 - a^2 = 0$$

The roots of the above equation are

$$\lambda_1 = \sqrt{a}$$
$$\lambda_2 = -\sqrt{a}$$
$$\lambda_3 = \sqrt{-a}$$
$$\lambda_4 = -\sqrt{-a}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{-\sqrt{-a}\,x}c_1 + \mathrm{e}^{\sqrt{a}\,x}c_2 + \mathrm{e}^{\sqrt{-a}\,x}c_3 + \mathrm{e}^{-\sqrt{a}\,x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-\sqrt{-a}\,x}$$
$$y_2 = \mathrm{e}^{\sqrt{a}\,x}$$
$$y_3 = \mathrm{e}^{\sqrt{-a}\,x}$$
$$y_4 = \mathrm{e}^{-\sqrt{a}\,x}$$

Summary

The solution(s) found are the following

$$y = e^{-\sqrt{-a}\,x}c_1 + e^{\sqrt{a}\,x}c_2 + e^{\sqrt{-a}\,x}c_3 + e^{-\sqrt{a}\,x}c_4 \tag{1}$$

Verification of solutions

$$y = e^{-\sqrt{-a}\,x}c_1 + e^{\sqrt{a}\,x}c_2 + e^{\sqrt{-a}\,x}c_3 + e^{-\sqrt{a}\,x}c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$4)-a^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\sqrt{a}\,x} + c_2 e^{-\sqrt{a}\,x} + c_3 \sin\left(\sqrt{a}\,x\right) + c_4 \cos\left(\sqrt{a}\,x\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 53

```
DSolve[y''''[x]-a^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 e^{-\sqrt{a}x} + c_4 e^{\sqrt{a}x} + c_1 \cos\left(\sqrt{a}x\right) + c_3 \sin\left(\sqrt{a}x\right)$$

7.10 problem Exercise 20.11, page 220

Internal problem ID [4581]
Internal file name [`OUTPUT/4074_Sunday_June_05_2022_12_18_58_PM_93956521/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.11, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2ky' - 2y = 0}$$

7.10.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2k, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2k\lambda\, e^{\lambda x} - 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$-2k\lambda + \lambda^2 - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2k, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2k}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2k^2 - (4)(1)(-2)} \\ &= k \pm \sqrt{k^2+2}\end{aligned}$$

Hence

$$\lambda_1 = k + \sqrt{k^2+2}$$
$$\lambda_2 = k - \sqrt{k^2+2}$$

Which simplifies to

$$\lambda_1 = k + \sqrt{k^2+2}$$
$$\lambda_2 = k - \sqrt{k^2+2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(k+\sqrt{k^2+2}\right)x} + c_2 e^{\left(k-\sqrt{k^2+2}\right)x}$$

Or

$$y = c_1 e^{\left(k+\sqrt{k^2+2}\right)x} + c_2 e^{\left(k-\sqrt{k^2+2}\right)x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\left(k+\sqrt{k^2+2}\right)x} + c_2 e^{\left(k-\sqrt{k^2+2}\right)x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{\left(k+\sqrt{k^2+2}\right)x} + c_2 e^{\left(k-\sqrt{k^2+2}\right)x}$$

Verified OK.

7.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2ky' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2k \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{k^2 + 2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= k^2 + 2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(k^2 + 2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 176: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = k^2 + 2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{x\sqrt{k^2+2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2k}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{kx} \\
&= z_1 \left(\mathrm{e}^{kx}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\left(k+\sqrt{k^2+2}\right)x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2k}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2kx}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2x\sqrt{k^2+2}}}{2\sqrt{k^2+2}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\left(k+\sqrt{k^2+2}\right)x}\right) + c_2 \left(\mathrm{e}^{\left(k+\sqrt{k^2+2}\right)x} \left(-\frac{\mathrm{e}^{-2x\sqrt{k^2+2}}}{2\sqrt{k^2+2}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\left(k+\sqrt{k^2+2}\right)x} - \frac{c_2 \mathrm{e}^{\left(k-\sqrt{k^2+2}\right)x}}{2\sqrt{k^2+2}} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{\left(k+\sqrt{k^2+2}\right)x} - \frac{c_2 \mathrm{e}^{\left(k-\sqrt{k^2+2}\right)x}}{2\sqrt{k^2+2}}
$$

Verified OK.

7.10.3 Maple step by step solution

Let's solve

$y'' - 2ky' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $-2kr + r^2 - 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(2k)\pm\left(\sqrt{4k^2+8}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(k - \sqrt{k^2+2}, k + \sqrt{k^2+2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{\left(k-\sqrt{k^2+2}\right)x}$

- 2nd solution of the ODE

  $y_2(x) = e^{\left(k+\sqrt{k^2+2}\right)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\left(k-\sqrt{k^2+2}\right)x} + c_2 e^{\left(k+\sqrt{k^2+2}\right)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)-2*k*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\left(k+\sqrt{k^2+2}\right)x} + c_2 e^{\left(k-\sqrt{k^2+2}\right)x}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 44

```
DSolve[y''[x]-2*k*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\left(k-\sqrt{k^2+2}\right)x} + c_2 e^{\left(\sqrt{k^2+2}+k\right)x}$$

7.11 problem Exercise 20.12, page 220

Internal problem ID [4582]
Internal file name [`OUTPUT/4075_Sunday_June_05_2022_12_19_05_PM_63797971/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.12, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4ky' - 12k^2y = 0}$$

7.11.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4k, C = -12k^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4k\lambda\, e^{\lambda x} - 12k^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$-12k^2 + 4k\lambda + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4k, C = -12k^2$ into the above gives

$$\lambda_{1,2} = \frac{-4k}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4k^2 - (4)(1)(-12k^2)}$$
$$= -2k \pm 4\sqrt{k^2}$$

Hence

$$\lambda_1 = -2k + 4\sqrt{k^2}$$
$$\lambda_2 = -2k - 4\sqrt{k^2}$$

Which simplifies to

$$\lambda_1 = -2k + 4\sqrt{k^2}$$
$$\lambda_2 = -2k - 4\sqrt{k^2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(-2k+4\sqrt{k^2}\right)x} + c_2 e^{\left(-2k-4\sqrt{k^2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\left(-2k+4\sqrt{k^2}\right)x} + c_2 \mathrm{e}^{\left(-2k-4\sqrt{k^2}\right)x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\left(-2k+4\sqrt{k^2}\right)x} + c_2 \mathrm{e}^{\left(-2k-4\sqrt{k^2}\right)x} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\left(-2k+4\sqrt{k^2}\right)x} + c_2 \mathrm{e}^{\left(-2k-4\sqrt{k^2}\right)x}$$

Verified OK.

7.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4ky' - 12k^2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4k \\ C &= -12k^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16k^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 16k^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(16k^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 178: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 16k^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{4x\sqrt{k^2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4k}{1}\,dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-2kx} \\
&= z_1 \left(\mathrm{e}^{-2kx}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2k(2 \,\mathrm{csgn}(k)-1)x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4k}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-4kx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{csgn}\left(k\right) \mathrm{e}^{-8kx \,\mathrm{csgn}(k)}}{8k}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2k(2 \,\mathrm{csgn}(k)-1)x}\right) + c_2 \left(\mathrm{e}^{2k(2 \,\mathrm{csgn}(k)-1)x} \left(-\frac{\mathrm{csgn}\left(k\right) \mathrm{e}^{-8kx \,\mathrm{csgn}(k)}}{8k}\right)\right)
\end{aligned}
$$

Simplifying the solution $y = c_1 \mathrm{e}^{2k(2 \,\mathrm{csgn}(k)-1)x} - \frac{c_2 \,\mathrm{csgn}(k) \mathrm{e}^{-2k(2 \,\mathrm{csgn}(k)+1)x}}{8k}$ to $y = c_1 \mathrm{e}^{2kx} - \frac{c_2 \mathrm{e}^{-6kx}}{8k}$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{2kx} - \frac{c_2 \mathrm{e}^{-6kx}}{8k} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{2kx} - \frac{c_2 \mathrm{e}^{-6kx}}{8k}
$$

Verified OK.

7.11.3 Maple step by step solution

Let's solve

$y'' + 4ky' - 12k^2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $-12k^2 + 4kr + r^2 = 0$

- Factor the characteristic polynomial

  $-(6k + r)(2k - r) = 0$

- Roots of the characteristic polynomial

  $r = (-6k, 2k)$

- 1st solution of the ODE

  $y_1(x) = e^{-6kx}$

- 2nd solution of the ODE

  $y_2(x) = e^{2kx}$

- General solution of the ODE

  $y = c_1y_1(x) + c_2y_2(x)$

- Substitute in solutions

  $y = c_1e^{-6kx} + c_2e^{2kx}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+4*k*diff(y(x),x)-12*k^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 \mathrm{e}^{8kx} + c_2\right) \mathrm{e}^{-6kx}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 24

```
DSolve[y''[x]+4*k*y'[x]-12*k^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-6kx}\left(c_2 e^{8kx} + c_1\right)$$

7.12 problem Exercise 20.13, page 220

Internal problem ID [4583]
Internal file name [OUTPUT/4076_Sunday_June_05_2022_12_19_12_PM_48847418/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.13, page 220.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _quadrature]]
```

$$\boxed{y'''' = 0}$$

The characteristic equation is

$$\lambda^4 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$
$$\lambda_4 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_4x^3 + c_3x^2 + c_2x + c_1$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = x^2$$
$$y_4 = x^3$$

Summary
The solution(s) found are the following

$$y = c_4x^3 + c_3x^2 + c_2x + c_1 \tag{1}$$

Verification of solutions

$$y = c_4x^3 + c_3x^2 + c_2x + c_1$$

Verified OK.

7.12.1 Maple step by step solution

Let's solve

$y'''' = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 0$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 0]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = \begin{bmatrix} c_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$4)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{6}c_1x^3 + \frac{1}{2}c_2x^2 + c_3x + c_4$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 22

```
DSolve[y''''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(x(c_4x + c_3) + c_2) + c_1$$

7.13 problem Exercise 20.14, page 220

Internal problem ID [4584]
Internal file name [`OUTPUT/4077_Sunday_June_05_2022_12_19_20_PM_99080352/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.14, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y' + 4y = 0}$$

7.13.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(4)^2 - (4)(1)(4)}\\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 e^{-2x} + c_2 x\, e^{-2x} \tag{1}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 x\, e^{-2x} \tag{1}$$

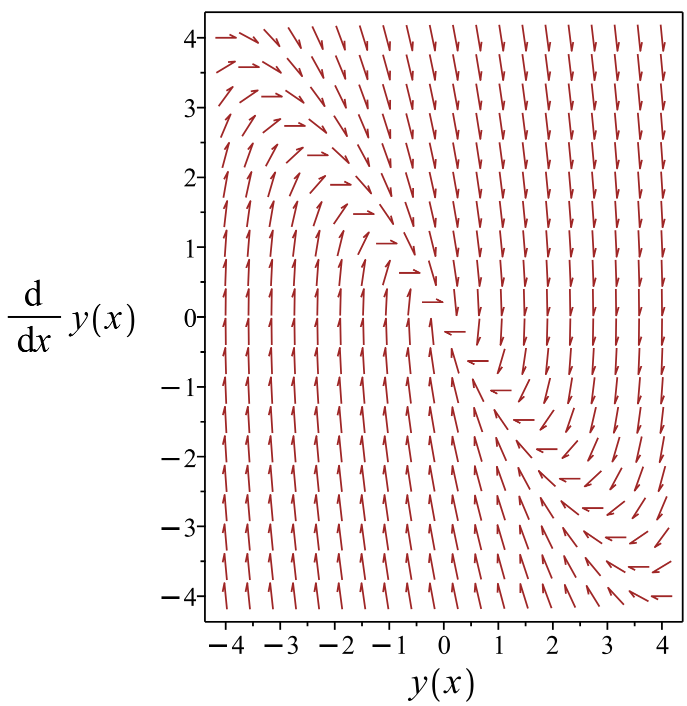


Figure 308: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-2x} + c_2 x\, e^{-2x}$$

Verified OK.

7.13.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 4\, dx} \\ &= \mathrm{e}^{2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\mathrm{e}^{2x} y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{2x} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{2x} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{2x}}$$

Or

$$y = c_1 x\, \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = c_1 x\, \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-2x} \tag{1}$$

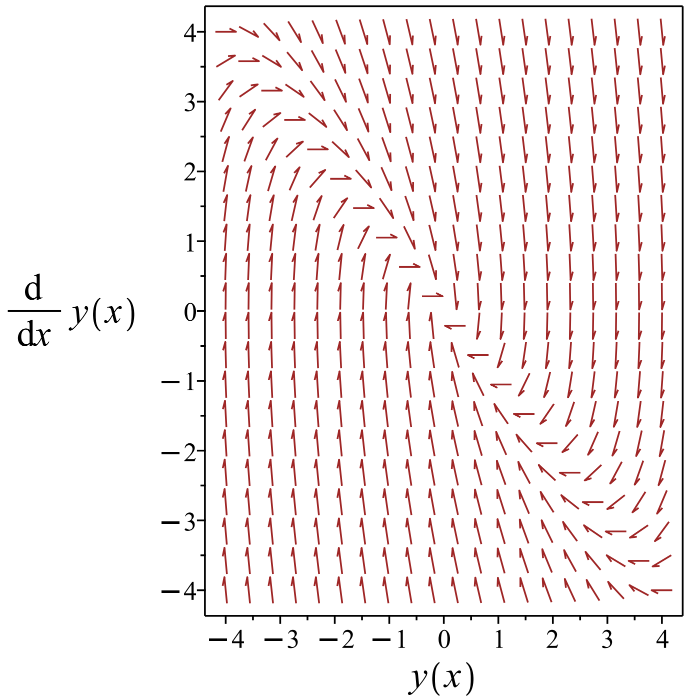


Figure 309: Slope field plot

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{-2x} + c_2\mathrm{e}^{-2x}$$

Verified OK.

7.13.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 181: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-2x}\right) + c_2 \left(e^{-2x}(x)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 e^{-2x} + c_2 x\, e^{-2x} \tag{1}
$$

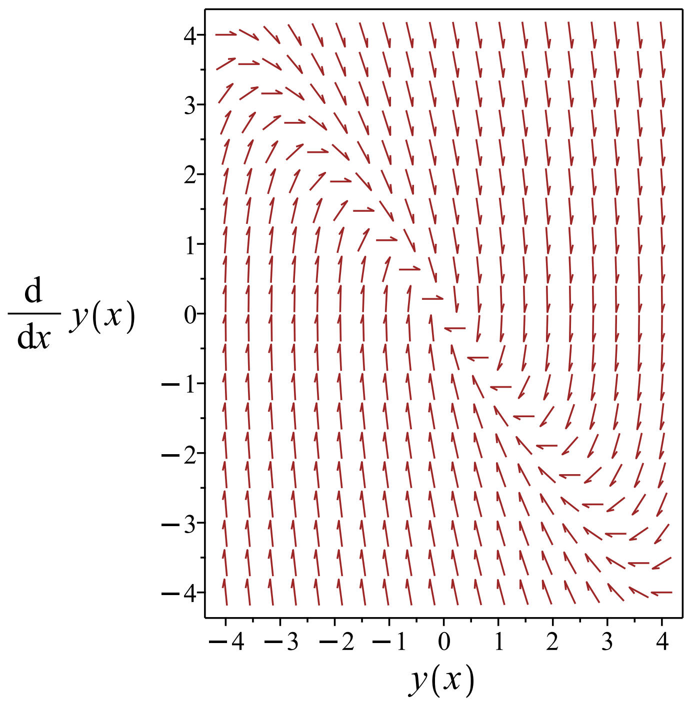


Figure 310: Slope field plot

Verification of solutions

$$
y = c_1 e^{-2x} + c_2 x\, e^{-2x}
$$

Verified OK.

7.13.4 Maple step by step solution

Let's solve

$y'' + 4y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r + 2)^2 = 0$

- Root of the characteristic polynomial

  $r = -2$

- 1st solution of the ODE

  $y_1(x) = e^{-2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-2x} + c_2 x\,e^{-2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-2x}(c_2 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 18

```
DSolve[y''[x]+4*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}(c_2 x + c_1)$$

7.14 problem Exercise 20.15, page 220

Internal problem ID [4585]
Internal file name [`OUTPUT/4078_Sunday_June_05_2022_12_19_27_PM_46574055/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.15, page 220.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{3y''' + 5y'' + y' - y = 0}$$

The characteristic equation is

$$3\lambda^3 + 5\lambda^2 + \lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = \frac{1}{3}$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \mathrm{e}^{\frac{x}{3}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = x\,\mathrm{e}^{-x}$$
$$y_3 = \mathrm{e}^{\frac{x}{3}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \mathrm{e}^{\frac{x}{3}} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \mathrm{e}^{\frac{x}{3}} c_3$$

Verified OK.

7.14.1 Maple step by step solution

Let's solve

$3y''' + 5y'' + y' - y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{5y''}{3} - \frac{y'}{3} + \frac{y}{3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{5y''}{3} + \frac{y'}{3} - \frac{y}{3} = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -\frac{5y_3(x)}{3} - \frac{y_2(x)}{3} + \frac{y_1(x)}{3}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -\frac{5y_3(x)}{3} - \frac{y_2(x)}{3} + \frac{y_1(x)}{3}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{5}{3} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{5}{3} \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[\frac{1}{3}, \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\vec{y}_1(x) = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{5}{3} \end{array}\right] - (-1) \cdot \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right) \cdot \overrightarrow{p} = \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue -1

$$\overrightarrow{y}_2(x) = \mathrm{e}^{-x} \cdot \left(x \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right] + \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right)$$

- Consider eigenpair

$$\left[\frac{1}{3}, \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{\frac{x}{3}} \cdot \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-x} \cdot \left(x \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + e^{\frac{x}{3}} c_3 \cdot \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \left(9 e^{\frac{4x}{3}} c_3 + c_2 x + c_1 + c_2\right) e^{-x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(3*diff(y(x),x$3)+5*diff(y(x),x$2)+diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 e^{\frac{4x}{3}} + c_3 x + c_2\right) e^{-x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 28

```
DSolve[3*y'''[x]+5*y''[x]+y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_1 e^{4x/3} + c_3 x + c_2\right)$$

7.15 problem Exercise 20.16, page 220

Internal problem ID [4586]
Internal file name [`OUTPUT/4079_Sunday_June_05_2022_12_19_34_PM_87531353/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.16, page 220.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 6y'' + 12y' - 8y = 0}$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{2x} + x\, e^{2x} c_2 + x^2 e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{2x}$$
$$y_2 = e^{2x} x$$
$$y_3 = x^2 e^{2x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{2x} + x\,e^{2x} c_2 + x^2 e^{2x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{2x} + x\,e^{2x} c_2 + x^2 e^{2x} c_3$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+12*diff(y(x),x)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{2x}(c_3 x^2 + c_2 x + c_1)$$

✓ Solution by Mathematica
Time used: 0.003 (sec). Leaf size: 23

```
DSolve[y'''[x]-6*y''[x]+12*y'[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(x(c_3 x + c_2) + c_1)$$

7.16 problem Exercise 20.17, page 220

Internal problem ID [4587]
Internal file name [`OUTPUT/4080_Sunday_June_05_2022_12_19_42_PM_82934179/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.17, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2ay' + ya^2 = 0}$$

7.16.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2a, C = a^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2a\lambda\,\mathrm{e}^{\lambda x} + a^2\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$a^2 - 2a\lambda + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2a, C = a^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2a}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2a)^2 - (4)(1)(a^2)} \\ &= a\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -a$. Therefore the solution is

$$y = c_1 \mathrm{e}^{ax} + c_2 x\, \mathrm{e}^{ax} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{ax} + c_2 x\, \mathrm{e}^{ax} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{ax} + c_2 x\, \mathrm{e}^{ax}$$

Verified OK.

7.16.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{(p(x)^2 + p'(x))\, y}{2} = f(x)$$

Where $p(x) = -2a$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -2a\, dx} \\ &= \mathrm{e}^{-ax}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}(M(x)y)'' &= 0 \\ \left(\mathrm{e}^{-ax} y\right)'' &= 0\end{aligned}$$

Integrating once gives

$$\left(e^{-ax} y\right)' = c_1$$

Integrating again gives

$$\left(e^{-ax} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{e^{-ax}}$$

Or

$$y = c_1 x\, e^{ax} + c_2 e^{ax}$$

Summary
The solution(s) found are the following

$$y = c_1 x\, e^{ax} + c_2 e^{ax} \tag{1}$$

Verification of solutions

$$y = c_1 x\, e^{ax} + c_2 e^{ax}$$

Verified OK.

7.16.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2ay' + ya^2 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2a \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 184: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2a}{1}\,dx} \\ &= z_1 e^{ax} \\ &= z_1 \left(\mathrm{e}^{ax}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{ax}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2a}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2ax}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{ax}) + c_2(\mathrm{e}^{ax}(x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{ax} + c_2 x\, \mathrm{e}^{ax} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{ax} + c_2 x\, \mathrm{e}^{ax}
$$

Verified OK.

7.16.4 Maple step by step solution

Let's solve

$y'' - 2ay' + ya^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $a^2 - 2ar + r^2 = 0$

- Factor the characteristic polynomial

  $(a - r)^2 = 0$

- Root of the characteristic polynomial

  $r = a$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{ax}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, \mathrm{e}^{ax}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

$$y = c_1 e^{ax} + c_2 x\, e^{ax}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)-2*a*diff(y(x),x)+a^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{ax}(c_2 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 18

```
DSolve[y''[x]-2*a*y'[x]+a^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{ax}(c_2 x + c_1)$$

7.17 problem Exercise 20.18, page 220

Internal problem ID [4588]
Internal file name [OUTPUT/4081_Sunday_June_05_2022_12_19_49_PM_78995689/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.18, page 220.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 3y''' = 0}$$

The characteristic equation is

$$\lambda^4 + 3\lambda^3 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= -3\\ \lambda_2 &= 0\\ \lambda_3 &= 0\\ \lambda_4 &= 0\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_3x^2 + c_2x + c_1 + \mathrm{e}^{-3x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= 1\\ y_2 &= x\\ y_3 &= x^2\\ y_4 &= \mathrm{e}^{-3x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_3x^2 + c_2x + c_1 + e^{-3x}c_4 \tag{1}$$

Verification of solutions

$$y = c_3x^2 + c_2x + c_1 + e^{-3x}c_4$$

Verified OK.

7.17.1 Maple step by step solution

Let's solve

$y'''' + 3y''' = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -3y_4(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -3y_4(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-3x} \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-3x} \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{c_1 e^{-3x}}{27} + c_2$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$4)+3*diff(y(x),x$3)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2x + c_3x^2 + c_4\mathrm{e}^{-3x}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 28

```
DSolve[y''''[x]+3*y'''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{27}c_1e^{-3x} + x(c_4x + c_3) + c_2$$

7.18 problem Exercise 20.19, page 220

Internal problem ID [4589]
Internal file name [`OUTPUT/4082_Sunday_June_05_2022_12_19_57_PM_33127888/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.19, page 220.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 2y'' = 0}$$

The characteristic equation is

$$\lambda^4 - 2\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = \sqrt{2}$$
$$\lambda_4 = -\sqrt{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + \mathrm{e}^{x\sqrt{2}} c_3 + \mathrm{e}^{-x\sqrt{2}} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = \mathrm{e}^{x\sqrt{2}}$$
$$y_4 = \mathrm{e}^{-x\sqrt{2}}$$

Summary
The solution(s) found are the following

$$y = c_2 x + c_1 + e^{x\sqrt{2}} c_3 + e^{-x\sqrt{2}} c_4 \tag{1}$$

Verification of solutions

$$y = c_2 x + c_1 + e^{x\sqrt{2}} c_3 + e^{-x\sqrt{2}} c_4$$

Verified OK.

7.18.1 Maple step by step solution

Let's solve

$y'''' - 2y'' = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 2y_3(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 2y_3(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[\sqrt{2}, \begin{bmatrix} \frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}\right], \left[-\sqrt{2}, \begin{bmatrix} -\frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[\sqrt{2}, \begin{bmatrix} \frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{x\sqrt{2}} \cdot \begin{bmatrix} \frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\sqrt{2}, \left[\begin{array}{c} -\frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_4 = e^{-x\sqrt{2}} \cdot \left[\begin{array}{c} -\frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3 + c_4 \overrightarrow{y}_4$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = e^{x\sqrt{2}} c_3 \cdot \left[\begin{array}{c} \frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{array}\right] + e^{-x\sqrt{2}} c_4 \cdot \left[\begin{array}{c} -\frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ 1 \end{array}\right] + \left[\begin{array}{c} c_1 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{\sqrt{2}\, e^{x\sqrt{2}} c_3}{4} - \frac{\sqrt{2}\, e^{-x\sqrt{2}} c_4}{4} + c_1$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$4)-2*diff(y(x),x$2)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 x + c_3 e^{x\sqrt{2}} + c_4 e^{-x\sqrt{2}}$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 42

```
DSolve[y''''[x]-2*y''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-\sqrt{2}x} \left(c_1 e^{2\sqrt{2}x} + c_2 \right) + c_4 x + c_3$$

7.19 problem Exercise 20.20, page 220

Internal problem ID [4590]
Internal file name [OUTPUT/4083_Sunday_June_05_2022_12_20_04_PM_55582033/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.20, page 220.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 2y''' - 11y'' - 12y' + 36y = 0}$$

The characteristic equation is

$$\lambda^4 + 2\lambda^3 - 11\lambda^2 - 12\lambda + 36 = 0$$

The roots of the above equation are

$$\lambda_1 = -3$$
$$\lambda_2 = -3$$
$$\lambda_3 = 2$$
$$\lambda_4 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-3x} + x\,\mathrm{e}^{-3x} c_2 + \mathrm{e}^{2x} c_3 + x\,\mathrm{e}^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-3x}$$
$$y_2 = x\,\mathrm{e}^{-3x}$$
$$y_3 = \mathrm{e}^{2x}$$
$$y_4 = \mathrm{e}^{2x} x$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} + x\,e^{-3x} c_2 + e^{2x} c_3 + x\,e^{2x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-3x} + x\,e^{-3x} c_2 + e^{2x} c_3 + x\,e^{2x} c_4$$

Verified OK.

7.19.1 Maple step by step solution

Let's solve

$y'''' + 2y''' - 11y'' - 12y' + 36y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -2y_4(x) + 11y_3(x) + 12y_2(x) - 36y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -2y_4(x) + 11y_3(x) + 12y_2(x) - 36y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 12 & 11 & -2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 12 & 11 & -2 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-3, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-3, \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -3

$$\overrightarrow{y}_1(x) = \mathrm{e}^{-3x} \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -3$ is the eigenvalue, a

$$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -3

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 12 & 11 & -2 \end{bmatrix} - (-3) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} -\frac{1}{81} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -3

$$\vec{y}_2(x) = \mathrm{e}^{-3x} \cdot \left(x \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{81} \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

$$\vec{y}_3(x) = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, an

$$\vec{y}_4(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_4(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \mathrm{e}^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_4(x)$ to be a solution to the homogeneous system

$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 12 & 11 & -2 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

$$\vec{y}_4(x) = \mathrm{e}^{2x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + c_4 \vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-3x} \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-3x} \cdot \left(x \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{81} \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) + \mathrm{e}^{2x} c_3 \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \mathrm{e}^{2x} c_4 \cdot \left(x \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right.$$

- First component of the vector is the solution to the ODE

$$y = -\frac{\mathrm{e}^{-3x} \left(\frac{27\left(\left(\frac{1}{2} - x\right) c_4 - c_3\right) \mathrm{e}^{5x}}{8} + \left(x + \frac{1}{3}\right) c_2 + c_1\right)}{27}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$3)-11*diff(y(x),x$2)-12*diff(y(x),x)+36*y(x)=0,y(x), sin
```

$$y(x) = \left((c_2x + c_1)\,\mathrm{e}^{5x} + xc_4 + c_3\right)\mathrm{e}^{-3x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 35

```
DSolve[y''''[x]+2*y'''[x]-11*y''[x]-12*y'[x]+36*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to e^{-3x}\left(c_3e^{5x} + x\left(c_4e^{5x} + c_2\right) + c_1\right)$$

7.20 problem Exercise 20.21, page 220

Internal problem ID [4591]
Internal file name [`OUTPUT/4084_Sunday_June_05_2022_12_20_12_PM_52029093/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.21, page 220.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{36y'''' - 37y'' + 4y' + 5y = 0}$$

The characteristic equation is

$$36\lambda^4 - 37\lambda^2 + 4\lambda + 5 = 0$$

The roots of the above equation are

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{3}$$
$$\lambda_3 = \frac{5}{6}$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{\frac{x}{2}} + e^{\frac{5x}{6}} c_3 + e^{-\frac{x}{3}} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{\frac{x}{2}}$$
$$y_3 = \mathrm{e}^{\frac{5x}{6}}$$
$$y_4 = \mathrm{e}^{-\frac{x}{3}}$$

Summary
The solution(s) found are the following

$$y = c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{\frac{x}{2}} + \mathrm{e}^{\frac{5x}{6}}c_3 + \mathrm{e}^{-\frac{x}{3}}c_4 \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{\frac{x}{2}} + \mathrm{e}^{\frac{5x}{6}}c_3 + \mathrm{e}^{-\frac{x}{3}}c_4$$

Verified OK.

7.20.1 Maple step by step solution

Let's solve

$36y'''' - 37y'' + 4y' + 5y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = \frac{37y''}{36} - \frac{y'}{9} - \frac{5y}{36}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' - \frac{37y''}{36} + \frac{y'}{9} + \frac{5y}{36} = 0$

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

$y_4(x) = y'''$

- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = \frac{37y_3(x)}{36} - \frac{y_2(x)}{9} - \frac{5y_1(x)}{36}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = \frac{37y_3(x)}{36} - \frac{y_2(x)}{9} - \frac{5y_1(x)}{36}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{5}{36} & -\frac{1}{9} & \frac{37}{36} & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{5}{36} & -\frac{1}{9} & \frac{37}{36} & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{3}, \begin{bmatrix} -27 \\ 9 \\ -3 \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \end{bmatrix}\right], \left[\frac{5}{6}, \begin{bmatrix} \frac{216}{125} \\ \frac{36}{25} \\ \frac{6}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{1}{3}, \begin{bmatrix} -27 \\ 9 \\ -3 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{-\frac{x}{3}} \cdot \begin{bmatrix} -27 \\ 9 \\ -3 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\frac{1}{2}, \begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{\frac{x}{2}} \cdot \begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\frac{5}{6}, \begin{bmatrix} \frac{216}{125} \\ \frac{36}{25} \\ \frac{6}{5} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{\frac{5x}{6}} \cdot \begin{bmatrix} \frac{216}{125} \\ \frac{36}{25} \\ \frac{6}{5} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + c_4\vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-\frac{x}{3}} \cdot \begin{bmatrix} -27 \\ 9 \\ -3 \\ 1 \end{bmatrix} + c_3 e^{\frac{x}{2}} \cdot \begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \end{bmatrix} + c_4 e^{\frac{5x}{6}} \cdot \begin{bmatrix} \frac{216}{125} \\ \frac{36}{25} \\ \frac{6}{5} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -e^{-x}\left(27c_2 e^{\frac{2x}{3}} - 8c_3 e^{\frac{3x}{2}} - \frac{216c_4 e^{\frac{11x}{6}}}{125} + c_1\right)$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(36*diff(y(x),x$4)-37*diff(y(x),x$2)+4*diff(y(x),x)+5*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_3 e^{\frac{11x}{6}} + c_1 e^{\frac{3x}{2}} + c_2 e^{\frac{2x}{3}} + c_4\right) e^{-x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 44

```
DSolve[36*y''''[x]-37*y''[x]+4*y'[x]+5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_1 e^{11x/6} + c_2 e^{2x/3} + c_3 e^{3x/2} + c_4\right)$$

7.21 problem Exercise 20.22, page 220

Internal problem ID [4592]
Internal file name [`OUTPUT/4085_Sunday_June_05_2022_12_20_19_PM_87601993/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.22, page 220.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 8y'' + 36y = 0}$$

The characteristic equation is

$$\lambda^4 - 8\lambda^2 + 36 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= -i + \sqrt{5}\\ \lambda_2 &= -\sqrt{5} + i\\ \lambda_3 &= i + \sqrt{5}\\ \lambda_4 &= -\sqrt{5} - i\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{\left(i+\sqrt{5}\right)x} c_1 + \mathrm{e}^{\left(-\sqrt{5}-i\right)x} c_2 + \mathrm{e}^{\left(-i+\sqrt{5}\right)x} c_3 + \mathrm{e}^{\left(-\sqrt{5}+i\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{\left(i+\sqrt{5}\right)x}$$
$$y_2 = e^{\left(-\sqrt{5}-i\right)x}$$
$$y_3 = e^{\left(-i+\sqrt{5}\right)x}$$
$$y_4 = e^{\left(-\sqrt{5}+i\right)x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{\left(i+\sqrt{5}\right)x} c_1 + e^{\left(-\sqrt{5}-i\right)x} c_2 + e^{\left(-i+\sqrt{5}\right)x} c_3 + e^{\left(-\sqrt{5}+i\right)x} c_4 \tag{1}$$

<u>Verification of solutions</u>

$$y = e^{\left(i+\sqrt{5}\right)x} c_1 + e^{\left(-\sqrt{5}-i\right)x} c_2 + e^{\left(-i+\sqrt{5}\right)x} c_3 + e^{\left(-\sqrt{5}+i\right)x} c_4$$

Verified OK.

7.21.1 Maple step by step solution

Let's solve

$y'''' - 8y'' + 36y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 8y_3(x) - 36y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 8y_3(x) - 36y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 0 & 8 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 0 & 8 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}+\sqrt{5}, \begin{bmatrix} \frac{1}{(-\mathrm{I}+\sqrt{5})^3} \\ \frac{1}{(-\mathrm{I}+\sqrt{5})^2} \\ \frac{1}{-\mathrm{I}+\sqrt{5}} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}+\sqrt{5}, \begin{bmatrix} \frac{1}{(\mathrm{I}+\sqrt{5})^3} \\ \frac{1}{(\mathrm{I}+\sqrt{5})^2} \\ \frac{1}{\mathrm{I}+\sqrt{5}} \\ 1 \end{bmatrix}\right], \left[-\sqrt{5}-\mathrm{I}, \begin{bmatrix} \frac{1}{(-\sqrt{5}-\mathrm{I})^3} \\ \frac{1}{(-\sqrt{5}-\mathrm{I})^2} \\ \frac{1}{-\sqrt{5}-\mathrm{I}} \\ 1 \end{bmatrix}\right], \left[-\sqrt{5}+\mathrm{I},\right.\right.$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}+\sqrt{5}, \left[\begin{array}{c}\frac{1}{\left(-\mathrm{I}+\sqrt{5}\right)^{3}} \\ \frac{1}{\left(-\mathrm{I}+\sqrt{5}\right)^{2}} \\ \frac{1}{-\mathrm{I}+\sqrt{5}} \\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\mathrm{I}+\sqrt{5}\right)x} \cdot \left[\begin{array}{c}\frac{1}{\left(-\mathrm{I}+\sqrt{5}\right)^{3}} \\ \frac{1}{\left(-\mathrm{I}+\sqrt{5}\right)^{2}} \\ \frac{1}{-\mathrm{I}+\sqrt{5}} \\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{x\sqrt{5}} \cdot (\cos(x) - \mathrm{I}\sin(x)) \cdot \left[\begin{array}{c}\frac{1}{\left(-\mathrm{I}+\sqrt{5}\right)^{3}} \\ \frac{1}{\left(-\mathrm{I}+\sqrt{5}\right)^{2}} \\ \frac{1}{-\mathrm{I}+\sqrt{5}} \\ 1\end{array}\right]$$

- Simplify expression

$$e^{x\sqrt{5}} \cdot \left[\begin{array}{c}\frac{\cos(x)-\mathrm{I}\sin(x)}{\left(-\mathrm{I}+\sqrt{5}\right)^{3}} \\ \frac{\cos(x)-\mathrm{I}\sin(x)}{\left(-\mathrm{I}+\sqrt{5}\right)^{2}} \\ \frac{\cos(x)-\mathrm{I}\sin(x)}{-\mathrm{I}+\sqrt{5}} \\ \cos(x) - \mathrm{I}\sin(x)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = e^{x\sqrt{5}} \cdot \begin{bmatrix} \frac{\cos(x)\sqrt{5}}{108} + \frac{7\sin(x)}{108} \\ \frac{\cos(x)}{9} + \frac{\sin(x)\sqrt{5}}{18} \\ \frac{\cos(x)\sqrt{5}}{6} + \frac{\sin(x)}{6} \\ \cos(x) \end{bmatrix}, \vec{y}_2(x) = e^{x\sqrt{5}} \cdot \begin{bmatrix} \frac{7\cos(x)}{108} - \frac{\sin(x)\sqrt{5}}{108} \\ \frac{\cos(x)\sqrt{5}}{18} - \frac{\sin(x)}{9} \\ \frac{\cos(x)}{6} - \frac{\sin(x)\sqrt{5}}{6} \\ -\sin(x) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\sqrt{5} - I, \begin{bmatrix} \frac{1}{\left(-\sqrt{5}-I\right)^3} \\ \frac{1}{\left(-\sqrt{5}-I\right)^2} \\ \frac{1}{-\sqrt{5}-I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(-\sqrt{5}-I\right)x} \cdot \begin{bmatrix} \frac{1}{\left(-\sqrt{5}-I\right)^3} \\ \frac{1}{\left(-\sqrt{5}-I\right)^2} \\ \frac{1}{-\sqrt{5}-I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-x\sqrt{5}} \cdot (\cos(x) - I\sin(x)) \cdot \begin{bmatrix} \frac{1}{\left(-\sqrt{5}-I\right)^3} \\ \frac{1}{\left(-\sqrt{5}-I\right)^2} \\ \frac{1}{-\sqrt{5}-I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-x\sqrt{5}} \cdot \begin{bmatrix} \frac{\cos(x)-\mathrm{I}\sin(x)}{\left(-\sqrt{5}-\mathrm{I}\right)^3} \\ \frac{\cos(x)-\mathrm{I}\sin(x)}{\left(-\sqrt{5}-\mathrm{I}\right)^2} \\ \frac{\cos(x)-\mathrm{I}\sin(x)}{-\sqrt{5}-\mathrm{I}} \\ \cos(x)-\mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{-x\sqrt{5}} \cdot \begin{bmatrix} -\frac{\cos(x)\sqrt{5}}{108}+\frac{7\sin(x)}{108} \\ \frac{\cos(x)}{9}-\frac{\sin(x)\sqrt{5}}{18} \\ -\frac{\cos(x)\sqrt{5}}{6}+\frac{\sin(x)}{6} \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = e^{-x\sqrt{5}} \cdot \begin{bmatrix} \frac{7\cos(x)}{108}+\frac{\sin(x)\sqrt{5}}{108} \\ -\frac{\cos(x)\sqrt{5}}{18}-\frac{\sin(x)}{9} \\ \frac{\cos(x)}{6}+\frac{\sin(x)\sqrt{5}}{6} \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(x)+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{x\sqrt{5}} \cdot \begin{bmatrix} \frac{\cos(x)\sqrt{5}}{108}+\frac{7\sin(x)}{108} \\ \frac{\cos(x)}{9}+\frac{\sin(x)\sqrt{5}}{18} \\ \frac{\cos(x)\sqrt{5}}{6}+\frac{\sin(x)}{6} \\ \cos(x) \end{bmatrix} + c_2 e^{x\sqrt{5}} \cdot \begin{bmatrix} \frac{7\cos(x)}{108}-\frac{\sin(x)\sqrt{5}}{108} \\ \frac{\cos(x)\sqrt{5}}{18}-\frac{\sin(x)}{9} \\ \frac{\cos(x)}{6}-\frac{\sin(x)\sqrt{5}}{6} \\ -\sin(x) \end{bmatrix} + c_3 e^{-x\sqrt{5}} \cdot \begin{bmatrix} -\frac{\cos(x)\sqrt{5}}{108}+\frac{7\sin(x)}{108} \\ \frac{\cos(x)}{9}-\frac{\sin(x)\sqrt{5}}{18} \\ -\frac{\cos(x)\sqrt{5}}{6}+\frac{\sin(x)}{6} \\ \cos(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left((-\cos(x)c_3+\sin(x)c_4)\sqrt{5}+7\sin(x)c_3+7\cos(x)c_4\right)e^{-x\sqrt{5}}}{108}+\frac{e^{x\sqrt{5}}\left((\cos(x)c_1-c_2\sin(x))\sqrt{5}+7\sin(x)c_1+7c_2\cos(x)\right)}{108}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 48

```
dsolve(diff(y(x),x$4)-8*diff(y(x),x$2)+36*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\sqrt{5}\, x} \sin(x) - c_2 e^{-\sqrt{5}\, x} \sin(x) + c_3 e^{\sqrt{5}\, x} \cos(x) + c_4 e^{-\sqrt{5}\, x} \cos(x)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 142

```
DSolve[y''''[x]-8*y''[x]+36*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\sqrt{6}x \cos\left(\frac{1}{2}\arctan\left(\frac{\sqrt{5}}{2}\right)\right)} \left(\left(c_3 e^{2\sqrt{6}x \cos\left(\frac{1}{2}\arctan\left(\frac{\sqrt{5}}{2}\right)\right)} + c_2\right) \cos\left(\sqrt{6}x \sin\left(\frac{1}{2}\arctan\left(\frac{\sqrt{5}}{2}\right)\right)\right) + \sin\left(\sqrt{6}x \sin\left(\frac{1}{2}\arctan\left(\frac{\sqrt{5}}{2}\right)\right)\right) \left(c_1 e^{2\sqrt{6}x \cos\left(\frac{1}{2}\arctan\left(\frac{\sqrt{5}}{2}\right)\right)} + c_4\right)\right)$$

7.22 problem Exercise 20.23, page 220

Internal problem ID [4593]
Internal file name [`OUTPUT/4086_Sunday_June_05_2022_12_20_27_PM_94473883/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.23, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 5y = 0}$$

7.22.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\, \mathrm{e}^{\lambda x} + 5\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(5)} \\ &= 1 \pm 2i\end{aligned}$$

Hence

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

Which simplifies to

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x}(c_1 \cos(2x) + c_2 \sin(2x)) \tag{1}$$

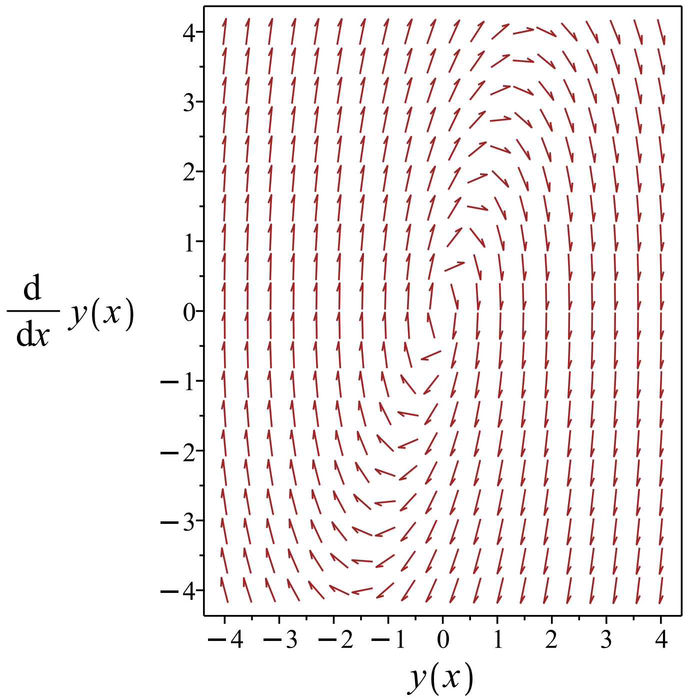


Figure 311: Slope field plot

Verification of solutions

$$y = e^{x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Verified OK.

7.22.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 191: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1(\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}\cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(\mathrm{e}^x \cos(2x)) + c_2\left(\mathrm{e}^x \cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^x \cos(2x) + \frac{c_2\mathrm{e}^x \sin(2x)}{2} \tag{1}
$$

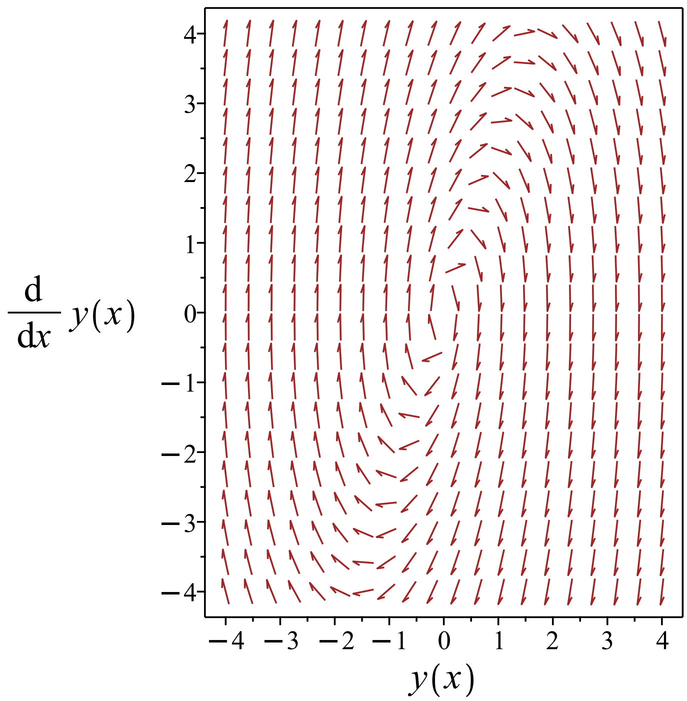


Figure 312: Slope field plot

Verification of solutions

$$
y = c_1\mathrm{e}^x \cos(2x) + \frac{c_2\mathrm{e}^x \sin(2x)}{2}
$$

Verified OK.

7.22.3 Maple step by step solution

Let's solve

$y'' - 2y' + 5y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - 2\,\mathrm{I}, 1 + 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^x \cos(2x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^x \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^x \cos(2x) + c_2 \mathrm{e}^x \sin(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+5*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{x}(c_1 \sin(2x) + c_2 \cos(2x))$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 24

```
DSolve[y''[x]-2*y'[x]+5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}(c_2 \cos(2x) + c_1 \sin(2x))$$

7.23 problem Exercise 20.24, page 220

Internal problem ID [4594]
Internal file name [`OUTPUT/4087_Sunday_June_05_2022_12_20_37_PM_36009604/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.24, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' + y = 0}$$

7.23.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-1^2 - (4)(1)(1)} \\ &= \frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \right) \tag{1}$$

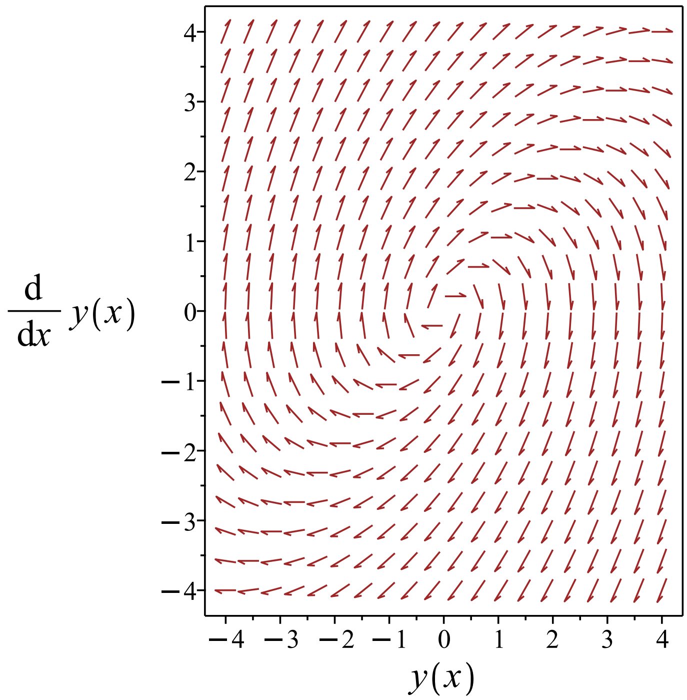


Figure 313: Slope field plot

Verification of solutions

$$y = e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Verified OK.

7.23.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 193: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)\right) + c_2 \left(\mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 \mathrm{e}^{\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} \tag{1}
$$

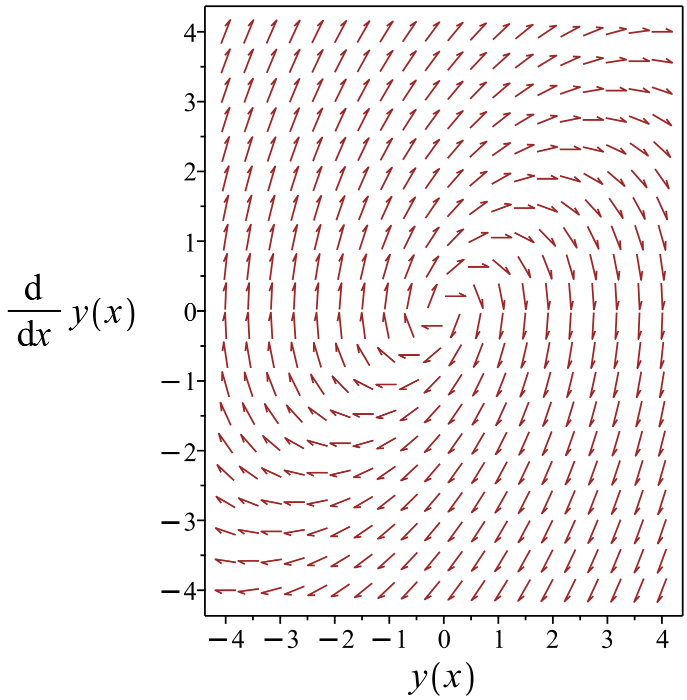


Figure 314: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2 e^{\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3}$$

Verified OK.

7.23.3 Maple step by step solution

Let's solve

$y'' - y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{1 \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - \frac{I\sqrt{3}}{2}, \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the ODE

  $y_2(x) = e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)-diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{\frac{x}{2}} \left(c_1 \sin\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 42

```
DSolve[y''[x]-y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x/2}\left(c_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

7.24 problem Exercise 20.25, page 220

Internal problem ID [4595]
Internal file name [OUTPUT/4088_Sunday_June_05_2022_12_20_44_PM_45160301/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.25, page 220.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 5y'' + 6y = 0}$$

The characteristic equation is

$$\lambda^4 + 5\lambda^2 + 6 = 0$$

The roots of the above equation are

$$\lambda_1 = i\sqrt{2}$$
$$\lambda_2 = -i\sqrt{2}$$
$$\lambda_3 = i\sqrt{3}$$
$$\lambda_4 = -i\sqrt{3}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{i\sqrt{2}\,x}c_1 + e^{-i\sqrt{2}\,x}c_2 + e^{i\sqrt{3}\,x}c_3 + e^{-i\sqrt{3}\,x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{i\sqrt{2}\,x}$$
$$y_2 = e^{-i\sqrt{2}\,x}$$
$$y_3 = e^{i\sqrt{3}\,x}$$
$$y_4 = e^{-i\sqrt{3}\,x}$$

Summary
The solution(s) found are the following

$$y = e^{i\sqrt{2}\,x} c_1 + e^{-i\sqrt{2}\,x} c_2 + e^{i\sqrt{3}\,x} c_3 + e^{-i\sqrt{3}\,x} c_4 \tag{1}$$

Verification of solutions

$$y = e^{i\sqrt{2}\,x} c_1 + e^{-i\sqrt{2}\,x} c_2 + e^{i\sqrt{3}\,x} c_3 + e^{-i\sqrt{3}\,x} c_4$$

Verified OK.

7.24.1 Maple step by step solution

Let's solve

$y'''' + 5y'' + 6y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -5y_3(x) - 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -5y_3(x) - 6y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 0 & -5 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 0 & -5 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}\sqrt{3}, \begin{bmatrix} -\frac{\mathrm{I}}{9}\sqrt{3} \\ -\frac{1}{3} \\ \frac{\mathrm{I}}{3}\sqrt{3} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{2}, \begin{bmatrix} \frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ -\frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{\mathrm{I}}{9}\sqrt{3} \\ -\frac{1}{3} \\ -\frac{\mathrm{I}}{3}\sqrt{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-I\sqrt{2}\,x}\cdot\left[\begin{array}{c} -\frac{I}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{I}{2}\sqrt{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$\left(\cos\left(x\sqrt{2}\right)-I\sin\left(x\sqrt{2}\right)\right)\cdot\left[\begin{array}{c} -\frac{I}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{I}{2}\sqrt{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\frac{I}{4}\left(\cos\left(x\sqrt{2}\right)-I\sin\left(x\sqrt{2}\right)\right)\sqrt{2} \\ -\frac{\cos\left(x\sqrt{2}\right)}{2}+\frac{I\sin\left(x\sqrt{2}\right)}{2} \\ \frac{I}{2}\left(\cos\left(x\sqrt{2}\right)-I\sin\left(x\sqrt{2}\right)\right)\sqrt{2} \\ \cos\left(x\sqrt{2}\right)-I\sin\left(x\sqrt{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x)=\left[\begin{array}{c} -\frac{\sqrt{2}\,\sin\left(x\sqrt{2}\right)}{4} \\ -\frac{\cos\left(x\sqrt{2}\right)}{2} \\ \frac{\sqrt{2}\,\sin\left(x\sqrt{2}\right)}{2} \\ \cos\left(x\sqrt{2}\right) \end{array}\right],\vec{y}_2(x)=\left[\begin{array}{c} -\frac{\sqrt{2}\,\cos\left(x\sqrt{2}\right)}{4} \\ \frac{\sin\left(x\sqrt{2}\right)}{2} \\ \frac{\sqrt{2}\,\cos\left(x\sqrt{2}\right)}{2} \\ -\sin\left(x\sqrt{2}\right) \end{array}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I\sqrt{3},\left[\begin{array}{c} -\frac{I}{9}\sqrt{3} \\ -\frac{1}{3} \\ \frac{I}{3}\sqrt{3} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-I\sqrt{3}\,x}\cdot\left[\begin{array}{c} -\frac{I}{9}\sqrt{3} \\ -\frac{1}{3} \\ \frac{I}{3}\sqrt{3} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$\left(\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)\right)\cdot\left[\begin{array}{c} -\frac{I}{9}\sqrt{3} \\ -\frac{1}{3} \\ \frac{I}{3}\sqrt{3} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\frac{I}{9}\left(\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)\right)\sqrt{3} \\ -\frac{\cos\left(\sqrt{3}\,x\right)}{3}+\frac{I\sin\left(\sqrt{3}\,x\right)}{3} \\ \frac{I}{3}\left(\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)\right)\sqrt{3} \\ \cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x)=\left[\begin{array}{c} -\frac{\sqrt{3}\,\sin\left(\sqrt{3}\,x\right)}{9} \\ -\frac{\cos\left(\sqrt{3}\,x\right)}{3} \\ \frac{\sqrt{3}\,\sin\left(\sqrt{3}\,x\right)}{3} \\ \cos\left(\sqrt{3}\,x\right) \end{array}\right], \vec{y}_4(x)=\left[\begin{array}{c} -\frac{\sqrt{3}\,\cos\left(\sqrt{3}\,x\right)}{9} \\ \frac{\sin\left(\sqrt{3}\,x\right)}{3} \\ \frac{\sqrt{3}\,\cos\left(\sqrt{3}\,x\right)}{3} \\ -\sin\left(\sqrt{3}\,x\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_1\vec{y}_1(x)+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = \begin{bmatrix} -\frac{c_4\sqrt{3}\cos(\sqrt{3}\,x)}{9} - \frac{c_3\sqrt{3}\sin(\sqrt{3}\,x)}{9} - \frac{c_2\sqrt{2}\cos(x\sqrt{2})}{4} - \frac{c_1\sqrt{2}\sin(x\sqrt{2})}{4} \\ \frac{c_4\sin(\sqrt{3}\,x)}{3} - \frac{c_3\cos(\sqrt{3}\,x)}{3} + \frac{c_2\sin(x\sqrt{2})}{2} - \frac{c_1\cos(x\sqrt{2})}{2} \\ \frac{c_4\sqrt{3}\cos(\sqrt{3}\,x)}{3} + \frac{c_3\sqrt{3}\sin(\sqrt{3}\,x)}{3} + \frac{c_2\sqrt{2}\cos(x\sqrt{2})}{2} + \frac{c_1\sqrt{2}\sin(x\sqrt{2})}{2} \\ -c_4\sin(\sqrt{3}\,x) + c_3\cos(\sqrt{3}\,x) - c_2\sin(x\sqrt{2}) + c_1\cos(x\sqrt{2}) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{c_4\sqrt{3}\cos(\sqrt{3}\,x)}{9} - \frac{c_3\sqrt{3}\sin(\sqrt{3}\,x)}{9} - \frac{c_2\sqrt{2}\cos(x\sqrt{2})}{4} - \frac{c_1\sqrt{2}\sin(x\sqrt{2})}{4}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$4)+5*diff(y(x),x$2)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sin\left(\sqrt{3}\,x\right) + c_2\cos\left(\sqrt{3}\,x\right) + c_3\sin\left(x\sqrt{2}\right) + c_4\cos\left(x\sqrt{2}\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 50

```
DSolve[y''''[x]+5*y''[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3\cos\left(\sqrt{2}x\right) + c_1\cos\left(\sqrt{3}x\right) + c_4\sin\left(\sqrt{2}x\right) + c_2\sin\left(\sqrt{3}x\right)$$

7.25 problem Exercise 20.26, page 220

Internal problem ID [4596]
Internal file name [`OUTPUT/4089_Sunday_June_05_2022_12_20_52_PM_48205543/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.26, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - 4y' + 20y = 0}$$

7.25.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 20$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 20 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 20 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 20$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-4^2 - (4)(1)(20)} \\
&= 2 \pm 4i
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= 2 + 4i \\
\lambda_2 &= 2 - 4i
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 2 + 4i \\
\lambda_2 &= 2 - 4i
\end{aligned}
$$

Since roots are complex conjugate of each others, then let the roots be

$$
\lambda_{1,2} = \alpha \pm i\beta
$$

Where $\alpha = 2$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$
y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))
$$

Which becomes

$$
y = e^{2x}(c_1 \cos(4x) + c_2 \sin(4x))
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = e^{2x}(c_1 \cos(4x) + c_2 \sin(4x)) \tag{1}
$$

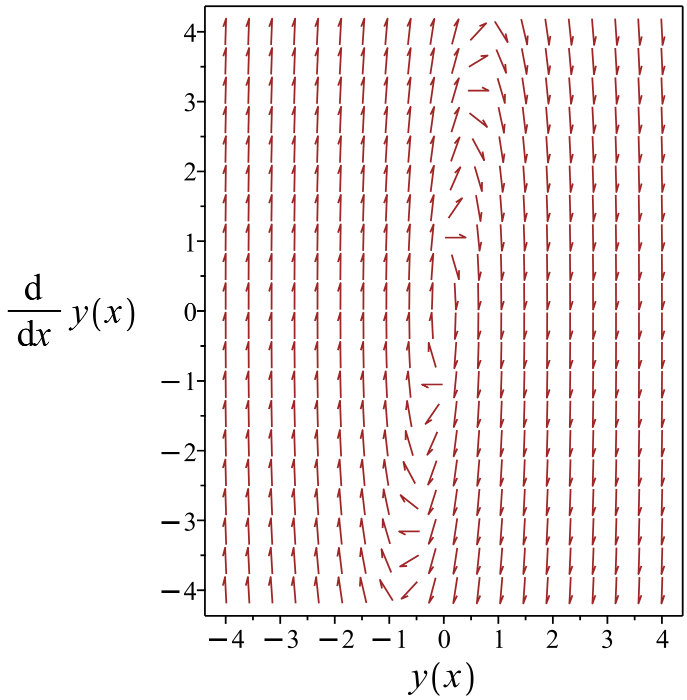


Figure 315: Slope field plot

Verification of solutions

$$y = e^{2x}(c_1 \cos(4x) + c_2 \sin(4x))$$

Verified OK.

7.25.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 20y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -16z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 196: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(4x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1\left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}\cos(4x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\tan(4x)}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{2x} \cos(4x)\right) + c_2 \left(e^{2x} \cos(4x) \left(\frac{\tan(4x)}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{2x} \cos(4x) + \frac{c_2 e^{2x} \sin(4x)}{4} \tag{1}
$$

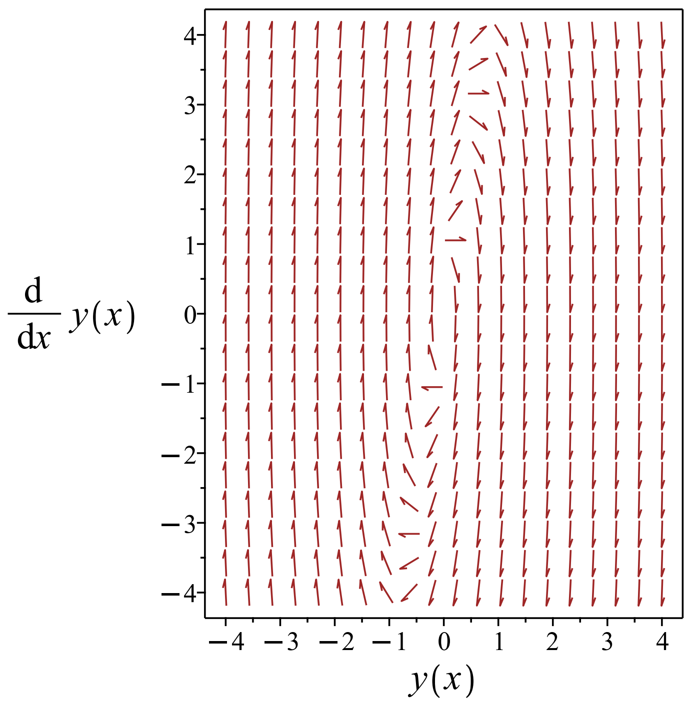


Figure 316: Slope field plot

Verification of solutions

$$
y = c_1 e^{2x} \cos(4x) + \frac{c_2 e^{2x} \sin(4x)}{4}
$$

Verified OK.

7.25.3 Maple step by step solution

Let's solve

$y'' - 4y' + 20y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 4r + 20 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4 \pm (\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (2 - 4\,\mathrm{I}, 2 + 4\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{2x} \cos(4x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{2x} \sin(4x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{2x} \cos(4x) + c_2 \mathrm{e}^{2x} \sin(4x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+20*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{2x}(c_1 \sin(4x) + c_2 \cos(4x))$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 26

```
DSolve[y''[x]-4*y'[x]+20*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(c_2 \cos(4x) + c_1 \sin(4x))$$

7.26 problem Exercise 20.27, page 220

Internal problem ID [4597]
Internal file name [`OUTPUT/4090_Sunday_June_05_2022_12_20_59_PM_45343476/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.27, page 220.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 4y'' + 4y = 0}$$

The characteristic equation is

$$\lambda^4 + 4\lambda^2 + 4 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= i\sqrt{2}\\ \lambda_2 &= -i\sqrt{2}\\ \lambda_3 &= i\sqrt{2}\\ \lambda_4 &= -i\sqrt{2}\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{i\sqrt{2}\,x}c_1 + x\,e^{i\sqrt{2}\,x}c_2 + e^{-i\sqrt{2}\,x}c_3 + x\,e^{-i\sqrt{2}\,x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= e^{i\sqrt{2}\,x}\\ y_2 &= x\,e^{i\sqrt{2}\,x}\\ y_3 &= e^{-i\sqrt{2}\,x}\\ y_4 &= x\,e^{-i\sqrt{2}\,x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{i\sqrt{2}\,x} c_1 + x\,e^{i\sqrt{2}\,x} c_2 + e^{-i\sqrt{2}\,x} c_3 + x\,e^{-i\sqrt{2}\,x} c_4 \tag{1}$$

Verification of solutions

$$y = e^{i\sqrt{2}\,x} c_1 + x\,e^{i\sqrt{2}\,x} c_2 + e^{-i\sqrt{2}\,x} c_3 + x\,e^{-i\sqrt{2}\,x} c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$4)+4*diff(y(x),x$2)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = (xc_4 + c_2)\cos\left(x\sqrt{2}\right) + \sin\left(x\sqrt{2}\right)(c_3 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 38

```
DSolve[y''''[x]+4*y''[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (c_2 x + c_1)\cos\left(\sqrt{2}x\right) + (c_4 x + c_3)\sin\left(\sqrt{2}x\right)$$

7.27 problem Exercise 20.28, page 220

Internal problem ID [4598]
Internal file name [OUTPUT/4091_Sunday_June_05_2022_12_21_06_PM_73745967/index.tex]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.28, page 220.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 8y = 0}$$

The characteristic equation is

$$\lambda^3 + 8 = 0$$

The roots of the above equation are

$$\lambda_1 = -2$$
$$\lambda_2 = 1 - i\sqrt{3}$$
$$\lambda_3 = 1 + i\sqrt{3}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{\left(1-i\sqrt{3}\right)x} c_2 + \mathrm{e}^{\left(1+i\sqrt{3}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \mathrm{e}^{\left(1-i\sqrt{3}\right)x}$$
$$y_3 = \mathrm{e}^{\left(1+i\sqrt{3}\right)x}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{\left(1-i\sqrt{3}\right)x} c_2 + \mathrm{e}^{\left(1+i\sqrt{3}\right)x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{\left(1-i\sqrt{3}\right)x} c_2 + \mathrm{e}^{\left(1+i\sqrt{3}\right)x} c_3$$

Verified OK.

7.27.1 Maple step by step solution

Let's solve

$y''' + 8y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ∘ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ∘ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ∘ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ∘ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -8y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 0 & 0 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[1-\mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(1-\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{1-\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[1+\mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(1+\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{1+\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-\mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(1-\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{1-\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(1-\mathrm{I}\sqrt{3}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(1-\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{1-\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{x} \cdot \left(\cos\left(\sqrt{3}\,x\right) - \mathrm{I}\sin\left(\sqrt{3}\,x\right)\right) \cdot \begin{bmatrix} \frac{1}{\left(1-\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{1-\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{x} \cdot \begin{bmatrix} \frac{\cos\left(\sqrt{3}\,x\right)-\mathrm{I}\sin\left(\sqrt{3}\,x\right)}{\left(1-\mathrm{I}\sqrt{3}\right)^2} \\ \frac{\cos\left(\sqrt{3}\,x\right)-\mathrm{I}\sin\left(\sqrt{3}\,x\right)}{1-\mathrm{I}\sqrt{3}} \\ \cos\left(\sqrt{3}\,x\right) - \mathrm{I}\sin\left(\sqrt{3}\,x\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{x} \cdot \begin{bmatrix} -\frac{\cos\left(\sqrt{3}\,x\right)}{8} + \frac{\sqrt{3}\,\sin\left(\sqrt{3}\,x\right)}{8} \\ \frac{\cos\left(\sqrt{3}\,x\right)}{4} + \frac{\sqrt{3}\,\sin\left(\sqrt{3}\,x\right)}{4} \\ \cos\left(\sqrt{3}\,x\right) \end{bmatrix}, \vec{y}_3(x) = e^{x} \cdot \begin{bmatrix} \frac{\sqrt{3}\,\cos\left(\sqrt{3}\,x\right)}{8} + \frac{\sin\left(\sqrt{3}\,x\right)}{8} \\ \frac{\sqrt{3}\,\cos\left(\sqrt{3}\,x\right)}{4} - \frac{\sin\left(\sqrt{3}\,x\right)}{4} \\ -\sin\left(\sqrt{3}\,x\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{x} \cdot \begin{bmatrix} -\frac{\cos\left(\sqrt{3}\,x\right)}{8} + \frac{\sqrt{3}\,\sin\left(\sqrt{3}\,x\right)}{8} \\ \frac{\cos\left(\sqrt{3}\,x\right)}{4} + \frac{\sqrt{3}\,\sin\left(\sqrt{3}\,x\right)}{4} \\ \cos\left(\sqrt{3}\,x\right) \end{bmatrix} + c_3 e^{x} \cdot \begin{bmatrix} \frac{\sqrt{3}\,\cos\left(\sqrt{3}\,x\right)}{8} + \frac{\sin\left(\sqrt{3}\,x\right)}{8} \\ \frac{\sqrt{3}\,\cos\left(\sqrt{3}\,x\right)}{4} - \frac{\sin\left(\sqrt{3}\,x\right)}{4} \\ -\sin\left(\sqrt{3}\,x\right) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(-\frac{e^{3x}\left(-c_3\sqrt{3}+c_2\right)\cos\left(\sqrt{3}\,x\right)}{2}+\frac{e^{3x}\left(c_2\sqrt{3}+c_3\right)\sin\left(\sqrt{3}\,x\right)}{2}+c_1\right)e^{-2x}}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$3)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2 e^{3x} \sin\left(\sqrt{3}\,x\right) + c_3 e^{3x} \cos\left(\sqrt{3}\,x\right) + c_1\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 42

```
DSolve[y'''[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-2x} + c_3 e^x \cos\left(\sqrt{3}x\right) + c_2 e^x \sin\left(\sqrt{3}x\right)$$

7.28 problem Exercise 20.29, page 220

Internal problem ID [4599]
Internal file name [`OUTPUT/4092_Sunday_June_05_2022_12_21_14_PM_20701212/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.29, page 220.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 4y'' = 0}$$

The characteristic equation is

$$\lambda^4 + 4\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 2i$$
$$\lambda_4 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_2x + c_1 + \mathrm{e}^{-2ix}c_3 + \mathrm{e}^{2ix}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = \mathrm{e}^{-2ix}$$
$$y_4 = \mathrm{e}^{2ix}$$

Summary
The solution(s) found are the following

$$y = c_2 x + c_1 + e^{-2ix} c_3 + e^{2ix} c_4 \tag{1}$$

Verification of solutions

$$y = c_2 x + c_1 + e^{-2ix} c_3 + e^{2ix} c_4$$

Verified OK.

7.28.1 Maple step by step solution

Let's solve

$y'''' + 4y'' = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -4y_3(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -4y_3(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-2\,\mathrm{I}x} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(2x) - \mathrm{I}\sin(2x)) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\mathrm{I}}{8}(\cos(2x) - \mathrm{I}\sin(2x)) \\ -\frac{\cos(2x)}{4} + \frac{\mathrm{I}\sin(2x)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2x) - \mathrm{I}\sin(2x)) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = \begin{bmatrix} -\frac{c_4\cos(2x)}{8} - \frac{c_3\sin(2x)}{8} + c_1 \\ \frac{c_4\sin(2x)}{4} - \frac{c_3\cos(2x)}{4} \\ \frac{c_4\cos(2x)}{2} + \frac{c_3\sin(2x)}{2} \\ -c_4\sin(2x) + c_3\cos(2x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{c_4\cos(2x)}{8} - \frac{c_3\sin(2x)}{8} + c_1$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$4)+4*diff(y(x),x$2)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 x + c_3 \sin(2x) + c_4 \cos(2x)$$

✓ Solution by Mathematica

Time used: 0.118 (sec). Leaf size: 32

```
DSolve[y''''[x]+4*y''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_4 x - \frac{1}{4} c_1 \cos(2x) - \frac{1}{4} c_2 \sin(2x) + c_3$$

7.29 problem Exercise 20.30, page 220

Internal problem ID [4600]
Internal file name [`OUTPUT/4093_Sunday_June_05_2022_12_21_21_PM_91188637/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20.30, page 220.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(5)}+2y'''+y'=0}$$

The characteristic equation is

$$\lambda^5+2\lambda^3+\lambda=0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1&=0\\ \lambda_2&=i\\ \lambda_3&=-i\\ \lambda_4&=i\\ \lambda_5&=-i\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x)=c_1+\mathrm{e}^{-ix}c_2+x\,\mathrm{e}^{-ix}c_3+\mathrm{e}^{ix}c_4+x\,\mathrm{e}^{ix}c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{-ix}$$
$$y_3 = x\,e^{-ix}$$
$$y_4 = e^{ix}$$
$$y_5 = x\,e^{ix}$$

Summary

The solution(s) found are the following

$$y = c_1 + e^{-ix} c_2 + x\,e^{-ix} c_3 + e^{ix} c_4 + x\,e^{ix} c_5 \tag{1}$$

Verification of solutions

$$y = c_1 + e^{-ix} c_2 + x\,e^{-ix} c_3 + e^{ix} c_4 + x\,e^{ix} c_5$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$5)+2*diff(y(x),x$3)+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (c_5 x + c_3) \cos(x) + (x c_4 + c_2) \sin(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 35

```
DSolve[y'''''[x]+2*y'''[x]+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (-c_4 x + c_2 - c_3) \cos(x) + (c_2 x + c_1 + c_4) \sin(x) + c_5$$

7.30 problem Exercise 20, problem 31, page 220

Internal problem ID [4601]
Internal file name [`OUTPUT/4094_Sunday_June_05_2022_12_21_29_PM_73806920/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20, problem 31, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y'' = 0}$$

With initial conditions

$$[y(1) = 2, y'(1) = -1]$$

7.30.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 0 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. Hence solution exists and is unique.

7.30.2 Solving as second order ode quadrature ode

Integrating twice gives the solution

$$y = c_1 x + c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 3$$

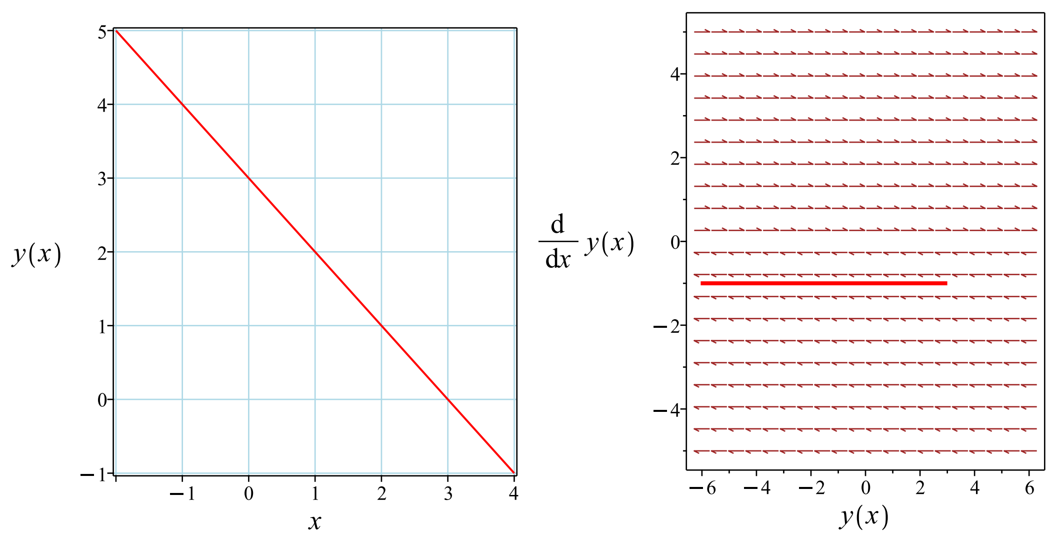
Substituting these values back in above solution results in

$$y = 3 - x$$

Summary

The solution(s) found are the following

$$y = 3 - x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 3 - x$$

Verified OK.

7.30.3 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(0)^2 - (4)\,(1)\,(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x + c_1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

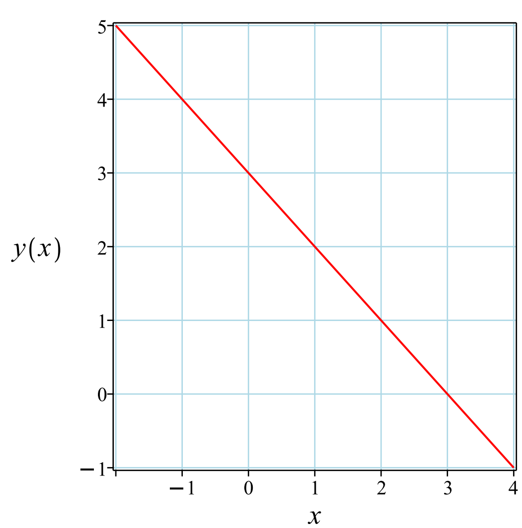
$$c_1 = 3$$
$$c_2 = -1$$

Substituting these values back in above solution results in

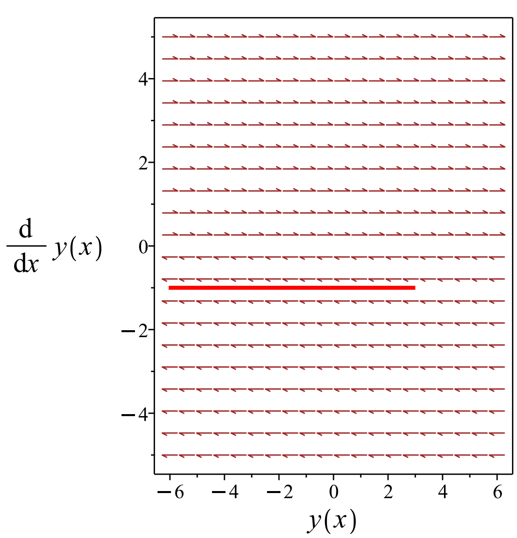
$$y = 3 - x$$

<u>Summary</u>
The solution(s) found are the following

$$y = 3 - x \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 3 - x$$

Verified OK.

7.30.4 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' = 0$$

Integrating the above w.r.t x gives

$$\int y'y''dx = 0$$
$$\frac{y'^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{2}\,\sqrt{c_1} \tag{1}$$
$$y' = -\sqrt{2}\,\sqrt{c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \sqrt{2}\,\sqrt{c_1}\,\mathrm{d}x \\ &= x\sqrt{2}\,\sqrt{c_1} + c_2 \end{aligned}$$

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\sqrt{2}\,\sqrt{c_1}\,\mathrm{d}x \\ &= -x\sqrt{2}\,\sqrt{c_1} + c_3 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$y = x\sqrt{2}\,\sqrt{c_1} + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$2 = \sqrt{2}\,\sqrt{c_1} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \sqrt{2}\,\sqrt{c_1}$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = \sqrt{2}\,\sqrt{c_1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$y = -x\sqrt{2}\,\sqrt{c_1} + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$2 = -\sqrt{2}\,\sqrt{c_1} + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sqrt{2}\,\sqrt{c_1}$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = -\sqrt{2}\,\sqrt{c_1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

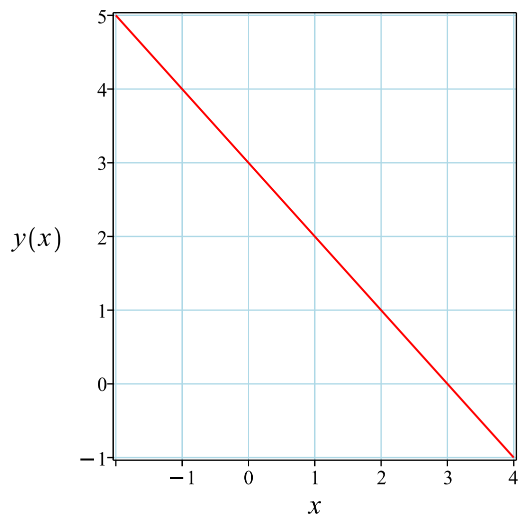
$$c_1 = \frac{1}{2}$$
$$c_3 = 3$$

Substituting these values back in above solution results in

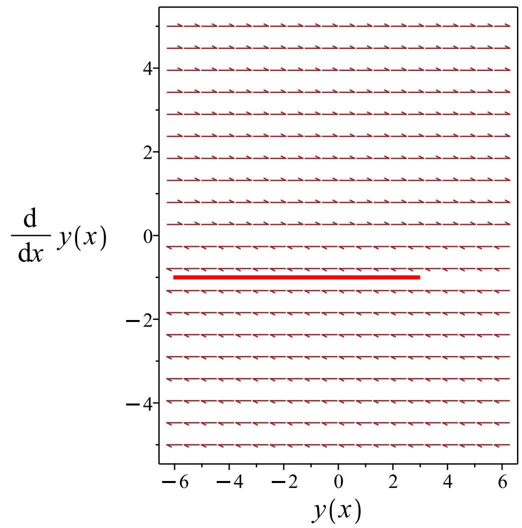
$$y = 3 - x$$

Summary
The solution(s) found are the following

$$y = 3 - x \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 3 - x$$

Verified OK.

7.30.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = 0$$
$$y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$y = \int c_1 \,\mathrm{d}x$$
$$= c_1 x + c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$

in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

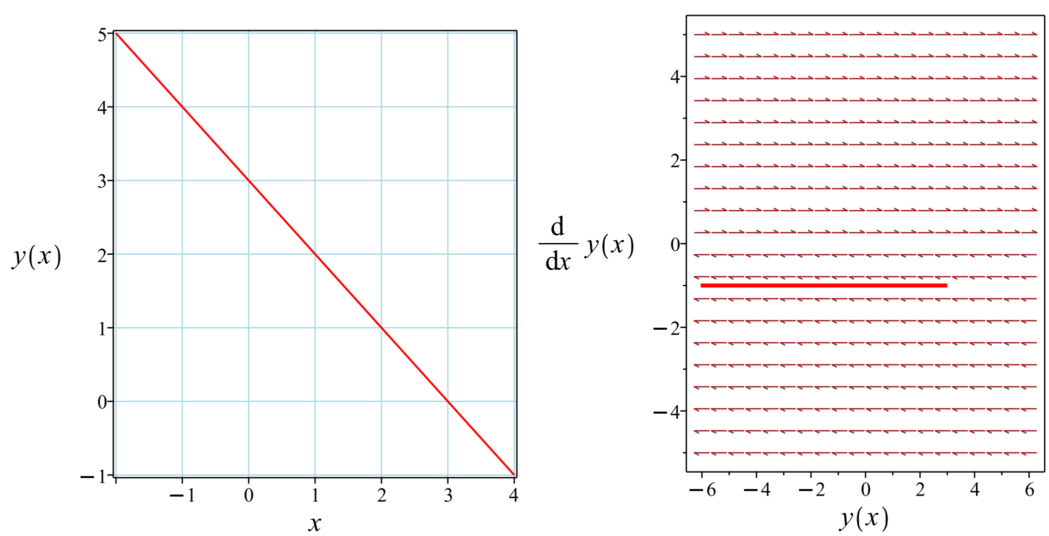
$$c_1 = -1$$
$$c_2 = 3$$

Substituting these values back in above solution results in

$$y = 3 - x$$

Summary
The solution(s) found are the following

$$y = 3 - x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 3 - x$$

Verified OK.

7.30.6 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $p = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$p(x) = -1$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -1$$

Integrating both sides gives

$$\begin{aligned} y &= \int -1 \, \mathrm{d}x \\ &= c_2 - x \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_2 - 1$$

$$c_2 = 3$$

Substituting c_2 found above in the general solution gives

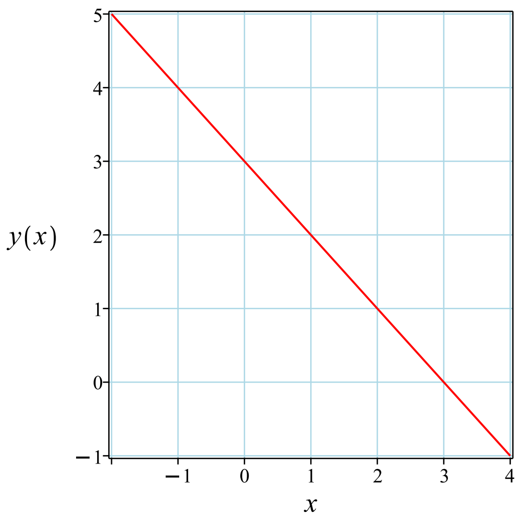
$$y = 3 - x$$

Initial conditions are used to solve for the constants of integration.

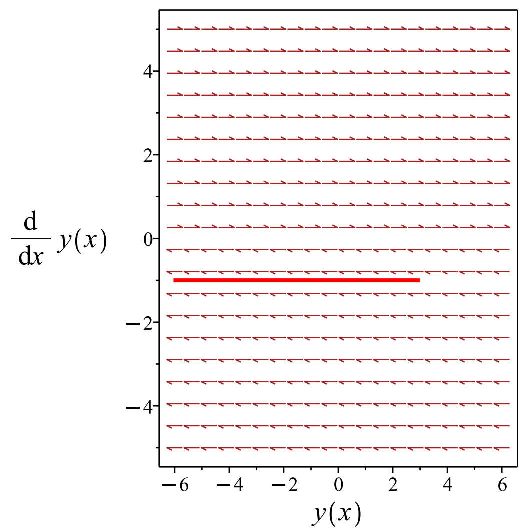
Summary

The solution(s) found are the following

$$y = 3 - x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3 - x$$

Verified OK.

7.30.7 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 200: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= 1 \int \frac{1}{1}\, dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x + c_1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = -1$$

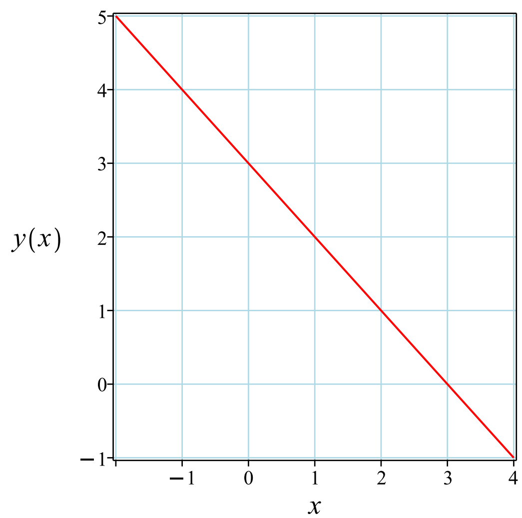
Substituting these values back in above solution results in

$$y = 3 - x$$

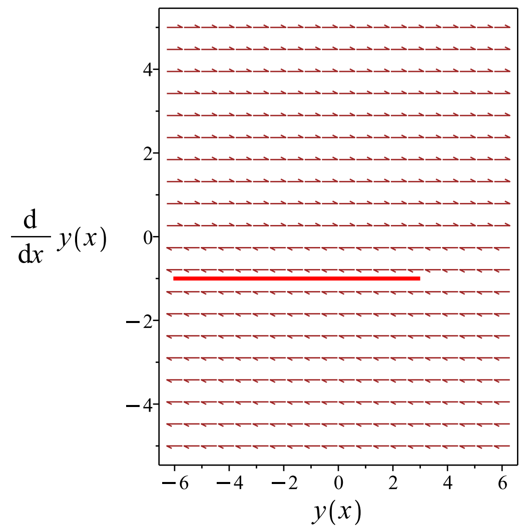
Summary

The solution(s) found are the following

$$y = 3 - x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3 - x$$

Verified OK.

7.30.8 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$y' = c_1$$

We now have a first order ode to solve which is

$$y' = c_1$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int c_1 \, \mathrm{d}x \\
&= c_1 x + c_2
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 x + c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$
2 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1
$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$
-1 = c_1 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

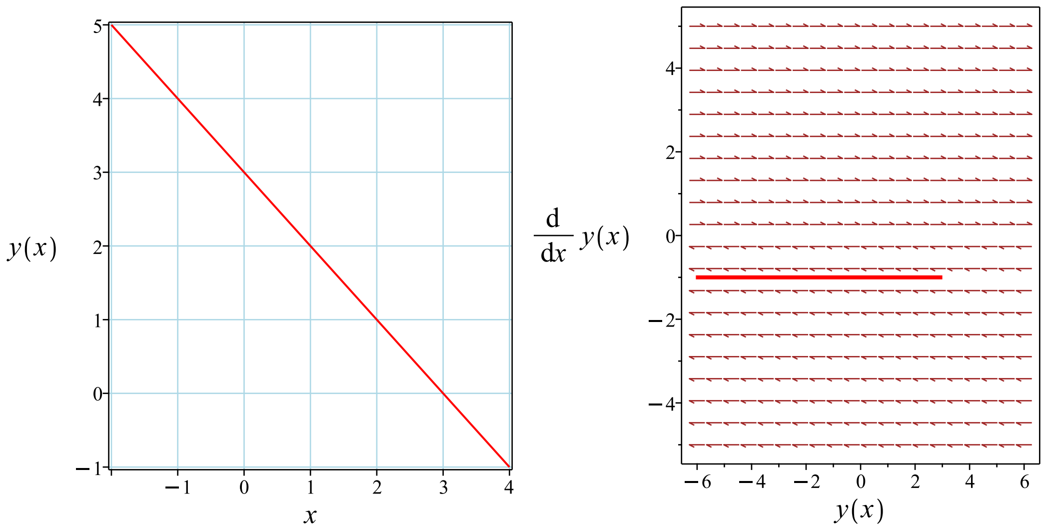
$$
\begin{aligned}
c_1 &= -1 \\
c_2 &= 3
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 3 - x
$$

Summary
The solution(s) found are the following

$$
y = 3 - x \tag{1}
$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = 3 - x$$

Verified OK.

7.30.9 Maple step by step solution

Let's solve

$$\left[y'' = 0, y(1) = 2, y'\Big|_{\{x=1\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the ODE

$$y_1(x) = 1$$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

$$y_2(x) = x$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_2 x + c_1$$

☐ Check validity of solution $y = c_2 x + c_1$

  ◦ Use initial condition $y(1) = 2$

$$2 = c_1 + c_2$$

  ◦ Compute derivative of the solution

$$y' = c_2$$

  ◦ Use the initial condition $y'\Big|_{\{x=1\}} = -1$

$$-1 = c_2$$

  ◦ Solve for c_1 and c_2

$$\{c_1 = 3, c_2 = -1\}$$

  ◦ Substitute constant values into general solution and simplify

$$y = 3 - x$$

- Solution to the IVP

$$y = 3 - x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve([diff(y(x),x$2)=0,y(1) = 2, D(y)(1) = -1],y(x), singsol=all)
```

$$y(x) = -x + 3$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 10

```
DSolve[{y''[x]==0,{y[1]==2,y'[1]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3 - x$$

7.31 problem Exercise 20, problem 32, page 220

Internal problem ID [4602]
Internal file name [`OUTPUT/4095_Sunday_June_05_2022_12_21_37_PM_17630807/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20, problem 32, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' + 4y' + 4y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

7.31.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 4 \\ q(x) &= 4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 4y = 0$$

The domain of $p(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.31.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\lambda\,\mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(4)^2 - (4)\,(1)\,(4)}\\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-2x} + c_2 x \,\mathrm{e}^{-2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-2x} - 2c_2 x \,\mathrm{e}^{-2x}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = -2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 3 \end{aligned}$$

Substituting these values back in above solution results in

$$y = 3x \,\mathrm{e}^{-2x} + \mathrm{e}^{-2x}$$

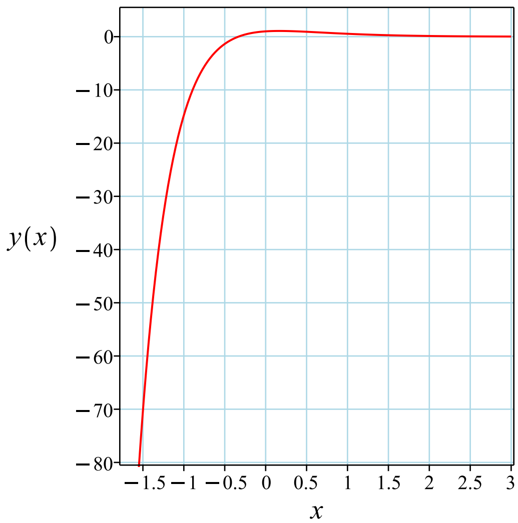
Which simplifies to

$$y = \mathrm{e}^{-2x}(3x + 1)$$

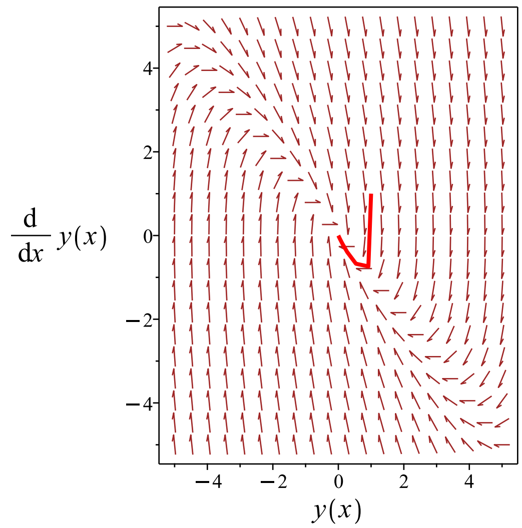
Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2x}(3x + 1) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{-2x}(3x + 1)$$

Verified OK.

7.31.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 4\,dx} \\ &= e^{2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 0$$
$$\left(e^{2x}y\right)'' = 0$$

Integrating once gives

$$\left(e^{2x}y\right)' = c_1$$

Integrating again gives

$$\left(e^{2x}y\right) = c_1x + c_2$$

Hence the solution is

$$y = \frac{c_1x + c_2}{e^{2x}}$$

Or

$$y = c_1x\,e^{-2x} + c_2e^{-2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1x\,e^{-2x} + c_2e^{-2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1e^{-2x} - 2c_1x\,e^{-2x} - 2c_2e^{-2x}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_1 - 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = 1$$

Substituting these values back in above solution results in

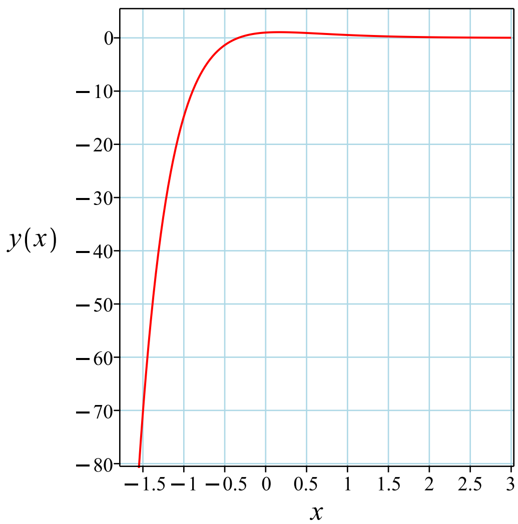
$$y = 3x\,e^{-2x} + e^{-2x}$$

Which simplifies to

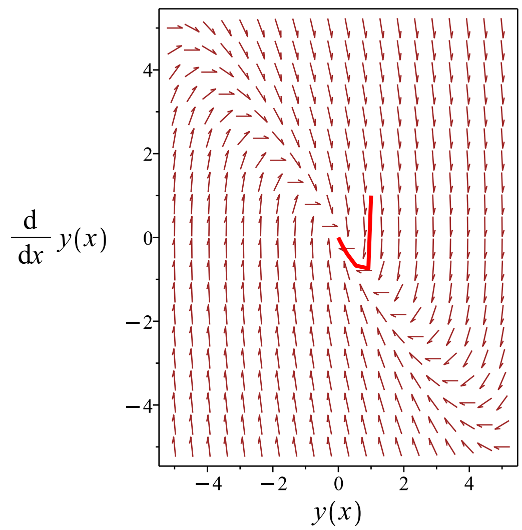
$$y = e^{-2x}(3x + 1)$$

Summary
The solution(s) found are the following

$$y = e^{-2x}(3x + 1) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-2x}(3x + 1)$$

Verified OK.

7.31.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 202: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\ &= z_1 e^{-2x} \\ &= z_1 \left(\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x}(x)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-2x} - 2c_2 x\,\mathrm{e}^{-2x}
$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$
1 = -2c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= 3
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 3x\,\mathrm{e}^{-2x} + \mathrm{e}^{-2x}
$$

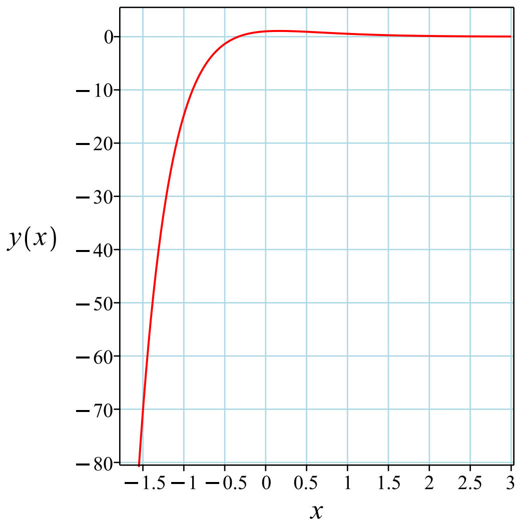
Which simplifies to

$$
y = \mathrm{e}^{-2x}(3x + 1)
$$

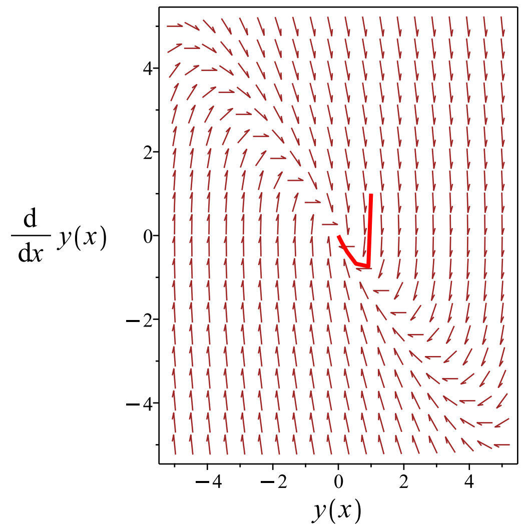
Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-2x}(3x + 1) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{-2x}(3x + 1)$$

Verified OK.

7.31.5 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 4y = 0, y(0) = 1, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r + 2)^2 = 0$

- Root of the characteristic polynomial

  $r = -2$

- 1st solution of the ODE

$y_1(x) = e^{-2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{-2x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 e^{-2x} + c_2 x\,e^{-2x}$

□ Check validity of solution $y = c_1 e^{-2x} + c_2 x e^{-2x}$

  - Use initial condition $y(0) = 1$

    $1 = c_1$
  - Compute derivative of the solution

    $y' = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x\,e^{-2x}$
  - Use the initial condition $y'\Big|_{\{x=0\}} = 1$

    $1 = -2c_1 + c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = 3\}$
  - Substitute constant values into general solution and simplify

    $y = e^{-2x}(3x + 1)$
- Solution to the IVP

  $y = e^{-2x}(3x + 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=0,y(0) = 1, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-2x}(1 + 3x)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 16

```
DSolve[{y''[x]+4*y'[x]+4*y[x]==0,{y[0]==1,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{-2x}(3x + 1)$$

7.32 problem Exercise 20, problem 33, page 220

Internal problem ID [4603]
Internal file name [`OUTPUT/4096_Sunday_June_05_2022_12_21_45_PM_85154291/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20, problem 33, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 5y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 1]$$

7.32.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 5 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 5y = 0$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.32.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(5)}\\ &= 1 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 2i\\ \lambda_2 &= 1 - 2i\end{aligned}$$

Which simplifies to

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{x}(c_1 \cos(2x) + c_2 \sin(2x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \mathrm{e}^{x}(c_1 \cos(2x) + c_2 \sin(2x)) + \mathrm{e}^{x}(-2c_1 \sin(2x) + 2c_2 \cos(2x))$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

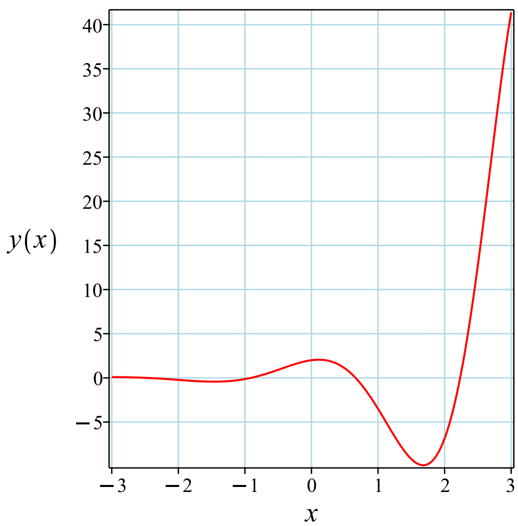
$$c_1 = 2$$
$$c_2 = -\frac{1}{2}$$

Substituting these values back in above solution results in

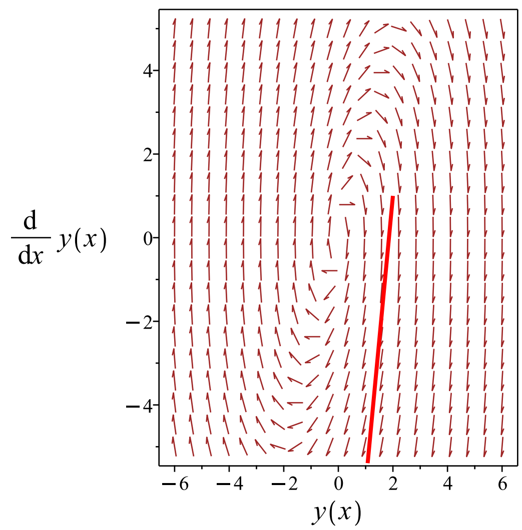
$$y = \frac{e^x(4\cos(2x) - \sin(2x))}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{e^x(4\cos(2x) - \sin(2x))}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^x(4\cos(2x) - \sin(2x))}{2}$$

Verified OK.

7.32.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -2 \\
C &= 5
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 204: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1 (\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = e^x \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2} dx \\ &= y_1 \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x \cos(2x)) + c_2 \left(e^x \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^x \cos(2x) + \frac{c_2 e^x \sin(2x)}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^x \cos(2x) - 2c_1 e^x \sin(2x) + \frac{c_2 e^x \sin(2x)}{2} + c_2 e^x \cos(2x)$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

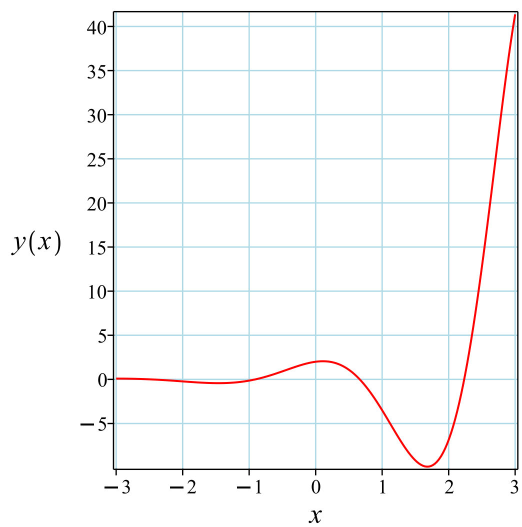
$$c_1 = 2$$
$$c_2 = -1$$

Substituting these values back in above solution results in

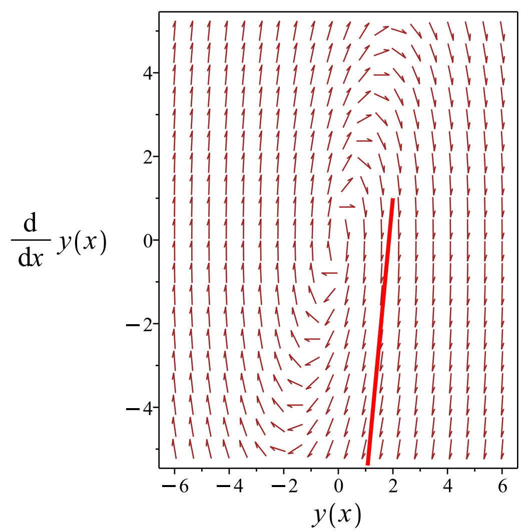
$$y = 2\,e^{x} \cos(2x) - \frac{e^{x} \sin(2x)}{2}$$

Summary
The solution(s) found are the following

$$y = 2\,e^{x} \cos(2x) - \frac{e^{x} \sin(2x)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,e^{x} \cos(2x) - \frac{e^{x} \sin(2x)}{2}$$

Verified OK.

7.32.4 Maple step by step solution

Let's solve

$\left[y''-2y'+5y=0, y(0)=2, y'\Big|_{\{x=0\}}=1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2-2r+5=0$

- Use quadratic formula to solve for r

  $r=\frac{2\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r=(1-2\,\mathrm{I}, 1+2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x)=\mathrm{e}^x\cos(2x)$

- 2nd solution of the ODE

  $y_2(x)=\mathrm{e}^x\sin(2x)$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)$

- Substitute in solutions

  $y=c_1\mathrm{e}^x\cos(2x)+c_2\mathrm{e}^x\sin(2x)$

□ Check validity of solution $y=c_1\mathrm{e}^x\cos(2x)+c_2\mathrm{e}^x\sin(2x)$

  - Use initial condition $y(0)=2$

    $2=c_1$

  - Compute derivative of the solution

    $y'=c_1\mathrm{e}^x\cos(2x)-2c_1\mathrm{e}^x\sin(2x)+c_2\mathrm{e}^x\sin(2x)+2c_2\mathrm{e}^x\cos(2x)$

  - Use the initial condition $y'\Big|_{\{x=0\}}=1$

    $1=c_1+2c_2$

  - Solve for c_1 and c_2

    $\left\{c_1=2, c_2=-\frac{1}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{e^x(4\cos(2x)-\sin(2x))}{2}$

- Solution to the IVP

  $y = \frac{e^x(4\cos(2x)-\sin(2x))}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+5*y(x)=0,y(0) = 2, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{e^x(\sin(2x) - 4\cos(2x))}{2}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 25

```
DSolve[{y''[x]-2*y'[x]+5*y[x]==0,{y[0]==2,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{1}{2}e^x(4\cos(2x) - \sin(2x))$$

7.33 problem Exercise 20, problem 34, page 220

Internal problem ID [4604]
Internal file name [`OUTPUT/4097_Sunday_June_05_2022_12_21_53_PM_7032412/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20, problem 34, page 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' + 20y = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 1, y'\left(\frac{\pi}{2}\right) = 1\right]$$

7.33.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= 20 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 20y = 0$$

The domain of $p(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The domain of $q(x) = 20$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is also inside this domain. Hence solution exists and is unique.

7.33.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 20$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\, \mathrm{e}^{\lambda x} + 20\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 20 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 20$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-4^2 - (4)(1)(20)}\\ &= 2 \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 2 + 4i\\ \lambda_2 &= 2 - 4i\end{aligned}$$

Which simplifies to

$$\lambda_1 = 2 + 4i$$
$$\lambda_2 = 2 - 4i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 2$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{2x}(c_1 \cos(4x) + c_2 \sin(4x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{2x}(c_1 \cos(4x) + c_2 \sin(4x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = \frac{\pi}{2}$ in the above gives

$$1 = c_1 e^{\pi} \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2\,e^{2x}(c_1 \cos(4x) + c_2 \sin(4x)) + e^{2x}(-4c_1 \sin(4x) + 4c_2 \cos(4x))$$

substituting $y' = 1$ and $x = \frac{\pi}{2}$ in the above gives

$$1 = 2(c_1 + 2c_2)\,e^{\pi} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = e^{-\pi}$$
$$c_2 = -\frac{e^{-\pi}}{4}$$

Substituting these values back in above solution results in

$$y = \cos(4x)\,e^{-\pi+2x} - \frac{\sin(4x)\,e^{-\pi+2x}}{4}$$

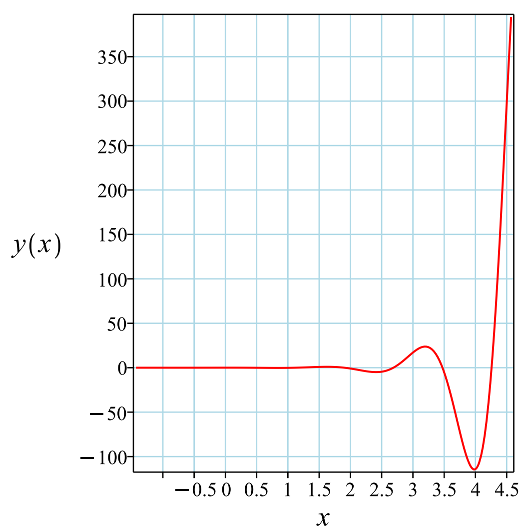
Which simplifies to

$$y = \frac{e^{-\pi+2x}(4\cos(4x) - \sin(4x))}{4}$$

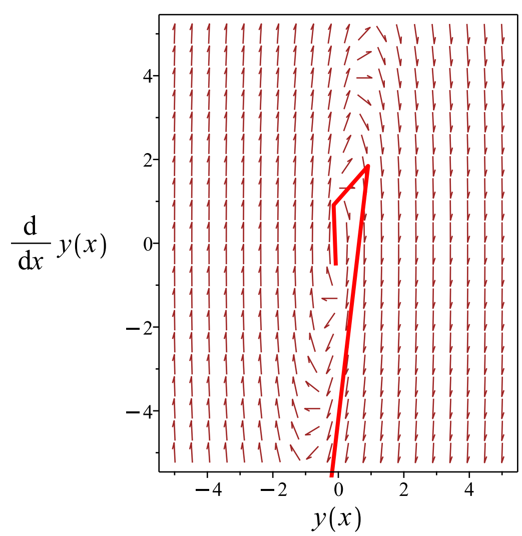
Summary

The solution(s) found are the following

$$y = \frac{e^{-\pi+2x}(4\cos(4x) - \sin(4x))}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-\pi+2x}(4\cos(4x) - \sin(4x))}{4}$$

Verified OK.

7.33.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 20y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -16 z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 206: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(4x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x} \cos(4x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(4x)}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x} \cos(4x)\right) + c_2 \left(\mathrm{e}^{2x} \cos(4x) \left(\frac{\tan(4x)}{4}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{2x} \cos(4x) + \frac{c_2 \mathrm{e}^{2x} \sin(4x)}{4} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = \frac{\pi}{2}$ in the above gives

$$
1 = c_1 \mathrm{e}^{\pi} \tag{1A}
$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} \cos(4x) - 4c_1 e^{2x} \sin(4x) + \frac{c_2 e^{2x} \sin(4x)}{2} + c_2 e^{2x} \cos(4x)$$

substituting $y' = 1$ and $x = \frac{\pi}{2}$ in the above gives

$$1 = (2c_1 + c_2) e^{\pi} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = e^{-\pi}$$
$$c_2 = -e^{-\pi}$$

Substituting these values back in above solution results in

$$y = \cos(4x) e^{-\pi+2x} - \frac{\sin(4x) e^{-\pi+2x}}{4}$$

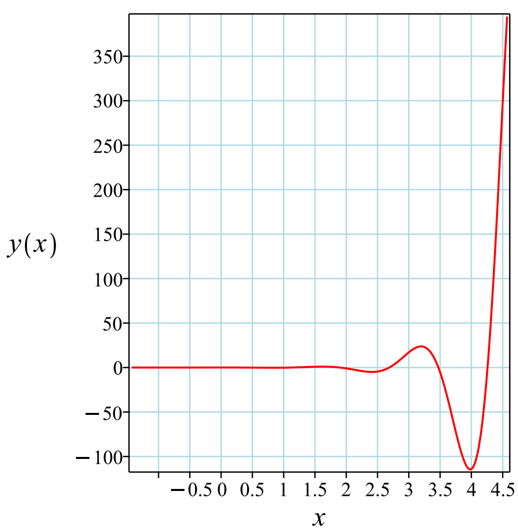
Which simplifies to

$$y = \frac{e^{-\pi+2x}(4\cos(4x) - \sin(4x))}{4}$$

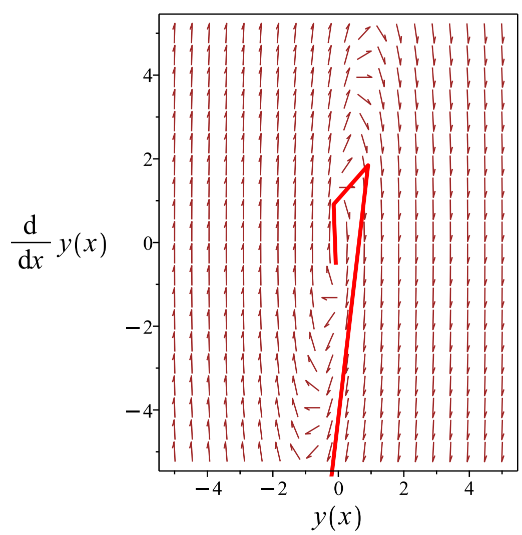
Summary

The solution(s) found are the following

$$y = \frac{e^{-\pi+2x}(4\cos(4x) - \sin(4x))}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-\pi+2x}(4\cos(4x) - \sin(4x))}{4}$$

Verified OK.

7.33.4 Maple step by step solution

Let's solve

$\left[y'' - 4y' + 20y = 0, y\left(\frac{\pi}{2}\right) = 1, y'\Big|_{\left\{x=\frac{\pi}{2}\right\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 4r + 20 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4\pm(\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (2 - 4\,\mathrm{I}, 2 + 4\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = e^{2x}\cos(4x)$

- 2nd solution of the ODE

  $y_2(x) = e^{2x}\sin(4x)$

- General solution of the ODE

  $y = c_1y_1(x) + c_2y_2(x)$

- Substitute in solutions

  $y = c_1e^{2x}\cos(4x) + c_2e^{2x}\sin(4x)$

□ Check validity of solution $y = c_1e^{2x}\cos(4x) + c_2e^{2x}\sin(4x)$

  ◦ Use initial condition $y\left(\frac{\pi}{2}\right) = 1$

    $1 = c_1e^{\pi}$

  ◦ Compute derivative of the solution

    $y' = 2c_1e^{2x}\cos(4x) - 4c_1e^{2x}\sin(4x) + 2c_2e^{2x}\sin(4x) + 4c_2e^{2x}\cos(4x)$

- Use the initial condition $y'\Big|_{\{x=\frac{\pi}{2}\}} = 1$

  $1 = 2c_1 e^{\pi} + 4\,e^{\pi} c_2$
- Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{e^{\pi}}, c_2 = -\frac{1}{4\,e^{\pi}}\right\}$
- Substitute constant values into general solution and simplify

  $y = \frac{e^{-\pi+2x}(4\cos(4x)-\sin(4x))}{4}$

- Solution to the IVP

  $y = \frac{e^{-\pi+2x}(4\cos(4x)-\sin(4x))}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 25

```
dsolve([diff(y(x),x$2)-4*diff(y(x),x)+20*y(x)=0,y(1/2*Pi) = 1, D(y)(1/2*Pi) = 1],y(x), sings
```

$$y(x) = -\frac{(\sin(4x) - 4\cos(4x))\,e^{-\pi+2x}}{4}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 31

```
DSolve[{y''[x]-4*y'[x]+20*y[x]==0,{y[Pi/2]==1,y'[Pi/2]==1}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{4}e^{2x-\pi}(4\cos(4x) - \sin(4x))$$

7.34 problem Exercise 20, problem 35, page 220

Internal problem ID [4605]
Internal file name [`OUTPUT/4098_Sunday_June_05_2022_12_22_02_PM_55432396/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 20. Constant coefficients
Problem number: Exercise 20, problem 35, page 220.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{3y''' + 5y'' + y' - y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1, y''(0) = -1]$$

The characteristic equation is

$$3\lambda^3 + 5\lambda^2 + \lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = \frac{1}{3}$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \mathrm{e}^{\frac{x}{3}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= x\,\mathrm{e}^{-x} \\
y_3 &= \mathrm{e}^{\frac{x}{3}}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \mathrm{e}^{\frac{x}{3}} c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 + c_3 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} c_2 + \frac{\mathrm{e}^{\frac{x}{3}} c_3}{3}
$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$
1 = -c_1 + c_2 + \frac{c_3}{3} \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = c_1 \mathrm{e}^{-x} - 2c_2 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \frac{\mathrm{e}^{\frac{x}{3}} c_3}{9}
$$

substituting $y'' = -1$ and $x = 0$ in the above gives

$$
-1 = c_1 - 2c_2 + \frac{c_3}{9} \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{9}{16} \\
c_2 &= \frac{1}{4} \\
c_3 &= \frac{9}{16}
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = -\frac{9\,\mathrm{e}^{-x}}{16} + \frac{x\,\mathrm{e}^{-x}}{4} + \frac{9\,\mathrm{e}^{\frac{x}{3}}}{16}$$

Which simplifies to

$$y = \frac{\left(9\,\mathrm{e}^{\frac{4x}{3}} + 4x - 9\right)\mathrm{e}^{-x}}{16}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(9\,\mathrm{e}^{\frac{4x}{3}} + 4x - 9\right)\mathrm{e}^{-x}}{16} \tag{1}$$

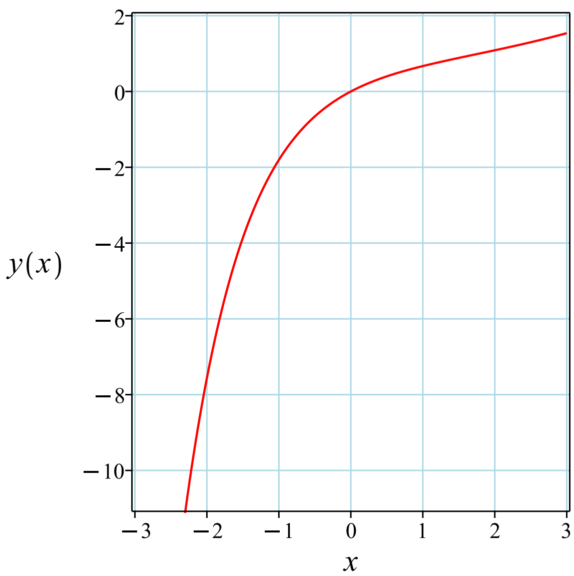


Figure 331: Solution plot

Verification of solutions

$$y = \frac{\left(9\,\mathrm{e}^{\frac{4x}{3}} + 4x - 9\right)\mathrm{e}^{-x}}{16}$$

Verified OK.

7.34.1 Maple step by step solution

Let's solve

$$\left[3y''' + 5y'' + y' - y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 1, y''\Big|_{\{x=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{5y''}{3} - \frac{y'}{3} + \frac{y}{3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{5y''}{3} + \frac{y'}{3} - \frac{y}{3} = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -\frac{5y_3(x)}{3} - \frac{y_2(x)}{3} + \frac{y_1(x)}{3}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -\frac{5y_3(x)}{3} - \frac{y_2(x)}{3} + \frac{y_1(x)}{3}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{5}{3} \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{5}{3} \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[\frac{1}{3}, \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\overrightarrow{y}_1(x) = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\overrightarrow{y}_2(x) = e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = \left(e^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda \mathrm{e}^{\lambda x}\left(x\overrightarrow{v}+\overrightarrow{p}\right)+\mathrm{e}^{\lambda x}\overrightarrow{v}=\mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v}+A\cdot \overrightarrow{p}\right)$

- Simplify equation

  $\lambda \overrightarrow{p}+\overrightarrow{v}=A\cdot \overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I)\cdot \overrightarrow{p}+\overrightarrow{v}=A\cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

  $(A-\lambda \cdot I)\cdot \overrightarrow{p}=\overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

  $$\left(\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{5}{3} \end{array}\right]-(-1)\cdot \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right)\cdot \overrightarrow{p}=\left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p}=\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue -1

  $$\overrightarrow{y}_2(x)=\mathrm{e}^{-x}\cdot \left(x\cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]+\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right)$$

- Consider eigenpair

  $$\left[\frac{1}{3}, \left[\begin{array}{c} 9 \\ 3 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{y}_3=\mathrm{e}^{\frac{x}{3}}\cdot \left[\begin{array}{c} 9 \\ 3 \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

  $\overrightarrow{y} = c_1 \overrightarrow{y}_1(x) + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3$
- Substitute solutions into the general solution

  $\overrightarrow{y} = c_1 \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-x} \cdot \left(x \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + \mathrm{e}^{\frac{x}{3}} c_3 \cdot \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix}$
- First component of the vector is the solution to the ODE

  $y = \left(9\,\mathrm{e}^{\frac{4x}{3}} c_3 + c_2 x + c_1 + c_2\right) \mathrm{e}^{-x}$
- Use the initial condition $y(0) = 0$

  $0 = 9c_3 + c_1 + c_2$
- Calculate the 1st derivative of the solution

  $y' = \left(12\,\mathrm{e}^{\frac{4x}{3}} c_3 + c_2\right) \mathrm{e}^{-x} - \left(9\,\mathrm{e}^{\frac{4x}{3}} c_3 + c_2 x + c_1 + c_2\right) \mathrm{e}^{-x}$
- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = 3c_3 - c_1$
- Calculate the 2nd derivative of the solution

  $y'' = 16\,\mathrm{e}^{\frac{4x}{3}} c_3 \mathrm{e}^{-x} - 2\left(12\,\mathrm{e}^{\frac{4x}{3}} c_3 + c_2\right) \mathrm{e}^{-x} + \left(9\,\mathrm{e}^{\frac{4x}{3}} c_3 + c_2 x + c_1 + c_2\right) \mathrm{e}^{-x}$
- Use the initial condition $y''\Big|_{\{x=0\}} = -1$

  $-1 = c_3 - c_2 + c_1$
- Solve for the unknown coefficients

  $\left\{c_1 = -\frac{13}{16}, c_2 = \frac{1}{4}, c_3 = \frac{1}{16}\right\}$
- Solution to the IVP

  $y = \frac{\left(9\,\mathrm{e}^{\frac{4x}{3}} + 4x - 9\right) \mathrm{e}^{-x}}{16}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([3*diff(y(x),x$3)+5*diff(y(x),x$2)+diff(y(x),x)-y(x)=0,y(0) = 0, D(y)(0) = 1, (D@@2)(
```

$$y(x) = \frac{\left(9\,\mathrm{e}^{\frac{4x}{3}} + 4x - 9\right)\mathrm{e}^{-x}}{16}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 28

```
DSolve[{3*y'''[x]+5*y''[x]+y'[x]-y[x]==0,{y[0]==0,y'[0]==1,y''[0]==-1}},y[x],x,IncludeSingul
```

$$y(x) \to \frac{1}{16}e^{-x}\left(4x + 9e^{4x/3} - 9\right)$$

8 Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients

8.1 problem Exercise 21.3, page 231

Internal problem ID [4606]
Internal file name [`OUTPUT/4099_Sunday_June_05_2022_12_22_11_PM_72331572/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.3, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 3y' + 2y = 4}$$

8.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = 4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 3\lambda\,\mathrm{e}^{\lambda x} + 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)} \\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x}\right) + (2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} + 2 \tag{1}$$

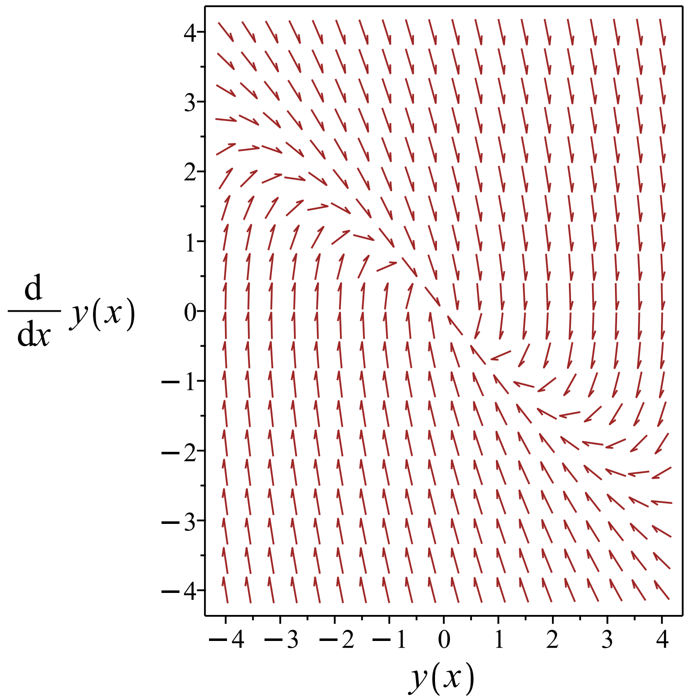


Figure 332: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} + 2$$

Verified OK.

8.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 209: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(\mathrm{e}^{x})\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{-2x}, \mathrm{e}^{-x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_1 = 4
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{-x}\right) + (2) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{-x} + 2 \tag{1}$$

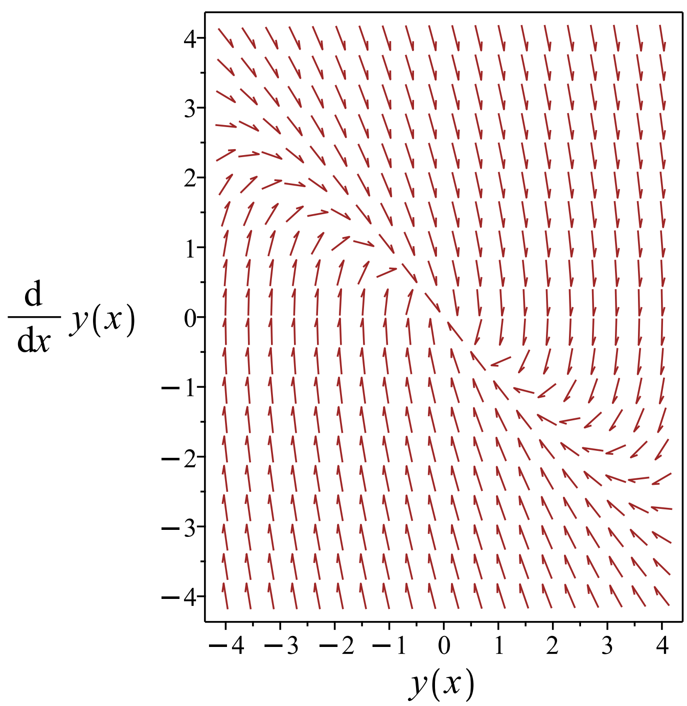


Figure 333: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-2x} + c_2 e^{-x} + 2$$

Verified OK.

8.1.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = 4$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4\right]$

- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{bmatrix}$

- ∘ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-3x}$

- ∘ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -4\,e^{-2x}\left(\int e^{2x} dx\right) + 4\,e^{-x}\left(\int e^{x} dx\right)$

- Compute integrals

  $y_p(x) = 2$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + 2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=4,y(x), singsol=all)
```

$$y(x) = -e^{-2x} c_1 + c_2 e^{-x} + 2$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 23

```
DSolve[y''[x]+3*y'[x]+2*y[x]==4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-2x} + c_2 e^{-x} + 2$$

8.2 problem Exercise 21.4, page 231

Internal problem ID [4607]
Internal file name [`OUTPUT/4100_Sunday_June_05_2022_12_22_19_PM_72302744/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.4, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 3y' + 2y = 12\,\mathrm{e}^{x}}$$

8.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = 12\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)} \\
&= -\frac{3}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{3}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -1 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}
\end{aligned}$$

Or

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12\,e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 e^{x} = 12\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,e^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left(2\,e^{x}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} + 2\,e^{x} \tag{1}$$

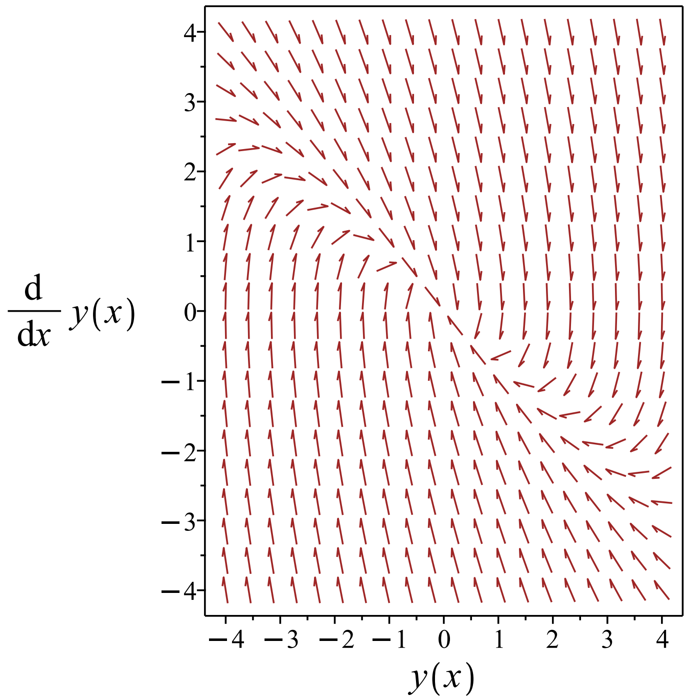


Figure 334: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} + 2\,e^{x}$$

Verified OK.

8.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 211: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} (\mathrm{e}^{x})\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x}, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 \mathrm{e}^{x} = 12\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,e^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{-x}\right) + \left(2\,e^{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{-x} + 2\,e^{x} \tag{1}$$

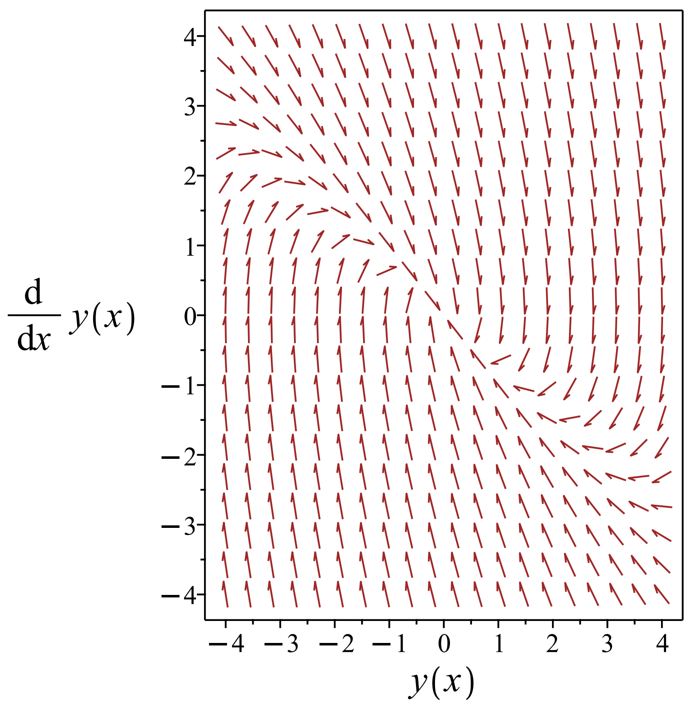


Figure 335: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-x} + 2\,e^{x}$$

Verified OK.

8.2.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = 12\,\mathrm{e}^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)\,(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 12\,\mathrm{e}^{x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-x} \\ -2\,\mathrm{e}^{-2x} & -\mathrm{e}^{-x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-3x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -12\,\mathrm{e}^{-2x}\left(\int \mathrm{e}^{3x}dx\right) + 12\,\mathrm{e}^{-x}\left(\int \mathrm{e}^{2x}dx\right)$

- Compute integrals

  $y_p(x) = 2\,\mathrm{e}^{x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + 2\,\mathrm{e}^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=12*exp(x),y(x), singsol=all)
```

$$y(x) = -\left(-2\,\mathrm{e}^{3x} - c_2 \mathrm{e}^{x} + c_1\right) \mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 27

```
DSolve[y''[x]+3*y'[x]+2*y[x]==12*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(2e^{3x} + c_2 e^{x} + c_1\right)$$

8.3 problem Exercise 21.5, page 231

Internal problem ID [4608]
Internal file name [`OUTPUT/4101_Sunday_June_05_2022_12_22_27_PM_96402999/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.5, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 3y' + 2y = e^{ix}}$$

8.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = e^{ix}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{3^2 - (4)(1)(2)} \\
&= -\frac{3}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{3}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -1 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}
\end{aligned}$$

Or

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-x}$$
$$y_2 = e^{-2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ \frac{d}{dx}(e^{-x}) & \frac{d}{dx}(e^{-2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2\,e^{-2x} \end{vmatrix}$$

Therefore

$$W = \left(e^{-x}\right)\left(-2\,e^{-2x}\right) - \left(e^{-2x}\right)\left(-e^{-x}\right)$$

Which simplifies to

$$W = -e^{-2x} e^{-x}$$

Which simplifies to

$$W = -\mathrm{e}^{-3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{-2x}\mathrm{e}^{ix}}{-\mathrm{e}^{-3x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\mathrm{e}^{(1+i)x}dx$$

Hence

$$u_1 = \left(\frac{1}{2} - \frac{i}{2}\right)\mathrm{e}^{(1+i)x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-x}\mathrm{e}^{ix}}{-\mathrm{e}^{-3x}}\,dx$$

Which simplifies to

$$u_2 = \int -\mathrm{e}^{(2+i)x}dx$$

Hence

$$u_2 = \left(-\frac{2}{5} + \frac{i}{5}\right)\mathrm{e}^{(2+i)x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{1}{2} - \frac{i}{2}\right)\mathrm{e}^{(1+i)x}\mathrm{e}^{-x} + \left(-\frac{2}{5} + \frac{i}{5}\right)\mathrm{e}^{(2+i)x}\mathrm{e}^{-2x}$$

Which simplifies to

$$y_p(x) = \left(\frac{1}{10} - \frac{3i}{10}\right)\mathrm{e}^{ix}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 e^{-x} + c_2 e^{-2x}) + \left(\left(\frac{1}{10} - \frac{3i}{10}\right) e^{ix}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-x} + c_2 e^{-2x} + \left(\frac{1}{10} - \frac{3i}{10}\right) e^{ix} \tag{1}
$$

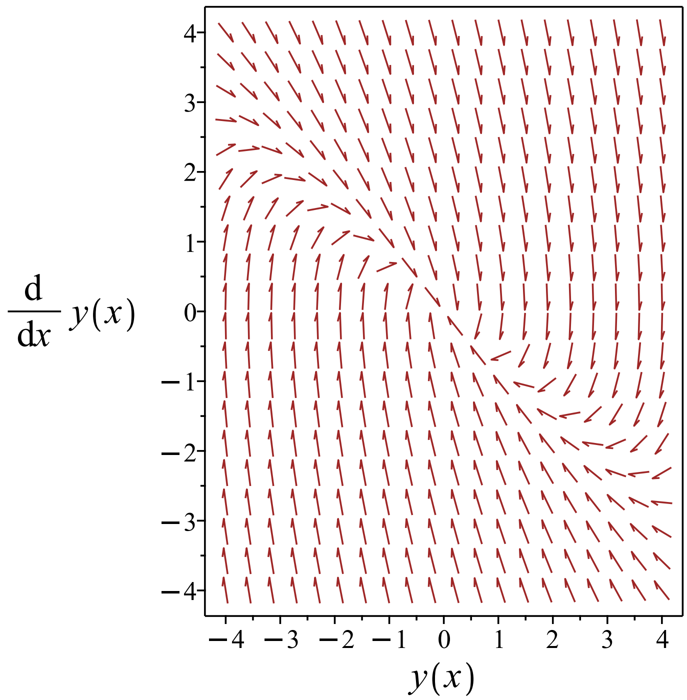


Figure 336: Slope field plot

Verification of solutions

$$
y = c_1 e^{-x} + c_2 e^{-2x} + \left(\frac{1}{10} - \frac{3i}{10}\right) e^{ix}
$$

Verified OK.

8.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 213: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3}{1} \, dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\, dx \\
&= y_1(\mathrm{e}^x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(\mathrm{e}^x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-2x}$$
$$y_2 = e^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-2x} & e^{-x} \\ \frac{d}{dx}(e^{-2x}) & \frac{d}{dx}(e^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{vmatrix}$$

Therefore

$$W = \left(e^{-2x}\right)\left(-e^{-x}\right) - \left(e^{-x}\right)\left(-2\,e^{-2x}\right)$$

Which simplifies to

$$W = e^{-2x} e^{-x}$$

Which simplifies to

$$W = e^{-3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-x} e^{ix}}{e^{-3x}} \, dx$$

Which simplifies to

$$u_1 = -\int e^{(2+i)x} dx$$

Hence

$$u_1 = \left(-\frac{2}{5} + \frac{i}{5}\right) e^{(2+i)x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-2x} e^{ix}}{e^{-3x}} \, dx$$

Which simplifies to

$$u_2 = \int e^{(1+i)x} dx$$

Hence

$$u_2 = \left(\frac{1}{2} - \frac{i}{2}\right) e^{(1+i)x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{1}{2} - \frac{i}{2}\right) e^{(1+i)x} e^{-x} + \left(-\frac{2}{5} + \frac{i}{5}\right) e^{(2+i)x} e^{-2x}$$

Which simplifies to

$$y_p(x) = \left(\frac{1}{10} - \frac{3i}{10}\right) e^{ix}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-2x} + c_2 e^{-x}\right) + \left(\left(\frac{1}{10} - \frac{3i}{10}\right) e^{ix}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-2x} + c_2 e^{-x} + \left(\frac{1}{10} - \frac{3i}{10}\right) e^{ix} \tag{1}
$$

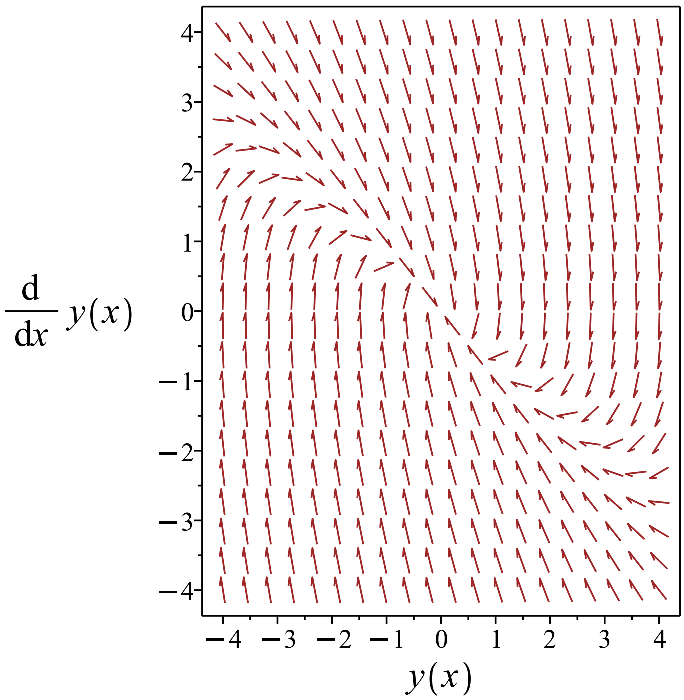


Figure 337: Slope field plot

Verification of solutions

$$
y = c_1 e^{-2x} + c_2 e^{-x} + \left(\frac{1}{10} - \frac{3i}{10}\right) e^{ix}
$$

Verified OK.

8.3.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = e^{Ix}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{Ix}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-3x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{-2x}\left(\int e^{(2+I)x} dx\right) + e^{-x}\left(\int e^{(1+I)x} dx\right)$

- Compute integrals

  $y_p(x) = \left(\frac{1}{10} - \frac{3\,\mathrm{I}}{10}\right) \mathrm{e}^{\mathrm{I}x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + \left(\frac{1}{10} - \frac{3\,\mathrm{I}}{10}\right) \mathrm{e}^{\mathrm{I}x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=exp(I*x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x}\left(\left(\frac{1}{10} - \frac{3i}{10}\right) \mathrm{e}^{(1+i)x} - \mathrm{e}^{-x} c_1 + c_2\right)$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 37

```
DSolve[y''[x]+3*y'[x]+2*y[x]==Exp[I*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\frac{1}{10} - \frac{3i}{10}\right) e^{ix} + c_1 e^{-2x} + c_2 e^{-x}$$

8.4 problem Exercise 21.6, page 231

Internal problem ID [4609]
Internal file name [`OUTPUT/4102_Sunday_June_05_2022_12_22_35_PM_59727082/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.6, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = \sin(x)}$$

8.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)} \\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \cos(x) + A_2 \sin(x) - 3A_1 \sin(x) + 3A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{10}, A_2 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(x)}{10} + \frac{\sin(x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left(-\frac{3\cos(x)}{10} + \frac{\sin(x)}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{3\cos(x)}{10} + \frac{\sin(x)}{10} \tag{1}$$

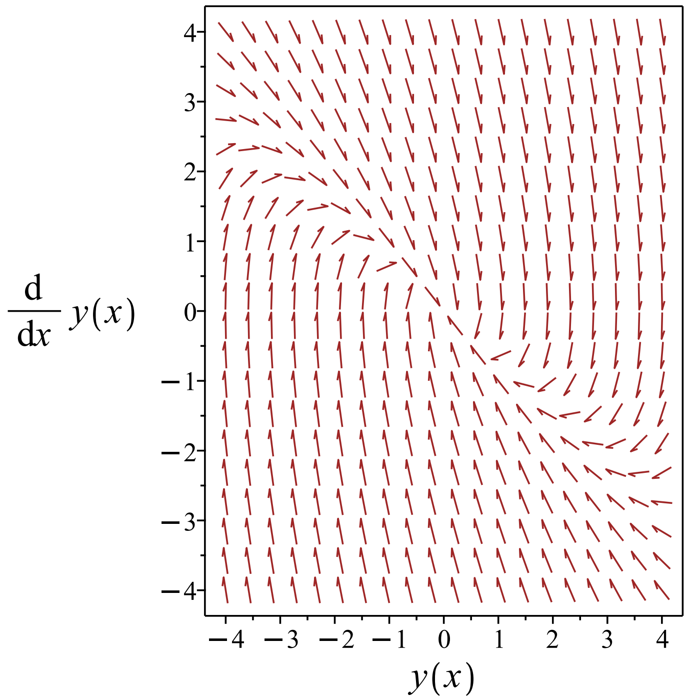


Figure 338: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{3\cos(x)}{10} + \frac{\sin(x)}{10}$$

Verified OK.

8.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 215: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\mathrm{e}^{x}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-2x}, \mathrm{e}^{-x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(x) + A_2 \sin(x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
A_1 \cos(x) + A_2 \sin(x) - 3A_1 \sin(x) + 3A_2 \cos(x) = \sin(x)
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{10}, A_2 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(x)}{10} + \frac{\sin(x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{-x}\right) + \left(-\frac{3\cos(x)}{10} + \frac{\sin(x)}{10}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{-x} - \frac{3\cos(x)}{10} + \frac{\sin(x)}{10} \tag{1}$$

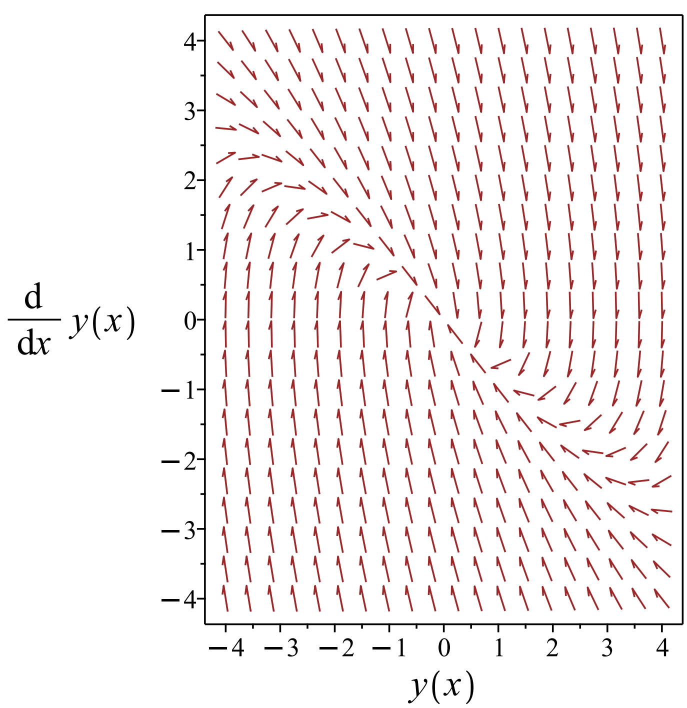


Figure 339: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-x} - \frac{3\cos(x)}{10} + \frac{\sin(x)}{10}$$

Verified OK.

8.4.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-3x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{-2x}\left(\int \sin(x) e^{2x} dx\right) + e^{-x}\left(\int \sin(x) e^{x} dx\right)$
- Compute integrals

  $y_p(x) = -\frac{3\cos(x)}{10} + \frac{\sin(x)}{10}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} - \frac{3\cos(x)}{10} + \frac{\sin(x)}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=sin(x),y(x), singsol=all)
```

$$y(x) = -e^{-2x}c_1 - \frac{3\cos(x)}{10} + \frac{\sin(x)}{10} + c_2 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 32

```
DSolve[y''[x]+3*y'[x]+2*y[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{10}\left(\sin(x) - 3\cos(x) + 10e^{-2x}(c_2e^x + c_1)\right)$$

8.5 problem Exercise 21.7, page 231

Internal problem ID [4610]
Internal file name [`OUTPUT/4103_Sunday_June_05_2022_12_22_43_PM_72801291/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.7, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = \cos(x)}$$

8.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \cos(x) + A_2 \sin(x) - 3A_1 \sin(x) + 3A_2 \cos(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{10}, A_2 = \frac{3}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)}{10} + \frac{3\sin(x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left(\frac{\cos(x)}{10} + \frac{3\sin(x)}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{\cos(x)}{10} + \frac{3\sin(x)}{10} \tag{1}$$

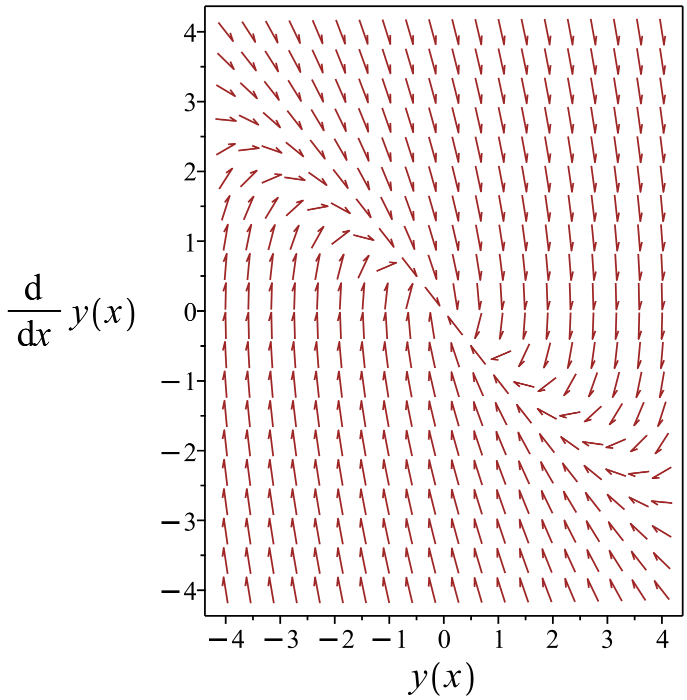


Figure 340: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{\cos(x)}{10} + \frac{3\sin(x)}{10}$$

Verified OK.

8.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 217: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\mathrm{e}^{x}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\cos(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-2x}, \mathrm{e}^{-x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1\cos(x) + A_2\sin(x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
A_1\cos(x) + A_2\sin(x) - 3A_1\sin(x) + 3A_2\cos(x) = \cos(x)
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{10}, A_2 = \frac{3}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)}{10} + \frac{3\sin(x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}\right) + \left(\frac{\cos(x)}{10} + \frac{3\sin(x)}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + \frac{\cos(x)}{10} + \frac{3\sin(x)}{10} \tag{1}$$

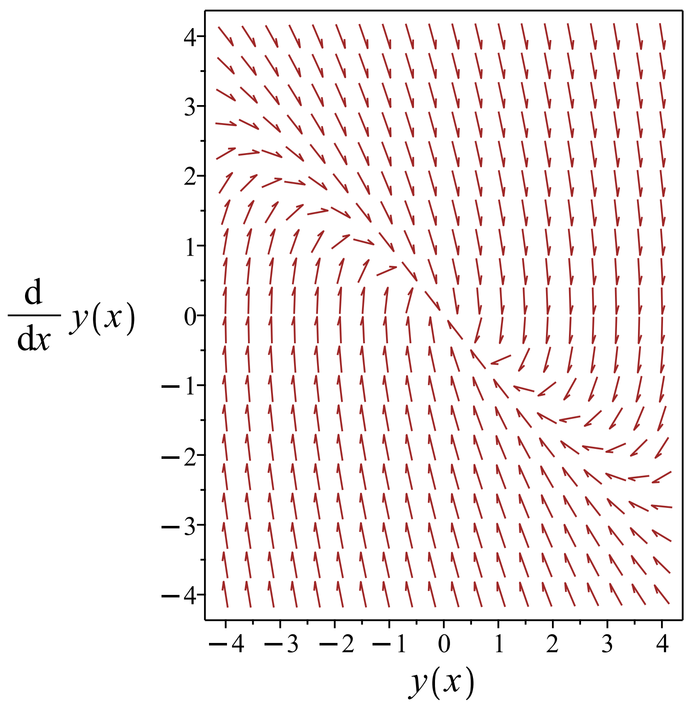


Figure 341: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-x} + \frac{\cos(x)}{10} + \frac{3\sin(x)}{10}$$

Verified OK.

8.5.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = \cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \cos(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x)\,,y_2(x))=\begin{bmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x)\,,y_2(x))=e^{-3x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x)=-e^{-2x}\left(\int\cos(x)\,e^{2x}dx\right)+e^{-x}\left(\int\cos(x)\,e^{x}dx\right)$

- Compute integrals

  $y_p(x)=\frac{\cos(x)}{10}+\frac{3\sin(x)}{10}$

- Substitute particular solution into general solution to ODE

  $y=c_1e^{-2x}+c_2e^{-x}+\frac{\cos(x)}{10}+\frac{3\sin(x)}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=cos(x),y(x), singsol=all)
```

$$y(x)=-e^{-2x}c_1+\frac{\cos(x)}{10}+\frac{3\sin(x)}{10}+c_2e^{-x}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 32

```
DSolve[y''[x]+3*y'[x]+2*y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{10}\left(3\sin(x) + \cos(x) + 10e^{-2x}(c_2e^x + c_1)\right)$$

8.6 problem Exercise 21.8, page 231

Internal problem ID [4611]
Internal file name [`OUTPUT/4104_Sunday_June_05_2022_12_22_51_PM_97957522/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.8, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = 8 + 6\,e^{x} + 2\sin(x)}$$

8.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = 8 + 6\,e^{x} + 2\sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 3\lambda\, \mathrm{e}^{\lambda x} + 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)} \\
&= -\frac{3}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{3}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -1 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8 + 6\,\mathrm{e}^{x} + 2\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\mathrm{e}^{x}\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x}, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2\mathrm{e}^{x} + A_3\cos(x) + A_4\sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_2\mathrm{e}^{x} + A_3\cos(x) + A_4\sin(x) - 3A_3\sin(x) + 3A_4\cos(x) + 2A_1 = 8 + 6\,\mathrm{e}^{x} + 2\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 4, A_2 = 1, A_3 = -\frac{3}{5}, A_4 = \frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4 + \mathrm{e}^{x} - \frac{3\cos(x)}{5} + \frac{\sin(x)}{5}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{-2x}\right) + \left(4 + \mathrm{e}^{x} - \frac{3\cos(x)}{5} + \frac{\sin(x)}{5}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} + 4 + e^{x} - \frac{3\cos(x)}{5} + \frac{\sin(x)}{5} \tag{1}$$

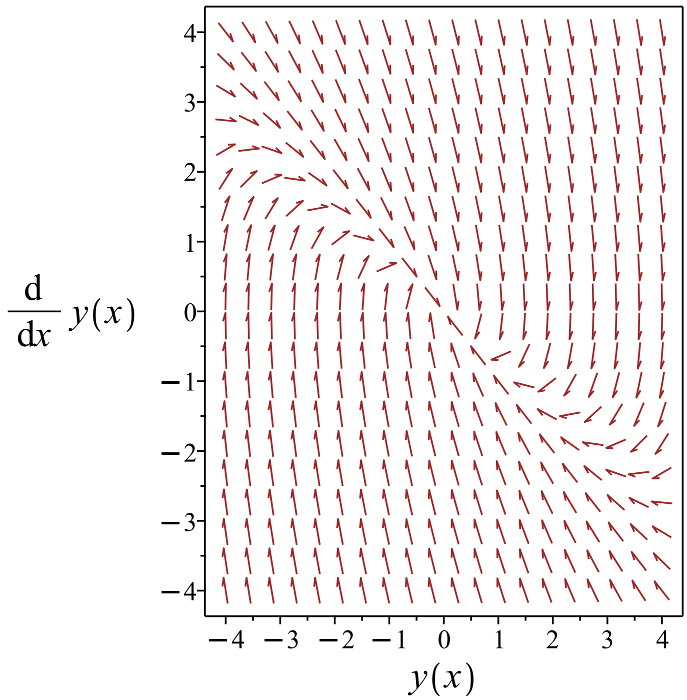


Figure 342: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} + 4 + e^{x} - \frac{3\cos(x)}{5} + \frac{\sin(x)}{5}$$

Verified OK.

8.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 3 \\
C &= 2
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 219: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2} \, dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x}(\mathrm{e}^{x})\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8 + 6\,\mathrm{e}^{x} + 2\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^{x}\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 e^{x} + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_2 e^{x} + A_3 \cos(x) + A_4 \sin(x) - 3A_3 \sin(x) + 3A_4 \cos(x) + 2A_1 = 8 + 6\,e^{x} + 2\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 4, A_2 = 1, A_3 = -\frac{3}{5}, A_4 = \frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4 + e^{x} - \frac{3\cos(x)}{5} + \frac{\sin(x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{-x}\right) + \left(4 + e^{x} - \frac{3\cos(x)}{5} + \frac{\sin(x)}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{-x} + 4 + e^{x} - \frac{3\cos(x)}{5} + \frac{\sin(x)}{5} \tag{1}$$

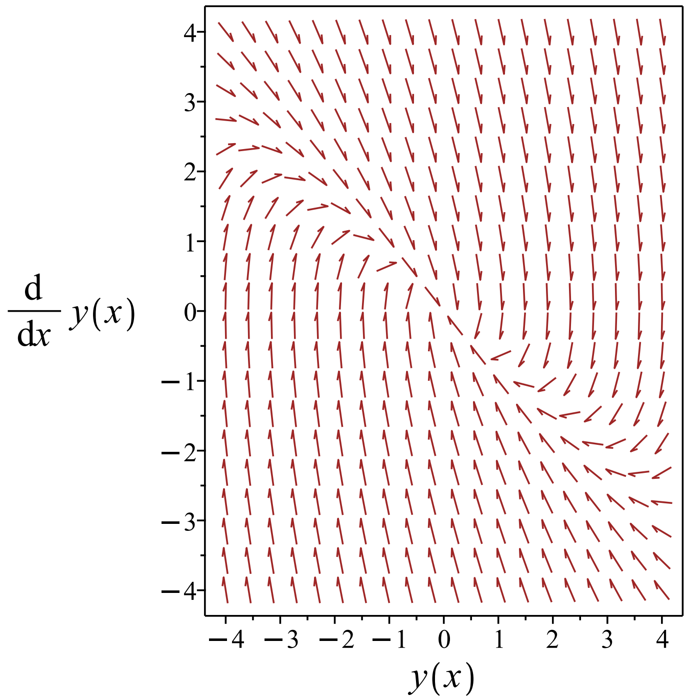


Figure 343: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-x} + 4 + e^x - \frac{3\cos(x)}{5} + \frac{\sin(x)}{5}$$

Verified OK.

8.6.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = 8 + 6\,e^x + 2\sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

$r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 8 + 6\,e^x + 2\sin(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{-3x}$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\,e^{-2x}\left(\int (4 + 3\,e^x + \sin(x))\,e^{2x}dx\right) + 2\,e^{-x}\left(\int (4 + 3\,e^x + \sin(x))\,e^x dx\right)$

  ◦ Compute integrals

    $y_p(x) = 4 + e^x - \frac{3\cos(x)}{5} + \frac{\sin(x)}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + 4 + e^x - \frac{3\cos(x)}{5} + \frac{\sin(x)}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=8+6*exp(x)+2*sin(x),y(x), singsol=all)
```

$$y(x) = -e^{-2x}\left(\left(-4+\frac{3\cos(x)}{5}-\frac{\sin(x)}{5}\right)e^{2x}-c_2e^{x}+c_1-e^{3x}\right)$$

✓ Solution by Mathematica

Time used: 0.165 (sec). Leaf size: 38

```
DSolve[y''[x]+3*y'[x]+2*y[x]==8+6*Exp[x]+2*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x + \frac{\sin(x)}{5} - \frac{3\cos(x)}{5} + c_1e^{-2x} + c_2e^{-x} + 4$$

8.7 problem Exercise 21.9, page 231

Internal problem ID [4612]
Internal file name [`OUTPUT/4105_Sunday_June_05_2022_12_23_00_PM_11199953/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.9, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' + y = x^2}$$

8.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda \, \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)} \sqrt{1^2 - (4)\,(1)\,(1)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_3 x^2 + A_2 x + 2x A_3 + A_1 + A_2 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -2, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 - 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)\right) + (x^2 - 2x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) + x^2 - 2x \tag{1}$$

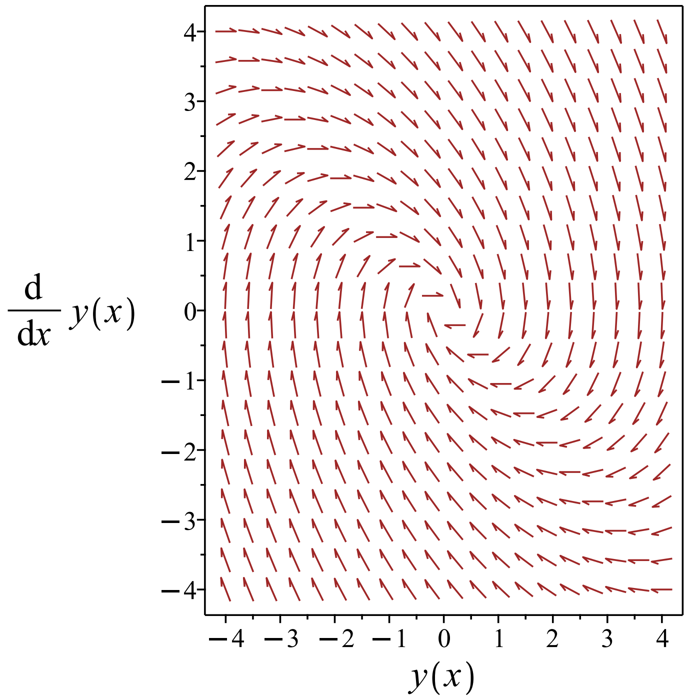


Figure 344: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) + x^2 - 2x$$

Verified OK.

8.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 221: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) c_1 + \frac{2c_2\mathrm{e}^{-\frac{x}{2}}\sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), \frac{2\,\mathrm{e}^{-\frac{x}{2}}\sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_3x^2 + A_2x + 2xA_3 + A_1 + A_2 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -2, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 - 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_1 + \frac{2c_2 \mathrm{e}^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right) + (x^2 - 2x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_1 + \frac{2c_2 \mathrm{e}^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + x^2 - 2x \tag{1}$$

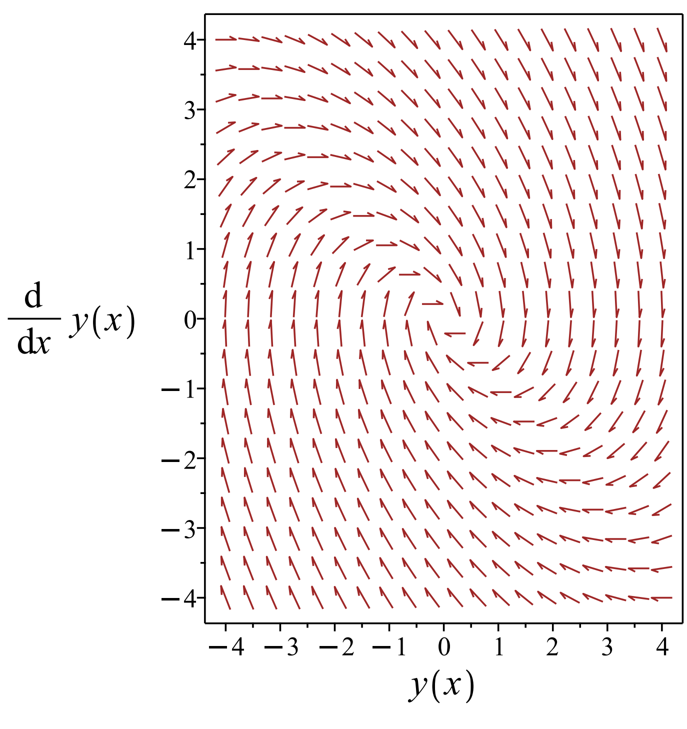


Figure 345: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) c_1 + \frac{2c_2 e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} + x^2 - 2x$$

Verified OK.

8.7.3 Maple step by step solution

Let's solve

$y'' + y' + y = x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) c_1 + e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}}\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,e^{-x}}{2}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2\,e^{-\frac{x}{2}}\sqrt{3}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int x^2 e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)dx\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int x^2 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)\right)}{3}$
- Compute integrals

  $y_p(x) = x(-2 + x)$

- Substitute particular solution into general solution to ODE

  $y = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_1 + e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) c_2 + x(-2 + x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=x^2,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) c_2 + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_1 + x^2 - 2x$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 54

```
DSolve[y''[x]+y'[x]+y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}\left(e^{x/2}(x-2)x + c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 \sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

8.8 problem Exercise 21.10, page 231

Internal problem ID [4613]
Internal file name [`OUTPUT/4106_Sunday_June_05_2022_12_23_08_PM_69846250/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.10, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' - 8y = 9\,\mathrm{e}^{x} x + 10\,\mathrm{e}^{-x}}$$

8.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = -8, f(x) = 9\,\mathrm{e}^{x} x + 10\,\mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' - 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -8$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\,\mathrm{e}^{\lambda x} - 8\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(-8)}\\ &= 1 \pm 3\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 3\\ \lambda_2 &= 1 - 3\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(4)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$9\,e^{x} x + 10\,e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}, \{e^{x} x, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{4x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-x} + A_2 e^{x} x + A_3 e^{x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 e^{-x} - 9A_2 e^{x} x - 9A_3 e^{x} = 9\,e^{x} x + 10\,e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = -1, A_3 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2\,e^{-x} - e^{x} x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{4x} + c_2 e^{-2x}\right) + \left(-2\,e^{-x} - e^{x} x\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 e^{4x} + c_2 e^{-2x} - 2\,e^{-x} - e^{x} x \tag{1}$$

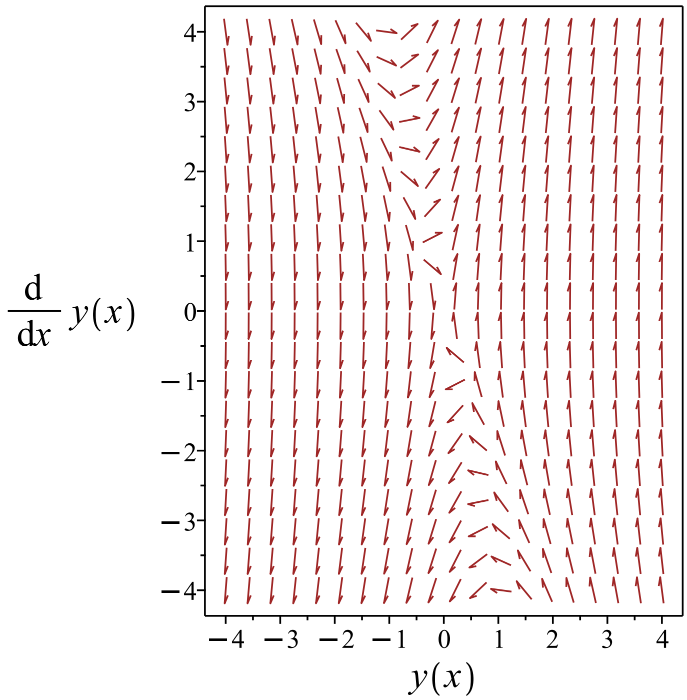


Figure 346: Slope field plot

Verification of solutions

$$y = c_1 e^{4x} + c_2 e^{-2x} - 2\,e^{-x} - e^{x} x$$

Verified OK.

8.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 223: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-3x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1(\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{6x}}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1y_1 + c_2y_2\\ &= c_1\left(e^{-2x}\right) + c_2\left(e^{-2x}\left(\frac{e^{6x}}{6}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' - 8y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1e^{-2x} + \frac{c_2e^{4x}}{6}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$9\,e^{x}x + 10\,e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}, \{e^{x}x, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{4x}}{6}, e^{-2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1e^{-x} + A_2e^{x}x + A_3e^{x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 e^{-x} - 9A_2 e^{x} x - 9A_3 e^{x} = 9\, e^{x} x + 10\, e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = -1, A_3 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2\, e^{-x} - e^{x} x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{4x}}{6}\right) + \left(-2\, e^{-x} - e^{x} x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{4x}}{6} - 2\, e^{-x} - e^{x} x \tag{1}$$

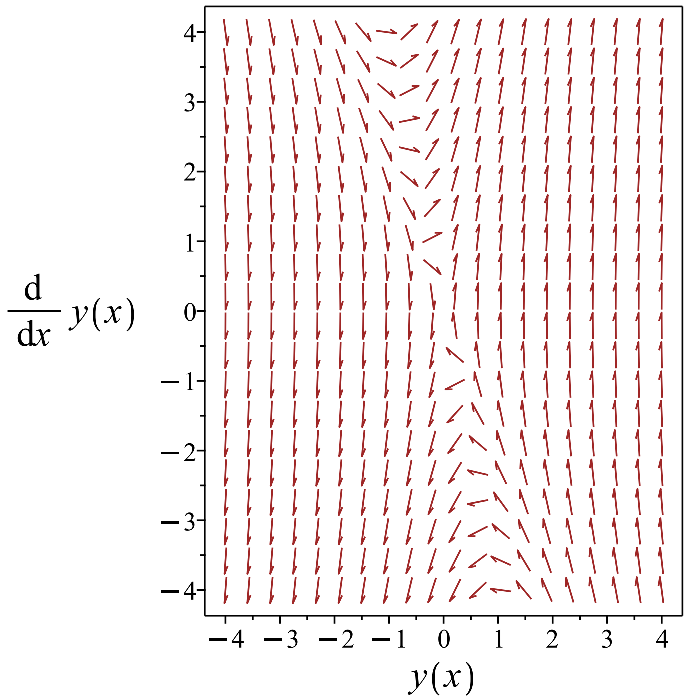


Figure 347: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-2x} + \frac{c_2 e^{4x}}{6} - 2\,e^{-x} - e^{x} x$$

Verified OK.

8.8.3 Maple step by step solution

Let's solve

$y'' - 2y' - 8y = 9\,e^{x} x + 10\,e^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r - 8 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-4) = 0$

- Roots of the characteristic polynomial

$r = (-2, 4)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{4x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 9\,e^{x}x + 10\,e^{-x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{4x} \\ -2\,e^{-2x} & 4\,e^{4x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 6\,e^{2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{e^{-2x}\left(\int (9\,e^{3x}x + 10\,e^{x})dx\right)}{6} + \frac{e^{4x}\left(\int (9\,e^{2x}x + 10)e^{-5x}dx\right)}{6}$

  ◦ Compute integrals

  $y_p(x) = -2\,e^{-x} - e^{x}x$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{4x} - 2\,e^{-x} - e^{x}x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-8*y(x)=9*x*exp(x)+10*exp(-x),y(x), singsol=all)
```

$$y(x) = \left(e^{6x} c_1 - e^{3x} x - 2\,e^{x} + c_2\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 35

```
DSolve[y''[x]-2*y'[x]-8*y[x]==9*x*Exp[x]+10*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(-e^{3x} x - 2e^{x} + c_2 e^{6x} + c_1\right)$$

8.9 problem Exercise 21.11, page 231

Internal problem ID [4614]
Internal file name [`OUTPUT/4107_Sunday_June_05_2022_12_23_16_PM_14616558/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.11, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - 3y' = 2\sin(x)\,\mathrm{e}^{2x}}$$

8.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 0, f(x) = 2\sin(x)\,\mathrm{e}^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-3y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=-3, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}-3\lambda\,\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2-3\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=-3, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{3}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{-3^2-(4)\,(1)\,(0)}\\&=\frac{3}{2}\pm\frac{3}{2}\end{aligned}$$

Hence

$$\lambda_1=\frac{3}{2}+\frac{3}{2}$$
$$\lambda_2=\frac{3}{2}-\frac{3}{2}$$

Which simplifies to

$$\lambda_1=3$$
$$\lambda_2=0$$

Since roots are real and distinct, then the solution is

$$y=c_1e^{\lambda_1x}+c_2e^{\lambda_2x}$$
$$y=c_1e^{(3)x}+c_2e^{(0)x}$$

Or

$$y = e^{3x}c_1 + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = e^{3x}c_1 + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\sin(x)e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)e^{2x}, \sin(x)e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(x)e^{2x} + A_2\sin(x)e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1\cos(x)e^{2x} - A_1\sin(x)e^{2x} - 3A_2\sin(x)e^{2x} + A_2\cos(x)e^{2x} = 2\sin(x)e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{5}, A_2 = -\frac{3}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)e^{2x}}{5} - \frac{3\sin(x)e^{2x}}{5}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{3x} c_1 + c_2\right) + \left(-\frac{\cos(x)\, e^{2x}}{5} - \frac{3 \sin(x)\, e^{2x}}{5}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = e^{3x} c_1 + c_2 - \frac{\cos(x)\, e^{2x}}{5} - \frac{3 \sin(x)\, e^{2x}}{5} \tag{1}
$$

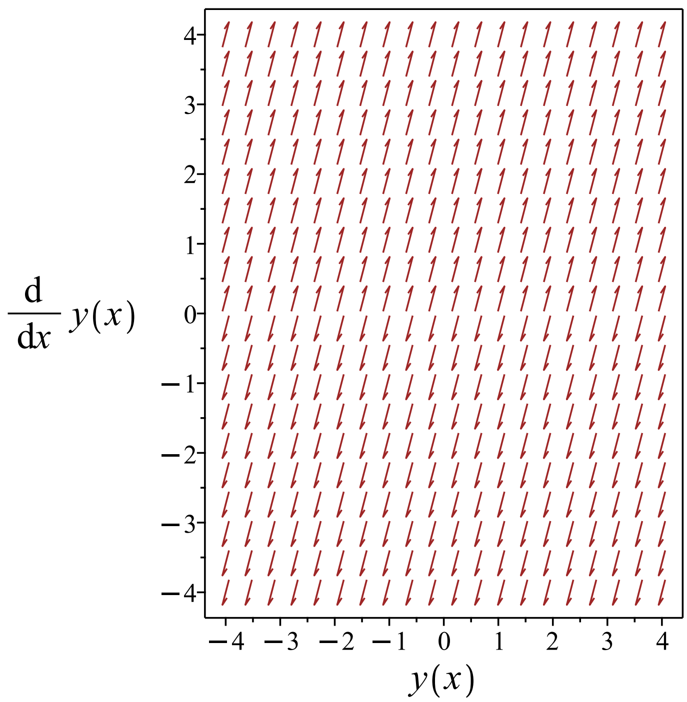


Figure 348: Slope field plot

Verification of solutions

$$
y = e^{3x} c_1 + c_2 - \frac{\cos(x)\, e^{2x}}{5} - \frac{3 \sin(x)\, e^{2x}}{5}
$$

Verified OK.

8.9.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 3y')\, dx = \int 2\sin(x)\, e^{2x} dx$$

$$-3y + y' = -\frac{2\cos(x)\, e^{2x}}{5} + \frac{4\sin(x)\, e^{2x}}{5} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= \frac{(4\sin(x) - 2\cos(x))\, e^{2x}}{5} + c_1 \end{aligned}$$

Hence the ode is

$$-3y + y' = \frac{(4\sin(x) - 2\cos(x))\, e^{2x}}{5} + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-3)dx} \\ &= e^{-3x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{(4\sin(x) - 2\cos(x))\, e^{2x}}{5} + c_1\right) \\ \frac{d}{dx}\left(e^{-3x} y\right) &= \left(e^{-3x}\right)\left(\frac{(4\sin(x) - 2\cos(x))\, e^{2x}}{5} + c_1\right) \\ d\left(e^{-3x} y\right) &= \left(\frac{(4\sin(x) - 2\cos(x))\, e^{-3x} e^{2x}}{5} + c_1 e^{-3x}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-3x} y &= \int \frac{(4\sin(x) - 2\cos(x))\, e^{-3x} e^{2x}}{5} + c_1 e^{-3x}\, dx \\ e^{-3x} y &= -\frac{e^{-x}\cos(x)}{5} - \frac{3\, e^{-x}\sin(x)}{5} - \frac{c_1 e^{-3x}}{3} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-3x}$ results in

$$y = e^{3x}\left(-\frac{e^{-x}\cos(x)}{5} - \frac{3\,e^{-x}\sin(x)}{5} - \frac{c_1 e^{-3x}}{3}\right) + c_2 e^{3x}$$

which simplifies to

$$y = \frac{(-\cos(x) - 3\sin(x))\,e^{2x}}{5} + c_2 e^{3x} - \frac{c_1}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{(-\cos(x) - 3\sin(x))\,e^{2x}}{5} + c_2 e^{3x} - \frac{c_1}{3} \tag{1}$$

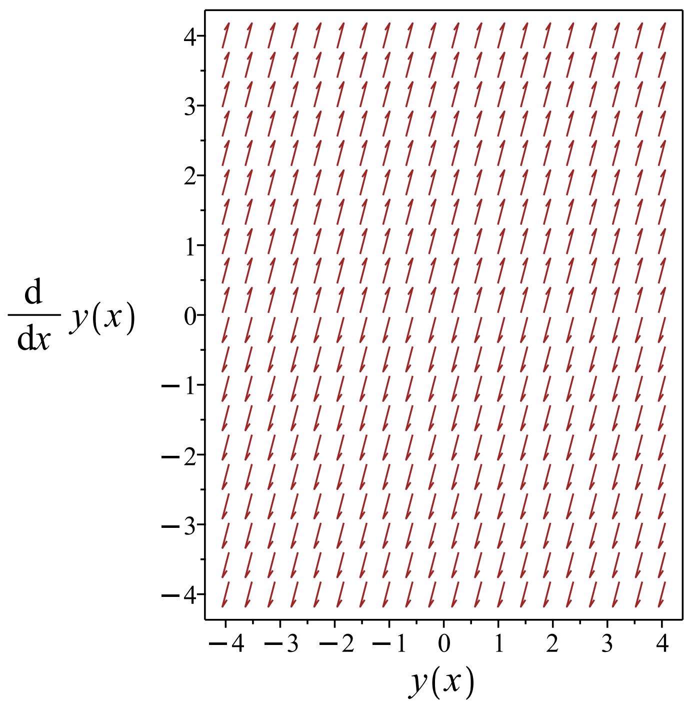


Figure 349: Slope field plot

Verification of solutions

$$y = \frac{(-\cos(x) - 3\sin(x))\,e^{2x}}{5} + c_2 e^{3x} - \frac{c_1}{3}$$

Verified OK.

8.9.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 3p(x) - 2\sin(x)\,e^{2x} = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= 2\sin(x)\,e^{2x} \end{aligned}$$

Hence the ode is

$$p'(x) - 3p(x) = 2\sin(x)\,e^{2x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-3)dx} \\ &= e^{-3x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)\left(2\sin(x)\,e^{2x}\right) \\ \frac{d}{dx}\left(e^{-3x}p\right) &= \left(e^{-3x}\right)\left(2\sin(x)\,e^{2x}\right) \\ d\left(e^{-3x}p\right) &= \left(2\,e^{-x}\sin(x)\right)\,dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-3x}p &= \int 2\,e^{-x}\sin(x)\,dx \\ e^{-3x}p &= -e^{-x}\cos(x) - e^{-x}\sin(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-3x}$ results in

$$p(x) = e^{3x}\left(-e^{-x}\cos(x) - e^{-x}\sin(x)\right) + e^{3x}c_1$$

which simplifies to

$$p(x) = (c_1 e^{x} - \sin(x) - \cos(x))\, e^{2x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = (c_1 e^{x} - \sin(x) - \cos(x))\, e^{2x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int -e^{2x}(-c_1 e^{x} + \sin(x) + \cos(x))\, dx \\ &= \frac{e^{3x}c_1}{3} - \frac{\cos(x)\, e^{2x}}{5} - \frac{3\sin(x)\, e^{2x}}{5} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{3x}c_1}{3} - \frac{\cos(x)\, e^{2x}}{5} - \frac{3\sin(x)\, e^{2x}}{5} + c_2 \tag{1}$$

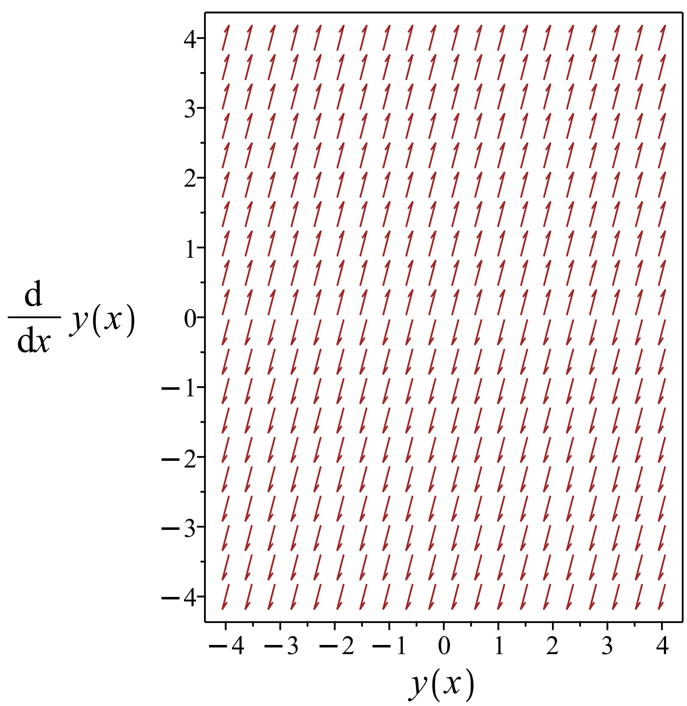


Figure 350: Slope field plot

Verification of solutions

$$y = \frac{e^{3x} c_1}{3} - \frac{\cos(x) e^{2x}}{5} - \frac{3\sin(x) e^{2x}}{5} + c_2$$

Verified OK.

8.9.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - 3y' = 2\sin(x) e^{2x}$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 3y')\, dx = \int 2\sin(x) e^{2x} dx$$

$$-3y + y' = -\frac{2\cos(x) e^{2x}}{5} + \frac{4\sin(x) e^{2x}}{5} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= \frac{(4\sin(x) - 2\cos(x)) e^{2x}}{5} + c_1 \end{aligned}$$

Hence the ode is

$$-3y + y' = \frac{(4\sin(x) - 2\cos(x)) e^{2x}}{5} + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-3) dx} \\ &= e^{-3x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{(4\sin(x) - 2\cos(x)) e^{2x}}{5} + c_1\right) \\ \frac{d}{dx}\left(e^{-3x} y\right) &= \left(e^{-3x}\right)\left(\frac{(4\sin(x) - 2\cos(x)) e^{2x}}{5} + c_1\right) \\ d\left(e^{-3x} y\right) &= \left(\frac{(4\sin(x) - 2\cos(x)) e^{-3x} e^{2x}}{5} + c_1 e^{-3x}\right) dx \end{aligned}$$

Integrating gives

$$e^{-3x}y = \int \frac{(4\sin(x) - 2\cos(x))\, e^{-3x}e^{2x}}{5} + c_1 e^{-3x}\, dx$$

$$e^{-3x}y = -\frac{e^{-x}\cos(x)}{5} - \frac{3\, e^{-x}\sin(x)}{5} - \frac{c_1 e^{-3x}}{3} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-3x}$ results in

$$y = e^{3x}\left(-\frac{e^{-x}\cos(x)}{5} - \frac{3\, e^{-x}\sin(x)}{5} - \frac{c_1 e^{-3x}}{3}\right) + c_2 e^{3x}$$

which simplifies to

$$y = \frac{(-\cos(x) - 3\sin(x))\, e^{2x}}{5} + c_2 e^{3x} - \frac{c_1}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{(-\cos(x) - 3\sin(x))\, e^{2x}}{5} + c_2 e^{3x} - \frac{c_1}{3} \tag{1}$$

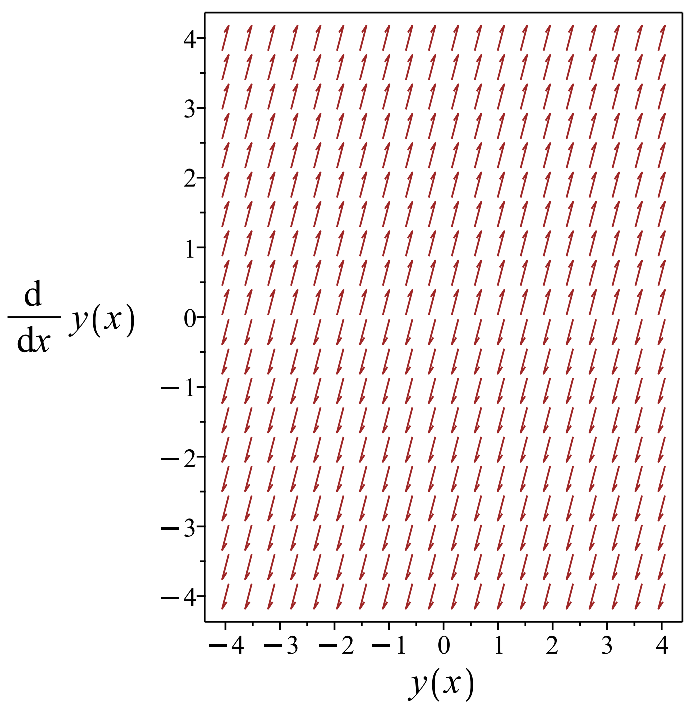


Figure 351: Slope field plot

Verification of solutions

$$y = \frac{(-\cos(x) - 3\sin(x))\,\mathrm{e}^{2x}}{5} + c_2\mathrm{e}^{3x} - \frac{c_1}{3}$$

Verified OK.

8.9.5 Solving using Kovacic algorithm

Writing the ode as

$$\begin{align*} y'' - 3y' &= 0 \tag{1} \\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= 1 \\ B &= -3 \tag{3} \\ C &= 0 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{align*} s &= 9 \\ t &= 4 \end{align*}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 225: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-3y'=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=c_1+\frac{c_2\,\mathrm{e}^{3x}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\sin(x)\,\mathrm{e}^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\,\mathrm{e}^{2x},\sin(x)\,\mathrm{e}^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1,\frac{\mathrm{e}^{3x}}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p=A_1\cos(x)\,\mathrm{e}^{2x}+A_2\sin(x)\,\mathrm{e}^{2x}$$

The unknowns $\{A_1,A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1\cos(x)\,\mathrm{e}^{2x}-A_1\sin(x)\,\mathrm{e}^{2x}-3A_2\sin(x)\,\mathrm{e}^{2x}+A_2\cos(x)\,\mathrm{e}^{2x}=2\sin(x)\,\mathrm{e}^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=-\frac{1}{5},A_2=-\frac{3}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\,e^{2x}}{5} - \frac{3\sin(x)\,e^{2x}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 e^{3x}}{3}\right) + \left(-\frac{\cos(x)\,e^{2x}}{5} - \frac{3\sin(x)\,e^{2x}}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \frac{c_2 e^{3x}}{3} - \frac{\cos(x)\,e^{2x}}{5} - \frac{3\sin(x)\,e^{2x}}{5} \tag{1}$$

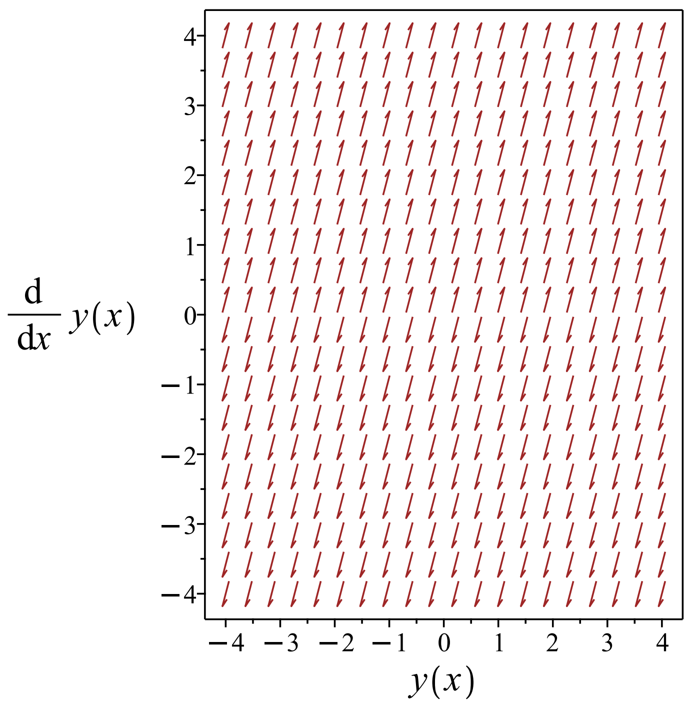


Figure 352: Slope field plot

Verification of solutions

$$y = c_1 + \frac{c_2 e^{3x}}{3} - \frac{\cos(x)\,e^{2x}}{5} - \frac{3\sin(x)\,e^{2x}}{5}$$

Verified OK.

8.9.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y''+q(x)\,y'+r(x)\,y=s(x)$$

is exact if

$$p''(x)-q'(x)+r(x)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x)&=1\\q(x)&=-3\\r(x)&=0\\s(x)&=2\sin\left(x\right)\mathrm{e}^{2x}\end{aligned}$$

Hence

$$\begin{aligned}p''(x)&=0\\q'(x)&=0\end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y'+\left(q(x)-p'(x)\right)y\right)'=s(x)$$

Integrating gives

$$p(x)\,y'+\left(q(x)-p'(x)\right)y=\int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$-3y+y'=\int 2\sin\left(x\right)\mathrm{e}^{2x}\,dx$$

We now have a first order ode to solve which is

$$-3y+y'=-\frac{2\cos\left(x\right)\mathrm{e}^{2x}}{5}+\frac{4\sin\left(x\right)\mathrm{e}^{2x}}{5}+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= \frac{(4\sin(x) - 2\cos(x))\,\mathrm{e}^{2x}}{5} + c_1 \end{aligned}$$

Hence the ode is

$$-3y + y' = \frac{(4\sin(x) - 2\cos(x))\,\mathrm{e}^{2x}}{5} + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int (-3)dx} \\ &= \mathrm{e}^{-3x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{(4\sin(x) - 2\cos(x))\,\mathrm{e}^{2x}}{5} + c_1\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-3x}y\right) &= \left(\mathrm{e}^{-3x}\right)\left(\frac{(4\sin(x) - 2\cos(x))\,\mathrm{e}^{2x}}{5} + c_1\right) \\ \mathrm{d}\left(\mathrm{e}^{-3x}y\right) &= \left(\frac{(4\sin(x) - 2\cos(x))\,\mathrm{e}^{-3x}\mathrm{e}^{2x}}{5} + c_1\mathrm{e}^{-3x}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{-3x}y &= \int \frac{(4\sin(x) - 2\cos(x))\,\mathrm{e}^{-3x}\mathrm{e}^{2x}}{5} + c_1\mathrm{e}^{-3x}\,\mathrm{d}x \\ \mathrm{e}^{-3x}y &= -\frac{\mathrm{e}^{-x}\cos(x)}{5} - \frac{3\,\mathrm{e}^{-x}\sin(x)}{5} - \frac{c_1\mathrm{e}^{-3x}}{3} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-3x}$ results in

$$y = \mathrm{e}^{3x}\left(-\frac{\mathrm{e}^{-x}\cos(x)}{5} - \frac{3\,\mathrm{e}^{-x}\sin(x)}{5} - \frac{c_1\mathrm{e}^{-3x}}{3}\right) + c_2\mathrm{e}^{3x}$$

which simplifies to

$$y = \frac{(-\cos(x) - 3\sin(x))\,\mathrm{e}^{2x}}{5} + c_2\mathrm{e}^{3x} - \frac{c_1}{3}$$

Summary
The solution(s) found are the following

$$y = \frac{(-\cos(x) - 3\sin(x))\,e^{2x}}{5} + c_2 e^{3x} - \frac{c_1}{3} \tag{1}$$

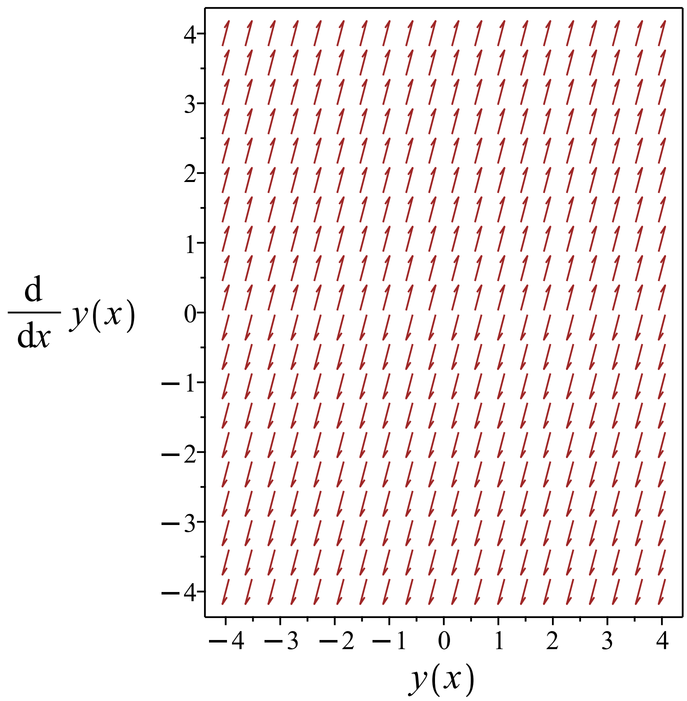


Figure 353: Slope field plot

Verification of solutions

$$y = \frac{(-\cos(x) - 3\sin(x))\,e^{2x}}{5} + c_2 e^{3x} - \frac{c_1}{3}$$

Verified OK.

8.9.7 Maple step by step solution

Let's solve

$y'' - 3y' = 2\sin(x)\,e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 3r = 0$

- Factor the characteristic polynomial

  $r(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (0, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 \mathrm{e}^{3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2\sin(x)\,\mathrm{e}^{2x}\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & \mathrm{e}^{3x} \\ 0 & 3\,\mathrm{e}^{3x} \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{3x}$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2\left(\int \sin(x)\mathrm{e}^{2x}dx\right)}{3} + \frac{2\,\mathrm{e}^{3x}\left(\int \mathrm{e}^{-x}\sin(x)dx\right)}{3}$

  ○ Compute integrals

  $y_p(x) = -\frac{(\cos(x)+3\sin(x))\mathrm{e}^{2x}}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 \mathrm{e}^{3x} - \frac{(\cos(x)+3\sin(x))\mathrm{e}^{2x}}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 2*sin(_a)*exp(2*_a)+3*_b(_a), _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)=2*exp(2*x)*sin(x),y(x), singsol=all)
```

$$y(x) = \frac{e^{2x}(-\cos(x) - 3\sin(x))}{5} + \frac{c_1 e^{3x}}{3} + c_2$$

✓ Solution by Mathematica

Time used: 0.245 (sec). Leaf size: 33

```
DSolve[y''[x]-3*y'[x]==2*Exp[2*x]*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{15} e^{2x}(-9\sin(x) - 3\cos(x) + 5c_1 e^x) + c_2$$

8.10 problem Exercise 21.13, page 231

Internal problem ID [4615]
Internal file name [`OUTPUT/4108_Sunday_June_05_2022_12_23_25_PM_80339459/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.13, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y' = x^2 + 2x}$$

8.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = x^2 + 2x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=1, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x}+\lambda\, e^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=1, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-1}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{1^2-(4)(1)(0)}\\&=-\frac{1}{2}\pm\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-\frac{1}{2}+\frac{1}{2}\\\lambda_2&=-\frac{1}{2}-\frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=0\\\lambda_2&=-1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}\\y&=c_1e^{(0)x}+c_2e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3 x^3 + A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3x^2 A_3 + 2x A_2 + 6x A_3 + A_1 + 2A_2 = x^2 + 2x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = 0, A_3 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^3}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 e^{-x}) + \left(\frac{x^3}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3} \tag{1}$$

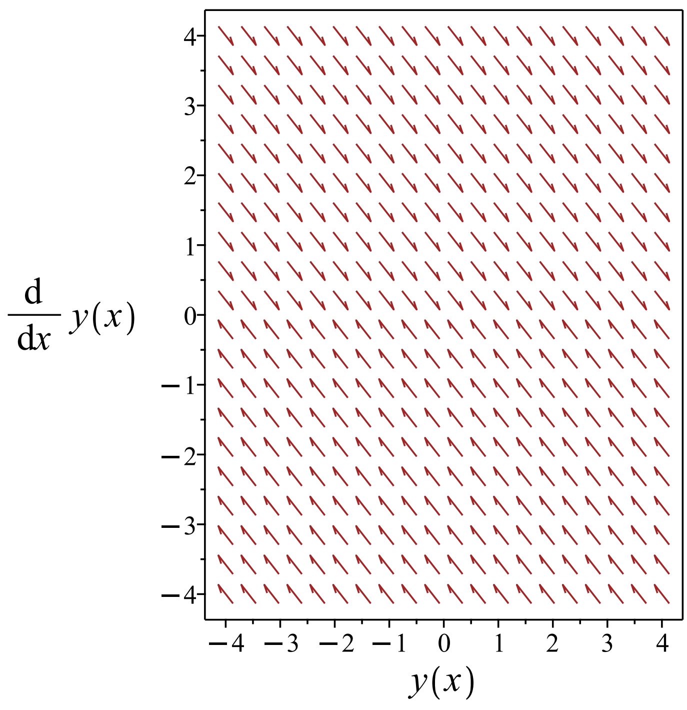


Figure 354: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3}$$

Verified OK.

8.10.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (y'' + y')\, dx &= \int (x^2 + 2x)\, dx \\
y' + y &= \frac{1}{3}x^3 + x^2 + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= \frac{1}{3}x^3 + x^2 + c_1
\end{aligned}
$$

Hence the ode is

$$
y' + y = \frac{1}{3}x^3 + x^2 + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1 dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{3}x^3 + x^2 + c_1\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x}) &= (\mathrm{e}^{x})\left(\frac{1}{3}x^3 + x^2 + c_1\right) \\
\mathrm{d}(y\,\mathrm{e}^{x}) &= \left(\frac{(x^3 + 3x^2 + 3c_1)\,\mathrm{e}^{x}}{3}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,\mathrm{e}^{x} &= \int \frac{(x^3 + 3x^2 + 3c_1)\,\mathrm{e}^{x}}{3}\,\mathrm{d}x \\
y\,\mathrm{e}^{x} &= \frac{(x^3 + 3c_1)\,\mathrm{e}^{x}}{3} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = \frac{e^{-x}(x^3 + 3c_1)\, e^x}{3} + c_2 e^{-x}$$

which simplifies to

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3} \tag{1}$$

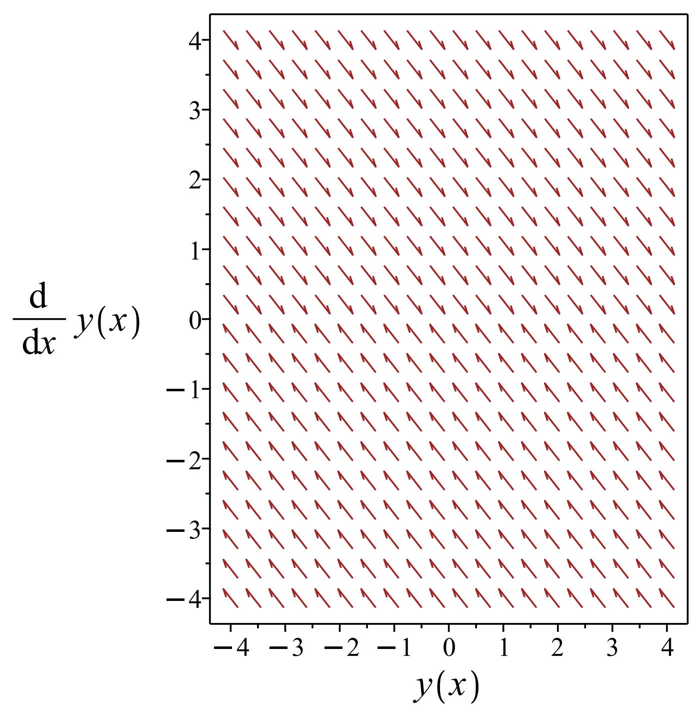


Figure 355: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3}$$

Verified OK.

8.10.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - x^2 - 2x = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x^2 + 2x \end{aligned}$$

Hence the ode is

$$p'(x) + p(x) = x^2 + 2x$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)\left(x^2 + 2x\right) \\ \frac{d}{dx}(e^x p) &= (e^x)\left(x^2 + 2x\right) \\ d(e^x p) &= \left(x(x+2)\, e^x\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^x p &= \int x(x+2)\, e^x \, dx \\ e^x p &= e^x x^2 + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$p(x) = e^{-x} e^x x^2 + c_1 e^{-x}$$

which simplifies to

$$p(x) = x^2 + c_1 e^{-x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x^2 + c_1 e^{-x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int x^2 + c_1 e^{-x} \, dx \\ &= \frac{x^3}{3} - c_1 e^{-x} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x^3}{3} - c_1 e^{-x} + c_2 \tag{1}$$

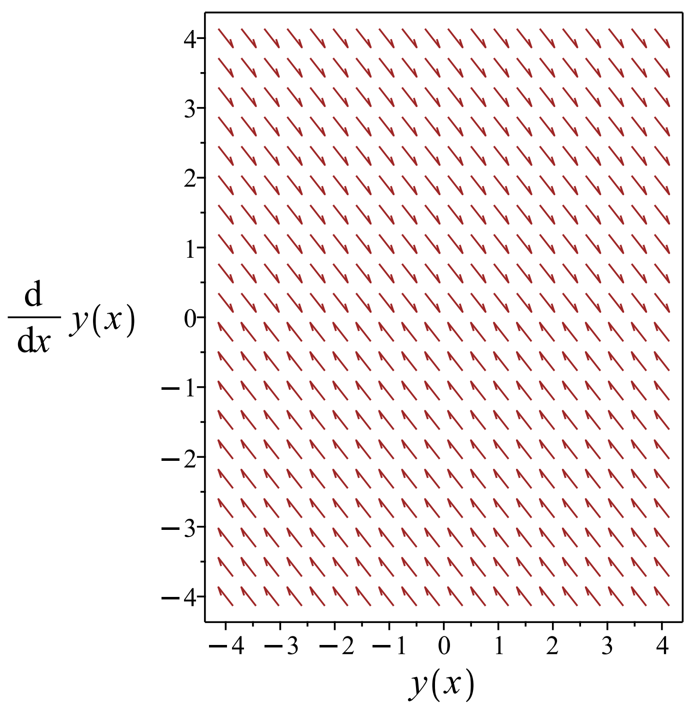


Figure 356: Slope field plot

Verification of solutions

$$y = \frac{x^3}{3} - c_1 \mathrm{e}^{-x} + c_2$$

Verified OK.

8.10.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = x^2 + 2x$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int (x^2 + 2x)\, dx$$
$$y' + y = \frac{1}{3}x^3 + x^2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = \frac{1}{3}x^3 + x^2 + c_1$$

Hence the ode is

$$y' + y = \frac{1}{3}x^3 + x^2 + c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int 1dx}$$
$$= \mathrm{e}^x$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{1}{3}x^3 + x^2 + c_1\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^x) = (\mathrm{e}^x)\left(\frac{1}{3}x^3 + x^2 + c_1\right)$$
$$\mathrm{d}(y\,\mathrm{e}^x) = \left(\frac{(x^3 + 3x^2 + 3c_1)\,\mathrm{e}^x}{3}\right)\mathrm{d}x$$

Integrating gives

$$y\,e^{x} = \int \frac{(x^3 + 3x^2 + 3c_1)\,e^{x}}{3}\,dx$$

$$y\,e^{x} = \frac{(x^3 + 3c_1)\,e^{x}}{3} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$y = \frac{e^{-x}(x^3 + 3c_1)\,e^{x}}{3} + c_2 e^{-x}$$

which simplifies to

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3} \tag{1}$$

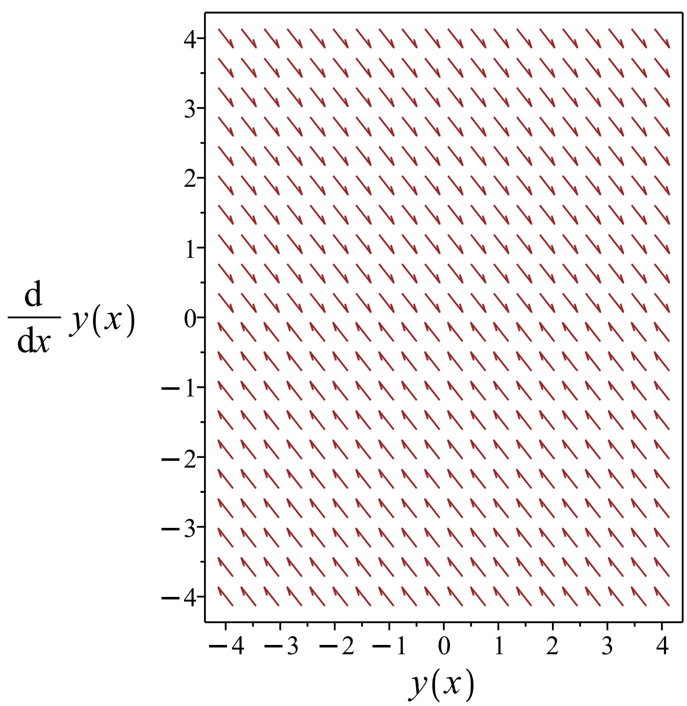


Figure 357: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3}$$

Verified OK.

8.10.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 227: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(\mathrm{e}^{x})\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3x^3 + A_2x^2 + A_1x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3x^2A_3 + 2xA_2 + 6xA_3 + A_1 + 2A_2 = x^2 + 2x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = 0, A_3 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^3}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2) + \left(\frac{x^3}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + \frac{x^3}{3} \tag{1}$$

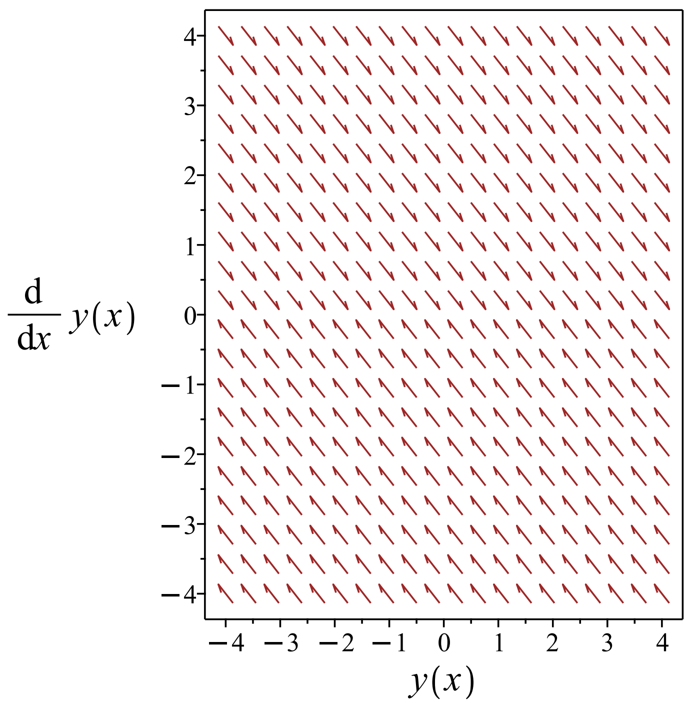


Figure 358: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 + \frac{x^3}{3}$$

Verified OK.

8.10.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= x^2 + 2x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int x^2 + 2x\, dx$$

We now have a first order ode to solve which is

$$y' + y = \frac{1}{3}x^3 + x^2 + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \frac{1}{3}x^3 + x^2 + c_1 \end{aligned}$$

Hence the ode is

$$y' + y = \frac{1}{3}x^3 + x^2 + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{1}{3}x^3 + x^2 + c_1\right) \\ \frac{d}{dx}(y\,e^x) &= (e^x)\left(\frac{1}{3}x^3 + x^2 + c_1\right) \\ d(y\,e^x) &= \left(\frac{(x^3 + 3x^2 + 3c_1)\,e^x}{3}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,e^x &= \int \frac{(x^3 + 3x^2 + 3c_1)\,e^x}{3}\, dx \\ y\,e^x &= \frac{(x^3 + 3c_1)\,e^x}{3} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = \frac{e^{-x}(x^3 + 3c_1)\,e^x}{3} + c_2 e^{-x}$$

which simplifies to

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3} \tag{1}$$

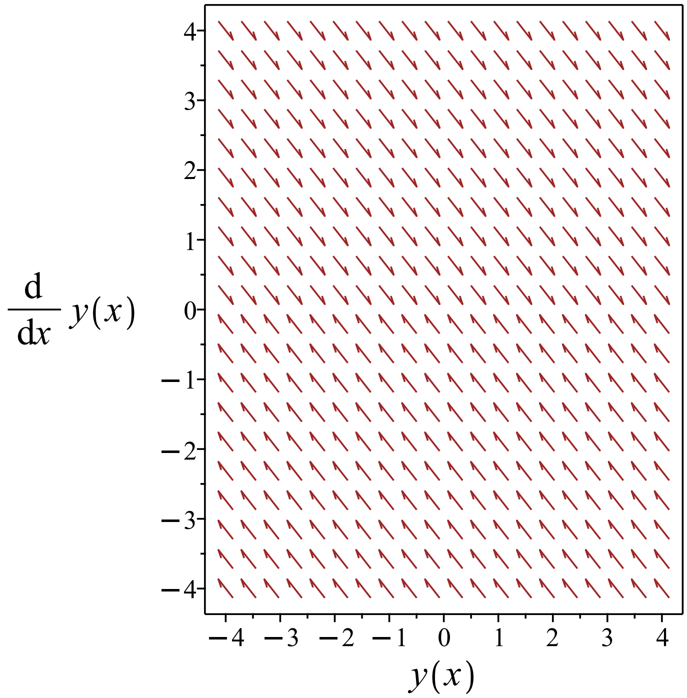


Figure 359: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3}$$

Verified OK.

8.10.7 Maple step by step solution

Let's solve

$y'' + y' = x^2 + 2x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2 + 2x\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\mathrm{e}^{-x} \left(\int x(x+2) \mathrm{e}^{x} dx\right) + \int (x^2 + 2x) \, dx$

- ○ Compute integrals

  $y_p(x) = \frac{x^3}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + \frac{x^3}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a^2-_b(_a)+2*_a, _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=x^2+2*x,y(x), singsol=all)
```

$$y(x) = \frac{x^3}{3} - e^{-x}c_1 + c_2$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 24

```
DSolve[y''[x]+y'[x]==x^2+2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{3} - c_1e^{-x} + c_2$$

8.11 problem Exercise 21.14, page 231

Internal problem ID [4616]
Internal file name [`OUTPUT/4109_Sunday_June_05_2022_12_23_33_PM_41338155/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.14, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y' = x + \sin(2x)}$$

8.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = x + \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=1, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x}+\lambda e^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=1, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-1}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{1^2-(4)(1)(0)}\\&=-\frac{1}{2}\pm\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-\frac{1}{2}+\frac{1}{2}\\\lambda_2&=-\frac{1}{2}-\frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=0\\\lambda_2&=-1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}\\y&=c_1e^{(0)x}+c_2e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}, \{\cos(2x), \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x + A_3 \cos(2x) + A_4 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 - 4A_3 \cos(2x) - 4A_4 \sin(2x) + 2A_2 x + A_1 - 2A_3 \sin(2x) + 2A_4 \cos(2x) = x + \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{1}{2}, A_3 = -\frac{1}{10}, A_4 = -\frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{2} - x - \frac{\cos(2x)}{10} - \frac{\sin(2x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 e^{-x}) + \left(\frac{x^2}{2} - x - \frac{\cos(2x)}{10} - \frac{\sin(2x)}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-x} + \frac{x^2}{2} - x - \frac{\cos(2x)}{10} - \frac{\sin(2x)}{5} \tag{1}$$

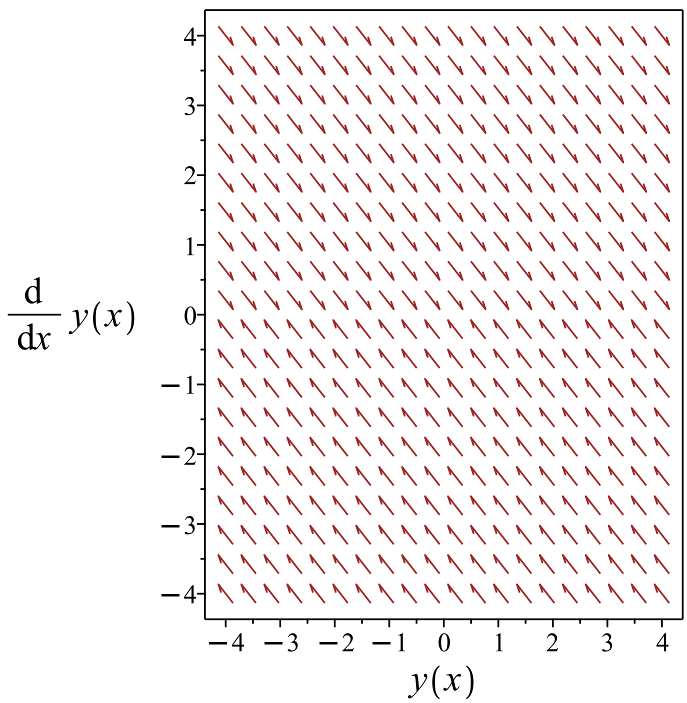


Figure 360: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-x} + \frac{x^2}{2} - x - \frac{\cos(2x)}{10} - \frac{\sin(2x)}{5}$$

Verified OK.

8.11.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int (x + \sin(2x))\, dx$$

$$y' + y = \frac{x^2}{2} - \frac{\cos(2x)}{2} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$

$$q(x) = \frac{x^2}{2} - \frac{\cos(2x)}{2} + c_1$$

Hence the ode is

$$y' + y = \frac{x^2}{2} - \frac{\cos(2x)}{2} + c_1$$

The integrating factor μ is

$$\mu = e^{\int 1 dx} = e^x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^2}{2} - \frac{\cos(2x)}{2} + c_1\right)$$

$$\frac{d}{dx}(y\, e^x) = (e^x)\left(\frac{x^2}{2} - \frac{\cos(2x)}{2} + c_1\right)$$

$$d(y\, e^x) = \left(-\frac{(-x^2 + \cos(2x) - 2c_1)\, e^x}{2}\right) dx$$

Integrating gives

$$y\, e^x = \int -\frac{(-x^2 + \cos(2x) - 2c_1)\, e^x}{2}\, dx$$

$$y\, e^x = \frac{e^x x^2}{2} - e^x x + \frac{11\, e^x}{10} - \frac{(2\sin(x) + \cos(x))\, e^x \cos(x)}{5} + c_1 e^x + c_2$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(\frac{e^x x^2}{2} - e^x x + \frac{11\,e^x}{10} - \frac{(2\sin(x) + \cos(x))\,e^x \cos(x)}{5} + c_1 e^x\right) + c_2 e^{-x}$$

which simplifies to

$$y = \frac{x^2}{2} - \frac{2\cos(x)\sin(x)}{5} - \frac{\cos(x)^2}{5} + c_1 - x + \frac{11}{10} + c_2 e^{-x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} - \frac{2\cos(x)\sin(x)}{5} - \frac{\cos(x)^2}{5} + c_1 - x + \frac{11}{10} + c_2 e^{-x} \tag{1}$$

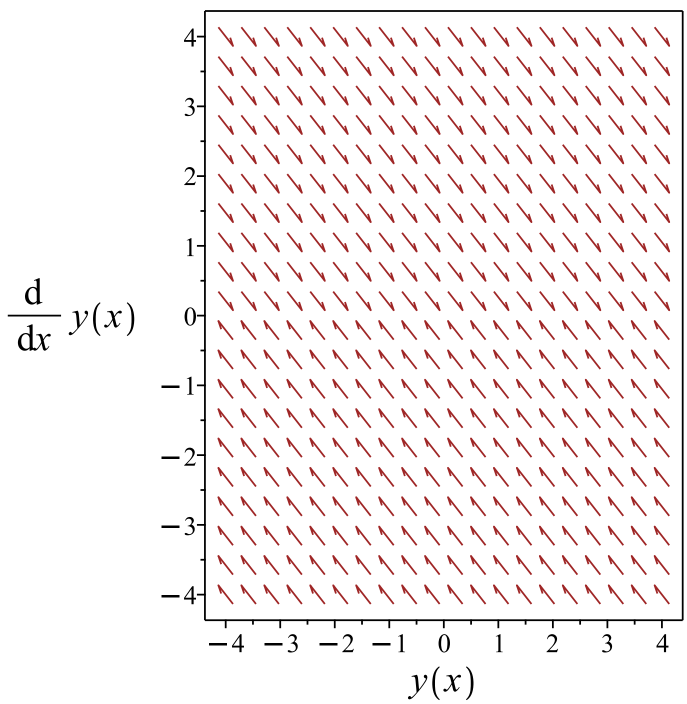


Figure 361: Slope field plot

Verification of solutions

$$y = \frac{x^2}{2} - \frac{2\cos(x)\sin(x)}{5} - \frac{\cos(x)^2}{5} + c_1 - x + \frac{11}{10} + c_2 e^{-x}$$

Verified OK.

8.11.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - x - \sin(2x) = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x + \sin(2x) \end{aligned}$$

Hence the ode is

$$p'(x) + p(x) = x + \sin(2x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)\,(x + \sin(2x)) \\ \frac{d}{dx}(e^{x} p) &= (e^{x})\,(x + \sin(2x)) \\ d(e^{x} p) &= ((x + \sin(2x))\, e^{x})\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{x} p &= \int (x + \sin(2x))\, e^{x}\, dx \\ e^{x} p &= e^{x} x - e^{x} + \frac{e^{x}(\sin(2x) - 2\cos(2x))}{5} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$p(x) = e^{-x}\left(e^x x - e^x + \frac{e^x(\sin(2x) - 2\cos(2x))}{5}\right) + c_1 e^{-x}$$

which simplifies to

$$p(x) = x + \frac{\sin(2x)}{5} - \frac{2\cos(2x)}{5} - 1 + c_1 e^{-x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x + \frac{\sin(2x)}{5} - \frac{2\cos(2x)}{5} - 1 + c_1 e^{-x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int x + \frac{\sin(2x)}{5} - \frac{2\cos(2x)}{5} - 1 + c_1 e^{-x}\, dx \\ &= -x + \frac{x^2}{2} - c_1 e^{-x} - \frac{\sin(2x)}{5} - \frac{\cos(2x)}{10} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -x + \frac{x^2}{2} - c_1 e^{-x} - \frac{\sin(2x)}{5} - \frac{\cos(2x)}{10} + c_2 \tag{1}$$

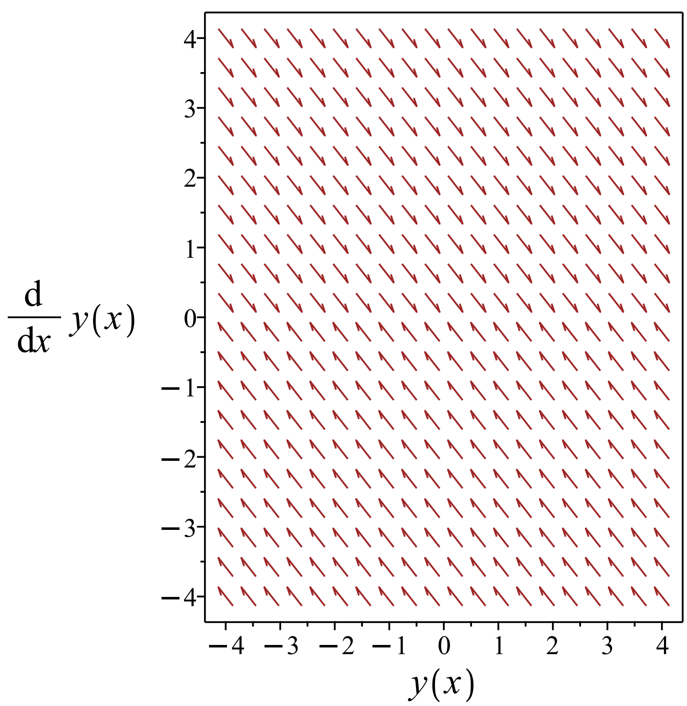


Figure 362: Slope field plot

Verification of solutions

$$y = -x + \frac{x^2}{2} - c_1 \mathrm{e}^{-x} - \frac{\sin(2x)}{5} - \frac{\cos(2x)}{10} + c_2$$

Verified OK.

8.11.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = x + \sin(2x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int (x + \sin(2x))\, dx$$

$$y' + y = \frac{x^2}{2} - \frac{\cos(2x)}{2} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = \frac{x^2}{2} - \frac{\cos(2x)}{2} + c_1$$

Hence the ode is

$$y' + y = \frac{x^2}{2} - \frac{\cos(2x)}{2} + c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int 1 dx}$$
$$= \mathrm{e}^{x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{x^2}{2} - \frac{\cos(2x)}{2} + c_1\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x}) = (\mathrm{e}^{x})\left(\frac{x^2}{2} - \frac{\cos(2x)}{2} + c_1\right)$$
$$\mathrm{d}(y\,\mathrm{e}^{x}) = \left(-\frac{(-x^2 + \cos(2x) - 2c_1)\,\mathrm{e}^{x}}{2}\right) \mathrm{d}x$$

Integrating gives

$$y\,\mathrm{e}^{x} = \int -\frac{\left(-x^{2}+\cos\left(2x\right)-2c_{1}\right)\mathrm{e}^{x}}{2}\,\mathrm{d}x$$
$$y\,\mathrm{e}^{x} = \frac{\mathrm{e}^{x}x^{2}}{2}-\mathrm{e}^{x}x+\frac{11\,\mathrm{e}^{x}}{10}-\frac{\left(2\sin\left(x\right)+\cos\left(x\right)\right)\mathrm{e}^{x}\cos\left(x\right)}{5}+c_{1}\mathrm{e}^{x}+c_{2}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$y = \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{x}x^{2}}{2}-\mathrm{e}^{x}x+\frac{11\,\mathrm{e}^{x}}{10}-\frac{\left(2\sin\left(x\right)+\cos\left(x\right)\right)\mathrm{e}^{x}\cos\left(x\right)}{5}+c_{1}\mathrm{e}^{x}\right)+c_{2}\mathrm{e}^{-x}$$

which simplifies to

$$y = \frac{x^{2}}{2}-\frac{2\cos\left(x\right)\sin\left(x\right)}{5}-\frac{\cos\left(x\right)^{2}}{5}+c_{1}-x+\frac{11}{10}+c_{2}\mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^{2}}{2}-\frac{2\cos\left(x\right)\sin\left(x\right)}{5}-\frac{\cos\left(x\right)^{2}}{5}+c_{1}-x+\frac{11}{10}+c_{2}\mathrm{e}^{-x} \tag{1}$$

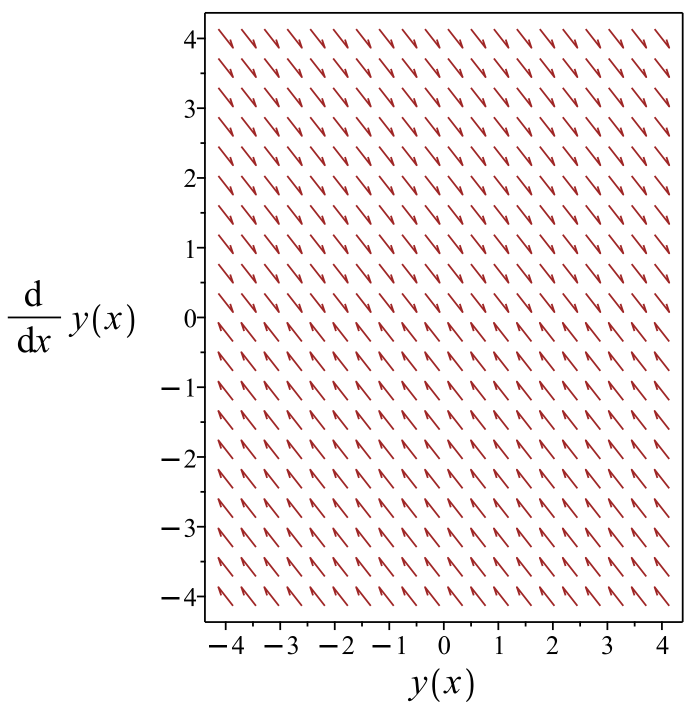


Figure 363: Slope field plot

Verification of solutions

$$y = \frac{x^2}{2} - \frac{2\cos(x)\sin(x)}{5} - \frac{\cos(x)^2}{5} + c_1 - x + \frac{11}{10} + c_2 e^{-x}$$

Verified OK.

8.11.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 229: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(\mathrm{e}^{x})\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}, \{\cos(2x), \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x + A_3 \cos(2x) + A_4 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 - 4A_3 \cos(2x) - 4A_4 \sin(2x) + 2A_2 x + A_1 - 2A_3 \sin(2x) + 2A_4 \cos(2x) = x + \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{1}{2}, A_3 = -\frac{1}{10}, A_4 = -\frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{2} - x - \frac{\cos(2x)}{10} - \frac{\sin(2x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2) + \left(\frac{x^2}{2} - x - \frac{\cos(2x)}{10} - \frac{\sin(2x)}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + \frac{x^2}{2} - x - \frac{\cos(2x)}{10} - \frac{\sin(2x)}{5} \tag{1}$$

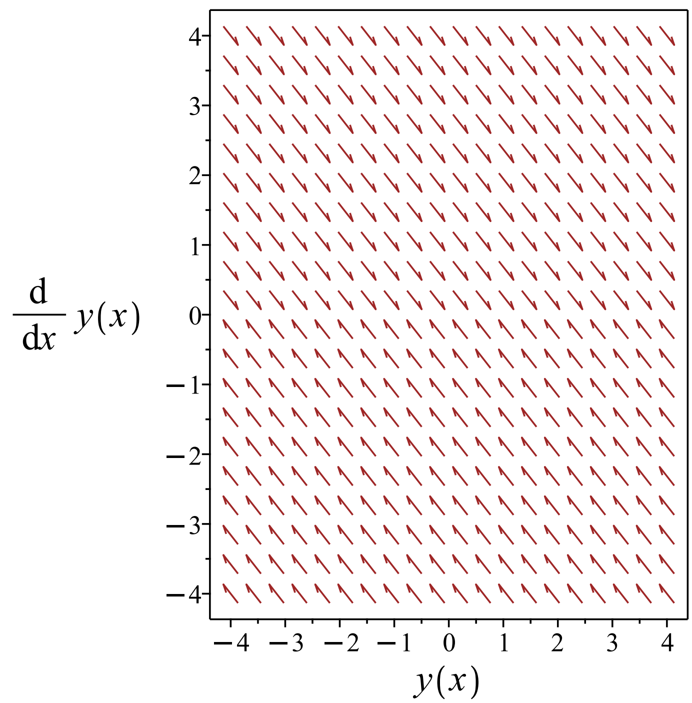


Figure 364: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 + \frac{x^2}{2} - x - \frac{\cos(2x)}{10} - \frac{\sin(2x)}{5}$$

Verified OK.

8.11.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= x + \sin(2x) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int x + \sin(2x)\; dx$$

We now have a first order ode to solve which is

$$y' + y = \frac{x^2}{2} - \frac{\cos(2x)}{2} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= 1\\ q(x) &= \frac{x^2}{2} - \frac{\cos(2x)}{2} + c_1\end{aligned}$$

Hence the ode is

$$y' + y = \frac{x^2}{2} - \frac{\cos(2x)}{2} + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1 dx}\\ &= e^x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^2}{2} - \frac{\cos(2x)}{2} + c_1\right)\\ \frac{d}{dx}(y\,e^x) &= (e^x)\left(\frac{x^2}{2} - \frac{\cos(2x)}{2} + c_1\right)\\ d(y\,e^x) &= \left(-\frac{(-x^2 + \cos(2x) - 2c_1)\,e^x}{2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,e^x &= \int -\frac{(-x^2 + \cos(2x) - 2c_1)\,e^x}{2}\,dx\\ y\,e^x &= \frac{e^x x^2}{2} - e^x x + \frac{11\,e^x}{10} - \frac{(2\sin(x) + \cos(x))\,e^x \cos(x)}{5} + c_1 e^x + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(\frac{e^x x^2}{2} - e^x x + \frac{11\,e^x}{10} - \frac{(2\sin(x) + \cos(x))\,e^x \cos(x)}{5} + c_1 e^x\right) + c_2 e^{-x}$$

which simplifies to

$$y = \frac{x^2}{2} - \frac{2\cos(x)\sin(x)}{5} - \frac{\cos(x)^2}{5} + c_1 - x + \frac{11}{10} + c_2 e^{-x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} - \frac{2\cos(x)\sin(x)}{5} - \frac{\cos(x)^2}{5} + c_1 - x + \frac{11}{10} + c_2 e^{-x} \tag{1}$$

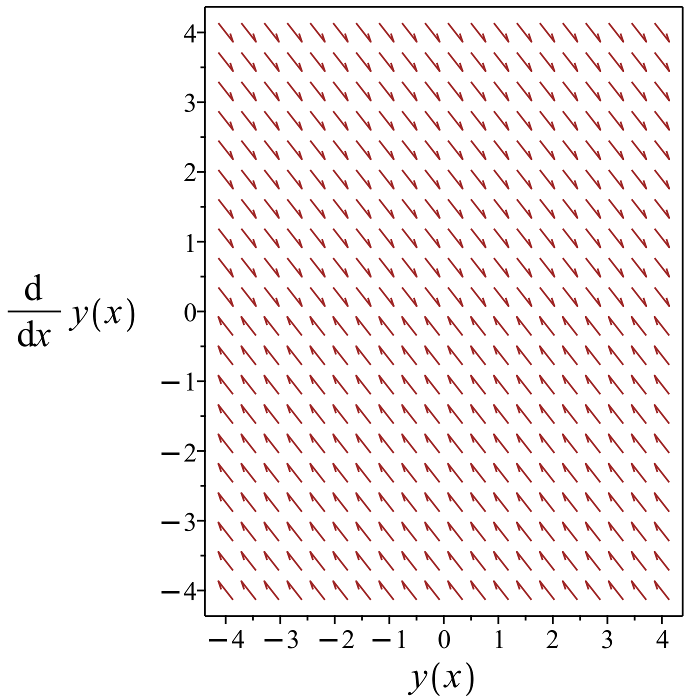


Figure 365: Slope field plot

Verification of solutions

$$y = \frac{x^2}{2} - \frac{2\cos(x)\sin(x)}{5} - \frac{\cos(x)^2}{5} + c_1 - x + \frac{11}{10} + c_2 e^{-x}$$

Verified OK.

8.11.7 Maple step by step solution

Let's solve

$y'' + y' = x + \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x + \sin(2x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & 1 \\ -e^{-x} & 0 \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{-x}\left(\int (x + \sin(2x)) e^x dx\right) + \int (x + \sin(2x)) dx$

  ◦ Compute integrals

  $y_p(x) = 1 - x - \frac{\sin(2x)}{5} - \frac{\cos(2x)}{10} + \frac{x^2}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 + 1 - x - \frac{\sin(2x)}{5} - \frac{\cos(2x)}{10} + \frac{x^2}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)+_a+sin(2*_a), _b(_a)`   *** Sub
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=x+sin(2*x),y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} - e^{-x} c_1 - \frac{\sin(2x)}{5} - \frac{\cos(2x)}{10} - x + c_2$$

✓ Solution by Mathematica

Time used: 0.359 (sec). Leaf size: 43

```
DSolve[y''[x]+y'[x]==x+Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2} - x - \frac{1}{5}\sin(2x) - \frac{1}{10}\cos(2x) - c_1 e^{-x} + c_2$$

8.12 problem Exercise 21.15, page 231

Internal problem ID [4617]
Internal file name [`OUTPUT/4110_Sunday_June_05_2022_12_23_41_PM_7426594/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.15, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = 4 \sin (x) x}$$

8.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 4 \sin (x) x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4 \sin(x) x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) x, \sin(x) x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \sin(x), \cos(x) x, \cos(x) x^2, \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \sin(x) + A_2 \cos(x) x + A_3 \cos(x) x^2 + A_4 \sin(x) x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \sin(x) + 4A_1 x \cos(x) - 2A_2 \sin(x) - 4A_3 \sin(x) x + 2A_3 \cos(x) + 2A_4 \cos(x) \\ = 4 \sin(x) x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = -1, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x) x^2 + \sin(x) x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (\cos(x) c_1 + c_2 \sin(x)) + (-\cos(x) x^2 + \sin(x) x)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x^2 + \sin(x) x \tag{1}
$$

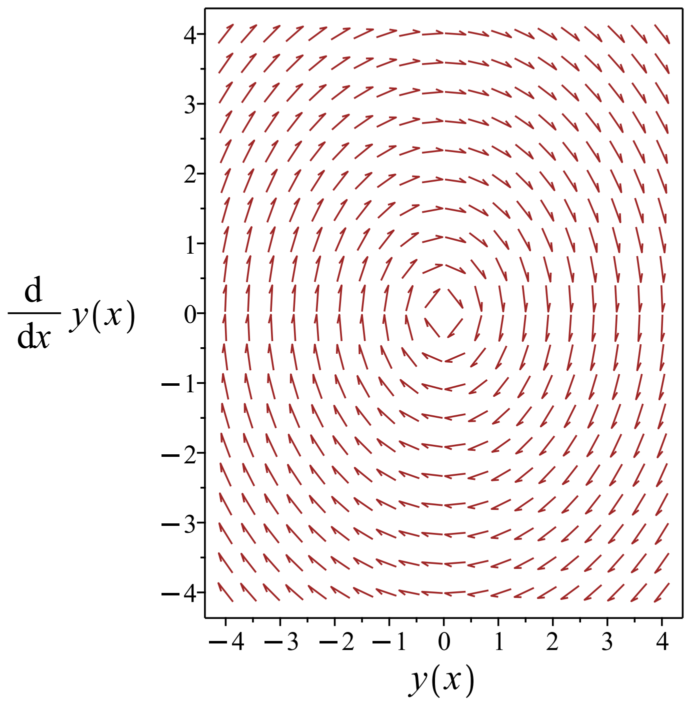


Figure 366: Slope field plot

<u>Verification of solutions</u>

$$
y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x^2 + \sin(x) x
$$

Verified OK.

8.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 231: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \cos(x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2}\,dx\\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx\\ &= \cos(x)\,(\tan(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x)))\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\,c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\sin(x)\,x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\,x, \sin(x)\,x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\sin(x), \cos(x)\,x, \cos(x)\,x^2, \sin(x)\,x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2\sin(x) + A_2\cos(x)\,x + A_3\cos(x)\,x^2 + A_4\sin(x)\,x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&2A_1\sin(x) + 4A_1x\cos(x) - 2A_2\sin(x) - 4A_3\sin(x)\,x + 2A_3\cos(x) + 2A_4\cos(x)\\&= 4\sin(x)\,x\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = -1, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x)\,x^2 + \sin(x)\,x$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= (\cos(x)\,c_1 + c_2\sin(x)) + \left(-\cos(x)\,x^2 + \sin(x)\,x\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x^2 + \sin(x) x \tag{1}$$

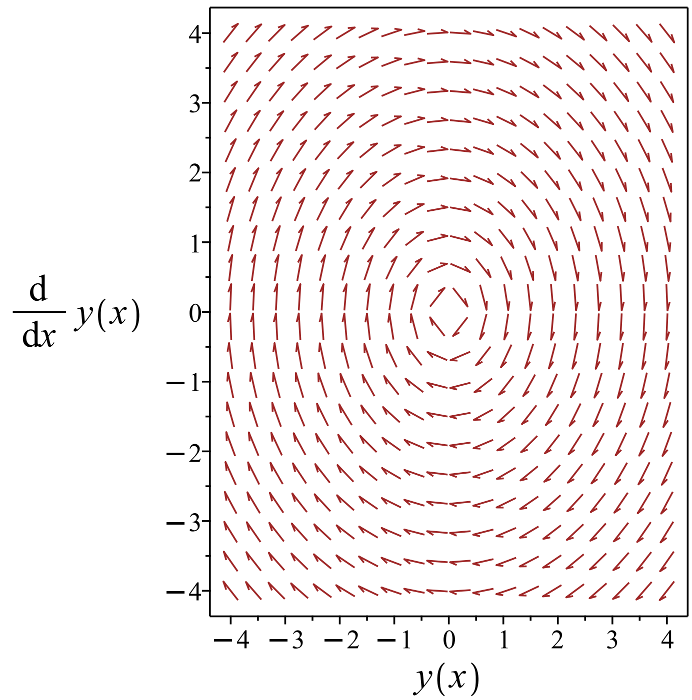


Figure 367: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x^2 + \sin(x) x$$

Verified OK.

8.12.3 Maple step by step solution

Let's solve

$y'' + y = 4\sin(x) x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4\sin(x) x\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$
- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -4\cos(x)\left(\int \sin(x)^2 x dx\right) + 2\sin(x)\left(\int x \sin(2x) dx\right)$
- ○ Compute integrals

  $y_p(x) = x(-\cos(x) x + \sin(x))$
- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + x(-\cos(x) x + \sin(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+y(x)=4*x*sin(x),y(x), singsol=all)
```

$$y(x) = (-x^2 + c_1)\cos(x) + \sin(x)(c_2 + x)$$

✓ Solution by Mathematica
Time used: 0.048 (sec). Leaf size: 27

```
DSolve[y''[x]+y[x]==4*x*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(-x^2 + \frac{1}{2} + c_1\right)\cos(x) + (x + c_2)\sin(x)$$

8.13 problem Exercise 21.16, page 231

Internal problem ID [4618]
Internal file name [`OUTPUT/4111_Sunday_June_05_2022_12_23_50_PM_57702987/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.16, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = x \sin(2x)}$$

8.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = x \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \cos(2x), x \sin(2x), \cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos(2x), x \sin(2x), x^2 \cos(2x), x^2 \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(2x) + A_2 x \sin(2x) + A_3 x^2 \cos(2x) + A_4 x^2 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&-4A_1 \sin(2x) + 4A_2 \cos(2x) + 2A_3 \cos(2x) - 8A_3 x \sin(2x) \\
&+ 2A_4 \sin(2x) + 8A_4 x \cos(2x) = x \sin(2x)
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{16}, A_3 = -\frac{1}{8}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x \sin(2x)}{16} - \frac{x^2 \cos(2x)}{8}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{x \sin(2x)}{16} - \frac{x^2 \cos(2x)}{8}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x \sin(2x)}{16} - \frac{x^2 \cos(2x)}{8} \tag{1}
$$

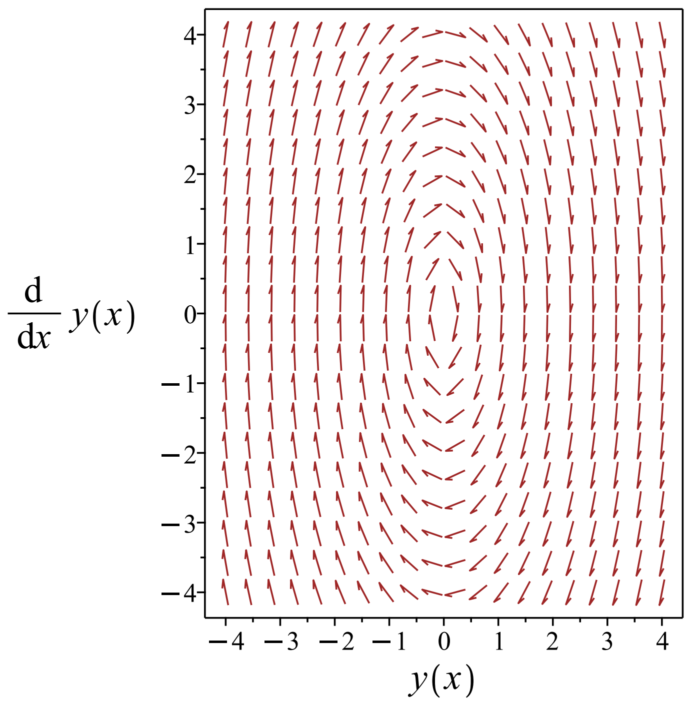


Figure 368: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x \sin(2x)}{16} - \frac{x^2 \cos(2x)}{8}
$$

Verified OK.

8.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 233: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \cos(2x), x \sin(2x), \cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos(2x), x \sin(2x), x^2 \cos(2x), x^2 \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(2x) + A_2 x \sin(2x) + A_3 x^2 \cos(2x) + A_4 x^2 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &-4A_1 \sin(2x) + 4A_2 \cos(2x) + 2A_3 \cos(2x) - 8A_3 x \sin(2x) \\ &+ 2A_4 \sin(2x) + 8A_4 x \cos(2x) = x \sin(2x) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{16}, A_3 = -\frac{1}{8}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\sin(2x)}{16} - \frac{x^2\cos(2x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(2x) + \frac{c_2\sin(2x)}{2}\right) + \left(\frac{x\sin(2x)}{16} - \frac{x^2\cos(2x)}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} + \frac{x\sin(2x)}{16} - \frac{x^2\cos(2x)}{8} \tag{1}$$

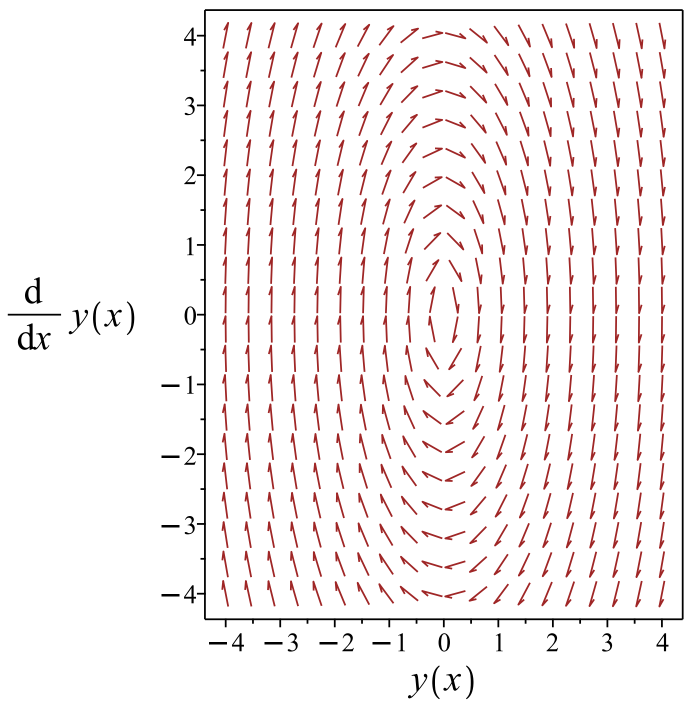


Figure 369: Slope field plot

Verification of solutions

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} + \frac{x\sin(2x)}{16} - \frac{x^2\cos(2x)}{8}$$

Verified OK.

8.13.3 Maple step by step solution

Let's solve

$y'' + 4y = x \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x \sin(2x)\right]$

- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

- ∘ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- ∘ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(2x)\left(\int \sin(2x)^2 x dx\right)}{2} + \frac{\sin(2x)\left(\int \sin(4x) x dx\right)}{4}$$

- Compute integrals

$$y_p(x) = \frac{x\sin(2x)}{16} - \frac{x^2\cos(2x)}{8}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x\sin(2x)}{16} - \frac{x^2\cos(2x)}{8}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)+4*y(x)=x*sin(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(-x^2 + 8c_1)\cos(2x)}{8} + \frac{\sin(2x)(16c_2 + x)}{16}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 38

```
DSolve[y''[x]+4*y[x]==x*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{64}\left((-8x^2 + 1 + 64c_1)\cos(2x) + 4(x + 16c_2)\sin(2x)\right)$$

8.14 problem Exercise 21.17, page 231

Internal problem ID [4619]
Internal file name [`OUTPUT/4112_Sunday_June_05_2022_12_23_58_PM_89488352/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.17, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + y = x^2 e^{-x}}$$

8.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = x^2 e^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)} \\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-x} + c_2 x\, e^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + x\, e^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^{-x}, x^2 e^{-x}, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{-x}, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}, x^3\mathrm{e}^{-x}\}]$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\mathrm{e}^{-x}, x^3\mathrm{e}^{-x}, x^4\mathrm{e}^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2\mathrm{e}^{-x} + A_2x^3\mathrm{e}^{-x} + A_3x^4\mathrm{e}^{-x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{-x} + 6A_2x\,\mathrm{e}^{-x} + 12A_3x^2\mathrm{e}^{-x} = x^2\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = 0, A_3 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^4\mathrm{e}^{-x}}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + x\,\mathrm{e}^{-x}c_2\right) + \left(\frac{x^4\mathrm{e}^{-x}}{12}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2x + c_1) + \frac{x^4\mathrm{e}^{-x}}{12}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + \frac{x^4 e^{-x}}{12} \tag{1}$$

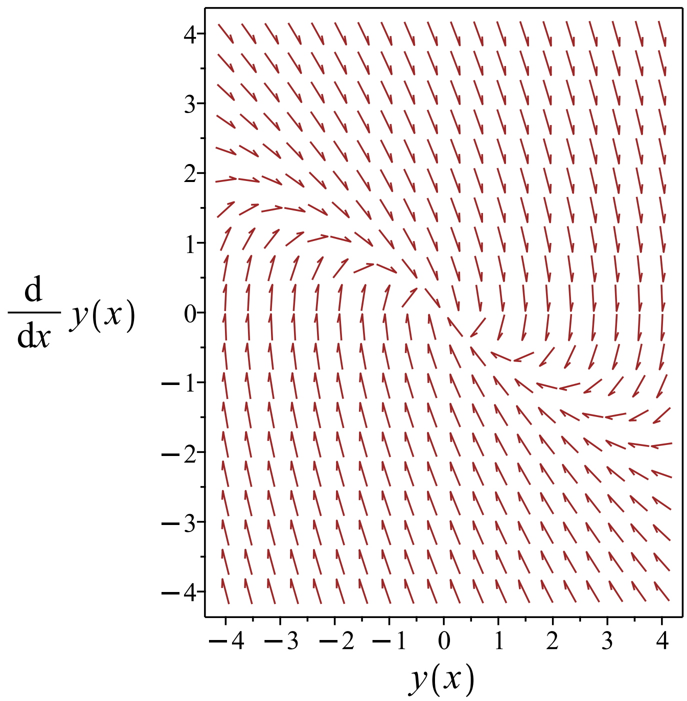


Figure 370: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + \frac{x^4 e^{-x}}{12}$$

Verified OK.

8.14.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= \mathrm{e}^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \mathrm{e}^{-x}\mathrm{e}^{x}x^{2} \\ (y\,\mathrm{e}^{x})'' &= \mathrm{e}^{-x}\mathrm{e}^{x}x^{2} \end{aligned}$$

Integrating once gives

$$(y\,\mathrm{e}^{x})' = \frac{x^{3}}{3} + c_1$$

Integrating again gives

$$(y\,\mathrm{e}^{x}) = \frac{1}{12}x^{4} + c_1 x + c_2$$

Hence the solution is

$$y = \frac{\frac{1}{12}x^{4} + c_1 x + c_2}{\mathrm{e}^{x}}$$

Or

$$y = \frac{x^{4}\mathrm{e}^{-x}}{12} + c_1 x\,\mathrm{e}^{-x} + c_2\mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^{4}\mathrm{e}^{-x}}{12} + c_1 x\,\mathrm{e}^{-x} + c_2\mathrm{e}^{-x} \tag{1}$$

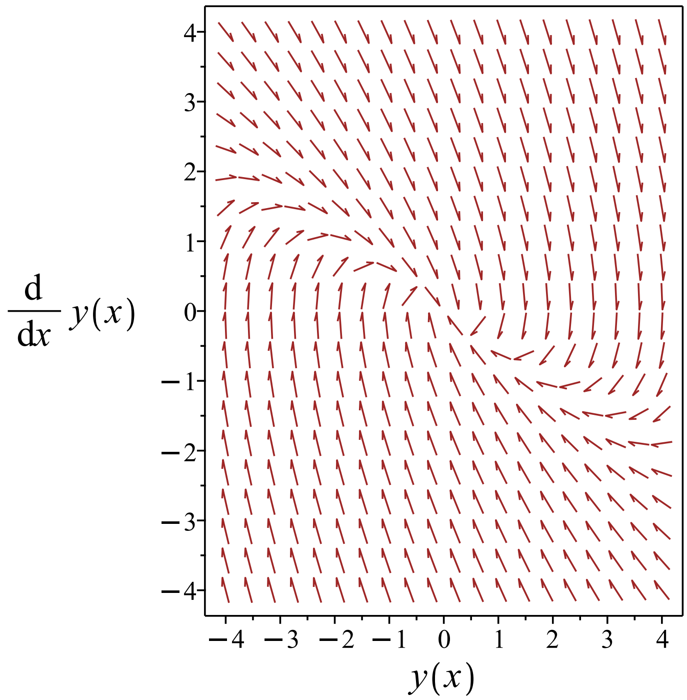


Figure 371: Slope field plot

Verification of solutions

$$y = \frac{x^4 e^{-x}}{12} + c_1 x \, e^{-x} + c_2 e^{-x}$$

Verified OK.

8.14.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 235: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1 \left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(x)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\mathrm{e}^{-x} + x\,\mathrm{e}^{-x}c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}, \mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}, x^3\mathrm{e}^{-x}\}]$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\mathrm{e}^{-x}, x^3\mathrm{e}^{-x}, x^4\mathrm{e}^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \mathrm{e}^{-x} + A_2 x^3 \mathrm{e}^{-x} + A_3 x^4 \mathrm{e}^{-x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^{-x} + 6A_2 x \,\mathrm{e}^{-x} + 12A_3 x^2 \mathrm{e}^{-x} = x^2 \mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = 0, A_3 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^4 \mathrm{e}^{-x}}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + x \,\mathrm{e}^{-x} c_2\right) + \left(\frac{x^4 \mathrm{e}^{-x}}{12}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \frac{x^4 \mathrm{e}^{-x}}{12}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \frac{x^4 \mathrm{e}^{-x}}{12} \tag{1}$$

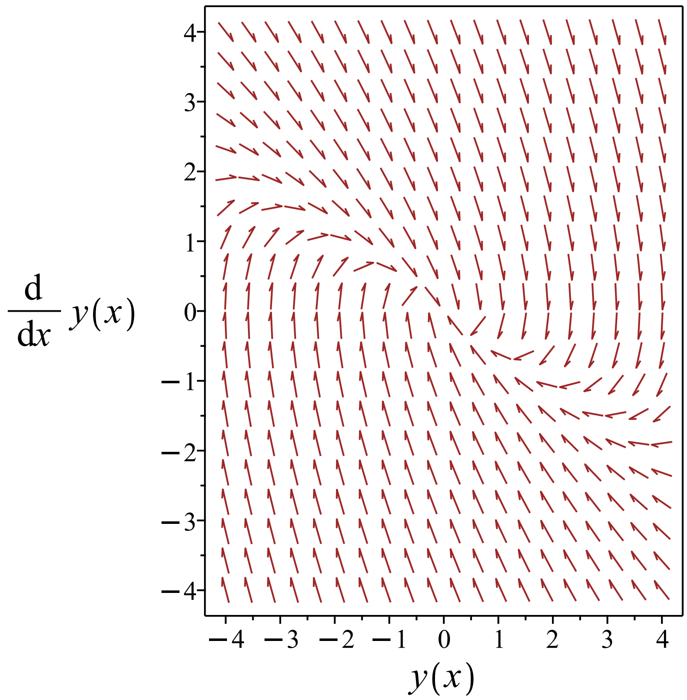


Figure 372: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + \frac{x^4 e^{-x}}{12}$$

Verified OK.

8.14.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = x^2 e^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2 \mathrm{e}^{-x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \mathrm{e}^{-x}\left(-\left(\int x^3 dx\right) + \left(\int x^2 dx\right) x\right)$

  - Compute integrals

    $y_p(x) = \frac{x^4 \mathrm{e}^{-x}}{12}$

- Substitute particular solution into general solution to ODE

  $y = x\,\mathrm{e}^{-x} c_2 + c_1 \mathrm{e}^{-x} + \frac{x^4 \mathrm{e}^{-x}}{12}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=x^2*exp(-x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x}\left(c_2 + c_1 x + \frac{1}{12}x^4\right)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 27

```
DSolve[y''[x]+2*y'[x]+y[x]==x^2*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{12}e^{-x}\left(x^4 + 12c_2 x + 12c_1\right)$$

8.15 problem Exercise 21.19, page 231

Internal problem ID [4620]
Internal file name [`OUTPUT/4113_Sunday_June_05_2022_12_24_07_PM_68501899/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.19, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = e^{-2x} + x^2}$$

8.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = e^{-2x} + x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-2x} + x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^{-2x}\}, \{1, x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\, e^{-2x} + A_2 + A_3 x + A_4 x^2$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{-2x} + 2A_4 + 3A_3 + 6A_4 x + 2A_2 + 2A_3 x + 2A_4 x^2 = e^{-2x} + x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{7}{4}, A_3 = -\frac{3}{2}, A_4 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\, e^{-2x} + \frac{7}{4} - \frac{3x}{2} + \frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left(-x\, e^{-2x} + \frac{7}{4} - \frac{3x}{2} + \frac{x^2}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} - x\,e^{-2x} + \frac{7}{4} - \frac{3x}{2} + \frac{x^2}{2} \tag{1}$$

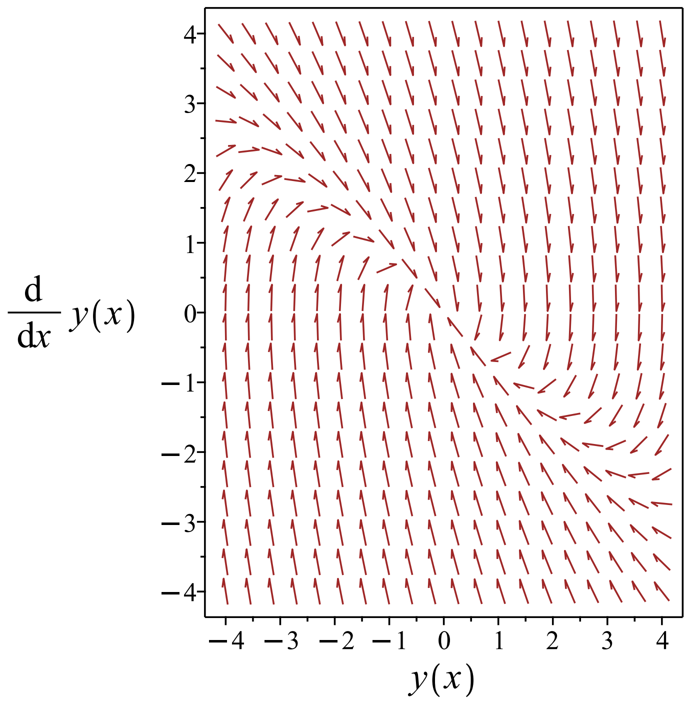


Figure 373: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} - x\,e^{-2x} + \frac{7}{4} - \frac{3x}{2} + \frac{x^2}{2}$$

Verified OK.

8.15.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 237: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\, dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x}(\mathrm{e}^{x})\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-2x} + x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-2x}\}, \{1, x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{-2x} + A_2 + A_3 x + A_4 x^2$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{-2x} + 2A_4 + 3A_3 + 6A_4 x + 2A_2 + 2A_3 x + 2A_4 x^2 = e^{-2x} + x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{7}{4}, A_3 = -\frac{3}{2}, A_4 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\,e^{-2x} + \frac{7}{4} - \frac{3x}{2} + \frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{-x}\right) + \left(-x\,e^{-2x} + \frac{7}{4} - \frac{3x}{2} + \frac{x^2}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{-x} - x\,e^{-2x} + \frac{7}{4} - \frac{3x}{2} + \frac{x^2}{2} \tag{1}$$

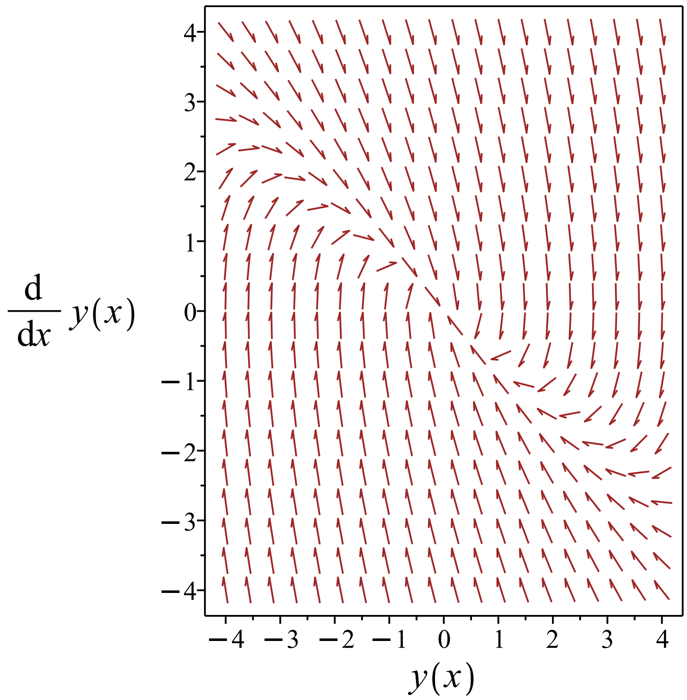


Figure 374: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-x} - x\,e^{-2x} + \frac{7}{4} - \frac{3x}{2} + \frac{x^2}{2}$$

Verified OK.

8.15.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = e^{-2x} + x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

$r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = e^{-2x} + x^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{-3x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -e^{-2x}\left(\int \left(x^2 e^{2x} + 1\right) dx\right) + e^{-x}\left(\int \left(e^{x} x^2 + e^{-x}\right) dx\right)$

  - Compute integrals

    $y_p(x) = \frac{7}{4} + (-1 - x)\,e^{-2x} + \frac{x^2}{2} - \frac{3x}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + \frac{7}{4} + (-1 - x)\,e^{-2x} + \frac{x^2}{2} - \frac{3x}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=exp(-2*x)+x^2,y(x), singsol=all)
```

$$y(x) = \frac{7}{4} + (-c_1 - x - 1)\,\mathrm{e}^{-2x} + \frac{x^2}{2} + c_2\mathrm{e}^{-x} - \frac{3x}{2}$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 41

```
DSolve[y''[x]+3*y'[x]+2*y[x]==Exp[-2*x]+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(2x^2 - 6x + 7\right) + e^{-2x}(-x - 1 + c_1) + c_2e^{-x}$$

8.16 problem Exercise 21.20, page 231

Internal problem ID [4621]
Internal file name [`OUTPUT/4114_Sunday_June_05_2022_12_24_15_PM_20617133/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.20, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 3y' + 2y = x\,\mathrm{e}^{-x}}$$

8.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = x\,\mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(2)} \\
&= \frac{3}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{3}{2} + \frac{1}{2} \\
\lambda_2 &= \frac{3}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= 1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(2)x} + c_2 e^{(1)x}
\end{aligned}$$

Or

$$y = c_1 e^{2x} + c_2 e^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{x}, \mathrm{e}^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-x} + A_2 \mathrm{e}^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1\mathrm{e}^{-x} + 6A_1 x\,\mathrm{e}^{-x} + 6A_2\mathrm{e}^{-x} = x\,\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = \frac{5}{36}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,\mathrm{e}^{-x}}{6} + \frac{5\,\mathrm{e}^{-x}}{36}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{2x} + c_2\mathrm{e}^{x}\right) + \left(\frac{x\,\mathrm{e}^{-x}}{6} + \frac{5\,\mathrm{e}^{-x}}{36}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{2x} + c_2\mathrm{e}^{x} + \frac{x\,\mathrm{e}^{-x}}{6} + \frac{5\,\mathrm{e}^{-x}}{36} \tag{1}$$

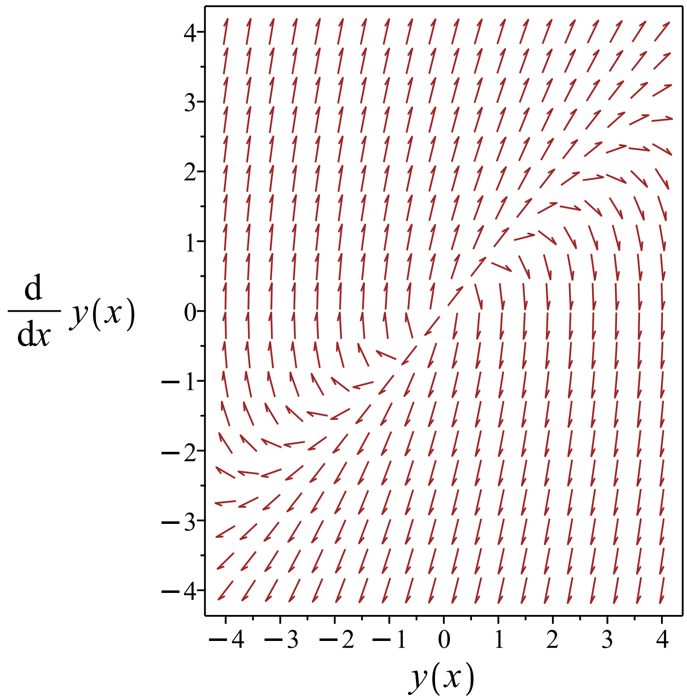


Figure 375: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^x + \frac{x\,e^{-x}}{6} + \frac{5\,e^{-x}}{36}$$

Verified OK.

8.16.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 239: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\mathrm{e}^x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{2x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x\,\mathrm{e}^{-x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^x, \mathrm{e}^{2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 x\,\mathrm{e}^{-x} + A_2 \mathrm{e}^{-x}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-5A_1 \mathrm{e}^{-x} + 6A_1 x\,\mathrm{e}^{-x} + 6A_2 \mathrm{e}^{-x} = x\,\mathrm{e}^{-x}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = \frac{5}{36}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,\mathrm{e}^{-x}}{6} + \frac{5\,\mathrm{e}^{-x}}{36}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x}\right) + \left(\frac{x\,\mathrm{e}^{-x}}{6} + \frac{5\,\mathrm{e}^{-x}}{36}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x} + \frac{x\,\mathrm{e}^{-x}}{6} + \frac{5\,\mathrm{e}^{-x}}{36} \tag{1}$$

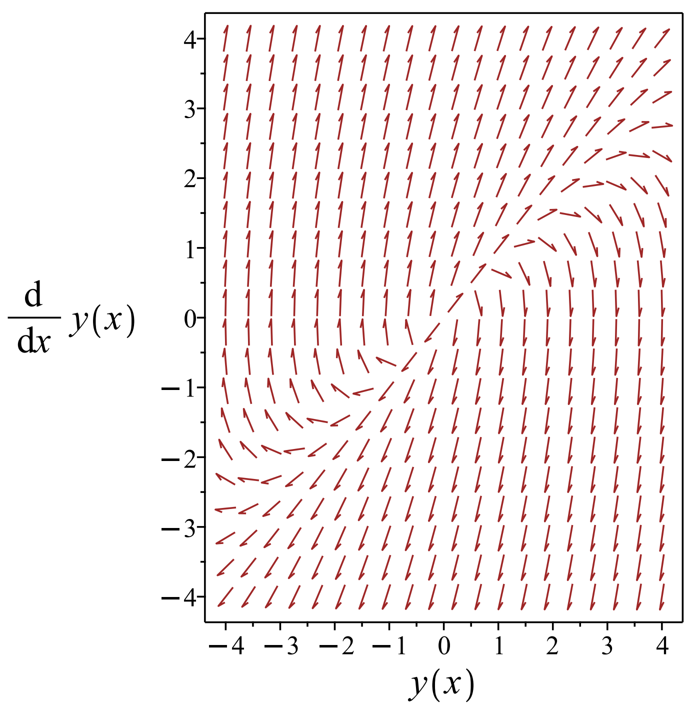


Figure 376: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{2x} + \frac{x\,e^{-x}}{6} + \frac{5\,e^{-x}}{36}$$

Verified OK.

8.16.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = x\,e^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{x} + c_2 e^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x\,e^{-x}\right]$

- ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2\,e^{2x} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{3x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^x\left(\int x\,e^{-2x}dx\right) + e^{2x}\left(\int x\,e^{-3x}dx\right)$

- Compute integrals

  $y_p(x) = \frac{(6x+5)e^{-x}}{36}$

- Substitute particular solution into general solution to ODE

  $y = \frac{(6x+5)e^{-x}}{36} + c_1 e^x + c_2 e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=x*exp(-x),y(x), singsol=all)
```

$$y(x) = \frac{(36c_1 e^{3x} + 36c_2 e^{2x} + 6x + 5)\,e^{-x}}{36}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 34

```
DSolve[y''[x]-3*y'[x]+2*y[x]==x*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{36}e^{-x}(6x+5)+c_1e^x+c_2e^{2x}$$

8.17 problem Exercise 21.21, page 231

Internal problem ID [4622]
Internal file name [`OUTPUT/4115_Sunday_June_05_2022_12_24_23_PM_40398780/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.21, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' - 6y = x + e^{2x}}$$

8.17.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -6, f(x) = x + e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda \,\mathrm{e}^{\lambda x} - 6 \,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -6$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(-6)} \\
&= -\frac{1}{2} \pm \frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{5}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{5}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -3
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(2)x} + c_2 e^{(-3)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x}x\}, \{1, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{2x} x + A_2 + A_3 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 e^{2x} + A_3 - 6A_2 - 6A_3 x = x + e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{5}, A_2 = -\frac{1}{36}, A_3 = -\frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{2x} x}{5} - \frac{1}{36} - \frac{x}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{-3x}\right) + \left(\frac{e^{2x} x}{5} - \frac{1}{36} - \frac{x}{6}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{-3x} + \frac{e^{2x} x}{5} - \frac{1}{36} - \frac{x}{6} \tag{1}$$

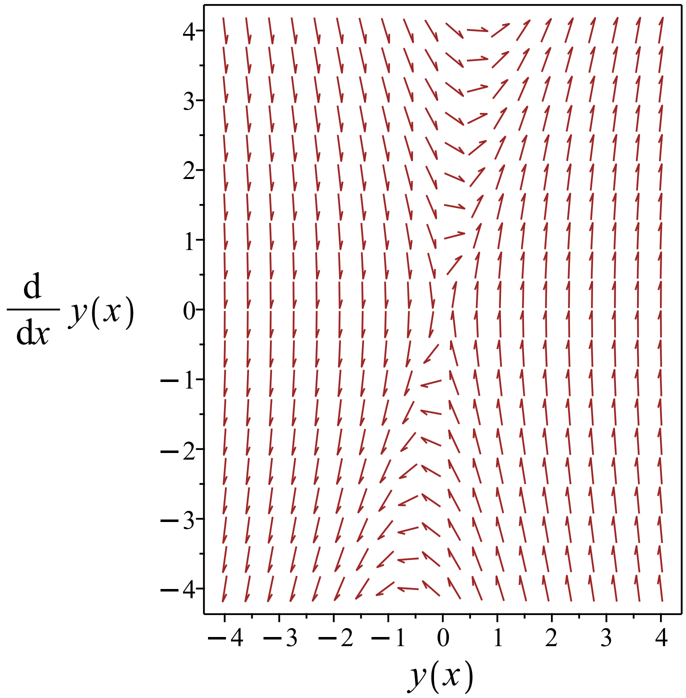


Figure 377: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-3x} + \frac{e^{2x} x}{5} - \frac{1}{36} - \frac{x}{6}$$

Verified OK.

8.17.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= -6
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{25}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 25 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{25z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 241: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\mathrm{e}^{5x}}{5}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{5x}}{5}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{2x}}{5}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-3x}$$
$$y_2 = \frac{\mathrm{e}^{2x}}{5}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & \frac{\mathrm{e}^{2x}}{5} \\ \frac{d}{dx}\left(\mathrm{e}^{-3x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{2x}}{5}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & \frac{\mathrm{e}^{2x}}{5} \\ -3\,\mathrm{e}^{-3x} & \frac{2\,\mathrm{e}^{2x}}{5} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-3x}\right)\left(\frac{2\,\mathrm{e}^{2x}}{5}\right) - \left(\frac{\mathrm{e}^{2x}}{5}\right)\left(-3\,\mathrm{e}^{-3x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-3x}\mathrm{e}^{2x}$$

Which simplifies to

$$W = \mathrm{e}^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{2x}(x+\mathrm{e}^{2x})}{5}}{\mathrm{e}^{-x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(x + \mathrm{e}^{2x})\,\mathrm{e}^{3x}}{5} dx$$

Hence

$$u_1 = -\frac{\mathrm{e}^{5x}}{25} - \frac{\mathrm{e}^{3x} x}{15} + \frac{\mathrm{e}^{3x}}{45}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-3x}(x + \mathrm{e}^{2x})}{\mathrm{e}^{-x}}\,dx$$

Which simplifies to

$$u_2 = \int \left(x\,\mathrm{e}^{-2x} + 1\right) dx$$

Hence

$$u_2 = x - \frac{x\,\mathrm{e}^{-2x}}{2} - \frac{\mathrm{e}^{-2x}}{4}$$

Which simplifies to

$$u_1 = \frac{(-3x + 1)\,\mathrm{e}^{3x}}{45} - \frac{\mathrm{e}^{5x}}{25}$$

$$u_2 = \frac{(-2x - 1)\,\mathrm{e}^{-2x}}{4} + x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{(-3x + 1)\,\mathrm{e}^{3x}}{45} - \frac{\mathrm{e}^{5x}}{25}\right)\mathrm{e}^{-3x} + \frac{\left(\frac{(-2x-1)\mathrm{e}^{-2x}}{4} + x\right)\mathrm{e}^{2x}}{5}$$

Which simplifies to

$$y_p(x) = -\frac{1}{36} + \frac{(-1+5x)\,e^{2x}}{25} - \frac{x}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + \frac{c_2 e^{2x}}{5}\right) + \left(-\frac{1}{36} + \frac{(-1+5x)\,e^{2x}}{25} - \frac{x}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} + \frac{c_2 e^{2x}}{5} - \frac{1}{36} + \frac{(-1+5x)\,e^{2x}}{25} - \frac{x}{6} \tag{1}$$

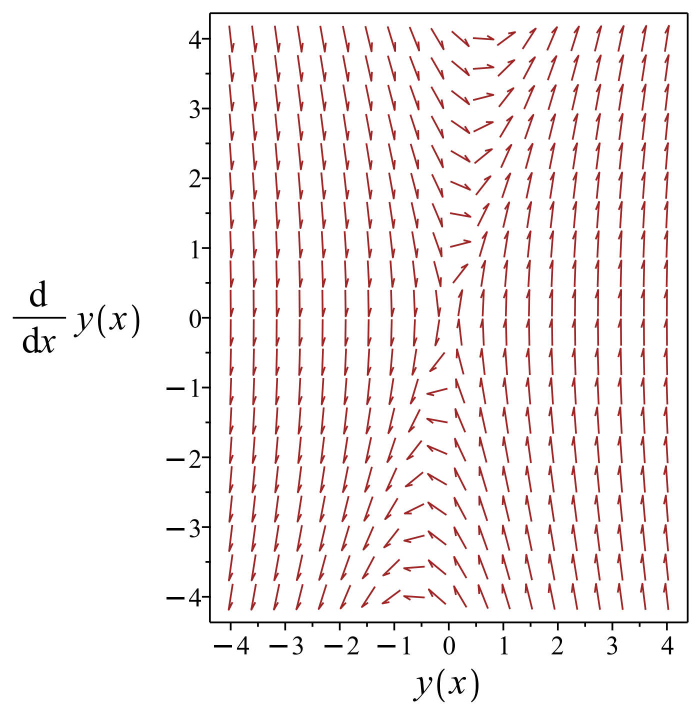


Figure 378: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + \frac{c_2 e^{2x}}{5} - \frac{1}{36} + \frac{(-1+5x)\,e^{2x}}{25} - \frac{x}{6}$$

Verified OK.

8.17.3 Maple step by step solution

Let's solve

$y'' + y' - 6y = x + e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 6 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-3x} + c_2 e^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x + e^{2x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-3x} & e^{2x} \\ -3\,e^{-3x} & 2\,e^{2x} \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = 5\,e^{-x}$

  ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{\left(\mathrm{e}^{5x}\left(\int \left(x\,\mathrm{e}^{-2x}+1\right)dx\right)-\left(\int \left(x+\mathrm{e}^{2x}\right)\mathrm{e}^{3x}dx\right)\right)\mathrm{e}^{-3x}}{5}$$

- Compute integrals

$$y_p(x) = -\frac{1}{36}+\frac{(-1+5x)\,\mathrm{e}^{2x}}{25}-\frac{x}{6}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\mathrm{e}^{-3x}+c_2\mathrm{e}^{2x}-\frac{1}{36}+\frac{(-1+5x)\,\mathrm{e}^{2x}}{25}-\frac{x}{6}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-6*y(x)=x+exp(2*x),y(x), singsol=all)
```

$$y(x) = -\frac{\left(\left(-\frac{6x}{5}-6c_2+\frac{6}{25}\right)\mathrm{e}^{5x}+\left(x+\frac{1}{6}\right)\mathrm{e}^{3x}-6c_1\right)\mathrm{e}^{-3x}}{6}$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 40

```
DSolve[y''[x]+y'[x]-6*y[x]==x+Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{36}(-6x-1)+c_1e^{-3x}+e^{2x}\left(\frac{x}{5}-\frac{1}{25}+c_2\right)$$

8.18 problem Exercise 21.22, page 231

Internal problem ID [4623]
Internal file name [`OUTPUT/4116_Sunday_June_05_2022_12_24_31_PM_29324005/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.22, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \sin(x) + e^{-x}}$$

8.18.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(x) + e^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x)\, c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x)\, c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x) + e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-x}\}, \{\cos(x) x, \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{-x} + A_2 \cos(x) x + A_3 \sin(x) x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{-x} - 2A_2 \sin(x) + 2A_3 \cos(x) = \sin(x) + e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = -\frac{1}{2}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-x}}{2} - \frac{\cos(x) x}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (\cos(x) c_1 + c_2 \sin(x)) + \left(\frac{e^{-x}}{2} - \frac{\cos(x) x}{2}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \cos(x) c_1 + c_2 \sin(x) + \frac{e^{-x}}{2} - \frac{\cos(x) x}{2} \tag{1}
$$

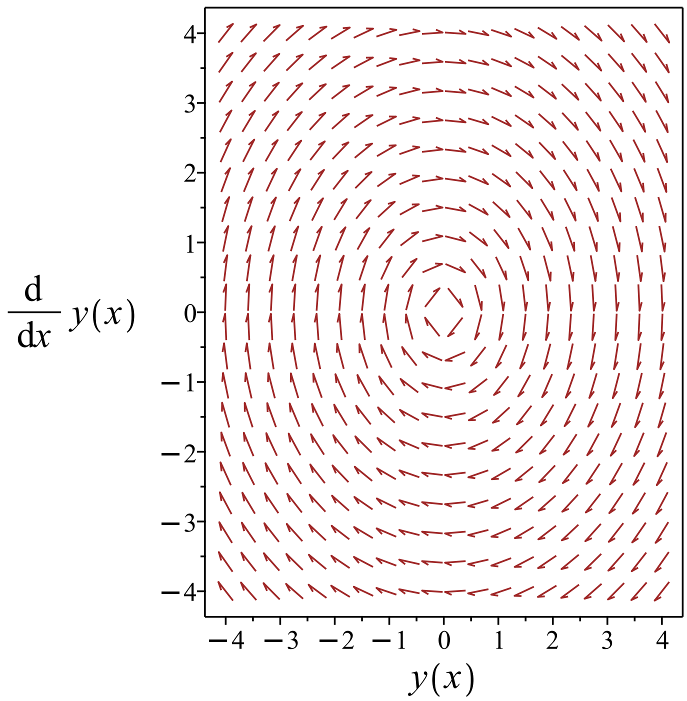


Figure 379: Slope field plot

Verification of solutions

$$
y = \cos(x) c_1 + c_2 \sin(x) + \frac{e^{-x}}{2} - \frac{\cos(x) x}{2}
$$

Verified OK.

8.18.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 243: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2} dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x) (\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x) + e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

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While the set of the basis functions for the homogeneous solution found earlier is

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The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{-x} - 2A_2 \sin(x) + 2A_3 \cos(x) = \sin(x) + e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = -\frac{1}{2}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-x}}{2} - \frac{\cos(x)\, x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\, c_1 + c_2 \sin(x)) + \left(\frac{e^{-x}}{2} - \frac{\cos(x)\, x}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) + \frac{e^{-x}}{2} - \frac{\cos(x) x}{2} \tag{1}$$

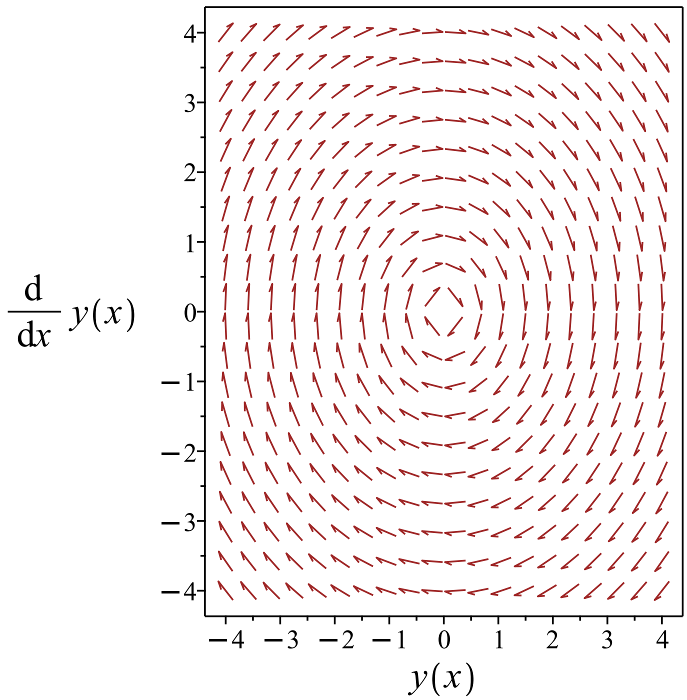

Figure 380: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + \frac{e^{-x}}{2} - \frac{\cos(x) x}{2}$$

Verified OK.

8.18.3 Maple step by step solution

Let's solve

$y'' + y = \sin(x) + e^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x) + \mathrm{e}^{-x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int \sin(x) (\sin(x) + \mathrm{e}^{-x}) dx\right) + \sin(x) \left(\int \cos(x) (\sin(x) + \mathrm{e}^{-x}) dx\right)$

  - Compute integrals

    $y_p(x) = \frac{\mathrm{e}^{-x}}{2} - \frac{\cos(x) x}{2}$

- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + \frac{\mathrm{e}^{-x}}{2} - \frac{\cos(x) x}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+y(x)=sin(x)+exp(-x),y(x), singsol=all)
```

$$y(x) = \frac{e^{-x}}{2} + \frac{(2c_1 - x)\cos(x)}{2} + c_2 \sin(x)$$

✓ Solution by Mathematica

Time used: 0.337 (sec). Leaf size: 36

```
DSolve[y''[x]+y[x]==Sin[x]+Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(2e^{-x} + \sin(x) - 2x\cos(x) + 4c_1\cos(x) + 4c_2\sin(x)\right)$$

8.19 problem Exercise 21.24, page 231

Internal problem ID [4624]
Internal file name [`OUTPUT/4117_Sunday_June_05_2022_12_24_40_PM_59635840/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.24, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \sin(x)^2}$$

8.19.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_2 \cos(2x) - 3A_3 \sin(2x) + A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = \frac{1}{6}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2} + \frac{\cos(2x)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x) c_1 + c_2 \sin(x)) + \left(\frac{1}{2} + \frac{\cos(2x)}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) + \frac{1}{2} + \frac{\cos(2x)}{6} \tag{1}$$

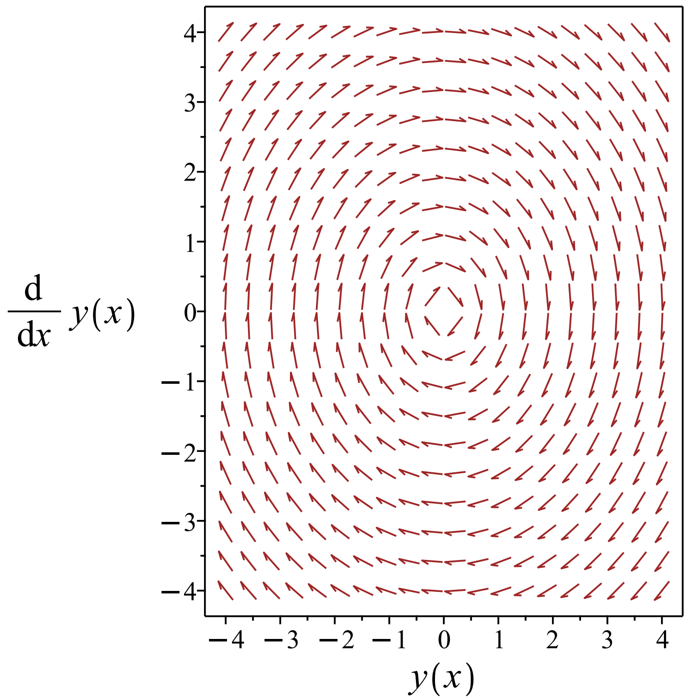

Figure 381: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + \frac{1}{2} + \frac{\cos(2x)}{6}$$

Verified OK.

8.19.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 245: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\, c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

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Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

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$$-3A_2 \cos(2x) - 3A_3 \sin(2x) + A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = \frac{1}{6}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2} + \frac{\cos(2x)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x) c_1 + c_2 \sin(x)) + \left(\frac{1}{2} + \frac{\cos(2x)}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) + \frac{1}{2} + \frac{\cos(2x)}{6} \tag{1}$$

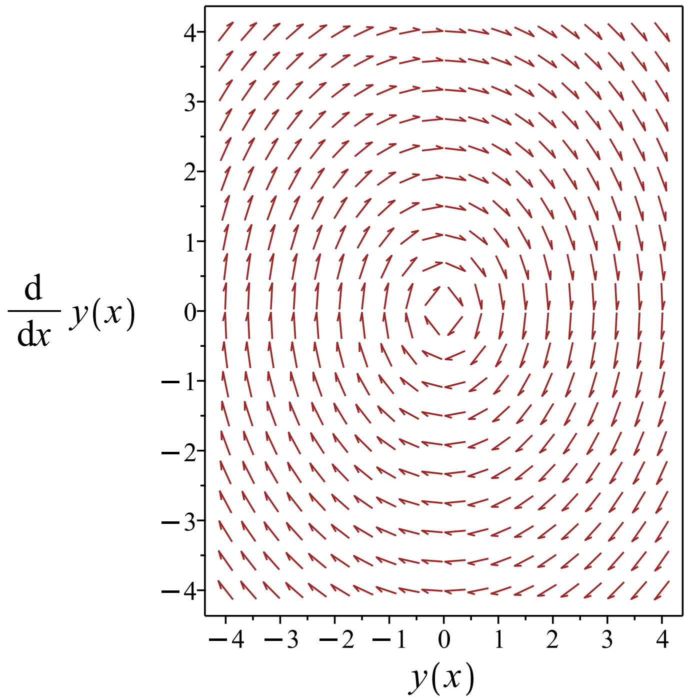


Figure 382: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + \frac{1}{2} + \frac{\cos(2x)}{6}$$

Verified OK.

8.19.3 Maple step by step solution

Let's solve

$y'' + y = \sin(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int \sin(x)^3 dx\right) + \sin(x) \left(\int \cos(x) \sin(x)^2 dx\right)$

  - Compute integrals

    $y_p(x) = \frac{1}{2} + \frac{\cos(2x)}{6}$

- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + \frac{1}{2} + \frac{\cos(2x)}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)+y(x)=sin(x)^2,y(x), singsol=all)
```

$$y(x) = c_2 \sin(x) + \cos(x) c_1 + \frac{\cos(x)^2}{3} + \frac{1}{3}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 27

```
DSolve[y''[x]+y[x]==Sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}(\cos(2x) + 6c_1 \cos(x) + 6c_2 \sin(x) + 3)$$

8.20 problem Exercise 21.27, page 231

Internal problem ID [4625]
Internal file name [`OUTPUT/4118_Sunday_June_05_2022_12_24_48_PM_57729369/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.27, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \sin(x)\sin(2x)}$$

8.20.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(x)\sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x) \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x) x, \sin(x) x\}, \{\cos(3x), \sin(3x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x) x + A_2 \sin(x) x + A_3 \cos(3x) + A_4 \sin(3x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) - 8A_3 \cos(3x) - 8A_4 \sin(3x) = \sin(x) \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{4}, A_3 = \frac{1}{16}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x) x}{4} + \frac{\cos(3x)}{16}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (\cos(x) c_1 + c_2 \sin(x)) + \left(\frac{\sin(x) x}{4} + \frac{\cos(3x)}{16}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \cos(x) c_1 + c_2 \sin(x) + \frac{\sin(x) x}{4} + \frac{\cos(3x)}{16} \tag{1}
$$

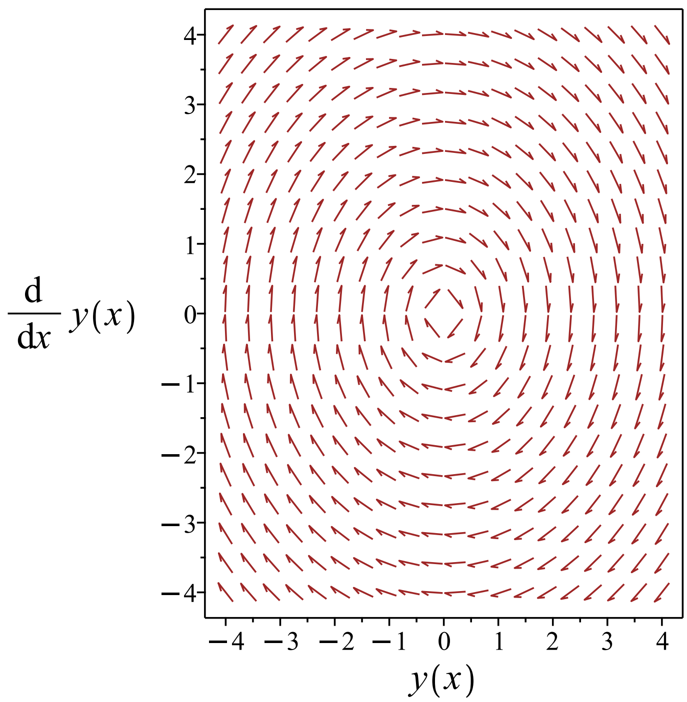


Figure 383: Slope field plot

Verification of solutions

$$
y = \cos(x) c_1 + c_2 \sin(x) + \frac{\sin(x) x}{4} + \frac{\cos(3x)}{16}
$$

Verified OK.

8.20.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 247: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x) (\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)x, \sin(x)x\}, \{\cos(3x), \sin(3x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1\cos(x)x + A_2\sin(x)x + A_3\cos(3x) + A_4\sin(3x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sin(x) + 2A_2\cos(x) - 8A_3\cos(3x) - 8A_4\sin(3x) = \sin(x)\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{4}, A_3 = \frac{1}{16}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)x}{4} + \frac{\cos(3x)}{16}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)c_1 + c_2\sin(x)) + \left(\frac{\sin(x)x}{4} + \frac{\cos(3x)}{16}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) + \frac{\sin(x) x}{4} + \frac{\cos(3x)}{16} \tag{1}$$

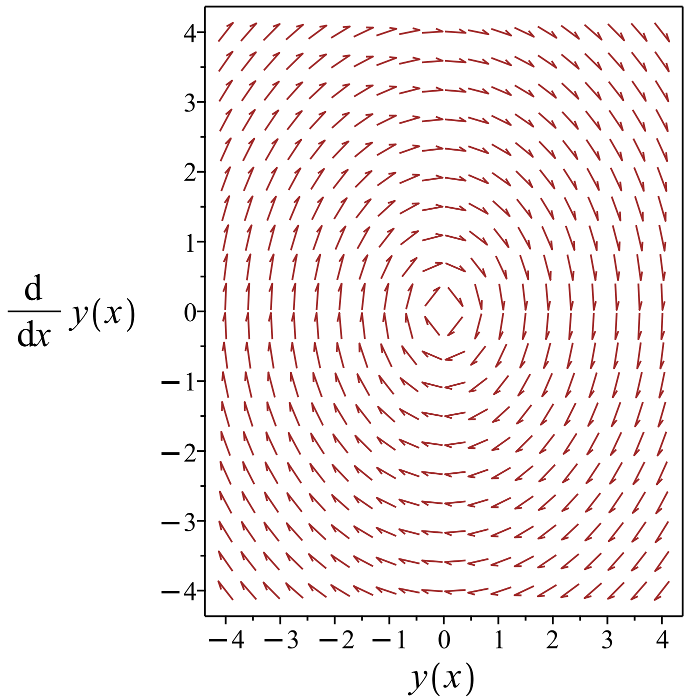


Figure 384: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + \frac{\sin(x) x}{4} + \frac{\cos(3x)}{16}$$

Verified OK.

8.20.3 Maple step by step solution

Let's solve

$y'' + y = \sin(x) \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x) \sin(2x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x) \left(\int \sin(x)^2 \sin(2x)\, dx\right) + \frac{\sin(x)\left(\int (1-\cos(4x)) dx\right)}{4}$

  ◦ Compute integrals

  $y_p(x) = \frac{\sin(x)(-\cos(x)\sin(x)+x)}{4}$

- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + \frac{\sin(x)(-\cos(x)\sin(x)+x)}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+y(x)=sin(2*x)*sin(x),y(x), singsol=all)
```

$$y(x) = -\frac{\sin(x)^2 \cos(x)}{4} + \frac{(4c_2 + x)\sin(x)}{4} + \cos(x) c_1$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 33

```
DSolve[y''[x]+y[x]==Sin[2*x]*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{16}(\cos(3x) + (-1 + 16c_1)\cos(x) + 4(x + 4c_2)\sin(x))$$

8.21 problem Exercise 21.28, page 231

Internal problem ID [4626]
Internal file name [`OUTPUT/4119_Sunday_June_05_2022_12_24_57_PM_12351430/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.28, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 5y' - 6y = e^{3x}}$$

With initial conditions

$$[y(0) = 2, y'(0) = 1]$$

8.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -5 \\ q(x) &= -6 \\ F &= e^{3x} \end{aligned}$$

Hence the ode is

$$y'' - 5y' - 6y = e^{3x}$$

The domain of $p(x) = -5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = e^{3x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.21.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -5, C = -6, f(x) = e^{3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' - 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -5, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 5\lambda e^{\lambda x} - 6 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 5\lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = -6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-5^2 - (4)(1)(-6)}\\ &= \frac{5}{2} \pm \frac{7}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{5}{2} + \frac{7}{2}\\ \lambda_2 &= \frac{5}{2} - \frac{7}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 6\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(6)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 e^{6x} + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{6x} + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{6x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-12A_1 e^{3x} = e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{3x}}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{6x} + c_2 e^{-x}\right) + \left(-\frac{e^{3x}}{12}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{6x} + c_2 e^{-x} - \frac{e^{3x}}{12} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + c_2 - \frac{1}{12} \tag{1A}$$

Taking derivative of the solution gives

$$y' = 6c_1 e^{6x} - c_2 e^{-x} - \frac{e^{3x}}{4}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = 6c_1 - c_2 - \frac{1}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{10}{21}$$
$$c_2 = \frac{45}{28}$$

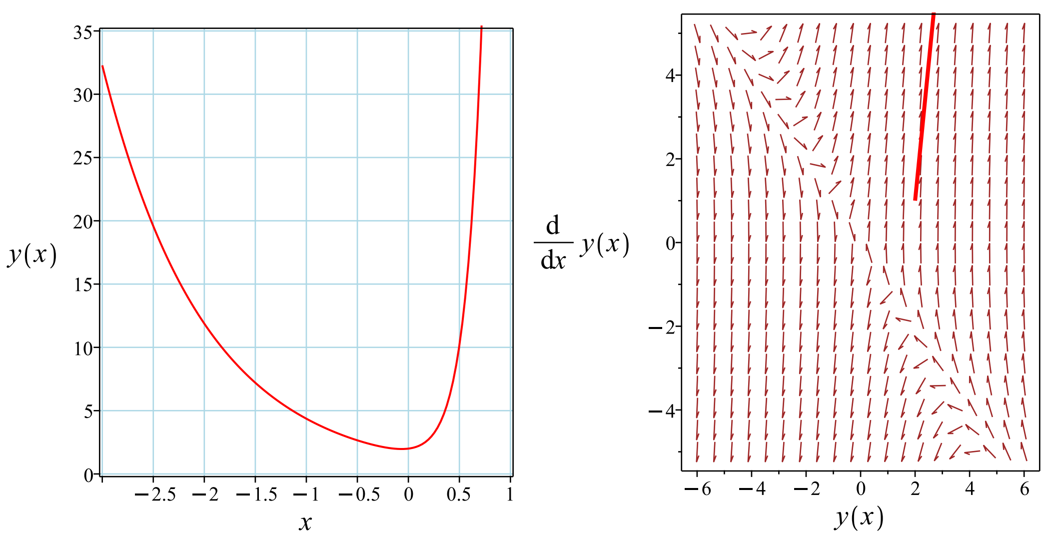
Substituting these values back in above solution results in

$$y = \frac{10\,e^{6x}}{21} + \frac{45\,e^{-x}}{28} - \frac{e^{3x}}{12}$$

Summary

The solution(s) found are the following

$$y = \frac{10\,e^{6x}}{21} + \frac{45\,e^{-x}}{28} - \frac{e^{3x}}{12} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{10\,e^{6x}}{21} + \frac{45\,e^{-x}}{28} - \frac{e^{3x}}{12}$$

Verified OK.

8.21.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{49}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 49 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{49z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 249: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{49}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{7x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dx} \\ &= z_1 e^{\frac{5x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{5x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{5x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{7x}}{7}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{7x}}{7}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' - 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{6x}}{7}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{6x}}{7}, \mathrm{e}^{-x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-12A_1 \mathrm{e}^{3x} = \mathrm{e}^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{3x}}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{6x}}{7}\right) + \left(-\frac{e^{3x}}{12}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + \frac{c_2 e^{6x}}{7} - \frac{e^{3x}}{12} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + \frac{c_2}{7} - \frac{1}{12} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-x} + \frac{6c_2 e^{6x}}{7} - \frac{e^{3x}}{4}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = -c_1 + \frac{6c_2}{7} - \frac{1}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{45}{28} \\ c_2 &= \frac{10}{3} \end{aligned}$$

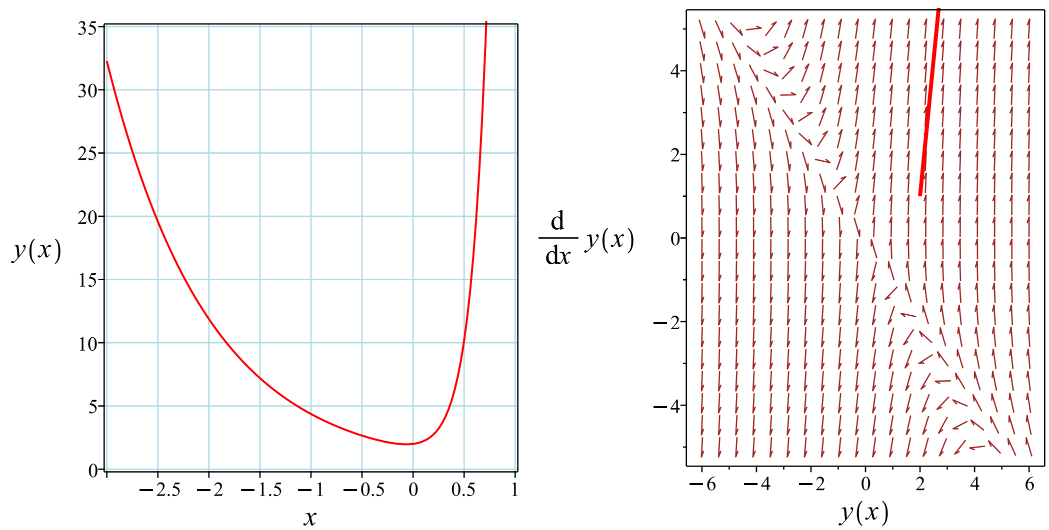
Substituting these values back in above solution results in

$$y = \frac{10\, e^{6x}}{21} + \frac{45\, e^{-x}}{28} - \frac{e^{3x}}{12}$$

Summary

The solution(s) found are the following

$$y = \frac{10\,e^{6x}}{21} + \frac{45\,e^{-x}}{28} - \frac{e^{3x}}{12} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{10\,e^{6x}}{21} + \frac{45\,e^{-x}}{28} - \frac{e^{3x}}{12}$$

Verified OK.

8.21.4 Maple step by step solution

Let's solve

$$\left[y'' - 5y' - 6y = e^{3x}, y(0) = 2, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 5r - 6 = 0$

- Factor the characteristic polynomial

$(r+1)(r-6)=0$

- Roots of the characteristic polynomial

  $r=(-1,6)$

- 1st solution of the homogeneous ODE

  $y_1(x)=e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x)=e^{6x}$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1e^{-x}+c_2e^{6x}+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=e^{3x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x))=\left[\begin{array}{cc} e^{-x} & e^{6x} \\ -e^{-x} & 6\,e^{6x} \end{array}\right]$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x))=7\,e^{5x}$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x)=-\frac{e^{-x}\left(\int e^{4x}dx\right)}{7}+\frac{e^{6x}\left(\int e^{-3x}dx\right)}{7}$

  ◦ Compute integrals

    $y_p(x)=-\frac{e^{3x}}{12}$

- Substitute particular solution into general solution to ODE

  $y=c_1e^{-x}+c_2e^{6x}-\frac{e^{3x}}{12}$

□ Check validity of solution $y=c_1e^{-x}+c_2e^{6x}-\frac{e^{3x}}{12}$

  ◦ Use initial condition $y(0)=2$

    $2=c_1+c_2-\frac{1}{12}$

  ◦ Compute derivative of the solution

$y' = -c_1 e^{-x} + 6c_2 e^{6x} - \frac{e^{3x}}{4}$

- Use the initial condition $y'\big|_{\{x=0\}} = 1$

  $1 = -c_1 + 6c_2 - \frac{1}{4}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{45}{28}, c_2 = \frac{10}{21}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{10\,e^{6x}}{21} + \frac{45\,e^{-x}}{28} - \frac{e^{3x}}{12}$

- Solution to the IVP

  $y = \frac{10\,e^{6x}}{21} + \frac{45\,e^{-x}}{28} - \frac{e^{3x}}{12}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$2)-5*diff(y(x),x)-6*y(x)=exp(3*x),y(0) = 2, D(y)(0) = 1],y(x), singsol=a
```

$$y(x) = \frac{45\,e^{-x}}{28} + \frac{10\,e^{6x}}{21} - \frac{e^{3x}}{12}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 30

```
DSolve[{y''[x]-5*y'[x]-6*y[x]==Exp[3*x],{y[0]==2,y'[0]==1}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{84} e^{-x} \left(-7 e^{4 x}+40 e^{7 x}+135\right)$$

8.22 problem Exercise 21.29, page 231

Internal problem ID [4627]
Internal file name [`OUTPUT/4120_Sunday_June_05_2022_12_25_06_PM_6452886/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.29, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y' - 2y = 5\sin(x)}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

8.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned}
p(x) &= -1 \\
q(x) &= -2 \\
F &= 5\sin(x)
\end{aligned}$$

Hence the ode is

$$y'' - y' - 2y = 5\sin(x)$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 5\sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.22.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -2, f(x) = 5\sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 2e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-2)} \\ &= \frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{3}{2} \\ \lambda_2 &= \frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5 \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(x) - 3A_2 \sin(x) + A_1 \sin(x) - A_2 \cos(x) = 5 \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = -\frac{3}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)}{2} - \frac{3\sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{-x}\right) + \left(\frac{\cos(x)}{2} - \frac{3\sin(x)}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + c_2 e^{-x} + \frac{\cos(x)}{2} - \frac{3\sin(x)}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 + \frac{1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} - c_2 e^{-x} - \frac{\sin(x)}{2} - \frac{3\cos(x)}{2}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = 2c_1 - c_2 - \frac{3}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{3}$$
$$c_2 = \frac{1}{6}$$

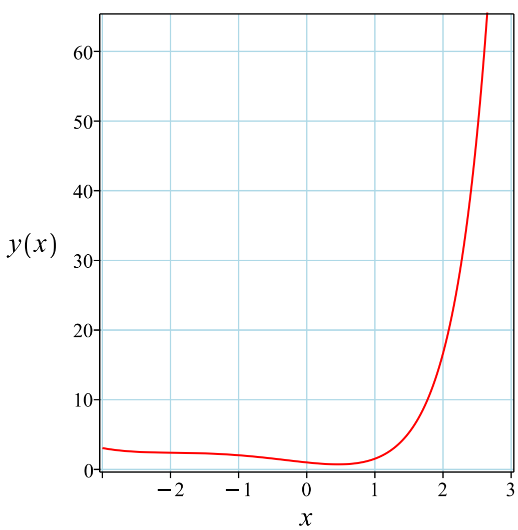
Substituting these values back in above solution results in

$$y = \frac{e^{2x}}{3} + \frac{e^{-x}}{6} + \frac{\cos(x)}{2} - \frac{3\sin(x)}{2}$$

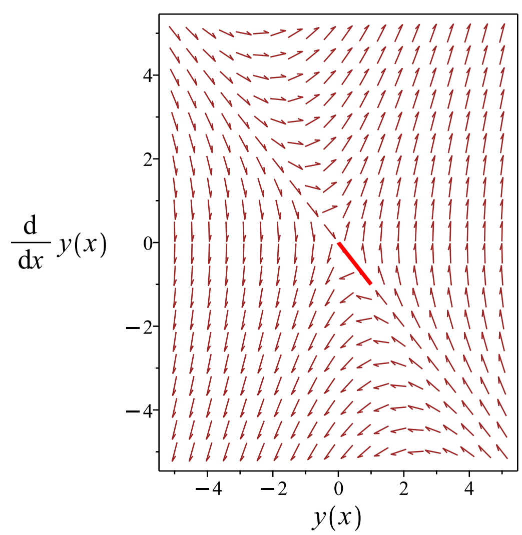
Summary

The solution(s) found are the following

$$y = \frac{e^{2x}}{3} + \frac{e^{-x}}{6} + \frac{\cos(x)}{2} - \frac{3\sin(x)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{2x}}{3} + \frac{e^{-x}}{6} + \frac{\cos(x)}{2} - \frac{3\sin(x)}{2}$$

Verified OK.

8.22.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 251: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(e^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{e^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{-x}\right) + c_2\left(e^{-x}\left(\frac{e^{3x}}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-y'-2y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=c_1\mathrm{e}^{-x}+\frac{c_2\mathrm{e}^{2x}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x),\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{2x}}{3},\mathrm{e}^{-x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p=A_1\cos(x)+A_2\sin(x)$$

The unknowns $\{A_1,A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1\cos(x)-3A_2\sin(x)+A_1\sin(x)-A_2\cos(x)=5\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=\frac{1}{2},A_2=-\frac{3}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p=\frac{\cos(x)}{2}-\frac{3\sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= \left(c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3}\right) + \left(\frac{\cos(x)}{2} - \frac{3\sin(x)}{2}\right)\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3} + \frac{\cos(x)}{2} - \frac{3\sin(x)}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + \frac{c_2}{3} + \frac{1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \mathrm{e}^{-x} + \frac{2c_2 \mathrm{e}^{2x}}{3} - \frac{\sin(x)}{2} - \frac{3\cos(x)}{2}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = -c_1 + \frac{2c_2}{3} - \frac{3}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}c_1 &= \frac{1}{6}\\ c_2 &= 1\end{aligned}$$

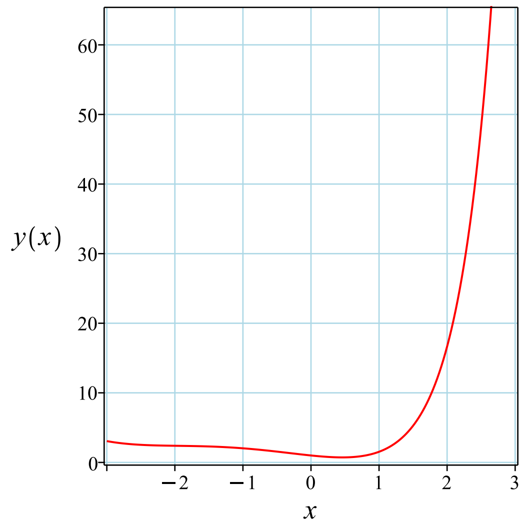
Substituting these values back in above solution results in

$$y = \frac{\mathrm{e}^{2x}}{3} + \frac{\mathrm{e}^{-x}}{6} + \frac{\cos(x)}{2} - \frac{3\sin(x)}{2}$$

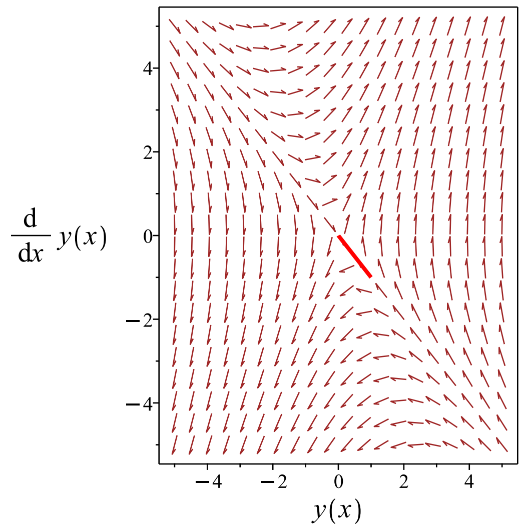
Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{2x}}{3} + \frac{\mathrm{e}^{-x}}{6} + \frac{\cos(x)}{2} - \frac{3\sin(x)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{2x}}{3} + \frac{e^{-x}}{6} + \frac{\cos(x)}{2} - \frac{3\sin(x)}{2}$$

Verified OK.

8.22.4 Maple step by step solution

Let's solve

$$\left[y'' - y' - 2y = 5\sin(x), y(0) = 1, y'\Big|_{\{x=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 2)$

- 1st solution of the homogeneous ODE

$y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 5\sin(x)\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{2x} \\ -\mathrm{e}^{-x} & 2\,\mathrm{e}^{2x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{5\,\mathrm{e}^{-x}\left(\int \sin(x)\mathrm{e}^{x}dx\right)}{3} + \frac{5\,\mathrm{e}^{2x}\left(\int \sin(x)\mathrm{e}^{-2x}dx\right)}{3}$

- Compute integrals

  $y_p(x) = \frac{\cos(x)}{2} - \frac{3\sin(x)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + \frac{\cos(x)}{2} - \frac{3\sin(x)}{2}$

□ Check validity of solution $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + \frac{\cos(x)}{2} - \frac{3\sin(x)}{2}$

- Use initial condition $y(0) = 1$

  $1 = c_1 + c_2 + \frac{1}{2}$

- Compute derivative of the solution

  $y' = -c_1 \mathrm{e}^{-x} + 2c_2 \mathrm{e}^{2x} - \frac{\sin(x)}{2} - \frac{3\cos(x)}{2}$

- Use the initial condition $y'\Big|_{\{x=0\}} = -1$

  $-1 = -c_1 + 2c_2 - \frac{3}{2}$

- Solve for c_1 and c_2

  $\{c_1 = \frac{1}{6}, c_2 = \frac{1}{3}\}$
- Substitute constant values into general solution and simplify

  $y = \frac{e^{2x}}{3} + \frac{e^{-x}}{6} + \frac{\cos(x)}{2} - \frac{3\sin(x)}{2}$

- Solution to the IVP

  $y = \frac{e^{2x}}{3} + \frac{e^{-x}}{6} + \frac{\cos(x)}{2} - \frac{3\sin(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 25

```
dsolve([diff(y(x),x$2)-diff(y(x),x)-2*y(x)=5*sin(x),y(0) = 1, D(y)(0) = -1],y(x), singsol=al
```

$$y(x) = \frac{e^{-x}}{6} + \frac{e^{2x}}{3} + \frac{\cos(x)}{2} - \frac{3\sin(x)}{2}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 30

```
DSolve[{y''[x]-y'[x]-2*y[x]==5*Sin[x],{y[0]==1,y'[0]==-1}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{1}{6}\left(e^{-x} + 2e^{2x} - 9\sin(x) + 3\cos(x)\right)$$

8.23 problem Exercise 21.31, page 231

Internal problem ID [4628]
Internal file name [`OUTPUT/4121_Sunday_June_05_2022_12_25_16_PM_63092722/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.31, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = 8\cos(x)}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = -1, y'\left(\frac{\pi}{2}\right) = 1\right]$$

8.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 9 \\ F &= 8\cos(x) \end{aligned}$$

Hence the ode is

$$y'' + 9y = 8\cos(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The domain of $q(x) = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is also inside this domain. The domain of $F = 8\cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is also inside this domain. Hence solution exists and is unique.

8.23.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = 8\cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 9\,e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)} \\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i \\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i \\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8 \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 \cos(x) + 8A_2 \sin(x) = 8\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3x) + c_2 \sin(3x)) + (\cos(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \cos(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = \frac{\pi}{2}$ in the above gives

$$-1 = -c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 \sin(3x) + 3c_2 \cos(3x) - \sin(x)$$

substituting $y' = 1$ and $x = \frac{\pi}{2}$ in the above gives

$$1 = 3c_1 - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{2}{3}$$
$$c_2 = 1$$

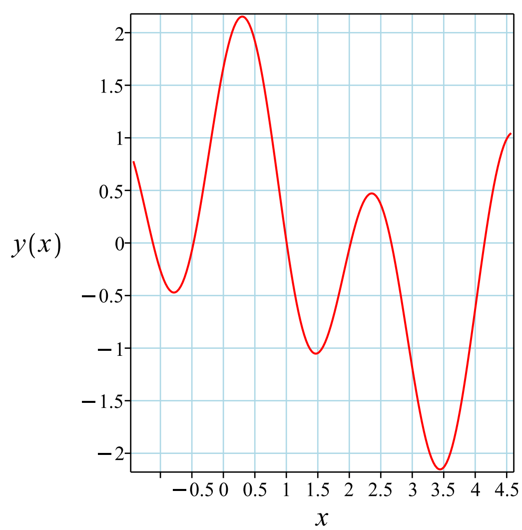
Substituting these values back in above solution results in

$$y = \frac{2\cos(3x)}{3} + \sin(3x) + \cos(x)$$

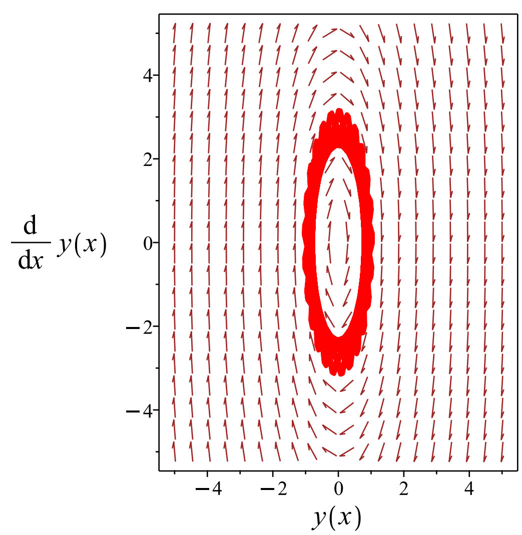
Summary

The solution(s) found are the following

$$y = \frac{2\cos(3x)}{3} + \sin(3x) + \cos(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2\cos(3x)}{3} + \sin(3x) + \cos(x)$$

Verified OK.

8.23.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 253: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2}\, dx \\ &= \cos(3x) \left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8 \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 \cos(x) + 8A_2 \sin(x) = 8 \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + (\cos(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \cos(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = \frac{\pi}{2}$ in the above gives

$$-1 = -\frac{c_2}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 \sin(3x) + c_2 \cos(3x) - \sin(x)$$

substituting $y' = 1$ and $x = \frac{\pi}{2}$ in the above gives

$$1 = 3c_1 - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{2}{3}$$
$$c_2 = 3$$

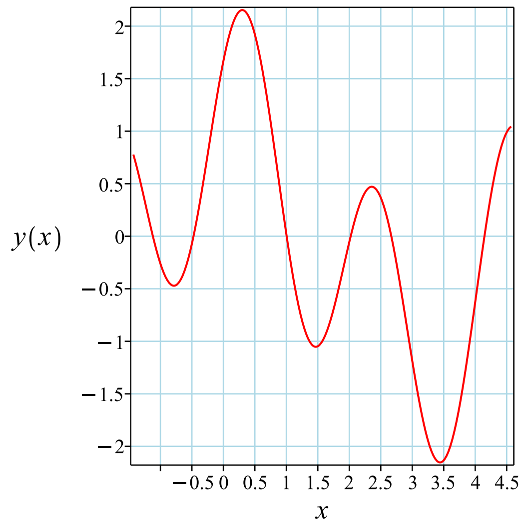
Substituting these values back in above solution results in

$$y = \frac{2\cos(3x)}{3} + \sin(3x) + \cos(x)$$

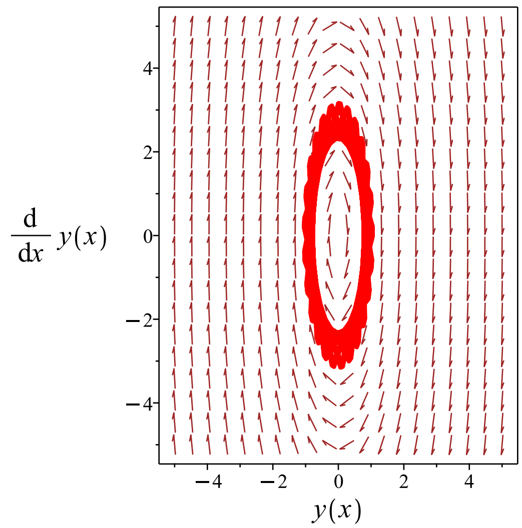
Summary

The solution(s) found are the following

$$y = \frac{2\cos(3x)}{3} + \sin(3x) + \cos(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{2\cos(3x)}{3} + \sin(3x) + \cos(x)$$

Verified OK.

8.23.4 Maple step by step solution

Let's solve

$$\left[y'' + 9y = 8\cos(x), y\left(\frac{\pi}{2}\right) = -1, y'\Big|_{\left\{x=\frac{\pi}{2}\right\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 8\cos(x)\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 3$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{8\cos(3x)\left(\int \cos(x)\sin(3x)dx\right)}{3} + \frac{8\sin(3x)\left(\int \cos(3x)\cos(x)dx\right)}{3}$

- Compute integrals

  $y_p(x) = \cos(x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + \cos(x)$

□ Check validity of solution $y = c_1 \cos(3x) + c_2 \sin(3x) + \cos(x)$

- Use initial condition $y\left(\frac{\pi}{2}\right) = -1$

  $-1 = -c_2$

- Compute derivative of the solution

  $y' = -3c_1 \sin(3x) + 3c_2 \cos(3x) - \sin(x)$

- Use the initial condition $y'\Big|_{\left\{x=\frac{\pi}{2}\right\}} = 1$

  $1 = 3c_1 - 1$

- Solve for c_1 and c_2

  $\{c_1 = \frac{2}{3}, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = \frac{2\cos(3x)}{3} + \sin(3x) + \cos(x)$

- Solution to the IVP

  $y = \frac{2\cos(3x)}{3} + \sin(3x) + \cos(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$2)+9*y(x)=8*cos(x),y(1/2*Pi) = -1, D(y)(1/2*Pi) = 1],y(x), singsol=all)
```

$$y(x) = \sin(3x) + \frac{2\cos(3x)}{3} + \cos(x)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 20

```
DSolve[{y''[x]+9*y[x]==8*Cos[x],{y[Pi/2]==-1,y'[Pi/2]==1}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \sin(3x) + \cos(x) + \frac{2}{3}\cos(3x)$$

8.24 problem Exercise 21.32, page 231

Internal problem ID [4629]
Internal file name [`OUTPUT/4122_Sunday_June_05_2022_12_25_28_PM_89086621/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.32, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 5y' + 6y = e^{x}(2x - 3)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 3]$$

8.24.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -5 \\ q(x) &= 6 \\ F &= e^{x}(2x - 3) \end{aligned}$$

Hence the ode is

$$y'' - 5y' + 6y = e^x(2x - 3)$$

The domain of $p(x) = -5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = e^x(2x - 3)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.24.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -5, C = 6, f(x) = e^x(2x - 3)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 5\lambda e^{\lambda x} + 6 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-5^2 - (4)(1)(6)} \\ &= \frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{5}{2} + \frac{1}{2} \\ \lambda_2 &= \frac{5}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3 \\ \lambda_2 &= 2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(3)x} + c_2 e^{(2)x}\end{aligned}$$

Or

$$y = e^{3x} c_1 + c_2 e^{2x}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{3x} c_1 + c_2 e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x (2x - 3)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x x + A_2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x x - 3A_1 e^x + 2A_2 e^x = e^x(2x - 3)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^x x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x} c_1 + c_2 e^{2x}\right) + (e^x x) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{3x} c_1 + c_2 e^{2x} + e^x x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3\,\mathrm{e}^{3x}c_1 + 2c_2\mathrm{e}^{2x} + \mathrm{e}^x x + \mathrm{e}^x$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = 3c_1 + 2c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 1$$

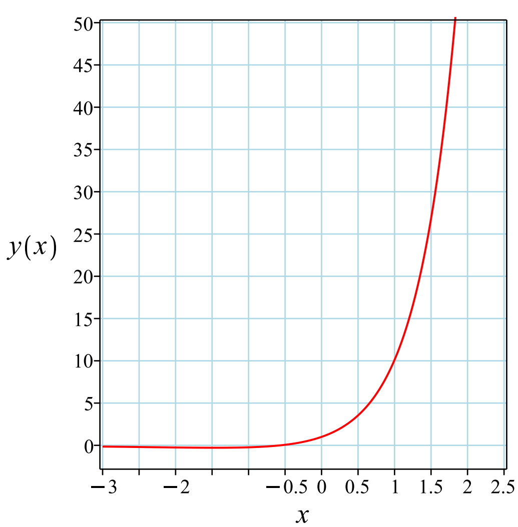
Substituting these values back in above solution results in

$$y = \mathrm{e}^x x + \mathrm{e}^{2x}$$

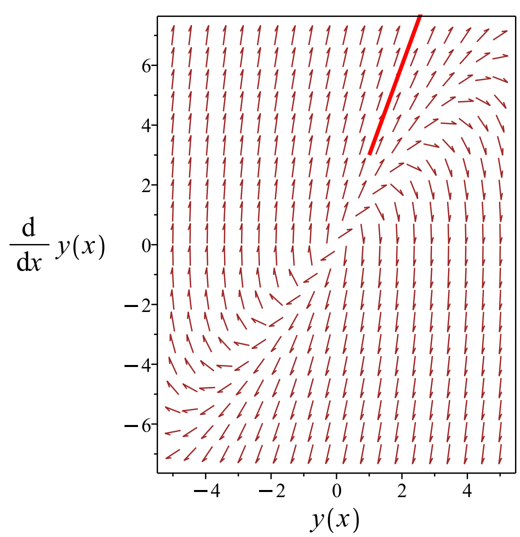
Summary

The solution(s) found are the following

$$y = \mathrm{e}^x x + \mathrm{e}^{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \mathrm{e}^x x + \mathrm{e}^{2x}$$

Verified OK.

8.24.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 255: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\frac{5x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{5x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{5x}}{(y_1)^2}\, dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}(\mathrm{e}^{x})\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 5y' + 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{3x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{x}(2x-3)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}x, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x}x + A_2 e^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{x}x - 3A_1 e^{x} + 2A_2 e^{x} = e^{x}(2x-3)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{x}x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{3x}\right) + \left(e^{x}x\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + c_2 e^{3x} + e^{x}x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} + 3c_2 e^{3x} + e^x x + e^x$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = 2c_1 + 3c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

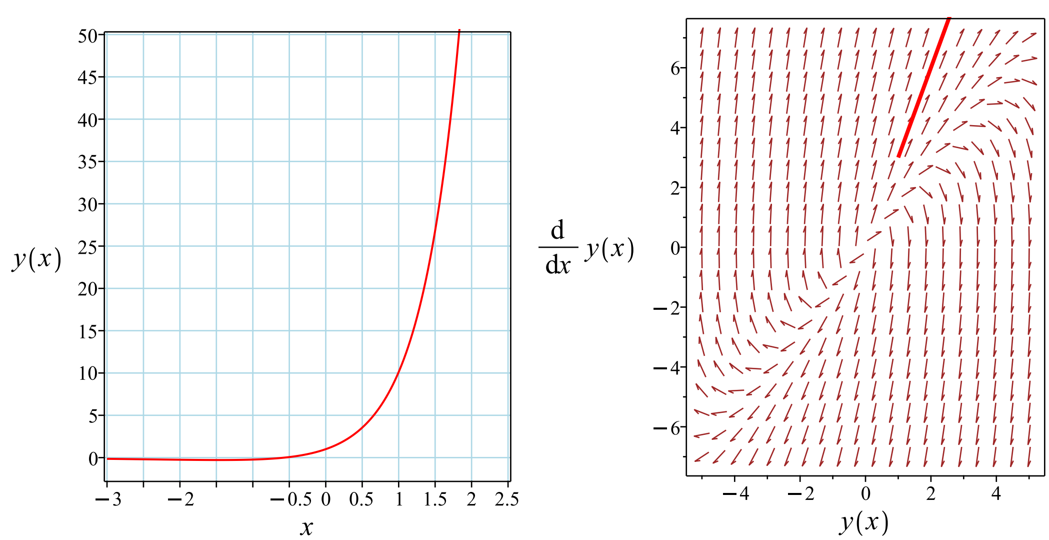
$$c_1 = 1$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = e^x x + e^{2x}$$

Summary
The solution(s) found are the following

$$y = e^x x + e^{2x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^x x + e^{2x}$$

Verified OK.

8.24.4 Maple step by step solution

Let's solve

$\left[y'' - 5y' + 6y = e^x (2x - 3), y(0) = 1, y'\Big|_{\{x=0\}} = 3\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 - 5r + 6 = 0$
- Factor the characteristic polynomial

  $(r - 2)(r - 3) = 0$
- Roots of the characteristic polynomial

  $r = (2, 3)$
- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{2x} + c_2 e^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^x (2x - 3)\right]$
- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & e^{3x} \\ 2\,e^{2x} & 3\,e^{3x} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{5x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{2x}\left(\int (2x-3)\,e^{-x}dx\right) + e^{3x}\left(\int (2x-3)\,e^{-2x}dx\right)$

- Compute integrals

  $y_p(x) = e^{x}x$

- Substitute particular solution into general solution to ODE

  $y = c_1e^{2x} + c_2e^{3x} + e^{x}x$

☐ Check validity of solution $y = c_1e^{2x} + c_2e^{3x} + e^{x}x$

- Use initial condition $y(0) = 1$

  $1 = c_1 + c_2$

- Compute derivative of the solution

  $y' = 2c_1e^{2x} + 3c_2e^{3x} + e^{x}x + e^{x}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 3$

  $3 = 2c_1 + 3c_2 + 1$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = e^{x}x + e^{2x}$

- Solution to the IVP

  $y = e^{x}x + e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve([diff(y(x),x$2)-5*diff(y(x),x)+6*y(x)=exp(x)*(2*x-3),y(0) = 1, D(y)(0) = 3],y(x), sin
```

$$y(x) = e^{2x} + x\,e^{x}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 35

```
DSolve[{y''[x]-5*y'[x]-6*y[x]==Exp[x]*(2*x-3),{y[0]==1,y'[0]==3}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{1}{175}e^{-x}\left(-7e^{2x}(5x-9)+87e^{7x}+25\right)$$

8.25 problem Exercise 21.33, page 231

Internal problem ID [4630]
Internal file name [`OUTPUT/4123_Sunday_June_05_2022_12_25_38_PM_24940963/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 21. Undetermined Coefficients
Problem number: Exercise 21.33, page 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 3y' + 2y = e^{-x}}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

8.25.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= 2 \\ F &= e^{-x} \end{aligned}$$

Hence the ode is

$$y'' - 3y' + 2y = e^{-x}$$

The domain of $p(x) = -3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = e^{-x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.25.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = e^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-3^2 - (4)(1)(2)} \\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= \frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= 1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(1)x}\end{aligned}$$

Or

$$y = c_1 e^{2x} + c_2 e^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 e^{-x} = e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-x}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{x}\right) + \left(\frac{e^{-x}}{6}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + c_2 e^{x} + \frac{e^{-x}}{6} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 + \frac{1}{6} \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x} - \frac{\mathrm{e}^{-x}}{6}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = 2c_1 + c_2 - \frac{1}{6} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{5}{3}$$
$$c_2 = \frac{5}{2}$$

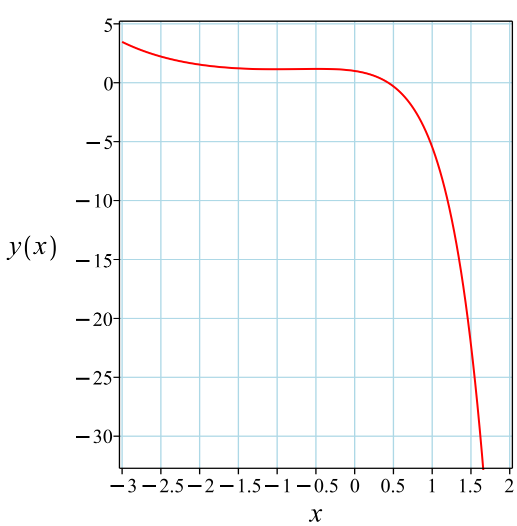
Substituting these values back in above solution results in

$$y = -\frac{5\,\mathrm{e}^{2x}}{3} + \frac{5\,\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{-x}}{6}$$

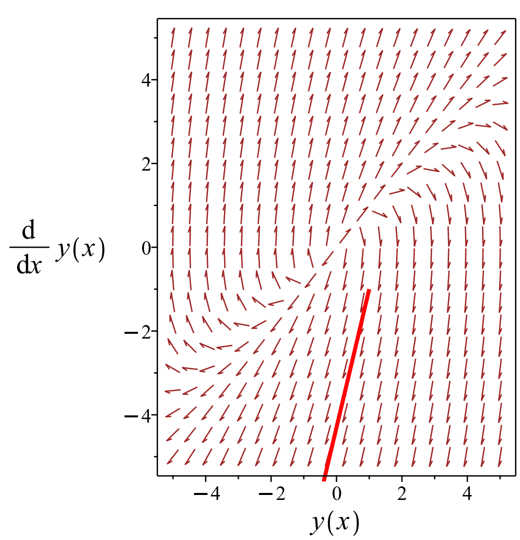
Summary

The solution(s) found are the following

$$y = -\frac{5\,\mathrm{e}^{2x}}{3} + \frac{5\,\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{-x}}{6} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{5\,\mathrm{e}^{2x}}{3} + \frac{5\,\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{-x}}{6}$$

Verified OK.

8.25.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 257: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\ &= z_1 e^{\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(\mathrm{e}^{x})) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-3y'+2y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=c_1\mathrm{e}^{x}+c_2\mathrm{e}^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{x},\mathrm{e}^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p=A_1\mathrm{e}^{-x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1\mathrm{e}^{-x}=\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=\frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p=\frac{\mathrm{e}^{-x}}{6}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x}\right) + \left(\frac{\mathrm{e}^{-x}}{6}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x} + \frac{\mathrm{e}^{-x}}{6} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + c_2 + \frac{1}{6} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 \mathrm{e}^{x} + 2c_2 \mathrm{e}^{2x} - \frac{\mathrm{e}^{-x}}{6}
$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$
-1 = c_1 + 2c_2 - \frac{1}{6} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{5}{2} \\
c_2 &= -\frac{5}{3}
\end{aligned}
$$

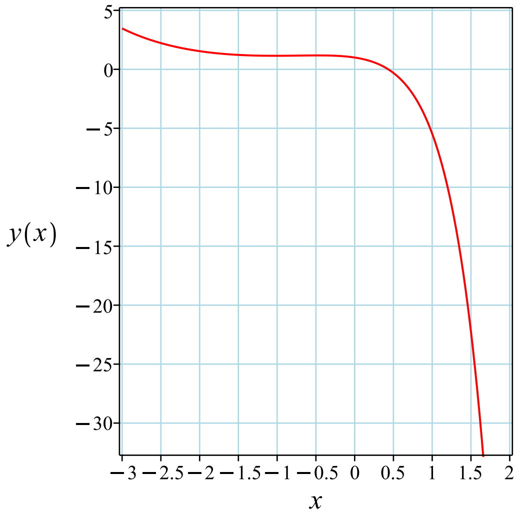
Substituting these values back in above solution results in

$$
y = -\frac{5\,\mathrm{e}^{2x}}{3} + \frac{5\,\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{-x}}{6}
$$

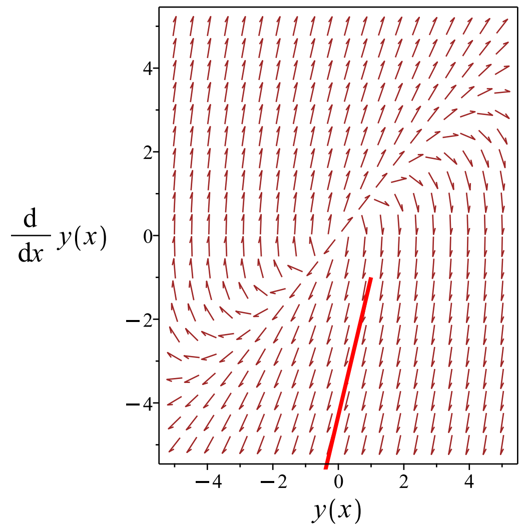
Summary

The solution(s) found are the following

$$
y = -\frac{5\,\mathrm{e}^{2x}}{3} + \frac{5\,\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{-x}}{6} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{5\,e^{2x}}{3} + \frac{5\,e^{x}}{2} + \frac{e^{-x}}{6}$$

Verified OK.

8.25.4 Maple step by step solution

Let's solve

$$\left[y'' - 3y' + 2y = e^{-x}, y(0) = 1, y'\Big|_{\{x=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

$y_1(x) = \mathrm{e}^{x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = \mathrm{e}^{-x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{x} & \mathrm{e}^{2x} \\ \mathrm{e}^{x} & 2\,\mathrm{e}^{2x} \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{3x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{x}\left(\int \mathrm{e}^{-2x}dx\right) + \mathrm{e}^{2x}\left(\int \mathrm{e}^{-3x}dx\right)$

  - Compute integrals

    $y_p(x) = \frac{\mathrm{e}^{-x}}{6}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x} + \frac{\mathrm{e}^{-x}}{6}$

□ Check validity of solution $y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x} + \frac{\mathrm{e}^{-x}}{6}$

  - Use initial condition $y(0) = 1$

    $1 = c_1 + c_2 + \frac{1}{6}$

  - Compute derivative of the solution

    $y' = c_1 \mathrm{e}^{x} + 2c_2 \mathrm{e}^{2x} - \frac{\mathrm{e}^{-x}}{6}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = -1$

    $-1 = c_1 + 2c_2 - \frac{1}{6}$

- Solve for c_1 and c_2

  $\{c_1 = \frac{5}{2}, c_2 = -\frac{5}{3}\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{5\,e^{2x}}{3} + \frac{5\,e^{x}}{2} + \frac{e^{-x}}{6}$

- Solution to the IVP

  $y = -\frac{5\,e^{2x}}{3} + \frac{5\,e^{x}}{2} + \frac{e^{-x}}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=exp(-x),y(0) = 1, D(y)(0) = -1],y(x), singsol=a
```

$$y(x) = -\frac{5\,e^{2x}}{3} + \frac{5\,e^{x}}{2} + \frac{e^{-x}}{6}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 31

```
DSolve[{y''[x]-3*y'[x]+2*y[x]==Exp[-x],{y[0]==1,y'[0]==-1}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{e^{-x}}{6} + \frac{5e^{x}}{2} - \frac{5e^{2x}}{3}$$

9 Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters

9.1 problem Exercise 22.1, page 240

Internal problem ID [4631]
Internal file name [`OUTPUT/4124_Sunday_June_05_2022_12_25_47_PM_34329004/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22.1, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \sec(x)}$$

9.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sec(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=0, C=1$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=1, B=0, C=1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\,dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x))\cos(x) + \sin(x)\,x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\,c_1 + c_2\sin(x)) + (\ln(\cos(x))\cos(x) + \sin(x)\,x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + \sin(x) x \tag{1}$$

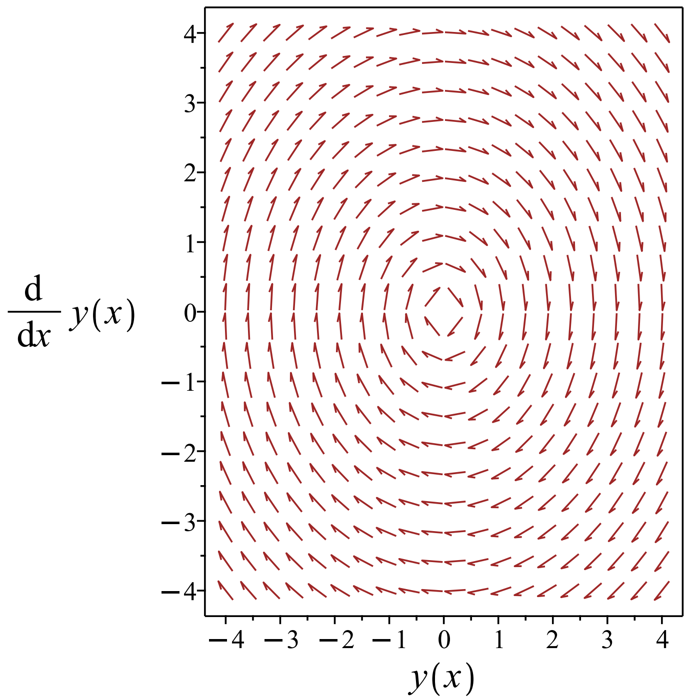


Figure 395: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + \sin(x) x$$

Verified OK.

9.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 259: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \cos(x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\, c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\,dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x))\cos(x) + \sin(x)\,x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\,c_1 + c_2\sin(x)) + (\ln(\cos(x))\cos(x) + \sin(x)\,x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \cos(x)\,c_1 + c_2\sin(x) + \ln(\cos(x))\cos(x) + \sin(x)\,x \tag{1}$$

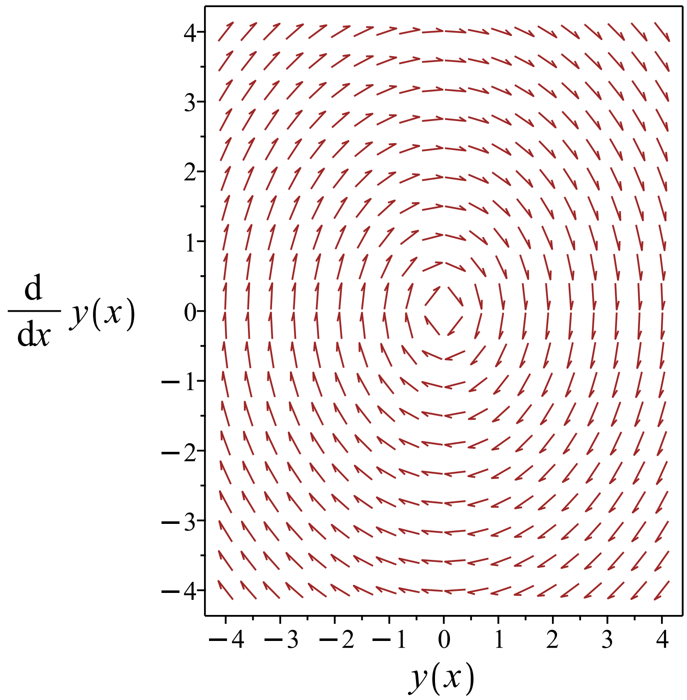


Figure 396: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + \sin(x) x$$

Verified OK.

9.1.3 Maple step by step solution

Let's solve

$y'' + y = \sec(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \cos(x)\, c_1 + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sec(x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int \tan(x)\, dx\right) + \sin(x) \left(\int 1 dx\right)$
  - Compute integrals

    $y_p(x) = \ln(\cos(x)) \cos(x) + \sin(x)\, x$
- Substitute particular solution into general solution to ODE

  $y = \cos(x)\, c_1 + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + \sin(x)\, x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+y(x)=sec(x),y(x), singsol=all)
```

$$y(x) = -\ln(\sec(x))\cos(x) + \cos(x) c_1 + \sin(x)(c_2 + x)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 22

```
DSolve[y''[x]+y[x]==Sec[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x + c_2)\sin(x) + \cos(x)(\log(\cos(x)) + c_1)$$

9.2 problem Exercise 22.2, page 240

Internal problem ID [4632]
Internal file name [`OUTPUT/4125_Sunday_June_05_2022_12_25_55_PM_77896584/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22.2, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \cot(x)}$$

9.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \cot(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\cot(x)}{1} \, dx$$

Which simplifies to

$$u_1 = -\int \cos(x) \, dx$$

Hence

$$u_1 = -\sin(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\cot(x)}{1} \, dx$$

Which simplifies to

$$u_2 = \int \cos(x)\cot(x) \, dx$$

Hence

$$u_2 = \cos(x) + \ln(\csc(x) - \cot(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)\sin(x) + (\cos(x) + \ln(\csc(x) - \cot(x)))\sin(x)$$

Which simplifies to

$$y_p(x) = \sin(x)\ln(\csc(x) - \cot(x))$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (\cos(x) c_1 + c_2 \sin(x)) + (\sin(x) \ln(\csc(x) - \cot(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \cos(x) c_1 + c_2 \sin(x) + \sin(x) \ln(\csc(x) - \cot(x)) \tag{1}
$$

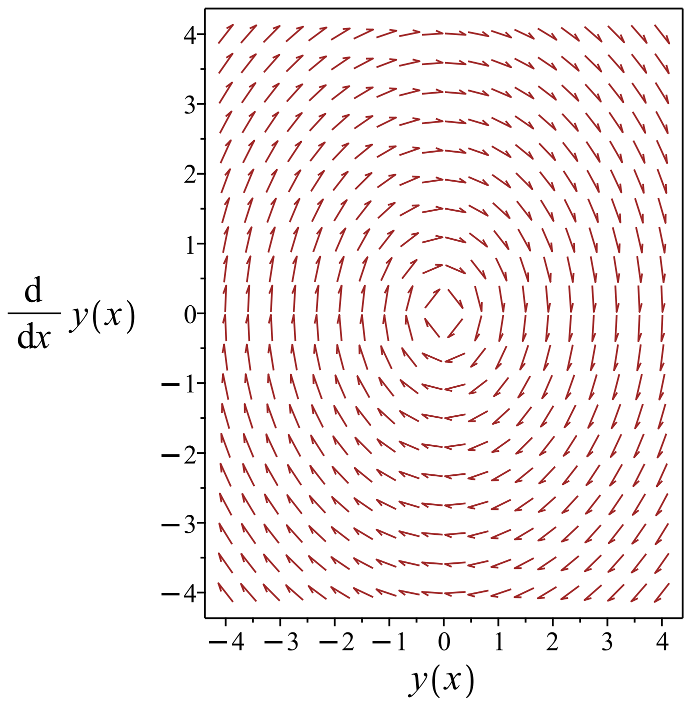


Figure 397: Slope field plot

Verification of solutions

$$
y = \cos(x) c_1 + c_2 \sin(x) + \sin(x) \ln(\csc(x) - \cot(x))
$$

Verified OK.

9.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 261: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x) (\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\, c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\cot(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \cos(x)\,dx$$

Hence

$$u_1 = -\sin(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\cot(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int \cos(x)\cot(x)\,dx$$

Hence

$$u_2 = \cos(x) + \ln(\csc(x) - \cot(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)\sin(x) + (\cos(x) + \ln(\csc(x) - \cot(x)))\sin(x)$$

Which simplifies to

$$y_p(x) = \sin(x)\ln(\csc(x) - \cot(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\,c_1 + c_2\sin(x)) + (\sin(x)\ln(\csc(x) - \cot(x))) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) + \sin(x) \ln(\csc(x) - \cot(x)) \tag{1}$$

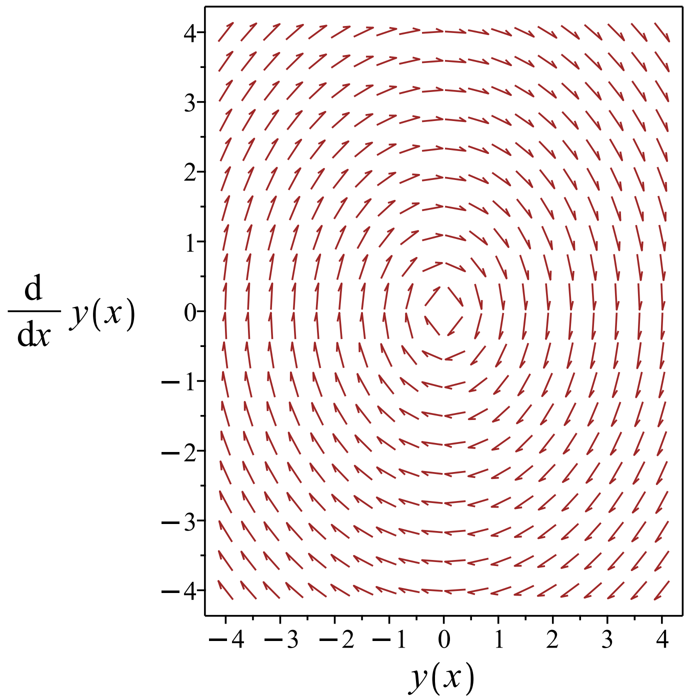


Figure 398: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + \sin(x) \ln(\csc(x) - \cot(x))$$

Verified OK.

9.2.3 Maple step by step solution

Let's solve

$y'' + y = \cot(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cot(x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int \cos(x) dx\right) + \sin(x) \left(\int \cos(x) \cot(x) dx\right)$
  - Compute integrals

    $y_p(x) = \sin(x) \ln(\csc(x) - \cot(x))$

- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + \sin(x) \ln(\csc(x) - \cot(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+y(x)=cot(x),y(x), singsol=all)
```

$$y(x) = c_2 \sin(x) + \cos(x) c_1 + \sin(x) \ln(\csc(x) - \cot(x))$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 33

```
DSolve[y''[x]+y[x]==Cot[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(x) + \sin(x) \left(\log\left(\sin\left(\frac{x}{2}\right)\right) - \log\left(\cos\left(\frac{x}{2}\right)\right) + c_2\right)$$

9.3 problem Exercise 22.3, page 240

Internal problem ID [4633]
Internal file name [`OUTPUT/4126_Sunday_June_05_2022_12_26_02_PM_86537023/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22.3, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \sec(x)^2}$$

9.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sec(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)^2}{1}\, dx$$

Which simplifies to

$$u_1 = -\int \sec(x)\tan(x)\, dx$$

Hence

$$u_1 = -\sec(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)^2}{1}\, dx$$

Which simplifies to

$$u_2 = \int \sec(x)\, dx$$

Hence

$$u_2 = \ln(\sec(x) + \tan(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)\sec(x) + \ln(\sec(x) + \tan(x))\sin(x)$$

Which simplifies to

$$y_p(x) = -1 + \ln(\sec(x) + \tan(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\, c_1 + c_2 \sin(x)) + (-1 + \ln(\sec(x) + \tan(x)) \sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x)\, c_1 + c_2 \sin(x) - 1 + \ln(\sec(x) + \tan(x)) \sin(x) \tag{1}$$

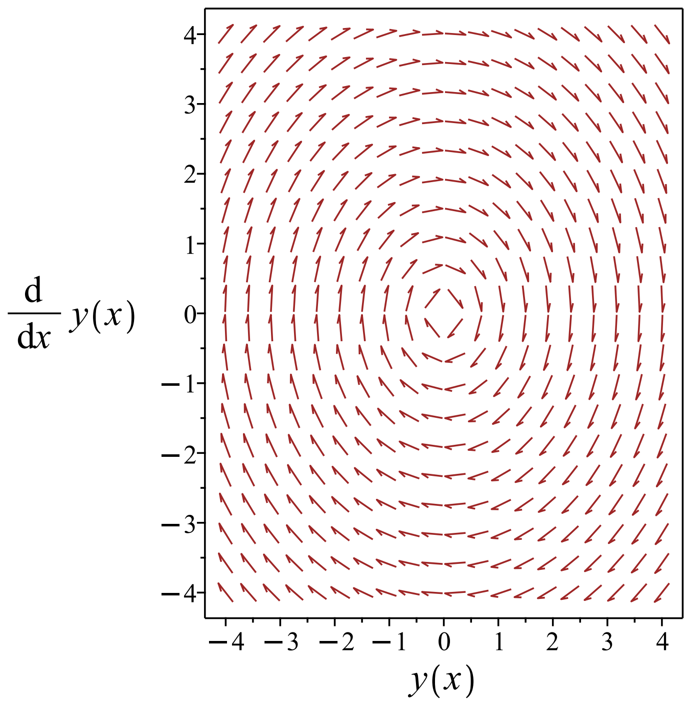


Figure 399: Slope field plot

Verification of solutions

$$y = \cos(x)\, c_1 + c_2 \sin(x) - 1 + \ln(\sec(x) + \tan(x)) \sin(x)$$

Verified OK.

9.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 263: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\, c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)^2}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \sec(x)\tan(x)\,dx$$

Hence

$$u_1 = -\sec(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)^2}{1}\,dx$$

Which simplifies to

$$u_2 = \int \sec(x)\,dx$$

Hence

$$u_2 = \ln(\sec(x) + \tan(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)\sec(x) + \ln(\sec(x) + \tan(x))\sin(x)$$

Which simplifies to

$$y_p(x) = -1 + \ln(\sec(x) + \tan(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\,c_1 + c_2\sin(x)) + (-1 + \ln(\sec(x) + \tan(x))\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) - 1 + \ln(\sec(x) + \tan(x)) \sin(x) \tag{1}$$

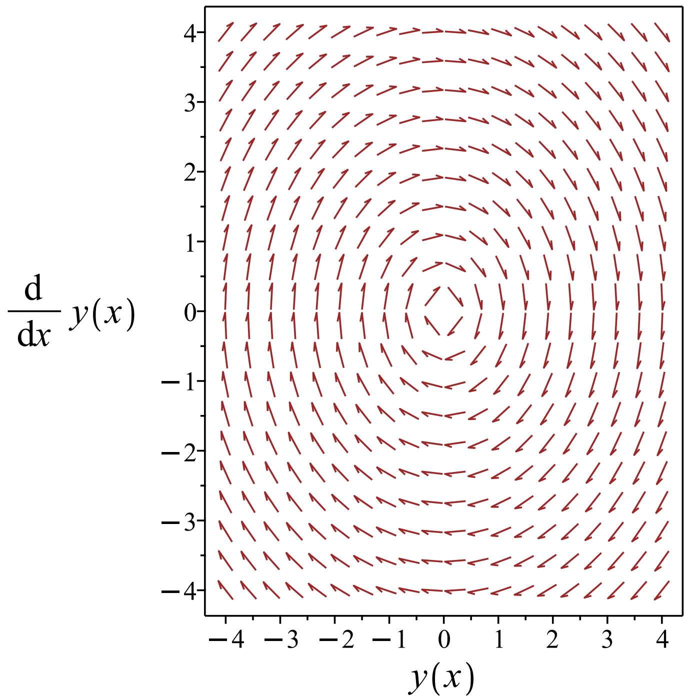


Figure 400: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - 1 + \ln(\sec(x) + \tan(x)) \sin(x)$$

Verified OK.

9.3.3 Maple step by step solution

Let's solve

$y'' + y = \sec(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$
- □ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sec(x)^2\right]$
  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int \sec(x) \tan(x) \, dx\right) + \sin(x) \left(\int \sec(x) \, dx\right)$
  - Compute integrals

    $y_p(x) = -1 + \ln(\sec(x) + \tan(x)) \sin(x)$
- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) - 1 + \ln(\sec(x) + \tan(x)) \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)=sec(x)^2,y(x), singsol=all)
```

$$y(x) = c_2 \sin(x) + \cos(x) c_1 + \ln(\sec(x) + \tan(x)) \sin(x) - 1$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 28

```
DSolve[y''[x]+y[x]==Sec[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\sin(x)\operatorname{arctanh}\left(\tan\left(\frac{x}{2}\right)\right) + c_1\cos(x) + c_2\sin(x) - 1$$

9.4 problem Exercise 22.4, page 240

Internal problem ID [4634]
Internal file name [`OUTPUT/4127_Sunday_June_05_2022_12_26_10_PM_9300741/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22.4, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = \sin(x)^2}$$

9.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = \sin(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1,B=0,C=-1$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=1,B=0,C=-1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_2 \cos(2x) - 5A_3 \sin(2x) - A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{10}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{2} + \frac{\cos(2x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 e^{-x}\right) + \left(-\frac{1}{2} + \frac{\cos(2x)}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} - \frac{1}{2} + \frac{\cos(2x)}{10} \tag{1}$$

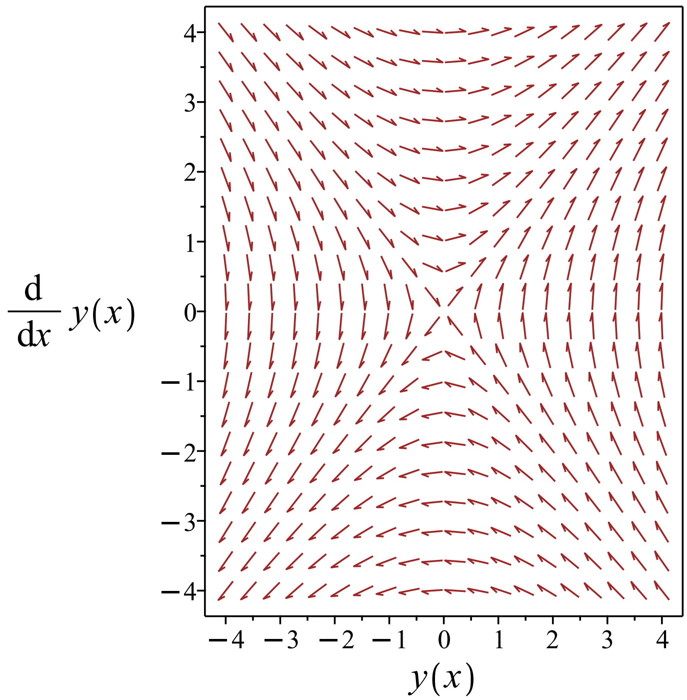


Figure 401: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} - \frac{1}{2} + \frac{\cos(2x)}{10}$$

Verified OK.

9.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 265: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\ &= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^x}{2}, e^{-x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_2 \cos(2x) - 5A_3 \sin(2x) - A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{10}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{2} + \frac{\cos(2x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^x}{2}\right) + \left(-\frac{1}{2} + \frac{\cos(2x)}{10}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - \frac{1}{2} + \frac{\cos(2x)}{10} \tag{1}$$

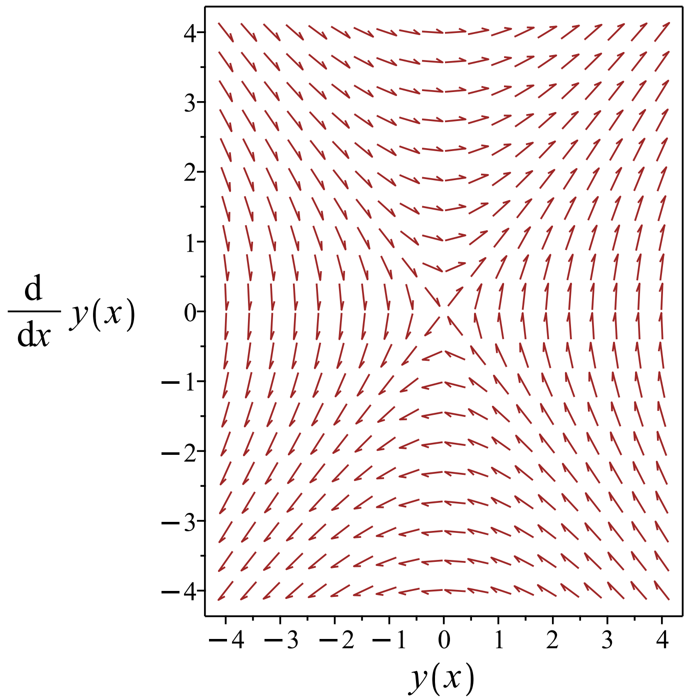


Figure 402: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - \frac{1}{2} + \frac{\cos(2x)}{10}$$

Verified OK.

9.4.3 Maple step by step solution

Let's solve

$y'' - y = \sin(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(x)^2\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\mathrm{e}^{-x}\left(\int \mathrm{e}^{x}\sin(x)^2 dx\right)}{2} + \frac{\mathrm{e}^{x}\left(\int \sin(x)^2 \mathrm{e}^{-x} dx\right)}{2}$
  - Compute integrals

    $y_p(x) = -\frac{1}{2} + \frac{\cos(2x)}{10}$
- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} - \frac{1}{2} + \frac{\cos(2x)}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-y(x)=sin(x)^2,y(x), singsol=all)
```

$$y(x) = c_2 e^x + e^{-x} c_1 + \frac{\cos(x)^2}{5} - \frac{3}{5}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 30

```
DSolve[y''[x]-y[x]==Sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{10}(\cos(2x) - 5) + c_1 e^x + c_2 e^{-x}$$

9.5 problem Exercise 22.5, page 240

Internal problem ID [4635]
Internal file name [`OUTPUT/4128_Sunday_June_05_2022_12_26_18_PM_8396343/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22.5, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \sin(x)^2}$$

9.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_2 \cos(2x) - 3A_3 \sin(2x) + A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = \frac{1}{6}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2} + \frac{\cos(2x)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x) c_1 + c_2 \sin(x)) + \left(\frac{1}{2} + \frac{\cos(2x)}{6}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) + \frac{1}{2} + \frac{\cos(2x)}{6} \tag{1}$$

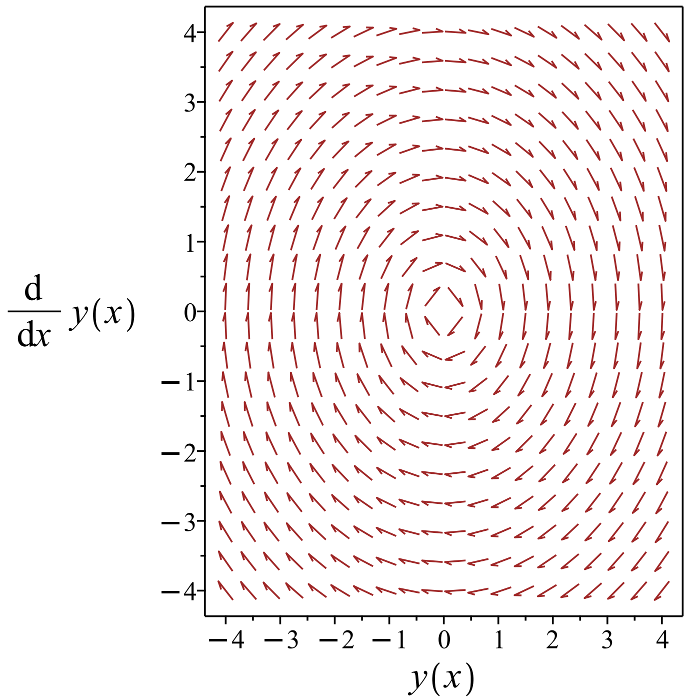


Figure 403: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + \frac{1}{2} + \frac{\cos(2x)}{6}$$

Verified OK.

9.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\,e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 267: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \cos(x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\,c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_2 \cos(2x) - 3A_3 \sin(2x) + A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = \frac{1}{6}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2} + \frac{\cos(2x)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x) c_1 + c_2 \sin(x)) + \left(\frac{1}{2} + \frac{\cos(2x)}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) + \frac{1}{2} + \frac{\cos(2x)}{6} \tag{1}$$

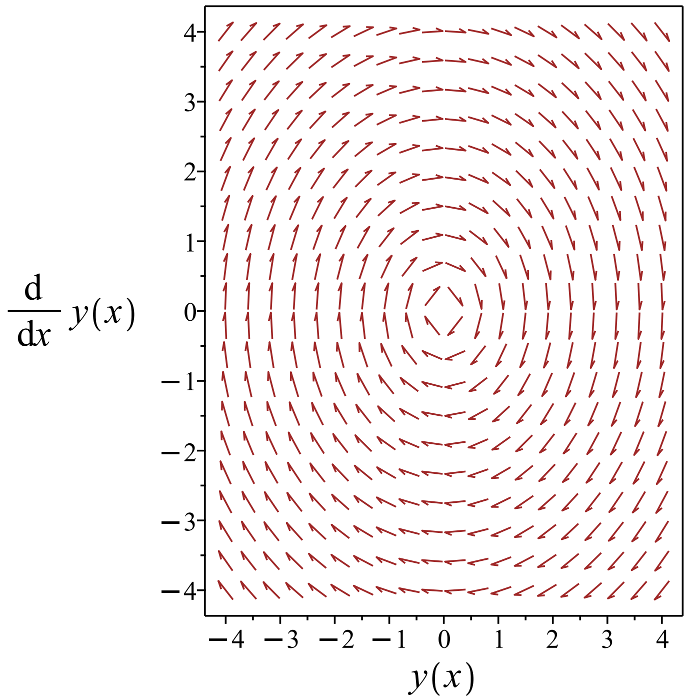


Figure 404: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + \frac{1}{2} + \frac{\cos(2x)}{6}$$

Verified OK.

9.5.3 Maple step by step solution

Let's solve

$y'' + y = \sin(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)^2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x) \left(\int \sin(x)^3 dx\right) + \sin(x) \left(\int \cos(x) \sin(x)^2 dx\right)$

  ◦ Compute integrals

  $y_p(x) = \frac{1}{2} + \frac{\cos(2x)}{6}$

- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + \frac{1}{2} + \frac{\cos(2x)}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)+y(x)=sin(x)^2,y(x), singsol=all)
```

$$y(x) = c_2 \sin(x) + \cos(x) c_1 + \frac{\cos(x)^2}{3} + \frac{1}{3}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 27

```
DSolve[y''[x]+y[x]==Sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}(\cos(2x) + 6c_1 \cos(x) + 6c_2 \sin(x) + 3)$$

9.6 problem Exercise 22.6, page 240

Internal problem ID [4636]
Internal file name [`OUTPUT/4129_Sunday_June_05_2022_12_26_26_PM_17895165/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22.6, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 3y' + 2y = 12\,\mathrm{e}^{x}}$$

9.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = 12\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 3\lambda\, \mathrm{e}^{\lambda x} + 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x}, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 \mathrm{e}^{x} = 12\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}\right) + \left(2\,\mathrm{e}^{x}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x} + 2\,\mathrm{e}^{x} \tag{1}$$

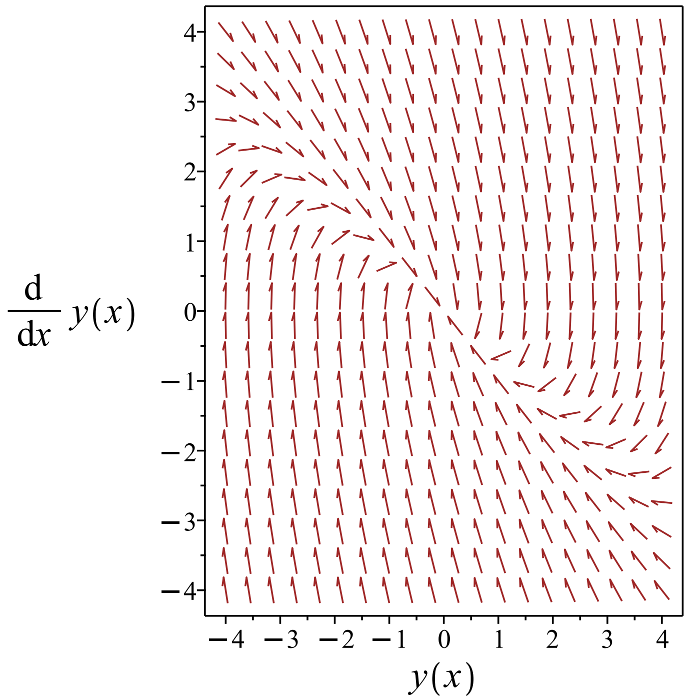


Figure 405: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} + 2\,e^{x}$$

Verified OK.

9.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 269: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-2x}\right) + c_2 \left(e^{-2x} (e^{x})\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 e^{-2x} + c_2 e^{-x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
12\, e^{x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{e^{x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{e^{-2x}, e^{-x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 e^{x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
6A_1 e^{x} = 12\, e^{x}
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-x}\right) + \left(2\,\mathrm{e}^{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-x} + 2\,\mathrm{e}^{x} \tag{1}$$

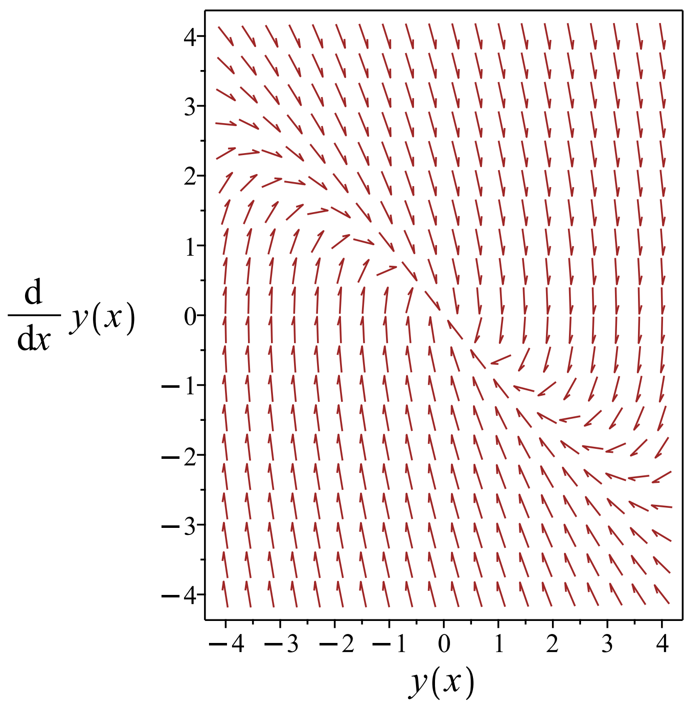


Figure 406: Slope field plot

Verification of solutions

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-x} + 2\,\mathrm{e}^{x}$$

Verified OK.

9.6.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = 12\,e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 12\,e^{x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-3x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -12\,e^{-2x}\left(\int e^{3x}dx\right) + 12\,e^{-x}\left(\int e^{2x}dx\right)$

- Compute integrals

  $y_p(x) = 2\,\mathrm{e}^{x}$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-x} + 2\,\mathrm{e}^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=12*exp(x),y(x), singsol=all)
```

$$y(x) = -\left(-2\,\mathrm{e}^{3x} - c_2\mathrm{e}^{x} + c_1\right)\mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 27

```
DSolve[y''[x]+3*y'[x]+2*y[x]==12*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(2e^{3x} + c_2e^{x} + c_1\right)$$

9.7 problem Exercise 22.7, page 240

Internal problem ID [4637]
Internal file name [`OUTPUT/4130_Sunday_June_05_2022_12_26_35_PM_51578414/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22.7, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + y = x^2 \mathrm{e}^{-x}}$$

9.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = x^2 \mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-x} + c_2 x\, e^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + x\, e^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^{-x}, x^2 e^{-x}, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{-x}, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-x}, x^2e^{-x}, x^3e^{-x}\}]$$

Since $x\,e^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2e^{-x}, x^3e^{-x}, x^4e^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2e^{-x} + A_2x^3e^{-x} + A_3x^4e^{-x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1e^{-x} + 6A_2x\,e^{-x} + 12A_3x^2e^{-x} = x^2e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = 0, A_3 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^4e^{-x}}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{-x} + x\,e^{-x}c_2\right) + \left(\frac{x^4e^{-x}}{12}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2x + c_1) + \frac{x^4e^{-x}}{12}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + \frac{x^4 e^{-x}}{12} \tag{1}$$

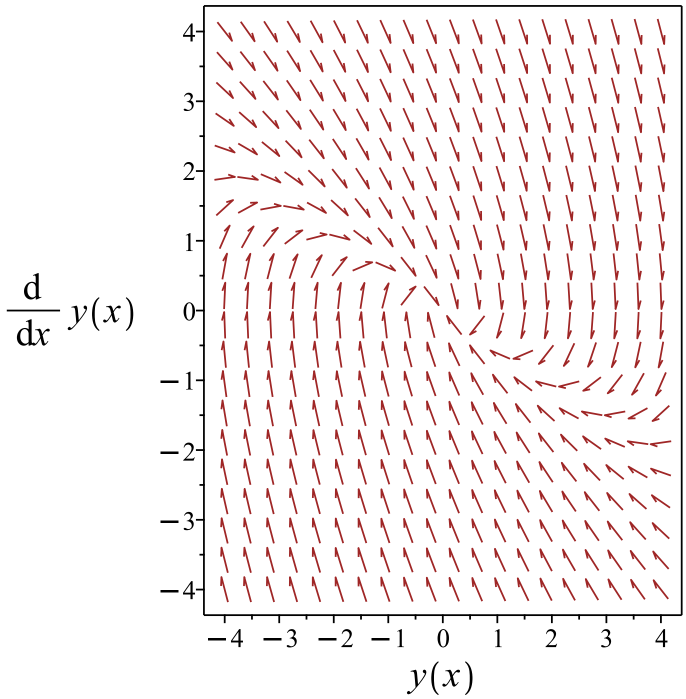


Figure 407: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + \frac{x^4 e^{-x}}{12}$$

Verified OK.

9.7.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= \mathrm{e}^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \mathrm{e}^{-x}\mathrm{e}^{x}x^2 \\ (y\,\mathrm{e}^{x})'' &= \mathrm{e}^{-x}\mathrm{e}^{x}x^2 \end{aligned}$$

Integrating once gives

$$(y\,\mathrm{e}^{x})' = \frac{x^3}{3} + c_1$$

Integrating again gives

$$(y\,\mathrm{e}^{x}) = \frac{1}{12}x^4 + c_1 x + c_2$$

Hence the solution is

$$y = \frac{\frac{1}{12}x^4 + c_1 x + c_2}{\mathrm{e}^{x}}$$

Or

$$y = \frac{x^4\mathrm{e}^{-x}}{12} + c_1 x\,\mathrm{e}^{-x} + c_2\mathrm{e}^{-x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{x^4\mathrm{e}^{-x}}{12} + c_1 x\,\mathrm{e}^{-x} + c_2\mathrm{e}^{-x} \tag{1}$$

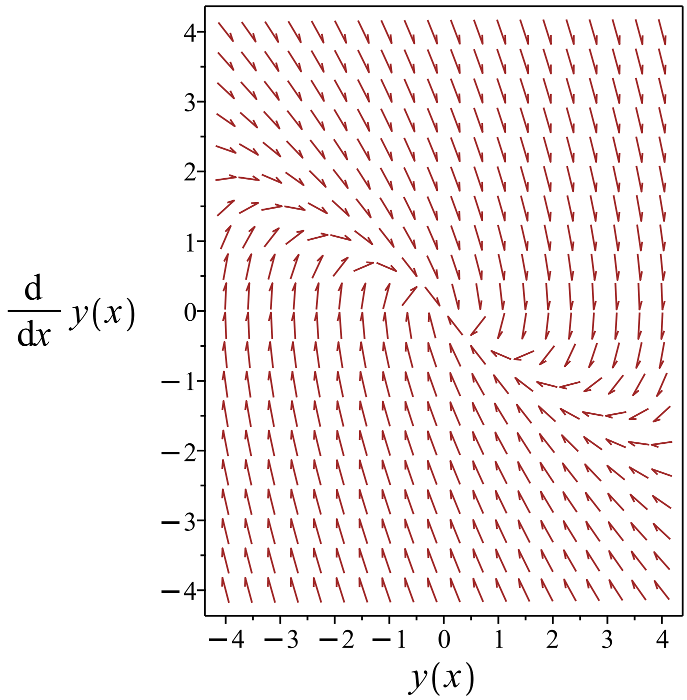


Figure 408: Slope field plot

Verification of solutions

$$y = \frac{x^4 e^{-x}}{12} + c_1 x e^{-x} + c_2 e^{-x}$$

Verified OK.

9.7.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 271: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2 \mathrm{e}^{-x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x\,\mathrm{e}^{-x}, x^2 \mathrm{e}^{-x}, \mathrm{e}^{-x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}
$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\,\mathrm{e}^{-x}, x^2 \mathrm{e}^{-x}, x^3 \mathrm{e}^{-x}\}]
$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x^2 \mathrm{e}^{-x}, x^3 \mathrm{e}^{-x}, x^4 \mathrm{e}^{-x}\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \mathrm{e}^{-x} + A_2 x^3 \mathrm{e}^{-x} + A_3 x^4 \mathrm{e}^{-x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^{-x} + 6A_2 x\,\mathrm{e}^{-x} + 12A_3 x^2 \mathrm{e}^{-x} = x^2 \mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = 0, A_3 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^4 \mathrm{e}^{-x}}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2\right) + \left(\frac{x^4 \mathrm{e}^{-x}}{12}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \frac{x^4 \mathrm{e}^{-x}}{12}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \frac{x^4 \mathrm{e}^{-x}}{12} \tag{1}$$

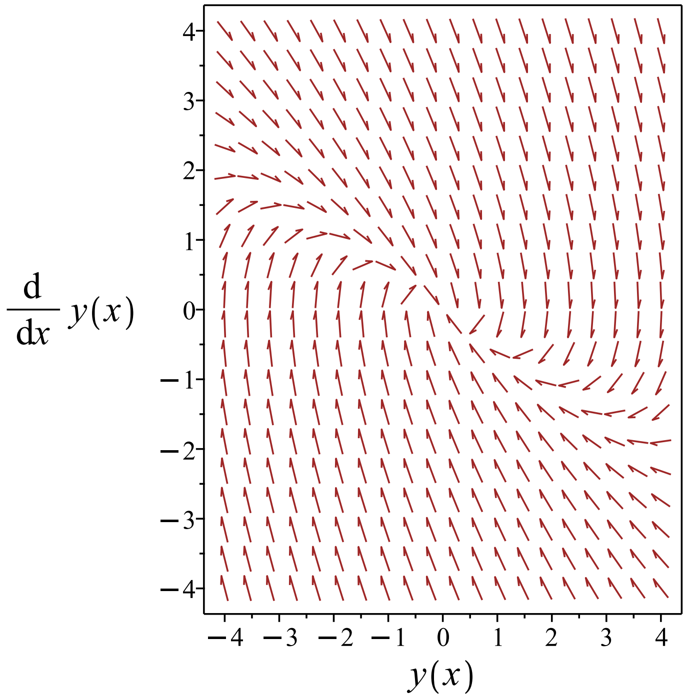


Figure 409: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2x + c_1) + \frac{x^4 e^{-x}}{12}$$

Verified OK.

9.7.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = x^2 e^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r + 1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = x^2 \mathrm{e}^{-x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \mathrm{e}^{-x}\left(-\left(\int x^3 dx\right) + \left(\int x^2 dx\right) x\right)$

  ◦ Compute integrals

  $y_p(x) = \frac{x^4 \mathrm{e}^{-x}}{12}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \frac{x^4 \mathrm{e}^{-x}}{12}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=x^2*exp(-x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x}\left(c_2 + c_1 x + \frac{1}{12}x^4\right)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 27

```
DSolve[y''[x]+2*y'[x]+y[x]==x^2*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{12}e^{-x}\left(x^4 + 12c_2 x + 12c_1\right)$$

9.8 problem Exercise 22.8, page 240

Internal problem ID [4638]
Internal file name [`OUTPUT/4131_Sunday_June_05_2022_12_26_43_PM_93146226/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22.8, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = 4 \sin(x) x}$$

9.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 4 \sin(x) x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4 \sin(x) x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) x, \sin(x) x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \sin(x), \cos(x) x, \cos(x) x^2, \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \sin(x) + A_2 \cos(x) x + A_3 \cos(x) x^2 + A_4 \sin(x) x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \sin(x) + 4A_1 x \cos(x) - 2A_2 \sin(x) - 4A_3 \sin(x) x + 2A_3 \cos(x) + 2A_4 \cos(x) = 4 \sin(x) x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = -1, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x) x^2 + \sin(x) x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (\cos(x) c_1 + c_2 \sin(x)) + \left(-\cos(x) x^2 + \sin(x) x\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x^2 + \sin(x) x \tag{1}
$$

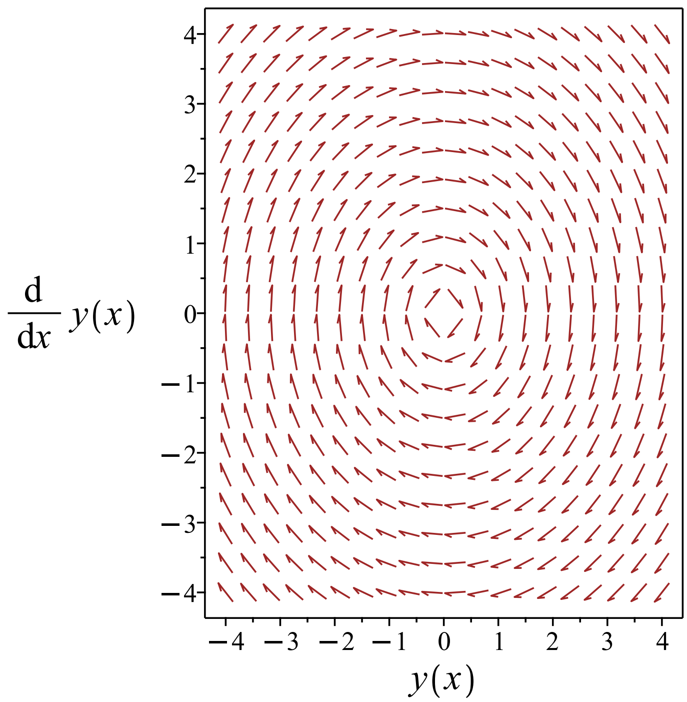


Figure 410: Slope field plot

Verification of solutions

$$
y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x^2 + \sin(x) x
$$

Verified OK.

9.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 273: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2} dx \\
&= \cos(x) (\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \cos(x) c_1 + c_2 \sin(x)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\sin(x)\,x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\,x, \sin(x)\,x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\sin(x), \cos(x)\,x, \cos(x)\,x^2, \sin(x)\,x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2\sin(x) + A_2\cos(x)\,x + A_3\cos(x)\,x^2 + A_4\sin(x)\,x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&2A_1\sin(x) + 4A_1x\cos(x) - 2A_2\sin(x) - 4A_3\sin(x)\,x + 2A_3\cos(x) + 2A_4\cos(x)\\&= 4\sin(x)\,x\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = -1, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x)\,x^2 + \sin(x)\,x$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= (\cos(x)\,c_1 + c_2\sin(x)) + \left(-\cos(x)\,x^2 + \sin(x)\,x\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x^2 + \sin(x) x \tag{1}$$

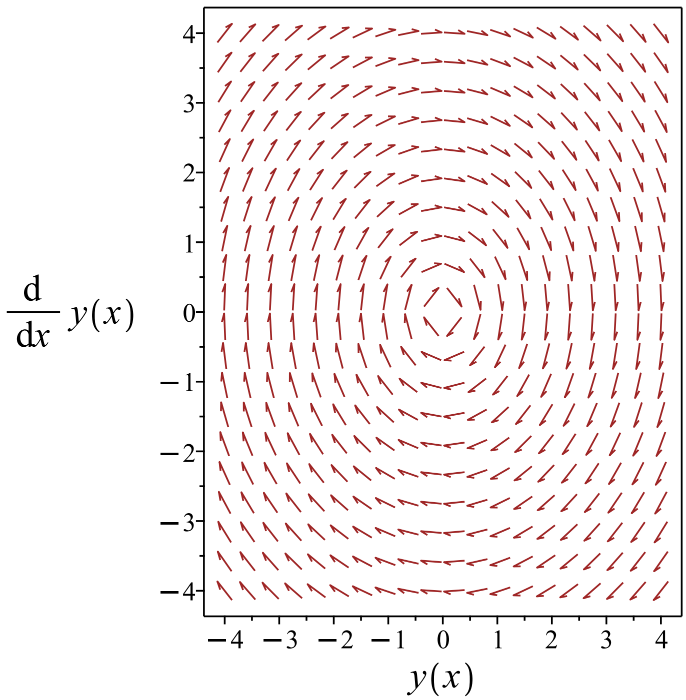


Figure 411: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x^2 + \sin(x) x$$

Verified OK.

9.8.3 Maple step by step solution

Let's solve

$y'' + y = 4\sin(x) x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4 \sin(x) x\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$
- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -4 \cos(x) \left(\int \sin(x)^2 x dx\right) + 2 \sin(x) \left(\int x \sin(2x) dx\right)$
- ○ Compute integrals

  $y_p(x) = x(-\cos(x) x + \sin(x))$

- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + x(-\cos(x) x + \sin(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+y(x)=4*x*sin(x),y(x), singsol=all)
```

$$y(x) = (-x^2 + c_1) \cos(x) + \sin(x) (c_2 + x)$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 27

```
DSolve[y''[x]+y[x]==4*x*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(-x^2 + \frac{1}{2} + c_1\right) \cos(x) + (x + c_2) \sin(x)$$

9.9 problem Exercise 22.9, page 240

Internal problem ID [4639]
Internal file name [`OUTPUT/4132_Sunday_June_05_2022_12_26_52_PM_85369126/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22.9, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__linear__constant__coeff", "linear__second__order__ode__solved__by__an__integrating__factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + y = \mathrm{e}^{-x} \ln (x)}$$

9.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = \mathrm{e}^{-x} \ln (x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda \, \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)} \\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-x} + c_2 x \, \mathrm{e}^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + x \, \mathrm{e}^{-x} c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x}$$

$$y_2 = x \, \mathrm{e}^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & x\,e^{-x} \\ \frac{d}{dx}\left(e^{-x}\right) & \frac{d}{dx}\left(x\,e^{-x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & x\,e^{-x} \\ -e^{-x} & e^{-x} - x\,e^{-x} \end{vmatrix}$$

Therefore

$$W = \left(e^{-x}\right)\left(e^{-x} - x\,e^{-x}\right) - \left(x\,e^{-x}\right)\left(-e^{-x}\right)$$

Which simplifies to

$$W = e^{-2x}$$

Which simplifies to

$$W = e^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\,e^{-2x} \ln(x)}{e^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \ln(x)\,x dx$$

Hence

$$u_1 = -\frac{\ln\left(x\right) x^{2}}{2}+\frac{x^{2}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{{\mathrm e}^{-2 x} \ln\left(x\right)}{{\mathrm e}^{-2 x}}\, dx$$

Which simplifies to

$$u_2 = \int \ln\left(x\right)dx$$

Hence

$$u_2 = \ln\left(x\right) x -x$$

Which simplifies to

$$\begin{aligned} u_1 &= -\frac{x^{2} \left(-1+2 \ln\left(x\right)\right)}{4} \\ u_2 &= x \left(\ln\left(x\right)-1\right) \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^{2} \left(-1+2 \ln\left(x\right)\right) {\mathrm e}^{-x}}{4}+x^{2} \left(\ln\left(x\right)-1\right) {\mathrm e}^{-x}$$

Which simplifies to

$$y_p(x) = \frac{x^{2} {\mathrm e}^{-x} \left(-3+2 \ln\left(x\right)\right)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_{1} {\mathrm e}^{-x}+x\,{\mathrm e}^{-x} c_{2}\right) + \left(\frac{x^{2} {\mathrm e}^{-x} \left(-3+2 \ln\left(x\right)\right)}{4}\right) \end{aligned}$$

Which simplifies to

$$y = {\mathrm e}^{-x} \left(c_{2} x +c_{1}\right)+\frac{x^{2} {\mathrm e}^{-x} \left(-3+2 \ln\left(x\right)\right)}{4}$$

Summary
The solution(s) found are the following

$$y = e^{-x}(c_2x + c_1) + \frac{x^2 e^{-x}(-3 + 2\ln(x))}{4} \tag{1}$$

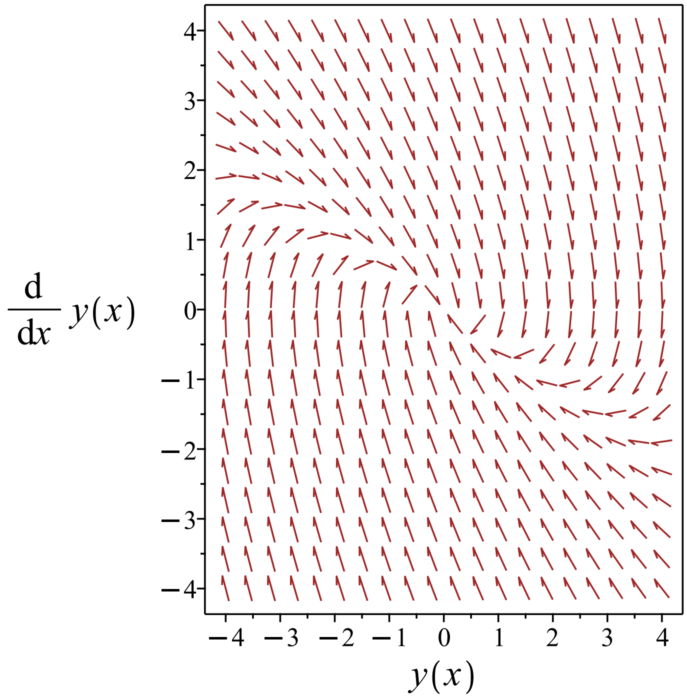


Figure 412: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2x + c_1) + \frac{x^2 e^{-x}(-3 + 2\ln(x))}{4}$$

Verified OK.

9.9.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= \mathrm{e}^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \mathrm{e}^{x}\mathrm{e}^{-x}\ln(x) \\ (y\,\mathrm{e}^{x})'' &= \mathrm{e}^{x}\mathrm{e}^{-x}\ln(x) \end{aligned}$$

Integrating once gives

$$(y\,\mathrm{e}^{x})' = x(\ln(x) - 1) + c_1$$

Integrating again gives

$$(y\,\mathrm{e}^{x}) = \frac{x(2\ln(x)\,x + 4c_1 - 3x)}{4} + c_2$$

Hence the solution is

$$y = \frac{\frac{x(2\ln(x)x+4c_1-3x)}{4} + c_2}{\mathrm{e}^{x}}$$

Or

$$y = \frac{x^2\mathrm{e}^{-x}\ln(x)}{2} + c_1 x\,\mathrm{e}^{-x} - \frac{3x^2\mathrm{e}^{-x}}{4} + c_2\mathrm{e}^{-x}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{x^2\mathrm{e}^{-x}\ln(x)}{2} + c_1 x\,\mathrm{e}^{-x} - \frac{3x^2\mathrm{e}^{-x}}{4} + c_2\mathrm{e}^{-x} \tag{1}$$

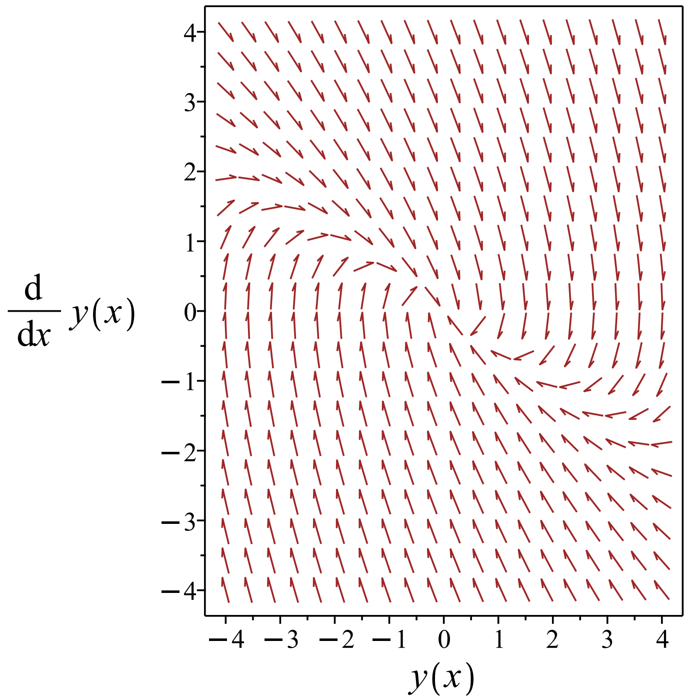


Figure 413: Slope field plot

Verification of solutions

$$y = \frac{x^2 e^{-x} \ln(x)}{2} + c_1 x\, e^{-x} - \frac{3x^2 e^{-x}}{4} + c_2 e^{-x}$$

Verified OK.

9.9.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 275: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx}\\ &= z_1 e^{-x}\\ &= z_1\left(\mathrm{e}^{-x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx\\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= x\,\mathrm{e}^{-x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & x\,e^{-x} \\ \frac{d}{dx}\left(e^{-x}\right) & \frac{d}{dx}\left(x\,e^{-x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & x\,e^{-x} \\ -e^{-x} & e^{-x} - x\,e^{-x} \end{vmatrix}$$

Therefore

$$W = \left(e^{-x}\right)\left(e^{-x} - x\,e^{-x}\right) - \left(x\,e^{-x}\right)\left(-e^{-x}\right)$$

Which simplifies to

$$W = e^{-2x}$$

Which simplifies to

$$W = e^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\,e^{-2x} \ln(x)}{e^{-2x}}\, dx$$

Which simplifies to

$$u_1 = -\int \ln(x)\, x dx$$

Hence

$$u_1 = -\frac{\ln(x)\, x^2}{2} + \frac{x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-2x} \ln(x)}{e^{-2x}}\, dx$$

Which simplifies to

$$u_2 = \int \ln(x)\, dx$$

Hence

$$u_2 = \ln(x)\, x - x$$

Which simplifies to

$$u_1 = -\frac{x^2(-1 + 2\ln(x))}{4}$$
$$u_2 = x(\ln(x) - 1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2(-1 + 2\ln(x))\, e^{-x}}{4} + x^2(\ln(x) - 1)\, e^{-x}$$

Which simplifies to

$$y_p(x) = \frac{x^2 e^{-x}(-3 + 2\ln(x))}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + x\, e^{-x} c_2\right) + \left(\frac{x^2 e^{-x}(-3 + 2\ln(x))}{4}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) + \frac{x^2 e^{-x}(-3 + 2\ln(x))}{4}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + \frac{x^2 e^{-x}(-3 + 2\ln(x))}{4} \tag{1}$$

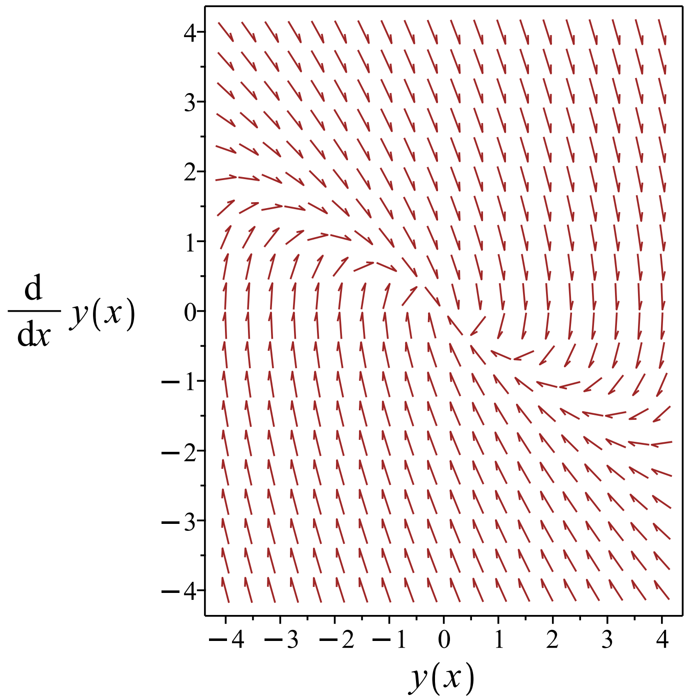


Figure 414: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + \frac{x^2 e^{-x}(-3 + 2\ln(x))}{4}$$

Verified OK.

9.9.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = e^{-x}\ln(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \mathrm{e}^{-x}\ln(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{array}\right]$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \mathrm{e}^{-x}\left(-\left(\int \ln(x)\, x dx\right) + \left(\int \ln(x)\, dx\right) x\right)$

  ◦ Compute integrals

  $y_p(x) = \frac{x^2 \mathrm{e}^{-x}(-3+2\ln(x))}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \frac{x^2 \mathrm{e}^{-x}(-3+2\ln(x))}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=exp(-x)*ln(x),y(x), singsol=all)
```

$$y(x) = \frac{e^{-x}(2\ln(x)x^2 + 4c_1x - 3x^2 + 4c_2)}{4}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 36

```
DSolve[y''[x]+2*y'[x]+y[x]==Exp[-x]*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-x}\left(-3x^2 + 2x^2\log(x) + 4c_2x + 4c_1\right)$$

9.10 problem Exercise 22.10, page 240

Internal problem ID [4640]
Internal file name [`OUTPUT/4133_Sunday_June_05_2022_12_26_59_PM_158880/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22.10, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \csc(x)}$$

9.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \csc(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\csc(x)}{1} dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\csc(x)}{1} dx$$

Which simplifies to

$$u_2 = \int \cot(x) dx$$

Hence

$$u_2 = \ln(\sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)x + \ln(\sin(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)c_1 + c_2\sin(x)) + (-\cos(x)x + \ln(\sin(x))\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x) \tag{1}$$

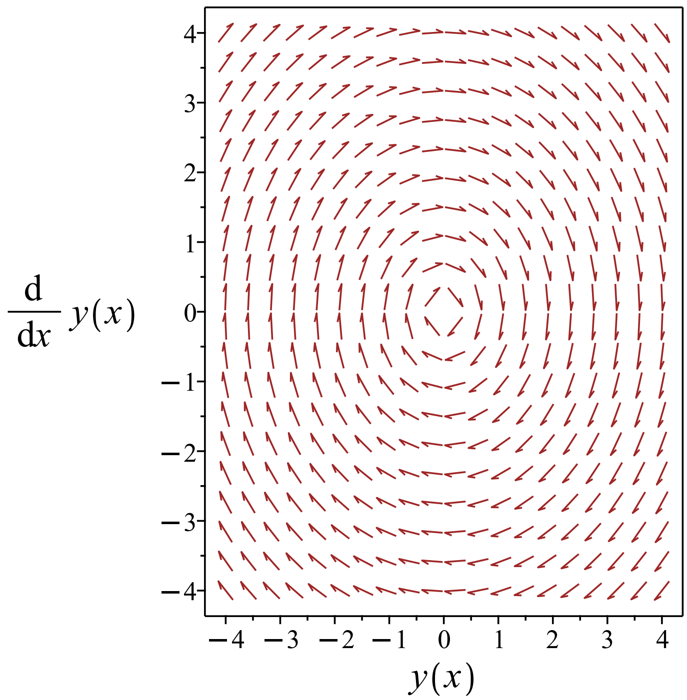


Figure 415: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x)$$

Verified OK.

9.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 277: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\, c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\csc(x)}{1}dx$$

Which simplifies to

$$u_1 = -\int 1dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\csc(x)}{1}dx$$

Which simplifies to

$$u_2 = \int \cot(x)\,dx$$

Hence

$$u_2 = \ln(\sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)\,x + \ln(\sin(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\,c_1 + c_2\sin(x)) + (-\cos(x)\,x + \ln(\sin(x))\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x)\,c_1 + c_2\sin(x) - \cos(x)\,x + \ln(\sin(x))\sin(x) \tag{1}$$

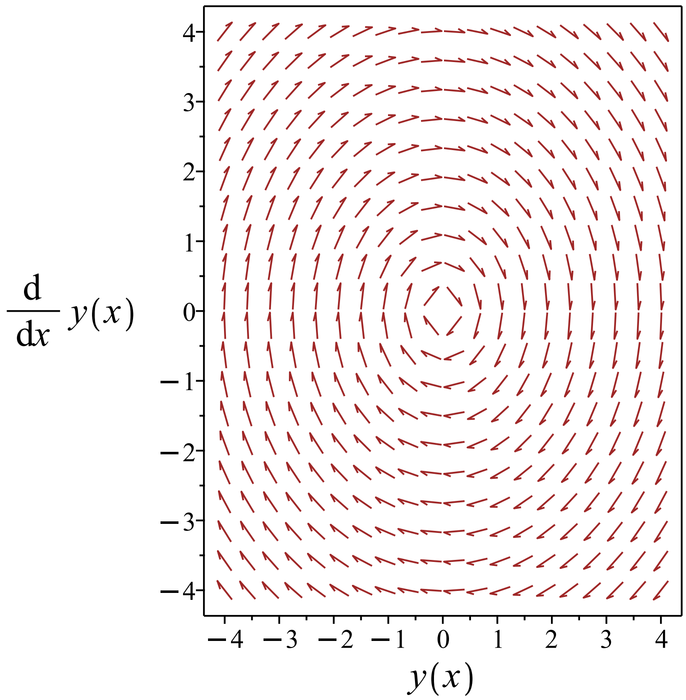


Figure 416: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x)$$

Verified OK.

9.10.3 Maple step by step solution

Let's solve

$y'' + y = \csc(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \csc(x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x)\left(\int 1dx\right) + \sin(x)\left(\int \cot(x)\, dx\right)$
  - Compute integrals

    $y_p(x) = -\cos(x)\, x + \ln(\sin(x)) \sin(x)$
- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) - \cos(x)\, x + \ln(\sin(x)) \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+y(x)=csc(x),y(x), singsol=all)
```

$$y(x) = -\ln(\csc(x))\sin(x) + (-x + c_1)\cos(x) + c_2 \sin(x)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 24

```
DSolve[y''[x]+y[x]==Csc[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (-x + c_1)\cos(x) + \sin(x)(\log(\sin(x)) + c_2)$$

9.11 problem Exercise 22.11, page 240

Internal problem ID [4641]
Internal file name [`OUTPUT/4134_Sunday_June_05_2022_12_27_07_PM_66142691/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22.11, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \tan(x)^2}$$

9.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \tan(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x)\, c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\tan(x)^2}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\tan(x)^2\,dx$$

Hence

$$u_1 = -\frac{\sin(x)^4}{\cos(x)} - \left(2 + \sin(x)^2\right)\cos(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\tan(x)^2}{1}\,dx$$

Which simplifies to

$$u_2 = \int \sin(x)\tan(x)\,dx$$

Hence

$$u_2 = -\sin(x) + \ln(\sec(x) + \tan(x))$$

Which simplifies to

$$u_1 = -\cos(x) - \sec(x)$$
$$u_2 = -\sin(x) + \ln(\sec(x) + \tan(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\cos(x) - \sec(x))\cos(x) + (-\sin(x) + \ln(\sec(x) + \tan(x)))\sin(x)$$

Which simplifies to

$$y_p(x) = -2 + \ln(\sec(x) + \tan(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\, c_1 + c_2 \sin(x)) + (-2 + \ln(\sec(x) + \tan(x))\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x)\, c_1 + c_2 \sin(x) - 2 + \ln(\sec(x) + \tan(x))\sin(x) \tag{1}$$

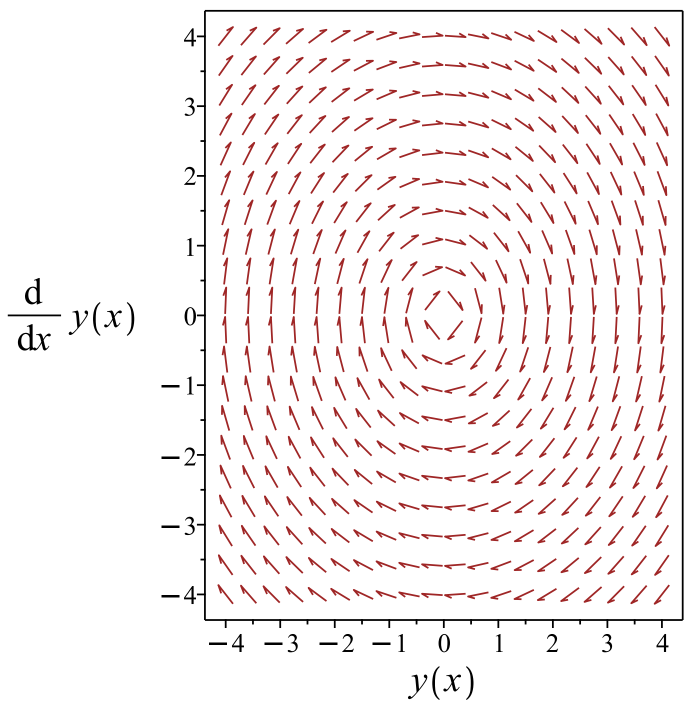


Figure 417: Slope field plot

Verification of solutions

$$y = \cos(x)\, c_1 + c_2 \sin(x) - 2 + \ln(\sec(x) + \tan(x))\sin(x)$$

Verified OK.

9.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 279: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\,c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\tan(x)^2}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\tan(x)^2\,dx$$

Hence

$$u_1 = -\frac{\sin(x)^4}{\cos(x)} - \left(2+\sin(x)^2\right)\cos(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\tan(x)^2}{1}\,dx$$

Which simplifies to

$$u_2 = \int \sin(x)\tan(x)\,dx$$

Hence

$$u_2 = -\sin(x) + \ln(\sec(x)+\tan(x))$$

Which simplifies to

$$u_1 = -\cos(x) - \sec(x)$$
$$u_2 = -\sin(x) + \ln(\sec(x)+\tan(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\cos(x)-\sec(x))\cos(x) + (-\sin(x)+\ln(\sec(x)+\tan(x)))\sin(x)$$

Which simplifies to

$$y_p(x) = -2 + \ln(\sec(x)+\tan(x))\sin(x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (\cos(x) c_1 + c_2 \sin(x)) + (-2 + \ln(\sec(x) + \tan(x)) \sin(x))
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \cos(x) c_1 + c_2 \sin(x) - 2 + \ln(\sec(x) + \tan(x)) \sin(x) \tag{1}
$$

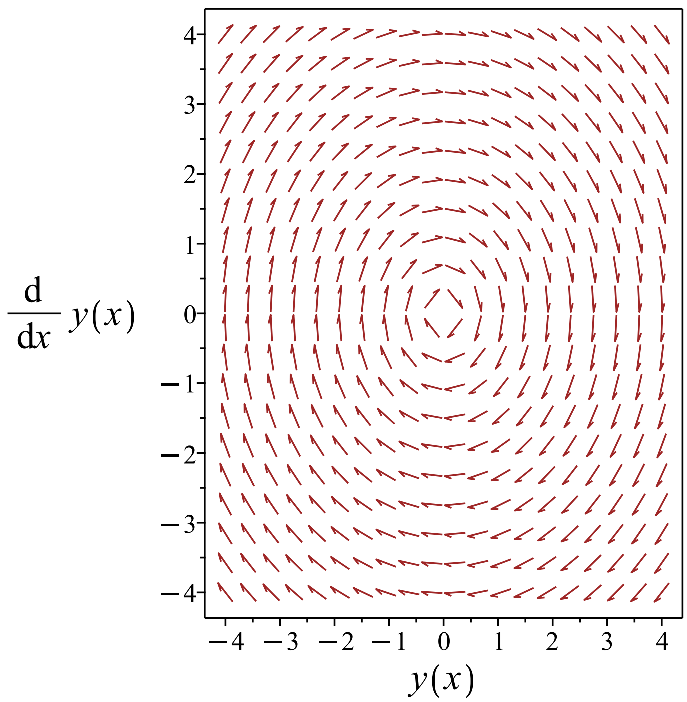


Figure 418: Slope field plot

Verification of solutions

$$
y = \cos(x) c_1 + c_2 \sin(x) - 2 + \ln(\sec(x) + \tan(x)) \sin(x)
$$

Verified OK.

9.11.3 Maple step by step solution

Let's solve

$y'' + y = \tan(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \tan(x)^2\right]$

- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

- ∘ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

- ∘ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\cos(x)\left(\int \sin(x)\tan(x)^2 dx\right) + \sin(x)\left(\int \sin(x)\tan(x) dx\right)$$

- Compute integrals

$$y_p(x) = -2 + \ln(\sec(x) + \tan(x))\sin(x)$$

- Substitute particular solution into general solution to ODE

$$y = \cos(x) c_1 + c_2 \sin(x) - 2 + \ln(\sec(x) + \tan(x))\sin(x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)=tan(x)^2,y(x), singsol=all)
```

$$y(x) = c_2 \sin(x) + \cos(x) c_1 - 2 + \ln(\sec(x) + \tan(x))\sin(x)$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 23

```
DSolve[y''[x]+y[x]==Tan[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x)\operatorname{arctanh}(\sin(x)) + c_1 \cos(x) + c_2 \sin(x) - 2$$

9.12 problem Exercise 22.12, page 240

Internal problem ID [4642]
Internal file name [`OUTPUT/4135_Sunday_June_05_2022_12_27_14_PM_79979444/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22.12, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''+2y'+y=\frac{e^{-x}}{x}}$$

9.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=1, B=2, C=1, f(x)=\frac{e^{-x}}{x}$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+2y'+y=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda \,\mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)} \\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-x} + c_2 x \,\mathrm{e}^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + x \,\mathrm{e}^{-x} c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \mathrm{e}^{-x} \\ y_2 &= x \,\mathrm{e}^{-x}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & x\,e^{-x} \\ \frac{d}{dx}\left(e^{-x}\right) & \frac{d}{dx}\left(x\,e^{-x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & x\,e^{-x} \\ -e^{-x} & e^{-x} - x\,e^{-x} \end{vmatrix}$$

Therefore

$$W = \left(e^{-x}\right)\left(e^{-x} - x\,e^{-x}\right) - \left(x\,e^{-x}\right)\left(-e^{-x}\right)$$

Which simplifies to

$$W = e^{-2x}$$

Which simplifies to

$$W = e^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-2x}}{e^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-2x}}{x}}{e^{-2x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x\, e^{-x} + \ln(x)\, x\, e^{-x}$$

Which simplifies to

$$y_p(x) = x\, e^{-x}(\ln(x) - 1)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + x\, e^{-x} c_2\right) + \left(x\, e^{-x}(\ln(x) - 1)\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) + x\, e^{-x}(\ln(x) - 1)$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + x\, e^{-x}(\ln(x) - 1) \tag{1}$$

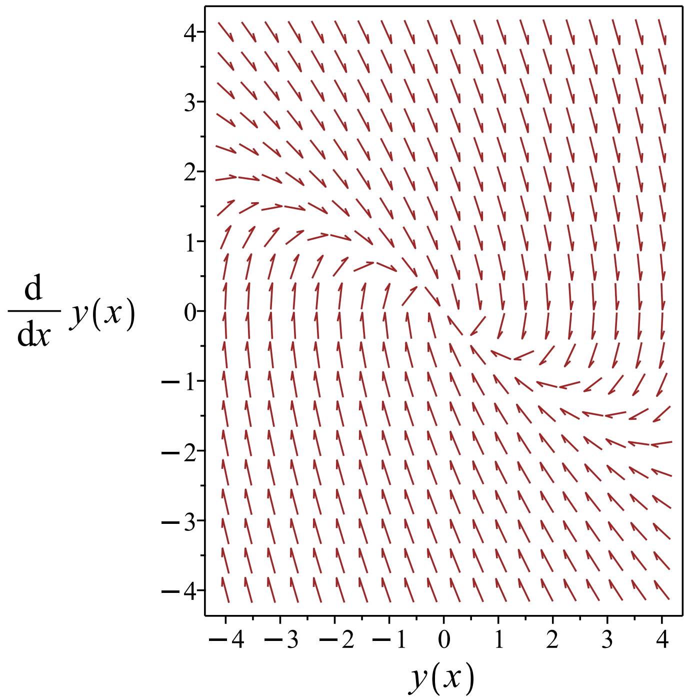


Figure 419: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2x + c_1) + x\,e^{-x}(\ln(x) - 1)$$

Verified OK.

9.12.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x) &= e^{\frac{1}{2}\int p\,dx}\\ &= e^{\int 2\,dx}\\ &= e^{x}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \frac{e^{x} e^{-x}}{x} \\
(y\, e^{x})'' &= \frac{e^{x} e^{-x}}{x}
\end{aligned}
$$

Integrating once gives

$$
(y\, e^{x})' = \ln(x) + c_1
$$

Integrating again gives

$$
(y\, e^{x}) = x(\ln(x) + c_1 - 1) + c_2
$$

Hence the solution is

$$
y = \frac{x(\ln(x) + c_1 - 1) + c_2}{e^{x}}
$$

Or

$$
y = c_1 x\, e^{-x} + \ln(x)\, x\, e^{-x} + c_2 e^{-x} - x\, e^{-x}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 x\, e^{-x} + \ln(x)\, x\, e^{-x} + c_2 e^{-x} - x\, e^{-x} \tag{1}
$$

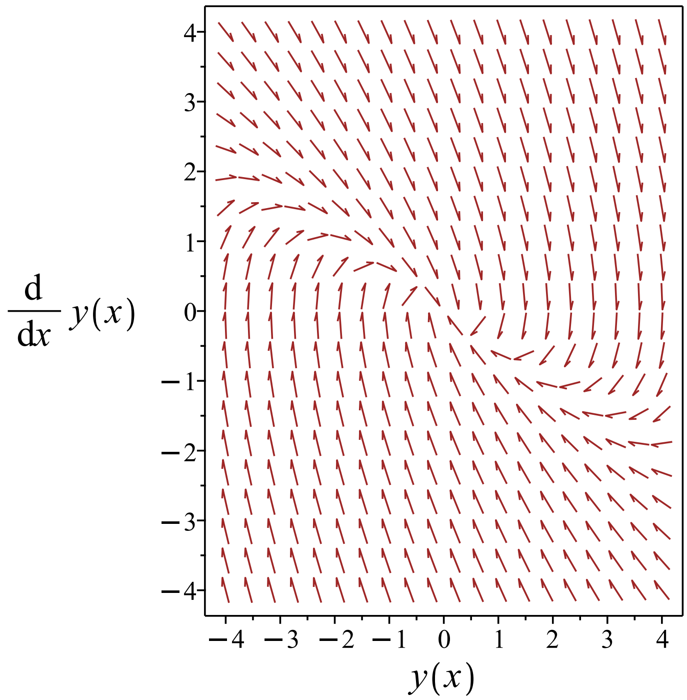


Figure 420: Slope field plot

Verification of solutions

$$y = c_1 x \, \mathrm{e}^{-x} + \ln(x) \, x \, \mathrm{e}^{-x} + c_2 \mathrm{e}^{-x} - x \, \mathrm{e}^{-x}$$

Verified OK.

9.12.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 281: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx}\\ &= z_1 e^{-x}\\ &= z_1\left(\mathrm{e}^{-x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx\\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= x\,\mathrm{e}^{-x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & x\,e^{-x} \\ \frac{d}{dx}(e^{-x}) & \frac{d}{dx}(x\,e^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & x\,e^{-x} \\ -e^{-x} & e^{-x} - x\,e^{-x} \end{vmatrix}$$

Therefore

$$W = (e^{-x})\,(e^{-x} - x\,e^{-x}) - (x\,e^{-x})\,(-e^{-x})$$

Which simplifies to

$$W = e^{-2x}$$

Which simplifies to

$$W = e^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-2x}}{e^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int 1dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-2x}}{x}}{e^{-2x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x}dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x\,e^{-x} + \ln(x)\,x\,e^{-x}$$

Which simplifies to

$$y_p(x) = x\,e^{-x}(\ln(x) - 1)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + x\,e^{-x} c_2\right) + \left(x\,e^{-x}(\ln(x) - 1)\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) + x\,e^{-x}(\ln(x) - 1)$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + x\,e^{-x}(\ln(x) - 1) \tag{1}$$

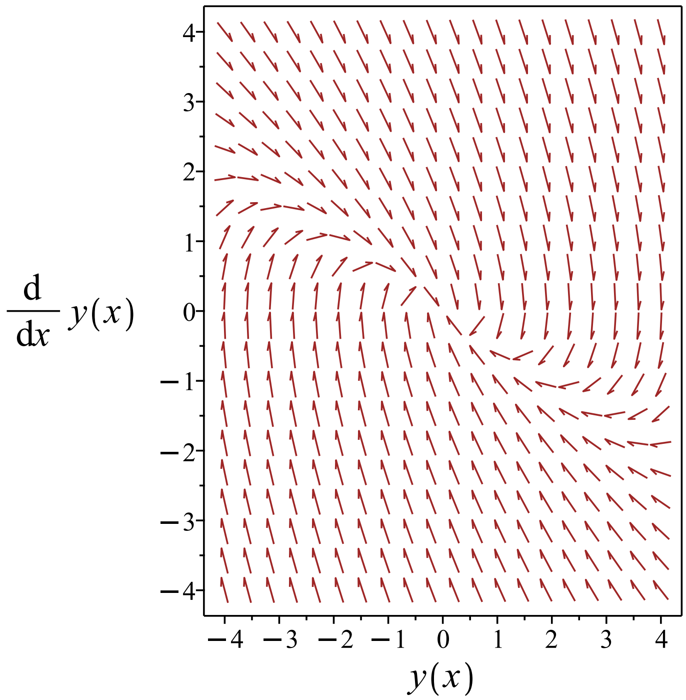


Figure 421: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2x + c_1) + x\,e^{-x}(\ln(x) - 1)$$

Verified OK.

9.12.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = \frac{e^{-x}}{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{\mathrm{e}^{-x}}{x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \mathrm{e}^{-x}\left(-\left(\int 1 dx\right) + \left(\int \frac{1}{x} dx\right) x\right)$

  ◦ Compute integrals

  $y_p(x) = x\,\mathrm{e}^{-x}(\ln(x) - 1)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + x\,\mathrm{e}^{-x}(\ln(x) - 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=exp(-x)/x,y(x), singsol=all)
```

$$y(x) = e^{-x}(\ln(x)\, x + x(c_1 - 1) + c_2)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 24

```
DSolve[y''[x]+2*y'[x]+y[x]==Exp[-x]/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{-x}(x \log(x) + (-1 + c_2)x + c_1)$$

9.13 problem Exercise 22.13, page 240

Internal problem ID [4643]
Internal file name [`OUTPUT/4136_Sunday_June_05_2022_12_27_21_PM_2119981/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22.13, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \sec(x)\csc(x)}$$

9.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sec(x)\csc(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x)\, c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)\csc(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \sec(x)\,dx$$

Hence

$$u_1 = -\ln(\sec(x) + \tan(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)\csc(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int \csc(x)\,dx$$

Hence

$$u_2 = -\ln(\csc(x) + \cot(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln(\sec(x) + \tan(x))\cos(x) - \ln(\csc(x) + \cot(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\,c_1 + c_2\sin(x)) + (-\ln(\sec(x) + \tan(x))\cos(x) - \ln(\csc(x) + \cot(x))\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) - \ln(\sec(x) + \tan(x)) \cos(x) - \ln(\csc(x) + \cot(x)) \sin(x) \tag{1}$$

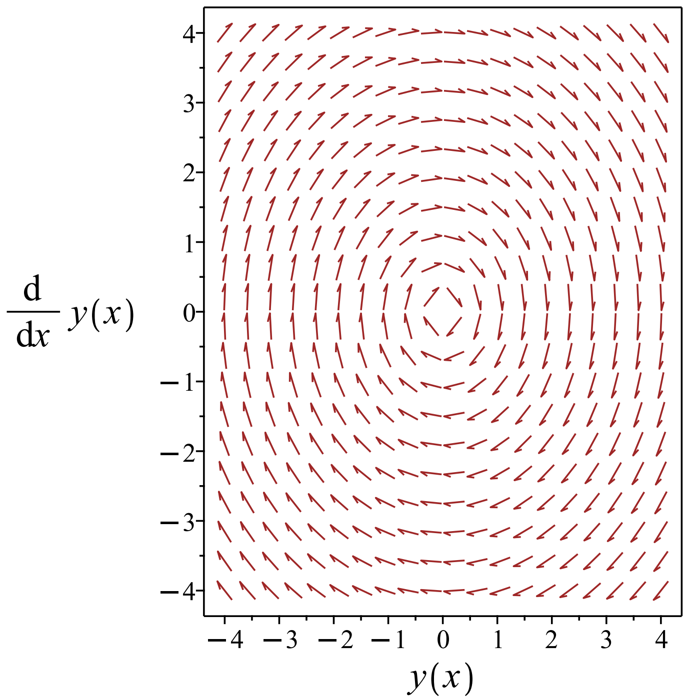


Figure 422: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - \ln(\sec(x) + \tan(x)) \cos(x) - \ln(\csc(x) + \cot(x)) \sin(x)$$

Verified OK.

9.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 283: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)\csc(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \sec(x)\,dx$$

Hence

$$u_1 = -\ln(\sec(x) + \tan(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)\csc(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int \csc(x)\,dx$$

Hence

$$u_2 = -\ln(\csc(x) + \cot(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln(\sec(x) + \tan(x))\cos(x) - \ln(\csc(x) + \cot(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\,c_1 + c_2\sin(x)) + (-\ln(\sec(x) + \tan(x))\cos(x) - \ln(\csc(x) + \cot(x))\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x)\,c_1 + c_2\sin(x) - \ln(\sec(x) + \tan(x))\cos(x) - \ln(\csc(x) + \cot(x))\sin(x) \tag{1}$$

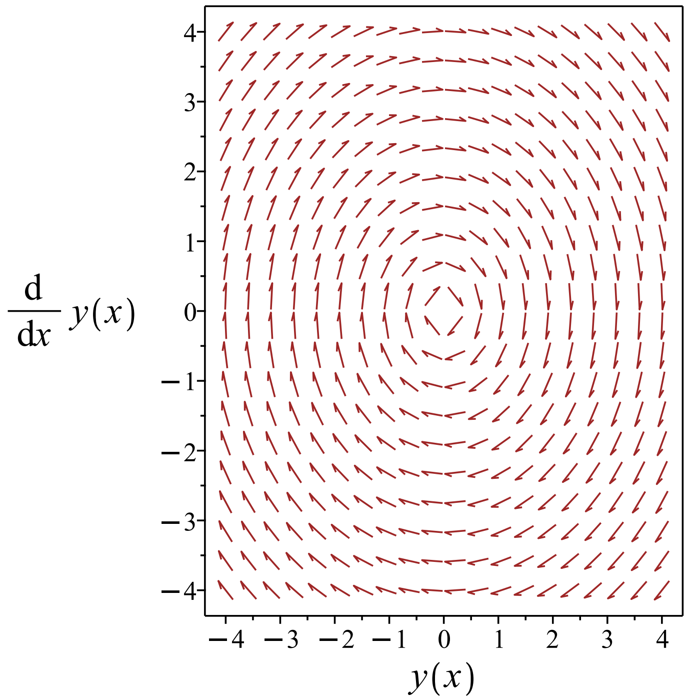


Figure 423: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - \ln(\sec(x) + \tan(x)) \cos(x) - \ln(\csc(x) + \cot(x)) \sin(x)$$

Verified OK.

9.13.3 Maple step by step solution

Let's solve

$y'' + y = \sec(x) \csc(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sec(x)\csc(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x)\left(\int \sec(x)\,dx\right) + \sin(x)\left(\int \csc(x)\,dx\right)$

  ◦ Compute integrals

  $y_p(x) = -\ln(\sec(x) + \tan(x))\cos(x) - \ln(\csc(x) + \cot(x))\sin(x)$

- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) - \ln(\sec(x) + \tan(x))\cos(x) - \ln(\csc(x) + \cot(x))\sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$2)+y(x)=sec(x)*csc(x),y(x), singsol=all)
```

$$y(x) = c_2 \sin(x) + \cos(x) c_1 + \sin(x) \ln(\csc(x) - \cot(x)) - \ln(\sec(x) + \tan(x)) \cos(x)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 30

```
DSolve[y''[x]+y[x]==Sec[x]*Csc[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sin(x)\operatorname{arctanh}(\cos(x)) + c_1 \cos(x) + c_2 \sin(x) + \cos(x) \left(-\coth^{-1}(\sin(x))\right)$$

9.14 problem Exercise 22.14, page 240

Internal problem ID [4644]
Internal file name [`OUTPUT/4137_Sunday_June_05_2022_12_27_29_PM_22887124/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22.14, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''-2y'+y=\mathrm{e}^{x}\ln(x)}$$

9.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=1,B=-2,C=1,f(x)=\mathrm{e}^{x}\ln(x)$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-2y'+y=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-2)^2 - (4)\,(1)\,(1)}\\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^{x} + c_2 e^{x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 x\, e^{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^{x}\\ y_2 &= e^{x} x\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{x} & \mathrm{e}^{x} x \\ \frac{d}{dx}(\mathrm{e}^{x}) & \frac{d}{dx}(\mathrm{e}^{x} x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{x} & \mathrm{e}^{x} x \\ \mathrm{e}^{x} & \mathrm{e}^{x} x + \mathrm{e}^{x} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{x})(\mathrm{e}^{x} x + \mathrm{e}^{x}) - (\mathrm{e}^{x} x)(\mathrm{e}^{x})$$

Which simplifies to

$$W = \mathrm{e}^{2x}$$

Which simplifies to

$$W = \mathrm{e}^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{2x} x \ln(x)}{\mathrm{e}^{2x}} \, dx$$

Which simplifies to

$$u_1 = -\int \ln(x) \, x dx$$

Hence

$$u_1 = -\frac{\ln(x) \, x^2}{2} + \frac{x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} \ln (x)}{e^{2x}} dx$$

Which simplifies to

$$u_2 = \int \ln (x) \, dx$$

Hence

$$u_2 = \ln (x) \, x - x$$

Which simplifies to

$$u_1 = -\frac{x^2(-1 + 2\ln (x))}{4}$$
$$u_2 = x(\ln (x) - 1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2(-1 + 2\ln (x)) e^x}{4} + x^2(\ln (x) - 1) e^x$$

Which simplifies to

$$y_p(x) = \frac{e^x x^2(-3 + 2\ln (x))}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 x \, e^x) + \left(\frac{e^x x^2(-3 + 2\ln (x))}{4}\right) \end{aligned}$$

Which simplifies to

$$y = e^x (c_2 x + c_1) + \frac{e^x x^2(-3 + 2\ln (x))}{4}$$

Summary
The solution(s) found are the following

$$y = e^x (c_2 x + c_1) + \frac{e^x x^2(-3 + 2\ln (x))}{4} \tag{1}$$

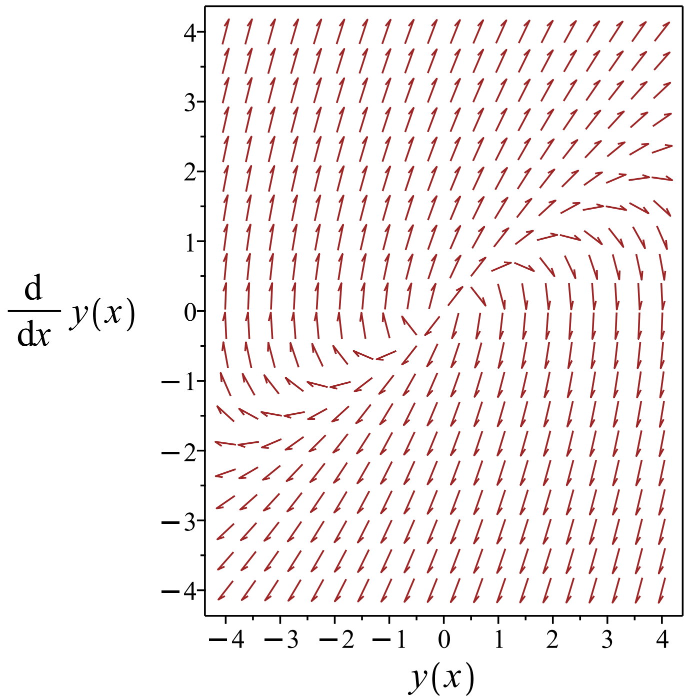


Figure 424: Slope field plot

Verification of solutions

$$y = e^{x}(c_2x + c_1) + \frac{e^{x}x^2(-3 + 2\ln(x))}{4}$$

Verified OK.

9.14.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\left(M(x)y\right)'' = e^x e^{-x} \ln(x)$$
$$\left(e^{-x}y\right)'' = e^x e^{-x} \ln(x)$$

Integrating once gives

$$\left(e^{-x}y\right)' = x(\ln(x) - 1) + c_1$$

Integrating again gives

$$\left(e^{-x}y\right) = \frac{x(2\ln(x)\,x + 4c_1 - 3x)}{4} + c_2$$

Hence the solution is

$$y = \frac{\frac{x(2\ln(x)x+4c_1-3x)}{4} + c_2}{e^{-x}}$$

Or

$$y = \frac{x^2 e^x \ln(x)}{2} + c_1 x\, e^x - \frac{3x^2 e^x}{4} + c_2 e^x$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 e^x \ln(x)}{2} + c_1 x\, e^x - \frac{3x^2 e^x}{4} + c_2 e^x \tag{1}$$

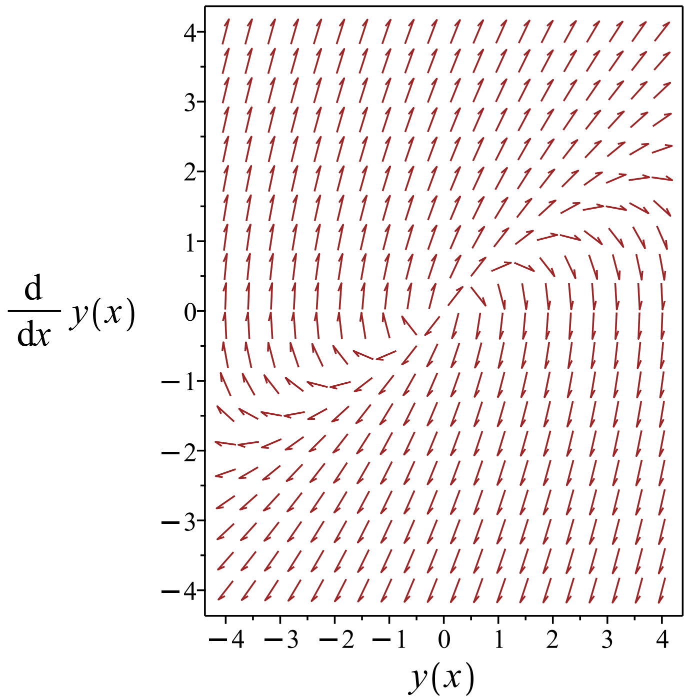


Figure 425: Slope field plot

Verification of solutions

$$y = \frac{x^2 e^x \ln(x)}{2} + c_1 x e^x - \frac{3x^2 e^x}{4} + c_2 e^x$$

Verified OK.

9.14.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 285: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}\\ &= z_1 e^{x}\\ &= z_1(\mathrm{e}^{x})\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx\\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^x + c_2 x \,\mathrm{e}^x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^x \\
y_2 &= \mathrm{e}^x x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & e^x x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^x x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & e^x x \\ e^x & e^x x + e^x \end{vmatrix}$$

Therefore

$$W = (e^x)(e^x x + e^x) - (e^x x)(e^x)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{2x} x \ln(x)}{e^{2x}} dx$$

Which simplifies to

$$u_1 = -\int \ln(x) x dx$$

Hence

$$u_1 = -\frac{\ln(x) x^2}{2} + \frac{x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} \ln(x)}{e^{2x}} dx$$

Which simplifies to

$$u_2 = \int \ln(x) dx$$

Hence

$$u_2 = \ln(x)\, x - x$$

Which simplifies to

$$u_1 = -\frac{x^2(-1+2\ln(x))}{4}$$
$$u_2 = x(\ln(x) - 1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2(-1+2\ln(x))\,e^x}{4} + x^2(\ln(x)-1)\,e^x$$

Which simplifies to

$$y_p(x) = \frac{e^x x^2(-3+2\ln(x))}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 x\, e^x) + \left(\frac{e^x x^2(-3+2\ln(x))}{4}\right) \end{aligned}$$

Which simplifies to

$$y = e^x(c_2 x + c_1) + \frac{e^x x^2(-3+2\ln(x))}{4}$$

Summary
The solution(s) found are the following

$$y = e^x(c_2 x + c_1) + \frac{e^x x^2(-3+2\ln(x))}{4} \tag{1}$$

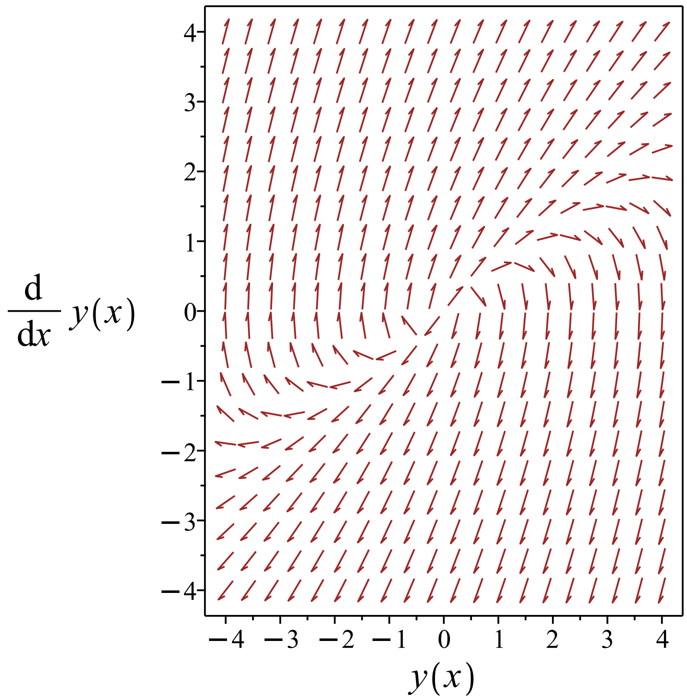


Figure 426: Slope field plot

Verification of solutions

$$y = e^{x}(c_2 x + c_1) + \frac{e^{x} x^2(-3 + 2\ln(x))}{4}$$

Verified OK.

9.14.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = e^{x} \ln(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

$r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^x x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^x + c_2 x\, e^x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^x \ln(x)\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^x x \\ e^x & e^x x + e^x \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{2x}$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = e^x \left(-\left(\int \ln(x)\, x dx\right) + \left(\int \ln(x)\, dx\right) x\right)$

  ○ Compute integrals

  $y_p(x) = \frac{e^x x^2 (-3 + 2\ln(x))}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^x + c_2 x\, e^x + \frac{e^x x^2 (-3 + 2\ln(x))}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=exp(x)*ln(x),y(x), singsol=all)
```

$$y(x) = \frac{e^{x}(2\ln(x)\,x^{2} + 4c_{1}x - 3x^{2} + 4c_{2})}{4}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 34

```
DSolve[y''[x]-2*y'[x]+y[x]==Exp[x]*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{x}\left(-3x^{2} + 2x^{2}\log(x) + 4c_{2}x + 4c_{1}\right)$$

9.15 problem Exercise 22.15, page 240

Internal problem ID [4645]
Internal file name [`OUTPUT/4138_Sunday_June_05_2022_12_27_36_PM_86339199/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22.15, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 3y' + 2y = \cos\left(\mathrm{e}^{-x}\right)}$$

9.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = \cos\left(\mathrm{e}^{-x}\right)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(2)}\\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= \frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= 1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(1)x}\end{aligned}$$

Or

$$y = c_1 e^{2x} + c_2 e^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= e^{2x} \\ y_2 &= e^{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & e^{x} \\ \frac{d}{dx}\left(e^{2x}\right) & \frac{d}{dx}\left(e^{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & e^{x} \\ 2\,e^{2x} & e^{x} \end{vmatrix}$$

Therefore

$$W = \left(e^{2x}\right)\left(e^{x}\right) - \left(e^{x}\right)\left(2\,e^{2x}\right)$$

Which simplifies to

$$W = -e^{2x} e^{x}$$

Which simplifies to

$$W = -\mathrm{e}^{3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{x} \cos\left(\mathrm{e}^{-x}\right)}{-\mathrm{e}^{3x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\cos\left(\mathrm{e}^{-x}\right) \mathrm{e}^{-2x} dx$$

Hence

$$u_1 = -\frac{2\,\mathrm{e}^{-x} \tan\left(\frac{\mathrm{e}^{-x}}{2}\right) + 2}{1 + \tan\left(\frac{\mathrm{e}^{-x}}{2}\right)^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{2x} \cos\left(\mathrm{e}^{-x}\right)}{-\mathrm{e}^{3x}}\,dx$$

Which simplifies to

$$u_2 = \int -\cos\left(\mathrm{e}^{-x}\right) \mathrm{e}^{-x} dx$$

Hence

$$u_2 = \sin\left(\mathrm{e}^{-x}\right)$$

Which simplifies to

$$u_1 = -\mathrm{e}^{-x} \sin\left(\mathrm{e}^{-x}\right) - \cos\left(\mathrm{e}^{-x}\right) - 1$$
$$u_2 = \sin\left(\mathrm{e}^{-x}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\mathrm{e}^{-x} \sin\left(\mathrm{e}^{-x}\right) - \cos\left(\mathrm{e}^{-x}\right) - 1\right) \mathrm{e}^{2x} + \sin\left(\mathrm{e}^{-x}\right) \mathrm{e}^{x}$$

Which simplifies to

$$y_p(x) = -\mathrm{e}^{2x}\left(1 + \cos\left(\mathrm{e}^{-x}\right)\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{2x} + c_2 e^{x}\right) + \left(-e^{2x}\left(1 + \cos\left(e^{-x}\right)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{2x} + c_2 e^{x} - e^{2x}\left(1 + \cos\left(e^{-x}\right)\right) \tag{1}
$$

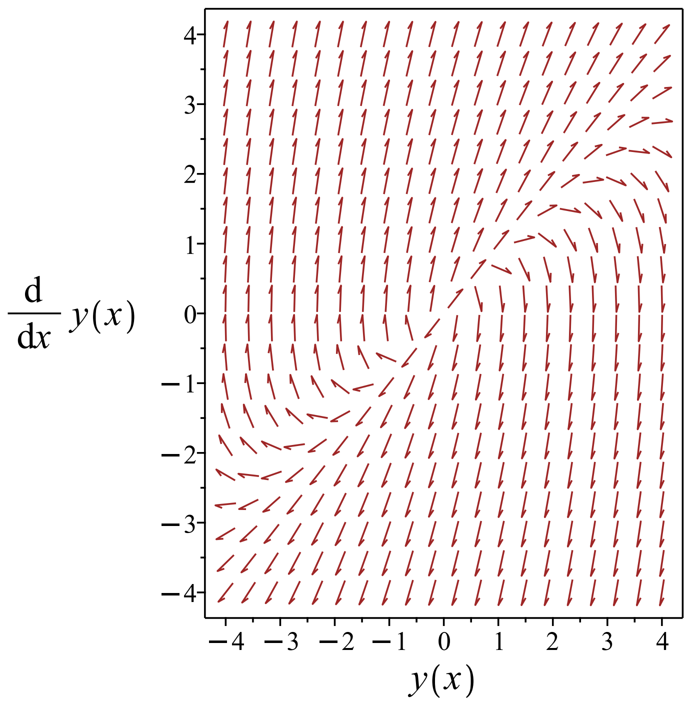


Figure 427: Slope field plot

Verification of solutions

$$
y = c_1 e^{2x} + c_2 e^{x} - e^{2x}\left(1 + \cos\left(e^{-x}\right)\right)
$$

Verified OK.

9.15.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 287: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3}{1} \, dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\, dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(\mathrm{e}^{x}))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x$$
$$y_2 = e^{2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & e^{2x} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^{2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2\,e^{2x} \end{vmatrix}$$

Therefore

$$W = (e^x)\left(2\,e^{2x}\right) - \left(e^{2x}\right)(e^x)$$

Which simplifies to

$$W = e^{2x} e^x$$

Which simplifies to

$$W = e^{3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{2x} \cos(e^{-x})}{e^{3x}} dx$$

Which simplifies to

$$u_1 = -\int \cos(e^{-x}) e^{-x} dx$$

Hence

$$u_1 = \sin(e^{-x})$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{x} \cos(e^{-x})}{e^{3x}} dx$$

Which simplifies to

$$u_2 = \int \cos(e^{-x}) e^{-2x} dx$$

Hence

$$u_2 = \frac{-2 e^{-x} \tan\left(\frac{e^{-x}}{2}\right) - 2}{1 + \tan\left(\frac{e^{-x}}{2}\right)^2}$$

Which simplifies to

$$\begin{aligned} u_1 &= \sin(e^{-x}) \\ u_2 &= -e^{-x} \sin(e^{-x}) - \cos(e^{-x}) - 1 \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-e^{-x} \sin(e^{-x}) - \cos(e^{-x}) - 1\right) e^{2x} + \sin(e^{-x}) e^{x}$$

Which simplifies to

$$y_p(x) = -e^{2x}(1 + \cos(e^{-x}))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 e^{2x}\right) + \left(-e^{2x}\left(1 + \cos\left(e^{-x}\right)\right)\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{2x} - e^{2x}\left(1 + \cos\left(e^{-x}\right)\right) \tag{1}$$

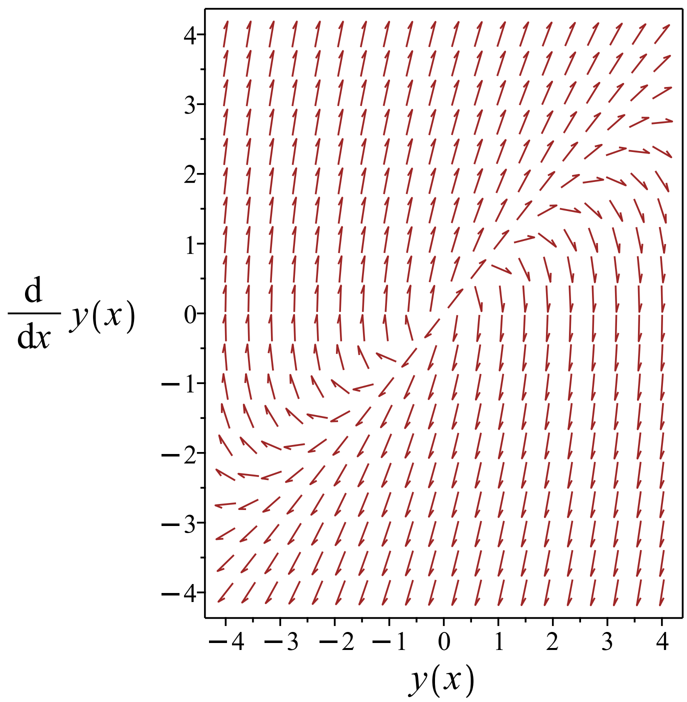


Figure 428: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^x + c_2 e^{2x} - e^{2x}\left(1 + \cos\left(e^{-x}\right)\right)$$

Verified OK.

9.15.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = \cos(\mathrm{e}^{-x})$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \cos(\mathrm{e}^{-x})\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^x & \mathrm{e}^{2x} \\ \mathrm{e}^x & 2\,\mathrm{e}^{2x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{3x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\mathrm{e}^x\left(\int \cos(\mathrm{e}^{-x})\,\mathrm{e}^{-x}dx\right) + \mathrm{e}^{2x}\left(\int \cos(\mathrm{e}^{-x})\,\mathrm{e}^{-2x}dx\right)$

- Compute integrals

  $y_p(x) = e^{2x}(-1 - \cos(e^{-x}))$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^x + c_2 e^{2x} + e^{2x}(-1 - \cos(e^{-x}))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=cos(exp(-x)),y(x), singsol=all)
```

$$y(x) = \left(-e^x \cos\left(e^{-x}\right) + (c_1 - 1)e^x + c_2\right)e^x$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 29

```
DSolve[y''[x]-3*y'[x]+2*y[x]==Cos[Exp[-x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x\left(-e^x \cos\left(e^{-x}\right) + c_2 e^x + c_1\right)$$

9.16 problem Exercise 22, problem 16, page 240

Internal problem ID [4646]
Internal file name [`OUTPUT/4139_Sunday_June_05_2022_12_27_44_PM_92191833/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22, problem 16, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - xy' + y = x}$$

9.16.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = 1, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - xy' + y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r + 1 = 0$$

Or

$$r^2 - 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 1$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1x + c_2x\ln(x)$$

Next, we find the particular solution to the ODE

$$x^2y'' - xy' + y = x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = \ln(x)\, x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \ln(x)\, x \\ \frac{d}{dx}(x) & \frac{d}{dx}(\ln(x)\, x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \ln(x)\, x \\ 1 & 1 + \ln(x) \end{vmatrix}$$

Therefore

$$W = (x)(1 + \ln(x)) - (\ln(x)\, x)(1)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\, x^2}{x^3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x \ln(x)^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x\left(\frac{\ln(x)^2}{2} + c_1 + c_2 \ln(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\left(\frac{\ln(x)^2}{2} + c_1 + c_2 \ln(x)\right) \tag{1}$$

Verification of solutions

$$y = x\left(\frac{\ln(x)^2}{2} + c_1 + c_2 \ln(x)\right)$$

Verified OK.

9.16.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - xy' + y = 0$$

In normal form the ode

$$x^2y'' - xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\,(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)} dx \\
&= \int e^{\ln(x)}\, dx \\
&= \int x dx \\
&= \frac{x^2}{2}
\end{aligned} \tag{6}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2}}{x^2} \\
&= \frac{1}{x^4}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{x^4} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{1}{x^4} = \frac{1}{4\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0
$$

Simplifying gives

$$4r(r-1)\,\tau^{r}+0\,\tau^{r}+\tau^{r}=0$$

Since $\tau^{r}\neq 0$ then dividing throughout by τ^{r} gives

$$4r(r-1)+0+1=0$$

Or

$$4r^{2}-4r+1=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1=\frac{1}{2}$$
$$r_2=\frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau)=c_1y_1+c_2y_2$$

Where $y_1=\tau^{r}$ and $y_2=\tau^{r}\ln(\tau)$. Hence

$$y(\tau)=c_1\sqrt{\tau}+c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y=\frac{x\sqrt{2}\,(c_1+2c_2\ln(x)-c_2\ln(2))}{2}$$

Therefore the homogeneous solution y_h is

$$y_h=\frac{x\sqrt{2}\,(c_1+2c_2\ln(x)-c_2\ln(2))}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2} \\ 1 & \sqrt{2}\,\ln(x) + \sqrt{2} - \frac{\sqrt{2}\,\ln(2)}{2} \end{vmatrix}$$

Therefore

$$W = (x)\left(\sqrt{2}\,\ln(x) + \sqrt{2} - \frac{\sqrt{2}\,\ln(2)}{2}\right) - \left(x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}\right)(1)$$

Which simplifies to

$$W = x\sqrt{2}$$

Which simplifies to

$$W = x\sqrt{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}\right)x}{x^3\sqrt{2}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2\ln(x) - \ln(2)}{2x}dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2} + \frac{\ln(2)\ln(x)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3\sqrt{2}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{2}}{2x}dx$$

Hence

$$u_2 = \frac{\sqrt{2}\,\ln(x)}{2}$$

Which simplifies to

$$u_1 = -\frac{\ln(x)(\ln(x) - \ln(2))}{2}$$

$$u_2 = \frac{\sqrt{2}\,\ln(x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)(\ln(x) - \ln(2))\,x}{2} + \frac{\sqrt{2}\,\ln(x)\left(x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}\right)}{2}$$

Which simplifies to

$$y_p(x) = \frac{x\ln(x)^2}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{x\sqrt{2}\,(c_1 + 2c_2 \ln(x) - c_2 \ln(2))}{2}\right) + \left(\frac{x \ln(x)^2}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x\sqrt{2}\,(c_1 + 2c_2 \ln(x) - c_2 \ln(2))}{2} + \frac{x \ln(x)^2}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{x\sqrt{2}\,(c_1 + 2c_2 \ln(x) - c_2 \ln(2))}{2} + \frac{x \ln(x)^2}{2}
$$

Verified OK.

9.16.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = x^2, B = -x, C = 1, f(x) = x$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
x^2y'' - xy' + y = 0
$$

In normal form the ode

$$
x^2y'' - xy' + y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{1}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -2c \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{c\tau} c_1$$

Now from (6)

$$\tau = \int \frac{1}{c}\sqrt{q}\,dx$$
$$= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c}$$
$$= \frac{\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}$$

Substituting the above into the solution obtained gives

$$y = c_1 x$$

Now the particular solution to this ODE is found

$$x^2y'' - xy' + y = x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2} \\ 1 & \sqrt{2}\,\ln(x) + \sqrt{2} - \frac{\sqrt{2}\,\ln(2)}{2} \end{vmatrix}$$

Therefore

$$W = (x)\left(\sqrt{2}\,\ln(x) + \sqrt{2} - \frac{\sqrt{2}\,\ln(2)}{2}\right) - \left(x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}\right)(1)$$

Which simplifies to

$$W = x\sqrt{2}$$

Which simplifies to

$$W = x\sqrt{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}\right)x}{x^3\sqrt{2}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2\ln(x) - \ln(2)}{2x}dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2} + \frac{\ln(2)\ln(x)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3\sqrt{2}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{2}}{2x} dx$$

Hence

$$u_2 = \frac{\sqrt{2}\, \ln (x)}{2}$$

Which simplifies to

$$u_1 = -\frac{\ln (x) (\ln (x) - \ln (2))}{2}$$

$$u_2 = \frac{\sqrt{2}\, \ln (x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln (x) (\ln (x) - \ln (2))\, x}{2} + \frac{\sqrt{2}\, \ln (x) \left(x\sqrt{2}\, \ln (x) - \frac{x\sqrt{2}\, \ln(2)}{2}\right)}{2}$$

Which simplifies to

$$y_p(x) = \frac{x \ln (x)^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x) + \left(\frac{x \ln (x)^2}{2}\right) \\ &= \frac{x \ln (x)^2}{2} + c_1 x \end{aligned}$$

Which simplifies to

$$y = x\left(\frac{\ln (x)^2}{2} + c_1\right)$$

Summary
The solution(s) found are the following

$$y = x\left(\frac{\ln\left(x\right)^{2}}{2}+c_{1}\right) \tag{1}$$

Verification of solutions

$$y = x\left(\frac{\ln\left(x\right)^{2}}{2}+c_{1}\right)$$

Verified OK.

9.16.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = 1, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - xy' + y = 0$$

In normal form the ode

$$x^2y'' - xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{1}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= e^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= (c_1 \ln(x) + c_2)\,x \\
&= (c_1 \ln(x) + c_2)\,x
\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' - x y' + y = x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x \\
y_2 &= \ln(x)\,x
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \ln(x)\, x \\ \frac{d}{dx}(x) & \frac{d}{dx}(\ln(x)\, x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \ln(x)\, x \\ 1 & 1 + \ln(x) \end{vmatrix}$$

Therefore

$$W = (x)(1 + \ln(x)) - (\ln(x)\, x)(1)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\, x^2}{x^3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x \ln(x)^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= ((c_1 \ln(x) + c_2)\, x) + \left(\frac{x \ln(x)^2}{2}\right) \\ &= \frac{x \ln(x)^2}{2} + (c_1 \ln(x) + c_2)\, x \end{aligned}$$

Which simplifies to

$$y = x\left(\frac{\ln(x)^2}{2} + c_1 \ln(x) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = x\left(\frac{\ln(x)^2}{2} + c_1 \ln(x) + c_2\right) \tag{1}$$

Verification of solutions

$$y = x\left(\frac{\ln(x)^2}{2} + c_1 \ln(x) + c_2\right)$$

Verified OK.

9.16.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^2 \\
B &= -x \\
C &= 1 \\
F &= x
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left(x^2\right)(0) + (-x)(-1) + (1)(-x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-x^3 v'' + (-x^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$-x^2(u'(x)\, x + u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{x} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x}\, \mathrm{d}x \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (-x)\,(c_1 \ln(x) + c_2) \\ &= -(c_1 \ln(x) + c_2)\, x \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= \ln(x)\, x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \ln(x)\, x \\ \frac{d}{dx}(x) & \frac{d}{dx}(\ln(x)\, x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \ln(x)\, x \\ 1 & 1 + \ln(x) \end{vmatrix}$$

Therefore

$$W = (x)(1 + \ln(x)) - (\ln(x)\, x)(1)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\, x^2}{x^3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x \ln(x)^2}{2}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (-(c_1 \ln(x) + c_2)\, x) + \left(\frac{x \ln(x)^2}{2}\right) \\ &= -\left(c_1 \ln(x) + c_2 - \frac{\ln(x)^2}{2}\right) x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\left(c_1 \ln(x) + c_2 - \frac{\ln(x)^2}{2}\right) x \tag{1}$$

Verification of solutions

$$y = -\left(c_1 \ln(x) + c_2 - \frac{\ln(x)^2}{2}\right) x$$

Verified OK.

9.16.6 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - xy' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 289: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$
\begin{gathered}
(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0 \\
0=0
\end{gathered}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2} \, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2(x(\ln(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - xy' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 x + c_2 x \ln(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \ln(x)\, x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \ln(x)\, x \\ \frac{d}{dx}(x) & \frac{d}{dx}(\ln(x)\, x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \ln(x)\, x \\ 1 & 1 + \ln(x) \end{vmatrix}$$

Therefore

$$W = (x)\,(1 + \ln(x)) - (\ln(x)\, x)\,(1)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\, x^2}{x^3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x \ln(x)^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x + c_2 x \ln(x)) + \left(\frac{x \ln(x)^2}{2}\right) \end{aligned}$$

Which simplifies to

$$y = x(c_2 \ln(x) + c_1) + \frac{x \ln(x)^2}{2}$$

Summary

The solution(s) found are the following

$$y = x(c_2 \ln(x) + c_1) + \frac{x \ln(x)^2}{2} \tag{1}$$

Verification of solutions

$$y = x(c_2 \ln(x) + c_1) + \frac{x \ln(x)^2}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=x,y(x), singsol=all)
```

$$y(x) = x\left(c_2 + c_1 \ln(x) + \frac{\ln(x)^2}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 25

```
DSolve[x^2*y''[x]-x*y'[x]+y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}x\left(\log^2(x) + 2c_2 \log(x) + 2c_1\right)$$

9.17 problem Exercise 22, problem 17, page 240

Internal problem ID [4647]
Internal file name [`OUTPUT/4140_Sunday_June_05_2022_12_27_52_PM_15031758/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22, problem 17, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - \frac{2y'}{x} + \frac{2y}{x^2} = \ln(x)\, x}$$

The ode can be written as

$$x^2y'' - 2xy' + 2y = \ln(x)\, x^3$$

Which shows it is a Euler ODE.

9.17.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -2x, C = 2, f(x) = \ln(x)\,x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 2xy' + 2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 2xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 2r\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 2r + 2 = 0$$

Or

$$r^2 - 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^2 + c_1x$$

Next, we find the particular solution to the ODE

$$x^2y'' - 2xy' + 2y = \ln(x)\,x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2)(1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5 \ln(x)}{x^4} dx$$

Which simplifies to

$$u_1 = -\int \ln(x) x dx$$

Hence

$$u_1 = -\frac{\ln(x) x^2}{2} + \frac{x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \ln(x)}{x^4} dx$$

Which simplifies to

$$u_2 = \int \ln(x) dx$$

Hence

$$u_2 = \ln(x) x - x$$

Which simplifies to

$$u_1 = -\frac{x^2(-1 + 2\ln(x))}{4}$$
$$u_2 = x(\ln(x) - 1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^3(-1 + 2\ln(x))}{4} + x^3(\ln(x) - 1)$$

Which simplifies to

$$y_p(x) = \frac{x^3(-3 + 2\ln(x))}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{x^3(-3 + 2\ln(x))}{4} + c_2 x^2 + c_1 x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x^3(-3 + 2\ln(x))}{4} + c_2 x^2 + c_1 x \tag{1}
$$

Verification of solutions

$$
y = \frac{x^3(-3 + 2\ln(x))}{4} + c_2 x^2 + c_1 x
$$

Verified OK.

9.17.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$
y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)
$$

Where $p(x) = -\frac{2}{x}$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\, dx} \\
&= e^{\int -\frac{2}{x}\, dx} \\
&= \frac{1}{x}
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \ln(x) \\
\left(\frac{y}{x}\right)'' &= \ln(x)
\end{aligned}
$$

Integrating once gives

$$
\left(\frac{y}{x}\right)' = x(\ln(x) - 1) + c_1
$$

Integrating again gives

$$\left(\frac{y}{x}\right) = \frac{x(2\ln(x)\,x + 4c_1 - 3x)}{4} + c_2$$

Hence the solution is

$$y = \frac{\frac{x(2\ln(x)x+4c_1-3x)}{4} + c_2}{\frac{1}{x}}$$

Or

$$y = \frac{x^3\ln(x)}{2} + c_1x^2 - \frac{3x^3}{4} + c_2x$$

Summary

The solution(s) found are the following

$$y = \frac{x^3\ln(x)}{2} + c_1x^2 - \frac{3x^3}{4} + c_2x \tag{1}$$

Verification of solutions

$$y = \frac{x^3\ln(x)}{2} + c_1x^2 - \frac{3x^3}{4} + c_2x$$

Verified OK.

9.17.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 2xy' + 2y = 0$$

In normal form the ode

$$x^2y'' - 2xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{2}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$
\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $p_1 = 0$. Eq (4) simplifies to

$$
\tau''(x) + p(x)\,\tau'(x) = 0
$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{2}{x}dx\right)}dx \\
&= \int e^{2\ln(x)}\,dx \\
&= \int x^2 dx \\
&= \frac{x^3}{3}
\end{aligned} \tag{6}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{2}{x^2}}{x^4} \\
&= \frac{2}{x^6}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{x^6} = 0$$

But in terms of τ

$$\frac{2}{x^6} = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 + 2 = 0$$

Or

$$9r^2 - 9r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{1}{3}} + c_2 \tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^3\right)^{\frac{1}{3}}$$

$$y_2 = \left(x^3\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^3\right)^{\frac{1}{3}} & \left(x^3\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(x^3\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(x^3\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^3\right)^{\frac{1}{3}} & \left(x^3\right)^{\frac{2}{3}} \\ \frac{x^2}{\left(x^3\right)^{\frac{2}{3}}} & \frac{2x^2}{\left(x^3\right)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^3\right)^{\frac{1}{3}}\right)\left(\frac{2x^2}{\left(x^3\right)^{\frac{1}{3}}}\right) - \left(\left(x^3\right)^{\frac{2}{3}}\right)\left(\frac{x^2}{\left(x^3\right)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x^3\right)^{\frac{2}{3}} x^3 \ln(x)}{x^4}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\left(x^3\right)^{\frac{2}{3}} \ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\left(x^3\right)^{\frac{2}{3}} \ln(x)}{2} + \frac{\left(x^3\right)^{\frac{2}{3}}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(x^3\right)^{\frac{1}{3}} x^3 \ln(x)}{x^4}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\left(x^3\right)^{\frac{1}{3}} \ln(x)}{x} dx$$

Hence

$$u_2 = \left(x^3\right)^{\frac{1}{3}} \ln(x) - \left(x^3\right)^{\frac{1}{3}}$$

Which simplifies to

$$u_1 = -\frac{(x^3)^{\frac{2}{3}} (-1 + 2\ln(x))}{4}$$

$$u_2 = (x^3)^{\frac{1}{3}} (\ln(x) - 1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^3(-1 + 2\ln(x))}{4} + x^3(\ln(x) - 1)$$

Which simplifies to

$$y_p(x) = \frac{x^3(-3 + 2\ln(x))}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3}\right) + \left(\frac{x^3(-3 + 2\ln(x))}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3} + \frac{x^3(-3 + 2\ln(x))}{4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3} + \frac{x^3(-3 + 2\ln(x))}{4}$$

Verified OK.

9.17.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -2x, C = 2, f(x) = x^3 \ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 2xy' + 2y = 0$$

In normal form the ode

$$x^2y'' - 2xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{2}{x^2}
\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c} \qquad (6)\\
\tau'' &= -\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{2}{x}\,\frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{3c\sqrt{2}}{2}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{3c\sqrt{2}\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{\frac{3\sqrt{2}\,c\tau}{4}}\left(c_1 \cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2 \sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{2}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Now the particular solution to this ODE is found

$$
x^2y'' - 2xy' + 2y = x^3\ln(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^3\right)^{\frac{1}{3}}$$
$$y_2 = \left(x^3\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^3\right)^{\frac{1}{3}} & \left(x^3\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(x^3\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(x^3\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^3\right)^{\frac{1}{3}} & \left(x^3\right)^{\frac{2}{3}} \\ \frac{x^2}{\left(x^3\right)^{\frac{2}{3}}} & \frac{2x^2}{\left(x^3\right)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^3\right)^{\frac{1}{3}}\right)\left(\frac{2x^2}{\left(x^3\right)^{\frac{1}{3}}}\right) - \left(\left(x^3\right)^{\frac{2}{3}}\right)\left(\frac{x^2}{\left(x^3\right)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x^3\right)^{\frac{2}{3}} x^3 \ln\left(x\right)}{x^4} dx$$

Which simplifies to

$$u_1 = -\int \frac{\left(x^3\right)^{\frac{2}{3}} \ln\left(x\right)}{x} dx$$

Hence

$$u_1 = -\frac{\left(x^3\right)^{\frac{2}{3}} \ln\left(x\right)}{2} + \frac{\left(x^3\right)^{\frac{2}{3}}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(x^3\right)^{\frac{1}{3}} x^3 \ln\left(x\right)}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{\left(x^3\right)^{\frac{1}{3}} \ln\left(x\right)}{x} dx$$

Hence

$$u_2 = \left(x^3\right)^{\frac{1}{3}} \ln\left(x\right) - \left(x^3\right)^{\frac{1}{3}}$$

Which simplifies to

$$u_1 = -\frac{\left(x^3\right)^{\frac{2}{3}} \left(-1 + 2\ln\left(x\right)\right)}{4}$$

$$u_2 = \left(x^3\right)^{\frac{1}{3}} \left(\ln\left(x\right) - 1\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^3(-1 + 2\ln\left(x\right))}{4} + x^3(\ln\left(x\right) - 1)$$

Which simplifies to

$$y_p(x) = \frac{x^3(-3 + 2\ln(x))}{4}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)\right) + \left(\frac{x^3(-3 + 2\ln(x))}{4}\right) \\
&= \frac{x^3(-3 + 2\ln(x))}{4} + x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)
\end{aligned}$$

Which simplifies to

$$y = \frac{x^3(-3 + 2\ln(x))}{4} + x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{x^3(-3 + 2\ln(x))}{4} + x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) \tag{1}$$

Verification of solutions

$$y = \frac{x^3(-3 + 2\ln(x))}{4} + x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)$$

Verified OK.

9.17.5 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 2xy' + 2y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(-\frac{2}{x}\right)'}{2} - \frac{\left(-\frac{2}{x}\right)^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\ &= \frac{2}{x^2} - \left(\frac{1}{x^2}\right) - \frac{1}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{-\frac{2}{x}}{2}} \\ &= x \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) = \ln(x)$$

Which is now solved for $v(x)$ Integrating once gives

$$v'(x) = \ln(x)\, x - x + c_1$$

Integrating again gives

$$v(x) = \frac{\ln(x)\, x^2}{2} - \frac{3x^2}{4} + c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(\frac{\ln(x)\, x^2}{2} + c_1 x - \frac{3x^2}{4} + c_2\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x$$

Hence (7) becomes

$$y = \left(\frac{\ln(x)\, x^2}{2} + c_1 x - \frac{3x^2}{4} + c_2\right) x$$

Therefore the homogeneous solution y_h is

$$y_h = \left(\frac{\ln(x)\, x^2}{2} + c_1 x - \frac{3x^2}{4} + c_2\right) x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^3\right)^{\frac{1}{3}}$$

$$y_2 = \left(x^3\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^3)^{\frac{1}{3}} & (x^3)^{\frac{2}{3}} \\ \frac{d}{dx}\left((x^3)^{\frac{1}{3}}\right) & \frac{d}{dx}\left((x^3)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^3)^{\frac{1}{3}} & (x^3)^{\frac{2}{3}} \\ \frac{x^2}{(x^3)^{\frac{2}{3}}} & \frac{2x^2}{(x^3)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left((x^3)^{\frac{1}{3}}\right)\left(\frac{2x^2}{(x^3)^{\frac{1}{3}}}\right) - \left((x^3)^{\frac{2}{3}}\right)\left(\frac{x^2}{(x^3)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^3)^{\frac{2}{3}}\, x^3 \ln(x)}{x^4}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^3)^{\frac{2}{3}} \ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{(x^3)^{\frac{2}{3}} \ln(x)}{2} + \frac{(x^3)^{\frac{2}{3}}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} x^3 \ln(x)}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} \ln(x)}{x} dx$$

Hence

$$u_2 = (x^3)^{\frac{1}{3}} \ln(x) - (x^3)^{\frac{1}{3}}$$

Which simplifies to

$$u_1 = -\frac{(x^3)^{\frac{2}{3}} (-1 + 2\ln(x))}{4}$$

$$u_2 = (x^3)^{\frac{1}{3}} (\ln(x) - 1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^3(-1 + 2\ln(x))}{4} + x^3(\ln(x) - 1)$$

Which simplifies to

$$y_p(x) = \frac{x^3(-3 + 2\ln(x))}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(\frac{\ln(x)\, x^2}{2} + c_1 x - \frac{3x^2}{4} + c_2\right) x\right) + \left(\frac{x^3(-3 + 2\ln(x))}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(\frac{\ln(x)\, x^2}{2} + c_1 x - \frac{3x^2}{4} + c_2\right) x + \frac{x^3(-3 + 2\ln(x))}{4} \tag{1}$$

Verification of solutions

$$y = \left(\frac{\ln(x)\, x^2}{2} + c_1 x - \frac{3x^2}{4} + c_2\right) x + \frac{x^3(-3 + 2\ln(x))}{4}$$

Verified OK.

9.17.6 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -2x, C = 2, f(x) = x^3 \ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 2xy' + 2y = 0$$

In normal form the ode

$$x^2y'' - 2xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{2}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{2n}{x^2} + \frac{2}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\
\ln(u) &= -2\ln(x) + c_1 \\
u &= \mathrm{e}^{-2\ln(x)+c_1} \\
&= \frac{c_1}{x^2}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{x} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{x} + c_2\right) x^2 \\
&= (c_2 x - c_1)\,x
\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' - 2xy' + 2y = x^3\ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x \\
y_2 &= x^2
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2)(1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5 \ln(x)}{x^4} \, dx$$

Which simplifies to

$$u_1 = -\int \ln(x) \, x dx$$

Hence

$$u_1 = -\frac{\ln(x) \, x^2}{2} + \frac{x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \ln(x)}{x^4} \, dx$$

Which simplifies to

$$u_2 = \int \ln(x) \, dx$$

Hence

$$u_2 = \ln(x) \, x - x$$

Which simplifies to

$$u_1 = -\frac{x^2(-1 + 2\ln(x))}{4}$$
$$u_2 = x(\ln(x) - 1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^3(-1 + 2\ln(x))}{4} + x^3(\ln(x) - 1)$$

Which simplifies to

$$y_p(x) = \frac{x^3(-3 + 2\ln(x))}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{x} + c_2\right) x^2\right) + \left(\frac{x^3(-3 + 2\ln(x))}{4}\right) \\ &= \frac{x^3(-3 + 2\ln(x))}{4} + \left(-\frac{c_1}{x} + c_2\right) x^2 \end{aligned}$$

Which simplifies to

$$y = \frac{x^3(-3 + 2\ln(x))}{4} + \left(-\frac{c_1}{x} + c_2\right) x^2$$

Summary

The solution(s) found are the following

$$y = \frac{x^3(-3 + 2\ln(x))}{4} + \left(-\frac{c_1}{x} + c_2\right) x^2 \tag{1}$$

Verification of solutions

$$y = \frac{x^3(-3 + 2\ln(x))}{4} + \left(-\frac{c_1}{x} + c_2\right) x^2$$

Verified OK.

9.17.7 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x^2 \\
B &= -2x \\
C &= 2 \\
F &= x^3 \ln(x)
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (x^2)(0) + (-2x)(-2) + (2)(-2x) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-2x^3 v'' + (0)\, v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
-2x^3 u'(x) = 0
$$

Which is now solved for u. Integrating both sides gives

$$
\begin{aligned}
u(x) &= \int 0 \, \mathrm{d}x \\
&= c_1
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int c_1 \, \mathrm{d}x \\
&= c_1 x + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (-2x)(c_1 x + c_2) \\
&= -2x(c_1 x + c_2)
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2)(1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5 \ln(x)}{x^4} dx$$

Which simplifies to

$$u_1 = -\int \ln(x)\, x dx$$

Hence

$$u_1 = -\frac{\ln(x)\, x^2}{2} + \frac{x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \ln(x)}{x^4} dx$$

Which simplifies to

$$u_2 = \int \ln(x)\, dx$$

Hence

$$u_2 = \ln(x)\, x - x$$

Which simplifies to

$$u_1 = -\frac{x^2(-1 + 2\ln(x))}{4}$$
$$u_2 = x(\ln(x) - 1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^3(-1 + 2\ln(x))}{4} + x^3(\ln(x) - 1)$$

Which simplifies to

$$y_p(x) = \frac{x^3(-3 + 2\ln(x))}{4}$$

Hence the complete solution is

$$
\begin{aligned}
y(x) &= y_h + y_p \\
&= (-2x(c_1x + c_2)) + \left(\frac{x^3(-3 + 2\ln(x))}{4}\right) \\
&= \frac{x^3 \ln(x)}{2} - \frac{3x^3}{4} - 2c_1x^2 - 2c_2x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x^3 \ln(x)}{2} - \frac{3x^3}{4} - 2c_1x^2 - 2c_2x \tag{1}
$$

Verification of solutions

$$
y = \frac{x^3 \ln(x)}{2} - \frac{3x^3}{4} - 2c_1x^2 - 2c_2x
$$

Verified OK.

9.17.8 Solving using Kovacic algorithm

Writing the ode as

$$
x^2y'' - 2xy' + 2y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= -2x \\
C &= 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 290: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2(x(x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 2xy' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2x^2 + c_1x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2)(1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5 \ln(x)}{x^4} \, dx$$

Which simplifies to

$$u_1 = -\int \ln(x) \, x dx$$

Hence

$$u_1 = -\frac{\ln(x) \, x^2}{2} + \frac{x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \ln(x)}{x^4} \, dx$$

Which simplifies to

$$u_2 = \int \ln(x) \, dx$$

Hence

$$u_2 = \ln(x) \, x - x$$

Which simplifies to

$$u_1 = -\frac{x^2(-1 + 2\ln(x))}{4}$$
$$u_2 = x(\ln(x) - 1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^3(-1 + 2\ln(x))}{4} + x^3(\ln(x) - 1)$$

Which simplifies to

$$y_p(x) = \frac{x^3(-3 + 2\ln(x))}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2x^2 + c_1x) + \left(\frac{x^3(-3 + 2\ln(x))}{4}\right) \end{aligned}$$

Which simplifies to

$$y = x(c_2x + c_1) + \frac{x^3(-3 + 2\ln(x))}{4}$$

Summary

The solution(s) found are the following

$$y = x(c_2x + c_1) + \frac{x^3(-3 + 2\ln(x))}{4} \tag{1}$$

Verification of solutions

$$y = x(c_2x + c_1) + \frac{x^3(-3 + 2\ln(x))}{4}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)-2/x*diff(y(x),x)+2/x^2*y(x)=x*ln(x),y(x), singsol=all)
```

$$y(x) = \frac{\ln(x)\,x^3}{2} - \frac{3x^3}{4} + c_2x^2 + c_1x$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 32

```
DSolve[y''[x]-2/x*y'[x]+2/x^2*y[x]==x*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}x\left(-3x^2 + 2x^2\log(x) + 4c_2x + 4c_1\right)$$

9.18 problem Exercise 22, problem 18, page 240

Internal problem ID [4648]
Internal file name [`OUTPUT/4141_Sunday_June_05_2022_12_27_59_PM_16309370/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22, problem 18, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' - 4y = x^3}$$

9.18.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -4, f(x) = x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - 4y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 4 = 0$$

Or

$$r^2 - 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -2$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^2} + c_2 x^2$$

Next, we find the particular solution to the ODE

$$x^2 y'' + xy' - 4y = x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ -\frac{2}{x^3} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(2x) - (x^2)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{4}{x}$$

Which simplifies to

$$W = \frac{4}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5}{4x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^4}{4} dx$$

Hence

$$u_1 = -\frac{x^5}{20}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x}{4x}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{4} dx$$

Hence

$$u_2 = \frac{x}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^3}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{x^3}{5} + \frac{c_1}{x^2} + c_2 x^2 \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{x^3}{5} + \frac{c_1}{x^2} + c_2 x^2 \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{x^3}{5} + \frac{c_1}{x^2} + c_2 x^2$$

Verified OK.

9.18.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + xy' - 4y = 0$$

In normal form the ode

$$x^2y'' + xy' - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'\left(x\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\left(x\right)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int \frac{1}{x}dx\right)} dx \\
&= \int e^{-\ln(x)}\, dx \\
&= \int \frac{1}{x} dx \\
&= \ln(x) \qquad (6)
\end{aligned}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{4}{x^2}}{\frac{1}{x^2}} \\
&= -4 \qquad (7)
\end{aligned}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 4y(\tau) &= 0
\end{aligned}
$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Ay''(\tau) + By'(\tau) + Cy(\tau) = 0
$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$
\lambda^2 e^{\lambda\tau} - 4\, e^{\lambda\tau} = 0 \qquad (1)
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$
\lambda^2 - 4 = 0 \qquad (2)
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-4)} \\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2 \\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\ y(\tau) &= c_1 e^{(2)\tau} + c_2 e^{(-2)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 \mathrm{e}^{2\tau} + c_2 \mathrm{e}^{-2\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^4 + c_2}{x^2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x^4 + c_2}{x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ -\frac{2}{x^3} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(2x) - (x^2)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{4}{x}$$

Which simplifies to

$$W = \frac{4}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5}{4x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^4}{4} dx$$

Hence

$$u_1 = -\frac{x^5}{20}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x}{4x}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{4} dx$$

Hence

$$u_2 = \frac{x}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^3}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x^4 + c_2}{x^2}\right) + \left(\frac{x^3}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^4 + c_2}{x^2} + \frac{x^3}{5} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^4 + c_2}{x^2} + \frac{x^3}{5}$$

Verified OK.

9.18.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -4, f(x) = x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - 4y = 0$$

In normal form the ode

$$x^2y'' + xy' - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{-\frac{1}{x^2}}}{c} \\ \tau'' &= \frac{2}{c\sqrt{-\frac{1}{x^2}}\, x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{2}{c\sqrt{-\frac{1}{x^2}}\, x^3} + \frac{1}{x}\frac{2\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\ &= \frac{\int 2\sqrt{-\frac{1}{x^2}}dx}{c} \\ &= \frac{2\sqrt{-\frac{1}{x^2}}\, x \ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cosh(2\ln(x)) + ic_2 \sinh(2\ln(x))$$

Now the particular solution to this ODE is found

$$x^2y'' + xy' - 4y = x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ -\frac{2}{x^3} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(2x) - (x^2)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{4}{x}$$

Which simplifies to

$$W = \frac{4}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5}{4x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x^4}{4}dx$$

Hence

$$u_1 = -\frac{x^5}{20}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x}{4x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{4}dx$$

Hence

$$u_2 = \frac{x}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^3}{5}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cosh (2 \ln (x)) + i c_2 \sinh (2 \ln (x))) + \left(\frac{x^3}{5}\right) \\
&= \frac{x^3}{5} + c_1 \cosh (2 \ln (x)) + i c_2 \sinh (2 \ln (x))
\end{aligned}
$$

Which simplifies to

$$
y = \frac{x^3}{5} + c_1 \cosh (2 \ln (x)) + i c_2 \sinh (2 \ln (x))
$$

Summary

The solution(s) found are the following

$$
y = \frac{x^3}{5} + c_1 \cosh (2 \ln (x)) + i c_2 \sinh (2 \ln (x)) \tag{1}
$$

Verification of solutions

$$
y = \frac{x^3}{5} + c_1 \cosh (2 \ln (x)) + i c_2 \sinh (2 \ln (x))
$$

Verified OK.

9.18.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
A y''(x) + B y'(x) + C y(x) = f(x)
$$

Where $A = x^2, B = x, C = -4, f(x) = x^3$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
x^2 y'' + x y' - 4y = 0
$$

In normal form the ode

$$x^2y'' + xy' - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{5v'(x)}{x} &= 0 \\ v''(x) + \frac{5v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{5u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{5u}{x}
\end{aligned}$$

Where $f(x) = -\frac{5}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{5}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{5}{x}\,dx \\
\ln(u) &= -5\ln(x) + c_1 \\
u &= \mathrm{e}^{-5\ln(x)+c_1} \\
&= \frac{c_1}{x^5}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{4x^4} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{4x^4} + c_2\right)x^2 \\
&= \frac{4c_2x^4 - c_1}{4x^2}
\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' + xy' - 4y = x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$

$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ -\frac{2}{x^3} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(2x) - \left(x^2\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{4}{x}$$

Which simplifies to

$$W = \frac{4}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5}{4x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^4}{4} dx$$

Hence

$$u_1 = -\frac{x^5}{20}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x}{4x}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{4} dx$$

Hence

$$u_2 = \frac{x}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^3}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{4x^4} + c_2\right) x^2\right) + \left(\frac{x^3}{5}\right) \\ &= \frac{x^3}{5} + \left(-\frac{c_1}{4x^4} + c_2\right) x^2 \end{aligned}$$

Which simplifies to

$$y = -\frac{-20c_2x^4 - 4x^5 + 5c_1}{20x^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{-20c_2x^4 - 4x^5 + 5c_1}{20x^2} \tag{1}$$

Verification of solutions

$$y = -\frac{-20c_2x^4 - 4x^5 + 5c_1}{20x^2}$$

Verified OK.

9.18.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + xy' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 291: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2x} + (-)\,(0) \\ &= -\frac{3}{2x} \\ &= -\frac{3}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2x}\right)(0) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x}\right)^2 - \left(\frac{15}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{3}{2x} dx} \\ &= \frac{1}{x^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^4}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x^2}\right) + c_2\left(\frac{1}{x^2}\left(\frac{x^4}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + xy' - 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x^2} + \frac{c_2 x^2}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \frac{1}{x^2} \\ y_2 &= \frac{x^2}{4} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{x^2}{4} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{x^2}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{x^2}{4} \\ -\frac{2}{x^3} & \frac{x}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(\frac{x}{2}\right) - \left(\frac{x^2}{4}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^5}{4}}{x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^4}{4} dx$$

Hence

$$u_1 = -\frac{x^5}{20}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x}{x}\, dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^3}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x^2} + \frac{c_2 x^2}{4}\right) + \left(\frac{x^3}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^2} + \frac{c_2 x^2}{4} + \frac{x^3}{5} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^2} + \frac{c_2 x^2}{4} + \frac{x^3}{5}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=x^3,y(x), singsol=all)
```

$$y(x) = \frac{c_2}{x^2} + c_1 x^2 + \frac{x^3}{5}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 25

```
DSolve[x^2*y''[x]+x*y'[x]-4*y[x]==x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{5} + c_2 x^2 + \frac{c_1}{x^2}$$

9.19 problem Exercise 22, problem 19, page 240

Internal problem ID [4649]
Internal file name [`OUTPUT/4142_Sunday_June_05_2022_12_28_07_PM_84125990/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22, problem 19, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' + xy' - y = x^2\mathrm{e}^{-x}}$$

9.19.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = x^2 e^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 1 = 0$$

Or

$$r^2 - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2x$$

Next, we find the particular solution to the ODE

$$x^2y'' + xy' - y = x^2 e^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & x \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x \\ -\frac{1}{x^2} & 1 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)(1) - (x)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{2}{x}$$

Which simplifies to

$$W = \frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3 e^{-x}}{2x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^2 e^{-x}}{2} dx$$

Hence

$$u_1 = \frac{(x^2 + 2x + 2)\, e^{-x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\, e^{-x}}{2x}\, dx$$

Which simplifies to

$$u_2 = \int \frac{e^{-x}}{2} dx$$

Hence

$$u_2 = -\frac{e^{-x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(x^2 + 2x + 2)\, e^{-x}}{2x} - \frac{x\, e^{-x}}{2}$$

Which simplifies to

$$y_p(x) = \frac{e^{-x}(x + 1)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{e^{-x}(x + 1)}{x} + \frac{c_1}{x} + c_2 x \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-x}(x+1)}{x} + \frac{c_1}{x} + c_2 x \tag{1}$$

Verification of solutions

$$y = \frac{e^{-x}(x+1)}{x} + \frac{c_1}{x} + c_2 x$$

Verified OK.

9.19.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + xy' - y = 0$$

In normal form the ode

$$x^2y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)} dx \\
&= \int e^{-\ln(x)}\, dx \\
&= \int \frac{1}{x} dx \\
&= \ln(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= -1
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) - y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} - e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1\tau} + c_2 e^{\lambda_2\tau} \\ y(\tau) &= c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{\tau} + c_2 e^{-\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^2 + c_2}{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x^2 + c_2}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\,\mathrm{e}^{-x}}{-2x}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\mathrm{e}^{-x}}{2}dx$$

Hence

$$u_1 = -\frac{\mathrm{e}^{-x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^3\mathrm{e}^{-x}}{-2x}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{x^2\mathrm{e}^{-x}}{2}dx$$

Hence

$$u_2 = \frac{(x^2+2x+2)\,\mathrm{e}^{-x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(x^2+2x+2)\,\mathrm{e}^{-x}}{2x} - \frac{x\,\mathrm{e}^{-x}}{2}$$

Which simplifies to

$$y_p(x) = \frac{e^{-x}(x+1)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x^2 + c_2}{x}\right) + \left(\frac{e^{-x}(x+1)}{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2 + c_2}{x} + \frac{e^{-x}(x+1)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2 + c_2}{x} + \frac{e^{-x}(x+1)}{x}$$

Verified OK.

9.19.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = x^2 e^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - y = 0$$

In normal form the ode

$$x^2y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -\frac{1}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$
\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{-\frac{1}{x^2}}}{c} \\
\tau'' &= \frac{1}{c\sqrt{-\frac{1}{x^2}}\, x^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{c\sqrt{-\frac{1}{x^2}}\, x^3} + \frac{1}{x}\frac{\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\ &= \frac{\int \sqrt{-\frac{1}{x^2}}dx}{c} \\ &= \frac{\sqrt{-\frac{1}{x^2}}\, x \ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}$$

Now the particular solution to this ODE is found

$$x^2 y'' + xy' - y = x^2 \mathrm{e}^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= x \\ y_2 &= \frac{1}{x}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\,\mathrm{e}^{-x}}{-2x}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\mathrm{e}^{-x}}{2}dx$$

Hence

$$u_1 = -\frac{\mathrm{e}^{-x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^3\mathrm{e}^{-x}}{-2x}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{x^2 e^{-x}}{2} dx$$

Hence

$$u_2 = \frac{(x^2 + 2x + 2)\, e^{-x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(x^2 + 2x + 2)\, e^{-x}}{2x} - \frac{x\, e^{-x}}{2}$$

Which simplifies to

$$y_p(x) = \frac{e^{-x}(x + 1)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}\right) + \left(\frac{e^{-x}(x + 1)}{x}\right) \\ &= \frac{e^{-x}(x + 1)}{x} + \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x} \end{aligned}$$

Which simplifies to

$$y = \frac{(2 + 2x)\, e^{-x} + (ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{(2 + 2x)\, e^{-x} + (ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{(2 + 2x)\, e^{-x} + (ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}$$

Verified OK.

9.19.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = x^2 e^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2 y'' + xy' - y = 0$$

In normal form the ode

$$x^2 y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}
v''(x) + \frac{3v'(x)}{x} &= 0 \\
v''(x) + \frac{3v'(x)}{x} &= 0 \qquad (7)
\end{aligned}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{3u}{x}
\end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{3}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\
\ln(u) &= -3\ln(x) + c_1 \\
u &= \mathrm{e}^{-3\ln(x)+c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(-\frac{c_1}{2x^2} + c_2\right) x \\ &= \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' + xy' - y = x^2\mathrm{e}^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= \frac{1}{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\,\mathrm{e}^{-x}}{-2x}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\mathrm{e}^{-x}}{2}dx$$

Hence

$$u_1 = -\frac{\mathrm{e}^{-x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^3 \mathrm{e}^{-x}}{-2x}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{x^2 \mathrm{e}^{-x}}{2}dx$$

Hence

$$u_2 = \frac{(x^2 + 2x + 2)\,\mathrm{e}^{-x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(x^2 + 2x + 2)\,e^{-x}}{2x} - \frac{x\,e^{-x}}{2}$$

Which simplifies to

$$y_p(x) = \frac{e^{-x}(x + 1)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x\right) + \left(\frac{e^{-x}(x+1)}{x}\right) \\ &= \frac{e^{-x}(x+1)}{x} + \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

Which simplifies to

$$y = \frac{(2 + 2x)\,e^{-x} + 2c_2x^2 - c_1}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{(2 + 2x)\,e^{-x} + 2c_2x^2 - c_1}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{(2 + 2x)\,e^{-x} + 2c_2x^2 - c_1}{2x}$$

Verified OK.

9.19.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y''+xy'-y)\,dx = \int x^2 e^{-x} dx$$
$$x^2y'-xy = -(x^2+2x+2)\,e^{-x}+c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{(-x^2-2x-2)\,e^{-x}+c_1}{x^2}$$

Hence the ode is

$$y'-\frac{y}{x} = \frac{(-x^2-2x-2)\,e^{-x}+c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{(-x^2-2x-2)\,e^{-x}+c_1}{x^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left(\frac{(-x^2-2x-2)\,e^{-x}+c_1}{x^2}\right)$$
$$d\left(\frac{y}{x}\right) = \left(\frac{(-x^2-2x-2)\,e^{-x}+c_1}{x^3}\right)\,dx$$

Integrating gives

$$\frac{y}{x} = \int \frac{(-x^2-2x-2)\,e^{-x}+c_1}{x^3}\,dx$$
$$\frac{y}{x} = -\frac{c_1}{2x^2}+\frac{e^{-x}}{x^2}+\frac{e^{-x}}{x}+c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(-\frac{c_1}{2x^2} + \frac{e^{-x}}{x^2} + \frac{e^{-x}}{x}\right) + c_2 x$$

which simplifies to

$$y = \frac{(2+2x)\, e^{-x} + 2c_2x^2 - c_1}{2x}$$

Summary
The solution(s) found are the following

$$y = \frac{(2+2x)\, e^{-x} + 2c_2x^2 - c_1}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{(2+2x)\, e^{-x} + 2c_2x^2 - c_1}{2x}$$

Verified OK.

9.19.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)\, u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^2 \\
B &= x \\
C &= -1 \\
F &= x^2 \mathrm{e}^{-x}
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= (x^2)\,(0) + (x)\,(1) + (-1)\,(x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$x^3 v'' + (3x^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$x^2 (u'(x)\, x + 3u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= -\frac{3u}{x}
\end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{3}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\
\ln(u) &= -3\ln(x) + c_1 \\
u &= \mathrm{e}^{-3\ln(x) + c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1}{x^3}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1}{x^3}\, \mathrm{d}x \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (x)\left(-\frac{c_1}{2x^2} + c_2\right) \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \frac{1}{x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\,\mathrm{e}^{-x}}{-2x}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\mathrm{e}^{-x}}{2}dx$$

Hence

$$u_1 = -\frac{\mathrm{e}^{-x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^3\mathrm{e}^{-x}}{-2x}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{x^2 \mathrm{e}^{-x}}{2} dx$$

Hence

$$u_2 = \frac{(x^2 + 2x + 2)\,\mathrm{e}^{-x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(x^2 + 2x + 2)\,\mathrm{e}^{-x}}{2x} - \frac{x\,\mathrm{e}^{-x}}{2}$$

Which simplifies to

$$y_p(x) = \frac{\mathrm{e}^{-x}(x + 1)}{x}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x\right) + \left(\frac{\mathrm{e}^{-x}(x + 1)}{x}\right) \\ &= \frac{(2 + 2x)\,\mathrm{e}^{-x} + 2c_2 x^2 - c_1}{2x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{(2 + 2x)\,\mathrm{e}^{-x} + 2c_2 x^2 - c_1}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{(2 + 2x)\,\mathrm{e}^{-x} + 2c_2 x^2 - c_1}{2x}$$

Verified OK.

9.19.7 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' + xy' - y = x^2 e^{-x}$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' + xy' - y)\, dx = \int x^2 e^{-x} dx$$
$$x^2y' - xy = -(x^2 + 2x + 2)\, e^{-x} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{(-x^2 - 2x - 2)\, e^{-x} + c_1}{x^2}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{(-x^2 - 2x - 2)\, e^{-x} + c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x} dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{(-x^2 - 2x - 2)\, e^{-x} + c_1}{x^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left(\frac{(-x^2 - 2x - 2)\, e^{-x} + c_1}{x^2}\right)$$
$$d\left(\frac{y}{x}\right) = \left(\frac{(-x^2 - 2x - 2)\, e^{-x} + c_1}{x^3}\right) dx$$

Integrating gives

$$\frac{y}{x} = \int \frac{(-x^2 - 2x - 2)\,\mathrm{e}^{-x} + c_1}{x^3}\,\mathrm{d}x$$
$$\frac{y}{x} = -\frac{c_1}{2x^2} + \frac{\mathrm{e}^{-x}}{x^2} + \frac{\mathrm{e}^{-x}}{x} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(-\frac{c_1}{2x^2} + \frac{\mathrm{e}^{-x}}{x^2} + \frac{\mathrm{e}^{-x}}{x}\right) + c_2 x$$

which simplifies to

$$y = \frac{(2 + 2x)\,\mathrm{e}^{-x} + 2c_2x^2 - c_1}{2x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{(2 + 2x)\,\mathrm{e}^{-x} + 2c_2x^2 - c_1}{2x} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{(2 + 2x)\,\mathrm{e}^{-x} + 2c_2x^2 - c_1}{2x}$$

Verified OK.

9.19.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + xy' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 292: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^{2}-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2x}\right)(0)+\left(\left(\frac{1}{2x^{2}}\right)+\left(-\frac{1}{2x}\right)^{2}-\left(\frac{3}{4x^{2}}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_{1}(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int -\frac{1}{2x}dx}\\&=\frac{1}{\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_{1}&=z_{1}e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_{1}e^{-\int \frac{1}{2}\frac{x}{x^{2}}\,dx}\\&=z_{1}e^{-\frac{\ln(x)}{2}}\\&=z_{1}\left(\frac{1}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_{1}=\frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_{2}=y_{1}\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_{1}^{2}}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{x^2}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^2}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + xy' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x} + \frac{c_2 x}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{x}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x}{2} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x}{2} \\ -\frac{1}{x^2} & \frac{1}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{2}\right) - \left(\frac{x}{2}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^3 e^{-x}}{2}}{x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^2 e^{-x}}{2} dx$$

Hence

$$u_1 = \frac{(x^2 + 2x + 2) e^{-x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\, e^{-x}}{x}\, dx$$

Which simplifies to

$$u_2 = \int e^{-x} dx$$

Hence

$$u_2 = -e^{-x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(x^2 + 2x + 2) e^{-x}}{2x} - \frac{x\, e^{-x}}{2}$$

Which simplifies to

$$y_p(x) = \frac{e^{-x}(x + 1)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2 x}{2}\right) + \left(\frac{e^{-x}(x + 1)}{x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} + \frac{e^{-x}(x+1)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} + \frac{e^{-x}(x+1)}{x}$$

Verified OK.

9.19.9 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 \\ q(x) &= x \\ r(x) &= -1 \\ s(x) &= x^2 e^{-x} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 1 \end{aligned}$$

Therefore (1) becomes

$$2 - (1) + (-1) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\, y' + (q(x) - p'(x))\, y\right)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$x^2y' - xy = \int x^2 \mathrm{e}^{-x}\, dx$$

We now have a first order ode to solve which is

$$x^2y' - xy = -\left(x^2 + 2x + 2\right) \mathrm{e}^{-x} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{\left(-x^2 - 2x - 2\right) \mathrm{e}^{-x} + c_1}{x^2} \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{\left(-x^2 - 2x - 2\right) \mathrm{e}^{-x} + c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu) \left(\frac{\left(-x^2 - 2x - 2\right) \mathrm{e}^{-x} + c_1}{x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right) \left(\frac{\left(-x^2 - 2x - 2\right) \mathrm{e}^{-x} + c_1}{x^2}\right) \\ \mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{\left(-x^2 - 2x - 2\right) \mathrm{e}^{-x} + c_1}{x^3}\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x} &= \int \frac{\left(-x^2 - 2x - 2\right) \mathrm{e}^{-x} + c_1}{x^3} \mathrm{d}x \\ \frac{y}{x} &= -\frac{c_1}{2x^2} + \frac{\mathrm{e}^{-x}}{x^2} + \frac{\mathrm{e}^{-x}}{x} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(-\frac{c_1}{2x^2} + \frac{e^{-x}}{x^2} + \frac{e^{-x}}{x}\right) + c_2 x$$

which simplifies to

$$y = \frac{(2+2x)\,e^{-x} + 2c_2x^2 - c_1}{2x}$$

Summary
The solution(s) found are the following

$$y = \frac{(2+2x)\,e^{-x} + 2c_2x^2 - c_1}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{(2+2x)\,e^{-x} + 2c_2x^2 - c_1}{2x}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 25

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=x^2*exp(-x),y(x), singsol=all)
```

$$y(x) = \frac{c_2x^2 + e^{-x}x + e^{-x} + c_1}{x}$$

✓ Solution by Mathematica
Time used: 0.024 (sec). Leaf size: 27

```
DSolve[x^2*y''[x]+x*y'[x]-y[x]==x^2*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2x^2 + e^{-x}(x+1) + c_1}{x}$$

9.20 problem Exercise 22, problem 20, page 240

Internal problem ID [4650]
Internal file name [`OUTPUT/4143_Sunday_June_05_2022_12_28_14_PM_78292739/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 4. Higher order linear differential equations. Lesson 22. Variation of Parameters
Problem number: Exercise 22, problem 20, page 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{2x^2y'' + 3xy' - y = \frac{1}{x}}$$

9.20.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2x^2, B = 3x, C = -1, f(x) = \frac{1}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$2x^2y'' + 3xy' - y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$2x^2(r(r-1))x^{r-2} + 3xrx^{r-1} - x^r = 0$$

Simplifying gives

$$2r(r-1)\,x^r + 3r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$2r(r-1) + 3r - 1 = 0$$

Or

$$2r^2 + r - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = \frac{1}{2}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2\sqrt{x}$$

Next, we find the particular solution to the ODE

$$2x^2y'' + 3xy' - y = \frac{1}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \sqrt{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \sqrt{x} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\sqrt{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \sqrt{x} \\ -\frac{1}{x^2} & \frac{1}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{2\sqrt{x}}\right) - \left(\sqrt{x}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{3}{2x^{\frac{3}{2}}}$$

Which simplifies to

$$W = \frac{3}{2x^{\frac{3}{2}}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{1}{\sqrt{x}}}{3\sqrt{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{3x}dx$$

Hence

$$u_1 = -\frac{\ln(x)}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x^2}}{3\sqrt{x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{3x^{\frac{5}{2}}}dx$$

Hence

$$u_2 = -\frac{2}{9x^{\frac{3}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)}{3x} - \frac{2}{9x}$$

Which simplifies to

$$y_p(x) = \frac{-3\ln(x) - 2}{9x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{9c_2 x^{\frac{3}{2}} - 3\ln(x) + 9c_1 - 2}{9x} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{9c_2 x^{\frac{3}{2}} - 3\ln(x) + 9c_1 - 2}{9x} \tag{1}$$

Verification of solutions

$$y = \frac{9c_2 x^{\frac{3}{2}} - 3\ln(x) + 9c_1 - 2}{9x}$$

Verified OK.

9.20.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$2x^2y'' + 3xy' - y = 0$$

In normal form the ode

$$2x^2y'' + 3xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{2x} \\ q(x) &= -\frac{1}{2x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{3}{2x}dx\right)}dx \\ &= \int e^{-\frac{3\ln(x)}{2}}\,dx \\ &= \int \frac{1}{x^{\frac{3}{2}}}dx \\ &= -\frac{2}{\sqrt{x}}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{2x^2}}{\frac{1}{x^3}} \\ &= -\frac{x}{2}\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{xy(\tau)}{2} &= 0\end{aligned}$$

But in terms of τ

$$-\frac{x}{2} = -\frac{2}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{2y(\tau)}{\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 2\tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r - 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau} + c_2\tau^2$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{-x^{\frac{3}{2}}c_1 + 8c_2}{2x}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{-x^{\frac{3}{2}}c_1 + 8c_2}{2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \sqrt{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \sqrt{x} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\sqrt{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \sqrt{x} \\ -\frac{1}{x^2} & \frac{1}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x})\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{3}{2x^{\frac{3}{2}}}$$

Which simplifies to

$$W = \frac{3}{2x^{\frac{3}{2}}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{1}{\sqrt{x}}}{3\sqrt{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{3x} dx$$

Hence

$$u_1 = -\frac{\ln(x)}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x^2}}{3\sqrt{x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{3x^{\frac{5}{2}}} dx$$

Hence

$$u_2 = -\frac{2}{9x^{\frac{3}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)}{3x} - \frac{2}{9x}$$

Which simplifies to

$$y_p(x) = \frac{-3\ln(x) - 2}{9x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{-x^{\frac{3}{2}}c_1 + 8c_2}{2x}\right) + \left(\frac{-3\ln(x) - 2}{9x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{-x^{\frac{3}{2}}c_1 + 8c_2}{2x} + \frac{-3\ln(x) - 2}{9x} \tag{1}$$

Verification of solutions

$$y = \frac{-x^{\frac{3}{2}}c_1 + 8c_2}{2x} + \frac{-3\ln(x) - 2}{9x}$$

Verified OK.

9.20.3 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2x^2, B = 3x, C = -1, f(x) = \frac{1}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$2x^2y'' + 3xy' - y = 0$$

In normal form the ode

$$2x^2y'' + 3xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{3}{2x} \\
q(x) &= -\frac{1}{2x^2}
\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{3n}{2x^2} - \frac{1}{2x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = \frac{1}{2} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}
v''(x) + \frac{5v'(x)}{2x} &= 0 \\
v''(x) + \frac{5v'(x)}{2x} &= 0
\end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{5u(x)}{2x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{5u}{2x} \end{aligned}$$

Where $f(x) = -\frac{5}{2x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{5}{2x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{5}{2x}\,dx \\ \ln(u) &= -\frac{5\ln(x)}{2} + c_1 \\ u &= \mathrm{e}^{-\frac{5\ln(x)}{2}+c_1} \\ &= \frac{c_1}{x^{\frac{5}{2}}} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{2c_1}{3x^{\frac{3}{2}}} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)\sqrt{x} \\ &= \frac{3c_2x^{\frac{3}{2}} - 2c_1}{3x} \end{aligned}$$

Now the particular solution to this ODE is found

$$2x^2y'' + 3xy' - y = \frac{1}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \sqrt{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \sqrt{x} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\sqrt{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \sqrt{x} \\ -\frac{1}{x^2} & \frac{1}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{2\sqrt{x}}\right) - \left(\sqrt{x}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{3}{2x^{\frac{3}{2}}}$$

Which simplifies to

$$W = \frac{3}{2x^{\frac{3}{2}}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{1}{\sqrt{x}}}{3\sqrt{x}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{3x} dx$$

Hence

$$u_1 = -\frac{\ln(x)}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x^2}}{3\sqrt{x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{3x^{\frac{5}{2}}} dx$$

Hence

$$u_2 = -\frac{2}{9x^{\frac{3}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)}{3x} - \frac{2}{9x}$$

Which simplifies to

$$y_p(x) = \frac{-3\ln(x) - 2}{9x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)\sqrt{x}\right) + \left(\frac{-3\ln\left(x\right) - 2}{9x}\right) \\
&= \frac{-3\ln\left(x\right) - 2}{9x} + \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)\sqrt{x}
\end{aligned}
$$

Which simplifies to

$$
y = \frac{9c_2 x^{\frac{3}{2}} - 3\ln\left(x\right) - 6c_1 - 2}{9x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{9c_2 x^{\frac{3}{2}} - 3\ln\left(x\right) - 6c_1 - 2}{9x} \tag{1}
$$

Verification of solutions

$$
y = \frac{9c_2 x^{\frac{3}{2}} - 3\ln\left(x\right) - 6c_1 - 2}{9x}
$$

Verified OK.

9.20.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int \left(2x^2y'' + 3xy' - y\right) dx &= \int \frac{1}{x} dx \\
-xy + 2x^2y' &= \ln\left(x\right) + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{2x} \\
q(x) &= \frac{\ln\left(x\right) + c_1}{2x^2}
\end{aligned}
$$

Hence the ode is

$$y' - \frac{y}{2x} = \frac{\ln(x) + c_1}{2x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{2x}dx} \\ &= \frac{1}{\sqrt{x}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{\ln(x) + c_1}{2x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{x}}\right) &= \left(\frac{1}{\sqrt{x}}\right)\left(\frac{\ln(x) + c_1}{2x^2}\right) \\ \mathrm{d}\left(\frac{y}{\sqrt{x}}\right) &= \left(\frac{\ln(x) + c_1}{2x^{\frac{5}{2}}}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{\sqrt{x}} &= \int \frac{\ln(x) + c_1}{2x^{\frac{5}{2}}}\,\mathrm{d}x \\ \frac{y}{\sqrt{x}} &= -\frac{\ln(x)}{3x^{\frac{3}{2}}} - \frac{2}{9x^{\frac{3}{2}}} - \frac{c_1}{3x^{\frac{3}{2}}} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x}}$ results in

$$y = \sqrt{x}\left(-\frac{\ln(x)}{3x^{\frac{3}{2}}} - \frac{2}{9x^{\frac{3}{2}}} - \frac{c_1}{3x^{\frac{3}{2}}}\right) + c_2\sqrt{x}$$

which simplifies to

$$y = \frac{9c_2 x^{\frac{3}{2}} - 3\ln(x) - 3c_1 - 2}{9x}$$

Summary

The solution(s) found are the following

$$y = \frac{9c_2 x^{\frac{3}{2}} - 3\ln(x) - 3c_1 - 2}{9x} \tag{1}$$

Verification of solutions

$$y = \frac{9c_2 x^{\frac{3}{2}} - 3\ln(x) - 3c_1 - 2}{9x}$$

Verified OK.

9.20.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$2x^2y'' + 3xy' - y = \frac{1}{x}$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(2x^2y'' + 3xy' - y\right) dx = \int \frac{1}{x} dx$$
$$-xy + 2x^2y' = \ln(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{2x}$$
$$q(x) = \frac{\ln(x) + c_1}{2x^2}$$

Hence the ode is

$$y' - \frac{y}{2x} = \frac{\ln(x) + c_1}{2x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{2x}dx}$$
$$= \frac{1}{\sqrt{x}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\ln(x) + c_1}{2x^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{\sqrt{x}}\right) = \left(\frac{1}{\sqrt{x}}\right)\left(\frac{\ln(x) + c_1}{2x^2}\right)$$
$$d\left(\frac{y}{\sqrt{x}}\right) = \left(\frac{\ln(x) + c_1}{2x^{\frac{5}{2}}}\right) dx$$

Integrating gives

$$
\begin{aligned}
\frac{y}{\sqrt{x}} &= \int \frac{\ln(x)+c_1}{2x^{\frac{5}{2}}}\,\mathrm{d}x \\
\frac{y}{\sqrt{x}} &= -\frac{\ln(x)}{3x^{\frac{3}{2}}} - \frac{2}{9x^{\frac{3}{2}}} - \frac{c_1}{3x^{\frac{3}{2}}} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x}}$ results in

$$
y = \sqrt{x}\left(-\frac{\ln(x)}{3x^{\frac{3}{2}}} - \frac{2}{9x^{\frac{3}{2}}} - \frac{c_1}{3x^{\frac{3}{2}}}\right) + c_2\sqrt{x}
$$

which simplifies to

$$
y = \frac{9c_2x^{\frac{3}{2}} - 3\ln(x) - 3c_1 - 2}{9x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{9c_2x^{\frac{3}{2}} - 3\ln(x) - 3c_1 - 2}{9x} \tag{1}
$$

Verification of solutions

$$
y = \frac{9c_2x^{\frac{3}{2}} - 3\ln(x) - 3c_1 - 2}{9x}
$$

Verified OK.

9.20.6 Solving using Kovacic algorithm

Writing the ode as

$$
2x^2y'' + 3xy' - y = 0 \tag{1}
$$
$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 2x^2 \\
B &= 3x \\
C &= -1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 293: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5}{16x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4x} + (-)\,(0) \\
&= -\frac{1}{4x} \\
&= -\frac{1}{4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4x}\right)(0) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x}\right)^2 - \left(\frac{5}{16x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int -\frac{1}{4x} dx} \\ &= \frac{1}{x^{\frac{1}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{2x^2}\, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{4}} \\ &= z_1 \left(\frac{1}{x^{\frac{3}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2x^{\frac{3}{2}}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{2x^{\frac{3}{2}}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
2x^2y'' + 3xy' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1}{x} + \frac{2c_2\sqrt{x}}{3}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{2\sqrt{x}}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{2\sqrt{x}}{3} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{2\sqrt{x}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{2\sqrt{x}}{3} \\ -\frac{1}{x^2} & \frac{1}{3\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{3\sqrt{x}}\right) - \left(\frac{2\sqrt{x}}{3}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x^{\frac{3}{2}}}$$

Which simplifies to

$$W = \frac{1}{x^{\frac{3}{2}}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2}{3\sqrt{x}}}{2\sqrt{x}}dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{3x}dx$$

Hence

$$u_1 = -\frac{\ln(x)}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x^2}}{2\sqrt{x}}dx$$

Which simplifies to

$$u_2 = \int \frac{1}{2x^{\frac{5}{2}}}dx$$

Hence

$$u_2 = -\frac{1}{3x^{\frac{3}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)}{3x} - \frac{2}{9x}$$

Which simplifies to

$$y_p(x) = \frac{-3\ln(x) - 2}{9x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{2c_2\sqrt{x}}{3}\right) + \left(\frac{-3\ln(x) - 2}{9x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{2c_2\sqrt{x}}{3} + \frac{-3\ln(x) - 2}{9x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{2c_2\sqrt{x}}{3} + \frac{-3\ln(x) - 2}{9x}$$

Verified OK.

9.20.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 2x^2 \\ q(x) &= 3x \\ r(x) &= -1 \\ s(x) &= \frac{1}{x} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 4 \\ q'(x) &= 3 \end{aligned}$$

Therefore (1) becomes

$$4 - (3) + (-1) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$-xy + 2x^2y' = \int \frac{1}{x}\, dx$$

We now have a first order ode to solve which is

$$-xy + 2x^2y' = \ln(x) + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{2x} \\ q(x) &= \frac{\ln(x) + c_1}{2x^2} \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{2x} = \frac{\ln(x) + c_1}{2x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\ln(x) + c_1}{2x^2}\right) \\ \frac{d}{dx}\left(\frac{y}{\sqrt{x}}\right) &= \left(\frac{1}{\sqrt{x}}\right)\left(\frac{\ln(x) + c_1}{2x^2}\right) \\ d\left(\frac{y}{\sqrt{x}}\right) &= \left(\frac{\ln(x) + c_1}{2x^{\frac{5}{2}}}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{\sqrt{x}} &= \int \frac{\ln(x) + c_1}{2x^{\frac{5}{2}}}\, dx \\ \frac{y}{\sqrt{x}} &= -\frac{\ln(x)}{3x^{\frac{3}{2}}} - \frac{2}{9x^{\frac{3}{2}}} - \frac{c_1}{3x^{\frac{3}{2}}} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x}}$ results in

$$y = \sqrt{x}\left(-\frac{\ln(x)}{3x^{\frac{3}{2}}} - \frac{2}{9x^{\frac{3}{2}}} - \frac{c_1}{3x^{\frac{3}{2}}}\right) + c_2\sqrt{x}$$

which simplifies to

$$y = \frac{9c_2x^{\frac{3}{2}} - 3\ln(x) - 3c_1 - 2}{9x}$$

Summary

The solution(s) found are the following

$$y = \frac{9c_2x^{\frac{3}{2}} - 3\ln(x) - 3c_1 - 2}{9x} \tag{1}$$

Verification of solutions

$$y = \frac{9c_2x^{\frac{3}{2}} - 3\ln(x) - 3c_1 - 2}{9x}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(2*x^2*diff(y(x),x$2)+3*x*diff(y(x),x)-y(x)=1/x,y(x), singsol=all)
```

$$y(x) = \frac{9x^{\frac{3}{2}}c_2 - 3\ln(x) + 9c_1 - 2}{9x}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 31

```
DSolve[2*x^2*y''[x]+3*x*y'[x]-y[x]==1/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{9c_2x^{3/2} - 3\log(x) - 2 + 9c_1}{9x}$$

10 Chapter 8. Special second order equations. Lesson 35. Independent variable x absent

10.1 problem Exercise 35.1, page 504

Internal problem ID [4651]
Internal file name [`OUTPUT/4144_Sunday_June_05_2022_12_28_22_PM_55259558/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.1, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
   _Lagerstrom, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
    _reducible, _mu_xy]]
```

$$\boxed{y'' - 2y'y = 0}$$

10.1.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 2y'y)\,dx = 0$$
$$-y^2 + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{y^2 + c_1} dy = c_2 + x$$
$$\frac{\arctan\left(\frac{y}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}$$

Summary
The solution(s) found are the following

$$y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1} \tag{1}$$

Verification of solutions

$$y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}$$

Verified OK.

10.1.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) - 2p(y)\,y = 0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned} p(y) &= \int 2y \,\mathrm{d}y \\ &= y^2 + c_1 \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = y^2 + c_1$$

Integrating both sides gives

$$
\int \frac{1}{y^2 + c_1} dy = c_2 + x
$$

$$
\frac{\arctan\left(\frac{y}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x
$$

Solving for y gives these solutions

$$
y_1 = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}
$$

Summary
The solution(s) found are the following

$$
y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1} \tag{1}
$$

Verification of solutions

$$
y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}
$$

Verified OK.

10.1.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
y'' - 2y'y = 0
$$

Integrating both sides of the ODE w.r.t x gives

$$
\int (y'' - 2y'y) \, dx = 0
$$

$$
-y^2 + y' = c_1
$$

Which is now solved for y. Integrating both sides gives

$$
\int \frac{1}{y^2 + c_1} dy = c_2 + x
$$

$$
\frac{\arctan\left(\frac{y}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x
$$

Solving for y gives these solutions

$$
y_1 = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}
$$

Summary

The solution(s) found are the following

$$y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1} \tag{1}$$

Verification of solutions

$$y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}$$

Verified OK.

10.1.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\frac{\partial a_2}{\partial y} = \frac{\partial a_1}{\partial y'}$$
$$\frac{\partial a_2}{\partial x} = \frac{\partial a_0}{\partial y'}$$
$$\frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial y}$$

Looking at the the ode given we see that

$$a_2 = 1$$
$$a_1 = -2y$$
$$a_0 = 0$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\int a_2\, dy' + \int a_1\, dy + \int a_0\, dx = c_1$$
$$\int 1\, dy' + \int -2y\, dy + \int 0\, dx = c_1$$

Which results in

$$-y^2 + y' = c_1$$

Which is now solved Integrating both sides gives

$$\int \frac{1}{y^2+c_1}dy = c_2 + x$$
$$\frac{\arctan\left(\frac{y}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}$$

Summary
The solution(s) found are the following

$$y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1} \tag{1}$$

Verification of solutions

$$y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}$$

Verified OK.

10.1.5 Maple step by step solution

Let's solve
$y'' - 2y'y = 0$

- Highest derivative means the order of the ODE is 2
  y''
- Define new dependent variable u
  $u(x) = y'$
- Compute y''
  $u'(x) = y''$
- Use chain rule on the lhs
  $y'\left(\frac{d}{dy}u(y)\right) = y''$
- Substitute in the definition of u
  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$
- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$$u(y)\left(\frac{d}{dy}u(y)\right)-2u(y)\,y=0$$

- Separate variables

$$\frac{d}{dy}u(y)=2y$$

- Integrate both sides with respect to y

$$\int\left(\frac{d}{dy}u(y)\right)dy=\int 2ydy+c_1$$

- Evaluate integral

$$u(y)=y^2+c_1$$

- Solve for $u(y)$

$$u(y)=y^2+c_1$$

- Solve 1st ODE for $u(y)$

$$u(y)=y^2+c_1$$

- Revert to original variables with substitution $u(y)=y',y=y$

$$y'=y^2+c_1$$

- Separate variables

$$\frac{y'}{y^2+c_1}=1$$

- Integrate both sides with respect to x

$$\int\frac{y'}{y^2+c_1}dx=\int 1dx+c_2$$

- Evaluate integral

$$\frac{\arctan\left(\frac{y}{\sqrt{c_1}}\right)}{\sqrt{c_1}}=c_2+x$$

- Solve for y

$$y=\tan\left(c_2\sqrt{c_1}+x\sqrt{c_1}\right)\sqrt{c_1}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-2*_a*_b(_a) = 0, _b(_a), HINT =
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 2*_b]
```

✓ Solution by Maple

Time used: 0.266 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)=2*y(x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{\tan\left(\frac{c_2+x}{c_1}\right)}{c_1}$$

✓ Solution by Mathematica

Time used: 9.872 (sec). Leaf size: 24

```
DSolve[y''[x]==2*y[x]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{c_1}\tan\left(\sqrt{c_1}(x+c_2)\right)$$

10.2 problem Exercise 35.2, page 504

Internal problem ID [4652]
Internal file name [`OUTPUT/4145_Sunday_June_05_2022_12_28_36_PM_89056019/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.2, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{y^3 y'' = k}$$

10.2.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$y^3 p(y) \left(\frac{d}{dy} p(y)\right) = k$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{k}{y^3 p} \end{aligned}$$

Where $f(y) = \frac{k}{y^3}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{1}{p}}\, dp &= \frac{k}{y^3}\, dy \\ \int \frac{1}{\frac{1}{p}}\, dp &= \int \frac{k}{y^3}\, dy \\ \frac{p^2}{2} &= -\frac{k}{2y^2} + c_1 \end{aligned}$$

The solution is

$$\frac{p(y)^2}{2} + \frac{k}{2y^2} - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{y'^2}{2} + \frac{k}{2y^2} - c_1 = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{2c_1 y^2 - k}}{y} \tag{1}$$

$$y' = -\frac{\sqrt{2c_1 y^2 - k}}{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} \int \frac{y}{\sqrt{2c_1 y^2 - k}} dy &= \int dx \\ \frac{\sqrt{2c_1 y^2 - k}}{2c_1} &= c_2 + x \end{aligned}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y}{\sqrt{2c_1y^2-k}}dy = \int dx$$
$$-\frac{\sqrt{2c_1y^2-k}}{2c_1} = x + c_3$$

Summary

The solution(s) found are the following

$$\frac{\sqrt{2c_1y^2-k}}{2c_1} = c_2 + x \tag{1}$$
$$-\frac{\sqrt{2c_1y^2-k}}{2c_1} = x + c_3 \tag{2}$$

Verification of solutions

$$\frac{\sqrt{2c_1y^2-k}}{2c_1} = c_2 + x$$

Verified OK.

$$-\frac{\sqrt{2c_1y^2-k}}{2c_1} = x + c_3$$

Verified OK.

10.2.2 Maple step by step solution

Let's solve

$y^3y'' = k$

- Highest derivative means the order of the ODE is 2

  y''
- Define new dependent variable u

  $u(x) = y'$
- Compute y''

  $u'(x) = y''$
- Use chain rule on the lhs

$y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

$u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$y^3 u(y)\left(\frac{d}{dy}u(y)\right) = k$

- Separate variables

$u(y)\left(\frac{d}{dy}u(y)\right) = \frac{k}{y^3}$

- Integrate both sides with respect to y

$\int u(y)\left(\frac{d}{dy}u(y)\right)dy = \int \frac{k}{y^3}dy + c_1$

- Evaluate integral

$\frac{u(y)^2}{2} = -\frac{k}{2y^2} + c_1$

- Solve for $u(y)$

$\left\{u(y) = \frac{\sqrt{2c_1y^2-k}}{y}, u(y) = -\frac{\sqrt{2c_1y^2-k}}{y}\right\}$

- Solve 1st ODE for $u(y)$

$u(y) = \frac{\sqrt{2c_1y^2-k}}{y}$

- Revert to original variables with substitution $u(y) = y', y = y$

$y' = \frac{\sqrt{2c_1y^2-k}}{y}$

- Separate variables

$\frac{yy'}{\sqrt{2c_1y^2-k}} = 1$

- Integrate both sides with respect to x

$\int \frac{yy'}{\sqrt{2c_1y^2-k}}dx = \int 1dx + c_2$

- Evaluate integral

$\frac{\sqrt{2c_1y^2-k}}{2c_1} = c_2 + x$

- Solve for y

$\left\{y = -\frac{\sqrt{2}\sqrt{c_1(4c_1^2c_2^2+8c_1^2c_2x+4x^2c_1^2+k)}}{2c_1}, y = \frac{\sqrt{2}\sqrt{c_1(4c_1^2c_2^2+8c_1^2c_2x+4x^2c_1^2+k)}}{2c_1}\right\}$

- Solve 2nd ODE for $u(y)$

  $u(y) = -\frac{\sqrt{2c_1y^2-k}}{y}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = -\frac{\sqrt{2c_1y^2-k}}{y}$

- Separate variables

  $\frac{yy'}{\sqrt{2c_1y^2-k}} = -1$

- Integrate both sides with respect to x

  $\int \frac{yy'}{\sqrt{2c_1y^2-k}}dx = \int (-1)\, dx + c_2$

- Evaluate integral

  $\frac{\sqrt{2c_1y^2-k}}{2c_1} = c_2 - x$

- Solve for y

  $\left\{y = -\frac{\sqrt{2}\,\sqrt{c_1\,(4c_1^2c_2^2-8c_1^2c_2x+4x^2c_1^2+k)}}{2c_1}, y = \frac{\sqrt{2}\,\sqrt{c_1\,(4c_1^2c_2^2-8c_1^2c_2x+4x^2c_1^2+k)}}{2c_1}\right\}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-k/_a^3 = 0, _b(_a), HINT = [[_a,
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, -_b]
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 46

```
dsolve(y(x)^3*diff(y(x),x$2)=k,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{\left(\left(c_2 + x\right)^2 c_1^2 + k\right) c_1}}{c_1}$$
$$y(x) = -\frac{\sqrt{\left(\left(c_2 + x\right)^2 c_1^2 + k\right) c_1}}{c_1}$$

✓ Solution by Mathematica

Time used: 2.878 (sec). Leaf size: 63

```
DSolve[y[x]^3*y''[x]==k,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{k + c_1{}^2(x + c_2)^2}}{\sqrt{c_1}}$$
$$y(x) \to \frac{\sqrt{k + c_1{}^2(x + c_2)^2}}{\sqrt{c_1}}$$
$$y(x) \to \text{Indeterminate}$$

10.3 problem Exercise 35.3, page 504

Internal problem ID [4653]
Internal file name [`OUTPUT/4146_Sunday_June_05_2022_12_28_50_PM_45313012/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.3, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{yy'' - y'^2 = -1}$$

10.3.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) - p(y)^2 = -1$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y, p) \\
&= f(y)g(p) \\
&= \frac{p^2 - 1}{yp}
\end{aligned}$$

Where $f(y) = \frac{1}{y}$ and $g(p) = \frac{p^2-1}{p}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{p^2-1}{p}}\, dp &= \frac{1}{y}\, dy \\
\int \frac{1}{\frac{p^2-1}{p}}\, dp &= \int \frac{1}{y}\, dy \\
\frac{\ln(p-1)}{2} + \frac{\ln(p+1)}{2} &= \ln(y) + c_1
\end{aligned}$$

The above can be written as

$$\begin{aligned}
\left(\frac{1}{2}\right)(\ln(p-1) + \ln(p+1)) &= \ln(y) + 2c_1 \\
\ln(p-1) + \ln(p+1) &= (2)(\ln(y) + 2c_1) \\
&= 2\ln(y) + 4c_1
\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(p-1)+\ln(p+1)} = e^{2\ln(y)+2c_1}$$

Which simplifies to

$$\begin{aligned}
p^2 - 1 &= 2c_1 y^2 \\
&= c_2 y^2
\end{aligned}$$

The solution is

$$p(y)^2 - 1 = c_2 y^2$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y'^2 - 1 = c_2 y^2$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{c_2 y^2 + 1} \tag{1}$$
$$y' = -\sqrt{c_2 y^2 + 1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{c_2 y^2 + 1}} dy = \int dx$$
$$\frac{\ln\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)}{\sqrt{c_2}} = x + c_3$$

Raising both side to exponential gives

$$e^{\frac{\ln\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)}{\sqrt{c_2}}} = e^{x + c_3}$$

Which simplifies to

$$\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)^{\frac{1}{\sqrt{c_2}}} = c_4 e^{x}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{c_2 y^2 + 1}} dy = \int dx$$
$$-\frac{\ln\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)}{\sqrt{c_2}} = x + c_5$$

Raising both side to exponential gives

$$e^{-\frac{\ln\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)}{\sqrt{c_2}}} = e^{x + c_5}$$

Which simplifies to

$$\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)^{-\frac{1}{\sqrt{c_2}}} = c_6 e^{x}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(\left(c_4 \mathrm{e}^x\right)^{2\sqrt{c_2}} - 1\right) \left(c_4 \mathrm{e}^x\right)^{-\sqrt{c_2}}}{2\sqrt{c_2}} \tag{1}$$

$$y = \frac{\left(\left(c_6 \mathrm{e}^x\right)^{-2\sqrt{c_2}} - 1\right) \left(c_6 \mathrm{e}^x\right)^{\sqrt{c_2}}}{2\sqrt{c_2}} \tag{2}$$

Verification of solutions

$$y = \frac{\left(\left(c_4 \mathrm{e}^x\right)^{2\sqrt{c_2}} - 1\right) \left(c_4 \mathrm{e}^x\right)^{-\sqrt{c_2}}}{2\sqrt{c_2}}$$

Verified OK.

$$y = \frac{\left(\left(c_6 \mathrm{e}^x\right)^{-2\sqrt{c_2}} - 1\right) \left(c_6 \mathrm{e}^x\right)^{\sqrt{c_2}}}{2\sqrt{c_2}}$$

Verified OK.

10.3.2 Maple step by step solution

Let's solve

$yy'' - y'^2 = -1$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$$yu(y)\left(\frac{d}{dy}u(y)\right)-u(y)^2=-1$$

- Separate variables

$$\frac{\left(\frac{d}{dy}u(y)\right)u(y)}{u(y)^2-1}=\frac{1}{y}$$

- Integrate both sides with respect to y

$$\int\frac{\left(\frac{d}{dy}u(y)\right)u(y)}{u(y)^2-1}dy=\int\frac{1}{y}dy+c_1$$

- Evaluate integral

$$\frac{\ln(u(y)-1)}{2}+\frac{\ln(u(y)+1)}{2}=\ln(y)+c_1$$

- Solve for $u(y)$

$$\left\{u(y)=\frac{\mathrm{e}^{2c_1}\left(-1+\sqrt{(\mathrm{e}^{c_1})^2y^2+1}\right)}{(\mathrm{e}^{c_1})^2}+1, u(y)=-\frac{\mathrm{e}^{2c_1}\left(1+\sqrt{(\mathrm{e}^{c_1})^2y^2+1}\right)}{(\mathrm{e}^{c_1})^2}+1\right\}$$

- Solve 1st ODE for $u(y)$

$$u(y)=\frac{\mathrm{e}^{2c_1}\left(-1+\sqrt{(\mathrm{e}^{c_1})^2y^2+1}\right)}{(\mathrm{e}^{c_1})^2}+1$$

- Revert to original variables with substitution $u(y)=y', y=y$

$$y'=\frac{\mathrm{e}^{2c_1}\left(-1+\sqrt{(\mathrm{e}^{c_1})^2y^2+1}\right)}{(\mathrm{e}^{c_1})^2}+1$$

- Separate variables

$$\frac{y'}{\frac{\mathrm{e}^{2c_1}\left(-1+\sqrt{(\mathrm{e}^{c_1})^2y^2+1}\right)}{(\mathrm{e}^{c_1})^2}+1}=1$$

- Integrate both sides with respect to x

$$\int\frac{y'}{\frac{\mathrm{e}^{2c_1}\left(-1+\sqrt{(\mathrm{e}^{c_1})^2y^2+1}\right)}{(\mathrm{e}^{c_1})^2}+1}dx=\int 1dx+c_2$$

- Evaluate integral

$$\left(\mathrm{e}^{c_1}\right)^2 \left(-\frac{\arctan\left(\frac{\mathrm{e}^{4c_1} y}{\sqrt{\mathrm{e}^{2c_1}\mathrm{e}^{4c_1}}}\right)}{\sqrt{\mathrm{e}^{2c_1}\mathrm{e}^{4c_1}}} + \frac{\mathrm{e}^{2c_1}\arctan\left(\frac{\mathrm{e}^{4c_1} y}{\sqrt{\mathrm{e}^{2c_1}\mathrm{e}^{4c_1}}}\right)}{\left(\mathrm{e}^{c_1}\right)^2\sqrt{\mathrm{e}^{2c_1}\mathrm{e}^{4c_1}}} - \frac{\mathrm{e}^{2c_1}\left(\frac{-\sqrt{\left(\mathrm{e}^{c_1}\right)^2\left(y-\sqrt{-\mathrm{e}^{6c_1}}\,\mathrm{e}^{-4c_1}\right)^2 + 2\left(\mathrm{e}^{c_1}\right)^2\sqrt{-\mathrm{e}^{6c_1}}\,\mathrm{e}^{-4c_1}\left(y-\sqrt{-\cdots}\right.}}{\cdots} \right.}{\cdots} \right.$$

- Solve for y

$$y = \frac{\left(\left(\mathrm{e}^{c_2 \mathrm{e}^{c_1}+x\,\mathrm{e}^{c_1}+c_1}\right)^2-\left(\mathrm{e}^{c_1}\right)^2\right)\mathrm{e}^{-3c_1-c_2\mathrm{e}^{c_1}-x\,\mathrm{e}^{c_1}}}{2}$$

- Solve 2nd ODE for $u(y)$

$$u(y) = -\frac{\mathrm{e}^{2c_1}\left(1+\sqrt{\left(\mathrm{e}^{c_1}\right)^2 y^2+1}\right)}{\left(\mathrm{e}^{c_1}\right)^2} + 1$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = -\frac{\mathrm{e}^{2c_1}\left(1+\sqrt{\left(\mathrm{e}^{c_1}\right)^2 y^2+1}\right)}{\left(\mathrm{e}^{c_1}\right)^2} + 1$$

- Separate variables

$$\frac{y'}{-\frac{\mathrm{e}^{2c_1}\left(1+\sqrt{\left(\mathrm{e}^{c_1}\right)^2 y^2+1}\right)}{\left(\mathrm{e}^{c_1}\right)^2}+1} = 1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{-\frac{\mathrm{e}^{2c_1}\left(1+\sqrt{\left(\mathrm{e}^{c_1}\right)^2 y^2+1}\right)}{\left(\mathrm{e}^{c_1}\right)^2}+1}dx = \int 1dx + c_2$$

- Evaluate integral

$$-\left(e^{c_1}\right)^2 \left(-\frac{e^{2c_1}\arctan\left(\frac{e^{4c_1}y}{\sqrt{e^{2c_1}e^{4c_1}}}\right)}{\left(e^{c_1}\right)^2\sqrt{e^{2c_1}e^{4c_1}}} + \frac{\arctan\left(\frac{e^{4c_1}y}{\sqrt{e^{2c_1}e^{4c_1}}}\right)}{\sqrt{e^{2c_1}e^{4c_1}}} - \frac{e^{2c_1}\left(\frac{-\sqrt{\left(e^{c_1}\right)^2\left(y-\sqrt{-e^{6c_1}}\,e^{-4c_1}\right)^2+2\left(e^{c_1}\right)^2\sqrt{-e^{6c_1}}\,e^{-4c_1}\left(y-\sqrt{\cdots}\right.}}{\cdots}\right.}{\cdots}\right.$$

- Solve for y

$$y = \frac{\left(\left(e^{c_1-c_2e^{c_1}-x\,e^{c_1}}\right)^2-\left(e^{c_1}\right)^2\right)e^{-3c_1+c_2e^{c_1}+x\,e^{c_1}}}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(_b(_a)^2-1)/_a = 0, _b(_a), HIN
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 0]
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 59

```
dsolve(y(x)*diff(y(x),x$2)=(diff(y(x),x))^2-1,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \left(-\mathrm{e}^{\frac{c_2+x}{c_1}} + \mathrm{e}^{\frac{-c_2-x}{c_1}}\right)}{2}$$
$$y(x) = -\frac{c_1 \left(-\mathrm{e}^{\frac{c_2+x}{c_1}} + \mathrm{e}^{\frac{-c_2-x}{c_1}}\right)}{2}$$

✓ Solution by Mathematica

Time used: 60.201 (sec). Leaf size: 85

```
DSolve[y[x]*y''[x]==(y'[x])^2-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{ie^{-c_1} \tanh\left(e^{c_1}(x + c_2)\right)}{\sqrt{-\operatorname{sech}^2\left(e^{c_1}(x + c_2)\right)}}$$
$$y(x) \to \frac{ie^{-c_1} \tanh\left(e^{c_1}(x + c_2)\right)}{\sqrt{-\operatorname{sech}^2\left(e^{c_1}(x + c_2)\right)}}$$

10.4 problem Exercise 35.4, page 504

Internal problem ID [4654]
Internal file name [`OUTPUT/4147_Sunday_June_05_2022_12_29_03_PM_61463731/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.4, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x^2y'' + xy' = 1}$$

10.4.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 0, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} + 0 = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r + 0 = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r + 0 = 0$$

Or

$$r^2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 0$$
$$r_2 = 0$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_2 \ln(x) + c_1$$

Next, we find the particular solution to the ODE

$$x^2 y'' + x y' = 1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \ln(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(x) \\ 0 & \frac{1}{x} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{x}\right) - (\ln(x))(0)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)}{x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{1}{x}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{\ln(x)^2}{2} + c_1 + c_2 \ln(x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^2}{2} + c_1 + c_2 \ln(x) \tag{1}$$

Verification of solutions

$$y = \frac{\ln(x)^2}{2} + c_1 + c_2 \ln(x)$$

Verified OK.

10.4.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$x^2 p'(x) + x p(x) - 1 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Hence the ode is

$$p'(x) + \frac{p(x)}{x} = \frac{1}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x}dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)\left(\frac{1}{x^2}\right) \\ \frac{d}{dx}(xp) &= (x)\left(\frac{1}{x^2}\right) \\ d(xp) &= \frac{1}{x}\, dx \end{aligned}$$

Integrating gives

$$
\begin{aligned}
xp &= \int \frac{1}{x}\,\mathrm{d}x \\
xp &= \ln(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
p(x) = \frac{\ln(x)}{x} + \frac{c_1}{x}
$$

which simplifies to

$$
p(x) = \frac{\ln(x) + c_1}{x}
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = \frac{\ln(x) + c_1}{x}
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{\ln(x) + c_1}{x}\,\mathrm{d}x \\
&= \frac{\ln(x)^2}{2} + c_1 \ln(x) + c_2
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{\ln(x)^2}{2} + c_1 \ln(x) + c_2 \tag{1}
$$

Verification of solutions

$$
y = \frac{\ln(x)^2}{2} + c_1 \ln(x) + c_2
$$

Verified OK.

10.4.3 Solving using Kovacic algorithm

Writing the ode as

$$
x^2 y'' + x y' = 0 \tag{1}
$$

$$
A y'' + B y' + C y = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= x \\
C &= 0
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A} dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 297: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(\ln(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' + xy' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2 \ln(x) + c_1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \ln(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(x) \\ 0 & \frac{1}{x} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{x}\right) - (\ln(x))(0)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)}{x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{1}{x}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 \ln(x) + c_1) + \left(\frac{\ln(x)^2}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^2}{2} + c_1 + c_2 \ln(x) \tag{1}$$

Verification of solutions

$$y = \frac{\ln(x)^2}{2} + c_1 + c_2 \ln(x)$$

Verified OK.

10.4.4 Maple step by step solution

Let's solve

$x^2 y'' + x y' = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $x^2 u'(x) + x u(x) = 1$

- Isolate the derivative

  $u'(x) = -\frac{u(x)}{x} + \frac{1}{x^2}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) + \frac{u(x)}{x} = \frac{1}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x) + \frac{u(x)}{x}\right) = \frac{\mu(x)}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, u(x))$

  $\mu(x)\left(u'(x) + \frac{u(x)}{x}\right) = \mu'(x)\, u(x) + \mu(x)\, u'(x)$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, u(x))\right) dx = \int \frac{\mu(x)}{x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, u(x) = \int \frac{\mu(x)}{x^2} dx + c_1$
- Solve for $u(x)$

  $u(x) = \frac{\int \frac{\mu(x)}{x^2} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = x$

  $u(x) = \frac{\int \frac{1}{x} dx + c_1}{x}$
- Evaluate the integrals on the rhs

  $u(x) = \frac{\ln(x) + c_1}{x}$
- Solve 1st ODE for $u(x)$

  $u(x) = \frac{\ln(x) + c_1}{x}$
- Make substitution $u = y'$

  $y' = \frac{\ln(x) + c_1}{x}$
- Integrate both sides to solve for y

  $\int y' dx = \int \frac{\ln(x) + c_1}{x} dx + c_2$
- Compute integrals

  $y = \frac{\ln(x)^2}{2} + c_1 \ln(x) + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(_a*_b(_a)-1)/_a^2, _b(_a)`   *** Subl
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x^2*diff(y(x),x$2)+x*(diff(y(x),x))=1,y(x), singsol=all)
```

$$y(x) = c_2 + c_1 \ln(x) + \frac{\ln(x)^2}{2}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 21

```
DSolve[x^2*y''[x]+x*y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\log^2(x)}{2} + c_1 \log(x) + c_2$$

10.5 problem Exercise 35.5, page 504

Internal problem ID [4655]
Internal file name [`OUTPUT/4148_Sunday_June_05_2022_12_29_10_PM_99842678/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.5, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{xy'' - y' = x^2}$$

10.5.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' - y')\, dx = \int x^2 dx$$
$$xy' - 2y = \frac{x^3}{3} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{x^3 + 3c_1}{3x} \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{x^3 + 3c_1}{3x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^3 + 3c_1}{3x}\right) \\ \frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{x^3 + 3c_1}{3x}\right) \\ d\left(\frac{y}{x^2}\right) &= \left(\frac{x^3 + 3c_1}{3x^3}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x^2} &= \int \frac{x^3 + 3c_1}{3x^3} dx \\ \frac{y}{x^2} &= \frac{x}{3} - \frac{c_1}{2x^2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(\frac{x}{3} - \frac{c_1}{2x^2}\right) + c_2x^2$$

which simplifies to

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2x^2$$

Summary
The solution(s) found are the following

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2x^2 \tag{1}$$

Verification of solutions

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2x^2$$

Verified OK.

10.5.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)\, x - p(x) - x^2 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = x$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = x$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) = (\mu)(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p}{x}\right) = \left(\frac{1}{x}\right)(x)$$
$$\mathrm{d}\left(\frac{p}{x}\right) = \mathrm{d}x$$

Integrating gives

$$\frac{p}{x} = \int \mathrm{d}x$$
$$\frac{p}{x} = x + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x + x^2$$

which simplifies to

$$p(x) = x(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x(x + c_1)$$

Integrating both sides gives

$$y = \int x(x + c_1)\, \mathrm{d}x$$
$$= \frac{1}{3}x^3 + \frac{1}{2}c_1 x^2 + c_2$$

Summary
The solution(s) found are the following

$$y = \frac{1}{3}x^3 + \frac{1}{2}c_1 x^2 + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{1}{3}x^3 + \frac{1}{2}c_1 x^2 + c_2$$

Verified OK.

10.5.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right)v' + (AB'' + BB' + CB)\,v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x \\ B &= -1 \\ C &= 0 \\ F &= x^2 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (x)\,(0) + (-1)\,(0) + (0)\,(-1) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-xv'' + (1)\,v' = 0$$

Now by applying $v' = u$ the above becomes

$$-xu'(x) + u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= \frac{1}{x}\,dx \\ \int \frac{1}{u}\,du &= \int \frac{1}{x}\,dx \\ \ln(u) &= \ln(x) + c_1 \\ u &= \mathrm{e}^{\ln(x)+c_1} \\ &= c_1 x \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= c_1 x \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int c_1 x \,\mathrm{d}x \\ &= \frac{c_1 x^2}{2} + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (-1)\left(\frac{c_1 x^2}{2} + c_2\right) \\ &= -\frac{c_1 x^2}{2} - c_2 \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = -1$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -1 & x^2 \\ \frac{d}{dx}(-1) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -1 & x^2 \\ 0 & 2x \end{vmatrix}$$

Therefore

$$W = (-1)(2x) - (x^2)(0)$$

Which simplifies to

$$W = -2x$$

Which simplifies to

$$W = -2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^4}{-2x^2}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{x^2}{2} dx$$

Hence

$$u_1 = \frac{x^3}{6}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-x^2}{-2x^2}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{2} dx$$

Hence

$$u_2 = \frac{x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^3}{3}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-\frac{c_1 x^2}{2} - c_2\right) + \left(\frac{x^3}{3}\right) \\ &= -\frac{1}{2} c_1 x^2 - c_2 + \frac{1}{3} x^3 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{2}c_1x^2 - c_2 + \frac{1}{3}x^3 \tag{1}$$

Verification of solutions

$$y = -\frac{1}{2}c_1x^2 - c_2 + \frac{1}{3}x^3$$

Verified OK.

10.5.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xy'' - y' = x^2$$

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' - y')\,dx = \int x^2 dx$$
$$xy' - 2y = \frac{x^3}{3} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{x^3 + 3c_1}{3x}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{x^3 + 3c_1}{3x}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{2}{x}dx}$$
$$= \frac{1}{x^2}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^3+3c_1}{3x}\right) \\
\frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{x^3+3c_1}{3x}\right) \\
d\left(\frac{y}{x^2}\right) &= \left(\frac{x^3+3c_1}{3x^3}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x^2} &= \int \frac{x^3+3c_1}{3x^3} dx \\
\frac{y}{x^2} &= \frac{x}{3} - \frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$
y = x^2\left(\frac{x}{3} - \frac{c_1}{2x^2}\right) + c_2 x^2
$$

which simplifies to

$$
y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2
$$

Summary

The solution(s) found are the following

$$
y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2 \tag{1}
$$

Verification of solutions

$$
y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2
$$

Verified OK.

10.5.5 Solving using Kovacic algorithm

Writing the ode as

$$
xy'' - y' = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x \\
B &= -1 \\
C &= 0
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 299: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (-)(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{x}\, dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\, dx \\
&= y_1\left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
xy'' - y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 + \frac{c_2 x^2}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= \frac{x^2}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(\frac{x^2}{2}\right) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ 0 & x \end{vmatrix}
$$

Therefore

$$W = (1)(x) - \left(\frac{x^2}{2}\right)(0)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^4}{2}}{x^2} dx$$

Which simplifies to

$$u_1 = -\int \frac{x^2}{2} dx$$

Hence

$$u_1 = -\frac{x^3}{6}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^2} dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^3}{3}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + \frac{c_2 x^2}{2}\right) + \left(\frac{x^3}{3}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 + \frac{1}{2}c_2 x^2 + \frac{1}{3}x^3 \tag{1}$$

Verification of solutions

$$y = c_1 + \frac{1}{2}c_2 x^2 + \frac{1}{3}x^3$$

Verified OK.

10.5.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x \\
q(x) &= -1 \\
r(x) &= 0 \\
s(x) &= x^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$xy' - 2y = \int x^2\,dx$$

We now have a first order ode to solve which is

$$xy' - 2y = \frac{x^3}{3} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{x^3 + 3c_1}{3x} \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{x^3 + 3c_1}{3x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^3 + 3c_1}{3x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{x^3 + 3c_1}{3x}\right) \\ \mathrm{d}\left(\frac{y}{x^2}\right) &= \left(\frac{x^3 + 3c_1}{3x^3}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\frac{y}{x^2} = \int \frac{x^3 + 3c_1}{3x^3}\,\mathrm{d}x$$
$$\frac{y}{x^2} = \frac{x}{3} - \frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(\frac{x}{3} - \frac{c_1}{2x^2}\right) + c_2 x^2$$

which simplifies to

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2$$

Summary

The solution(s) found are the following

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2 \tag{1}$$

Verification of solutions

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2$$

Verified OK.

10.5.7 Maple step by step solution

Let's solve

$y''x - y' = x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x)\,x - u(x) = x^2$

- Isolate the derivative

  $u'(x) = \frac{u(x)}{x} + x$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) - \frac{u(x)}{x} = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x)-\frac{u(x)}{x}\right)=\mu(x)\,x$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x)-\frac{u(x)}{x}\right)=\mu'(x)\,u(x)+\mu(x)\,u'(x)$
- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x)=\frac{1}{x}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,u(x))\right)dx=\int\mu(x)\,xdx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,u(x)=\int\mu(x)\,xdx+c_1$
- Solve for $u(x)$

  $u(x)=\frac{\int\mu(x)xdx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\frac{1}{x}$

  $u(x)=x\left(\int 1dx+c_1\right)$
- Evaluate the integrals on the rhs

  $u(x)=x(x+c_1)$
- Solve 1st ODE for $u(x)$

  $u(x)=x(x+c_1)$
- Make substitution $u=y'$

  $y'=x(x+c_1)$
- Integrate both sides to solve for y

  $\int y'dx=\int x(x+c_1)\,dx+c_2$
- Compute integrals

  $y=\frac{1}{3}x^3+\frac{1}{2}c_1x^2+c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (_a^2+_b(_a))/_a, _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x$2)-diff(y(x),x)=x^2,y(x), singsol=all)
```

$$y(x) = \frac{1}{3}x^3 + \frac{1}{2}c_1x^2 + c_2$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 24

```
DSolve[x*y''[x]-y'[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{3} + \frac{c_1x^2}{2} + c_2$$

10.6 problem Exercise 35.6, page 504

Internal problem ID [4656]
Internal file name [`OUTPUT/4149_Sunday_June_05_2022_12_29_20_PM_35294735/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.6, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{(1+y)\,y''-3{y'}^2=0}$$

10.6.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y'=p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$(1+y)\,p(y)\left(\frac{d}{dy}p(y)\right)-3p(y)^2=0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{3p}{1+y} \end{aligned}$$

Where $f(y) = \frac{3}{1+y}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned} \frac{1}{p}\, dp &= \frac{3}{1+y}\, dy \\ \int \frac{1}{p}\, dp &= \int \frac{3}{1+y}\, dy \\ \ln(p) &= 3\ln(1+y) + c_1 \\ p &= \mathrm{e}^{3\ln(1+y)+c_1} \\ &= c_1(1+y)^3 \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1(1+y)^3$$

Integrating both sides gives

$$\begin{aligned} \int \frac{1}{c_1(1+y)^3} dy &= c_2 + x \\ -\frac{1}{2c_1(1+y)^2} &= c_2 + x \end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned} y_1 &= -\frac{\sqrt{-2c_1c_2 - 2c_1x} - 1}{\sqrt{-2c_1c_2 - 2c_1x}} \\ y_2 &= -\frac{\sqrt{-2c_1c_2 - 2c_1x} + 1}{\sqrt{-2c_1c_2 - 2c_1x}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{-2c_1c_2 - 2c_1x} - 1}{\sqrt{-2c_1c_2 - 2c_1x}} \tag{1}$$

$$y = -\frac{\sqrt{-2c_1c_2 - 2c_1x} + 1}{\sqrt{-2c_1c_2 - 2c_1x}} \tag{2}$$

Verification of solutions

$$y = -\frac{\sqrt{-2c_1c_2 - 2c_1x} - 1}{\sqrt{-2c_1c_2 - 2c_1x}}$$

Verified OK.

$$y = -\frac{\sqrt{-2c_1c_2 - 2c_1x} + 1}{\sqrt{-2c_1c_2 - 2c_1x}}$$

Verified OK.

10.6.2 Maple step by step solution

Let's solve

$(1 + y)\, y'' - 3y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y' \left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y) \left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y) \left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $(1 + y)\, u(y) \left(\frac{d}{dy}u(y)\right) - 3u(y)^2 = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)} = \frac{3}{1+y}$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy}u(y)}{u(y)} dy = \int \frac{3}{1+y} dy + c_1$

- Evaluate integral

$\ln(u(y)) = 3\ln(1+y) + c_1$

- Solve for $u(y)$

  $u(y) = e^{c_1}(1+y)^3$

- Solve 1st ODE for $u(y)$

  $u(y) = e^{c_1}(1+y)^3$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = e^{c_1}(1+y)^3$

- Separate variables

  $\frac{y'}{(1+y)^3} = e^{c_1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{(1+y)^3}dx = \int e^{c_1}dx + c_2$

- Evaluate integral

  $-\frac{1}{2(1+y)^2} = x\,e^{c_1} + c_2$

- Solve for y

  $\left\{y = -\frac{\sqrt{-2x\,e^{c_1}-2c_2}-1}{\sqrt{-2x\,e^{c_1}-2c_2}}, y = -\frac{\sqrt{-2x\,e^{c_1}-2c_2}+1}{\sqrt{-2x\,e^{c_1}-2c_2}}\right\}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve((y(x)+1)*diff(y(x),x$2)=3*(diff(y(x),x))^2,y(x), singsol=all)
```

$$y(x) = -1$$
$$y(x) = -\frac{\sqrt{-2c_1x - 2c_2} - 1}{\sqrt{-2c_1x - 2c_2}}$$
$$y(x) = -\frac{\sqrt{-2c_1x - 2c_2} + 1}{\sqrt{-2c_1x - 2c_2}}$$

✓ Solution by Mathematica

Time used: 1.485 (sec). Leaf size: 107

```
DSolve[(y[x]+1)*y''[x]==3*(y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2c_1x + \sqrt{2}\sqrt{-c_1(x + c_2)} + 2c_2c_1}{2c_1(x + c_2)}$$
$$y(x) \to \frac{-2c_1x + \sqrt{2}\sqrt{-c_1(x + c_2)} - 2c_2c_1}{2c_1(x + c_2)}$$
$$y(x) \to -1$$
$$y(x) \to \text{Indeterminate}$$

10.7 problem Exercise 35.7, page 504

Internal problem ID [4657]
Internal file name [`OUTPUT/4150_Sunday_June_05_2022_12_29_25_PM_80642096/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.7, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{r'' + \frac{k}{r^2} = 0}$$

10.7.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable r an independent variable. Using

$$r' = p(r)$$

Then

$$\begin{aligned} r'' &= \frac{dp}{dt} \\ &= \frac{dr}{dt}\frac{dp}{dr} \\ &= p\frac{dp}{dr} \end{aligned}$$

Hence the ode becomes

$$p(r)\left(\frac{d}{dr}p(r)\right)r^2 = -k$$

Which is now solved as first order ode for $p(r)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(r,p) \\ &= f(r)g(p) \\ &= -\frac{k}{p\,r^2} \end{aligned}$$

Where $f(r) = -\frac{k}{r^2}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\frac{1}{\frac{1}{p}}\,dp = -\frac{k}{r^2}\,dr$$

$$\int \frac{1}{\frac{1}{p}}\,dp = \int -\frac{k}{r^2}\,dr$$

$$\frac{p^2}{2} = \frac{k}{r} + c_1$$

The solution is

$$\frac{p(r)^2}{2} - \frac{k}{r} - c_1 = 0$$

For solution (1) found earlier, since $p = r'$ then we now have a new first order ode to solve which is

$$\frac{r'^2}{2} - \frac{k}{r} - c_1 = 0$$

Solving the given ode for r' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$r' = \frac{\sqrt{2}\,\sqrt{r\,(c_1 r + k)}}{r} \tag{1}$$

$$r' = -\frac{\sqrt{2}\,\sqrt{r\,(c_1 r + k)}}{r} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{r\sqrt{2}}{2\sqrt{r\,(c_1 r + k)}}dr = \int dt$$

$$\frac{\sqrt{2}\,\sqrt{r^2 c_1 + rk}}{2c_1} - \frac{\sqrt{2}\,k \ln\left(\frac{\frac{k}{2}+c_1 r}{\sqrt{c_1}} + \sqrt{r^2 c_1 + rk}\right)}{4c_1^{\frac{3}{2}}} = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{r\sqrt{2}}{2\sqrt{r\left(c_1 r+k\right)}}dr = \int dt$$

$$-\frac{\sqrt{2}\,\sqrt{r^2 c_1+rk}}{2c_1}+\frac{\sqrt{2}\,k\ln\left(\frac{\frac{k}{2}+c_1 r}{\sqrt{c_1}}+\sqrt{r^2 c_1+rk}\right)}{4c_1^{\frac{3}{2}}} = t+c_3$$

Summary
The solution(s) found are the following

$$\frac{\sqrt{2}\,\sqrt{r^2 c_1+rk}}{2c_1}-\frac{\sqrt{2}\,k\ln\left(\frac{\frac{k}{2}+c_1 r}{\sqrt{c_1}}+\sqrt{r^2 c_1+rk}\right)}{4c_1^{\frac{3}{2}}} = t+c_2 \tag{1}$$

$$-\frac{\sqrt{2}\,\sqrt{r^2 c_1+rk}}{2c_1}+\frac{\sqrt{2}\,k\ln\left(\frac{\frac{k}{2}+c_1 r}{\sqrt{c_1}}+\sqrt{r^2 c_1+rk}\right)}{4c_1^{\frac{3}{2}}} = t+c_3 \tag{2}$$

Verification of solutions

$$\frac{\sqrt{2}\,\sqrt{r^2 c_1+rk}}{2c_1}-\frac{\sqrt{2}\,k\ln\left(\frac{\frac{k}{2}+c_1 r}{\sqrt{c_1}}+\sqrt{r^2 c_1+rk}\right)}{4c_1^{\frac{3}{2}}} = t+c_2$$

Verified OK.

$$-\frac{\sqrt{2}\,\sqrt{r^2 c_1+rk}}{2c_1}+\frac{\sqrt{2}\,k\ln\left(\frac{\frac{k}{2}+c_1 r}{\sqrt{c_1}}+\sqrt{r^2 c_1+rk}\right)}{4c_1^{\frac{3}{2}}} = t+c_3$$

Verified OK.

10.7.2 Maple step by step solution

Let's solve

$r'' r^2 = -k$

- Highest derivative means the order of the ODE is 2

  r''

- Define new dependent variable u

  $u(t) = r'$

- Compute r''

  $u'(t) = r''$
- Use chain rule on the lhs

  $r'\left(\frac{d}{dr}u(r)\right) = r''$
- Substitute in the definition of u

  $u(r)\left(\frac{d}{dr}u(r)\right) = r''$
- Make substitutions $r' = u(r), r'' = u(r)\left(\frac{d}{dr}u(r)\right)$ to reduce order of ODE

  $u(r)\left(\frac{d}{dr}u(r)\right)r^2 = -k$
- Separate variables

  $u(r)\left(\frac{d}{dr}u(r)\right) = -\frac{k}{r^2}$
- Integrate both sides with respect to r

  $\int u(r)\left(\frac{d}{dr}u(r)\right)dr = \int -\frac{k}{r^2}dr + c_1$
- Evaluate integral

  $\frac{u(r)^2}{2} = \frac{k}{r} + c_1$
- Solve for $u(r)$

  $\left\{u(r) = \frac{\sqrt{2}\,\sqrt{r(c_1r+k)}}{r}, u(r) = -\frac{\sqrt{2}\,\sqrt{r(c_1r+k)}}{r}\right\}$
- Solve 1st ODE for $u(r)$

  $u(r) = \frac{\sqrt{2}\,\sqrt{r(c_1r+k)}}{r}$
- Revert to original variables with substitution $u(r) = r', r = r$

  $r' = \frac{\sqrt{2}\,\sqrt{r(c_1r+k)}}{r}$
- Separate variables

  $\frac{r'r}{\sqrt{r(c_1r+k)}} = \sqrt{2}$
- Integrate both sides with respect to t

  $\int \frac{r'r}{\sqrt{r(c_1r+k)}}dt = \int \sqrt{2}dt + c_2$
- Evaluate integral

  $\frac{\sqrt{r^2c_1+rk}}{c_1} - \frac{k\ln\left(\frac{\frac{k}{2}+c_1r}{\sqrt{c_1}} + \sqrt{r^2c_1+rk}\right)}{2c_1^{\frac{3}{2}}} = \sqrt{2}\,t + c_2$
- Solve for r

$$\left\{ -\frac{4\sqrt{c_1}\, k\, \mathrm{e}^{\mathit{RootOf}\left(64 c_1^{\frac{5}{2}} \sqrt{2} \left(\mathrm{e}^{\_Z}\right)^2 \_Z k t + 64 c_1^{\frac{5}{2}} \left(\mathrm{e}^{\_Z}\right)^2 c_2 \_Z k + 128 c_1^4 \sqrt{2} \left(\mathrm{e}^{\_Z}\right)^2 c_2 t + 64 \left(\mathrm{e}^{\_Z}\right)^2 c_1^4 c_2^2 + 128 \left(\mathrm{e}^{\_Z}\right)^2 c_1^4 t^2 + 16 \left(\mathrm{e}^{\_Z}\right)^2 c_1 \_Z^2 k^2 - 16 \cdots\right.}}{8\, \mathrm{e}^{\mathit{RootOf}\left(64 c_1^{\frac{5}{2}} \sqrt{2} \left(\mathrm{e}^{\_Z}\right)^2 \_Z k t + 64 c_1^{\frac{5}{2}} \left(\mathrm{e}^{\_Z}\right)^2 c_2 \cdots\right.}} \right.$$

- Solve 2nd ODE for $u(r)$

  $u(r) = -\frac{\sqrt{2}\,\sqrt{r(c_1 r + k)}}{r}$

- Revert to original variables with substitution $u(r) = r', r = r$

  $r' = -\frac{\sqrt{2}\,\sqrt{r(c_1 r + k)}}{r}$

- Separate variables

  $\frac{r' r}{\sqrt{r(c_1 r + k)}} = -\sqrt{2}$

- Integrate both sides with respect to t

  $\int \frac{r' r}{\sqrt{r(c_1 r + k)}} dt = \int -\sqrt{2} dt + c_2$

- Evaluate integral

  $\frac{\sqrt{r^2 c_1 + r k}}{c_1} - \frac{k \ln\left(\frac{\frac{k}{2} + c_1 r}{\sqrt{c_1}} + \sqrt{r^2 c_1 + r k}\right)}{2 c_1^{\frac{3}{2}}} = -\sqrt{2}\, t + c_2$

- Solve for r

$$\left\{ -\frac{4\sqrt{c_1}\, k\, \mathrm{e}^{\mathit{RootOf}\left(64 c_1^{\frac{5}{2}} \sqrt{2} \left(\mathrm{e}^{\_Z}\right)^2 \_Z k t - 64 c_1^{\frac{5}{2}} \left(\mathrm{e}^{\_Z}\right)^2 c_2 \_Z k + 128 c_1^4 \sqrt{2} \left(\mathrm{e}^{\_Z}\right)^2 c_2 t - 64 \left(\mathrm{e}^{\_Z}\right)^2 c_1^4 c_2^2 - 128 \left(\mathrm{e}^{\_Z}\right)^2 c_1^4 t^2 - 16 \left(\mathrm{e}^{\_Z}\right)^2 c_1 \_Z^2 k^2 + 16 \cdots\right.}}{8\, \mathrm{e}^{\mathit{RootOf}\left(64 c_1^{\frac{5}{2}} \sqrt{2} \left(\mathrm{e}^{\_Z}\right)^2 \_Z k t - 64 c_1^{\frac{5}{2}} \left(\mathrm{e}^{\_Z}\right)^2 c_2 \cdots\right.}} \right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+k/_a^2 = 0, _b(_a), HINT = [[_a,
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, -1/2*_b]
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 369

```
dsolve(diff(r(t),t$2)=-k/(r(t)^2),r(t), singsol=all)
```

$$r(t) = \frac{c_1\left(c_1^2k^2 - 2kc_1 e^{\mathrm{RootOf}\left(\mathrm{csgn}\left(\frac{1}{c_1}\right)c_1^4k^2+2\_Z c_1^3 k\, e^{-\_Z}-\mathrm{csgn}\left(\frac{1}{c_1}\right)e^{2\_Z}c_1^2-2\,\mathrm{csgn}\left(\frac{1}{c_1}\right)e^{\_Z}c_2-2\,\mathrm{csgn}\left(\frac{1}{c_1}\right)e^{\_Z}t\right)} + e^{2\,\mathrm{RootOf}\left(\mathrm{csgn}\left(\frac{1}{c}\right.\right.}\right.}{}$$

$$r(t) = \frac{c_1\left(c_1^2k^2 - 2kc_1 e^{\mathrm{RootOf}\left(\mathrm{csgn}\left(\frac{1}{c_1}\right)c_1^4k^2+2\_Z c_1^3 k\, e^{-\_Z}-\mathrm{csgn}\left(\frac{1}{c_1}\right)e^{2\_Z}c_1^2+2\,\mathrm{csgn}\left(\frac{1}{c_1}\right)e^{\_Z}c_2+2\,\mathrm{csgn}\left(\frac{1}{c_1}\right)e^{\_Z}t\right)} + e^{2\,\mathrm{RootOf}\left(\mathrm{csgn}\left(\frac{1}{c}\right.\right.}\right.}{}$$

✓ Solution by Mathematica

Time used: 0.169 (sec). Leaf size: 65

```
DSolve[r''[t]==-k/(r[t]^2),r[t],t,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\left(\frac{r(t)\sqrt{\frac{2k}{r(t)}+c_1}}{c_1} - \frac{2k\operatorname{arctanh}\left(\frac{\sqrt{\frac{2k}{r(t)}+c_1}}{\sqrt{c_1}}\right)}{c_1{}^{3/2}}\right)^2 = (t+c_2)^2, r(t)\right]$$

10.8 problem Exercise 35.8, page 504

Internal problem ID [4658]
Internal file name [`OUTPUT/4151_Sunday_June_05_2022_12_29_37_PM_65462958/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.8, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{y'' - \frac{3k y^2}{2} = 0}$$

10.8.1 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - \frac{3k y^2 y'}{2} = 0$$

Integrating the above w.r.t x gives

$$\int \left(y'y'' - \frac{3k y^2 y'}{2}\right) dx = 0$$
$$\frac{y'^2}{2} - \frac{k y^3}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{k y^3 + 2c_1} \tag{1}$$
$$y' = -\sqrt{k y^3 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{k\,y^3+2c_1}}dy = \int dx$$
$$\int^y \frac{1}{\sqrt{\_a^3k+2c_1}}d\_a = c_2+x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{k\,y^3+2c_1}}dy = \int dx$$
$$\int^y -\frac{1}{\sqrt{\_a^3k+2c_1}}d\_a = x+c_3$$

Summary

The solution(s) found are the following

$$\int^y \frac{1}{\sqrt{\_a^3k+2c_1}}d\_a = c_2+x \tag{1}$$
$$\int^y -\frac{1}{\sqrt{\_a^3k+2c_1}}d\_a = x+c_3 \tag{2}$$

Verification of solutions

$$\int^y \frac{1}{\sqrt{\_a^3k+2c_1}}d\_a = c_2+x$$

Verified OK.

$$\int^y -\frac{1}{\sqrt{\_a^3k+2c_1}}d\_a = x+c_3$$

Verified OK.

10.8.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) - \frac{3k\,y^2}{2} = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{3k\,y^2}{2p} \end{aligned}$$

Where $f(y) = \frac{3k\,y^2}{2}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{1}{p}}\,dp &= \frac{3k\,y^2}{2}\,dy \\ \int \frac{1}{\frac{1}{p}}\,dp &= \int \frac{3k\,y^2}{2}\,dy \\ \frac{p^2}{2} &= \frac{k\,y^3}{2} + c_1 \end{aligned}$$

The solution is

$$\frac{p(y)^2}{2} - \frac{k\,y^3}{2} - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{y'^2}{2} - \frac{k y^3}{2} - c_1 = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{k y^3 + 2c_1} \tag{1}$$
$$y' = -\sqrt{k y^3 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{k\, y^3 + 2c_1}} dy = \int dx$$
$$\int^y \frac{1}{\sqrt{\_a^3 k + 2c_1}} d\_a = c_2 + x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{k\, y^3 + 2c_1}} dy = \int dx$$
$$\int^y -\frac{1}{\sqrt{\_a^3 k + 2c_1}} d\_a = x + c_3$$

Summary

The solution(s) found are the following

$$\int^y \frac{1}{\sqrt{\_a^3 k + 2c_1}} d\_a = c_2 + x \tag{1}$$
$$\int^y -\frac{1}{\sqrt{\_a^3 k + 2c_1}} d\_a = x + c_3 \tag{2}$$

Verification of solutions

$$\int^y \frac{1}{\sqrt{\_a^3 k + 2c_1}} d\_a = c_2 + x$$

Verified OK.

$$\int^y -\frac{1}{\sqrt{\_a^3 k + 2c_1}} d\_a = x + c_3$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
<- 2nd_order WeierstrassP successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)=3/2*k*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{4\,\text{WeierstrassP}\left(x + c_1, 0, c_2\right)}{k}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]==3/2*(k*y[x]^2),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

10.9 problem Exercise 35.9, page 504

Internal problem ID [4659]
Internal file name [`OUTPUT/4152_Sunday_June_05_2022_12_29_48_PM_70386169/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.9, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{y'' - 2k y^3 = 0}$$

10.9.1 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - 2ky^3y' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - 2ky^3y')\, dx = 0$$
$$\frac{y'^2}{2} - \frac{ky^4}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{ky^4 + 2c_1} \tag{1}$$
$$y' = -\sqrt{ky^4 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{k\,y^4+2c_1}}dy = \int dx$$
$$\int^{y} \frac{1}{\sqrt{\_a^4k+2c_1}}d\_a = c_2+x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{k\,y^4+2c_1}}dy = \int dx$$
$$\int^{y} -\frac{1}{\sqrt{\_a^4k+2c_1}}d\_a = x+c_3$$

Summary

The solution(s) found are the following

$$\int^{y} \frac{1}{\sqrt{\_a^4k+2c_1}}d\_a = c_2+x \tag{1}$$
$$\int^{y} -\frac{1}{\sqrt{\_a^4k+2c_1}}d\_a = x+c_3 \tag{2}$$

Verification of solutions

$$\int^{y} \frac{1}{\sqrt{\_a^4k+2c_1}}d\_a = c_2+x$$

Verified OK.

$$\int^{y} -\frac{1}{\sqrt{\_a^4k+2c_1}}d\_a = x+c_3$$

Verified OK.

10.9.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) - 2k\,y^3 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{2k\,y^3}{p} \end{aligned}$$

Where $f(y) = 2k\,y^3$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{1}{p}}\,dp &= 2k\,y^3\,dy \\ \int \frac{1}{\frac{1}{p}}\,dp &= \int 2k\,y^3\,dy \\ \frac{p^2}{2} &= \frac{k\,y^4}{2} + c_1 \end{aligned}$$

The solution is

$$\frac{p(y)^2}{2} - \frac{k\,y^4}{2} - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{y'^2}{2} - \frac{k y^4}{2} - c_1 = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{k y^4 + 2c_1} \tag{1}$$

$$y' = -\sqrt{k y^4 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{k\,y^4 + 2c_1}} dy = \int dx$$

$$\int^y \frac{1}{\sqrt{\_a^4 k + 2c_1}} d\_a = c_2 + x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{k\,y^4 + 2c_1}} dy = \int dx$$

$$\int^y -\frac{1}{\sqrt{\_a^4 k + 2c_1}} d\_a = x + c_3$$

Summary

The solution(s) found are the following

$$\int^y \frac{1}{\sqrt{\_a^4 k + 2c_1}} d\_a = c_2 + x \tag{1}$$

$$\int^y -\frac{1}{\sqrt{\_a^4 k + 2c_1}} d\_a = x + c_3 \tag{2}$$

Verification of solutions

$$\int^y \frac{1}{\sqrt{\_a^4 k + 2c_1}} d\_a = c_2 + x$$

Verified OK.

$$\int^y -\frac{1}{\sqrt{\_a^4 k + 2c_1}} d\_a = x + c_3$$

Verified OK.

10.9.3 Maple step by step solution

Let's solve

$y'' - 2ky^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y), y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $u(y)\left(\frac{d}{dy}u(y)\right) - 2k\,y^3 = 0$

- Integrate both sides with respect to y

  $\int \left(u(y)\left(\frac{d}{dy}u(y)\right) - 2k\,y^3\right) dy = \int 0dy + c_1$

- Evaluate integral

  $\frac{u(y)^2}{2} - \frac{k\,y^4}{2} = c_1$

- Solve for $u(y)$

  $\left\{u(y) = \sqrt{k\,y^4 + 2c_1}, u(y) = -\sqrt{k\,y^4 + 2c_1}\right\}$

- Solve 1st ODE for $u(y)$

  $u(y) = \sqrt{k\,y^4 + 2c_1}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = \sqrt{ky^4 + 2c_1}$

- Separate variables

  $\frac{y'}{\sqrt{ky^4+2c_1}} = 1$

- Integrate both sides with respect to x

$$\int \frac{y'}{\sqrt{k y^4 + 2c_1}} dx = \int 1 dx + c_2$$

- Evaluate integral

$$\frac{\sqrt{2}\,\sqrt{4 - \frac{2\,\mathrm{I}\sqrt{2}\,\sqrt{k}\,y^2}{\sqrt{c_1}}}\,\sqrt{4 + \frac{2\,\mathrm{I}\sqrt{2}\,\sqrt{k}\,y^2}{\sqrt{c_1}}}\,\mathit{EllipticF}\left(\frac{y\sqrt{2}\,\sqrt{\frac{\mathrm{I}\sqrt{2}\,\sqrt{k}}{\sqrt{c_1}}}}{2}, \mathrm{I}\right)}{4\sqrt{\frac{\mathrm{I}\sqrt{2}\,\sqrt{k}}{\sqrt{c_1}}}\,\sqrt{k y^4 + 2c_1}} = c_2 + x$$

- Solve for y

$$\left\{\frac{\mathrm{JacobiSN}\left(\sqrt{\mathrm{I}\sqrt{c_1}\,\sqrt{k}\,\sqrt{2}}\,(c_2 + x), \mathrm{I}\right)\sqrt{2}}{\sqrt{\frac{\mathrm{I}\sqrt{2}\,\sqrt{k}}{\sqrt{c_1}}}}, -\frac{\mathrm{JacobiSN}\left(\sqrt{\mathrm{I}\sqrt{c_1}\,\sqrt{k}\,\sqrt{2}}\,(c_2 + x), \mathrm{I}\right)\sqrt{2}}{\sqrt{\frac{\mathrm{I}\sqrt{2}\,\sqrt{k}}{\sqrt{c_1}}}}\right\}$$

- Solve 2nd ODE for $u(y)$

$$u(y) = -\sqrt{k\,y^4 + 2c_1}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = -\sqrt{k y^4 + 2c_1}$$

- Separate variables

$$\frac{y'}{\sqrt{k y^4 + 2c_1}} = -1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{\sqrt{k y^4 + 2c_1}} dx = \int (-1)\, dx + c_2$$

- Evaluate integral

$$\frac{\sqrt{2}\,\sqrt{4 - \frac{2\,\mathrm{I}\sqrt{2}\,\sqrt{k}\,y^2}{\sqrt{c_1}}}\,\sqrt{4 + \frac{2\,\mathrm{I}\sqrt{2}\,\sqrt{k}\,y^2}{\sqrt{c_1}}}\,\mathit{EllipticF}\left(\frac{y\sqrt{2}\,\sqrt{\frac{\mathrm{I}\sqrt{2}\,\sqrt{k}}{\sqrt{c_1}}}}{2}, \mathrm{I}\right)}{4\sqrt{\frac{\mathrm{I}\sqrt{2}\,\sqrt{k}}{\sqrt{c_1}}}\,\sqrt{k y^4 + 2c_1}} = c_2 - x$$

- Solve for y

$$\left\{\frac{\mathrm{JacobiSN}\left(\sqrt{\mathrm{I}\sqrt{c_1}\,\sqrt{k}\,\sqrt{2}}\,(c_2 - x), \mathrm{I}\right)\sqrt{2}}{\sqrt{\frac{\mathrm{I}\sqrt{2}\,\sqrt{k}}{\sqrt{c_1}}}}, -\frac{\mathrm{JacobiSN}\left(\sqrt{\mathrm{I}\sqrt{c_1}\,\sqrt{k}\,\sqrt{2}}\,(c_2 - x), \mathrm{I}\right)\sqrt{2}}{\sqrt{\frac{\mathrm{I}\sqrt{2}\,\sqrt{k}}{\sqrt{c_1}}}}\right\}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
<- 2nd_order JacobiSN successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)=2*k*y(x)^3,y(x), singsol=all)
```

$$y(x) = c_2 \operatorname{JacobiSN}\left(\left(\sqrt{-k}\, x + c_1\right) c_2, i\right)$$

✓ Solution by Mathematica

Time used: 61.304 (sec). Leaf size: 115

```
DSolve[y''[x]==2*k*y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{i \operatorname{sn}\left((-1)^{3/4} \sqrt{\sqrt{k}\sqrt{c_1}(x+c_2)^2} \middle| -1\right)}{\sqrt{\frac{i\sqrt{k}}{\sqrt{c_1}}}}$$

$$y(x) \to \frac{i \operatorname{sn}\left((-1)^{3/4} \sqrt{\sqrt{k}\sqrt{c_1}(x+c_2)^2} \middle| -1\right)}{\sqrt{\frac{i\sqrt{k}}{\sqrt{c_1}}}}$$

10.10 problem Exercise 35.10, page 504

Internal problem ID [4660]
Internal file name [`OUTPUT/4153_Sunday_June_05_2022_12_29_57_PM_34490220/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.10, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear], [
  _2nd_order, _reducible, _mu_x_y1], [_2nd_order, _reducible,
  _mu_xy]]
```

$$\boxed{yy'' + {y'}^2 - y' = 0}$$

10.10.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (yy'' + (y' - 1)\, y')\, dx = 0$$

$$y'y - y = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{y}{y + c_1} dy = c_2 + x$$

$$y - c_1 \ln (y + c_1) = c_2 + x$$

Solving for y gives these solutions

$$
\begin{aligned}
y_1 &= -c_1\left(\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-\frac{c_1+c_2+x}{c_1}}}{c_1}\right)+1\right) \\
&= -c_1\left(\operatorname{LambertW}\left(-\frac{\mathrm{e}^{\frac{-c_1-x}{c_1}}}{c_2c_1}\right)+1\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -c_1\left(\operatorname{LambertW}\left(-\frac{\mathrm{e}^{\frac{-c_1-x}{c_1}}}{c_2c_1}\right)+1\right) \tag{1}
$$

Verification of solutions

$$
y = -c_1\left(\operatorname{LambertW}\left(-\frac{\mathrm{e}^{\frac{-c_1-x}{c_1}}}{c_2c_1}\right)+1\right)
$$

Verified OK.

10.10.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$
y' = p(y)
$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
yp(y)\left(\frac{d}{dy}p(y)\right) + (p(y)-1)\,p(y) = 0
$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{-p+1}{y} \end{aligned}$$

Where $f(y) = \frac{1}{y}$ and $g(p) = -p+1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-p+1}\,dp &= \frac{1}{y}\,dy \\ \int \frac{1}{-p+1}\,dp &= \int \frac{1}{y}\,dy \\ -\ln(p-1) &= \ln(y) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{p-1} = e^{\ln(y)+c_1}$$

Which simplifies to

$$\frac{1}{p-1} = c_2 y$$

Which simplifies to

$$p(y) = \frac{(c_2 e^{c_1} y + 1)\,e^{-c_1}}{c_2 y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{(c_2 e^{c_1} y + 1)\,e^{-c_1}}{c_2 y}$$

Integrating both sides gives

$$\begin{aligned} \int \frac{c_2 e^{c_1} y}{c_2 e^{c_1} y + 1} dy &= x + c_3 \\ y - \frac{e^{-c_1} \ln(c_2 e^{c_1} y + 1)}{c_2} &= x + c_3 \end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned}
y_1 &= -\frac{\left(\operatorname{LambertW}\left(-\mathrm{e}^{-c_3 c_2 \mathrm{e}^{c_1}-c_2 x \,\mathrm{e}^{c_1}-1}\right)+1\right)\mathrm{e}^{-c_1}}{c_2} \\
&= -\frac{\left(\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-c_2 x \,\mathrm{e}^{c_1}-1}}{c_3}\right)+1\right)\mathrm{e}^{-c_1}}{c_2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-c_2 x \,\mathrm{e}^{c_1}-1}}{c_3}\right)+1\right)\mathrm{e}^{-c_1}}{c_2} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-c_2 x \,\mathrm{e}^{c_1}-1}}{c_3}\right)+1\right)\mathrm{e}^{-c_1}}{c_2}$$

Verified OK.

10.10.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$yy'' + (y'-1)\,y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}
\int (yy'' + (y'-1)\,y')\,dx &= 0 \\
y'y - y &= c_1
\end{aligned}$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned}
\int \frac{y}{y+c_1}dy &= c_2 + x \\
y - c_1 \ln(y+c_1) &= c_2 + x
\end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned}
y_1 &= -c_1\left(\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-\frac{c_1+c_2+x}{c_1}}}{c_1}\right)+1\right) \\
&= -c_1\left(\operatorname{LambertW}\left(-\frac{\mathrm{e}^{\frac{-c_1-x}{c_1}}}{c_2 c_1}\right)+1\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = -c_1 \left(\operatorname{LambertW}\left(-\frac{e^{\frac{-c_1-x}{c_1}}}{c_2 c_1}\right)+1\right) \tag{1}$$

Verification of solutions

$$y = -c_1 \left(\operatorname{LambertW}\left(-\frac{e^{\frac{-c_1-x}{c_1}}}{c_2 c_1}\right)+1\right)$$

Verified OK.

10.10.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x,y,y')\,y'' + a_1(x,y,y')\,y' + a_0(x,y,y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}
\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'} \\
\frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'} \\
\frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}
\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}
a_2 &= y \\
a_1 &= y' - 1 \\
a_0 &= 0
\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}
\int a_2\,dy' + \int a_1\,dy + \int a_0\,dx &= c_1 \\
\int y\,dy' + \int y' - 1\,dy + \int 0\,dx &= c_1
\end{aligned}$$

Which results in

$$y'y + (y'-1)\,y = c_1$$

Which is now solved Integrating both sides gives

$$\int \frac{2y}{y+c_1}dy = c_2 + x$$
$$2y - 2c_1 \ln (y+c_1) = c_2 + x$$

Solving for y gives these solutions

$$y_1 = -c_1 \left(\text{LambertW}\left(-\frac{e^{-\frac{2c_1+c_2+x}{2c_1}}}{c_1}\right)+1\right)$$
$$= -c_1 \left(\text{LambertW}\left(-\frac{c_2 e^{-\frac{2c_1+x}{2c_1}}}{c_1}\right)+1\right)$$

Summary

The solution(s) found are the following

$$y = -c_1 \left(\text{LambertW}\left(-\frac{c_2 e^{-\frac{2c_1+x}{2c_1}}}{c_1}\right)+1\right) \tag{1}$$

Verification of solutions

$$y = -c_1 \left(\text{LambertW}\left(-\frac{c_2 e^{-\frac{2c_1+x}{2c_1}}}{c_1}\right)+1\right)$$

Verified OK.

10.10.5 Maple step by step solution

Let's solve

$yy'' + (y'-1)\,y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

$$y'\left(\frac{d}{dy}u(y)\right) = y''$$

- Substitute in the definition of u

$$u(y)\left(\frac{d}{dy}u(y)\right) = y''$$

- Make substitutions $y' = u(y), y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$$yu(y)\left(\frac{d}{dy}u(y)\right) + (u(y) - 1)\, u(y) = 0$$

- Separate variables

$$\frac{\frac{d}{dy}u(y)}{u(y)-1} = -\frac{1}{y}$$

- Integrate both sides with respect to y

$$\int \frac{\frac{d}{dy}u(y)}{u(y)-1} dy = \int -\frac{1}{y} dy + c_1$$

- Evaluate integral

$$\ln(u(y) - 1) = -\ln(y) + c_1$$

- Solve for $u(y)$

$$u(y) = \frac{e^{c_1} + y}{y}$$

- Solve 1st ODE for $u(y)$

$$u(y) = \frac{e^{c_1} + y}{y}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = \frac{e^{c_1} + y}{y}$$

- Separate variables

$$\frac{y'y}{e^{c_1} + y} = 1$$

- Integrate both sides with respect to x

$$\int \frac{y'y}{e^{c_1} + y} dx = \int 1 dx + c_2$$

- Evaluate integral

$$y - e^{c_1} \ln(e^{c_1} + y) = c_2 + x$$

- Solve for y

$$y = -\mathit{LambertW}\left(-e^{-\frac{c_1 e^{c_1} + e^{c_1} + c_2 + x}{e^{c_1}}}\right) e^{c_1} - e^{c_1}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)*(_b(_a)-1)/_a = 0, _b(_a)
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 0]
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 34

```
dsolve(y(x)*diff(y(x),x$2)+(diff(y(x),x))^2-diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = -c_1 \left(\operatorname{LambertW}\left(-\frac{e^{\frac{-c_1-c_2-x}{c_1}}}{c_1}\right)+1\right)$$

✓ Solution by Mathematica

Time used: 60.084 (sec). Leaf size: 32

```
DSolve[y[x]*y''[x]+(y'[x])^2-y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -c_1 \left(1 + W\left(-\frac{e^{-\frac{x+c_1+c_2}{c_1}}}{c_1}\right)\right)$$

10.11 problem Exercise 35.11, page 504

Internal problem ID [4661]
Internal file name [`OUTPUT/4154_Sunday_June_05_2022_12_30_08_PM_76444680/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.11, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{r'' - \frac{h^2}{r^3} + \frac{k}{r^2} = 0}$$

10.11.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable r an independent variable. Using

$$r' = p(r)$$

Then

$$\begin{aligned} r'' &= \frac{dp}{dt} \\ &= \frac{dr}{dt}\frac{dp}{dr} \\ &= p\frac{dp}{dr} \end{aligned}$$

Hence the ode becomes

$$p(r)\left(\frac{d}{dr}p(r)\right)r^3 + kr = h^2$$

Which is now solved as first order ode for $p(r)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(r,p) \\ &= f(r)g(p) \\ &= \frac{h^2 - kr}{p\,r^3} \end{aligned}$$

Where $f(r) = \frac{h^2-kr}{r^3}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{1}{p}}\,dp &= \frac{h^2 - kr}{r^3}\,dr \\ \int \frac{1}{\frac{1}{p}}\,dp &= \int \frac{h^2 - kr}{r^3}\,dr \\ \frac{p^2}{2} &= -\frac{h^2}{2r^2} + \frac{k}{r} + c_1 \end{aligned}$$

The solution is

$$\frac{p(r)^2}{2} + \frac{h^2}{2r^2} - \frac{k}{r} - c_1 = 0$$

For solution (1) found earlier, since $p = r'$ then we now have a new first order ode to solve which is

$$\frac{r'^2}{2} + \frac{h^2}{2r^2} - \frac{k}{r} - c_1 = 0$$

Solving the given ode for r' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$r' = \frac{\sqrt{2r^2c_1 + 2rk - h^2}}{r} \tag{1}$$

$$r' = -\frac{\sqrt{2r^2c_1 + 2rk - h^2}}{r} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{r}{\sqrt{2c_1r^2 - h^2 + 2kr}}dr = \int dt$$

$$\frac{\sqrt{2r^2c_1 + 2rk - h^2}}{2c_1} - \frac{k\ln\left(\frac{(2c_1r+k)\sqrt{2}}{2\sqrt{c_1}} + \sqrt{2r^2c_1 + 2rk - h^2}\right)\sqrt{2}}{4c_1^{\frac{3}{2}}} = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{r}{\sqrt{2c_1r^2 - h^2 + 2kr}}dr = \int dt$$

$$-\frac{\sqrt{2r^2c_1 + 2rk - h^2}}{2c_1} + \frac{k\ln\left(\frac{(2c_1r+k)\sqrt{2}}{2\sqrt{c_1}} + \sqrt{2r^2c_1 + 2rk - h^2}\right)\sqrt{2}}{4c_1^{\frac{3}{2}}} = t + c_3$$

Summary

The solution(s) found are the following

$$\frac{\sqrt{2r^2c_1 + 2rk - h^2}}{2c_1} - \frac{k\ln\left(\frac{(2c_1r+k)\sqrt{2}}{2\sqrt{c_1}} + \sqrt{2r^2c_1 + 2rk - h^2}\right)\sqrt{2}}{4c_1^{\frac{3}{2}}} = t + c_2 \tag{1}$$

$$-\frac{\sqrt{2r^2c_1 + 2rk - h^2}}{2c_1} + \frac{k\ln\left(\frac{(2c_1r+k)\sqrt{2}}{2\sqrt{c_1}} + \sqrt{2r^2c_1 + 2rk - h^2}\right)\sqrt{2}}{4c_1^{\frac{3}{2}}} = t + c_3 \tag{2}$$

Verification of solutions

$$\frac{\sqrt{2r^2c_1 + 2rk - h^2}}{2c_1} - \frac{k\ln\left(\frac{(2c_1r+k)\sqrt{2}}{2\sqrt{c_1}} + \sqrt{2r^2c_1 + 2rk - h^2}\right)\sqrt{2}}{4c_1^{\frac{3}{2}}} = t + c_2$$

Verified OK.

$$-\frac{\sqrt{2r^2c_1 + 2rk - h^2}}{2c_1} + \frac{k\ln\left(\frac{(2c_1r+k)\sqrt{2}}{2\sqrt{c_1}} + \sqrt{2r^2c_1 + 2rk - h^2}\right)\sqrt{2}}{4c_1^{\frac{3}{2}}} = t + c_3$$

Verified OK.

10.11.2 Maple step by step solution

Let's solve

$r''r^3 + rk = h^2$

- Highest derivative means the order of the ODE is 2

  r''

- Define new dependent variable u

  $u(t) = r'$

- Compute r''

  $u'(t) = r''$
- Use chain rule on the lhs

  $r'\left(\frac{d}{dr}u(r)\right) = r''$
- Substitute in the definition of u

  $u(r)\left(\frac{d}{dr}u(r)\right) = r''$
- Make substitutions $r' = u(r), r'' = u(r)\left(\frac{d}{dr}u(r)\right)$ to reduce order of ODE

  $u(r)\left(\frac{d}{dr}u(r)\right) r^3 + kr = h^2$
- Separate variables

  $u(r)\left(\frac{d}{dr}u(r)\right) = \frac{h^2-kr}{r^3}$
- Integrate both sides with respect to r

  $\int u(r)\left(\frac{d}{dr}u(r)\right) dr = \int \frac{h^2-kr}{r^3} dr + c_1$
- Evaluate integral

  $\frac{u(r)^2}{2} = -\frac{h^2}{2r^2} + \frac{k}{r} + c_1$
- Solve for $u(r)$

  $\left\{u(r) = \frac{\sqrt{2r^2c_1-h^2+2kr}}{r}, u(r) = -\frac{\sqrt{2r^2c_1-h^2+2kr}}{r}\right\}$
- Solve 1st ODE for $u(r)$

  $u(r) = \frac{\sqrt{2r^2c_1-h^2+2kr}}{r}$
- Revert to original variables with substitution $u(r) = r', r = r$

  $r' = \frac{\sqrt{2r^2c_1+2rk-h^2}}{r}$
- Separate variables

  $\frac{r'r}{\sqrt{2r^2c_1+2rk-h^2}} = 1$
- Integrate both sides with respect to t

  $\int \frac{r'r}{\sqrt{2r^2c_1+2rk-h^2}} dt = \int 1 dt + c_2$
- Evaluate integral

  $\frac{\sqrt{2r^2c_1+2rk-h^2}}{2c_1} - \frac{k \ln\left(\frac{(2c_1r+k)\sqrt{2}}{2\sqrt{c_1}} + \sqrt{2r^2c_1+2rk-h^2}\right)\sqrt{2}}{4c_1^{\frac{3}{2}}} = t + c_2$
- Solve for r

$$\left\{\frac{RootOf\left(-4\sqrt{2}\,c_1^{\frac{7}{2}}\,c_2 k RootOf\left(-8 RootOf\left(8\sqrt{2}\,c_1^{\frac{7}{2}}\,c_2\_\_Zk+8\sqrt{2}\,c_1^{\frac{7}{2}}\_\_Zkt+8\sqrt{2}\,c_1^{\frac{7}{2}}\,c_2k+8\sqrt{2}\,c_1^{\frac{7}{2}}\,kt-4\sqrt{2}\,e^{-Z}c_1^{\frac{5}{2}}\_\_Zk-4\sqrt{2}\,e^{-Z}c_1^{\frac{5}{2}}\right.\right.\right.}{}\right.$$

- Solve 2nd ODE for $u(r)$

$$u(r) = -\frac{\sqrt{2r^2c_1-h^2+2kr}}{r}$$

- Revert to original variables with substitution $u(r) = r', r = r$

$$r' = -\frac{\sqrt{2r^2c_1+2rk-h^2}}{r}$$

- Separate variables

$$\frac{r'r}{\sqrt{2r^2c_1+2rk-h^2}} = -1$$

- Integrate both sides with respect to t

$$\int \frac{r'r}{\sqrt{2r^2c_1+2rk-h^2}}dt = \int (-1)\, dt + c_2$$

- Evaluate integral

$$\frac{\sqrt{2r^2c_1+2rk-h^2}}{2c_1} - \frac{k\ln\left(\frac{(2c_1r+k)\sqrt{2}}{2\sqrt{c_1}}+\sqrt{2r^2c_1+2rk-h^2}\right)\sqrt{2}}{4c_1^{\frac{3}{2}}} = -t + c_2$$

- Solve for r

$$\left\{\frac{RootOf\left(-4\sqrt{2}\,c_1^{\frac{7}{2}}\,c_2 k RootOf\left(-8 RootOf\left(8\sqrt{2}\,c_1^{\frac{7}{2}}\,c_2\_\_Zk-8\sqrt{2}\,c_1^{\frac{7}{2}}\_\_Zkt+8\sqrt{2}\,c_1^{\frac{7}{2}}\,c_2k-8\sqrt{2}\,c_1^{\frac{7}{2}}\,kt-4\sqrt{2}\,e^{-Z}c_1^{\frac{5}{2}}\_\_Zk-4\sqrt{2}\,e^{-Z}c_1^{\frac{5}{2}}\right.\right.\right.}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+(_a*k-h^2)/_a^3 = 0, _b(_a)`   *
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 441

```
dsolve(diff(r(t),t$2)= h^2/r(t)^3-k/r(t)^2,r(t), singsol=all)
```

$$r(t) = \frac{c_1\left(c_1^2k^2 - 2kc_1 e^{\text{RootOf}\left(\text{csgn}\left(\frac{1}{c_1}\right)c_1^4k^2+2\_Zc_1^3k\,e^{-\_Z}-\text{csgn}\left(\frac{1}{c_1}\right)e^{2\_Z}c_1^2+\text{csgn}\left(\frac{1}{c_1}\right)c_1^2h^2-2\,\text{csgn}\left(\frac{1}{c_1}\right)e^{-\_Z}c_2-2\,\text{csgn}\left(\frac{1}{c_1}\right)e^{-\_Z}t\right)} + e^{2}\right.}{}$$

$$r(t) = \frac{c_1\left(c_1^2k^2 - 2kc_1 e^{\text{RootOf}\left(\text{csgn}\left(\frac{1}{c_1}\right)c_1^4k^2+2\_Zc_1^3k\,e^{-\_Z}-\text{csgn}\left(\frac{1}{c_1}\right)e^{2\_Z}c_1^2+\text{csgn}\left(\frac{1}{c_1}\right)c_1^2h^2+2\,\text{csgn}\left(\frac{1}{c_1}\right)e^{-\_Z}c_2+2\,\text{csgn}\left(\frac{1}{c_1}\right)e^{-\_Z}t\right)} + e^{2}\right.}{}$$

✓ Solution by Mathematica

Time used: 1.099 (sec). Leaf size: 130

```
DSolve[r''[t]==h^2/r[t]^3-k/r[t]^2,r[t],t,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{\left(\sqrt{c_1}(-h^2+r(t)(2k+c_1r(t)))-k\sqrt{-h^2+r(t)(2k+c_1r(t))}\text{arctanh}\left(\frac{k+c_1r(t)}{\sqrt{c_1}\sqrt{-h^2+r(t)(2k+c_1r(t))}}\right)\right)^2}{c_1{}^3r(t)^2\left(-\frac{h^2}{r(t)^2}+\frac{2k}{r(t)}+c_1\right)}=\right.$$

$$\left.+c_2)^2,r(t)\right]$$

10.12 problem Exercise 35.12, page 504

Internal problem ID [4662]
Internal file name [`OUTPUT/4155_Sunday_June_05_2022_12_30_20_PM_45097950/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.12, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1],
   [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{yy'' + y'^3 - y'^2 = 0}$$

10.12.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) + \left(p(y)^2 - p(y)\right)p(y) = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= -\frac{p(p-1)}{y}
\end{aligned}$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = p(p-1)$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p(p-1)}\,dp &= -\frac{1}{y}\,dy \\
\int \frac{1}{p(p-1)}\,dp &= \int -\frac{1}{y}\,dy \\
\ln(p-1) - \ln(p) &= -\ln(y) + c_1
\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(p-1)-\ln(p)} = e^{-\ln(y)+c_1}$$

Which simplifies to

$$\frac{p-1}{p} = \frac{c_2}{y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{y}{c_2 - y}$$

Integrating both sides gives

$$\begin{aligned}
\int \frac{-c_2 + y}{y}\,dy &= x + c_3 \\
y - c_2 \ln(y) &= x + c_3
\end{aligned}$$

Solving for y gives these solutions

Summary

The solution(s) found are the following

$$y = e^{-\frac{c_2 \operatorname{LambertW}\left(-\frac{e^{-\frac{x+c_3}{c_2}}}{c_2}\right)+c_3+x}{c_2}} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-\frac{c_2 \,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-\frac{x+c_3}{c_2}}}{c_2}\right)+c_3+x}{c_2}}$$

Verified OK.

10.12.2 Maple step by step solution

Let's solve

$yy'' + \left(y'^2 - y'\right) y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y' \left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y) \left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y) \left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $yu(y) \left(\frac{d}{dy}u(y)\right) + \left(u(y)^2 - u(y)\right) u(y) = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)^2-u(y)} = -\frac{1}{y}$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy}u(y)}{u(y)^2-u(y)}dy = \int -\frac{1}{y}dy + c_1$

- Evaluate integral

  $\ln\left(u(y) - 1\right) - \ln\left(u(y)\right) = -\ln\left(y\right) + c_1$

- Solve for $u(y)$

$u(y) = -\frac{y}{e^{c_1} - y}$

- Solve 1st ODE for $u(y)$

  $u(y) = -\frac{y}{e^{c_1} - y}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = -\frac{y}{e^{c_1} - y}$

- Separate variables

  $\frac{y'(e^{c_1} - y)}{y} = -1$

- Integrate both sides with respect to x

  $\int \frac{y'(e^{c_1} - y)}{y} dx = \int (-1) \, dx + c_2$

- Evaluate integral

  $-y + e^{c_1} \ln(y) = c_2 - x$

- Solve for y

  $y = e^{-\frac{\mathit{LambertW}\left(-e^{-\frac{c_1 e^{c_1} - c_2 + x}{e^{c_1}}}\right) e^{c_1} - c_2 + x}{e^{c_1}}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)^2*(_b(_a)-1)/_a = 0, _b(_
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve(y(x)*diff(y(x),x$2)+(diff(y(x),x))^3-diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1$$
$$y(x) = e^{\frac{-c_1 \operatorname{LambertW}\left(\frac{e^{\frac{c_2+x}{c_1}}}{c_1}\right)+c_2+x}{c_1}}$$

✓ Solution by Mathematica

Time used: 22.229 (sec). Leaf size: 32

```
DSolve[y[x]*y''[x]+(y'[x])^3-(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{c_1} W\left(e^{e^{-c_1}(x - e^{c_1} c_1 + c_2)}\right)$$

10.13 problem Exercise 35.13, page 504

Internal problem ID [4663]
Internal file name [`OUTPUT/4156_Sunday_June_05_2022_12_30_25_PM_2046285/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.13, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{yy'' - 3y'^2 = 0}$$

10.13.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) - 3p(y)^2 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{3p}{y} \end{aligned}$$

Where $f(y) = \frac{3}{y}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned} \frac{1}{p}\,dp &= \frac{3}{y}\,dy \\ \int \frac{1}{p}\,dp &= \int \frac{3}{y}\,dy \\ \ln(p) &= 3\ln(y) + c_1 \\ p &= \mathrm{e}^{3\ln(y)+c_1} \\ &= c_1 y^3 \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = y^3 c_1$$

Integrating both sides gives

$$\begin{aligned} \int \frac{1}{c_1 y^3} dy &= c_2 + x \\ -\frac{1}{2c_1 y^2} &= c_2 + x \end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned} y_1 &= \frac{1}{\sqrt{-2c_1c_2 - 2c_1x}} \\ y_2 &= -\frac{1}{\sqrt{-2c_1c_2 - 2c_1x}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{-2c_1c_2 - 2c_1x}} \tag{1}$$

$$y = -\frac{1}{\sqrt{-2c_1c_2 - 2c_1x}} \tag{2}$$

Verification of solutions

$$y = \frac{1}{\sqrt{-2c_1c_2 - 2c_1x}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{-2c_1c_2 - 2c_1x}}$$

Verified OK.

10.13.2 Maple step by step solution

Let's solve

$y y'' - 3y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y' \left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y) \left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y) \left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $y u(y) \left(\frac{d}{dy}u(y)\right) - 3u(y)^2 = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)} = \frac{3}{y}$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy}u(y)}{u(y)} dy = \int \frac{3}{y} dy + c_1$

- Evaluate integral

$\ln(u(y)) = 3\ln(y) + c_1$

- Solve for $u(y)$

$u(y) = e^{c_1} y^3$

- Solve 1st ODE for $u(y)$

$u(y) = e^{c_1} y^3$

- Revert to original variables with substitution $u(y) = y', y = y$

$y' = e^{c_1} y^3$

- Separate variables

$\frac{y'}{y^3} = e^{c_1}$

- Integrate both sides with respect to x

$\int \frac{y'}{y^3} dx = \int e^{c_1} dx + c_2$

- Evaluate integral

$-\frac{1}{2y^2} = x\,e^{c_1} + c_2$

- Solve for y

$\left\{y = \frac{1}{\sqrt{-2x\,e^{c_1} - 2c_2}}, y = -\frac{1}{\sqrt{-2x\,e^{c_1} - 2c_2}}\right\}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(y(x)*diff(y(x),x$2)-3*(diff(y(x),x))^2=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = \frac{1}{\sqrt{-2c_1x - 2c_2}}$$
$$y(x) = -\frac{1}{\sqrt{-2c_1x - 2c_2}}$$

✓ Solution by Mathematica

Time used: 0.106 (sec). Leaf size: 14

```
DSolve[y[x]*y''[x]-(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 e^{c_1 x}$$

10.14 problem Exercise 35.14, page 504

Internal problem ID [4664]
Internal file name [`OUTPUT/4157_Sunday_June_05_2022_12_30_43_PM_53909131/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.14, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{(x^2+1)\,y''+{y'}^2=-1}$$

10.14.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x)=y'$$

Then

$$p'(x)=y''$$

Hence the ode becomes

$$(x^2+1)\,p'(x)+p(x)^2+1=0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned}p' &= F(x,p)\\ &= f(x)g(p)\\ &= \frac{-p^2-1}{x^2+1}\end{aligned}$$

Where $f(x) = \frac{1}{x^2+1}$ and $g(p) = -p^2 - 1$. Integrating both sides gives

$$\frac{1}{-p^2-1}\,dp = \frac{1}{x^2+1}\,dx$$
$$\int \frac{1}{-p^2-1}\,dp = \int \frac{1}{x^2+1}\,dx$$
$$-\arctan(p) = \arctan(x) + c_1$$

The solution is

$$-\arctan(p(x)) - \arctan(x) - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\arctan(y') - \arctan(x) - c_1 = 0$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\tan(\arctan(x) + c_1)\,\mathrm{d}x \\ &= \frac{i\,\mathrm{e}^{4ic_1} x}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{ix}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{4\,\mathrm{e}^{2ic_1}\ln\left((-\mathrm{e}^{2ic_1}+1)\,x + i\,\mathrm{e}^{2ic_1} + i\right)}{(\mathrm{e}^{2ic_1}-1)^2} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{i\,\mathrm{e}^{4ic_1} x}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{ix}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{4\,\mathrm{e}^{2ic_1}\ln\left((-\mathrm{e}^{2ic_1}+1)\,x + i\,\mathrm{e}^{2ic_1} + i\right)}{(\mathrm{e}^{2ic_1}-1)^2} + c_2 \qquad (1)$$

Verification of solutions

$$y = \frac{i\,\mathrm{e}^{4ic_1} x}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{ix}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{4\,\mathrm{e}^{2ic_1}\ln\left((-\mathrm{e}^{2ic_1}+1)\,x + i\,\mathrm{e}^{2ic_1} + i\right)}{(\mathrm{e}^{2ic_1}-1)^2} + c_2$$

Verified OK.

10.14.2 Maple step by step solution

Let's solve

$(x^2+1)\,y''+y'^2=-1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u=y'$ to reduce order of ODE

  $(x^2+1)\,u'(x)+u(x)^2=-1$

- Separate variables

  $\frac{u'(x)}{-u(x)^2-1}=\frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{-u(x)^2-1}dx=\int \frac{1}{x^2+1}dx+c_1$

- Evaluate integral

  $-\arctan\left(u(x)\right)=\arctan\left(x\right)+c_1$

- Solve for $u(x)$

  $u(x)=-\tan\left(\arctan\left(x\right)+c_1\right)$

- Solve 1st ODE for $u(x)$

  $u(x)=-\tan\left(\arctan\left(x\right)+c_1\right)$

- Make substitution $u=y'$

  $y'=-\tan\left(\arctan\left(x\right)+c_1\right)$

- Integrate both sides to solve for y

  $\int y'dx=\int -\tan\left(\arctan\left(x\right)+c_1\right)dx+c_2$

- Compute integrals

  $y=\frac{\mathrm{I}\,\mathrm{e}^{4\,\mathrm{I}c_1}x}{\left(\mathrm{e}^{2\,\mathrm{I}c_1}-1\right)^2}-\frac{\mathrm{I}x}{\left(\mathrm{e}^{2\,\mathrm{I}c_1}-1\right)^2}-\frac{4\,\mathrm{e}^{2\,\mathrm{I}c_1}\ln\left(\left(-\mathrm{e}^{2\,\mathrm{I}c_1}+1\right)x+\mathrm{I}\,\mathrm{e}^{2\,\mathrm{I}c_1}+\mathrm{I}\right)}{\left(\mathrm{e}^{2\,\mathrm{I}c_1}-1\right)^2}+c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(_b(_a)^2+1)/(_a^2+1), _b(_a)`   *** S
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   <- separable successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve((1+x^2)*diff(y(x),x$2)+(diff(y(x),x))^2+1=0,y(x), singsol=all)
```

$$y(x) = \frac{\ln (c_1x-1)\, c_1^2 + c_2c_1^2 + c_1x + \ln (c_1x-1)}{c_1^2}$$

✓ Solution by Mathematica

Time used: 7.091 (sec). Leaf size: 33

```
DSolve[(1+x^2)*y''[x]+(y'[x])^2+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\cot(c_1) + \csc^2(c_1)\log(-x\sin(c_1) - \cos(c_1)) + c_2$$

10.15 problem Exercise 35.15, page 504

Internal problem ID [4665]
Internal file name [`OUTPUT/4158_Sunday_June_05_2022_12_30_57_PM_29148347/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.15, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{(x^2+1)\,y''+2x(1+y')=0}$$

10.15.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int \left((x^2+1)\,y''+2xy'\right)dx &= \int -2x dx\\ (x^2+1)\,y' &= -x^2+c_1\end{aligned}$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned}y &= \int \frac{-x^2+c_1}{x^2+1}\,\mathrm{d}x\\ &= -x+(c_1+1)\arctan(x)+c_2\end{aligned}$$

Summary

The solution(s) found are the following

$$y = -x + (c_1 + 1)\arctan(x) + c_2 \tag{1}$$

Verification of solutions

$$y = -x + (c_1 + 1)\arctan(x) + c_2$$

Verified OK.

10.15.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$(x^2 + 1)\,p'(x) + 2xp(x) + 2x = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x, p) \\ &= f(x)g(p) \\ &= \frac{x(-2p - 2)}{x^2 + 1} \end{aligned}$$

Where $f(x) = \frac{x}{x^2+1}$ and $g(p) = -2p - 2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-2p - 2}\,dp &= \frac{x}{x^2 + 1}\,dx \\ \int \frac{1}{-2p - 2}\,dp &= \int \frac{x}{x^2 + 1}\,dx \\ -\frac{\ln(p + 1)}{2} &= \frac{\ln(x^2 + 1)}{2} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{p + 1}} = \mathrm{e}^{\frac{\ln(x^2+1)}{2} + c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{p+1}} = c_2\sqrt{x^2+1}$$

Which simplifies to

$$p(x) = -\frac{\left(c_2^2 \mathrm{e}^{2c_1}\left(x^2+1\right)-1\right)\mathrm{e}^{-2c_1}}{c_2^2\left(x^2+1\right)}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{\left(c_2^2 \mathrm{e}^{2c_1}\left(x^2+1\right)-1\right)\mathrm{e}^{-2c_1}}{c_2^2\left(x^2+1\right)}$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{\left(c_2^2 x^2 \mathrm{e}^{2c_1} + \mathrm{e}^{2c_1} c_2^2 - 1\right)\mathrm{e}^{-2c_1}}{c_2^2\left(x^2+1\right)}\,\mathrm{d}x \\ &= -\frac{\mathrm{e}^{-2c_1}\left(\mathrm{e}^{2c_1} c_2^2 x - \arctan\left(x\right)\right)}{c_2^2} + c_3 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-2c_1}\left(\mathrm{e}^{2c_1} c_2^2 x - \arctan\left(x\right)\right)}{c_2^2} + c_3 \tag{1}$$

Verification of solutions

$$y = -\frac{\mathrm{e}^{-2c_1}\left(\mathrm{e}^{2c_1} c_2^2 x - \arctan\left(x\right)\right)}{c_2^2} + c_3$$

Verified OK.

10.15.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$\left(x^2+1\right)y'' + 2xy' = -2x$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned} \int \left(\left(x^2+1\right)y'' + 2xy'\right)dx &= \int -2x dx \\ \left(x^2+1\right)y' &= -x^2 + c_1 \end{aligned}$$

Which is now solved for y. Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{-x^2 + c_1}{x^2 + 1}\, \mathrm{d}x \\
&= -x + (c_1 + 1) \arctan(x) + c_2
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = -x + (c_1 + 1) \arctan(x) + c_2 \tag{1}
$$

Verification of solutions

$$
y = -x + (c_1 + 1) \arctan(x) + c_2
$$

Verified OK.

10.15.4 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
(x^2 + 1) y'' + 2xy' &= 0 && (1) \\
Ay'' + By' + Cy &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 + 1 \\
B &= 2x \\
C &= 0
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 309: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{i}{4(x-i)} + \frac{i}{4x+4i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} + (-)(0) \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} \\
&= \frac{x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)(0) + \left(\left(-\frac{1}{2(x-i)^2} - \frac{1}{2(x+i)^2}\right) + \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)^2 - \left(\frac{1}{(x^2+1)^2}\right.\right.$$
$$0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right) dx} \\ &= \sqrt{x^2+1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2x}{x^2+1} \, dx} \\ &= z_1 e^{-\frac{\ln(x^2+1)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x^2+1}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln(x^2+1)}}{(y_1)^2} \, dx \\ &= y_1(\arctan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(\arctan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$(x^2 + 1)\, y'' + 2xy' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + c_2 \arctan(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 1 \\ y_2 &= \arctan(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \arctan(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\arctan(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \arctan(x) \\ 0 & \frac{1}{x^2+1} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{x^2+1}\right) - (\arctan(x))(0)$$

Which simplifies to

$$W = \frac{1}{x^2+1}$$

Which simplifies to

$$W = \frac{1}{x^2+1}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2x\arctan(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int -2x\arctan(x)\,dx$$

Hence

$$u_1 = \arctan(x)\,x^2 - x + \arctan(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{-2x}{1}\,dx$$

Which simplifies to

$$u_2 = \int -2x dx$$

Hence

$$u_2 = -x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x + \arctan(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 \arctan(x)) + (-x + \arctan(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 \arctan(x) - x + \arctan(x) \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 \arctan(x) - x + \arctan(x)$$

Verified OK.

10.15.5 Solving as exact linear second order ode ode

An ode of the form

$$p(x) y'' + q(x) y' + r(x) y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 + 1 \\ q(x) &= 2x \\ r(x) &= 0 \\ s(x) &= -2x \end{aligned}$$

Hence

$$p''(x) = 2$$
$$q'(x) = 2$$

Therefore (1) becomes

$$2 - (2) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx$$

Substituting the above values for p, q, r, s gives

$$\left(x^2+1\right) y' = \int -2x\, dx$$

We now have a first order ode to solve which is

$$\left(x^2+1\right) y' = -x^2 + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{-x^2+c_1}{x^2+1}\, \mathrm{d}x \\ &= -x + (c_1+1)\arctan(x) + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -x + (c_1+1)\arctan(x) + c_2 \tag{1}$$

Verification of solutions

$$y = -x + (c_1+1)\arctan(x) + c_2$$

Verified OK.

10.15.6 Maple step by step solution

Let's solve

$(x^2+1)\,y''+2xy'=-2x$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u=y'$ to reduce order of ODE

  $(x^2+1)\,u'(x)+2xu(x)=-2x$

- Integrate both sides with respect to x

  $\int \left((x^2+1)\,u'(x)+2xu(x)\right)dx=\int -2xdx+c_1$

- Evaluate integral

  $(x^2+1)\,u(x)=-x^2+c_1$

- Solve for $u(x)$

  $u(x)=\frac{-x^2+c_1}{x^2+1}$

- Solve 1st ODE for $u(x)$

  $u(x)=\frac{-x^2+c_1}{x^2+1}$

- Make substitution $u=y'$

  $y'=\frac{-x^2+c_1}{x^2+1}$

- Integrate both sides to solve for y

  $\int y'dx=\int \frac{-x^2+c_1}{x^2+1}dx+c_2$

- Compute integrals

  $y=-x+(c_1+1)\arctan(x)+c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -2*_a*(_b(_a)+1)/(_a^2+1), _b(_a)`   **
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((1+x^2)*diff(y(x),x$2)+2*x*(diff(y(x),x)+1)=0,y(x), singsol=all)
```

$$y(x) = -x + (1 + c_1) \arctan(x) + c_2$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 18

```
DSolve[(1+x^2)*y''[x]+2*x*(y'[x]+1)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow (1 + c_1) \arctan(x) - x + c_2$$

10.16 problem Exercise 35.16, page 504

Internal problem ID [4666]
Internal file name [`OUTPUT/4159_Sunday_June_05_2022_12_31_07_PM_75810893/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.16, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{(1+y)\,y''-3{y'}^2=0}$$

With initial conditions

$$\left[y(1)=0, y'(1)=-\frac{1}{2}\right]$$

10.16.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y'=p(y)$$

Then

$$\begin{aligned}y''&=\frac{dp}{dx}\\&=\frac{dy}{dx}\frac{dp}{dy}\\&=p\frac{dp}{dy}\end{aligned}$$

Hence the ode becomes

$$(1+y)\, p(y) \left(\frac{d}{dy} p(y)\right) - 3 p(y)^2 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= \frac{3p}{1+y}
\end{aligned}$$

Where $f(y) = \frac{3}{1+y}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\, dp &= \frac{3}{1+y}\, dy \\
\int \frac{1}{p}\, dp &= \int \frac{3}{1+y}\, dy \\
\ln(p) &= 3\ln(1+y) + c_1 \\
p &= e^{3\ln(1+y)+c_1} \\
&= c_1 (1+y)^3
\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $y = 0$ and $p = -\frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = c_1$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$p(y) = -\frac{(1+y)^3}{2}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{(1+y)^3}{2}$$

Integrating both sides gives

$$\begin{aligned}
\int -\frac{2}{(1+y)^3} dy &= c_2 + x \\
\frac{1}{(1+y)^2} &= c_2 + x
\end{aligned}$$

Solving for y gives these solutions

$$y_1 = -\frac{\sqrt{c_2+x}-1}{\sqrt{c_2+x}}$$
$$y_2 = -\frac{\sqrt{c_2+x}+1}{\sqrt{c_2+x}}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-\sqrt{c_2+1}-1}{\sqrt{c_2+1}}$$

Warning: Unable to solve for c_2. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid. Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-\sqrt{c_2+1}+1}{\sqrt{c_2+1}}$$

$$c_2 = 0$$

Substituting c_2 found above in the general solution gives

$$y = \frac{-\sqrt{x}+1}{\sqrt{x}}$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{-\sqrt{x}+1}{\sqrt{x}} \tag{1}$$

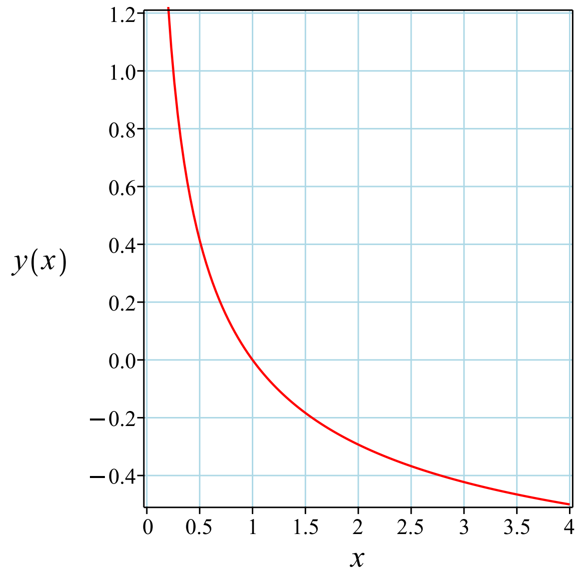


Figure 429: Solution plot

Verification of solutions

$$y = \frac{-\sqrt{x}+1}{\sqrt{x}}$$

Verified OK.

10.16.2 Maple step by step solution

Let's solve

$$\left[(1+y)\, y'' - 3{y'}^{2} = 0, y(1) = 0, y'\Big|_{\{x=1\}} = -\frac{1}{2}\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right)=y''$

- Make substitutions $y'=u(y)\,,y''=u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $(1+y)\,u(y)\left(\frac{d}{dy}u(y)\right)-3u(y)^2=0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)}=\frac{3}{1+y}$

- Integrate both sides with respect to y

  $\int\frac{\frac{d}{dy}u(y)}{u(y)}dy=\int\frac{3}{1+y}dy+c_1$

- Evaluate integral

  $\ln(u(y))=3\ln(1+y)+c_1$

- Solve for $u(y)$

  $u(y)=\mathrm{e}^{c_1}(1+y)^3$

- Solve 1st ODE for $u(y)$

  $u(y)=\mathrm{e}^{c_1}(1+y)^3$

- Revert to original variables with substitution $u(y)=y',y=y$

  $y'=\mathrm{e}^{c_1}(1+y)^3$

- Separate variables

  $\frac{y'}{(1+y)^3}=\mathrm{e}^{c_1}$

- Integrate both sides with respect to x

  $\int\frac{y'}{(1+y)^3}dx=\int\mathrm{e}^{c_1}dx+c_2$

- Evaluate integral

  $-\frac{1}{2(1+y)^2}=x\,\mathrm{e}^{c_1}+c_2$

- Solve for y

  $\left\{y=-\frac{\sqrt{-2x\,\mathrm{e}^{c_1}-2c_2}-1}{\sqrt{-2x\,\mathrm{e}^{c_1}-2c_2}},y=-\frac{\sqrt{-2x\,\mathrm{e}^{c_1}-2c_2}+1}{\sqrt{-2x\,\mathrm{e}^{c_1}-2c_2}}\right\}$

- [ ] Check validity of solution $y=-\frac{\sqrt{-2x\mathrm{e}^{c_1}-2c_2}-1}{\sqrt{-2x\mathrm{e}^{c_1}-2c_2}}$

  - Use initial condition $y(1)=0$

    $0=-\frac{\sqrt{-2\,\mathrm{e}^{c_1}-2c_2}-1}{\sqrt{-2\,\mathrm{e}^{c_1}-2c_2}}$

- Compute derivative of the solution

  $y' = \frac{e^{c_1}}{-2x\, e^{c_1}-2c_2} - \frac{\left(\sqrt{-2x\, e^{c_1}-2c_2}-1\right)e^{c_1}}{\left(-2x\, e^{c_1}-2c_2\right)^{\frac{3}{2}}}$

- Use the initial condition $y'\Big|_{\{x=1\}} = -\frac{1}{2}$

  $-\frac{1}{2} = \frac{e^{c_1}}{-2\, e^{c_1}-2c_2} - \frac{\left(\sqrt{-2\, e^{c_1}-2c_2}-1\right)e^{c_1}}{\left(-2\, e^{c_1}-2c_2\right)^{\frac{3}{2}}}$

- Solve for c_1 and c_2

  $\{c_1 = -\ln(2) + I\pi, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{\sqrt{x}-1}{\sqrt{x}}$

□ Check validity of solution $y = -\frac{\sqrt{-2x e^{c_1}-2c_2}+1}{\sqrt{-2x e^{c_1}-2c_2}}$

- Use initial condition $y(1) = 0$

  $0 = -\frac{\sqrt{-2\, e^{c_1}-2c_2}+1}{\sqrt{-2\, e^{c_1}-2c_2}}$

- Compute derivative of the solution

  $y' = \frac{e^{c_1}}{-2x\, e^{c_1}-2c_2} - \frac{\left(\sqrt{-2x\, e^{c_1}-2c_2}+1\right)e^{c_1}}{\left(-2x\, e^{c_1}-2c_2\right)^{\frac{3}{2}}}$

- Use the initial condition $y'\Big|_{\{x=1\}} = -\frac{1}{2}$

  $-\frac{1}{2} = \frac{e^{c_1}}{-2\, e^{c_1}-2c_2} - \frac{\left(\sqrt{-2\, e^{c_1}-2c_2}+1\right)e^{c_1}}{\left(-2\, e^{c_1}-2c_2\right)^{\frac{3}{2}}}$

- Solve for c_1 and c_2
- The solution does not satisfy the initial conditions

- Solution to the IVP

  $y = -\frac{\sqrt{x}-1}{\sqrt{x}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.344 (sec). Leaf size: 15

```
dsolve([(y(x)+1)*diff(y(x),x$2)=3*(diff(y(x),x))^2,y(1) = 0, D(y)(1) = -1/2],y(x), singsol=a
```

$$y(x) = \frac{-x + \sqrt{x}}{x}$$

✓ Solution by Mathematica

Time used: 1.693 (sec). Leaf size: 572

```
DSolve[{(y[x]+1)*y''[x]==3*(y'[x])^2,{y[1]==0,y'[0]==-1/2}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{6\left(\left(-12 + 3\ 2^{2/3}\sqrt[3]{27-3\sqrt{69}} - \sqrt[3]{2}\left(27-3\sqrt{69}\right)^{2/3} + 3\ 2^{2/3}\sqrt[3]{3\left(9+\sqrt{69}\right)} - \sqrt[3]{2}\left(3\left(9+\sqrt{69}\right)\right)^{2/3}\right)\right.}{}$$

10.17 problem Exercise 35.17, page 504

Internal problem ID [4667]
Internal file name [`OUTPUT/4160_Sunday_June_05_2022_12_31_11_PM_88433444/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.17, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear], [
   _2nd_order, _reducible, _mu_xy]]
```

$$\boxed{y'' - y'\mathrm{e}^{y} = 0}$$

With initial conditions

$$[y(3) = 0, y'(3) = 1]$$

10.17.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - y'\mathrm{e}^{y})\, dx = 0$$
$$-\mathrm{e}^{y} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{e^y + c_1} dy = \int dx$$
$$\frac{\ln(e^y)}{c_1} - \frac{\ln(e^y + c_1)}{c_1} = c_2 + x$$

The above can be written as

$$\left(\frac{1}{c_1}\right)(\ln(e^y) - \ln(e^y + c_1)) = c_2 + x$$
$$\ln(e^y) - \ln(e^y + c_1) = (c_1)(c_2 + x)$$
$$= c_1(c_2 + x)$$

Raising both side to exponential gives

$$e^{\ln(e^y) - \ln(e^y + c_1)} = c_1 c_2 e^{c_1 x}$$

Which simplifies to

$$\frac{e^y}{e^y + c_1} = c_3 e^{c_1 x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \ln\left(-\frac{c_3 c_1}{-1 + c_3 e^{c_1 x}}\right) + c_1 x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 3$ in the above gives

$$0 = \ln\left(-\frac{c_3 c_1}{-1 + c_3 e^{3c_1}}\right) + 3c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_3 e^{c_1 x} c_1}{-1 + c_3 e^{c_1 x}} + c_1$$

substituting $y' = 1$ and $x = 3$ in the above gives

$$1 = -\frac{c_1}{-1 + c_3 e^{3c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

10.17.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) - p(y)\,\mathrm{e}^{y} = 0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned} p(y) &= \int \mathrm{e}^{y}\,\mathrm{d}y \\ &= \mathrm{e}^{y} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $y = 0$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 + 1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$p(y) = \mathrm{e}^{y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \mathrm{e}^{y}$$

Integrating both sides gives

$$\int e^{-y} dy = c_2 + x$$

$$-e^{-y} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \ln\left(-\frac{1}{c_2 + x}\right)$$

Initial conditions are used to solve for c_2. Substituting $x = 3$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln\left(-\frac{1}{c_2 + 3}\right)$$

$$c_2 = -4$$

Substituting c_2 found above in the general solution gives

$$y = \ln\left(-\frac{1}{-4 + x}\right)$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \ln\left(-\frac{1}{-4 + x}\right) \tag{1}$$

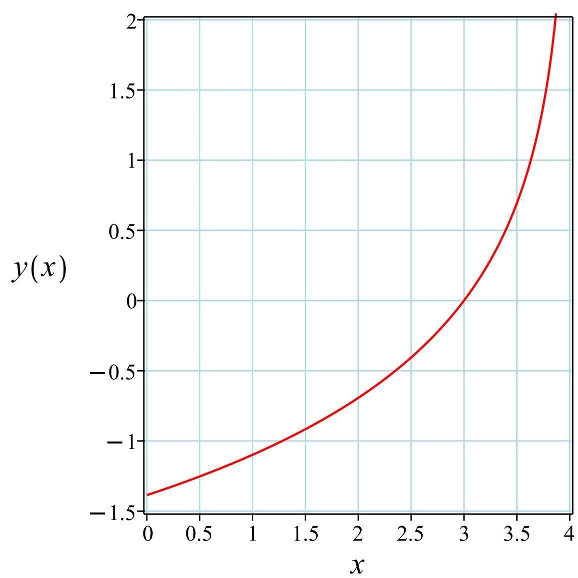


Figure 430: Solution plot

Verification of solutions

$$y = \ln\left(-\frac{1}{-4+x}\right)$$

Verified OK.

10.17.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - y'e^{y} = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - y'e^{y})\, dx = 0$$
$$-e^{y} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{e^{y} + c_1} dy = \int dx$$
$$\frac{\ln(e^{y})}{c_1} - \frac{\ln(e^{y} + c_1)}{c_1} = c_2 + x$$

The above can be written as

$$\left(\frac{1}{c_1}\right)(\ln(e^{y}) - \ln(e^{y} + c_1)) = c_2 + x$$
$$\ln(e^{y}) - \ln(e^{y} + c_1) = (c_1)(c_2 + x)$$
$$= c_1(c_2 + x)$$

Raising both side to exponential gives

$$e^{\ln(e^{y}) - \ln(e^{y} + c_1)} = c_1 c_2 e^{c_1 x}$$

Which simplifies to

$$\frac{e^{y}}{e^{y} + c_1} = c_3 e^{c_1 x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \ln\left(-\frac{c_3 c_1}{-1 + c_3 e^{c_1 x}}\right) + c_1 x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 3$ in the above gives

$$0 = \ln\left(-\frac{c_3 c_1}{-1 + c_3 e^{3c_1}}\right) + 3c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_3 e^{c_1 x} c_1}{-1 + c_3 e^{c_1 x}} + c_1$$

substituting $y' = 1$ and $x = 3$ in the above gives

$$1 = -\frac{c_1}{-1 + c_3 e^{3c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

10.17.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'}\\ \frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'}\\ \frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}a_2 &= 1\\ a_1 &= -e^y\\ a_0 &= 0\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}\int a_2\, dy' + \int a_1\, dy + \int a_0\, dx &= c_1\\ \int 1\, dy' + \int -e^y\, dy + \int 0\, dx &= c_1\end{aligned}$$

Which results in

$$-e^{y}+y'=c_1$$

Which is now solved Integrating both sides gives

$$\int \frac{1}{e^{y}+c_1}dy = \int dx$$
$$\frac{\ln(e^{y})}{c_1}-\frac{\ln(e^{y}+c_1)}{c_1} = c_2+x$$

The above can be written as

$$\left(\frac{1}{c_1}\right)\left(\ln(e^{y})-\ln(e^{y}+c_1)\right) = c_2+x$$
$$\ln(e^{y})-\ln(e^{y}+c_1) = (c_1)(c_2+x)$$
$$= c_1(c_2+x)$$

Raising both side to exponential gives

$$e^{\ln(e^{y})-\ln(e^{y}+c_1)} = c_1 c_2 e^{c_1 x}$$

Which simplifies to

$$\frac{e^{y}}{e^{y}+c_1} = c_3 e^{c_1 x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \ln\left(-\frac{c_3 c_1}{-1+c_3 e^{c_1 x}}\right)+c_1 x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=0$ and $x=3$ in the above gives

$$0 = \ln\left(-\frac{c_3 c_1}{-1+c_3 e^{3c_1}}\right)+3c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_3 e^{c_1 x} c_1}{-1+c_3 e^{c_1 x}}+c_1$$

substituting $y'=1$ and $x=3$ in the above gives

$$1 = -\frac{c_1}{-1+c_3 e^{3c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

10.17.5 Maple step by step solution

Let's solve

$\left[y'' - y' e^{y} = 0, y(3) = 0, y'\Big|_{\{x=3\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y), y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $u(y)\left(\frac{d}{dy}u(y)\right) - u(y)\,e^{y} = 0$

- Separate variables

  $\frac{d}{dy}u(y) = e^{y}$

- Integrate both sides with respect to y

  $\int \left(\frac{d}{dy}u(y)\right) dy = \int e^{y} dy + c_1$

- Evaluate integral

  $u(y) = e^{y} + c_1$

- Solve for $u(y)$

  $u(y) = e^{y} + c_1$

- Solve 1st ODE for $u(y)$

  $u(y) = e^{y} + c_1$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = e^{y} + c_1$

- Separate variables

  $\frac{y'}{e^y+c_1} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{e^y+c_1}dx = \int 1dx + c_2$

- Evaluate integral

  $\frac{\ln(e^y)}{c_1} - \frac{\ln(e^y+c_1)}{c_1} = c_2 + x$

- Solve for y

  $y = c_2c_1 + c_1x + \ln\left(-\frac{c_1}{-1+e^{c_2c_1+c_1x}}\right)$

□ Check validity of solution $y = c_2c_1 + c_1x + \ln\left(-\frac{c_1}{-1+e^{c_2c_1+c_1x}}\right)$

  - Use initial condition $y(3) = 0$

    $0 = c_2c_1 + 3c_1 + \ln\left(-\frac{c_1}{-1+e^{c_2c_1+3c_1}}\right)$

  - Compute derivative of the solution

    $y' = -\frac{c_1e^{c_2c_1+c_1x}}{-1+e^{c_2c_1+c_1x}} + c_1$

  - Use the initial condition $y'\Big|_{\{x=3\}} = 1$

    $1 = -\frac{c_1e^{c_2c_1+3c_1}}{-1+e^{c_2c_1+3c_1}} + c_1$

  - Solve for c_1 and c_2
  - The solution does not satisfy the initial conditions

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-_b(_a)*exp(_a) = 0, _b(_a), HINT
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, _b]
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2)=diff(y(x),x)*exp(y(x)),y(3) = 0, D(y)(3) = 1],y(x), singsol=all)
```

$$y(x) = -\ln(-x+4)$$

✓ Solution by Mathematica

Time used: 7.673 (sec). Leaf size: 13

```
DSolve[{y''[x]==y'[x]*Exp[y[x]],{y[3]==0,y'[3]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\log(4-x)$$

10.18 problem Exercise 35.18, page 504

Internal problem ID [4668]
Internal file name [`OUTPUT/4161_Sunday_June_05_2022_12_31_22_PM_23437431/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.18, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
   _Lagerstrom, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
    _reducible, _mu_xy]]
```

$$\boxed{y'' - 2y'y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 2]$$

10.18.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 2y'y)\,dx = 0$$
$$-y^2 + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{y^2 + c_1} dy = c_2 + x$$
$$\frac{\arctan\left(\frac{y}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = \tan\left(c_2\sqrt{c_1}\right)\sqrt{c_1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1 + \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)^2\right)$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = c_1 \sec\left(c_2\sqrt{c_1}\right)^2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Warning, unable to solve for constants of integrations.

Verification of solutions N/A

10.18.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y)\left(\frac{d}{dy}p(y)\right) - 2p(y)\, y = 0
$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$
\begin{aligned}
p(y) &= \int 2y \; \mathrm{d}y \\
&= y^2 + c_1
\end{aligned}
$$

Initial conditions are used to solve for c_1. Substituting $y = 1$ and $p = 2$ in the above solution gives an equation to solve for the constant of integration.

$$
2 = c_1 + 1
$$

$$
c_1 = 1
$$

Substituting c_1 found above in the general solution gives

$$
p(y) = y^2 + 1
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = 1 + y^2
$$

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{y^2 + 1} dy &= c_2 + x \\
\arctan(y) &= c_2 + x
\end{aligned}
$$

Solving for y gives these solutions

$$
y_1 = \tan(c_2 + x)
$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \tan(c_2)$$

$$c_2 = \frac{\pi}{4}$$

Substituting c_2 found above in the general solution gives

$$y = \tan\left(\frac{\pi}{4} + x\right)$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{\pi}{4} + x\right) \tag{1}$$

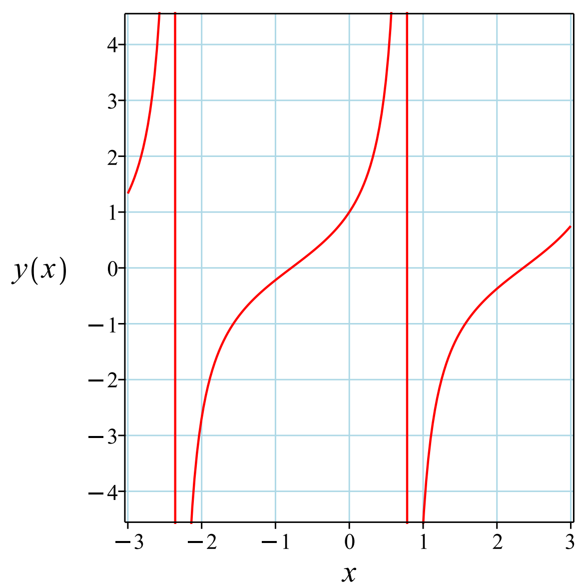


Figure 431: Solution plot

Verification of solutions

$$y = \tan\left(\frac{\pi}{4} + x\right)$$

Verified OK.

10.18.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - 2y'y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 2y'y)\, dx = 0$$
$$-y^2 + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{y^2 + c_1} dy = c_2 + x$$
$$\frac{\arctan\left(\frac{y}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = \tan\left(c_2\sqrt{c_1}\right)\sqrt{c_1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1 + \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)^2\right)$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = c_1 \sec\left(c_2\sqrt{c_1}\right)^2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Warning, unable to solve for constants of integrations.

10.18.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x,y,y')\,y'' + a_1(x,y,y')\,y' + a_0(x,y,y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'}\\ \frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'}\\ \frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}a_2 &= 1\\ a_1 &= -2y\\ a_0 &= 0\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}\int a_2\,dy' + \int a_1\,dy + \int a_0\,dx &= c_1\\ \int 1\,dy' + \int -2y\,dy + \int 0\,dx &= c_1\end{aligned}$$

Which results in

$$-y^2 + y' = c_1$$

Which is now solved Integrating both sides gives

$$\begin{aligned}\int \frac{1}{y^2 + c_1}dy &= c_2 + x\\ \frac{\arctan\left(\frac{y}{\sqrt{c_1}}\right)}{\sqrt{c_1}} &= c_2 + x\end{aligned}$$

Solving for y gives these solutions

$$y_1 = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = \tan\left(c_2\sqrt{c_1}\right)\sqrt{c_1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1 + \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)^2\right)$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = c_1 \sec\left(c_2\sqrt{c_1}\right)^2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Warning, unable to solve for constants of integrations.

Verification of solutions N/A

10.18.5 Maple step by step solution

Let's solve

$\left[y'' - 2y'y = 0, y(0) = 1, y'\Big|_{\{x=0\}} = 2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right)=y''$
- Make substitutions $y'=u(y)\,,y''=u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $u(y)\left(\frac{d}{dy}u(y)\right)-2u(y)\,y=0$
- Separate variables

  $\frac{d}{dy}u(y)=2y$
- Integrate both sides with respect to y

  $\int\left(\frac{d}{dy}u(y)\right)dy=\int 2ydy+c_1$
- Evaluate integral

  $u(y)=y^2+c_1$
- Solve for $u(y)$

  $u(y)=y^2+c_1$
- Solve 1st ODE for $u(y)$

  $u(y)=y^2+c_1$
- Revert to original variables with substitution $u(y)=y',y=y$

  $y'=y^2+c_1$
- Separate variables

  $\frac{y'}{y^2+c_1}=1$
- Integrate both sides with respect to x

  $\int\frac{y'}{y^2+c_1}dx=\int 1dx+c_2$
- Evaluate integral

  $\frac{\arctan\left(\frac{y}{\sqrt{c_1}}\right)}{\sqrt{c_1}}=c_2+x$
- Solve for y

  $y=\tan\left(c_2\sqrt{c_1}+x\sqrt{c_1}\right)\sqrt{c_1}$

□ Check validity of solution $y=\tan\left(c_2\sqrt{c_1}+x\sqrt{c_1}\right)\sqrt{c_1}$

  - Use initial condition $y(0)=1$

    $1=\tan\left(c_2\sqrt{c_1}\right)\sqrt{c_1}$

- Compute derivative of the solution

  $$y' = c_1\left(1+\tan\left(c_2\sqrt{c_1}+x\sqrt{c_1}\right)^2\right)$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 2$

  $$2 = c_1\left(1+\tan\left(c_2\sqrt{c_1}\right)^2\right)$$

- Solve for c_1 and c_2

  $$\left\{c_1 = 1, c_2 = \frac{\pi}{4}\right\}$$

- Substitute constant values into general solution and simplify

  $$y = \tan\left(\frac{\pi}{4}+x\right)$$

- Solution to the IVP

  $$y = \tan\left(\frac{\pi}{4}+x\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 10

```
dsolve([diff(y(x),x$2)=2*y(x)*diff(y(x),x),y(0) = 1, D(y)(0) = 2],y(x), singsol=all)
```

$$y(x) = \tan\left(x+\frac{\pi}{4}\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]==2*y[x]*y'[x],{y[0]==1,y'[0]==2}},y[x],x,IncludeSingularSolutions -> True]
```

{}

10.19 problem Exercise 35.19, page 504

Internal problem ID [4669]
Internal file name [`OUTPUT/4162_Sunday_June_05_2022_12_31_33_PM_72566539/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.19, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{2y'' - e^{y} = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

10.19.1 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$2y'y'' - y'e^{y} = 0$$

Integrating the above w.r.t x gives

$$\int (2y'y'' - y'e^{y})\, dx = 0$$
$$y'^{2} - e^{y} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{e^{y} + c_1} \tag{1}$$
$$y' = -\sqrt{e^{y} + c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{e^y + c_1}} dy = \int dx$$

$$-\frac{2 \operatorname{arctanh}\left(\frac{\sqrt{e^y+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{e^y + c_1}} dy = \int dx$$

$$\frac{2 \operatorname{arctanh}\left(\frac{\sqrt{e^y+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$-\frac{2 \operatorname{arctanh}\left(\frac{\sqrt{e^y+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$-\frac{2 \operatorname{arctanh}\left(\frac{\sqrt{c_1+1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\tanh\left(\frac{c_2\sqrt{c_1}}{2} + \frac{x\sqrt{c_1}}{2}\right) c_1^{\frac{3}{2}} \left(1 - \tanh\left(\frac{c_2\sqrt{c_1}}{2} + \frac{x\sqrt{c_1}}{2}\right)^2\right)}{\tanh\left(\frac{c_2\sqrt{c_1}}{2} + \frac{x\sqrt{c_1}}{2}\right)^2 c_1 - c_1}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \frac{\left(-\mathrm{e}^{c_2\sqrt{c_1}} + 1\right)\sqrt{c_1}}{\mathrm{e}^{c_2\sqrt{c_1}} + 1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$\frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{\mathrm{e}^y + c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$\frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{c_1+1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\tanh\left(\frac{c_3\sqrt{c_1}}{2} + \frac{x\sqrt{c_1}}{2}\right) c_1^{\frac{3}{2}}\left(1 - \tanh\left(\frac{c_3\sqrt{c_1}}{2} + \frac{x\sqrt{c_1}}{2}\right)^2\right)}{\tanh\left(\frac{c_3\sqrt{c_1}}{2} + \frac{x\sqrt{c_1}}{2}\right)^2 c_1 - c_1}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \frac{\left(-\mathrm{e}^{c_3\sqrt{c_1}} + 1\right)\sqrt{c_1}}{\mathrm{e}^{c_3\sqrt{c_1}} + 1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

10.19.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
2p(y)\left(\frac{d}{dy}p(y)\right) = \mathrm{e}^{y}
$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= \frac{\mathrm{e}^{y}}{2p}
\end{aligned}
$$

Where $f(y) = \frac{\mathrm{e}^{y}}{2}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{p}}\, dp &= \frac{\mathrm{e}^{y}}{2}\, dy \\
\int \frac{1}{\frac{1}{p}}\, dp &= \int \frac{\mathrm{e}^{y}}{2}\, dy \\
\frac{p^{2}}{2} &= \frac{\mathrm{e}^{y}}{2} + c_1
\end{aligned}
$$

The solution is

$$
\frac{p(y)^{2}}{2} - \frac{\mathrm{e}^{y}}{2} - c_1 = 0
$$

Initial conditions are used to solve for c_1. Substituting $y = 0$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
-c_1 = 0
$$

$$
c_1 = 0
$$

Substituting c_1 found above in the general solution gives

$$\frac{p^2}{2} - \frac{e^y}{2} = 0$$

Solving for $p(y)$ from the above gives

$$p(y) = e^{\frac{y}{2}}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = e^{\frac{y}{2}}$$

Integrating both sides gives

$$\int e^{-\frac{y}{2}} dy = c_2 + x$$
$$-2\,e^{-\frac{y}{2}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = 2\ln\left(-\frac{2}{c_2 + x}\right)$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 2\ln(2) + 2\ln\left(-\frac{1}{c_2}\right)$$

$$c_2 = -2$$

Substituting c_2 found above in the general solution gives

$$y = 2\ln(2) + 2\ln\left(-\frac{1}{-2 + x}\right)$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = 2\ln(2) + 2\ln\left(-\frac{1}{-2 + x}\right) \tag{1}$$

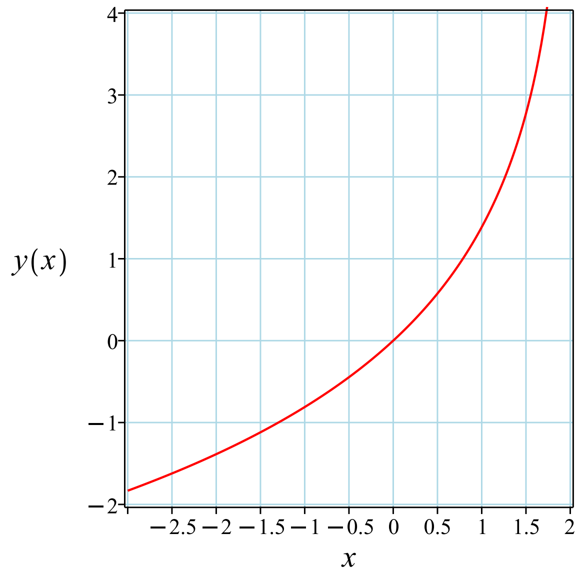


Figure 432: Solution plot

Verification of solutions

$$y = 2\ln(2) + 2\ln\left(-\frac{1}{-2+x}\right)$$

Verified OK.

10.19.3 Maple step by step solution

Let's solve

$$\left[2y'' = e^{y}, y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right)=y''$

- Make substitutions $y'=u(y)\,,y''=u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $2u(y)\left(\frac{d}{dy}u(y)\right)=\mathrm{e}^{y}$

- Integrate both sides with respect to y

  $\int 2u(y)\left(\frac{d}{dy}u(y)\right)dy=\int \mathrm{e}^{y}dy+c_1$

- Evaluate integral

  $u(y)^2=\mathrm{e}^{y}+c_1$

- Solve for $u(y)$

  $\left\{u(y)=\sqrt{\mathrm{e}^{y}+c_1},u(y)=-\sqrt{\mathrm{e}^{y}+c_1}\right\}$

- Solve 1st ODE for $u(y)$

  $u(y)=\sqrt{\mathrm{e}^{y}+c_1}$

- Revert to original variables with substitution $u(y)=y',y=y$

  $y'=\sqrt{\mathrm{e}^{y}+c_1}$

- Separate variables

  $\frac{y'}{\sqrt{\mathrm{e}^{y}+c_1}}=1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{\mathrm{e}^{y}+c_1}}dx=\int 1dx+c_2$

- Evaluate integral

  $-\frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{\mathrm{e}^{y}+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}}=c_2+x$

- Solve for y

  $y=\ln\left(\tanh\left(\frac{c_2\sqrt{c_1}}{2}+\frac{x\sqrt{c_1}}{2}\right)^2c_1-c_1\right)$

- Solve 2nd ODE for $u(y)$

  $u(y)=-\sqrt{\mathrm{e}^{y}+c_1}$

- Revert to original variables with substitution $u(y)=y',y=y$

  $y'=-\sqrt{\mathrm{e}^{y}+c_1}$

- Separate variables

$\frac{y'}{\sqrt{e^y+c_1}} = -1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{e^y+c_1}} dx = \int (-1)\, dx + c_2$

- Evaluate integral

  $-\frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{e^y+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 - x$

- Solve for y

  $y = \ln\left(\tanh\left(\frac{c_2\sqrt{c_1}}{2} - \frac{x\sqrt{c_1}}{2}\right)^2 c_1 - c_1\right)$

□ Check validity of solution $y = \ln\left(\tanh\left(\frac{c_2\sqrt{c_1}}{2} - \frac{x\sqrt{c_1}}{2}\right)^2 c_1 - c_1\right)$

  ○ Use initial condition $y(0) = 0$

  $0 = \ln\left(\tanh\left(\frac{c_2\sqrt{c_1}}{2}\right)^2 c_1 - c_1\right)$

  ○ Compute derivative of the solution

  $y' = -\frac{\tanh\left(\frac{c_2\sqrt{c_1}}{2} - \frac{x\sqrt{c_1}}{2}\right) c_1^{\frac{3}{2}} \left(1 - \tanh\left(\frac{c_2\sqrt{c_1}}{2} - \frac{x\sqrt{c_1}}{2}\right)^2\right)}{\tanh\left(\frac{c_2\sqrt{c_1}}{2} - \frac{x\sqrt{c_1}}{2}\right)^2 c_1 - c_1}$

  ○ Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = -\frac{\tanh\left(\frac{c_2\sqrt{c_1}}{2}\right) c_1^{\frac{3}{2}} \left(1 - \tanh\left(\frac{c_2\sqrt{c_1}}{2}\right)^2\right)}{\tanh\left(\frac{c_2\sqrt{c_1}}{2}\right)^2 c_1 - c_1}$

  ○ Solve for c_1 and c_2

  ○ The solution does not satisfy the initial conditions

□ Check validity of solution $y = \ln\left(\tanh\left(\frac{c_2\sqrt{c_1}}{2} + \frac{x\sqrt{c_1}}{2}\right)^2 c_1 - c_1\right)$

  ○ Use initial condition $y(0) = 0$

  $0 = \ln\left(\tanh\left(\frac{c_2\sqrt{c_1}}{2}\right)^2 c_1 - c_1\right)$

  ○ Compute derivative of the solution

  $y' = \frac{\tanh\left(\frac{c_2\sqrt{c_1}}{2} + \frac{x\sqrt{c_1}}{2}\right) c_1^{\frac{3}{2}} \left(1 - \tanh\left(\frac{c_2\sqrt{c_1}}{2} + \frac{x\sqrt{c_1}}{2}\right)^2\right)}{\tanh\left(\frac{c_2\sqrt{c_1}}{2} + \frac{x\sqrt{c_1}}{2}\right)^2 c_1 - c_1}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

$$1 = \frac{\tanh\left(\frac{c_2\sqrt{c_1}}{2}\right)c_1^{\frac{3}{2}}\left(1-\tanh\left(\frac{c_2\sqrt{c_1}}{2}\right)^2\right)}{\tanh\left(\frac{c_2\sqrt{c_1}}{2}\right)^2 c_1-c_1}$$

- Solve for c_1 and c_2
- The solution does not satisfy the initial conditions

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(1/2)*exp(_a) = 0, _b(_a), HINT
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 1/2*_b]
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 15

```
dsolve([2*diff(y(x),x$2)=exp(y(x)),y(0) = 0, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = 2\ln(2) + \ln\left(\frac{1}{(x-2)^2}\right)$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 15

```
DSolve[{2*y''[x]==Exp[y[x]],{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2\log\left(1-\frac{x}{2}\right)$$

10.20 problem Exercise 35.20, page 504

Internal problem ID [4670]
Internal file name [`OUTPUT/4163_Sunday_June_05_2022_12_31_41_PM_46645455/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.20, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x^2y'' + xy' = 1}$$

With initial conditions

$$[y(1) = 1, y'(1) = 2]$$

10.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = 0$$
$$F = \frac{1}{x^2}$$

Hence the ode is

$$y'' + \frac{y'}{x} = \frac{1}{x^2}$$

The domain of $p(x) = \frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $F = \frac{1}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

10.20.2 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 0, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} + 0 = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r + 0 = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r + 0 = 0$$

Or

$$r^2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 0$$
$$r_2 = 0$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_2 \ln(x) + c_1$$

Next, we find the particular solution to the ODE

$$x^2 y'' + x y' = 1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \ln(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(x) \\ 0 & \frac{1}{x} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{x}\right) - (\ln(x))(0)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)}{x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{1}{x}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{\ln(x)^2}{2} + c_1 + c_2 \ln(x) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{\ln(x)^2}{2} + c_1 + c_2 \ln(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\ln(x)}{x} + \frac{c_2}{x}$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$2 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 2 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{\ln(x)^2}{2} + 1 + 2\ln(x)$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^2}{2} + 1 + 2\ln(x) \tag{1}$$

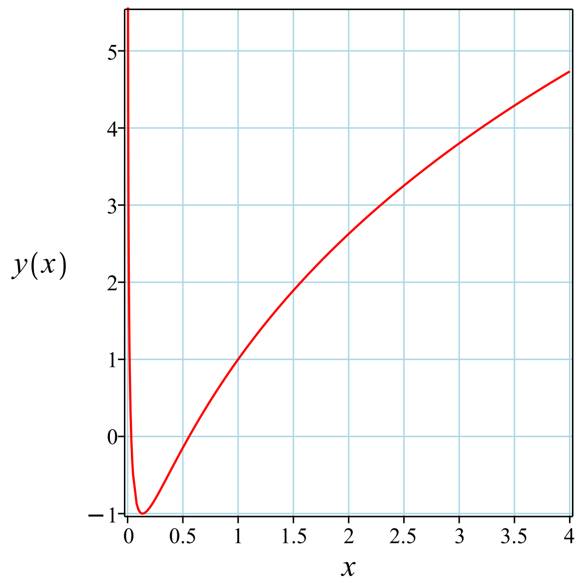


Figure 433: Solution plot

Verification of solutions

$$y = \frac{\ln(x)^2}{2} + 1 + 2\ln(x)$$

Verified OK.

10.20.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$x^2 p'(x) + x p(x) - 1 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int \frac{1}{x} dx}$$
$$= x$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(\frac{1}{x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(x p) &= (x)\left(\frac{1}{x^2}\right) \\
\mathrm{d}(x p) &= \frac{1}{x}\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x p &= \int \frac{1}{x}\,\mathrm{d}x \\
x p &= \ln(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
p(x) = \frac{\ln(x)}{x} + \frac{c_1}{x}
$$

which simplifies to

$$
p(x) = \frac{\ln(x) + c_1}{x}
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $p = 2$ in the above solution gives an equation to solve for the constant of integration.

$$
2 = c_1
$$

$$
c_1 = 2
$$

Substituting c_1 found above in the general solution gives

$$
p(x) = \frac{\ln(x) + 2}{x}
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = \frac{\ln(x) + 2}{x}
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{\ln(x) + 2}{x}\,\mathrm{d}x \\
&= \frac{\ln(x)^2}{2} + 2\ln(x) + c_2
\end{aligned}
$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_2$$

$$c_2 = 1$$

Substituting c_2 found above in the general solution gives

$$y = \frac{\ln(x)^2}{2} + 1 + 2\ln(x)$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^2}{2} + 1 + 2\ln(x) \tag{1}$$

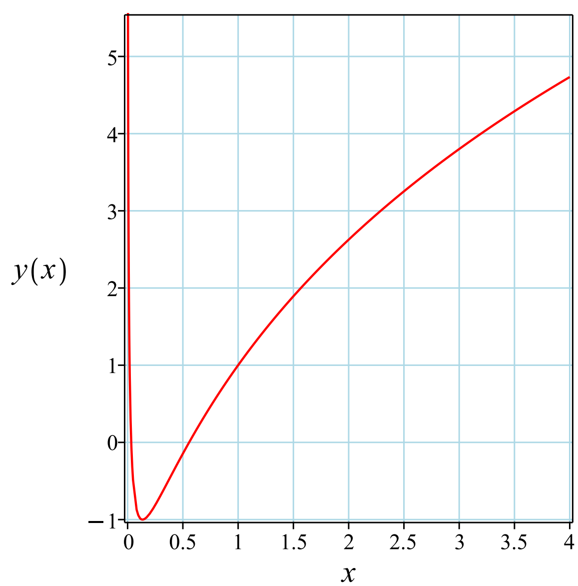


Figure 434: Solution plot

Verification of solutions

$$y = \frac{\ln(x)^2}{2} + 1 + 2\ln(x)$$

Verified OK.

10.20.4 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + xy' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 315: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(\ln(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
x^2y'' + xy' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2 \ln(x) + c_1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \ln(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(x) \\ 0 & \frac{1}{x} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{x}\right) - (\ln(x))(0)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\left(\int \frac{\ln(x)}{x} dx\right)$$

Which simplifies to

$$u_1 = -\left(\int \frac{\ln(x)}{x} dx\right)$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{1}{x} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 \ln(x) + c_1) + \left(\frac{\ln(x)^2}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{\ln(x)^2}{2} + c_1 + c_2 \ln(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\ln(x)}{x} + \frac{c_2}{x}$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$2 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = \frac{\ln(x)^2}{2} + 1 + 2\ln(x)$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^2}{2} + 1 + 2\ln(x) \tag{1}$$

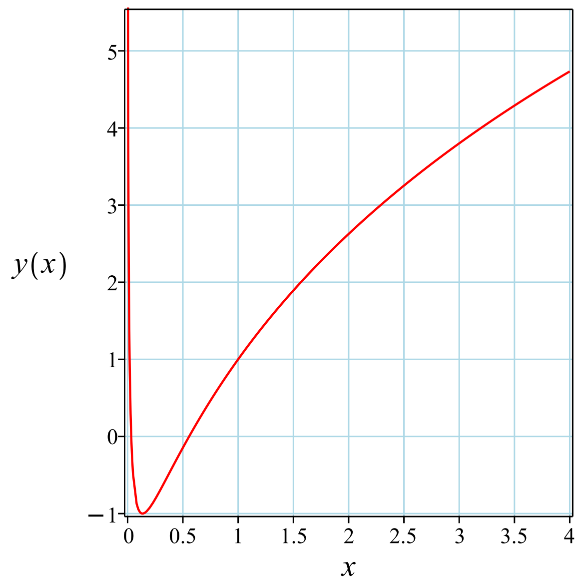


Figure 435: Solution plot

Verification of solutions

$$y = \frac{\ln(x)^2}{2} + 1 + 2\ln(x)$$

Verified OK.

10.20.5 Maple step by step solution

Let's solve

$$\left[x^2y'' + xy' = 1, y(1) = 1, y'\Big|_{\{x=1\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $x^2u'(x) + xu(x) = 1$

- Isolate the derivative

  $u'(x) = -\frac{u(x)}{x} + \frac{1}{x^2}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) + \frac{u(x)}{x} = \frac{1}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x)+\frac{u(x)}{x}\right)=\frac{\mu(x)}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x)+\frac{u(x)}{x}\right)=\mu'(x)\,u(x)+\mu(x)\,u'(x)$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x)=x$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,u(x))\right)dx=\int\frac{\mu(x)}{x^2}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,u(x)=\int\frac{\mu(x)}{x^2}dx+c_1$

- Solve for $u(x)$

  $u(x)=\frac{\int\frac{\mu(x)}{x^2}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=x$

  $u(x)=\frac{\int\frac{1}{x}dx+c_1}{x}$

- Evaluate the integrals on the rhs

  $u(x)=\frac{\ln(x)+c_1}{x}$

- Solve 1st ODE for $u(x)$

  $u(x)=\frac{\ln(x)+c_1}{x}$

- Make substitution $u=y'$

  $y'=\frac{\ln(x)+c_1}{x}$

- Integrate both sides to solve for y

  $\int y'dx=\int\frac{\ln(x)+c_1}{x}dx+c_2$

- Compute integrals

  $y=\frac{\ln(x)^2}{2}+c_1\ln(x)+c_2$

□ Check validity of solution $y=\frac{\ln(x)^2}{2}+c_1\ln(x)+c_2$

  - Use initial condition $y(1)=1$

$1 = c_2$

- Compute derivative of the solution

  $y' = \frac{\ln(x)}{x} + \frac{c_1}{x}$

- Use the initial condition $y'\Big|_{\{x=1\}} = 2$

  $2 = c_1$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = \frac{\ln(x)^2}{2} + 1 + 2\ln(x)$

- Solution to the IVP

  $y = \frac{\ln(x)^2}{2} + 1 + 2\ln(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([x^2*diff(y(x),x$2)+x*diff(y(x),x)=1,y(1) = 1, D(y)(1) = 2],y(x), singsol=all)
```

$$y(x) = 1 + 2\ln(x) + \frac{\ln(x)^2}{2}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 19

```
DSolve[{x^2*y''[x]+x*y'[x]==1,{y[1]==1,y'[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(\log^2(x) + 4\log(x) + 2\right)$$

10.21 problem Exercise 35.21, page 504

Internal problem ID [4671]
Internal file name [`OUTPUT/4164_Sunday_June_05_2022_12_31_50_PM_54637956/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.21, page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{xy'' - y' = x^2}$$

With initial conditions

$$[y(1) = 0, y'(1) = -1]$$

10.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 0 \\ F &= x \end{aligned}$$

Hence the ode is

$$y'' - \frac{y'}{x} = x$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $F = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

10.21.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned} \int (xy'' - y')\, dx &= \int x^2 dx \\ xy' - 2y &= \frac{x^3}{3} + c_1 \end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{x^3 + 3c_1}{3x} \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{x^3 + 3c_1}{3x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^3 + 3c_1}{3x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{x^3 + 3c_1}{3x}\right) \\ \mathrm{d}\left(\frac{y}{x^2}\right) &= \left(\frac{x^3 + 3c_1}{3x^3}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2} &= \int \frac{x^3 + 3c_1}{3x^3}\,\mathrm{d}x \\ \frac{y}{x^2} &= \frac{x}{3} - \frac{c_1}{2x^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(\frac{x}{3} - \frac{c_1}{2x^2}\right) + c_2x^2$$

which simplifies to

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = \frac{1}{3} - \frac{c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_2x + x^2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = 2c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{4}{3}$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2$$

Summary
The solution(s) found are the following

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2 \tag{1}$$

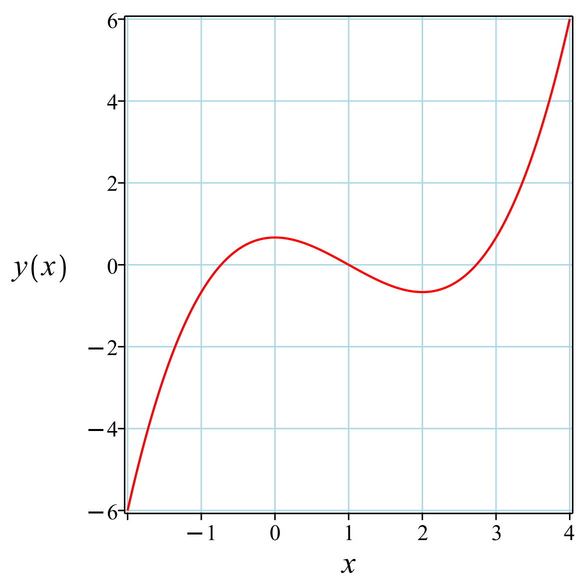


Figure 436: Solution plot

Verification of solutions

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2$$

Verified OK.

10.21.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)\, x - p(x) - x^2 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx}\\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu p) &= (\mu)\,(x)\\ \frac{d}{dx}\left(\frac{p}{x}\right) &= \left(\frac{1}{x}\right)(x)\\ d\left(\frac{p}{x}\right) &= dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{p}{x} &= \int dx\\ \frac{p}{x} &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x + x^2$$

which simplifies to

$$p(x) = x(x + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $p = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1 + 1$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$p(x) = x(-2 + x)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x(-2 + x)$$

Integrating both sides gives

$$\begin{aligned} y &= \int x(-2 + x)\,\mathrm{d}x \\ &= \frac{1}{3}x^3 - x^2 + c_2 \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{2}{3} + c_2$$

$$c_2 = \frac{2}{3}$$

Substituting c_2 found above in the general solution gives

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2 \tag{1}$$

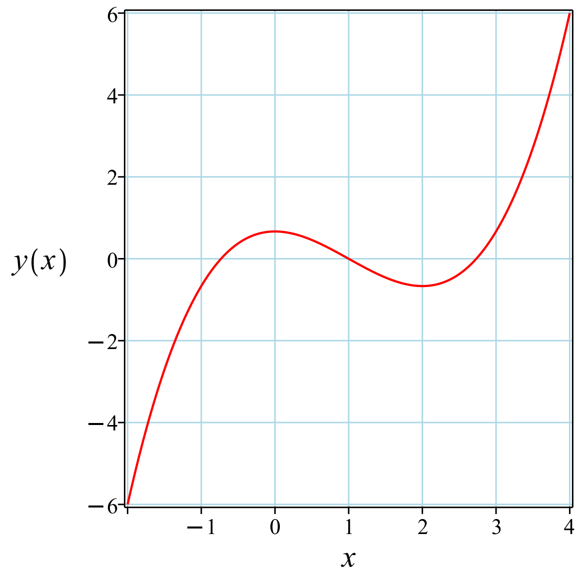


Figure 437: Solution plot

Verification of solutions

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2$$

Verified OK.

10.21.4 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x \\
B &= -1 \\
C &= 0 \\
F &= x^2
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= (x)(0) + (-1)(0) + (0)(-1) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-xv'' + (1) v' = 0$$

Now by applying $v' = u$ the above becomes

$$-xu'(x) + u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= \frac{u}{x}
\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u} \,du &= \frac{1}{x} \,dx \\
\int \frac{1}{u} \,du &= \int \frac{1}{x} \,dx \\
\ln(u) &= \ln(x) + c_1 \\
u &= e^{\ln(x) + c_1} \\
&= c_1 x
\end{aligned}$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1 x
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int c_1 x \, \mathrm{d}x \\
&= \frac{c_1 x^2}{2} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (-1) \left(\frac{c_1 x^2}{2} + c_2\right) \\
&= -\frac{c_1 x^2}{2} - c_2
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= -1 \\
y_2 &= x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -1 & x^2 \\ \frac{d}{dx}(-1) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -1 & x^2 \\ 0 & 2x \end{vmatrix}$$

Therefore

$$W = (-1)(2x) - (x^2)(0)$$

Which simplifies to

$$W = -2x$$

Which simplifies to

$$W = -2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^4}{-2x^2}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{x^2}{2}dx$$

Hence

$$u_1 = \frac{x^3}{6}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-x^2}{-2x^2}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{2}dx$$

Hence

$$u_2 = \frac{x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^3}{3}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-\frac{c_1 x^2}{2} - c_2\right) + \left(\frac{x^3}{3}\right) \\ &= -\frac{1}{2} c_1 x^2 - c_2 + \frac{1}{3} x^3 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{1}{2} c_1 x^2 - c_2 + \frac{1}{3} x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = -\frac{c_1}{2} - c_2 + \frac{1}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 x + x^2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = 1 - c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 2 \\ c_2 &= -\frac{2}{3} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2$$

Summary

The solution(s) found are the following

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2 \tag{1}$$

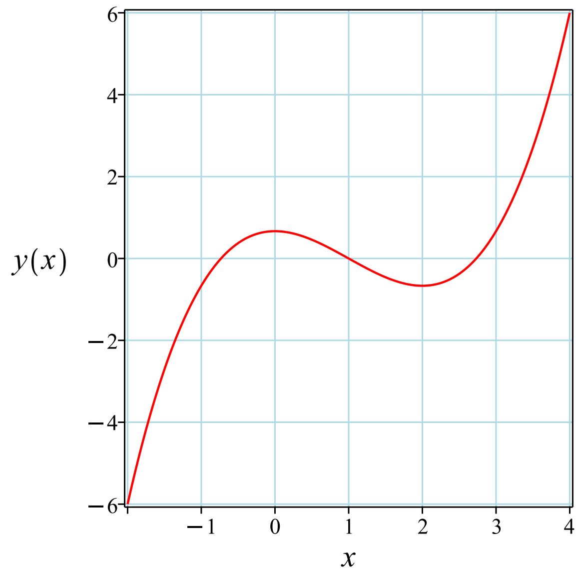


Figure 438: Solution plot

Verification of solutions

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2$$

Verified OK.

10.21.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xy'' - y' = x^2$$

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' - y')\, dx = \int x^2 dx$$
$$xy' - 2y = \frac{x^3}{3} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{x^3 + 3c_1}{3x}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{x^3 + 3c_1}{3x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x} dx}$$
$$= \frac{1}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^3 + 3c_1}{3x}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \left(\frac{1}{x^2}\right)\left(\frac{x^3 + 3c_1}{3x}\right)$$
$$d\left(\frac{y}{x^2}\right) = \left(\frac{x^3 + 3c_1}{3x^3}\right) dx$$

Integrating gives

$$\frac{y}{x^2} = \int \frac{x^3 + 3c_1}{3x^3} dx$$
$$\frac{y}{x^2} = \frac{x}{3} - \frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(\frac{x}{3} - \frac{c_1}{2x^2}\right) + c_2x^2$$

which simplifies to

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = \frac{1}{3} - \frac{c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_2x + x^2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = 2c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{4}{3}$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2$$

Summary

The solution(s) found are the following

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2 \tag{1}$$

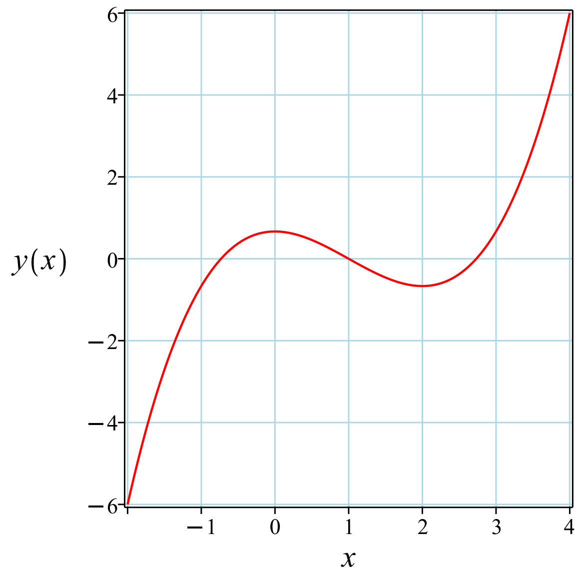


Figure 439: Solution plot

Verification of solutions

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2$$

Verified OK.

10.21.6 Solving using Kovacic algorithm

Writing the ode as

$$xy'' - y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 317: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{x}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
xy'' - y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 + \frac{c_2 x^2}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \frac{x^2}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(\frac{x^2}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ 0 & x \end{vmatrix}$$

Therefore

$$W = (1)(x) - \left(\frac{x^2}{2}\right)(0)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^4}{2}}{x^2} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^2}{2} dx$$

Hence

$$u_1 = -\frac{x^3}{6}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^2} dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^3}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 x^2}{2}\right) + \left(\frac{x^3}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + \frac{1}{2} c_2 x^2 + \frac{1}{3} x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = c_1 + \frac{c_2}{2} + \frac{1}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2 x + x^2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{2}{3}$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2$$

Summary

The solution(s) found are the following

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2 \tag{1}$$

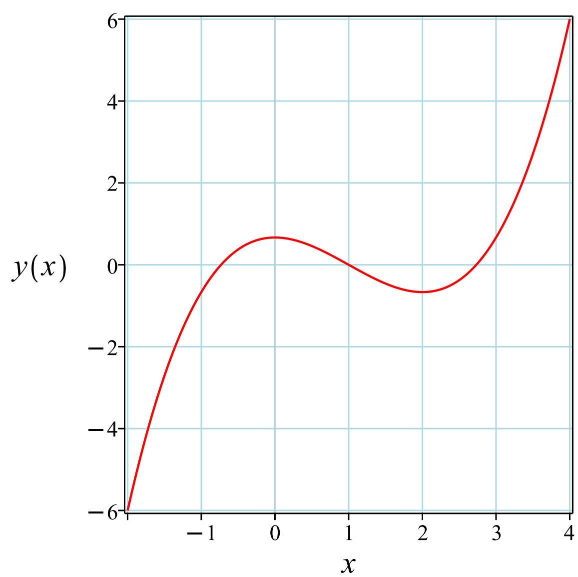


Figure 440: Solution plot

Verification of solutions

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2$$

Verified OK.

10.21.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x \\ q(x) &= -1 \\ r(x) &= 0 \\ s(x) &= x^2 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$xy' - 2y = \int x^2\,dx$$

We now have a first order ode to solve which is

$$xy' - 2y = \frac{x^3}{3} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{x^3 + 3c_1}{3x} \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{x^3 + 3c_1}{3x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^3 + 3c_1}{3x}\right) \\ \frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{x^3 + 3c_1}{3x}\right) \\ d\left(\frac{y}{x^2}\right) &= \left(\frac{x^3 + 3c_1}{3x^3}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x^2} &= \int \frac{x^3 + 3c_1}{3x^3} dx \\ \frac{y}{x^2} &= \frac{x}{3} - \frac{c_1}{2x^2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(\frac{x}{3} - \frac{c_1}{2x^2}\right) + c_2x^2$$

which simplifies to

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = \frac{1}{3} - \frac{c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_2 x + x^2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = 2c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{4}{3} \\ c_2 &= -1 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2 \tag{1}$$

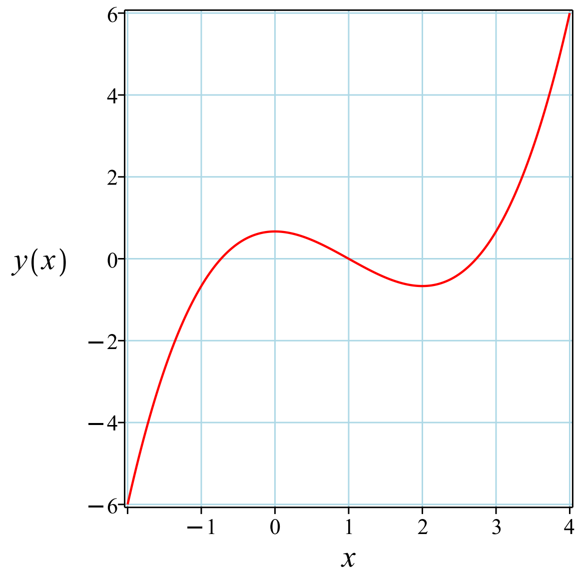


Figure 441: Solution plot

Verification of solutions

$$y = \frac{1}{3}x^3 + \frac{2}{3} - x^2$$

Verified OK.

10.21.8 Maple step by step solution

Let's solve

$$\left[y''x - y' = x^2, y(1) = 0, y'\Big|_{\{x=1\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x)\,x - u(x) = x^2$

- Isolate the derivative

  $u'(x) = \frac{u(x)}{x} + x$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) - \frac{u(x)}{x} = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(u'(x)-\frac{u(x)}{x}\right)=\mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x)-\frac{u(x)}{x}\right)=\mu'(x)\,u(x)+\mu(x)\,u'(x)$
- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x)=\frac{1}{x}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,u(x))\right)dx=\int\mu(x)\,xdx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,u(x)=\int\mu(x)\,xdx+c_1$
- Solve for $u(x)$

  $u(x)=\frac{\int\mu(x)xdx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\frac{1}{x}$

  $u(x)=x\left(\int 1dx+c_1\right)$
- Evaluate the integrals on the rhs

  $u(x)=x(x+c_1)$
- Solve 1st ODE for $u(x)$

  $u(x)=x(x+c_1)$
- Make substitution $u=y'$

  $y'=x(x+c_1)$
- Integrate both sides to solve for y

  $\int y'dx=\int x(x+c_1)\,dx+c_2$
- Compute integrals

  $y=\frac{1}{3}x^3+\frac{1}{2}c_1x^2+c_2$

☐ Check validity of solution $y=\frac{1}{3}x^3+\frac{1}{2}c_1x^2+c_2$

- ○ Use initial condition $y(1)=0$

  $0=\frac{1}{3}+\frac{c_1}{2}+c_2$

- Compute derivative of the solution

  $y' = c_1 x + x^2$

- Use the initial condition $y'\Big|_{\{x=1\}} = -1$

  $-1 = c_1 + 1$

- Solve for c_1 and c_2

  $\{c_1 = -2, c_2 = \frac{2}{3}\}$

- Substitute constant values into general solution and simplify

  $y = \frac{1}{3}x^3 + \frac{2}{3} - x^2$

- Solution to the IVP

  $y = \frac{1}{3}x^3 + \frac{2}{3} - x^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve([x*diff(y(x),x$2)-diff(y(x),x)=x^2,y(1) = 0, D(y)(1) = -1],y(x), singsol=all)
```

$$y(x) = \frac{1}{3}x^3 - x^2 + \frac{2}{3}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 19

```
DSolve[{x*y''[x]-y'[x]==x^2,{y[1]==0,y'[1]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}(x^3 - 3x^2 + 2)$$

10.22 problem Exercise 35.23(a), page 504

Internal problem ID [4672]
Internal file name [`OUTPUT/4165_Sunday_June_05_2022_12_32_02_PM_91977113/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.23(a), page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_nonlinear_solved_by_mainardi_lioville_method"**

Maple gives the following as the ode type

```
[_Liouville, [_2nd_order, _with_linear_symmetries], [_2nd_order
   , _reducible, _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{xyy'' - 2xy'^2 + y'y = 0}$$

10.22.1 Solving as second order nonlinear solved by mainardi lioville method ode

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y)y'^2 = 0 \tag{1A}$$

Where in this problem

$$f(x) = \frac{1}{x}$$
$$g(y) = -\frac{2}{y}$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx} \ln(y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned} g\frac{dy}{dx} &= \left(\frac{d}{dy} \int g dy\right) \frac{dy}{dx} \\ &= \frac{d}{dx} \int g dy \end{aligned} \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx} \ln(y') + \frac{d}{dx} \int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln(y') + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy} e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = -\frac{2}{y}$ and $f = \frac{1}{x}$, then

$$\begin{aligned} \int -g dy &= \int \frac{2}{y} dy \\ &= 2\ln(y) \\ \int -f dx &= \int -\frac{1}{x} dx \\ &= -\ln(x) \end{aligned}$$

Substituting the above into Eq(6A) gives

$$y' = \frac{c_2 y^2}{x}$$

Which is now solved as first order separable ode. In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{c_2 y^2}{x} \end{aligned}$$

Where $f(x) = \frac{c_2}{x}$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = \frac{c_2}{x}\,dx$$
$$\int \frac{1}{y^2}\,dy = \int \frac{c_2}{x}\,dx$$
$$-\frac{1}{y} = c_2 \ln(x) + c_3$$

The solution is

$$-\frac{1}{y} - c_2 \ln(x) - c_3 = 0$$

Summary

The solution(s) found are the following

$$-\frac{1}{y} - c_2 \ln(x) - c_3 = 0 \tag{1}$$

Verification of solutions

$$-\frac{1}{y} - c_2 \ln(x) - c_3 = 0$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve(x*y(x)*diff(y(x),x$2)-2*x*(diff(y(x),x))^2+y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = -\frac{1}{c_1 \ln(x) + c_2}$$

✓ Solution by Mathematica

Time used: 0.243 (sec). Leaf size: 22

```
DSolve[x*y[x]*y''[x]-2*x*(y'[x])^2+y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2}{-\log(x) + c_1}$$
$$y(x) \to 0$$

10.23 problem Exercise 35.23(b), page 504

Internal problem ID [4673]
Internal file name [`OUTPUT/4166_Sunday_June_05_2022_12_32_06_PM_94626859/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.23(b), page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_nonlinear_solved_by_mainardi_lioville_method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _nonlinear], _Liouville, [_2nd_order,
  _with_linear_symmetries], [_2nd_order, _reducible, _mu_x_y1],
   [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{xyy'' + xy'^2 - y'y = 0}$$

10.23.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (xyy'' + (-y + xy')\, y')\, dx = 0$$
$$xyy' - y^2 = c_1$$

Which is now solved for y. In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2 + c_1}{xy} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \frac{y^2+c_1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^2+c_1}{y}}\,dy = \frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{y^2+c_1}{y}}\,dy = \int \frac{1}{x}\,dx$$

$$\frac{\ln(y^2+c_1)}{2} = \ln(x) + c_2$$

Raising both side to exponential gives

$$\sqrt{y^2+c_1} = e^{\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{y^2+c_1} = c_3 x$$

Which simplifies to

$$\sqrt{y^2+c_1} = c_3 x\,e^{c_2}$$

The solution is

$$\sqrt{y^2+c_1} = c_3 x\,e^{c_2}$$

Summary

The solution(s) found are the following

$$\sqrt{y^2+c_1} = c_3 x\,e^{c_2} \tag{1}$$

Verification of solutions

$$\sqrt{y^2+c_1} = c_3 x\,e^{c_2}$$

Verified OK.

10.23.2 Solving as second order nonlinear solved by mainardi lioville method ode

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y)y'^2 = 0 \tag{1A}$$

Where in this problem

$$\begin{aligned}f(x) &= -\frac{1}{x}\\ g(y) &= \frac{1}{y}\end{aligned}$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx}\ln(y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned}g\frac{dy}{dx} &= \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx}\\ &= \frac{d}{dx}\int g dy \end{aligned}\tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx}\ln(y') + \frac{d}{dx}\int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln(y') + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy}\, e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = \frac{1}{y}$ and $f = -\frac{1}{x}$, then

$$\int -g dy = \int -\frac{1}{y} dy$$
$$= -\ln(y)$$
$$\int -f dx = \int \frac{1}{x} dx$$
$$= \ln(x)$$

Substituting the above into Eq(6A) gives

$$y' = \frac{c_2 x}{y}$$

Which is now solved as first order separable ode. In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{c_2 x}{y} \end{aligned}$$

Where $f(x) = c_2 x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}} \, dy = c_2 x \, dx$$

$$\int \frac{1}{\frac{1}{y}} \, dy = \int c_2 x \, dx$$

$$\frac{y^2}{2} = \frac{c_2 x^2}{2} + c_3$$

The solution is

$$\frac{y^2}{2} - \frac{c_2 x^2}{2} - c_3 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2} - \frac{c_2 x^2}{2} - c_3 = 0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2} - \frac{c_2 x^2}{2} - c_3 = 0$$

Verified OK.

10.23.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xyy'' + (-y + xy')\, y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (xyy'' + (-y + xy')\, y')\, dx = 0$$
$$xyy' - y^2 = c_1$$

Which is now solved for y. In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y^2 + c_1}{xy} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \frac{y^2+c_1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^2+c_1}{y}}\, dy = \frac{1}{x}\, dx$$
$$\int \frac{1}{\frac{y^2+c_1}{y}}\, dy = \int \frac{1}{x}\, dx$$
$$\frac{\ln(y^2 + c_1)}{2} = \ln(x) + c_2$$

Raising both side to exponential gives

$$\sqrt{y^2 + c_1} = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{y^2 + c_1} = c_3 x$$

Which simplifies to

$$\sqrt{y^2 + c_1} = c_3 x\, \mathrm{e}^{c_2}$$

The solution is

$$\sqrt{y^2 + c_1} = c_3 x\, \mathrm{e}^{c_2}$$

Summary
The solution(s) found are the following

$$\sqrt{y^2+c_1} = c_3 x\, e^{c_2} \tag{1}$$

Verification of solutions

$$\sqrt{y^2+c_1} = c_3 x\, e^{c_2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 35

```
dsolve(x*y(x)*diff(y(x),x$2)+x*(diff(y(x),x))^2-y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = \sqrt{c_1 x^2 + 2c_2}$$
$$y(x) = -\sqrt{c_1 x^2 + 2c_2}$$

✓ Solution by Mathematica
Time used: 0.241 (sec). Leaf size: 18

```
DSolve[x*y[x]*y''[x]+x*(y'[x])^2-y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \sqrt{x^2 + c_1}$$

10.24 problem Exercise 35.23(c), page 504

Internal problem ID [4674]
Internal file name [`OUTPUT/4167_Sunday_June_05_2022_12_32_17_PM_91608857/index.tex`]

Book: Ordinary Differential Equations, By Tenenbaum and Pollard. Dover, NY 1963
Section: Chapter 8. Special second order equations. Lesson 35. Independent variable x absent
Problem number: Exercise 35.23(c), page 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _reducible
   , _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

Unable to solve or complete the solution.

$$\boxed{xyy'' - 2xy'^2 + (1 + y)\, y' = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integrating f
   --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for dynam
   -> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integratin
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cases
trying symmetries linear in x and y(x)
`, `-> Computing symmetries using: way = 3
Try integration with the canonical coordinates of the symmetry [x, 0]
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _b(_a)*(_b(_a)-2)/_a, _b(_a), explicit,
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 0]
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 22

```
dsolve(x*y(x)*diff(y(x),x$2)-2*x*(diff(y(x),x))^2+(1+y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1 \tanh\left(\frac{\ln(x) - c_2}{2c_1}\right)$$

✓ Solution by Mathematica

Time used: 20.549 (sec). Leaf size: 52

```
DSolve[x*y[x]*y''[x]-2*x*(y'[x])^2+(1+y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{\tan\left(\frac{\sqrt{c_1}(\log(x)-c_2)}{\sqrt{2}}\right)}{\sqrt{2}\sqrt{c_1}}$$
$$y(x) \to \frac{1}{2}(\log(x)-c_2)$$