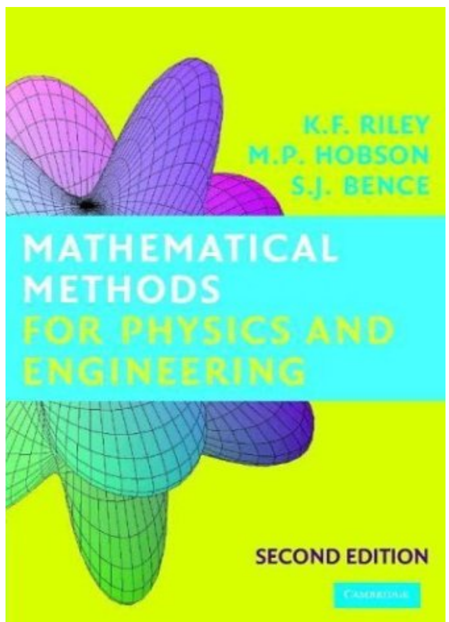
A Solution Manual For

Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002



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1 Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490

1.1 problem Problem 14.2 (a)

Internal problem ID [2486]
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Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.2 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - xy^3 = 0}$$

1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= x\,y^3 \end{aligned}$$

Where $f(x) = x$ and $g(y) = y^3$. Integrating both sides gives

$$\frac{1}{y^3}\,dy = x\,dx$$

$$\int \frac{1}{y^3}\,dy = \int x\,dx$$

$$-\frac{1}{2y^2} = \frac{x^2}{2} + c_1$$

Which results in

$$y = -\frac{1}{\sqrt{-x^2 - 2c_1}}$$

$$y = \frac{1}{\sqrt{-x^2 - 2c_1}}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{\sqrt{-x^2 - 2c_1}} \tag{1}$$

$$y = \frac{1}{\sqrt{-x^2 - 2c_1}} \tag{2}$$

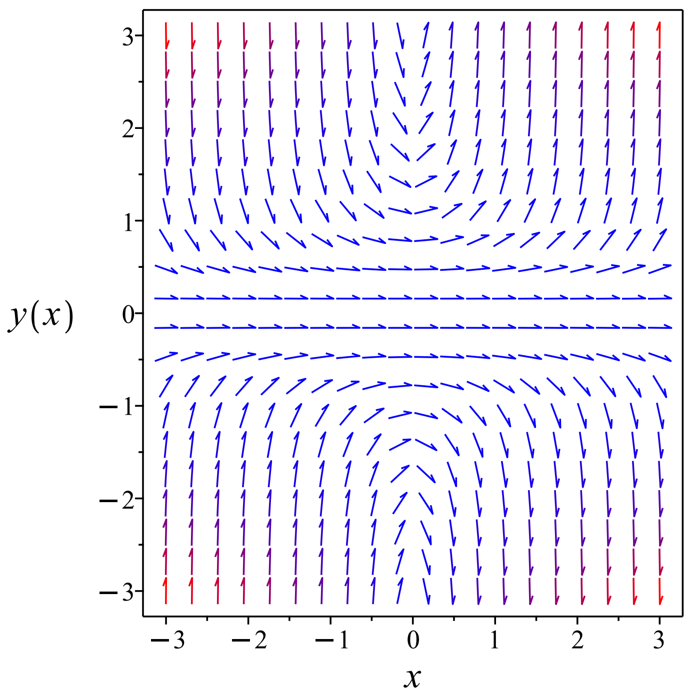


Figure 1: Slope field plot

Verification of solutions

$$y = -\frac{1}{\sqrt{-x^2 - 2c_1}}$$

Verified OK.

$$y = \frac{1}{\sqrt{-x^2 - 2c_1}}$$

Verified OK.

1.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= x\, y^3 \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x\, y^3$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \tag{4}$$

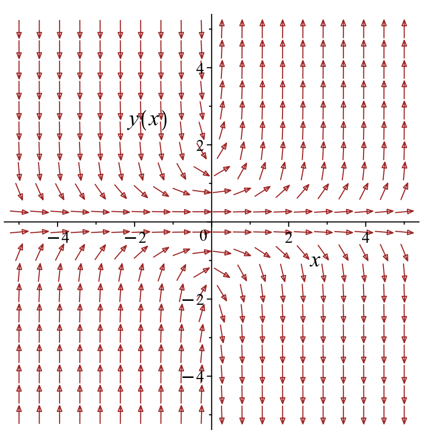
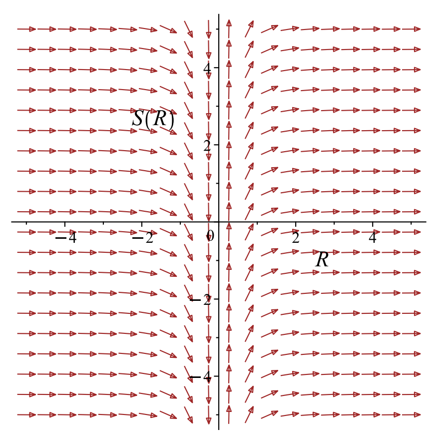
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x\, y^3$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Summary

The solution(s) found are the following

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1 \tag{1}$$

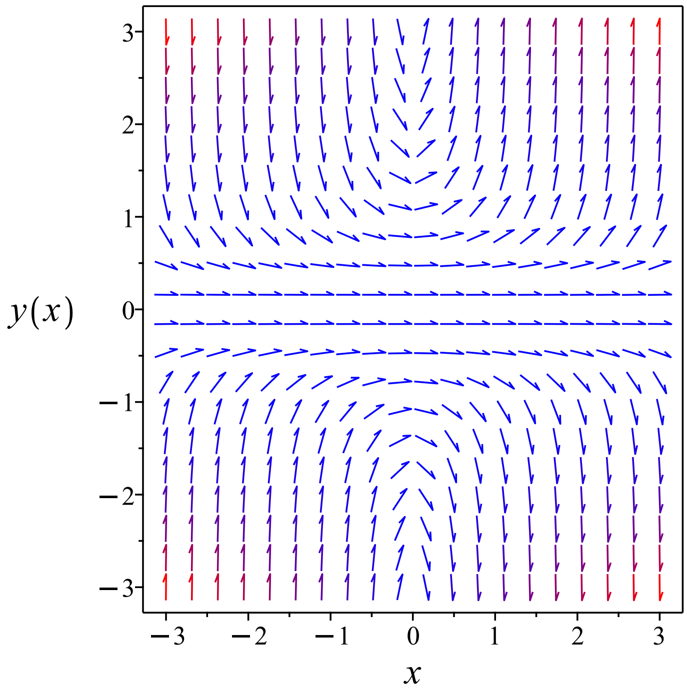


Figure 2: Slope field plot

Verification of solutions

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1$$

Verified OK.

1.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^3}\right)\mathrm{d}y &= (x)\,\mathrm{d}x \\ (-x)\,\mathrm{d}x + \left(\frac{1}{y^3}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x \\ N(x,y) &= \frac{1}{y^3}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^3}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -x \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^3}$. Therefore equation (4) becomes

$$
\frac{1}{y^3} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^3}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^3}\right) \mathrm{d}y$$

$$f(y) = -\frac{1}{2y^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{1}{2y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{1}{2y^2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} - \frac{1}{2y^2} = c_1 \tag{1}$$

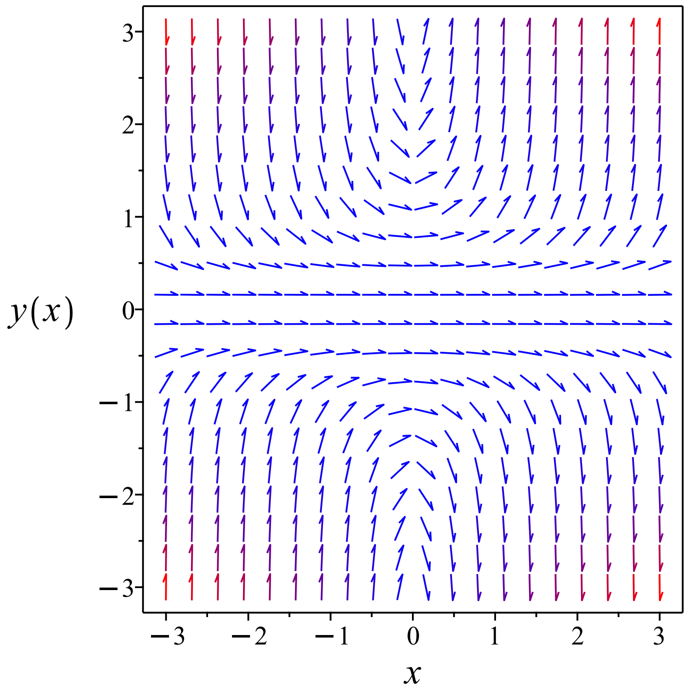


Figure 3: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} - \frac{1}{2y^2} = c_1$$

Verified OK.

1.1.4 Maple step by step solution

Let's solve

$y' - xy^3 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3} dx = \int x dx + c_1$

- Evaluate integral

$$-\frac{1}{2y^2} = \frac{x^2}{2} + c_1$$

- Solve for y

$$\left\{y = \frac{1}{\sqrt{-x^2 - 2c_1}}, y = -\frac{1}{\sqrt{-x^2 - 2c_1}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x)-x*y(x)^3=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{-x^2 + c_1}}$$
$$y(x) = -\frac{1}{\sqrt{-x^2 + c_1}}$$

✓ Solution by Mathematica

Time used: 0.17 (sec). Leaf size: 44

```
DSolve[y'[x]-x*y[x]^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{1}{\sqrt{-x^2 - 2c_1}}$$
$$y(x) \rightarrow \frac{1}{\sqrt{-x^2 - 2c_1}}$$
$$y(x) \rightarrow 0$$

1.2 problem Problem 14.2 (b)

Internal problem ID [2487]
Internal file name [`OUTPUT/1979_Sunday_June_05_2022_02_42_00_AM_28281360/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.2 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\frac{y'}{\tan(x)}-\frac{y}{x^2+1}=0}$$

1.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= \frac{y\tan(x)}{x^2+1}
\end{aligned}$$

Where $f(x) = \frac{\tan(x)}{x^2+1}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\, dy &= \frac{\tan(x)}{x^2+1}\, dx \\
\int \frac{1}{y}\, dy &= \int \frac{\tan(x)}{x^2+1}\, dx \\
\ln(y) &= \int \frac{\tan(x)}{x^2+1} dx + c_1 \\
y &= e^{\int \frac{\tan(x)}{x^2+1} dx + c_1} \\
&= c_1 e^{\int \frac{\tan(x)}{x^2+1} dx}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\int \frac{\tan(x)}{x^2+1} dx} \tag{1}$$

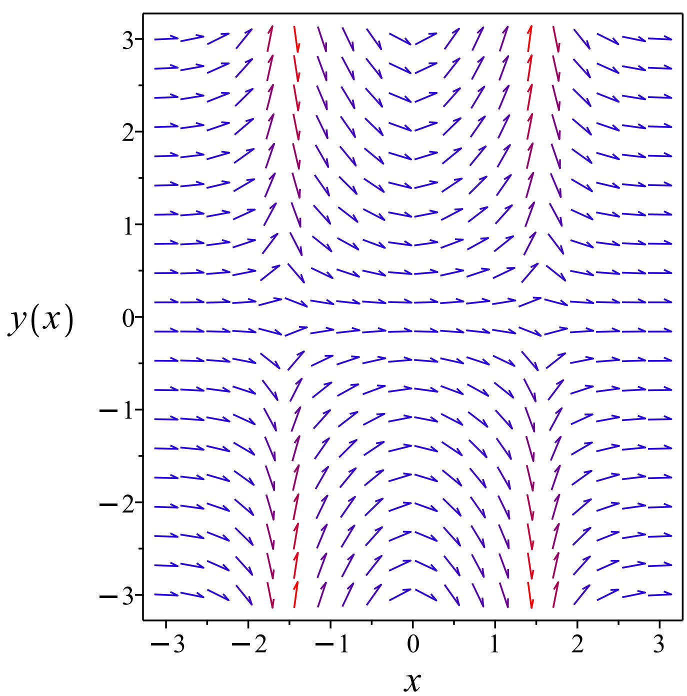


Figure 4: Slope field plot

Verification of solutions

$$y = c_1 e^{\int \frac{\tan(x)}{x^2+1} dx}$$

Verified OK.

1.2.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{\tan(x)}{x^2+1} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{y\tan(x)}{x^2+1} = 0$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{\tan(x)}{x^2+1}dx}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\int -\frac{\tan(x)}{x^2+1}dx} y\right) &= 0 \end{aligned}$$

Integrating gives

$$\mathrm{e}^{\int -\frac{\tan(x)}{x^2+1}dx} y = c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int -\frac{\tan(x)}{x^2+1}dx}$ results in

$$y = c_1 \mathrm{e}^{\int \frac{\tan(x)}{x^2+1}dx}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\int \frac{\tan(x)}{x^2+1}dx} \tag{1}$$

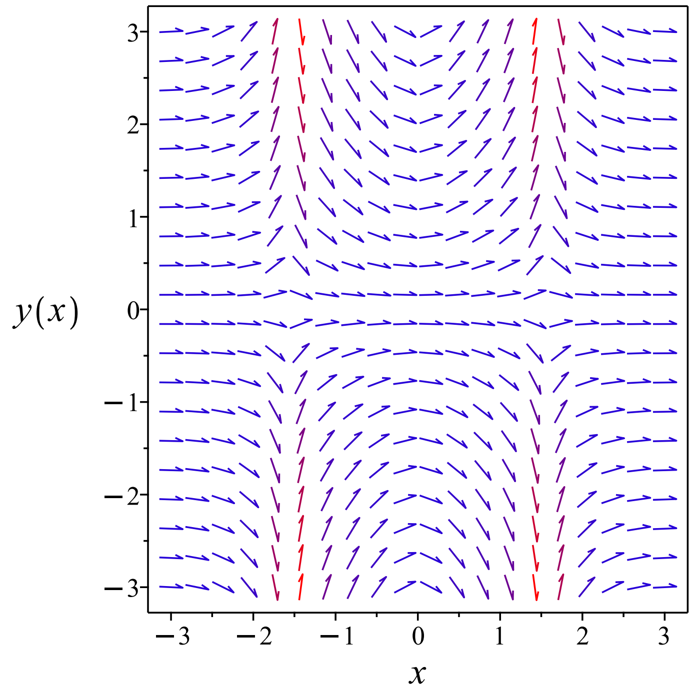


Figure 5: Slope field plot

Verification of solutions

$$y = c_1 e^{\int \frac{\tan(x)}{x^2+1}dx}$$

Verified OK.

1.2.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$\frac{u'(x)\, x + u(x)}{\tan(x)} - \frac{u(x)\, x}{x^2+1} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(\tan(x)\, x - x^2 - 1)}{x\,(x^2+1)} \end{aligned}$$

Where $f(x) = \frac{\tan(x)x - x^2 - 1}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{\tan(x)\,x - x^2 - 1}{x\,(x^2+1)}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{\tan(x)\,x - x^2 - 1}{x\,(x^2+1)}\,dx \\
\ln(u) &= \int \frac{\tan(x)\,x - x^2 - 1}{x\,(x^2+1)}dx + c_2 \\
u &= \mathrm{e}^{\int \frac{\tan(x)x - x^2 - 1}{x\left(x^2+1\right)}dx + c_2} \\
&= c_2\mathrm{e}^{\int \frac{\tan(x)x - x^2 - 1}{x\left(x^2+1\right)}dx}
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= xc_2\mathrm{e}^{\int \frac{\tan(x)x - x^2 - 1}{x\left(x^2+1\right)}dx}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = xc_2\mathrm{e}^{\int \frac{\tan(x)x - x^2 - 1}{x\left(x^2+1\right)}dx} \tag{1}
$$

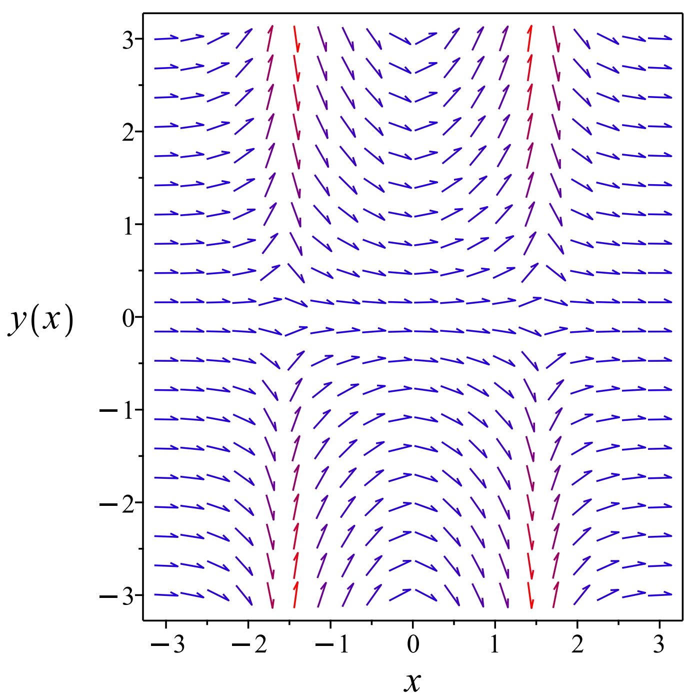


Figure 6: Slope field plot

Verification of solutions

$$y = xc_2 e^{\int \frac{\tan(x)x - x^2 - 1}{x(x^2+1)} dx}$$

Verified OK.

1.2.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y \tan(x)}{x^2 + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 4: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \mathrm{e}^{\frac{\ln(x+i)}{2} - \frac{\ln(x-i)}{2} - i\left(\int -\frac{2}{\left(\mathrm{e}^{2ix}+1\right)\left(x^2+1\right)}dx\right)} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\frac{\ln(x+i)}{2} - \frac{\ln(x-i)}{2} - i\left(\int -\frac{2}{\left(\mathrm{e}^{2ix}+1\right)\left(x^2+1\right)} dx\right)}} dy \end{aligned}$$

1.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y &= \left(\frac{\tan(x)}{x^2+1}\right)\mathrm{d}x\\ \left(-\frac{\tan(x)}{x^2+1}\right)\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{\tan(x)}{x^2+1}\\ N(x,y) &= \frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\tan(x)}{x^2+1}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{\tan(x)}{x^2+1} \mathrm{d}x$$
$$\phi = \int^x -\frac{\tan(\_a)}{\_a^2+1} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^x -\frac{\tan(\_a)}{\_a^2+1} d\_a + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^x -\frac{\tan(\_a)}{\_a^2+1} d\_a + \ln(y)$$

The solution becomes

$$y = e^{-\left(\int^{x} -\frac{\tan(\_a)}{\_a^{2}+1}d\_a\right)+c_{1}}$$

Summary

The solution(s) found are the following

$$y = e^{-\left(\int^{x} -\frac{\tan(\_a)}{\_a^{2}+1}d\_a\right)+c_{1}} \tag{1}$$

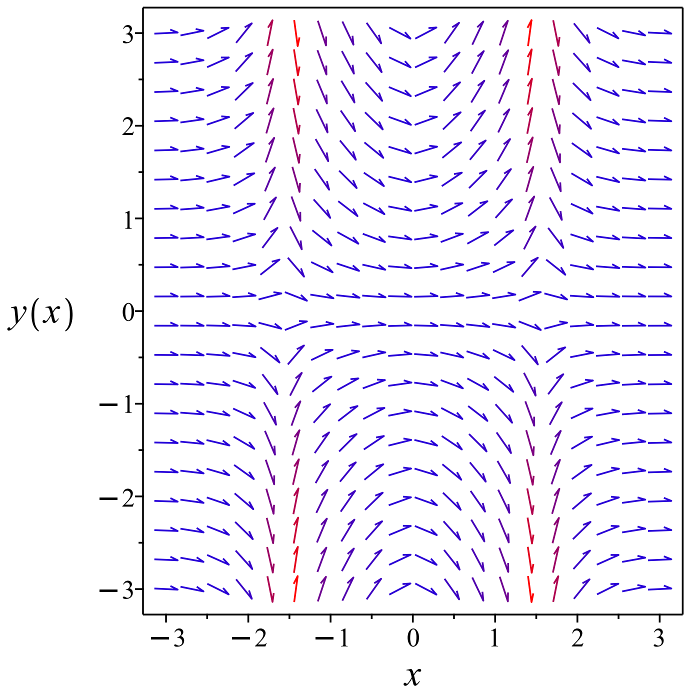


Figure 7: Slope field plot

Verification of solutions

$$y = e^{-\left(\int^{x} -\frac{\tan(\_a)}{\_a^{2}+1}d\_a\right)+c_{1}}$$

Verified OK.

1.2.6 Maple step by step solution

Let's solve

$\frac{y'}{\tan(x)} - \frac{y}{x^2+1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{\tan(x)}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{\tan(x)}{x^2+1}dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{\ln(x+\mathrm{I})}{2} - \frac{\ln(x-\mathrm{I})}{2} - \mathrm{I}\left(\int -\frac{2}{\left((\mathrm{e}^{\mathrm{I}x})^2+1\right)(x^2+1)}dx\right) + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)/tan(x)-y(x)/(1+x^2)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{\int \frac{\tan(x)}{x^2+1}dx}$$

✓ Solution by Mathematica

Time used: 9.987 (sec). Leaf size: 34

```
DSolve[y'[x]/Tan[x]-y[x]/(1+x^2)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \exp \left(\int_1^x \frac{\tan(K[1])}{K[1]^2 + 1} dK[1] \right)$$
$$y(x) \to 0$$

1.3 problem Problem 14.2 (c)

Internal problem ID [2488]
Internal file name [`OUTPUT/1980_Sunday_June_05_2022_02_42_03_AM_34099516/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.2 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'x^2 + y^2x - 4y^2 = 0}$$

1.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y^2(x-4)}{x^2} \end{aligned}$$

Where $f(x) = -\frac{x-4}{x^2}$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = -\frac{x-4}{x^2}\,dx$$

$$\int \frac{1}{y^2}\,dy = \int -\frac{x-4}{x^2}\,dx$$

$$-\frac{1}{y} = -\frac{4}{x} - \ln(x) + c_1$$

Which results in

$$y = \frac{x}{\ln(x)\, x - c_1 x + 4}$$

Summary
The solution(s) found are the following

$$y = \frac{x}{\ln(x)\, x - c_1 x + 4} \tag{1}$$

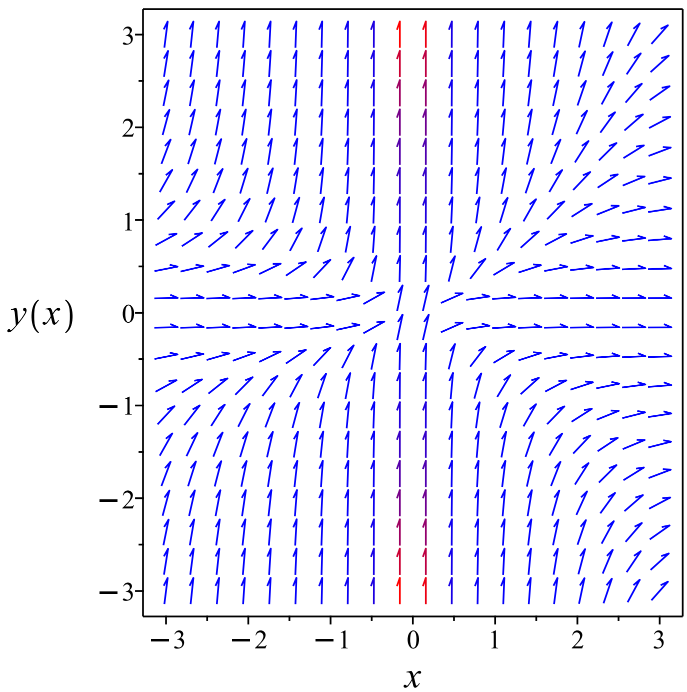


Figure 8: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x)\, x - c_1 x + 4}$$

Verified OK.

1.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y^2(x-4)}{x^2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 7: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = -\frac{x^2}{x-4}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{x^2}{x-4}} dx \end{aligned}$$

Which results in

$$S = -\frac{4}{x} - \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y^2(x-4)}{x^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{-x+4}{x^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{1}{R} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{-\ln(x)\,x-4}{x} = -\frac{1}{y} + c_1
$$

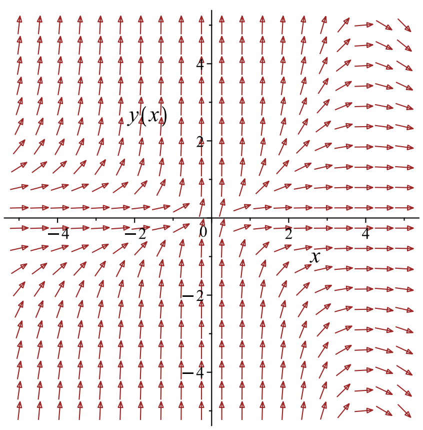
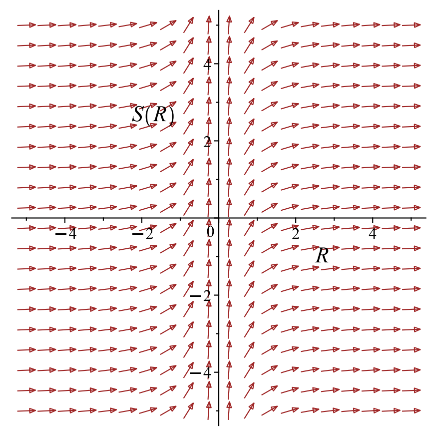
Which simplifies to

$$
\frac{-\ln(x)\,x-4}{x} = -\frac{1}{y} + c_1
$$

Which gives

$$
y = \frac{x}{4+\ln(x)\,x+c_1 x}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2(x-4)}{x^2}$ 	$R = y$ $S = \frac{-\ln(x)\, x - 4}{x}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{x}{4 + \ln(x)\, x + c_1 x} \tag{1}$$

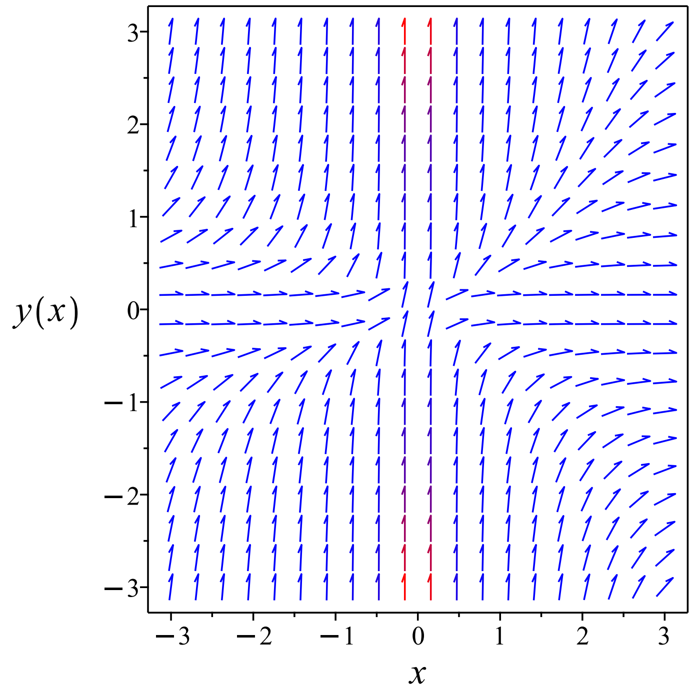


Figure 9: Slope field plot

Verification of solutions

$$y = \frac{x}{4 + \ln(x)\, x + c_1 x}$$

Verified OK.

1.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{y^2}\right) \mathrm{d}y &= \left(\frac{x-4}{x^2}\right) \mathrm{d}x \\
\left(-\frac{x-4}{x^2}\right) \mathrm{d}x + \left(-\frac{1}{y^2}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{x-4}{x^2} \\
N(x, y) &= -\frac{1}{y^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x-4}{x^2}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x-4}{x^2} \, \mathrm{d}x \\
\phi &= -\frac{4}{x} - \ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y^2}$. Therefore equation (4) becomes

$$
-\frac{1}{y^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y^2}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y^2}\right) \mathrm{d}y$$
$$f(y) = \frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{4}{x} - \ln(x) + \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{4}{x} - \ln(x) + \frac{1}{y}$$

The solution becomes

$$y = \frac{x}{4 + \ln(x)\,x + c_1 x}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{4 + \ln(x)\,x + c_1 x} \tag{1}$$

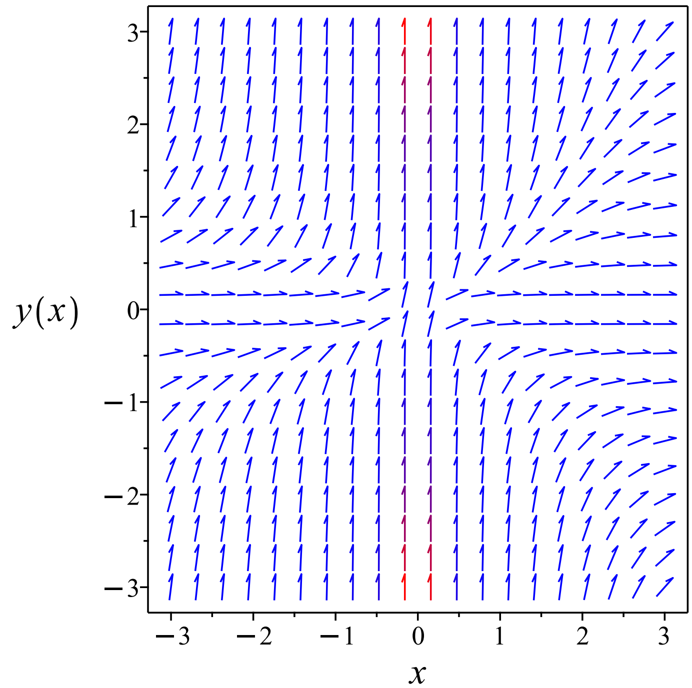


Figure 10: Slope field plot

Verification of solutions

$$y = \frac{x}{4 + \ln(x)x + c_1 x}$$

Verified OK.

1.3.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y^2(x-4)}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2}{x} + \frac{4y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = -\frac{x-4}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{(x-4)u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{x^2} + \frac{2x-8}{x^3} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{(x-4)\, u''(x)}{x^2} - \left(-\frac{1}{x^2} + \frac{2x-8}{x^3}\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \left(\frac{4}{x} + \ln(x)\right) c_2$$

The above shows that

$$u'(x) = \frac{(x-4)\, c_2}{x^2}$$

Using the above in (1) gives the solution

$$y = \frac{c_2}{c_1 + \left(\frac{4}{x} + \ln(x)\right) c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x}{\ln(x)\, x + c_3 x + 4}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{\ln(x) x + c_3 x + 4} \tag{1}$$

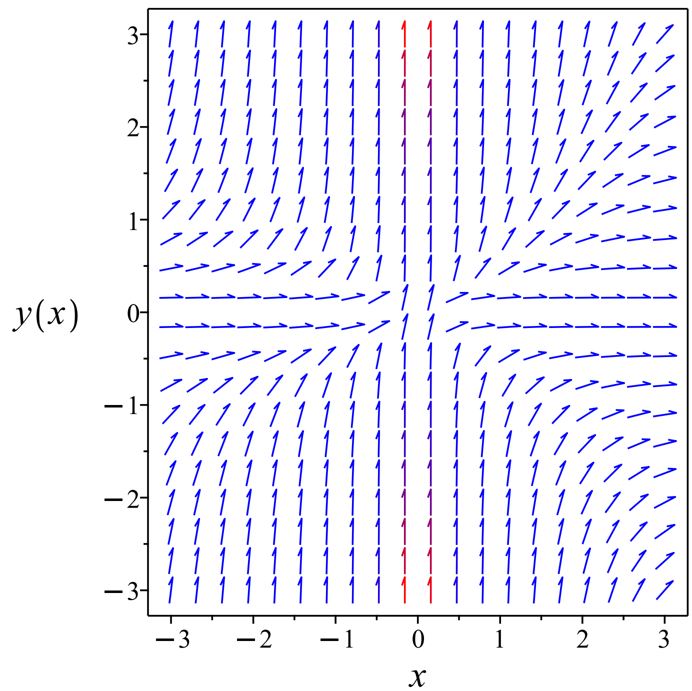


Figure 11: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) x + c_3 x + 4}$$

Verified OK.

1.3.5 Maple step by step solution

Let's solve

$y'x^2 + y^2x - 4y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y^2} = -\frac{x-4}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int -\frac{x-4}{x^2} dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = -\frac{4}{x} - \ln(x) + c_1$

- Solve for y

  $y = \frac{x}{\ln(x)x - c_1 x + 4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x)+x*y(x)^2=4*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{x}{4 + x\ln(x) + c_1 x}$$

✓ Solution by Mathematica

Time used: 0.146 (sec). Leaf size: 25

```
DSolve[y'[x]+x*y[x]^2==4*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{x^2 - 8x - 2c_1}$$
$$y(x) \to 0$$

1.4 problem Problem 14.3 (a)

Internal problem ID [2489]
Internal file name [`OUTPUT/1981_Sunday_June_05_2022_02_42_06_AM_58394202/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.3 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational, [_1st_order, `_with_symmetry_[F(x)*G(y)
    ,0]`]]
```

$$\boxed{y(2y^2x^2+1)\,y'+x(y^4+1)=0}$$

1.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(y\left(2y^2x^2+1\right)\right) \mathrm{d}y = \left(-x\left(y^4+1\right)\right) \mathrm{d}x$$
$$\left(x\left(y^4+1\right)\right) \mathrm{d}x + \left(y\left(2y^2x^2+1\right)\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = x\left(y^4+1\right)$$
$$N(x, y) = y\left(2y^2x^2+1\right)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x\left(y^4+1\right)\right) \\ &= 4x\,y^3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y\left(2y^2x^2+1\right)\right) \\ &= 4x\,y^3\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int x\left(y^4+1\right)\mathrm{d}x$$
$$\phi = \frac{x^2(y^4+1)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2y^3x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y(2y^2x^2+1)$. Therefore equation (4) becomes

$$y\left(2y^2x^2+1\right) = 2y^3x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y)\,\mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2(y^4+1)}{2} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2(y^4+1)}{2} + \frac{y^2}{2}$$

<u>Summary</u>

The solution(s) found are the following

$$\frac{x^2(y^4+1)}{2} + \frac{y^2}{2} = c_1 \tag{1}$$

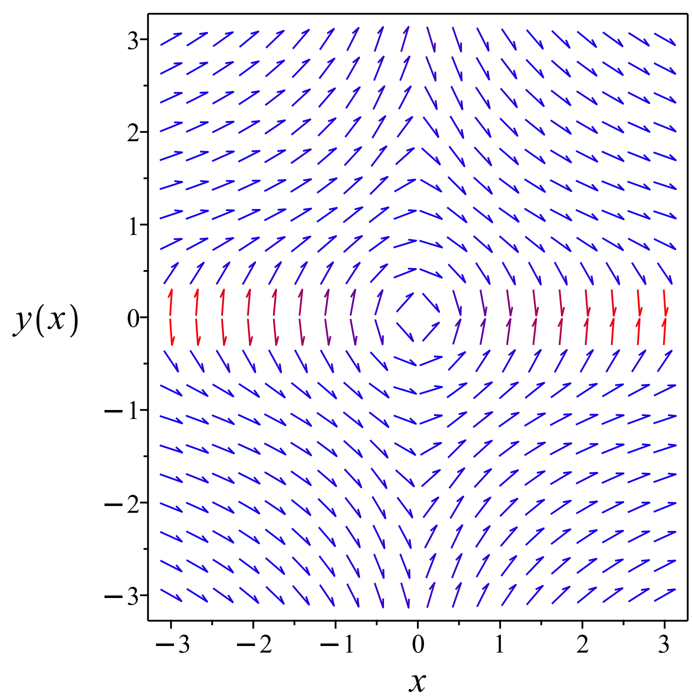


Figure 12: Slope field plot

<u>Verification of solutions</u>

$$\frac{x^2(y^4+1)}{2} + \frac{y^2}{2} = c_1$$

Verified OK.

1.4.2 Maple step by step solution

Let's solve

$y(2y^2x^2+1)\, y'+x(y^4+1)=0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $4x\, y^3=4x\, y^3$

  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y)\, , N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int x(y^4+1)\, dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{x^2(y^4+1)}{2}+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $y(2y^2x^2+1)=2y^3x^2+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=-2y^3x^2+y(2y^2x^2+1)$

- Solve for $f_1(y)$

  $f_1(y)=\frac{y^2}{2}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$F(x, y) = \frac{x^2(y^4+1)}{2} + \frac{y^2}{2}$

- Substitute $F(x, y)$ into the solution of the ODE

  $\frac{x^2(y^4+1)}{2} + \frac{y^2}{2} = c_1$

- Solve for y

  $\left\{y = -\frac{\sqrt{-2-2\sqrt{-4x^4+8c_1x^2+1}}}{2x}, y = \frac{\sqrt{-2-2\sqrt{-4x^4+8c_1x^2+1}}}{2x}, y = -\frac{\sqrt{2}\sqrt{-1+\sqrt{-4x^4+8c_1x^2+1}}}{2x}, y = \frac{\sqrt{2}\sqrt{-1+ \ldots\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 119

```
dsolve(y(x)*(2*x^2*y(x)^2+1)*diff(y(x),x)+x*(y(x)^4+1)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-2-2\sqrt{-4x^4-8c_1x^2+1}}}{2x}$$
$$y(x) = \frac{\sqrt{-2-2\sqrt{-4x^4-8c_1x^2+1}}}{2x}$$
$$y(x) = -\frac{\sqrt{2}\sqrt{-1+\sqrt{-4x^4-8c_1x^2+1}}}{2x}$$
$$y(x) = \frac{\sqrt{2}\sqrt{-1+\sqrt{-4x^4-8c_1x^2+1}}}{2x}$$

✓ Solution by Mathematica

Time used: 10.416 (sec). Leaf size: 197

```
DSolve[y[x]*(2*x^2*y[x]^2+1)*y'[x]+x*(y[x]^4+1)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-\frac{1+\sqrt{-4x^4+8c_1x^2+1}}{x^2}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-\frac{1+\sqrt{-4x^4+8c_1x^2+1}}{x^2}}}{\sqrt{2}}$$
$$y(x) \to -\frac{\sqrt{\frac{-1+\sqrt{-4x^4+8c_1x^2+1}}{x^2}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{\frac{-1+\sqrt{-4x^4+8c_1x^2+1}}{x^2}}}{\sqrt{2}}$$
$$y(x) \to -\sqrt[4]{-1}$$
$$y(x) \to \sqrt[4]{-1}$$
$$y(x) \to -(-1)^{3/4}$$
$$y(x) \to (-1)^{3/4}$$

1.5 problem Problem 14.3 (b)

Internal problem ID [2490]
Internal file name [`OUTPUT/1982_Sunday_June_05_2022_02_42_10_AM_24305801/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.3 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2xy' + y = -3x}$$

1.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{2x}$$
$$q(x) = -\frac{3}{2}$$

Hence the ode is

$$y' + \frac{y}{2x} = -\frac{3}{2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{2x}dx} \\
&= \sqrt{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{3}{2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\, y\right) &= \left(\sqrt{x}\right)\left(-\frac{3}{2}\right) \\
\mathrm{d}\left(\sqrt{x}\, y\right) &= \left(-\frac{3\sqrt{x}}{2}\right)\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\sqrt{x}\, y &= \int -\frac{3\sqrt{x}}{2}\, \mathrm{d}x \\
\sqrt{x}\, y &= -x^{\frac{3}{2}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sqrt{x}$ results in

$$
y = -x + \frac{c_1}{\sqrt{x}}
$$

Summary

The solution(s) found are the following

$$
y = -x + \frac{c_1}{\sqrt{x}} \tag{1}
$$

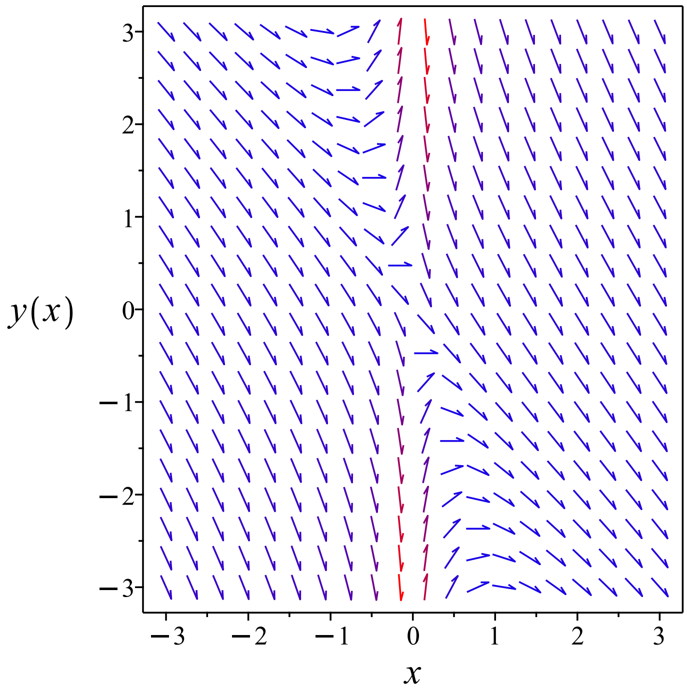


Figure 13: Slope field plot

Verification of solutions

$$y = -x + \frac{c_1}{\sqrt{x}}$$

Verified OK.

1.5.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2x(u'(x)\,x + u(x)) + u(x)\,x = -3x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-\frac{3u}{2} - \frac{3}{2}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -\frac{3u}{2} - \frac{3}{2}$. Integrating both sides gives

$$\frac{1}{-\frac{3u}{2} - \frac{3}{2}}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{-\frac{3u}{2} - \frac{3}{2}} \, du = \int \frac{1}{x} \, dx$$

$$-\frac{2 \ln (u + 1)}{3} = \ln (x) + c_2$$

Raising both side to exponential gives

$$\frac{1}{(u + 1)^{\frac{2}{3}}} = e^{\ln(x) + c_2}$$

Which simplifies to

$$\frac{1}{(u + 1)^{\frac{2}{3}}} = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x\left(-1 + \left(\frac{e^{-c_2}}{c_3 x}\right)^{\frac{3}{2}}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\left(-1 + \left(\frac{e^{-c_2}}{c_3 x}\right)^{\frac{3}{2}}\right) \tag{1}$$

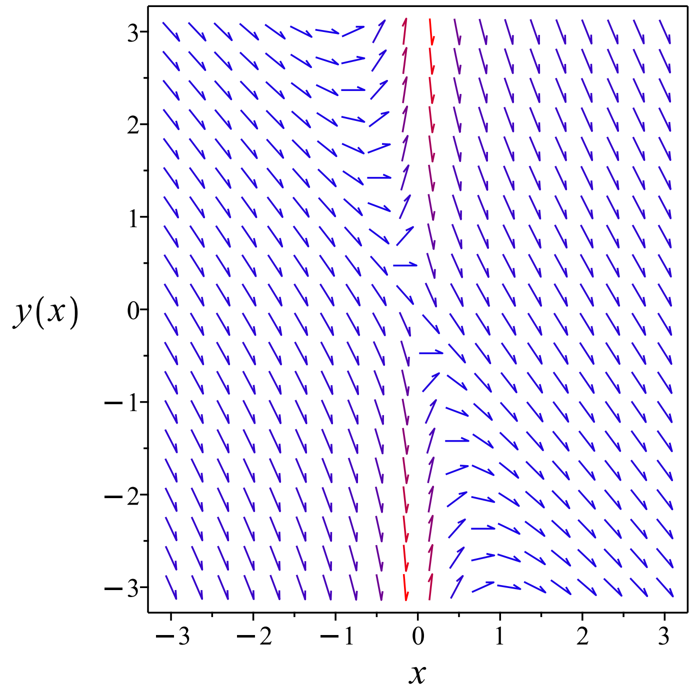


Figure 14: Slope field plot

Verification of solutions

$$y = x\left(-1 + \left(\frac{e^{-c_2}}{c_3 x}\right)^{\frac{3}{2}}\right)$$

Verified OK.

1.5.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3x + y}{2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 11: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{\sqrt{x}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sqrt{x}}} dy \end{aligned}$$

Which results in

$$S = \sqrt{x}\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3x + y}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{2\sqrt{x}} \\ S_y &= \sqrt{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{3\sqrt{x}}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{3\sqrt{R}}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -R^{\frac{3}{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sqrt{x}\, y = -x^{\frac{3}{2}} + c_1$$

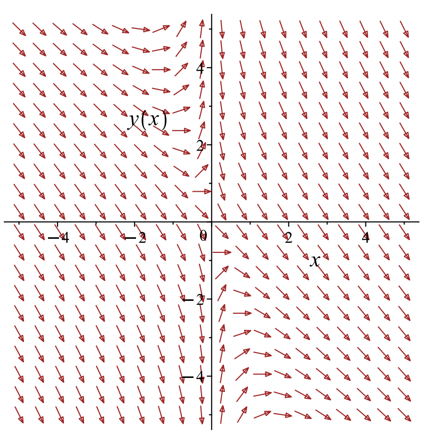
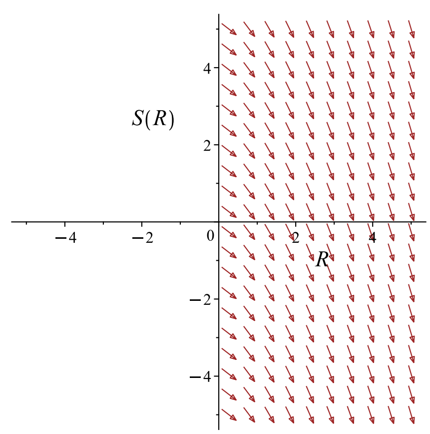
Which simplifies to

$$\sqrt{x}\, y = -x^{\frac{3}{2}} + c_1$$

Which gives

$$y = -\frac{x^{\frac{3}{2}} - c_1}{\sqrt{x}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dy}{dx} = -\frac{3x+y}{2x}$
</td><td>$R = x$
$S = \sqrt{x}\, y$</td><td>$\frac{dS}{dR} = -\frac{3\sqrt{R}}{2}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\frac{x^{\frac{3}{2}} - c_1}{\sqrt{x}} \tag{1}$$

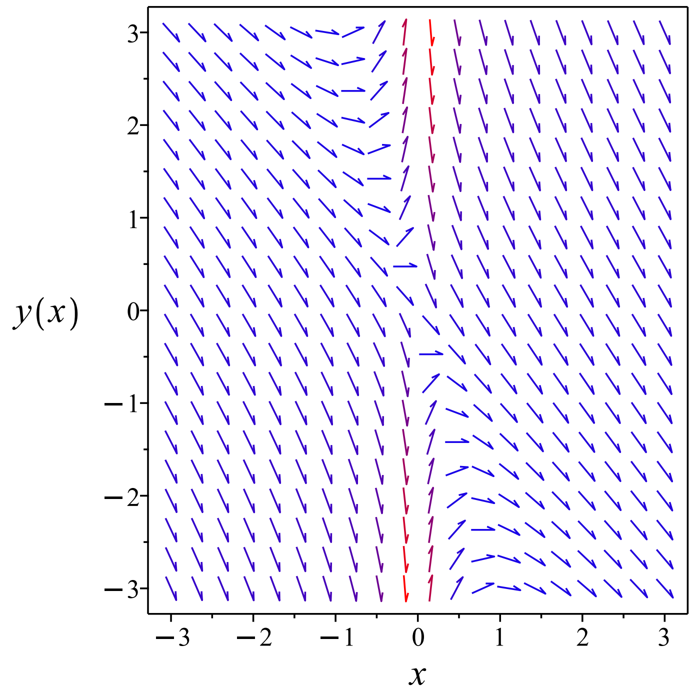


Figure 15: Slope field plot

Verification of solutions

$$y = -\frac{x^{\frac{3}{2}} - c_1}{\sqrt{x}}$$

Verified OK.

1.5.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x)\,\mathrm{d}y &= (-3x - y)\,\mathrm{d}x\\ (3x + y)\,\mathrm{d}x + (2x)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 3x + y\\ N(x,y) &= 2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x+y)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x)\\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2x}((1)-(2))\\ &= -\frac{1}{2x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{1}{2x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{\ln(x)}{2}}\\ &= \frac{1}{\sqrt{x}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{\sqrt{x}}(3x+y)\\ &= \frac{3x+y}{\sqrt{x}}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sqrt{x}}(2x) \\
&= 2\sqrt{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{3x+y}{\sqrt{x}}\right) + \left(2\sqrt{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{3x+y}{\sqrt{x}} \mathrm{d}x
\end{aligned}
$$

$$
\phi = 2\sqrt{x}\,(y+x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 2\sqrt{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2\sqrt{x}$. Therefore equation (4) becomes

$$
2\sqrt{x} = 2\sqrt{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 2\sqrt{x}\,(y + x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2\sqrt{x}\,(y + x)$$

The solution becomes

$$y = -\frac{2x^{\frac{3}{2}} - c_1}{2\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = -\frac{2x^{\frac{3}{2}} - c_1}{2\sqrt{x}} \tag{1}$$

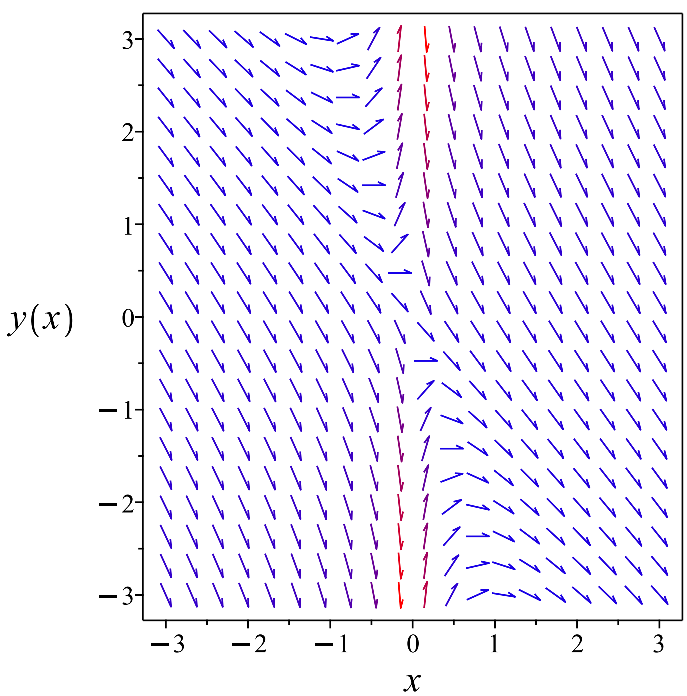


Figure 16: Slope field plot

Verification of solutions

$$y = -\frac{2x^{\frac{3}{2}} - c_1}{2\sqrt{x}}$$

Verified OK.

1.5.5 Maple step by step solution

Let's solve

$2xy' + y = -3x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{3}{2} - \frac{y}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{2x} = -\frac{3}{2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{2x}\right) = -\frac{3\mu(x)}{2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{2x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{2x}$

- Solve to find the integrating factor

  $\mu(x) = \sqrt{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\frac{3\mu(x)}{2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int -\frac{3\mu(x)}{2} dx + c_1$

- Solve for y

  $y = \frac{\int -\frac{3\mu(x)}{2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \sqrt{x}$

$$y = \frac{\int -\frac{3\sqrt{x}}{2}dx + c_1}{\sqrt{x}}$$

- Evaluate the integrals on the rhs

$$y = \frac{-x^{\frac{3}{2}} + c_1}{\sqrt{x}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve(2*x*diff(y(x),x)+3*x+y(x)=0,y(x), singsol=all)
```

$$y(x) = -x + \frac{c_1}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 17

```
DSolve[2*x*y'[x]+3*x+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x + \frac{c_1}{\sqrt{x}}$$

1.6 problem Problem 14.3 (c)

Internal problem ID [2491]
Internal file name [`OUTPUT/1983_Sunday_June_05_2022_02_42_13_AM_33645999/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.3 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`], [_Abel, `2nd
   type`, `class B`]]
```

$$\boxed{\left(\cos \left(x\right)^{2}+y \sin \left(2 x\right)\right) y^{\prime}+y^{2}=0}$$

1.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\cos(x)^2 + y\sin(2x)\right)\mathrm{d}y &= \left(-y^2\right)\mathrm{d}x\\ \left(y^2\right)\mathrm{d}x + \left(\cos(x)^2 + y\sin(2x)\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y^2\\ N(x, y) &= \cos(x)^2 + y\sin(2x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2\right)\\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\cos(x)^2 + y\sin(2x)\right)\\ &= -\sin(2x) + 2y\cos(2x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{\sec(x)}{2\sin(x)\,y + \cos(x)}\left((2y) - (-2\sin(x)\cos(x) + 2y\cos(2x))\right)\\ &= 2\tan(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 2\tan(x) \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(\cos(x))} \\ &= \sec(x)^2\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \sec(x)^2 \left(y^2\right) \\ &= y^2 \sec(x)^2\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \sec(x)^2 \left(\cos(x)^2 + y\sin(2x)\right) \\ &= 2\tan(x)\, y + 1\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(y^2 \sec(x)^2\right) + (2\tan(x)\, y + 1)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int y^2 \sec(x)^2 \, \mathrm{d}x\end{aligned}$$

$$\phi = y^2 \tan(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2\tan(x)\, y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2\tan(x)\, y + 1$. Therefore equation (4) becomes

$$2\tan(x)\, y + 1 = 2\tan(x)\, y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y)\, dy = \int (1)\, dy$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y^2 \tan(x) + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^2 \tan(x) + y$$

Summary
The solution(s) found are the following

$$y^2 \tan(x) + y = c_1 \tag{1}$$

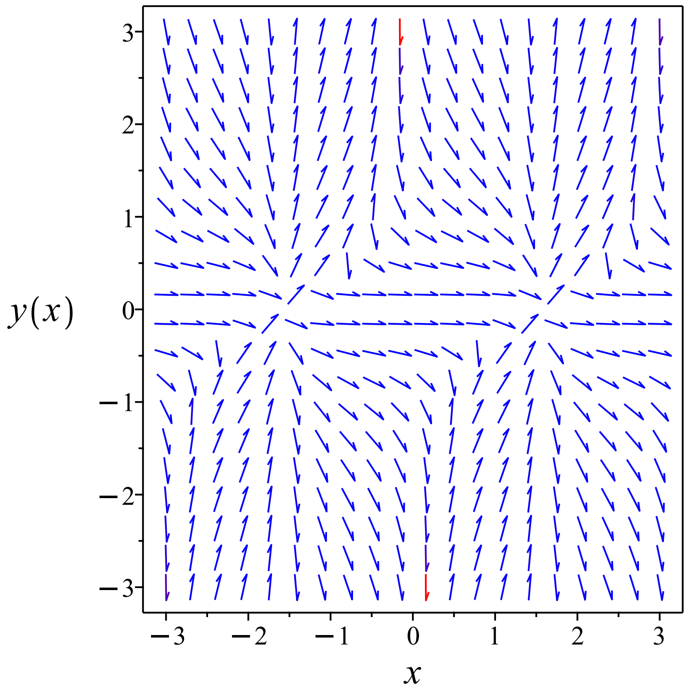


Figure 17: Slope field plot

Verification of solutions

$$y^2 \tan(x) + y = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel AIR successful: ODE belongs to the 1F1 2-parameter class`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve((cos(x)^2+y(x)*sin(2*x))*diff(y(x),x)+y(x)^2=0,y(x), singsol=all)
```

$$c_1 + y(x)^2 \tan(x) + y(x) = 0$$

✓ Solution by Mathematica

Time used: 23.536 (sec). Leaf size: 170

```
DSolve[(Cos[x]^2+y[x]*Sin[2*x])*y'[x]+y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\cot(x)}{2} - \frac{\csc(2x)\sqrt{e^{-\operatorname{arctanh}(\cos(2x))}\left(4c_1\sin(2x)e^{\operatorname{arctanh}(\cos(2x))} + \csc(2x) + (\cos(2x)+2)\cot(2x)\right)}}{2\sqrt{\csc(2x)e^{-\operatorname{arctanh}(\cos(2x))}}}$$

$$y(x) \to -\frac{\cot(x)}{2} + \frac{\csc(2x)\sqrt{e^{-\operatorname{arctanh}(\cos(2x))}\left(4c_1\sin(2x)e^{\operatorname{arctanh}(\cos(2x))} + \csc(2x) + (\cos(2x)+2)\cot(2x)\right)}}{2\sqrt{\csc(2x)e^{-\operatorname{arctanh}(\cos(2x))}}}$$

$$y(x) \to 0$$

1.7 problem Problem 14.5 (a)

Internal problem ID [2492]
Internal file name [`OUTPUT/1984_Sunday_June_05_2022_02_42_16_AM_25912747/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.5 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\left(-x^2+1\right) y'+4yx=\left(-x^2+1\right)^{\frac{3}{2}}}$$

1.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned} p(x)&=-\frac{4x}{x^2-1}\\ q(x)&=\sqrt{-x^2+1} \end{aligned}$$

Hence the ode is

$$y'-\frac{4xy}{x^2-1}=\sqrt{-x^2+1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{4x}{x^2-1}dx} \\ &= e^{-2\ln(x-1)-2\ln(x+1)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{1}{(x-1)^2 (x+1)^2}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\sqrt{-x^2+1}\right) \\ \frac{d}{dx}\left(\frac{y}{(x-1)^2 (x+1)^2}\right) &= \left(\frac{1}{(x-1)^2 (x+1)^2}\right)\left(\sqrt{-x^2+1}\right) \\ d\left(\frac{y}{(x-1)^2 (x+1)^2}\right) &= \left(\frac{\sqrt{-x^2+1}}{(x-1)^2 (x+1)^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{(x-1)^2 (x+1)^2} &= \int \frac{\sqrt{-x^2+1}}{(x-1)^2 (x+1)^2} dx \\ \frac{y}{(x-1)^2 (x+1)^2} &= -\frac{x\sqrt{-x^2+1}}{(x+1)(x-1)} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(x-1)^2(x+1)^2}$ results in

$$y = -(x-1)(x+1)x\sqrt{-x^2+1} + c_1(x-1)^2 (x+1)^2$$

Summary

The solution(s) found are the following

$$y = -(x-1)(x+1)x\sqrt{-x^2+1} + c_1(x-1)^2 (x+1)^2 \tag{1}$$

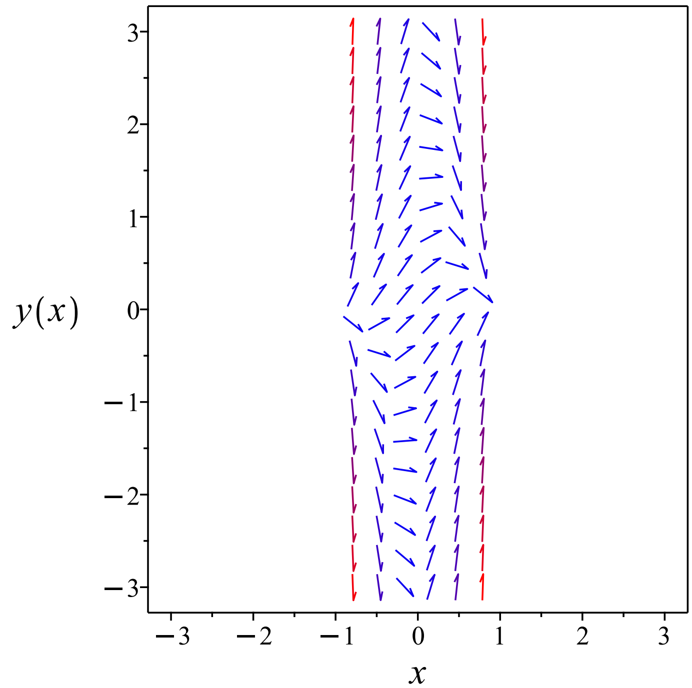


Figure 18: Slope field plot

Verification of solutions

$$y = -(x-1)(x+1)x\sqrt{-x^2+1} + c_1(x-1)^2(x+1)^2$$

Verified OK.

1.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-4xy + (-x^2+1)^{\frac{3}{2}}}{x^2-1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 14: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{2\ln(x-1)+2\ln(x+1)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{2\ln(x-1)+2\ln(x+1)}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{(x-1)^2 (x+1)^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-4xy + (-x^2+1)^{\frac{3}{2}}}{x^2-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{4xy}{(x-1)^3 (x+1)^3} \\ S_y &= \frac{1}{(x-1)^2 (x+1)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(-x^2+1)^{\frac{3}{2}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(-R^2+1)^{\frac{3}{2}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{(R-1)(R+1)R}{(-R^2+1)^{\frac{3}{2}}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{(x-1)^2 (x+1)^2} = -\frac{(x-1)(x+1)x}{(-x^2+1)^{\frac{3}{2}}} + c_1$$

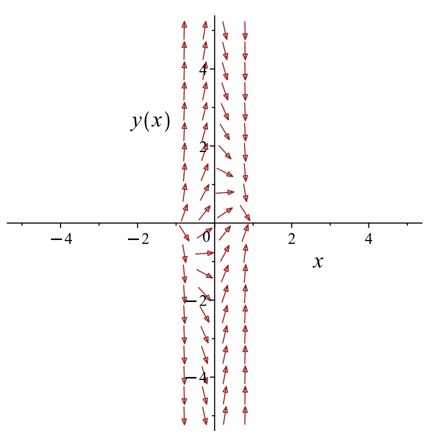
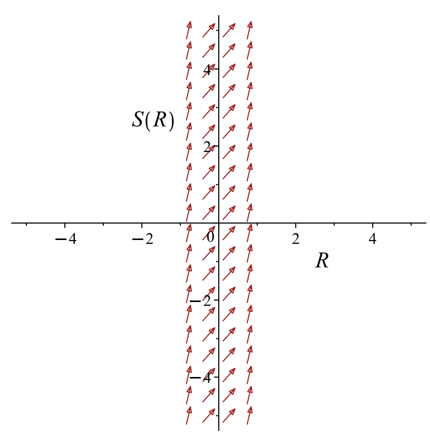
Which simplifies to

$$\frac{y}{(x-1)^2 (x+1)^2} = -\frac{(x-1)(x+1)x}{(-x^2+1)^{\frac{3}{2}}} + c_1$$

Which gives

$$y = \frac{\left(c_1(-x^2+1)^{\frac{3}{2}} - x^3 + x\right)(x-1)^2 (x+1)^2}{(-x^2+1)^{\frac{3}{2}}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-4xy+(-x^2+1)^{\frac{3}{2}}}{x^2-1}$ 	$R = x$ $S = \frac{y}{(x-1)^2 (x+1)^2}$	$\frac{dS}{dR} = \frac{1}{(-R^2+1)^{\frac{3}{2}}}$ 

Summary
The solution(s) found are the following

$$y = \frac{\left(c_1(-x^2+1)^{\frac{3}{2}} - x^3 + x\right)(x-1)^2 (x+1)^2}{(-x^2+1)^{\frac{3}{2}}} \tag{1}$$

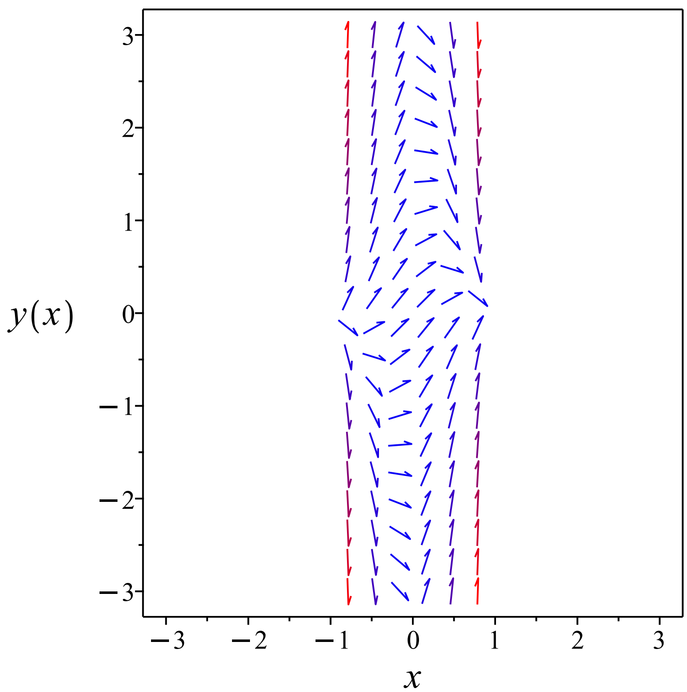


Figure 19: Slope field plot

Verification of solutions

$$y = \frac{\left(c_1(-x^2+1)^{\frac{3}{2}} - x^3 + x\right)(x-1)^2 (x+1)^2}{(-x^2+1)^{\frac{3}{2}}}$$

Verified OK.

1.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(-x^2+1\right)\mathrm{d}y=\left(-4xy+\left(-x^2+1\right)^{\frac{3}{2}}\right)\mathrm{d}x$$

$$\left(-\left(-x^2+1\right)^{\frac{3}{2}}+4xy\right)\mathrm{d}x+\left(-x^2+1\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\left(-x^2+1\right)^{\frac{3}{2}}+4xy\\ N(x,y)&=-x^2+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\left(-x^2+1\right)^{\frac{3}{2}} + 4xy\right)\\ &= 4x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^2+1\right)\\ &= -2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{x^2-1}((4x) - (-2x))\\ &= -\frac{6x}{x^2-1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int -\frac{6x}{x^2-1} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(x-1)-3\ln(x+1)}\\ &= \frac{1}{\left(x-1\right)^3\left(x+1\right)^3}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{\left(x-1\right)^3\left(x+1\right)^3}\left(-\left(-x^2+1\right)^{\frac{3}{2}} + 4xy\right)\\ &= \frac{\sqrt{-x^2+1}\,x^2 + 4xy - \sqrt{-x^2+1}}{\left(x-1\right)^3\left(x+1\right)^3}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\left(x-1\right)^{3} \left(x+1\right)^{3}}\left(-x^{2}+1\right) \\
&= -\frac{1}{\left(x-1\right)^{2} \left(x+1\right)^{2}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0
$$

$$
\left(\frac{\sqrt{-x^{2}+1}\, x^{2}+4 x y -\sqrt{-x^{2}+1}}{\left(x-1\right)^{3} \left(x+1\right)^{3}}\right) + \left(-\frac{1}{\left(x-1\right)^{2} \left(x+1\right)^{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathop{\mathrm{d}x} &= \int \overline{M}\mathop{\mathrm{d}x} \\
\int \frac{\partial \phi}{\partial x} \mathop{\mathrm{d}x} &= \int \frac{\sqrt{-x^{2}+1}\, x^{2}+4 x y -\sqrt{-x^{2}+1}}{\left(x-1\right)^{3} \left(x+1\right)^{3}}\mathop{\mathrm{d}x} \\
\phi &= \frac{\left(-x^{3}+x \right) \sqrt{-x^{2}+1}-y}{\left(x-1\right)^{2} \left(x+1\right)^{2}} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{1}{\left(x-1\right)^{2} \left(x+1\right)^{2}}+f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{\left(x-1\right)^{2} \left(x+1\right)^{2}}$. Therefore equation (4) becomes

$$
-\frac{1}{\left(x-1\right)^{2} \left(x+1\right)^{2}} = -\frac{1}{\left(x-1\right)^{2} \left(x+1\right)^{2}}+f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(-x^3 + x)\sqrt{-x^2+1} - y}{(x-1)^2 (x+1)^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(-x^3 + x)\sqrt{-x^2+1} - y}{(x-1)^2 (x+1)^2}$$

The solution becomes

$$y = -\left(c_1 x^2 + \sqrt{-x^2+1}\, x - c_1\right)(x-1)(x+1)$$

Summary

The solution(s) found are the following

$$y = -\left(c_1 x^2 + \sqrt{-x^2+1}\, x - c_1\right)(x-1)(x+1) \tag{1}$$

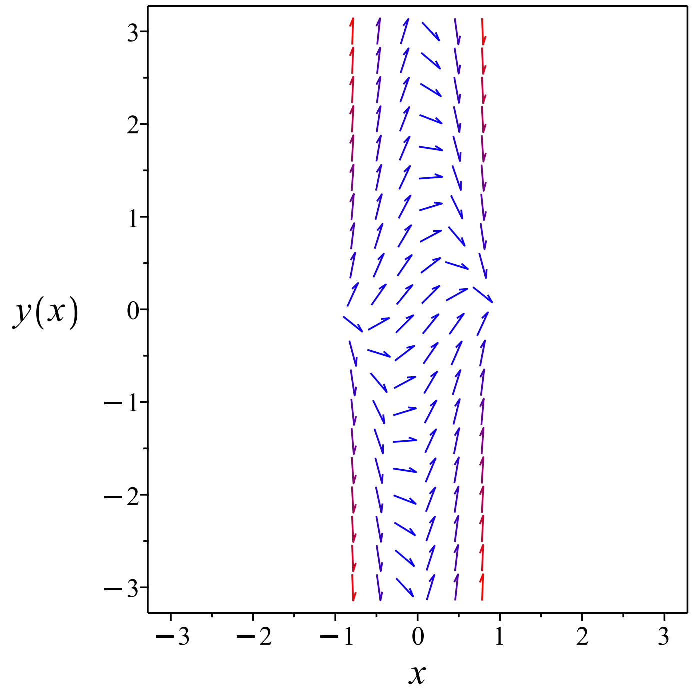


Figure 20: Slope field plot

Verification of solutions

$$y = -\left(c_1x^2 + \sqrt{-x^2+1}\,x - c_1\right)(x-1)(x+1)$$

Verified OK.

1.7.4 Maple step by step solution

Let's solve

$(-x^2+1)\,y' + 4yx = (-x^2+1)^{\frac{3}{2}}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{4xy}{x^2-1} + \sqrt{-x^2+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{4xy}{x^2-1} = \sqrt{-x^2+1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y' - \frac{4xy}{x^2-1}\right) = \mu(x)\sqrt{-x^2+1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{4xy}{x^2-1}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{4\mu(x)x}{x^2-1}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{(x-1)^2(x+1)^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\sqrt{-x^2+1}dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\sqrt{-x^2+1}dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\sqrt{-x^2+1}dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{(x-1)^2(x+1)^2}$

  $y = (x-1)^2(x+1)^2\left(\int \frac{\sqrt{-x^2+1}}{(x-1)^2(x+1)^2}dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = (x-1)^2(x+1)^2\left(-\frac{x\sqrt{-x^2+1}}{(x+1)(x-1)} + c_1\right)$

- Simplify

  $y = \left(c_1x^2 - \sqrt{-x^2+1}\,x - c_1\right)(x^2-1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve((1-x^2)*diff(y(x),x)+2*x*y(x)+2*x*y(x)=(1-x^2)^(3/2),y(x), singsol=all)
```

$$y(x) = c_1 x^4 - x^3\sqrt{-x^2+1} - 2c_1 x^2 + x\sqrt{-x^2+1} + c_1$$

✓ Solution by Mathematica

Time used: 0.116 (sec). Leaf size: 29

```
DSolve[(1-x^2)*y'[x]+2*x*y[x]+2*x*y[x]==(1-x^2)^(3/2),y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to (x^2-1)^2 \left(\frac{x}{\sqrt{1-x^2}} + c_1\right)$$

1.8 problem Problem 14.5 (b)

Internal problem ID [2493]
Internal file name [`OUTPUT/1985_Sunday_June_05_2022_02_42_18_AM_32262369/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.5 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - y\cot(x) = -\frac{1}{\sin(x)}}$$

1.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\cot(x)$$
$$q(x) = -\csc(x)$$

Hence the ode is

$$y' - y\cot(x) = -\csc(x)$$

The integrating factor μ is

$$\mu = e^{\int -\cot(x)dx}$$
$$= \frac{1}{\sin(x)}$$

Which simplifies to

$$\mu = \csc(x)$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(-\csc(x))$$
$$\frac{d}{dx}(\csc(x)\, y) = (\csc(x))(-\csc(x))$$
$$d(\csc(x)\, y) = \left(-\csc(x)^2\right)\, dx$$

Integrating gives

$$\csc(x)\, y = \int -\csc(x)^2\, dx$$
$$\csc(x)\, y = \cot(x) + c_1$$

Dividing both sides by the integrating factor $\mu = \csc(x)$ results in

$$y = \cot(x)\sin(x) + c_1 \sin(x)$$

which simplifies to

$$y = c_1 \sin(x) + \cos(x)$$

Summary

The solution(s) found are the following

$$y = c_1 \sin(x) + \cos(x) \tag{1}$$

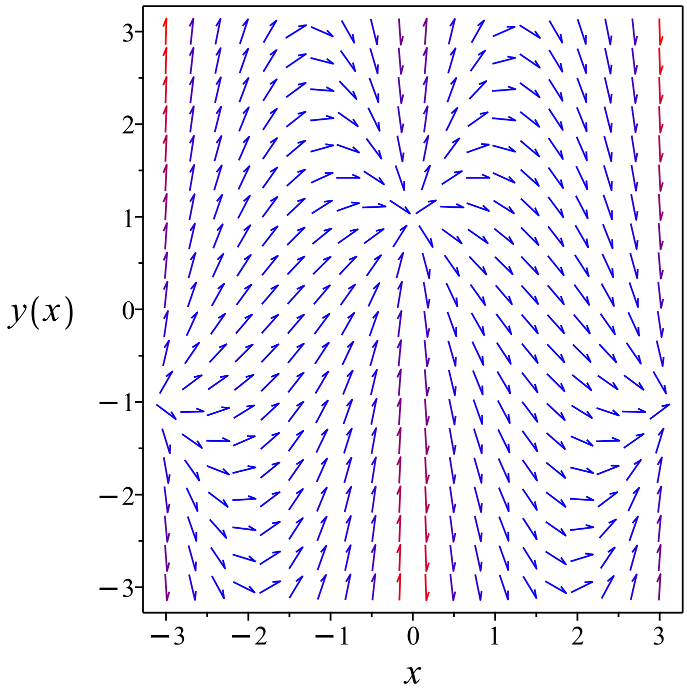


Figure 21: Slope field plot

Verification of solutions

$$y = c_1 \sin(x) + \cos(x)$$

Verified OK.

1.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y \cot(x) \sin(x) - 1}{\sin(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 17: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \sin(x) \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sin(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\sin(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y \cot(x) \sin(x) - 1}{\sin(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\csc(x) \cot(x) y \\ S_y &= \csc(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\csc(x)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\csc(R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \cot(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\csc(x)\, y = \cot(x) + c_1$$

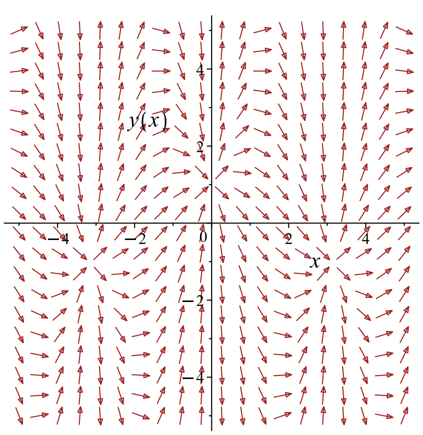
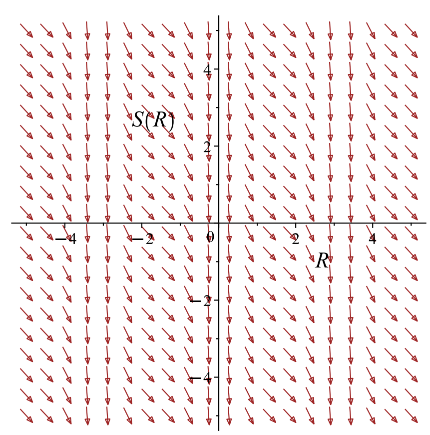
Which simplifies to

$$\csc(x)\, y = \cot(x) + c_1$$

Which gives

$$y = \frac{\cot(x) + c_1}{\csc(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y \cot(x) \sin(x) - 1}{\sin(x)}$ 	$R = x$ $S = \csc(x)\, y$	$\frac{dS}{dR} = -\csc(R)^2$ 

Summary

The solution(s) found are the following

$$y = \frac{\cot(x) + c_1}{\csc(x)} \tag{1}$$

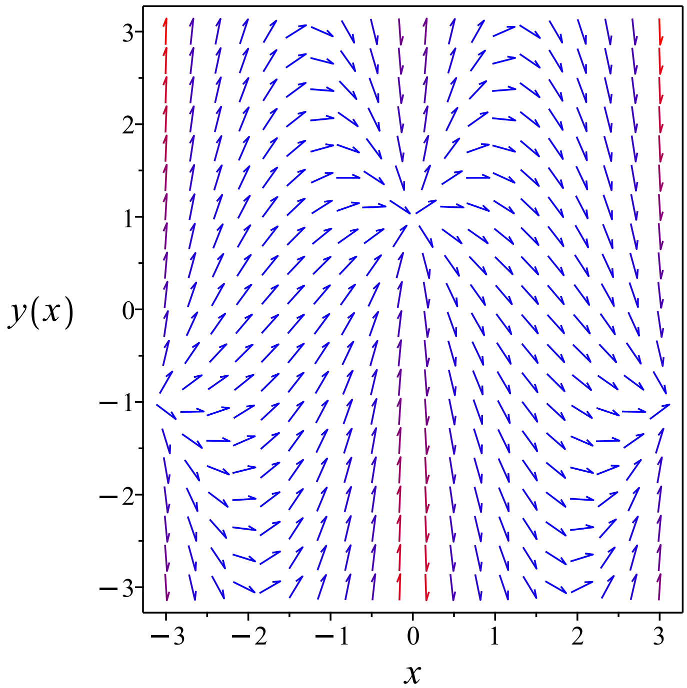


Figure 22: Slope field plot

Verification of solutions

$$y = \frac{\cot(x) + c_1}{\csc(x)}$$

Verified OK.

1.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(y \cot (x) - \frac{1}{\sin (x)}\right) \mathrm{d}x \\
\left(-y \cot (x) + \frac{1}{\sin (x)}\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -y \cot (x) + \frac{1}{\sin (x)} \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y \cot (x) + \frac{1}{\sin (x)}\right) \\
&= -\cot (x)
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-\cot(x)) - (0)) \\ &= -\cot(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\cot(x) \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(\sin(x))} \\ &= \csc(x)\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \csc(x)\left(-y\cot(x) + \frac{1}{\sin(x)}\right) \\ &= \csc(x)^2(-\cos(x)\, y + 1)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \csc(x)(1) \\ &= \csc(x)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\csc(x)^2(-\cos(x)\, y + 1)\right) + (\csc(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \overline{M} \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int \csc(x)^2 (-\cos(x) y + 1) \, dx$$
$$\phi = \csc(x) y - \cot(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \csc(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \csc(x)$. Therefore equation (4) becomes

$$\csc(x) = \csc(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \csc(x) y - \cot(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \csc(x) y - \cot(x)$$

The solution becomes

$$y = \frac{\cot(x) + c_1}{\csc(x)}$$

Summary
The solution(s) found are the following

$$y = \frac{\cot(x) + c_1}{\csc(x)} \tag{1}$$

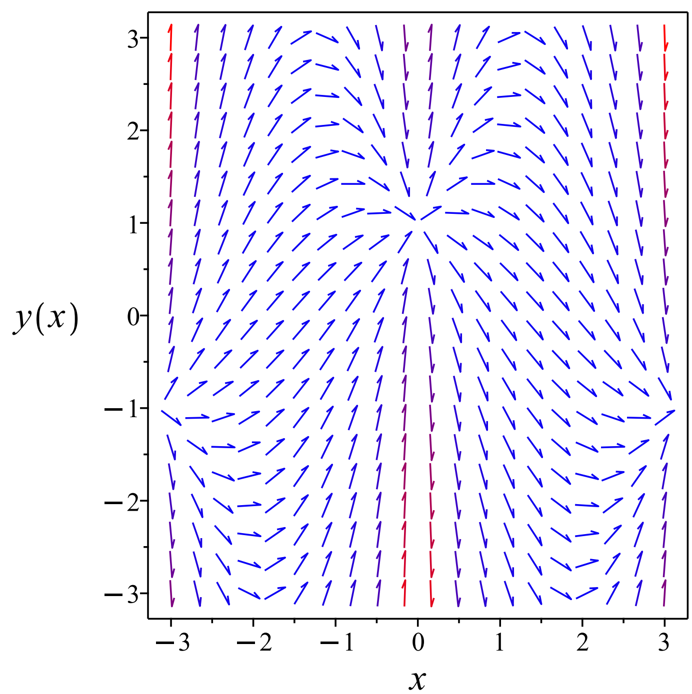


Figure 23: Slope field plot

Verification of solutions

$$y = \frac{\cot(x) + c_1}{\csc(x)}$$

Verified OK.

1.8.4 Maple step by step solution

Let's solve

$y' - y\cot(x) = -\frac{1}{\sin(x)}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y\cot(x) - \frac{1}{\sin(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y\cot(x) = -\frac{1}{\sin(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - y\cot(x)\right) = -\frac{\mu(x)}{\sin(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - y\cot(x)\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)\cot(x)$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{\sin(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\frac{\mu(x)}{\sin(x)} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int -\frac{\mu(x)}{\sin(x)} dx + c_1$

- Solve for y

  $y = \frac{\int -\frac{\mu(x)}{\sin(x)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\sin(x)}$

  $y = \sin(x)\left(\int -\frac{1}{\sin(x)^2} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = \sin(x)\left(\cot(x) + c_1\right)$

- Simplify

  $y = c_1 \sin(x) + \cos(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)-y(x)*cot(x)+1/sin(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(x) + \cos(x)$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 13

```
DSolve[y'[x]-y[x]*Cot[x]+1/Sin[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x) + c_1 \sin(x)$$

1.9 problem Problem 14.5 (c)

Internal problem ID [2494]
Internal file name [`OUTPUT/1986_Sunday_June_05_2022_02_42_21_AM_26596396/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.5 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{(x+y^3)\,y'-y=0}$$

1.9.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y}{y^3+x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{y(b_3 - a_2)}{y^3 + x} - \frac{y^2 a_3}{(y^3 + x)^2} + \frac{y(x a_2 + y a_3 + a_1)}{(y^3 + x)^2} - \left(\frac{1}{y^3 + x} - \frac{3y^3}{(y^3 + x)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{y^6 b_2 + 4x\, y^3 b_2 - y^4 a_2 + 3y^4 b_3 + 2y^3 b_1 - x b_1 + y a_1}{(y^3 + x)^2} = 0$$

Setting the numerator to zero gives

$$y^6 b_2 + 4x\, y^3 b_2 - y^4 a_2 + 3y^4 b_3 + 2y^3 b_1 - x b_1 + y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$b_2 v_2^6 - a_2 v_2^4 + 4b_2 v_1 v_2^3 + 3b_3 v_2^4 + 2b_1 v_2^3 + a_1 v_2 - b_1 v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$4b_2 v_1 v_2^3 - b_1 v_1 + b_2 v_2^6 + (-a_2 + 3b_3)\, v_2^4 + 2b_1 v_2^3 + a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_2 &= 0 \\ -b_1 &= 0 \\ 2b_1 &= 0 \\ 4b_2 &= 0 \\ -a_2 + 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 3b_3 \\ a_3 &= a_3 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= y \\ \eta &= 0 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= 0 - \left(\frac{y}{y^3 + x}\right)(y) \\ &= -\frac{y^2}{y^3 + x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta}dy\\ &= \int \frac{1}{-\frac{y^2}{y^3+x}}dy\end{aligned}$$

Which results in

$$S = -\frac{y^2}{2}+\frac{x}{y}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{y^3+x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= \frac{1}{y}\\ S_y &= -y-\frac{x}{y^2}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

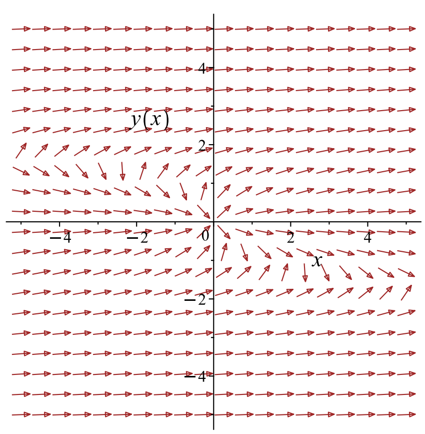
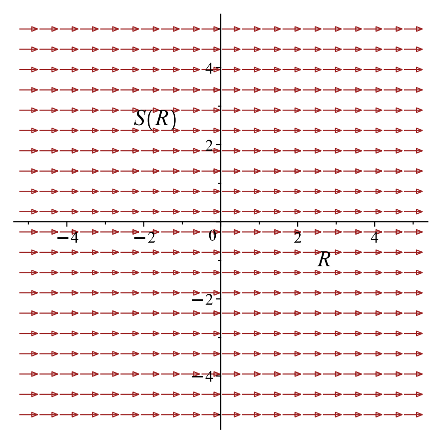
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{y^2}{2} + \frac{x}{y} = c_1$$

Which simplifies to

$$-\frac{y^2}{2} + \frac{x}{y} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{y^3+x}$ 	$R = x$ $S = -\frac{y^2}{2} + \frac{x}{y}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$-\frac{y^2}{2} + \frac{x}{y} = c_1 \tag{1}$$

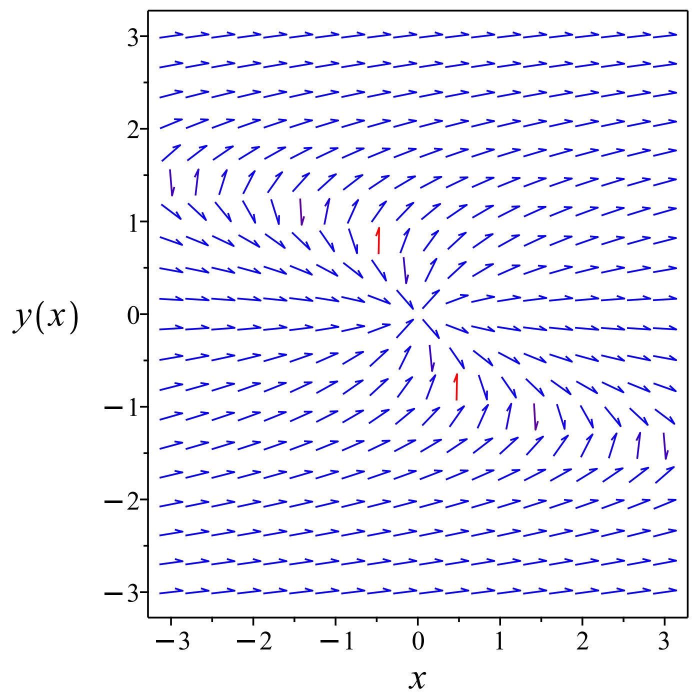


Figure 24: Slope field plot

Verification of solutions

$$-\frac{y^2}{2}+\frac{x}{y}=c_1$$

Verified OK.

1.9.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(y^3 + x\right) \mathrm{d}y &= (y)\,\mathrm{d}x \\
(-y)\,\mathrm{d}x + \left(y^3 + x\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -y \\
N(x, y) &= y^3 + x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y^3 + x\right) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{y^3 + x}((-1) - (1)) \\ &= -\frac{2}{y^3 + x} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{y}((1) - (-1)) \\ &= -\frac{2}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{2}{y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(y)} \\ &= \frac{1}{y^2} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^2}(-y) \\ &= -\frac{1}{y} \end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}\left(y^3+x\right) \\
&= \frac{y^3+x}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{1}{y}\right)+\left(\frac{y^3+x}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{y}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x}{y}+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{x}{y^2}+f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{y^3+x}{y^2}$. Therefore equation (4) becomes

$$
\frac{y^3+x}{y^2} = \frac{x}{y^2}+f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y)\,\mathrm{d}y$$

$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x}{y} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x}{y} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2} - \frac{x}{y} = c_1 \tag{1}$$

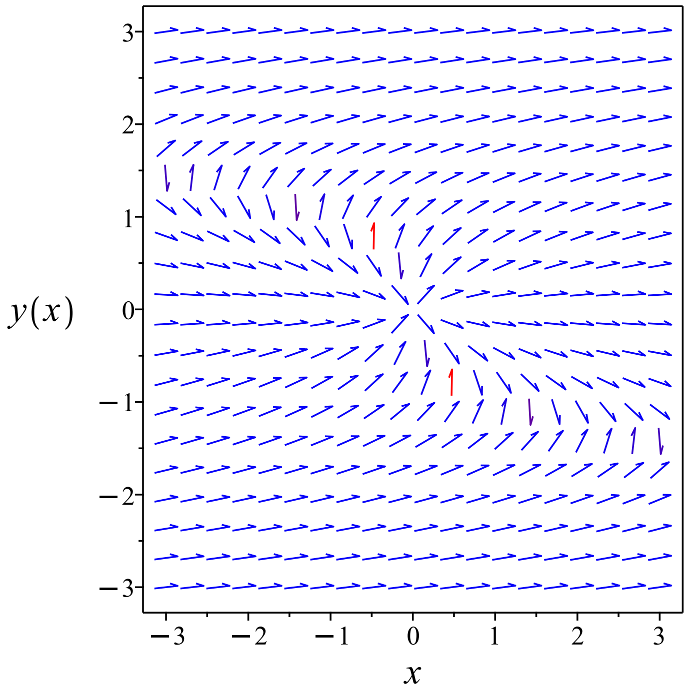


Figure 25: Slope field plot

Verification of solutions

$$\frac{y^2}{2} - \frac{x}{y} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 224

```
dsolve((x+y(x)^3)*diff(y(x),x)=y(x),y(x), singsol=all)
```

$$y(x) = \frac{\left(27x+3\sqrt{24c_1^3+81x^2}\right)^{\frac{2}{3}}-6c_1}{3\left(27x+3\sqrt{24c_1^3+81x^2}\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{i\sqrt{3}\left(27x+3\sqrt{24c_1^3+81x^2}\right)^{\frac{2}{3}}+6i\sqrt{3}\,c_1+\left(27x+3\sqrt{24c_1^3+81x^2}\right)^{\frac{2}{3}}-6c_1}{6\left(27x+3\sqrt{24c_1^3+81x^2}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{i\sqrt{3}\left(27x+3\sqrt{24c_1^3+81x^2}\right)^{\frac{2}{3}}+6i\sqrt{3}\,c_1-\left(27x+3\sqrt{24c_1^3+81x^2}\right)^{\frac{2}{3}}+6c_1}{6\left(27x+3\sqrt{24c_1^3+81x^2}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 1.757 (sec). Leaf size: 263

```
DSolve[(x+y[x]^3)*y'[x]==y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2\,3^{2/3}c_1-\sqrt[3]{3}\left(-9x+\sqrt{81x^2+24c_1{}^3}\right)^{2/3}}{3\sqrt[3]{-9x+\sqrt{81x^2+24c_1{}^3}}}$$

$$y(x) \to \frac{\sqrt[3]{3}\left(1-i\sqrt{3}\right)\left(-9x+\sqrt{81x^2+24c_1{}^3}\right)^{2/3}-2\sqrt[6]{3}\left(\sqrt{3}+3i\right)c_1}{6\sqrt[3]{-9x+\sqrt{81x^2+24c_1{}^3}}}$$

$$y(x) \to \frac{\sqrt[3]{3}\left(1+i\sqrt{3}\right)\left(-9x+\sqrt{81x^2+24c_1{}^3}\right)^{2/3}-2\sqrt[6]{3}\left(\sqrt{3}-3i\right)c_1}{6\sqrt[3]{-9x+\sqrt{81x^2+24c_1{}^3}}}$$

$$y(x) \to 0$$

1.10 problem Problem 14.6

Internal problem ID [2495]
Internal file name [`OUTPUT/1987_Sunday_June_05_2022_02_42_24_AM_32295778/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{y' + \frac{2x^2 + y^2 + x}{yx} = 0}$$

1.10.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= -\frac{2x^2 + y^2 + x}{yx} \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 20: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^2 y}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2 y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2 x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2x^2 + y^2 + x}{yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= x\,y^2 \\ S_y &= x^2 y \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -2x^3 - x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -2R^3 - R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2}R^4 - \frac{1}{3}R^3 + c_1 \tag{4}$$

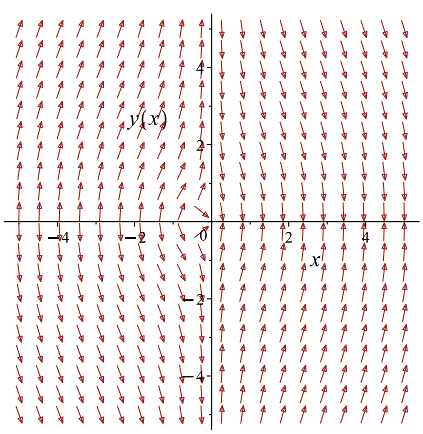
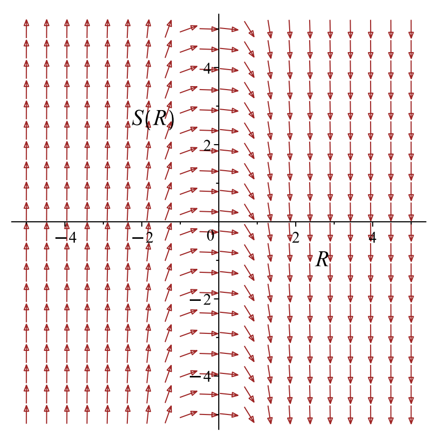
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2x^2}{2} = -\frac{1}{2}x^4 - \frac{1}{3}x^3 + c_1$$

Which simplifies to

$$\frac{y^2x^2}{2} = -\frac{1}{2}x^4 - \frac{1}{3}x^3 + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x^2+y^2+x}{yx}$ 	$R = x$ $S = \frac{y^2x^2}{2}$	$\frac{dS}{dR} = -2R^3 - R^2$ 

Summary
The solution(s) found are the following

$$\frac{y^2x^2}{2} = -\frac{1}{2}x^4 - \frac{1}{3}x^3 + c_1 \tag{1}$$

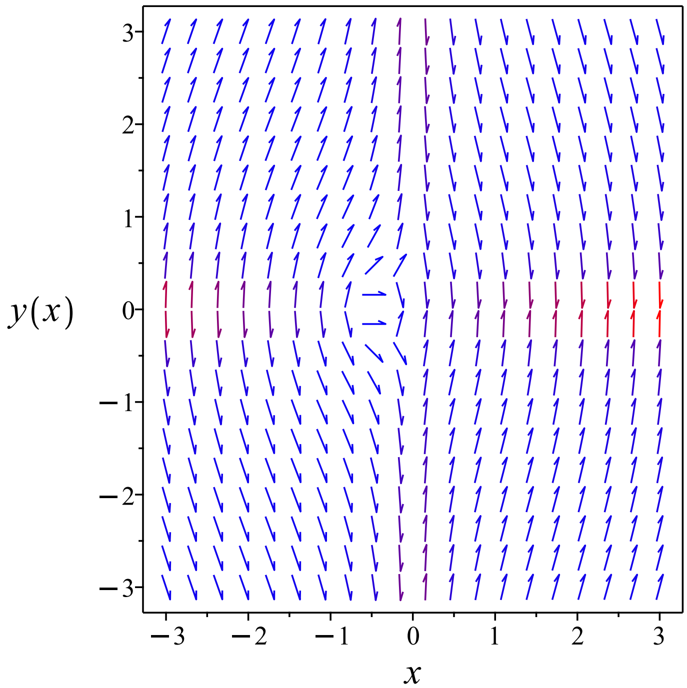


Figure 26: Slope field plot

Verification of solutions

$$\frac{y^2x^2}{2} = -\frac{1}{2}x^4 - \frac{1}{3}x^3 + c_1$$

Verified OK.

1.10.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{2x^2 + y^2 + x}{yx} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - \frac{2x^2 + x}{x}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= -\frac{2x^2+x}{x} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{y^2}{x} - \frac{2x^2+x}{x} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= -\frac{w(x)}{x} - \frac{2x^2+x}{x} \\ w' &= -\frac{2w}{x} - \frac{2(2x^2+x)}{x} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= -2 - 4x \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x} = -2 - 4x$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{2}{x}dx} \\
&= x^2
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)(-2 - 4x) \\
\frac{d}{dx}(x^2 w) &= (x^2)(-2 - 4x) \\
d(x^2 w) &= (-4x^3 - 2x^2)\, dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
x^2 w &= \int -4x^3 - 2x^2\, dx \\
x^2 w &= -x^4 - \frac{2}{3}x^3 + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = \frac{-x^4 - \frac{2}{3}x^3}{x^2} + \frac{c_1}{x^2}$$

which simplifies to

$$w(x) = \frac{-3x^4 - 2x^3 + 3c_1}{3x^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{-3x^4 - 2x^3 + 3c_1}{3x^2}$$

Solving for y gives

$$\begin{aligned}
y(x) &= \frac{\sqrt{-9x^4 - 6x^3 + 9c_1}}{3x} \\
y(x) &= -\frac{\sqrt{-9x^4 - 6x^3 + 9c_1}}{3x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-9x^4 - 6x^3 + 9c_1}}{3x} \tag{1}$$

$$y = -\frac{\sqrt{-9x^4 - 6x^3 + 9c_1}}{3x} \tag{2}$$

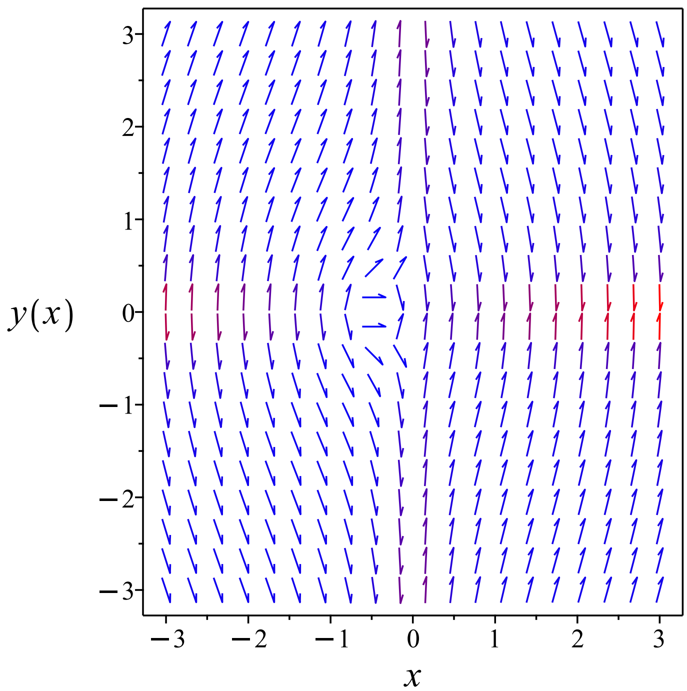


Figure 27: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-9x^4 - 6x^3 + 9c_1}}{3x}$$

Verified OK.

$$y = -\frac{\sqrt{-9x^4 - 6x^3 + 9c_1}}{3x}$$

Verified OK.

1.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(xy)\,\mathrm{d}y &= \left(-2x^2 - y^2 - x\right)\mathrm{d}x\\ \left(2x^2 + y^2 + x\right)\mathrm{d}x + (xy)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2x^2 + y^2 + x\\ N(x,y) &= xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x^2 + y^2 + x) \\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xy) \\ &= y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{yx}((2y) - (y)) \\ &= \frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{1}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)} \\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x(2x^2 + y^2 + x) \\ &= 2x^3 + x\,y^2 + x^2\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x(xy) \\ &= x^2 y\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(2x^3 + x\,y^2 + x^2\right) + \left(x^2 y\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 2x^3 + x\,y^2 + x^2\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{1}{2}x^4 + \frac{1}{2}y^2x^2 + \frac{1}{3}x^3 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 y$. Therefore equation (4) becomes

$$x^2 y = x^2 y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{2}x^4 + \frac{1}{2}y^2x^2 + \frac{1}{3}x^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{2}x^4 + \frac{1}{2}y^2x^2 + \frac{1}{3}x^3$$

Summary

The solution(s) found are the following

$$\frac{y^2x^2}{2} + \frac{x^4}{2} + \frac{x^3}{3} = c_1 \tag{1}$$

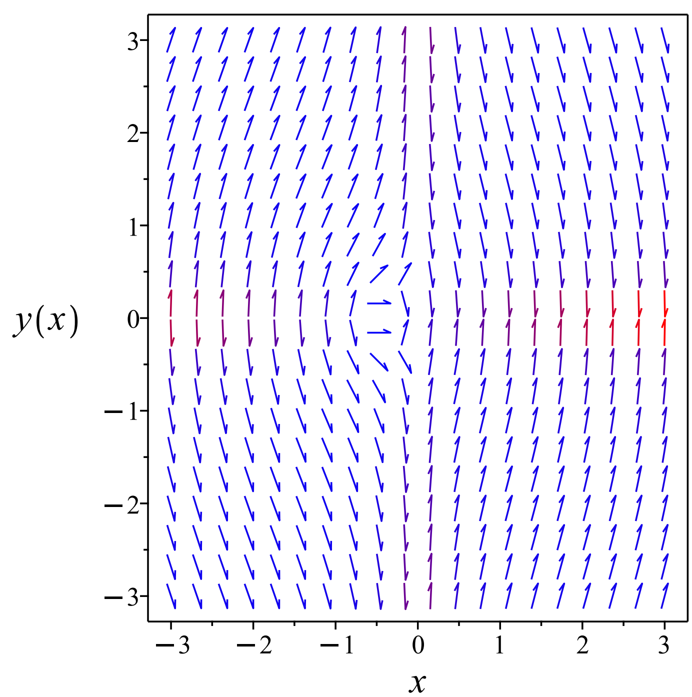


Figure 28: Slope field plot

Verification of solutions

$$\frac{y^2x^2}{2}+\frac{x^4}{2}+\frac{x^3}{3}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
dsolve(diff(y(x),x) = - (2*x^2+y(x)^2+x)/(x*y(x)),y(x), singsol=all)
```

$$y(x)=-\frac{\sqrt{-9x^4-6x^3+9c_1}}{3x}$$
$$y(x)=\frac{\sqrt{-9x^4-6x^3+9c_1}}{3x}$$

✓ Solution by Mathematica

Time used: 0.251 (sec). Leaf size: 56

```
DSolve[y'[x] == - (2*x^2+y[x]^2+x)/(x*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to-\frac{\sqrt{-x^4-\frac{2x^3}{3}+c_1}}{x}$$
$$y(x)\to\frac{\sqrt{-x^4-\frac{2x^3}{3}+c_1}}{x}$$

1.11 problem Problem 14.11

Internal problem ID [2496]
Internal file name [`OUTPUT/1988_Sunday_June_05_2022_02_42_29_AM_20341962/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(y-x)\,y'+3y=-2x}$$

1.11.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u(x)\,x-x)\,(u'(x)\,x+u(x))+3u(x)\,x=-2x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2+2u+2}{x\,(u-1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+2u+2}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+2u+2}{u-1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2+2u+2}{u-1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln\left(u^2+2u+2\right)}{2} - 2\arctan\left(u+1\right) = -\ln\left(x\right) + c_2$$

The solution is

$$\frac{\ln\left(u(x)^2+2u(x)+2\right)}{2} - 2\arctan\left(u(x)+1\right) + \ln\left(x\right) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{y^2}{x^2}+\frac{2y}{x}+2\right)}{2} - 2\arctan\left(\frac{y}{x}+1\right) + \ln\left(x\right) - c_2 = 0$$

$$\frac{\ln\left(\frac{y^2}{x^2}+\frac{2y}{x}+2\right)}{2} - 2\arctan\left(\frac{y+x}{x}\right) + \ln\left(x\right) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{x^2}+\frac{2y}{x}+2\right)}{2} - 2\arctan\left(\frac{y+x}{x}\right) + \ln\left(x\right) - c_2 = 0 \tag{1}$$

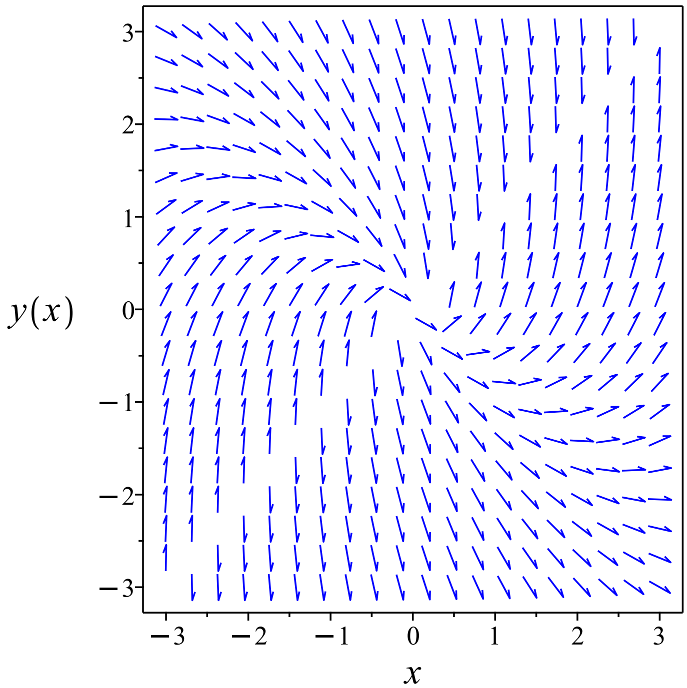


Figure 29: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+\frac{2y}{x}+2\right)}{2}-2\arctan\left(\frac{y+x}{x}\right)+\ln(x)-c_2=0$$

Verified OK.

1.11.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{2x+3y}{y-x}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 - \frac{(2x+3y)(b_3-a_2)}{y-x} - \frac{(2x+3y)^2 a_3}{(y-x)^2} \\ - \left(-\frac{2}{y-x} - \frac{2x+3y}{(y-x)^2}\right)(xa_2+ya_3+a_1) \\ - \left(-\frac{3}{y-x} + \frac{2x+3y}{(y-x)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^2a_2 + 4x^2a_3 + 4x^2b_2 - 2x^2b_3 - 4xya_2 + 12xya_3 + 2xyb_2 + 4xyb_3 - 3y^2a_2 + 4y^2a_3 - y^2b_2 + 3y^2b_3 + 5}{(-y+x)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} -2x^2a_2 - 4x^2a_3 - 4x^2b_2 + 2x^2b_3 + 4xya_2 - 12xya_3 - 2xyb_2 \\ - 4xyb_3 + 3y^2a_2 - 4y^2a_3 + y^2b_2 - 3y^2b_3 - 5xb_1 + 5ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} -2a_2v_1^2 + 4a_2v_1v_2 + 3a_2v_2^2 - 4a_3v_1^2 - 12a_3v_1v_2 - 4a_3v_2^2 - 4b_2v_1^2 \\ - 2b_2v_1v_2 + b_2v_2^2 + 2b_3v_1^2 - 4b_3v_1v_2 - 3b_3v_2^2 + 5a_1v_2 - 5b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-2a_2 - 4a_3 - 4b_2 + 2b_3)\, v_1^2 + (4a_2 - 12a_3 - 2b_2 - 4b_3)\, v_1 v_2 \\ - 5b_1 v_1 + (3a_2 - 4a_3 + b_2 - 3b_3)\, v_2^2 + 5a_1 v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}5a_1 &= 0\\ -5b_1 &= 0\\ -2a_2 - 4a_3 - 4b_2 + 2b_3 &= 0\\ 3a_2 - 4a_3 + b_2 - 3b_3 &= 0\\ 4a_2 - 12a_3 - 2b_2 - 4b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= 2a_3 + b_3\\ a_3 &= a_3\\ b_1 &= 0\\ b_2 &= -2a_3\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi\\ &= y - \left(-\frac{2x + 3y}{y - x}\right)(x)\\ &= \frac{-2x^2 - 2xy - y^2}{-y + x}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2x^2-2xy-y^2}{-y+x}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(2x^2 + 2xy + y^2)}{2} - 2\arctan\left(\frac{2x + 2y}{2x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2x + 3y}{y - x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x + 3y}{2x^2 + 2xy + y^2} \\ S_y &= \frac{y - x}{2x^2 + 2xy + y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

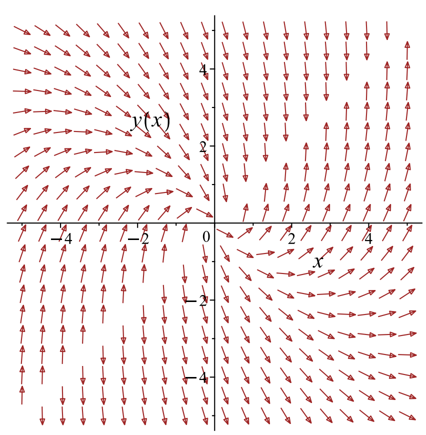
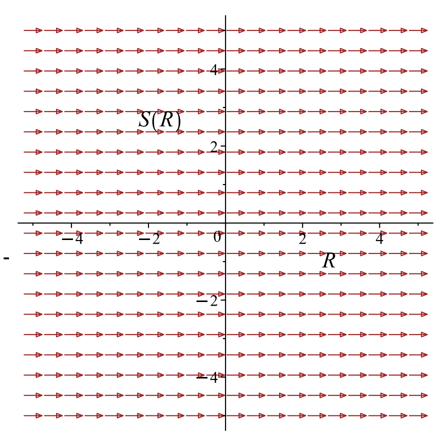
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2 + 2yx + 2x^2)}{2} - 2\arctan\left(\frac{y+x}{x}\right) = c_1$$

Which simplifies to

$$\frac{\ln(y^2 + 2yx + 2x^2)}{2} - 2\arctan\left(\frac{y+x}{x}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+3y}{y-x}$ 	$R = x$ $S = \frac{\ln(2x^2 + 2xy + y^2)}{2}$ -	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln\left(y^{2}+2yx+2x^{2}\right)}{2}-2\arctan\left(\frac{y+x}{x}\right)=c_{1} \tag{1}$$

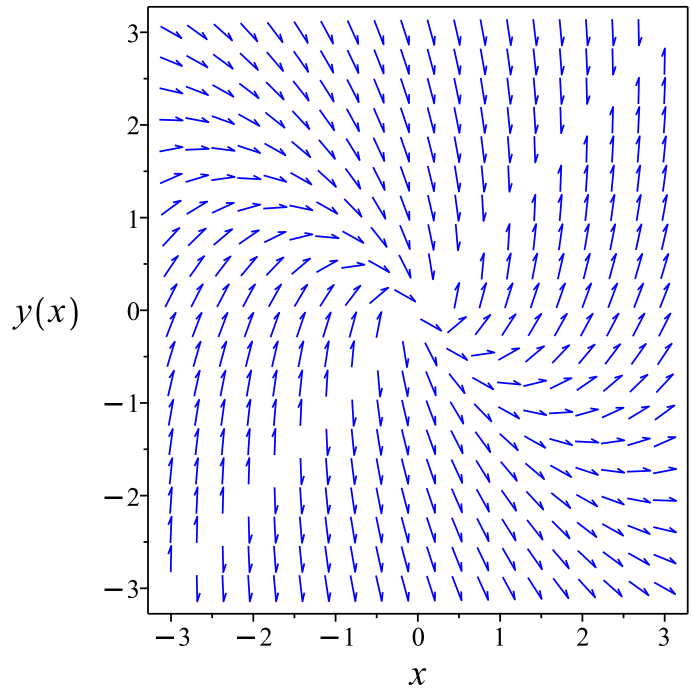


Figure 30: Slope field plot

Verification of solutions

$$\frac{\ln\left(y^{2}+2yx+2x^{2}\right)}{2}-2\arctan\left(\frac{y+x}{x}\right)=c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve((y(x)-x)*diff(y(x),x)+2*x+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = x\left(-1 + \tan\left(\mathrm{RootOf}\left(-4\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x\right) + 2c_1\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 45

```
DSolve[(y[x]-x)*y'[x]+2*x+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\frac{1}{2}\log\left(\frac{y(x)^2}{x^2} + \frac{2y(x)}{x} + 2\right) - 2\arctan\left(\frac{y(x)}{x} + 1\right) = -\log(x) + c_1, y(x)\right]$$

1.12 problem Problem 14.14

Internal problem ID [2497]
Internal file name [`OUTPUT/1989_Sunday_June_05_2022_02_42_33_AM_67771648/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeC", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], [_Abel, `2nd type`, `class C`],
   _dAlembert]
```

$$\boxed{y' - \frac{1}{x+2y+1} = 0}$$

1.12.1 Solving as homogeneousTypeC ode

Let

$$z = x + 2y + 1 \tag{1}$$

Then

$$z'(x) = 1 + 2y'$$

Therefore

$$y' = \frac{z'(x)}{2} - \frac{1}{2}$$

Hence the given ode can now be written as

$$\frac{z'(x)}{2} - \frac{1}{2} = \frac{1}{z}$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{\frac{2}{z} + 1} dz$$

$$x + c_1 = z - 2\ln(2 + z)$$

Replacing z back by its value from (1) then the above gives the solution as

$$y = -\frac{3}{2} - \text{LambertW}\left(-\frac{e^{-1-\frac{x}{2}-\frac{c_1}{2}}}{2}\right) - \frac{x}{2}$$

$$y = -\frac{3}{2} - \text{LambertW}\left(-\frac{e^{-1-\frac{x}{2}-\frac{c_1}{2}}}{2}\right) - \frac{x}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{3}{2} - \text{LambertW}\left(-\frac{e^{-1-\frac{x}{2}-\frac{c_1}{2}}}{2}\right) - \frac{x}{2} \tag{1}$$

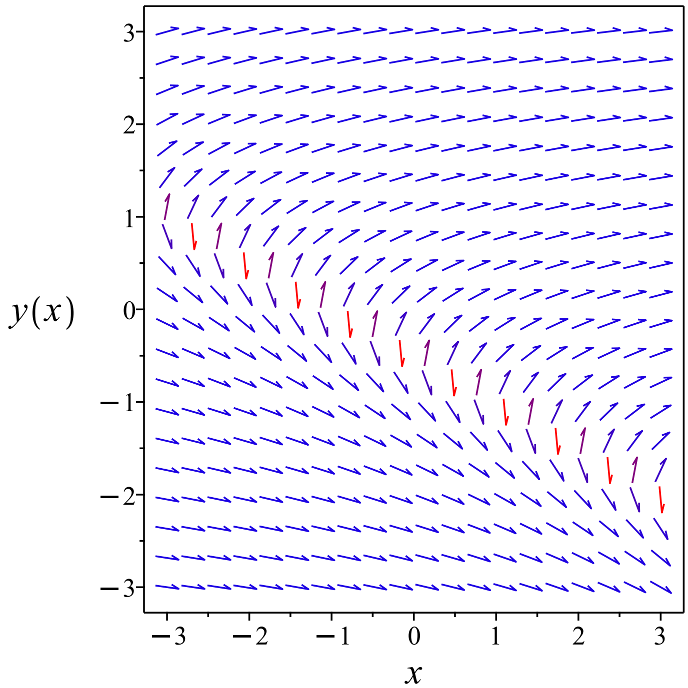


Figure 31: Slope field plot

Verification of solutions

$$y = -\frac{3}{2} - \text{LambertW}\left(-\frac{e^{-1-\frac{x}{2}-\frac{c_1}{2}}}{2}\right) - \frac{x}{2}$$

Verified OK.

1.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{1}{x+2y+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 22: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 1 \\
\eta(x, y) &= -\frac{1}{2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{-\frac{1}{2}}{1}\\ &= -\frac{1}{2}\end{aligned}$$

This is easily solved to give

$$y = -\frac{x}{2} + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{x}{2} + y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{1}{x + 2y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= \frac{1}{2}\\ R_y &= 1\\ S_x &= 1\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2x + 4y + 2}{3 + x + 2y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{4R + 2}{3 + 2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2R - 2\ln(3 + 2R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = x + 2y - 2\ln(3 + x + 2y) + c_1$$

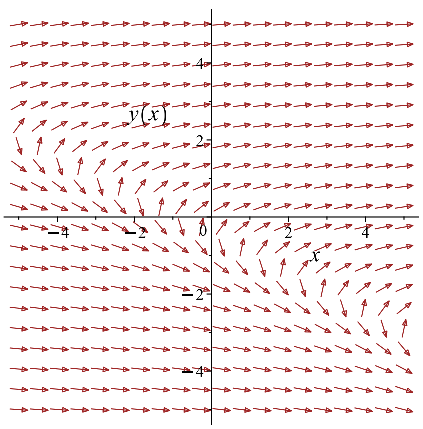
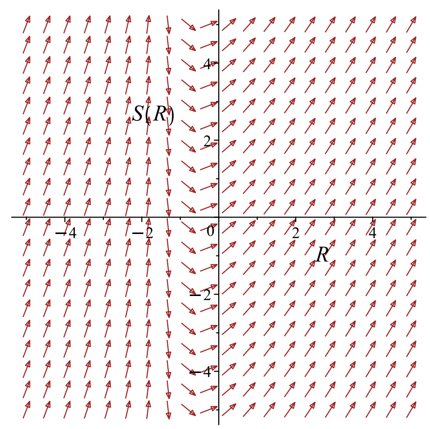
Which simplifies to

$$x = x + 2y - 2\ln(3 + x + 2y) + c_1$$

Which gives

$$y = -\operatorname{LambertW}\left(-\frac{e^{-\frac{3}{2}-\frac{x}{2}+\frac{c_1}{2}}}{2}\right) - \frac{3}{2} - \frac{x}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1}{x+2y+1}$ 	$R = \frac{x}{2} + y$ $S = x$	$\frac{dS}{dR} = \frac{4R+2}{3+2R}$ 

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{e^{-\frac{3}{2}-\frac{x}{2}+\frac{c_1}{2}}}{2}\right) - \frac{3}{2} - \frac{x}{2} \tag{1}$$

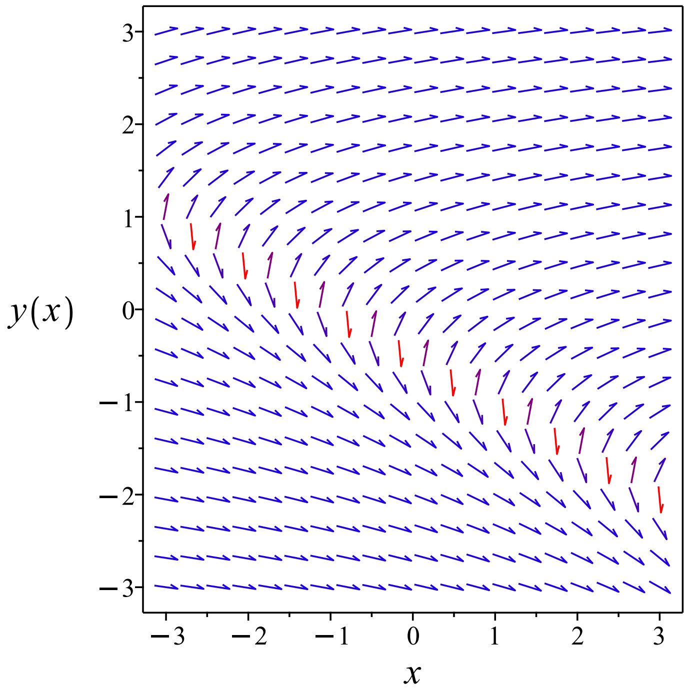


Figure 32: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-\frac{e^{-\frac{3}{2}-\frac{x}{2}+\frac{c_1}{2}}}{2}\right) - \frac{3}{2} - \frac{x}{2}$$

Verified OK.

1.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x + 2y + 1)\,\mathrm{d}y &= \mathrm{d}x \\ -\,\mathrm{d}x + (x + 2y + 1)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -1 \\ N(x, y) &= x + 2y + 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x + 2y + 1) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x+2y+1}((0)-(1)) \\
&= -\frac{1}{x+2y+1}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -1((1)-(0)) \\
&= -1
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -1 \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-y} \\
&= \mathrm{e}^{-y}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-y}(-1) \\
&= -\mathrm{e}^{-y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-y}(x+2y+1) \\
&= (x+2y+1)\,\mathrm{e}^{-y}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-y}\right) + \left((x+2y+1)\,\mathrm{e}^{-y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\mathrm{e}^{-y}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\mathrm{e}^{-y}x + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{-y}x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x+2y+1)\,\mathrm{e}^{-y}$. Therefore equation (4) becomes

$$
(x+2y+1)\,\mathrm{e}^{-y} = \mathrm{e}^{-y}x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= 2\,\mathrm{e}^{-y}y + \mathrm{e}^{-y} \\
&= \mathrm{e}^{-y}(2y+1)
\end{aligned}
$$

Integrating the above w.r.t y results in

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\mathrm{e}^{-y}(2y+1)\right)\mathrm{d}y \\
f(y) &= -(2y+3)\,\mathrm{e}^{-y} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\mathrm{e}^{-y} x - (2y+3)\,\mathrm{e}^{-y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\mathrm{e}^{-y} x - (2y+3)\,\mathrm{e}^{-y}$$

The solution becomes

$$y = -\frac{x}{2} - \mathrm{LambertW}\left(\frac{c_1 \mathrm{e}^{-\frac{x}{2}-\frac{3}{2}}}{2}\right) - \frac{3}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{2} - \mathrm{LambertW}\left(\frac{c_1 \mathrm{e}^{-\frac{x}{2}-\frac{3}{2}}}{2}\right) - \frac{3}{2} \tag{1}$$

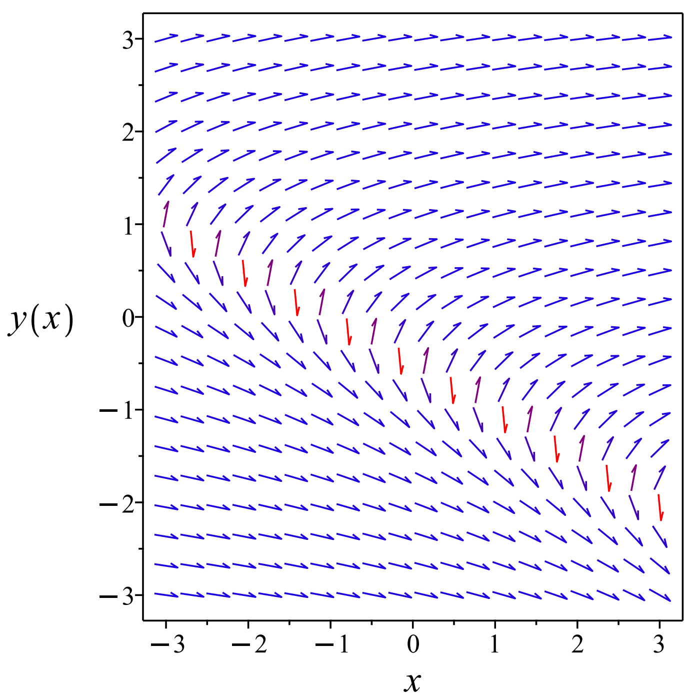


Figure 33: Slope field plot

Verification of solutions

$$y = -\frac{x}{2} - \mathrm{LambertW}\left(\frac{c_1 \mathrm{e}^{-\frac{x}{2}-\frac{3}{2}}}{2}\right) - \frac{3}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(x),x) = 1/(x+2*y(x)+1),y(x), singsol=all)
```

$$y(x) = -\mathrm{LambertW}\left(-\frac{c_1 \mathrm{e}^{-\frac{x}{2}-\frac{3}{2}}}{2}\right) - \frac{x}{2} - \frac{3}{2}$$

✓ Solution by Mathematica

Time used: 60.047 (sec). Leaf size: 34

```
DSolve[y'[x] == 1/(x+2*y[x]+1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-2W\left(-\frac{1}{2}c_1 e^{-\frac{x}{2}-\frac{3}{2}}\right) - x - 3\right)$$

1.13 problem Problem 14.15

Internal problem ID [2498]
Internal file name [`OUTPUT/1990_Sunday_June_05_2022_02_42_40_AM_88204930/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' + \frac{y + x}{3x + 3y - 4} = 0}$$

1.13.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y + x}{3x + 3y - 4}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(y+x)(b_3-a_2)}{3x+3y-4} - \frac{(y+x)^2 a_3}{(3x+3y-4)^2} - \left(-\frac{1}{3x+3y-4} + \frac{3x+3y}{(3x+3y-4)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{1}{3x+3y-4} + \frac{3x+3y}{(3x+3y-4)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^2a_2 - x^2a_3 + 9x^2b_2 - 3x^2b_3 + 6xya_2 - 2xya_3 + 18xyb_2 - 6xyb_3 + 3y^2a_2 - y^2a_3 + 9y^2b_2 - 3y^2b_3 - 8xa_2}{(3x+3y-4)^2} = 0$$

Setting the numerator to zero gives

$$3x^2a_2 - x^2a_3 + 9x^2b_2 - 3x^2b_3 + 6xya_2 - 2xya_3 + 18xyb_2 - 6xyb_3 + 3y^2a_2 - y^2a_3 + 9y^2b_2 - 3y^2b_3 - 8xa_2 - 28xb_2 + 4xb_3 - 4ya_2 - 4ya_3 - 24yb_2 - 4a_1 - 4b_1 + 16b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$3a_2v_1^2 + 6a_2v_1v_2 + 3a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - a_3v_2^2 + 9b_2v_1^2 + 18b_2v_1v_2 + 9b_2v_2^2 - 3b_3v_1^2 - 6b_3v_1v_2 - 3b_3v_2^2 - 8a_2v_1 - 4a_2v_2 - 4a_3v_2 - 28b_2v_1 - 24b_2v_2 + 4b_3v_1 - 4a_1 - 4b_1 + 16b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(3a_2 - a_3 + 9b_2 - 3b_3) v_1^2 + (6a_2 - 2a_3 + 18b_2 - 6b_3) v_1 v_2 + (-8a_2 - 28b_2 + 4b_3) v_1 + (3a_2 - a_3 + 9b_2 - 3b_3) v_2^2 + (-4a_2 - 4a_3 - 24b_2) v_2 - 4a_1 - 4b_1 + 16b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-4a_1 - 4b_1 + 16b_2 &= 0 \\
-8a_2 - 28b_2 + 4b_3 &= 0 \\
-4a_2 - 4a_3 - 24b_2 &= 0 \\
3a_2 - a_3 + 9b_2 - 3b_3 &= 0 \\
6a_2 - 2a_3 + 18b_2 - 6b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -b_1 + 4b_2 \\
a_2 &= -3b_2 \\
a_3 &= -3b_2 \\
b_1 &= b_1 \\
b_2 &= b_2 \\
b_3 &= b_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -1 \\
\eta &= 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 1 - \left(-\frac{y + x}{3x + 3y - 4}\right)(-1) \\
&= \frac{2x + 2y - 4}{3x + 3y - 4} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2x+2y-4}{3x+3y-4}} dy \end{aligned}$$

Which results in

$$S = \frac{3y}{2} + \ln (x + y - 2)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y + x}{3x + 3y - 4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x + y - 2} \\ S_y &= \frac{3}{2} + \frac{1}{x + y - 2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3y}{2} + \ln(x + y - 2) = -\frac{x}{2} + c_1$$

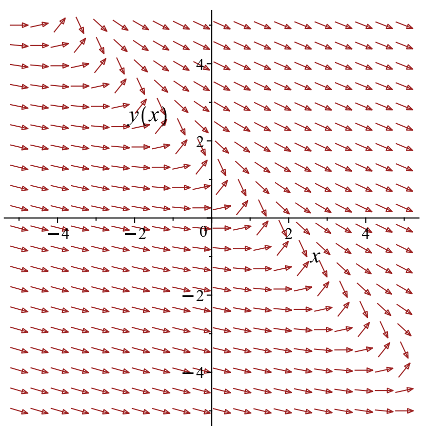
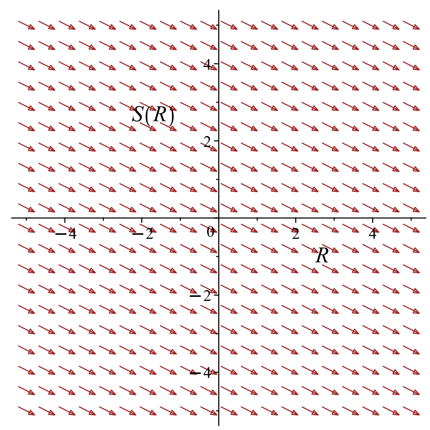
Which simplifies to

$$\frac{3y}{2} + \ln(x + y - 2) = -\frac{x}{2} + c_1$$

Which gives

$$y = \frac{2\,\mathrm{LambertW}\left(\frac{3\,\mathrm{e}^{x-3+c_1}}{2}\right)}{3} - x + 2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y+x}{3x+3y-4}$ 	$R = x$ $S = \frac{3y}{2} + \ln(x + y - 2)$	$\frac{dS}{dR} = -\frac{1}{2}$ 

Summary

The solution(s) found are the following

$$y = \frac{2\,\mathrm{LambertW}\left(\frac{3\,\mathrm{e}^{x-3+c_1}}{2}\right)}{3} - x + 2 \tag{1}$$

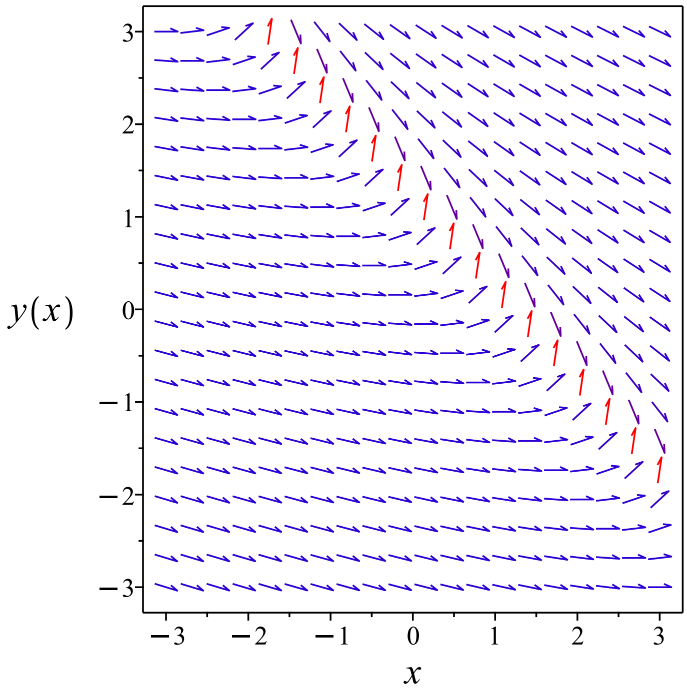


Figure 34: Slope field plot

Verification of solutions

$$y = \frac{2\,\mathrm{LambertW}\left(\frac{3\,\mathrm{e}^{x-3+c_1}}{2}\right)}{3} - x + 2$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -1, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 21

```
dsolve(diff(y(x),x) = - (x+y(x))/(3*x+3*y(x)-4),y(x), singsol=all)
```

$$y(x) = \frac{2\,\mathrm{LambertW}\left(\frac{3\,\mathrm{e}^{x-3-c_1}}{2}\right)}{3} - x + 2$$

✓ Solution by Mathematica

Time used: 3.788 (sec). Leaf size: 33

```
DSolve[y'[x] == - (x+y[x])/(3*x+3*y[x]-4),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{3}W\left(-e^{x-1+c_1}\right) - x + 2$$
$$y(x) \to 2 - x$$

1.14 problem Problem 14.16

Internal problem ID [2499]
Internal file name [`OUTPUT/1991_Sunday_June_05_2022_02_42_42_AM_8312713/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \tan(x)\cos(y)(\cos(y) + \sin(y)) = 0}$$

1.14.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \tan(x)\cos(y)(\cos(y) + \sin(y)) \end{aligned}$$

Where $f(x) = \tan(x)$ and $g(y) = \cos(y)(\cos(y) + \sin(y))$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\cos(y)(\cos(y) + \sin(y))}\, dy &= \tan(x)\, dx \\ \int \frac{1}{\cos(y)(\cos(y) + \sin(y))}\, dy &= \int \tan(x)\, dx \\ \ln(\tan(y) + 1) &= -\ln(\cos(x)) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\tan(y) + 1 = e^{-\ln(\cos(x))+c_1}$$

Which simplifies to

$$\tan(y) + 1 = \frac{c_2}{\cos(x)}$$

Summary

The solution(s) found are the following

$$y = -\arctan\left(\frac{-e^{c_1}c_2 + \cos(x)}{\cos(x)}\right) \tag{1}$$

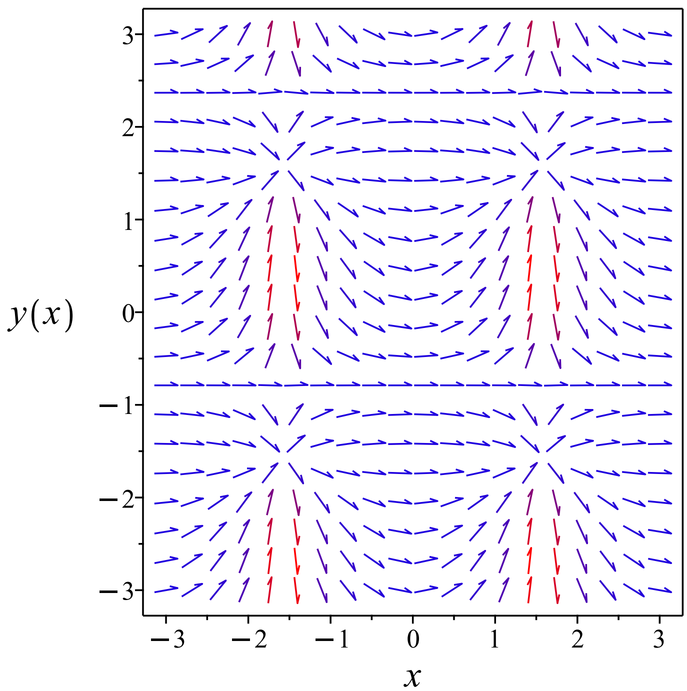


Figure 35: Slope field plot

Verification of solutions

$$y = -\arctan\left(\frac{-e^{c_1}c_2 + \cos(x)}{\cos(x)}\right)$$

Verified OK.

1.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \tan(x)\cos(y)(\cos(y)+\sin(y)) \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 24: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{1}{\tan(x)}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{\tan(x)}} dx \end{aligned}$$

Which results in

$$S = -\ln(\cos(x))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \tan(x)\cos(y)(\cos(y) + \sin(y))$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \tan(x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{\sec(y)}{\cos(y) + \sin(y)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{\sec(R)}{\cos(R) + \sin(R)}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(\tan(R) + 1) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln(\cos(x)) = \ln(\tan(y) + 1) + c_1
$$

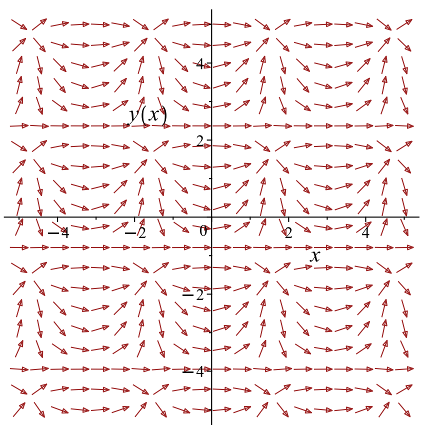
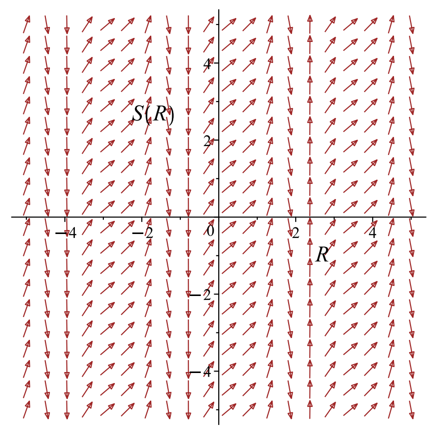
Which simplifies to

$$
-\ln(\cos(x)) = \ln(\tan(y) + 1) + c_1
$$

Which gives

$$
y = -\arctan\left(\frac{(\cos(x)\,\mathrm{e}^{c_1} - 1)\,\mathrm{e}^{-c_1}}{\cos(x)}\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \tan(x)\cos(y)(\cos(y)+\sin(y))$ 	$R = y$ $S = -\ln(\cos(x))$	$\frac{dS}{dR} = \frac{\sec(R)}{\cos(R)+\sin(R)}$ 

Summary
The solution(s) found are the following

$$y = -\arctan\left(\frac{(\cos(x)\,e^{c_1} - 1)\,e^{-c_1}}{\cos(x)}\right) \tag{1}$$

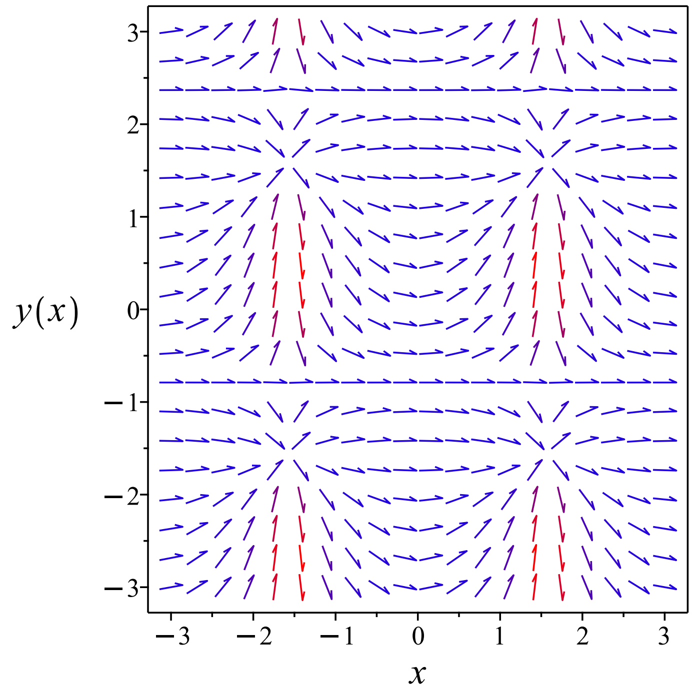


Figure 36: Slope field plot

Verification of solutions

$$y = -\arctan\left(\frac{(\cos(x)\,\mathrm{e}^{c_1} - 1)\,\mathrm{e}^{-c_1}}{\cos(x)}\right)$$

Verified OK.

1.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{(\cos(y) + \sin(y))\cos(y)}\right) \mathrm{d}y &= (\tan(x)) \,\mathrm{d}x \\
(-\tan(x)) \,\mathrm{d}x + \left(\frac{1}{(\cos(y) + \sin(y))\cos(y)}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\tan(x) \\
N(x, y) &= \frac{1}{(\cos(y) + \sin(y))\cos(y)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\tan(x)) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{(\cos(y)+\sin(y))\cos(y)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\tan(x)\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \ln(\cos(x)) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{1}{(\cos(y)+\sin(y))\cos(y)}$. Therefore equation (4) becomes

$$
\frac{1}{(\cos(y)+\sin(y))\cos(y)} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= \frac{1}{(\cos(y)+\sin(y))\cos(y)} \\
&= \frac{\sec(y)}{\cos(y)+\sin(y)}
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{\sec(y)}{\cos(y)+\sin(y)}\right)\mathrm{d}y$$
$$f(y) = \ln(\tan(y)+1) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(\cos(x)) + \ln(\tan(y)+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(\cos(x)) + \ln(\tan(y)+1)$$

Summary
The solution(s) found are the following

$$\ln(\cos(x)) + \ln(\tan(y)+1) = c_1 \tag{1}$$

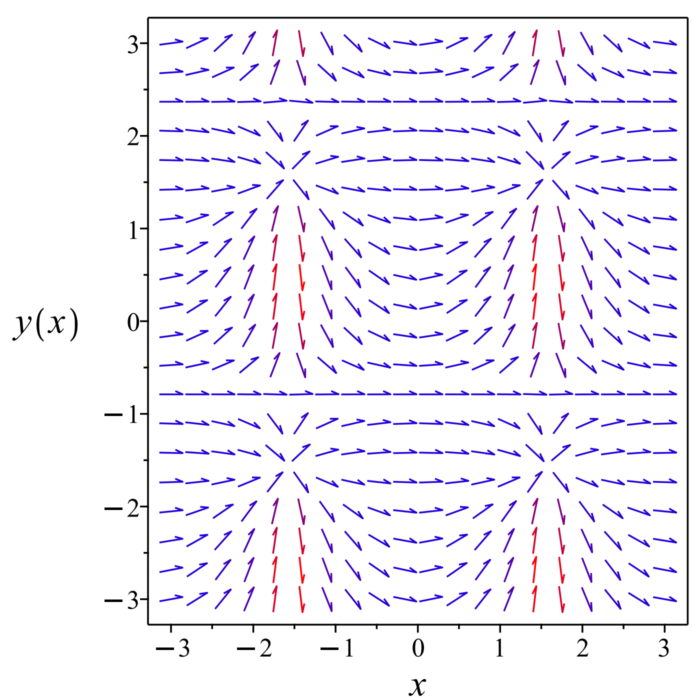


Figure 37: Slope field plot

Verification of solutions

$$\ln(\cos(x)) + \ln(\tan(y) + 1) = c_1$$

Verified OK.

1.14.4 Maple step by step solution

Let's solve

$y' - \tan(x)\cos(y)(\cos(y) + \sin(y)) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(\cos(y)+\sin(y))\cos(y)} = \tan(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{(\cos(y)+\sin(y))\cos(y)} dx = \int \tan(x)\, dx + c_1$

- Evaluate integral

  $\ln(\tan(y) + 1) = -\ln(\cos(x)) + c_1$

- Solve for y

  $y = -\arctan\left(\frac{-e^{c_1} + \cos(x)}{\cos(x)}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 11

```
dsolve(diff(y(x),x) = tan(x)*cos(y(x))*( cos(y(x)) + sin(y(x)) ),y(x), singsol=all)
```

$$y(x) = \arctan\left(-1 + \sec\left(x\right) c_1\right)$$

✓ Solution by Mathematica

Time used: 60.547 (sec). Leaf size: 143

```
DSolve[y'[x]==Tan[x]*Cos[y[x]]*( Cos[y[x]] + Sin[y[x]] ),y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\arccos\left(-\frac{\cos(x)}{\sqrt{\cos(2x) - 2e^{\frac{c_1}{2}}\cos(x) + 1 + e^{c_1}}}\right)$$

$$y(x) \to \arccos\left(-\frac{\cos(x)}{\sqrt{\cos(2x) - 2e^{\frac{c_1}{2}}\cos(x) + 1 + e^{c_1}}}\right)$$

$$y(x) \to -\arccos\left(\frac{\cos(x)}{\sqrt{\cos(2x) - 2e^{\frac{c_1}{2}}\cos(x) + 1 + e^{c_1}}}\right)$$

$$y(x) \to \arccos\left(\frac{\cos(x)}{\sqrt{\cos(2x) - 2e^{\frac{c_1}{2}}\cos(x) + 1 + e^{c_1}}}\right)$$

1.15 problem Problem 14.17

Internal problem ID [2500]
Internal file name [`OUTPUT/1992_Sunday_June_05_2022_02_43_03_AM_95188742/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _exact, _rational, [_Abel, `2nd
   type`, `class B`]]
```

$$\boxed{x(1-2x^2y)\,y'+y-3y^2x^2=0}$$

With initial conditions

$$\left[y(1)=\frac{1}{2}\right]$$

1.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y)\\ &= -\frac{y(3x^2y-1)}{x\,(2x^2y-1)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=\frac{1}{2}$ is

$$\{-\infty \le x < -1, -1 < x < 0, 0 < x < 1, 1 < x \le \infty\}$$

But the point $x_0 = 1$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

1.15.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y(3x^2y-1)}{x\,(2x^2y-1)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2 + ya_3 + a_1 \tag{1E}
$$

$$
\eta = xb_2 + yb_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{y(3x^2y-1)\,(b_3-a_2)}{x\,(2x^2y-1)} - \frac{y^2(3x^2y-1)^2\,a_3}{x^2\,(2x^2y-1)^2} \\
&- \left(-\frac{6y^2}{2x^2y-1} + \frac{y(3x^2y-1)}{x^2\,(2x^2y-1)} + \frac{4y^2(3x^2y-1)}{(2x^2y-1)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{3x^2y-1}{x\,(2x^2y-1)} - \frac{3yx}{2x^2y-1} + \frac{2y(3x^2y-1)\,x}{(2x^2y-1)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{10x^6y^2b_2 - 15x^4y^4a_3 + 6x^5y^2b_1 - 6x^4y^3a_1 - 10x^4yb_2 - 2x^3y^2a_2 - x^3y^2b_3 + 9x^2y^3a_3 - 6x^3yb_1 + 3x^2y^2a_1 +}{(2x^2y-1)^2\,x^2}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$10x^6y^2b_2 - 15x^4y^4a_3 + 6x^5y^2b_1 - 6x^4y^3a_1 - 10x^4yb_2 - 2x^3y^2a_2 - x^3y^2b_3 + 9x^2y^3a_3 - 6x^3yb_1 + 3x^2y^2a_1 + 2b_2x^2 - 2y^2a_3 + xb_1 - ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-15a_3v_1^4v_2^4 + 10b_2v_1^6v_2^2 - 6a_1v_1^4v_2^3 + 6b_1v_1^5v_2^2 - 2a_2v_1^3v_2^2 + 9a_3v_1^2v_2^3 - 10b_2v_1^4v_2 - b_3v_1^3v_2^2 + 3a_1v_1^2v_2^2 - 6b_1v_1^3v_2 - 2a_3v_2^2 + 2b_2v_1^2 - a_1v_2 + b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$10b_2v_1^6v_2^2 + 6b_1v_1^5v_2^2 - 15a_3v_1^4v_2^4 - 6a_1v_1^4v_2^3 - 10b_2v_1^4v_2 + (-2a_2 - b_3)\,v_1^3v_2^2 - 6b_1v_1^3v_2 + 9a_3v_1^2v_2^3 + 3a_1v_1^2v_2^2 + 2b_2v_1^2 + b_1v_1 - 2a_3v_2^2 - a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_1 &= 0 \\
-6a_1 &= 0 \\
-a_1 &= 0 \\
3a_1 &= 0 \\
-15a_3 &= 0 \\
-2a_3 &= 0 \\
9a_3 &= 0 \\
-6b_1 &= 0 \\
6b_1 &= 0 \\
-10b_2 &= 0 \\
2b_2 &= 0 \\
10b_2 &= 0 \\
-2a_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -2y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -2y - \left(-\frac{y(3x^2y-1)}{x\,(2x^2y-1)}\right)(x) \\
&= \frac{-y^2x^2+y}{2x^2y-1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-y^2x^2+y}{2x^2y-1}} dy \end{aligned}$$

Which results in

$$S = -\ln\left(y\left(x^2y - 1\right)\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(3x^2y - 1)}{x\,(2x^2y - 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2xy}{x^2y - 1} \\ S_y &= -\frac{1}{y} - \frac{x^2}{x^2y - 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

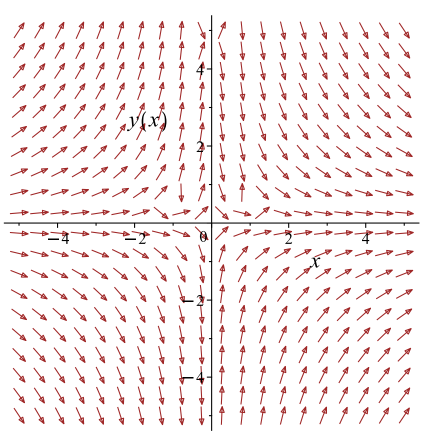
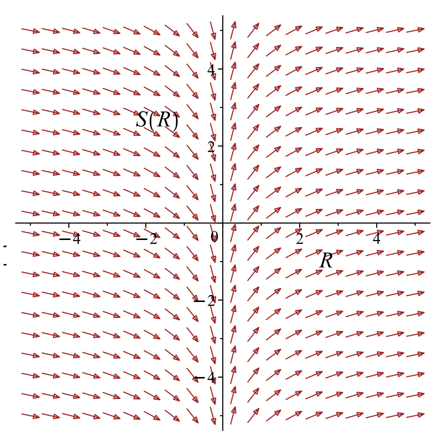
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(y) - \ln\left(x^2y - 1\right) = \ln(x) + c_1$$

Which simplifies to

$$-\ln(y) - \ln\left(x^2y - 1\right) = \ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(3x^2y-1)}{x(2x^2y-1)}$  	$R = x$ $S = -\ln(y) - \ln\left(x^2y - \right.$	$\frac{dS}{dR} = \frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$2\ln(2) - i\pi = c_1$$

$$c_1 = 2\ln(2) - i\pi$$

Substituting c_1 found above in the general solution gives

$$-\ln(y) - \ln\left(x^2 y - 1\right) = \ln(x) + 2\ln(2) - i\pi$$

Summary
The solution(s) found are the following

$$-\ln(y) - \ln\left(x^2 y - 1\right) = \ln(x) + 2\ln(2) - i\pi \tag{1}$$

Verification of solutions

$$-\ln(y) - \ln\left(x^2 y - 1\right) = \ln(x) + 2\ln(2) - i\pi$$

Verified OK.

1.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(x\left(-2x^2y+1\right)\right)\mathrm{d}y &= \left(3y^2x^2-y\right)\mathrm{d}x \\
\left(-3y^2x^2+y\right)\mathrm{d}x + \left(x\left(-2x^2y+1\right)\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= -3y^2x^2+y \\
N(x,y) &= x\left(-2x^2y+1\right)
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3y^2x^2+y\right) \\
&= -6x^2y+1
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(-2x^2y+1\right)\right) \\
&= -6x^2y+1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int M \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int -3y^2x^2 + y \,\mathrm{d}x$$

$$\phi = -xy\left(x^2y - 1\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= -x\left(x^2y - 1\right) - x^3y + f'(y) \qquad (4)\\ &= -2x^3y + x + f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x(-2x^2y + 1)$. Therefore equation (4) becomes

$$x\left(-2x^2y + 1\right) = -2x^3y + x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -xy\left(x^2y - 1\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -xy\left(x^2y - 1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{4} = c_1$$

$$c_1 = \frac{1}{4}$$

Substituting c_1 found above in the general solution gives

$$-xy\left(x^2y-1\right) = \frac{1}{4}$$

Summary
The solution(s) found are the following

$$-xy\left(x^2y-1\right) = \frac{1}{4} \tag{1}$$

Verification of solutions

$$-xy\left(x^2y-1\right) = \frac{1}{4}$$

Verified OK.

1.15.4 Maple step by step solution

Let's solve

$\left[x(1-2x^2y)\, y' + y - 3y^2x^2 = 0, y(1) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$
  - Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right) y' = 0$
  - Evaluate derivatives

    $-6x^2y + 1 = -6x^2y + 1$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y)\,, N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

$F(x,y)=\int \left(-3y^2x^2+y\right)dx+f_1(y)$

- Evaluate integral

  $F(x,y)=-y(x^3y-x)+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x(-2x^2y+1)=-2x^3y+x+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=2x^3y-x+x(-2x^2y+1)$

- Solve for $f_1(y)$

  $f_1(y)=0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=-y(x^3y-x)$

- Substitute $F(x,y)$ into the solution of the ODE

  $-y(x^3y-x)=c_1$

- Solve for y

  $\left\{y=\frac{1+\sqrt{-4c_1x+1}}{2x^2}, y=-\frac{-1+\sqrt{-4c_1x+1}}{2x^2}\right\}$

- Use initial condition $y(1)=\frac{1}{2}$

  $\frac{1}{2}=\frac{1}{2}+\frac{\sqrt{-4c_1+1}}{2}$

- Solve for c_1

  $c_1=\frac{1}{4}$

- Substitute $c_1=\frac{1}{4}$ into general solution and simplify

  $y=\frac{1+\sqrt{1-x}}{2x^2}$

- Use initial condition $y(1)=\frac{1}{2}$

  $\frac{1}{2}=\frac{1}{2}-\frac{\sqrt{-4c_1+1}}{2}$

- Solve for c_1

  $c_1=\frac{1}{4}$

- Substitute $c_1=\frac{1}{4}$ into general solution and simplify

$y = \frac{1-\sqrt{1-x}}{2x^2}$

- Solutions to the IVP

$\left\{y = \frac{1+\sqrt{1-x}}{2x^2}, y = \frac{1-\sqrt{1-x}}{2x^2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 35

```
dsolve([x*(1-2*x^2*y(x))*diff(y(x),x) +y(x) = 3*x^2*y(x)^2,y(1) = 1/2],y(x), singsol=all)
```

$$y(x) = \frac{1-\sqrt{1-x}}{2x^2}$$
$$y(x) = \frac{1+\sqrt{1-x}}{2x^2}$$

✓ Solution by Mathematica

Time used: 0.599 (sec). Leaf size: 53

```
DSolve[{x*(1-2*x^2*y[x])*y'[x] +y[x] == 3*x^2*y[x]^2,y[1]==1/2},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{x-\sqrt{-((x-1)x^2)}}{2x^3}$$
$$y(x) \to \frac{\sqrt{-((x-1)x^2)}+x}{2x^3}$$

1.16 problem Problem 14.23 (a)

Internal problem ID [2501]
Internal file name [`OUTPUT/1993_Sunday_June_05_2022_02_43_05_AM_19041325/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.23 (a) .
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' + \frac{xy}{a^2 + x^2} = x}$$

1.16.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{x}{a^2 + x^2}$$
$$q(x) = x$$

Hence the ode is

$$y' + \frac{xy}{a^2 + x^2} = x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{x}{a^2+x^2}dx} \\
&= \sqrt{a^2+x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{a^2+x^2}\, y\right) &= \left(\sqrt{a^2+x^2}\right)(x) \\
\mathrm{d}\left(\sqrt{a^2+x^2}\, y\right) &= \left(x\sqrt{a^2+x^2}\right)\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\sqrt{a^2+x^2}\, y &= \int x\sqrt{a^2+x^2}\, \mathrm{d}x \\
\sqrt{a^2+x^2}\, y &= \frac{(a^2+x^2)^{\frac{3}{2}}}{3} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sqrt{a^2+x^2}$ results in

$$
y = \frac{a^2}{3} + \frac{x^2}{3} + \frac{c_1}{\sqrt{a^2+x^2}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{a^2}{3} + \frac{x^2}{3} + \frac{c_1}{\sqrt{a^2+x^2}} \tag{1}
$$

Verification of solutions

$$
y = \frac{a^2}{3} + \frac{x^2}{3} + \frac{c_1}{\sqrt{a^2+x^2}}
$$

Verified OK.

1.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{x(-a^2 - x^2 + y)}{a^2 + x^2} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 28: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \frac{1}{\sqrt{a^2+x^2}} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sqrt{a^2+x^2}}} dy \end{aligned}$$

Which results in

$$S = \sqrt{a^2+x^2}\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x(-a^2-x^2+y)}{a^2+x^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{yx}{\sqrt{a^2+x^2}} \\
S_y &= \sqrt{a^2+x^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = x\sqrt{a^2+x^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = R\sqrt{R^2+a^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{(R^2+a^2)^{\frac{3}{2}}}{3} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\sqrt{a^2+x^2}\, y = \frac{(a^2+x^2)^{\frac{3}{2}}}{3} + c_1
$$

Which simplifies to

$$
\sqrt{a^2+x^2}\, y = \frac{(a^2+x^2)^{\frac{3}{2}}}{3} + c_1
$$

Which gives

$$
y = \frac{(a^2+x^2)^{\frac{3}{2}} + 3c_1}{3\sqrt{a^2+x^2}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(a^2+x^2)^{\frac{3}{2}} + 3c_1}{3\sqrt{a^2+x^2}} \tag{1}
$$

Verification of solutions

$$y = \frac{(a^2+x^2)^{\frac{3}{2}} + 3c_1}{3\sqrt{a^2+x^2}}$$

Verified OK.

1.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-\frac{xy}{a^2+x^2} + x\right)\mathrm{d}x$$

$$\left(\frac{xy}{a^2+x^2} - x\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= \frac{xy}{a^2+x^2} - x \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{xy}{a^2+x^2} - x\right) \\ &= \frac{x}{a^2+x^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(\frac{x}{a^2+x^2}\right) - (0)\right) \\ &= \frac{x}{a^2+x^2} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int \frac{x}{a^2+x^2}\,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{\ln\left(a^2+x^2\right)}{2}} \\ &= \sqrt{a^2+x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \sqrt{a^2+x^2}\left(\frac{xy}{a^2+x^2} - x\right) \\ &= \frac{x(-a^2-x^2+y)}{\sqrt{a^2+x^2}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \sqrt{a^2+x^2}(1) \\ &= \sqrt{a^2+x^2}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{x(-a^2-x^2+y)}{\sqrt{a^2+x^2}}\right) + \left(\sqrt{a^2+x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x(-a^2-x^2+y)}{\sqrt{a^2+x^2}}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{(a^2+x^2-3y)\sqrt{a^2+x^2}}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sqrt{a^2 + x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sqrt{a^2 + x^2}$. Therefore equation (4) becomes

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(a^2 + x^2 - 3y)\sqrt{a^2 + x^2}}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(a^2 + x^2 - 3y)\sqrt{a^2 + x^2}}{3}$$

The solution becomes

$$y = \frac{a^2\sqrt{a^2 + x^2} + x^2\sqrt{a^2 + x^2} + 3c_1}{3\sqrt{a^2 + x^2}}$$

Summary

The solution(s) found are the following

$$y = \frac{a^2\sqrt{a^2 + x^2} + x^2\sqrt{a^2 + x^2} + 3c_1}{3\sqrt{a^2 + x^2}} \tag{1}$$

Verification of solutions

$$y = \frac{a^2\sqrt{a^2 + x^2} + x^2\sqrt{a^2 + x^2} + 3c_1}{3\sqrt{a^2 + x^2}}$$

Verified OK.

1.16.4 Maple step by step solution

Let's solve

$y' + \frac{xy}{a^2+x^2} = x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{xy}{a^2+x^2} + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{xy}{a^2+x^2} = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{xy}{a^2+x^2}\right) = \mu(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{xy}{a^2+x^2}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)x}{a^2+x^2}$

- Solve to find the integrating factor

  $\mu(x) = \sqrt{a^2+x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \sqrt{a^2+x^2}$

  $y = \frac{\int x\sqrt{a^2+x^2}dx + c_1}{\sqrt{a^2+x^2}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{(a^2+x^2)^{\frac{3}{2}}}{3} + c_1}{\sqrt{a^2+x^2}}$

- Simplify

$$y = \frac{(a^2+x^2)^{\frac{3}{2}}+3c_1}{3\sqrt{a^2+x^2}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(diff(y(x),x)+ (x*y(x))/(a^2+x^2)=x,y(x), singsol=all)
```

$$y(x) = \frac{a^2}{3} + \frac{x^2}{3} + \frac{c_1}{\sqrt{a^2+x^2}}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 31

```
DSolve[y'[x]+ (x*y[x])/(a^2+x^2)==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}(a^2+x^2) + \frac{c_1}{\sqrt{a^2+x^2}}$$

1.17 problem Problem 14.23 (b)

Internal problem ID [2502]
Internal file name [`OUTPUT/1994_Sunday_June_05_2022_02_43_08_AM_6025860/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.23 (b) .
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{4y^2}{x^2} + y^2 = 0}$$

1.17.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{y^2(x^2-4)}{x^2} \end{aligned}$$

Where $f(x) = -\frac{x^2-4}{x^2}$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\, dy = -\frac{x^2-4}{x^2}\, dx$$

$$\int \frac{1}{y^2}\, dy = \int -\frac{x^2-4}{x^2}\, dx$$
$$-\frac{1}{y} = -x - \frac{4}{x} + c_1$$

Which results in

$$y = -\frac{x}{c_1 x - x^2 - 4}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{c_1 x - x^2 - 4} \tag{1}$$

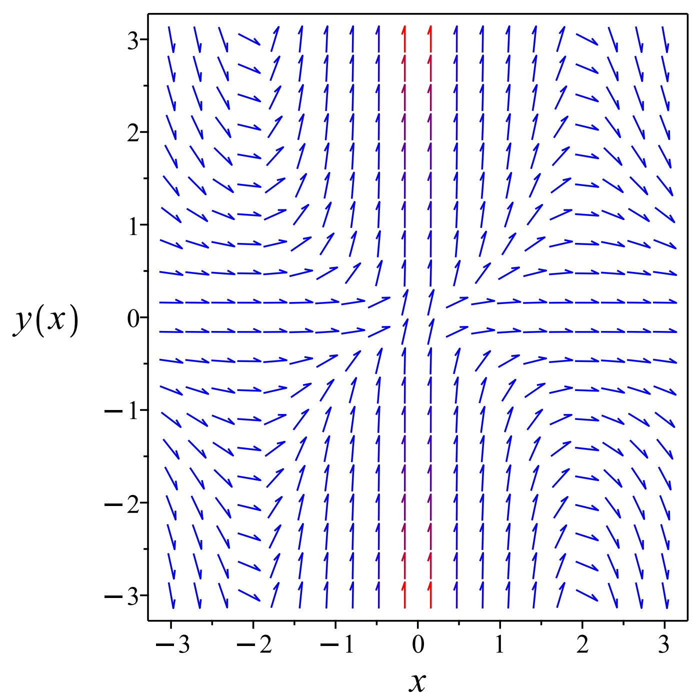


Figure 38: Slope field plot

Verification of solutions

$$y = -\frac{x}{c_1 x - x^2 - 4}$$

Verified OK.

1.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y^2(x^2-4)}{x^2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 31: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{x^2}{x^2-4} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{x^2}{x^2-4}} dx
\end{aligned}
$$

Which results in

$$
S = -x - \frac{4}{x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y^2(x^2-4)}{x^2}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -1 + \frac{4}{x^2} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-x - \frac{4}{x} = -\frac{1}{y} + c_1$$

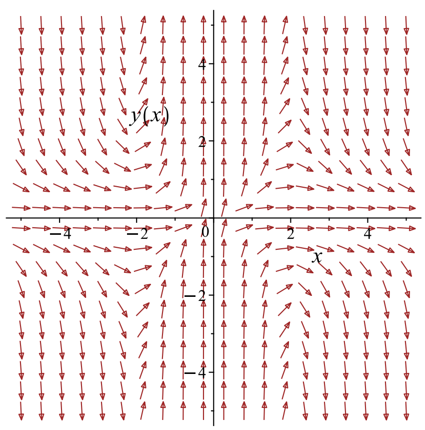
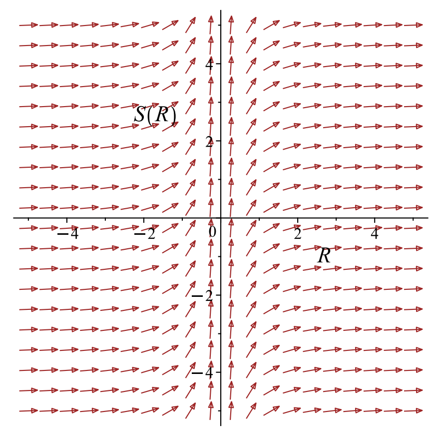
Which simplifies to

$$-x - \frac{4}{x} = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{x}{c_1 x + x^2 + 4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2(x^2-4)}{x^2}$ 	$R = y$ $S = -x - \frac{4}{x}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{x}{c_1 x + x^2 + 4} \tag{1}$$

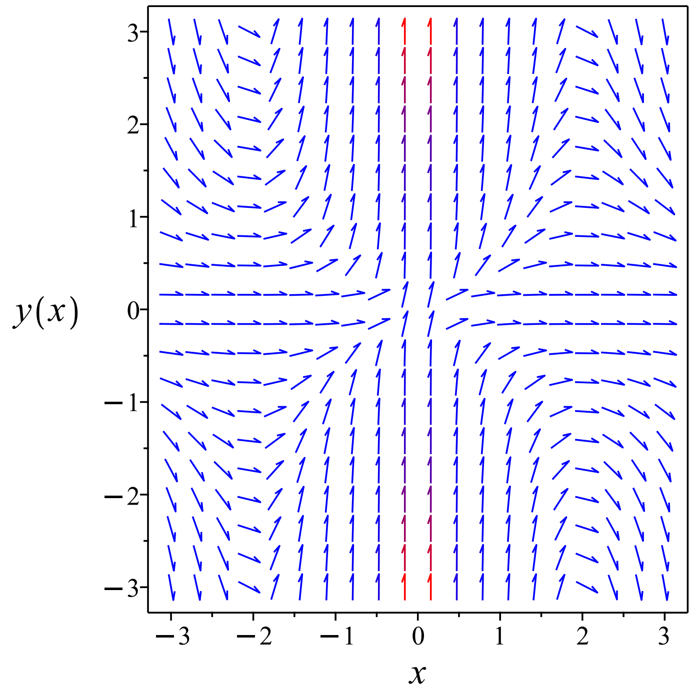


Figure 39: Slope field plot

Verification of solutions

$$y = \frac{x}{c_1 x + x^2 + 4}$$

Verified OK.

1.17.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y^2}\right)\mathrm{d}y &= \left(\frac{x^2-4}{x^2}\right)\mathrm{d}x\\ \left(-\frac{x^2-4}{x^2}\right)\mathrm{d}x + \left(-\frac{1}{y^2}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{x^2-4}{x^2}\\ N(x,y) &= -\frac{1}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x^2-4}{x^2}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y^2}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1)\\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{x^2-4}{x^2}\, \mathrm{d}x\\
\phi &= -x - \frac{4}{x} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y^2}$. Therefore equation (4) becomes

$$
-\frac{1}{y^2} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y^2}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y^2}\right)\mathrm{d}y$$

$$f(y) = \frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - \frac{4}{x} + \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \frac{4}{x} + \frac{1}{y}$$

The solution becomes

$$y = \frac{x}{c_1 x + x^2 + 4}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{x}{c_1 x + x^2 + 4} \tag{1}$$

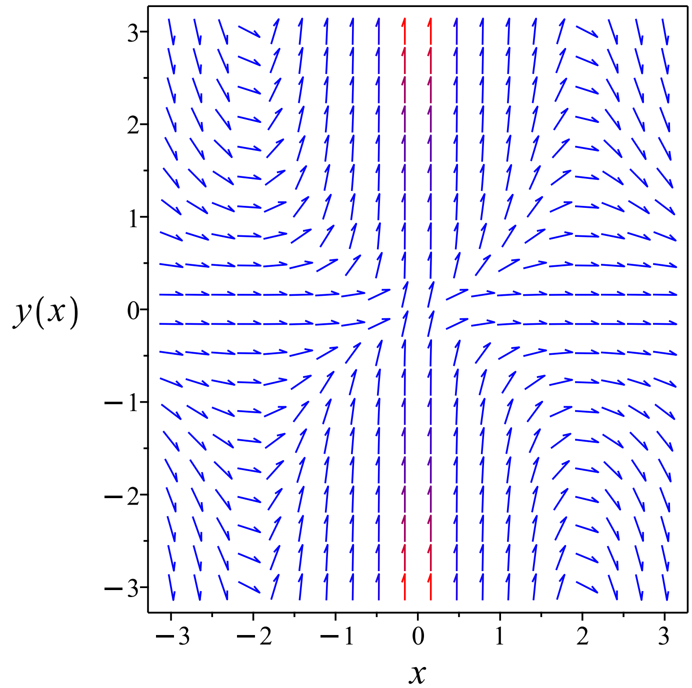


Figure 40: Slope field plot

Verification of solutions

$$y = \frac{x}{c_1 x + x^2 + 4}$$

Verified OK.

1.17.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y^2(x^2 - 4)}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{4y^2}{x^2} - y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = -\frac{x^2-4}{x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{(x^2-4)u}{x^2}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{2}{x} + \frac{2x^2-8}{x^3} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-\frac{(x^2-4)\, u''(x)}{x^2} - \left(-\frac{2}{x} + \frac{2x^2-8}{x^3}\right) u'(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 + \frac{(x^2+4)\, c_2}{x}
$$

The above shows that

$$
u'(x) = \frac{c_2(x^2-4)}{x^2}
$$

Using the above in (1) gives the solution

$$
y = \frac{c_2}{c_1 + \frac{(x^2+4)c_2}{x}}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{x}{c_3 x + x^2 + 4}
$$

Summary

The solution(s) found are the following

$$y = \frac{x}{c_3 x + x^2 + 4} \tag{1}$$

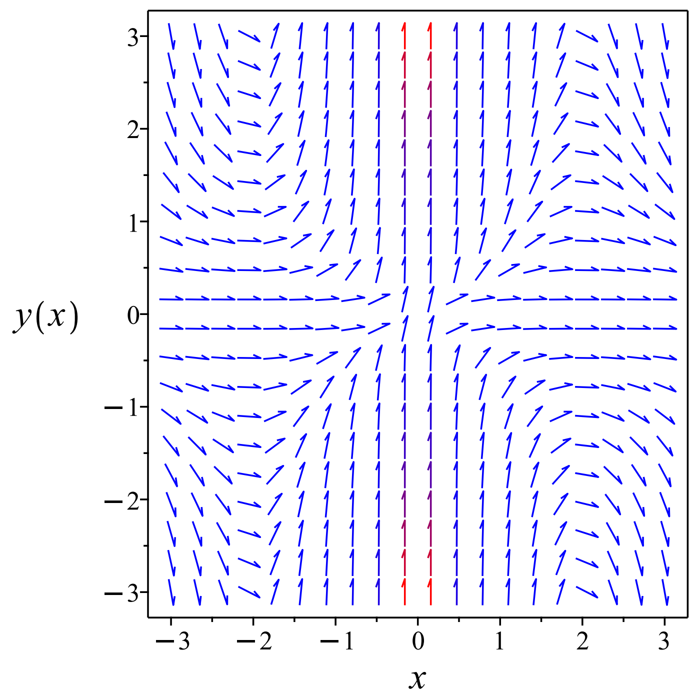


Figure 41: Slope field plot

Verification of solutions

$$y = \frac{x}{c_3 x + x^2 + 4}$$

Verified OK.

1.17.5 Maple step by step solution

Let's solve

$y' - \frac{4y^2}{x^2} + y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$$\frac{y'}{y^2} = -\frac{(x+2)(x-2)}{x^2}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{y^2} dx = \int -\frac{(x+2)(x-2)}{x^2} dx + c_1$$

- Evaluate integral

$$-\frac{1}{y} = -x - \frac{4}{x} + c_1$$

- Solve for y

$$y = -\frac{x}{c_1 x - x^2 - 4}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)= 4*y(x)^2/x^2 - y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{x}{c_1 x + x^2 + 4}$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 24

```
DSolve[y'[x]== 4*y[x]^2/x^2 - y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{x^2 - c_1 x + 4}$$
$$y(x) \to 0$$

1.18 problem Problem 14.24 (a)

Internal problem ID [2503]
Internal file name [`OUTPUT/1995_Sunday_June_05_2022_02_43_10_AM_7420785/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.24 (a) .
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{y}{x} = 1}$$

With initial conditions

$$[y(1) = -1]$$

1.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 1$$

Hence the ode is

$$y' - \frac{y}{x} = 1$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

1.18.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= \mu \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \frac{1}{x} \\ d\left(\frac{y}{x}\right) &= \frac{1}{x}dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{1}{x}\, dx \\ \frac{y}{x} &= \ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x + \ln(x)\, x$$

which simplifies to

$$y = x(\ln(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

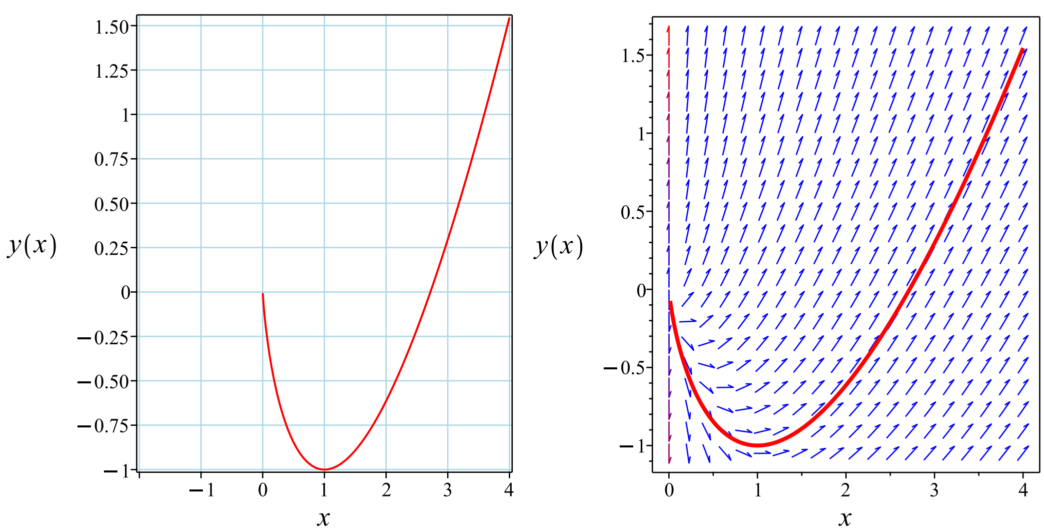
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$y = \ln(x)\, x - x$$

Summary
The solution(s) found are the following

$$y = \ln(x)\, x - x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \ln(x)\, x - x$$

Verified OK.

1.18.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x = 1$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \frac{1}{x}\,\mathrm{d}x \\ &= \ln(x) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x(\ln(x) + c_2) \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_2$$

$$c_2 = -1$$

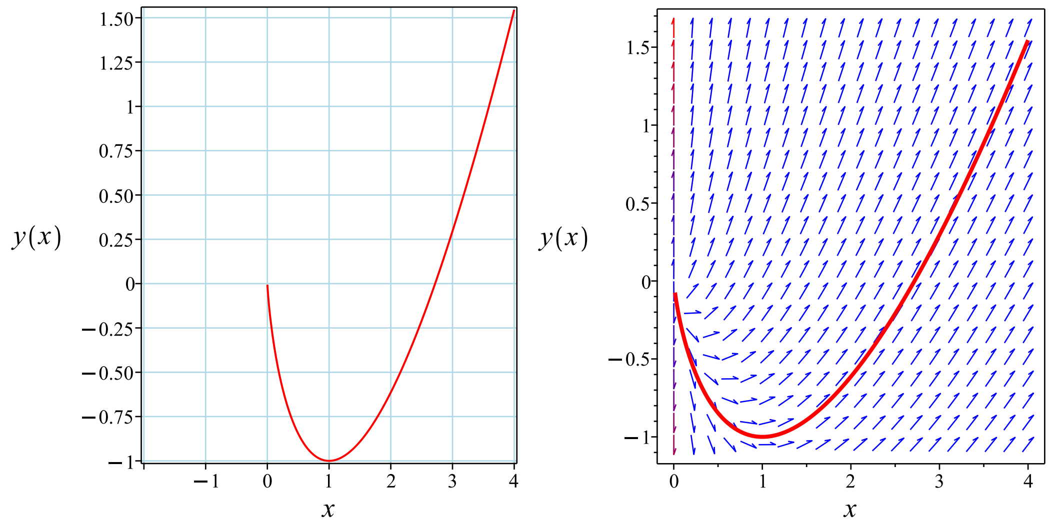
Substituting c_2 found above in the general solution gives

$$y = \ln(x)\,x - x$$

Summary

The solution(s) found are the following

$$y = \ln(x)\,x - x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \ln(x)\,x - x$$

Verified OK.

1.18.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + x}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 34: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y + x}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \ln(x) + c_1$$

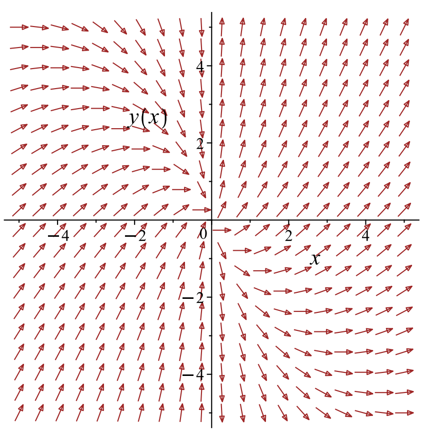
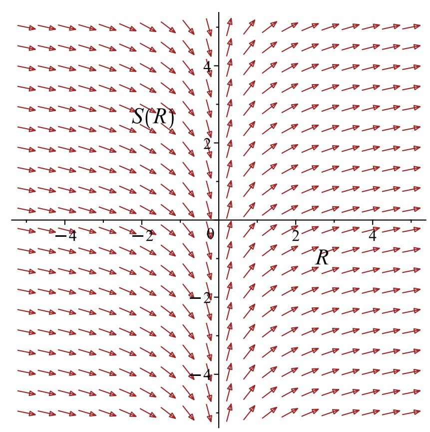
Which simplifies to

$$\frac{y}{x} = \ln(x) + c_1$$

Which gives

$$y = x(\ln(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+x}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

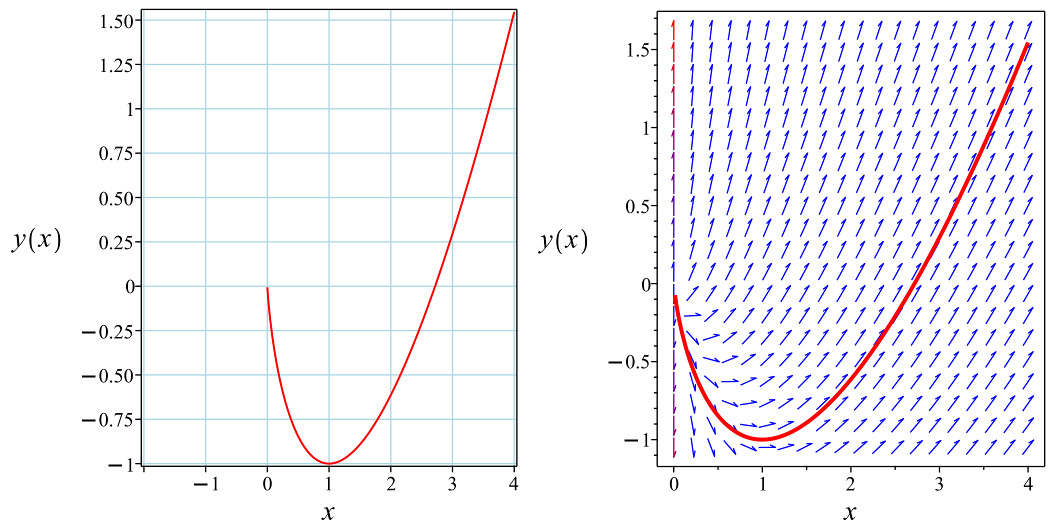
Substituting c_1 found above in the general solution gives

$$y = \ln(x)\,x - x$$

Summary

The solution(s) found are the following

$$y = \ln(x)\,x - x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \ln(x)\,x - x$$

Verified OK.

1.18.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(1+\frac{y}{x}\right)\mathrm{d}x\\ \left(-\frac{y}{x}-1\right)\mathrm{d}x+\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{y}{x}-1\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{y}{x}-1\right)\\ &=-\frac{1}{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(-\frac{1}{x}\right) - (0)\right)\\ &= -\frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{1}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(x)}\\ &= \frac{1}{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x}\left(-\frac{y}{x} - 1\right)\\ &= \frac{-y - x}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x}(1)\\ &= \frac{1}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-y - x}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \frac{-y - x}{x^2} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{y}{x} - \ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$
\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y}{x} - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y}{x} - \ln(x)$$

The solution becomes

$$y = x(\ln(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

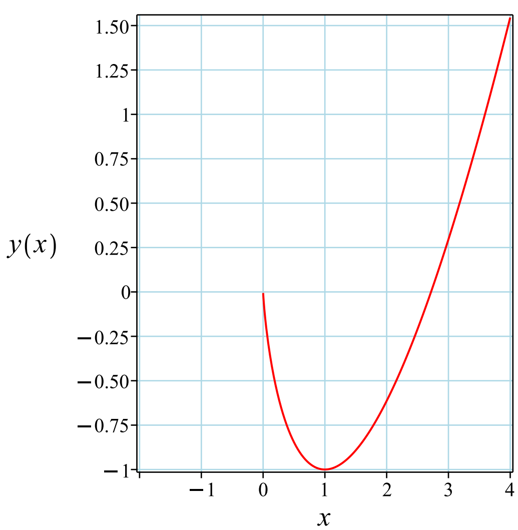
Substituting c_1 found above in the general solution gives

$$y = \ln(x)\,x - x$$

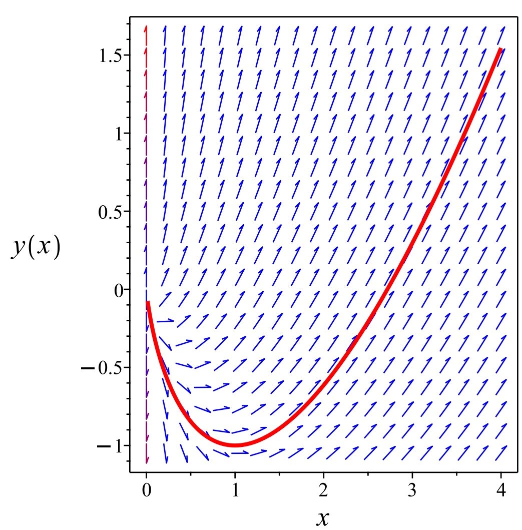
Summary

The solution(s) found are the following

$$y = \ln(x)\,x - x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \ln(x)\, x - x$$

Verified OK.

1.18.6 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{x} = 1, y(1) = -1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \frac{1}{x} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(\ln(x) + c_1)$

- Use initial condition $y(1) = -1$

  $-1 = c_1$

- Solve for c_1

  $c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

  $y = (\ln(x) - 1)\, x$

- Solution to the IVP

  $y = (\ln(x) - 1)\, x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve([diff(y(x),x)-y(x)/x=1,y(1) = -1],y(x), singsol=all)
```

$$y(x) = x(-1 + \ln(x))$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 11

```
DSolve[{y'[x]-y[x]/x==1,y[1]==-1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(\log(x) - 1)$$

1.19 problem Problem 14.24 (b)

Internal problem ID [2504]
Internal file name [`OUTPUT/1996_Sunday_June_05_2022_02_43_12_AM_67962190/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.24 (b) .
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - y\tan(x) = 1}$$

With initial conditions

$$\left[y\left(\frac{\pi}{4}\right) = 3\right]$$

1.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\tan(x)$$
$$q(x) = 1$$

Hence the ode is

$$y' - y \tan(x) = 1$$

The domain of $p(x) = -\tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z136 \lor \frac{1}{2}\pi + \pi\_Z136 < x\right\}$$

And the point $x_0 = \frac{\pi}{4}$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{4}$ is also inside this domain. Hence solution exists and is unique.

1.19.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\tan(x)dx} \\ &= \cos(x)\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= \mu \\ \frac{d}{dx}(\cos(x)\, y) &= \cos(x) \\ d(\cos(x)\, y) &= \cos(x)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\cos(x)\, y &= \int \cos(x)\, dx \\ \cos(x)\, y &= \sin(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \cos(x)$ results in

$$y = \sec(x)\sin(x) + c_1 \sec(x)$$

which simplifies to

$$y = \tan(x) + c_1 \sec(x)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = 1 + \sqrt{2}\, c_1$$

$$c_1 = \sqrt{2}$$

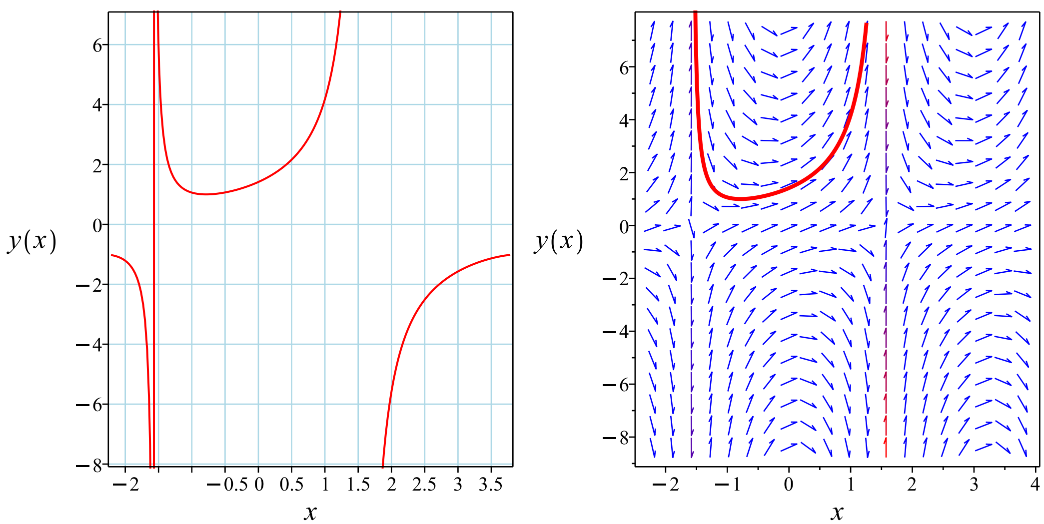
Substituting c_1 found above in the general solution gives

$$y = \sec(x)\sin(x) + \sec(x)\sqrt{2}$$

Summary

The solution(s) found are the following

$$y = \sec(x)\sin(x) + \sec(x)\sqrt{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \sec(x)\sin(x) + \sec(x)\sqrt{2}$$

Verified OK.

1.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= y\tan(x) + 1 \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 37: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\cos(x)}
\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{\cos(x)}} dy
\end{aligned}$$

Which results in

$$S = \cos(x)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = y \tan(x) + 1$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\sin(x)\, y \\
S_y &= \cos(x)
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \cos(x) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \cos(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \sin(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\cos(x)\, y = \sin(x) + c_1
$$

Which simplifies to

$$
\cos(x)\, y = \sin(x) + c_1
$$

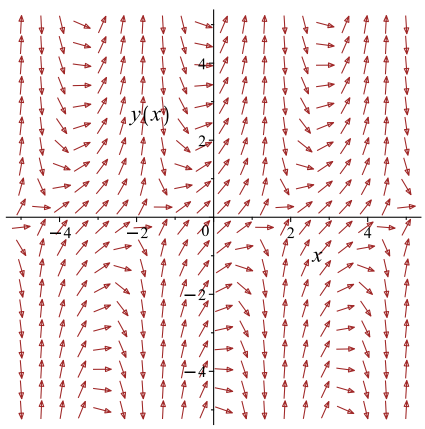
Which gives

$$
y = \frac{\sin(x) + c_1}{\cos(x)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = y \tan(x) + 1$



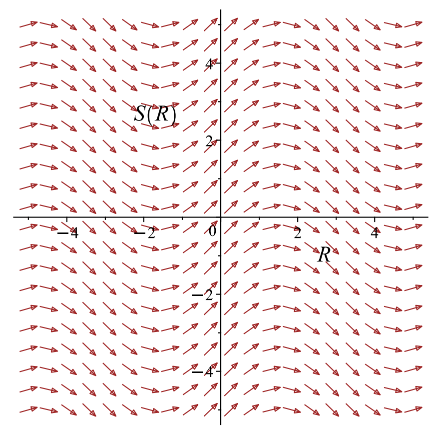
</td><td>

$R = x$

$S = \cos(x)\, y$

</td><td>

$\frac{dS}{dR} = \cos(R)$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = 1 + \sqrt{2}\, c_1$$

$$c_1 = \sqrt{2}$$

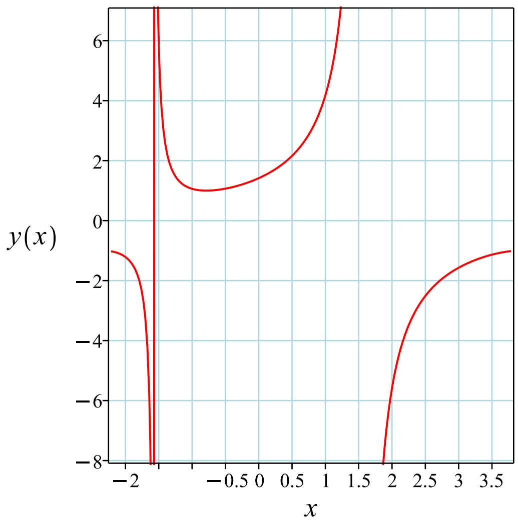
Substituting c_1 found above in the general solution gives

$$y = \sec(x)\sin(x) + \sec(x)\sqrt{2}$$

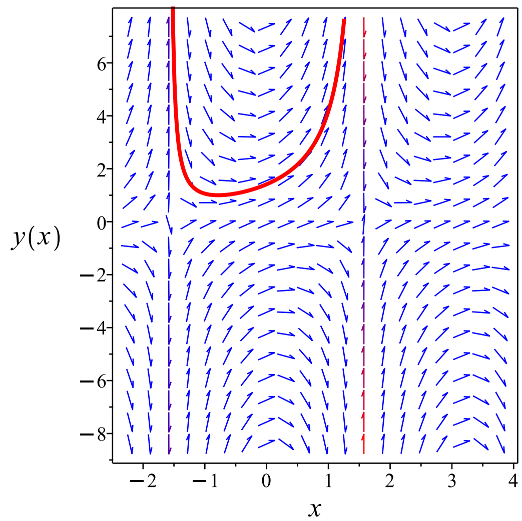
Summary

The solution(s) found are the following

$$y = \sec(x)\sin(x) + \sec(x)\sqrt{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sec(x)\sin(x) + \sec(x)\sqrt{2}$$

Verified OK.

1.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= (y\tan(x) + 1)\,\mathrm{d}x \\ (-y\tan(x) - 1)\,\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -y\tan(x) - 1 \\ N(x, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y\tan(x) - 1) \\ &= -\tan(x) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-\tan(x)) - (0)) \\ &= -\tan(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int -\tan(x)\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(\cos(x))} \\ &= \cos(x)\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \cos(x)\left(-y\tan(x) - 1\right) \\ &= -\sin(x)\,y - \cos(x)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \cos(x)\,(1) \\ &= \cos(x)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\sin(x)\,y - \cos(x)\right) + \left(\cos(x)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int -\sin(x)\,y - \cos(x)\,\mathrm{d}x\end{aligned}$$

$$\phi = \cos(x)\,y - \sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cos(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(x)$. Therefore equation (4) becomes

$$\cos(x) = \cos(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \cos(x)\, y - \sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(x)\, y - \sin(x)$$

The solution becomes

$$y = \frac{\sin(x) + c_1}{\cos(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = 1 + \sqrt{2}\, c_1$$

$$c_1 = \sqrt{2}$$

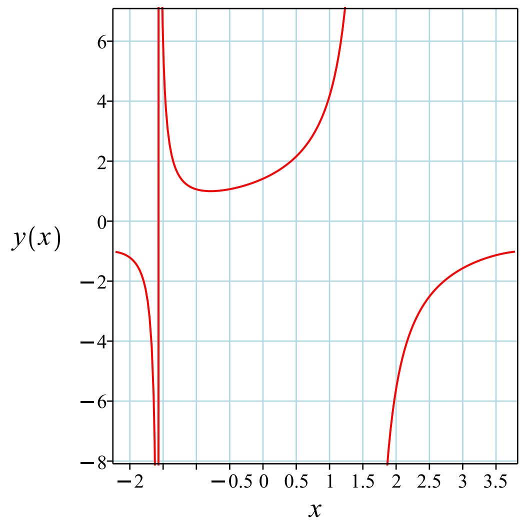
Substituting c_1 found above in the general solution gives

$$y = \sec(x)\sin(x) + \sec(x)\sqrt{2}$$

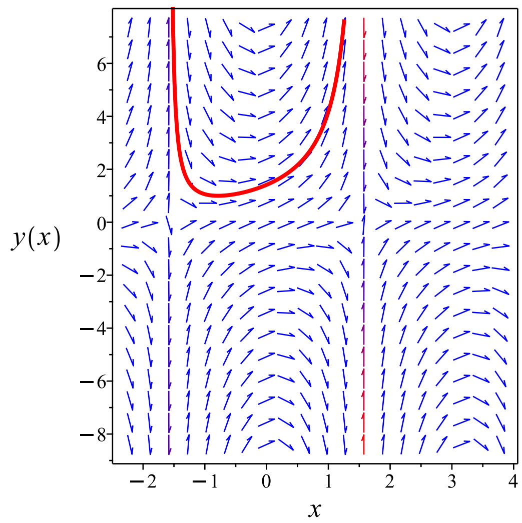
Summary

The solution(s) found are the following

$$y = \sec(x)\sin(x) + \sec(x)\sqrt{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sec(x)\sin(x) + \sec(x)\sqrt{2}$$

Verified OK.

1.19.5 Maple step by step solution

Let's solve

$\left[y' - y\tan(x) = 1, y\left(\frac{\pi}{4}\right) = 3\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y\tan(x) + 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y\tan(x) = 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x) \left(y' - y \tan(x)\right) = \mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' - y \tan(x)\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x) \tan(x)$

- Solve to find the integrating factor

  $\mu(x) = \cos(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \cos(x)$

  $y = \frac{\int \cos(x) dx + c_1}{\cos(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{\sin(x) + c_1}{\cos(x)}$

- Simplify

  $y = \tan(x) + c_1 \sec(x)$

- Use initial condition $y\left(\frac{\pi}{4}\right) = 3$

  $3 = 1 + \sqrt{2}\, c_1$

- Solve for c_1

  $c_1 = \sqrt{2}$

- Substitute $c_1 = \sqrt{2}$ into general solution and simplify

  $y = \tan(x) + \sec(x) \sqrt{2}$

- Solution to the IVP

  $y = \tan(x) + \sec(x) \sqrt{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)-y(x)*tan(x)=1,y(1/4*Pi) = 3],y(x), singsol=all)
```

$$y(x) = \tan(x) + \sec(x)\sqrt{2}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 16

```
DSolve[{y'[x]-y[x]*Tan[x]==1,y[Pi/4]==3},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\sin(x) + \sqrt{2}\right)\sec(x)$$

1.20 problem Problem 14.24 (c)

Internal problem ID [2505]
Internal file name [`OUTPUT/1997_Sunday_June_05_2022_02_43_15_AM_20113648/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.24 (c) .
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y' - \frac{y^2}{x^2} = \frac{1}{4}}$$

With initial conditions

$$[y(1) = 1]$$

1.20.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{x^2 + 4y^2}{4x^2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x^2 + 4y^2}{4x^2}\right) \\ &= \frac{2y}{x^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.20.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - u(x)^2 = \frac{1}{4}$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{-u + u^2 + \frac{1}{4}}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -u + u^2 + \frac{1}{4}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{-u + u^2 + \frac{1}{4}}\, du &= \frac{1}{x}\, dx \\ \int \frac{1}{-u + u^2 + \frac{1}{4}}\, du &= \int \frac{1}{x}\, dx \\ -\frac{2}{2u - 1} &= \ln(x) + c_2\end{aligned}$$

The solution is

$$-\frac{2}{2u(x)-1}-\ln(x)-c_2=0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{2}{\frac{2y}{x}-1}-\ln(x)-c_2=0$$
$$\frac{(2c_2+2\ln(x))\,y-x(c_2+\ln(x)-2)}{-2y+x}=0$$

Substituting initial conditions and solving for c_2 gives $c_2=-2$. Hence the solution be-
Summary
The solution(s) found are the following
comes

$$\frac{(-4+2\ln(x))\,y-x(-4+\ln(x))}{-2y+x}=0 \tag{1}$$

Verification of solutions

$$\frac{(-4+2\ln(x))\,y-x(-4+\ln(x))}{-2y+x}=0$$

Verified OK.

1.20.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{x^2+4y^2}{4x^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 + \frac{(x^2+4y^2)(b_3-a_2)}{4x^2} - \frac{(x^2+4y^2)^2 a_3}{16x^4} \\
& - \left(\frac{1}{2x} - \frac{x^2+4y^2}{2x^3}\right)(xa_2+ya_3+a_1) - \frac{2y(xb_2+yb_3+b_1)}{x^2} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{4x^4a_2 + x^4a_3 - 16b_2x^4 - 4x^4b_3 + 32x^3yb_2 - 16x^2y^2a_2 + 8x^2y^2a_3 + 16x^2y^2b_3 - 32x\,y^3a_3 + 16y^4a_3 + 32x}{16x^4}
$$
$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& -4x^4a_2 - x^4a_3 + 16b_2x^4 + 4x^4b_3 - 32x^3yb_2 + 16x^2y^2a_2 - 8x^2y^2a_3 \\
& - 16x^2y^2b_3 + 32x\,y^3a_3 - 16y^4a_3 - 32x^2yb_1 + 32x\,y^2a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& -4a_2v_1^4 + 16a_2v_1^2v_2^2 - a_3v_1^4 - 8a_3v_1^2v_2^2 + 32a_3v_1v_2^3 - 16a_3v_2^4 + 16b_2v_1^4 \\
& - 32b_2v_1^3v_2 + 4b_3v_1^4 - 16b_3v_1^2v_2^2 + 32a_1v_1v_2^2 - 32b_1v_1^2v_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
& (-4a_2 - a_3 + 16b_2 + 4b_3)\,v_1^4 - 32b_2v_1^3v_2 + (16a_2 - 8a_3 - 16b_3)\,v_1^2v_2^2 \\
& - 32b_1v_1^2v_2 + 32a_3v_1v_2^3 + 32a_1v_1v_2^2 - 16a_3v_2^4 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
32a_1 &= 0 \\
-16a_3 &= 0 \\
32a_3 &= 0 \\
-32b_1 &= 0 \\
-32b_2 &= 0 \\
16a_2 - 8a_3 - 16b_3 &= 0 \\
-4a_2 - a_3 + 16b_2 + 4b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{x^2 + 4y^2}{4x^2}\right)(x) \\
&= \frac{-x^2 + 4xy - 4y^2}{4x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2+4xy-4y^2}{4x}} dy \end{aligned}$$

Which results in

$$S = \frac{2x}{-x+2y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2+4y^2}{4x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{4y}{(x-2y)^2} \\ S_y &= -\frac{4x}{(x-2y)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2x}{-2y + x} = -\ln(x) + c_1$$

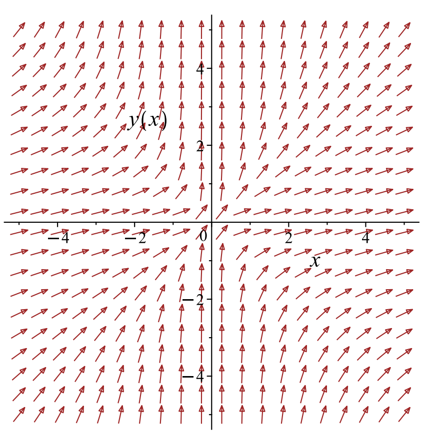
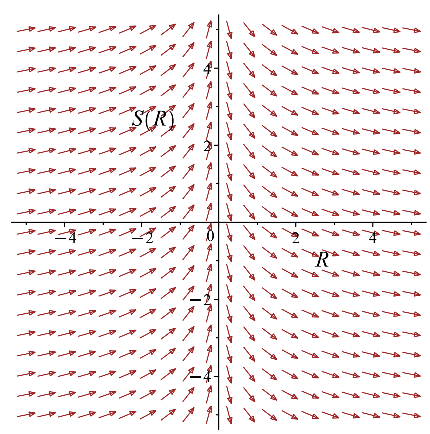
Which simplifies to

$$-\frac{2x}{-2y + x} = -\ln(x) + c_1$$

Which gives

$$y = \frac{x(\ln(x) - c_1 - 2)}{2\ln(x) - 2c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+4y^2}{4x^2}$ 	$R = x$ $S = -\frac{2x}{x - 2y}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{2 + c_1}{2c_1}$$

$$c_1 = 2$$

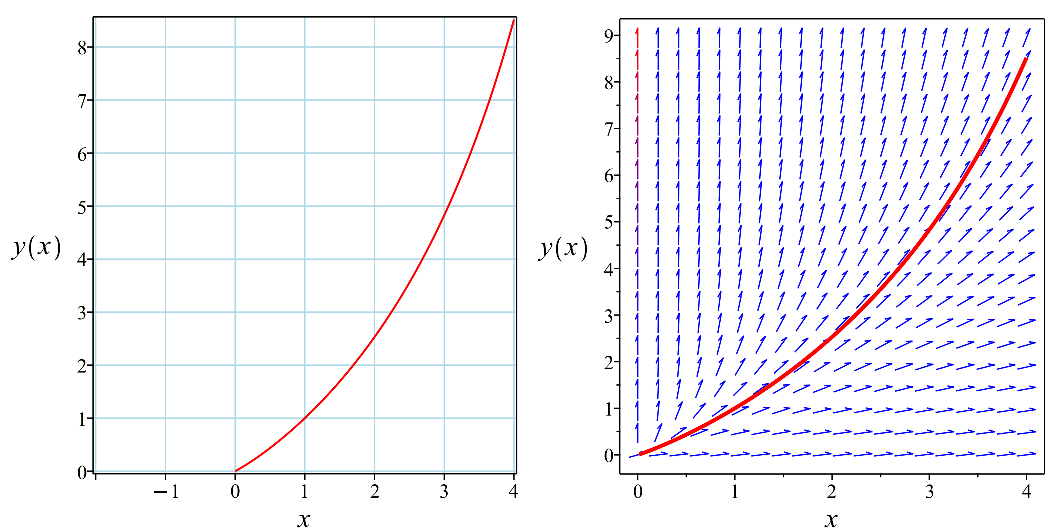
Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(x)\, x - 4x}{-4 + 2\ln(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)\, x - 4x}{-4 + 2\ln(x)} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\ln(x)\, x - 4x}{-4 + 2\ln(x)}$$

Verified OK.

1.20.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^2+4y^2}{4x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{x^2} + \frac{1}{4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{1}{4}$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{1}{4x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{2u'(x)}{x^3} + \frac{u(x)}{4x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_2 \ln(x) + c_1}{\sqrt{x}}$$

The above shows that

$$u'(x) = -\frac{c_2 \ln(x) + c_1 - 2c_2}{2x^{\frac{3}{2}}}$$

Using the above in (1) gives the solution

$$y = \frac{(c_2 \ln(x) + c_1 - 2c_2)\, x}{2c_2 \ln(x) + 2c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(\ln(x) + c_3 - 2)\, x}{2\ln(x) + 2c_3}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-2 + c_3}{2c_3}$$

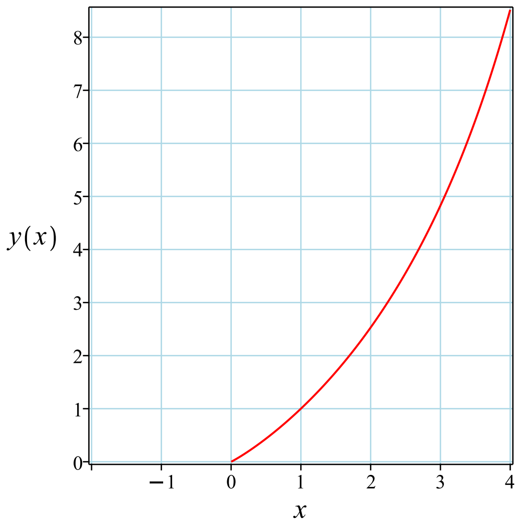
$$c_3 = -2$$

Substituting c_3 found above in the general solution gives

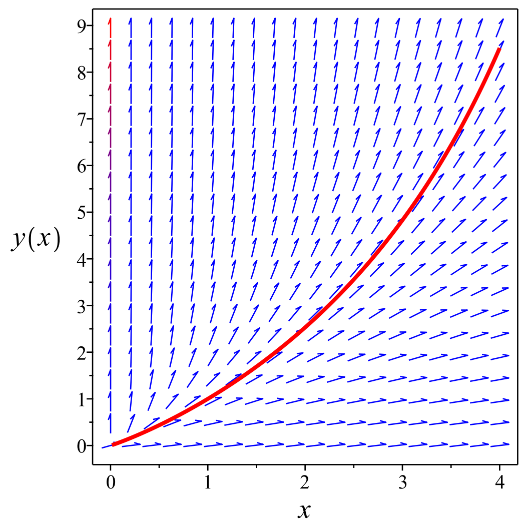
$$y = \frac{\ln(x)\, x - 4x}{-4 + 2\ln(x)}$$

Summary
The solution(s) found are the following

$$y = \frac{\ln(x)\, x - 4x}{-4 + 2\ln(x)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\ln(x)\, x - 4x}{-4 + 2\ln(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)-y(x)^2/x^2=1/4,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{x(\ln(x) - 4)}{2\ln(x) - 4}$$

✓ Solution by Mathematica

Time used: 0.132 (sec). Leaf size: 20

```
DSolve[{y'[x]-y[x]^2/x^2==1/4,y[1]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x(\log(x) - 4)}{2(\log(x) - 2)}$$

1.21 problem Problem 14.24 (d)

Internal problem ID [2506]
Internal file name [`OUTPUT/1998_Sunday_June_05_2022_02_43_18_AM_8787990/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.24 (d) .
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y' - \frac{y^2}{x^2} = \frac{1}{4}}$$

1.21.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - u(x)^2 = \frac{1}{4}$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-u + u^2 + \frac{1}{4}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -u + u^2 + \frac{1}{4}$. Integrating both sides gives

$$\frac{1}{-u + u^2 + \frac{1}{4}} \, du = \frac{1}{x} \, dx$$

$$\int \frac{1}{-u + u^2 + \frac{1}{4}} \, du = \int \frac{1}{x} \, dx$$

$$-\frac{2}{2u - 1} = \ln(x) + c_2$$

The solution is

$$-\frac{2}{2u(x) - 1} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{2}{\frac{2y}{x} - 1} - \ln(x) - c_2 = 0$$

$$\frac{(2c_2 + 2\ln(x))\,y - x(c_2 + \ln(x) - 2)}{-2y + x} = 0$$

Summary

The solution(s) found are the following

$$\frac{(2c_2 + 2\ln(x))\,y - x(c_2 + \ln(x) - 2)}{-2y + x} = 0 \tag{1}$$

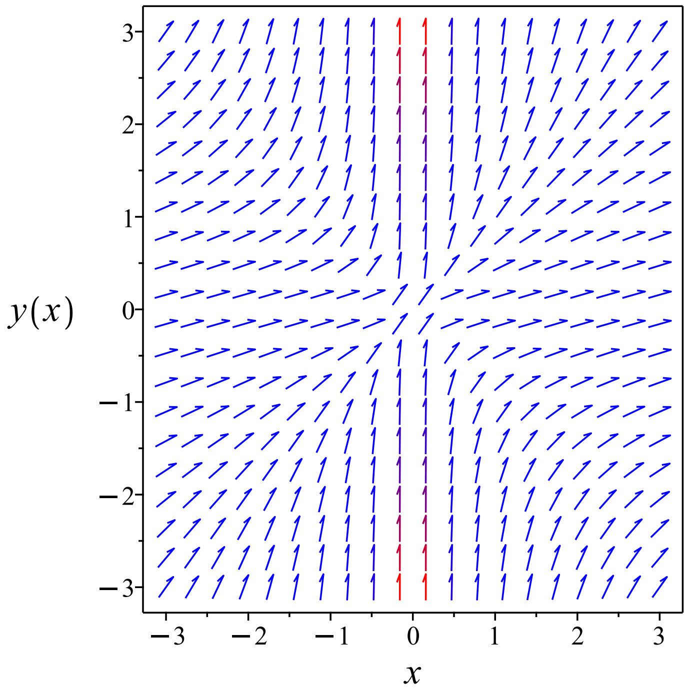


Figure 51: Slope field plot

Verification of solutions

$$\frac{(2c_2 + 2\ln(x))\,y - x(c_2 + \ln(x) - 2)}{-2y + x} = 0$$

Verified OK.

1.21.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x^2 + 4y^2}{4x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2+\frac{\left(x^2+4y^2\right)\left(b_3-a_2\right)}{4x^2}-\frac{\left(x^2+4y^2\right)^2 a_3}{16x^4} \\ & -\left(\frac{1}{2x}-\frac{x^2+4y^2}{2x^3}\right)\left(x a_2+y a_3+a_1\right)-\frac{2y\left(x b_2+y b_3+b_1\right)}{x^2}=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4x^4a_2+x^4a_3-16b_2x^4-4x^4b_3+32x^3yb_2-16x^2y^2a_2+8x^2y^2a_3+16x^2y^2b_3-32x\,y^3a_3+16y^4a_3+32x}{16x^4} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -4x^4a_2-x^4a_3+16b_2x^4+4x^4b_3-32x^3yb_2+16x^2y^2a_2-8x^2y^2a_3 \\ & -16x^2y^2b_3+32x\,y^3a_3-16y^4a_3-32x^2yb_1+32x\,y^2a_1=0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -4a_2v_1^4+16a_2v_1^2v_2^2-a_3v_1^4-8a_3v_1^2v_2^2+32a_3v_1v_2^3-16a_3v_2^4+16b_2v_1^4 \\ & -32b_2v_1^3v_2+4b_3v_1^4-16b_3v_1^2v_2^2+32a_1v_1v_2^2-32b_1v_1^2v_2=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\left(-4a_2-a_3+16b_2+4b_3\right)v_1^4-32b_2v_1^3v_2+\left(16a_2-8a_3-16b_3\right)v_1^2v_2^2 \\ -32b_1v_1^2v_2+32a_3v_1v_2^3+32a_1v_1v_2^2-16a_3v_2^4=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 32a_1 &= 0 \\ -16a_3 &= 0 \\ 32a_3 &= 0 \\ -32b_1 &= 0 \\ -32b_2 &= 0 \\ 16a_2-8a_3-16b_3 &= 0 \\ -4a_2-a_3+16b_2+4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi=0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta-\omega(x,y)\,\xi \\ &= y-\left(\frac{x^2+4y^2}{4x^2}\right)(x) \\ &= \frac{-x^2+4xy-4y^2}{4x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2+4xy-4y^2}{4x}} dy \end{aligned}$$

Which results in

$$S = \frac{2x}{-x + 2y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 + 4y^2}{4x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{4y}{(x - 2y)^2} \\ S_y &= -\frac{4x}{(x - 2y)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2x}{-2y + x} = -\ln(x) + c_1$$

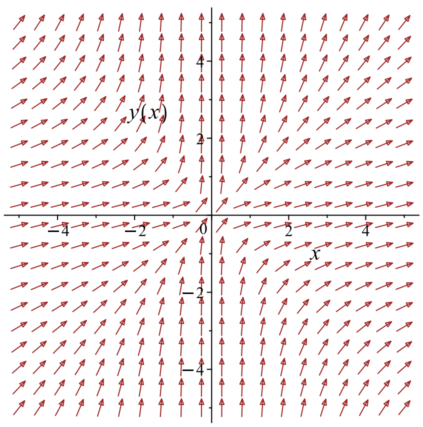
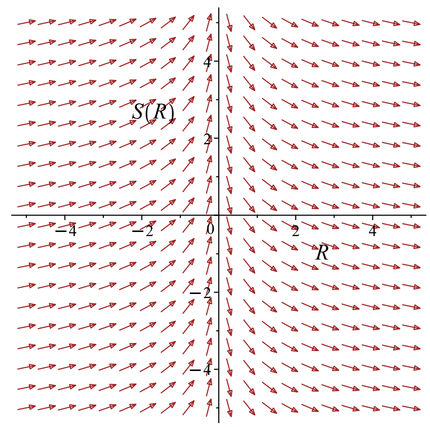
Which simplifies to

$$-\frac{2x}{-2y + x} = -\ln(x) + c_1$$

Which gives

$$y = \frac{x(\ln(x) - c_1 - 2)}{2\ln(x) - 2c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+4y^2}{4x^2}$ 	$R = x$ $S = -\frac{2x}{x-2y}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{x(\ln(x) - c_1 - 2)}{2\ln(x) - 2c_1} \tag{1}$$

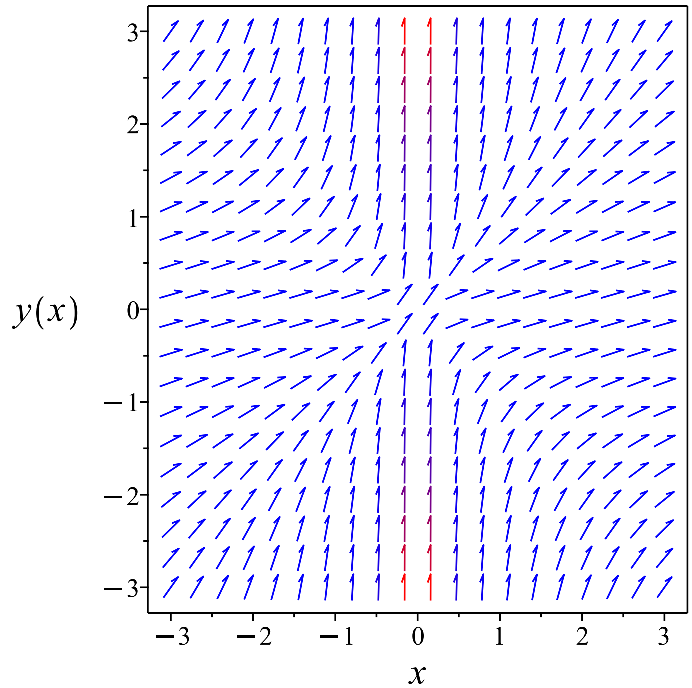


Figure 52: Slope field plot

Verification of solutions

$$y = \frac{x(\ln(x) - c_1 - 2)}{2\ln(x) - 2c_1}$$

Verified OK.

1.21.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + 4y^2}{4x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{x^2} + \frac{1}{4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{1}{4}$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{x^2}} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= -\frac{2}{x^3} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= \frac{1}{4x^4}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{x^2} + \frac{2u'(x)}{x^3} + \frac{u(x)}{4x^4} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = \frac{c_2 \ln(x) + c_1}{\sqrt{x}}
$$

The above shows that

$$
u'(x) = -\frac{c_2 \ln(x) + c_1 - 2c_2}{2x^{\frac{3}{2}}}
$$

Using the above in (1) gives the solution

$$
y = \frac{(c_2 \ln(x) + c_1 - 2c_2) x}{2c_2 \ln(x) + 2c_1}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{(\ln(x) + c_3 - 2) x}{2 \ln(x) + 2c_3}
$$

Summary

The solution(s) found are the following

$$y = \frac{(\ln(x) + c_3 - 2)\, x}{2\ln(x) + 2c_3} \tag{1}$$

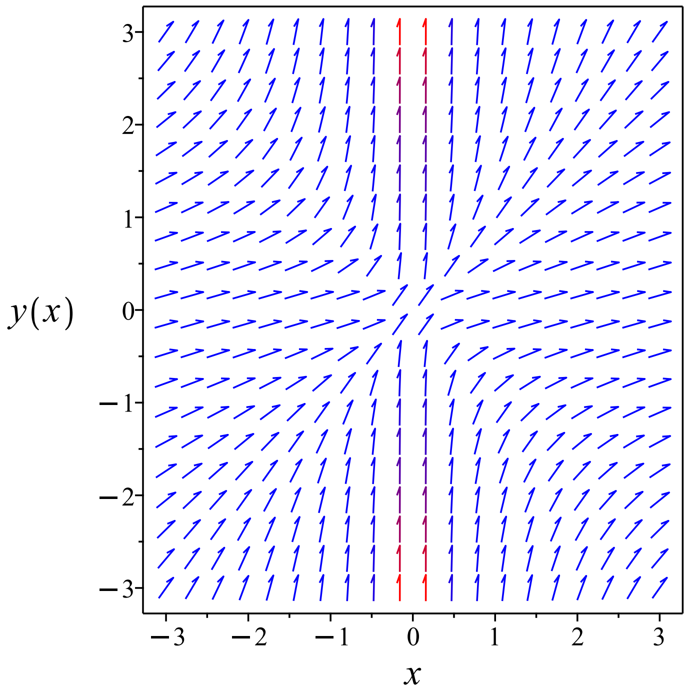


Figure 53: Slope field plot

Verification of solutions

$$y = \frac{(\ln(x) + c_3 - 2)\, x}{2\ln(x) + 2c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x)-y(x)^2/x^2=1/4,y(x), singsol=all)
```

$$y(x) = \frac{x(\ln(x) + c_1 - 2)}{2\ln(x) + 2c_1}$$

✓ Solution by Mathematica

Time used: 0.096 (sec). Leaf size: 36

```
DSolve[y'[x]-y[x]^2/x^2==1/4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x(\log(x) - 2 + 4c_1)}{2(\log(x) + 4c_1)}$$
$$y(x) \to \frac{x}{2}$$

1.22 problem Problem 14.26

Internal problem ID [2507]
Internal file name [`OUTPUT/1999_Sunday_June_05_2022_02_43_21_AM_89003149/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' \sin(x) + 2\cos(x) y = 1}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 1\right]$$

1.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2\cot(x)$$
$$q(x) = \csc(x)$$

Hence the ode is

$$y' + 2y\cot(x) = \csc(x)$$

The domain of $p(x) = 2\cot(x)$ is

$$\{x < \pi\_Z137 \lor \pi\_Z137 < x\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The domain of $q(x) = \csc(x)$ is

$$\{x < \pi\_Z137 \lor \pi\_Z137 < x\}$$

And the point $x_0 = \frac{\pi}{2}$ is also inside this domain. Hence solution exists and is unique.

1.22.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 2\cot(x)dx} \\ &= \sin(x)^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(\csc(x)) \\ \frac{d}{dx}\left(\sin(x)^2 y\right) &= \left(\sin(x)^2\right)(\csc(x)) \\ d\left(\sin(x)^2 y\right) &= \sin(x)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\sin(x)^2 y &= \int \sin(x)\, dx \\ \sin(x)^2 y &= -\cos(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sin(x)^2$ results in

$$y = -\csc(x)^2\cos(x) + c_1\csc(x)^2$$

which simplifies to

$$y = \csc(x)^2(-\cos(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

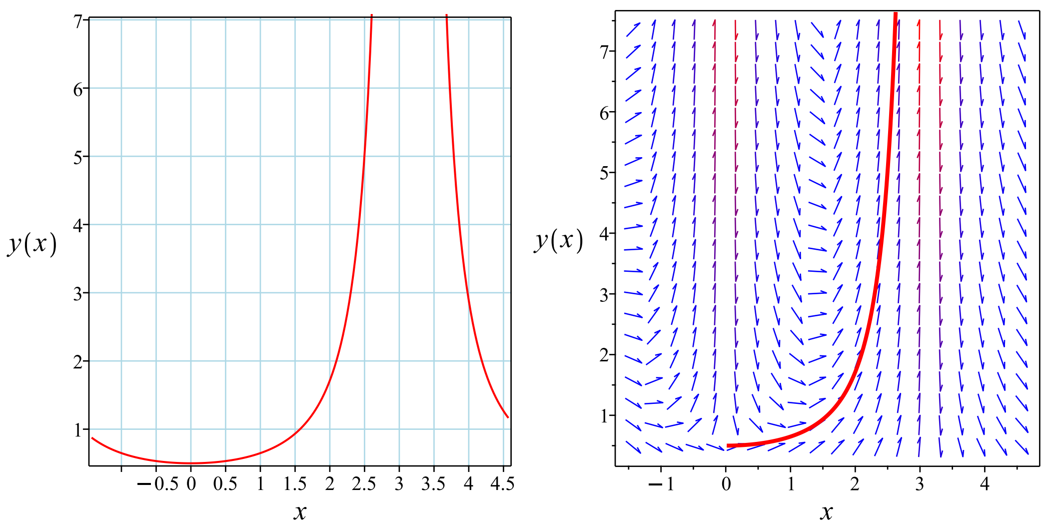
Substituting c_1 found above in the general solution gives

$$y = -\csc(x)^2 \cos(x) + \csc(x)^2$$

Summary

The solution(s) found are the following

$$y = -\csc(x)^2 \cos(x) + \csc(x)^2 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\csc(x)^2 \cos(x) + \csc(x)^2$$

Verified OK.

1.22.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2\cos(x)\, y - 1}{\sin(x)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 40: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{\sin(x)^2} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sin(x)^2}} dy \end{aligned}$$

Which results in

$$S = \sin(x)^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2\cos(x)\,y - 1}{\sin(x)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= y \sin(2x) \\
S_y &= \sin(x)^2
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \sin(x) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \sin(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\cos(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\sin(x)^2 y = -\cos(x) + c_1
$$

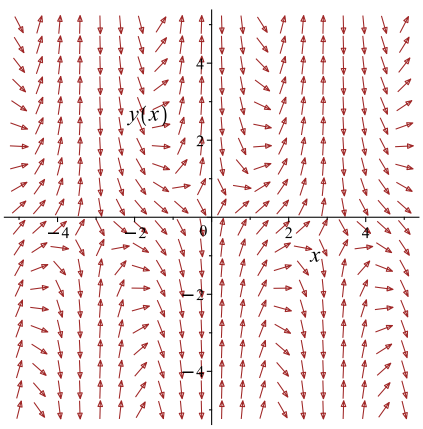
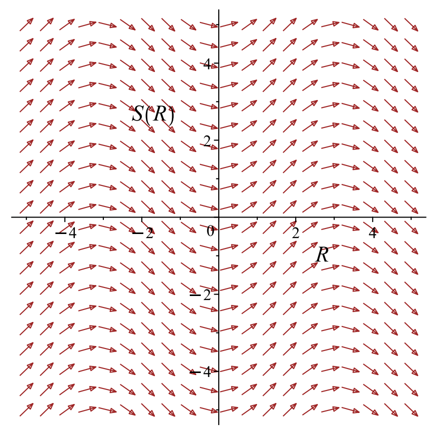
Which simplifies to

$$
\sin(x)^2 y = -\cos(x) + c_1
$$

Which gives

$$
y = -\frac{\cos(x) - c_1}{\sin(x)^2}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2\cos(x)y-1}{\sin(x)}$ 	$R = x$ $S = \sin(x)^2 y$	$\frac{dS}{dR} = \sin(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

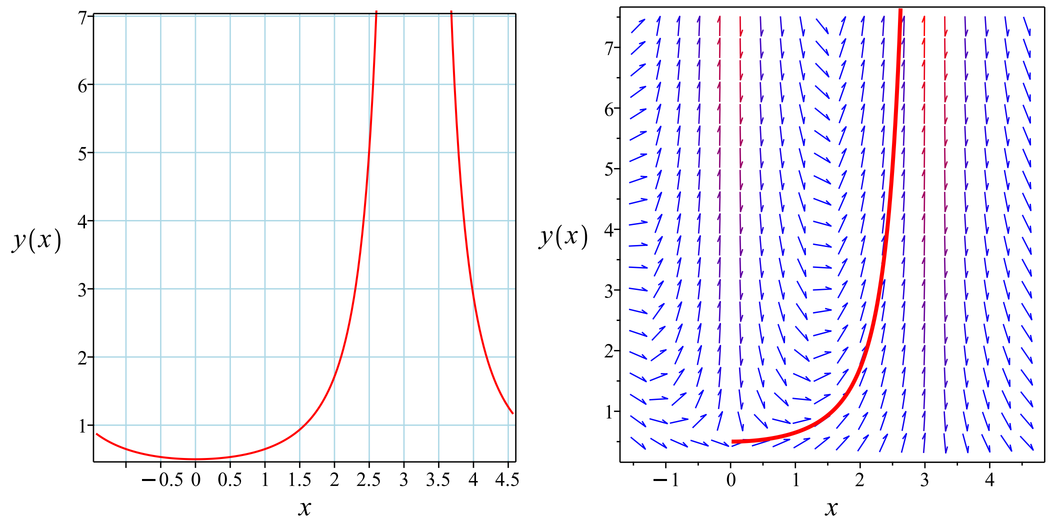
Substituting c_1 found above in the general solution gives

$$y = -\csc(x)^2 \cos(x) + \csc(x)^2$$

Summary

The solution(s) found are the following

$$y = -\csc(x)^2 \cos(x) + \csc(x)^2 \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\csc(x)^2 \cos(x) + \csc(x)^2$$

Verified OK.

1.22.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(\sin(x))\,\mathrm{d}y &= (-2\cos(x)\,y + 1)\,\mathrm{d}x \\
(2\cos(x)\,y - 1)\,\mathrm{d}x + (\sin(x))\,\mathrm{d}y &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= 2\cos(x)\,y - 1 \\
N(x, y) &= \sin(x)
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2\cos(x)\,y - 1) \\
&= 2\cos(x)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\sin(x)) \\
&= \cos(x)
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \csc(x)\,((2\cos(x)) - (\cos(x))) \\
&= \cot(x)
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \cot(x) \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(\sin(x))} \\ &= \sin(x)\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \sin(x)\,(2\cos(x)\,y - 1) \\ &= 2y\sin(x)\cos(x) - \sin(x)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \sin(x)\,(\sin(x)) \\ &= \sin(x)^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (2y\sin(x)\cos(x) - \sin(x)) + \left(\sin(x)^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int 2y\sin(x)\cos(x) - \sin(x) \, \mathrm{d}x\end{aligned}$$

$$\phi = \sin(x)^2 y + \cos(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin(x)^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)^2$. Therefore equation (4) becomes

$$\sin(x)^2 = \sin(x)^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin(x)^2 y + \cos(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin(x)^2 y + \cos(x)$$

The solution becomes

$$y = -\frac{\cos(x) - c_1}{\sin(x)^2}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

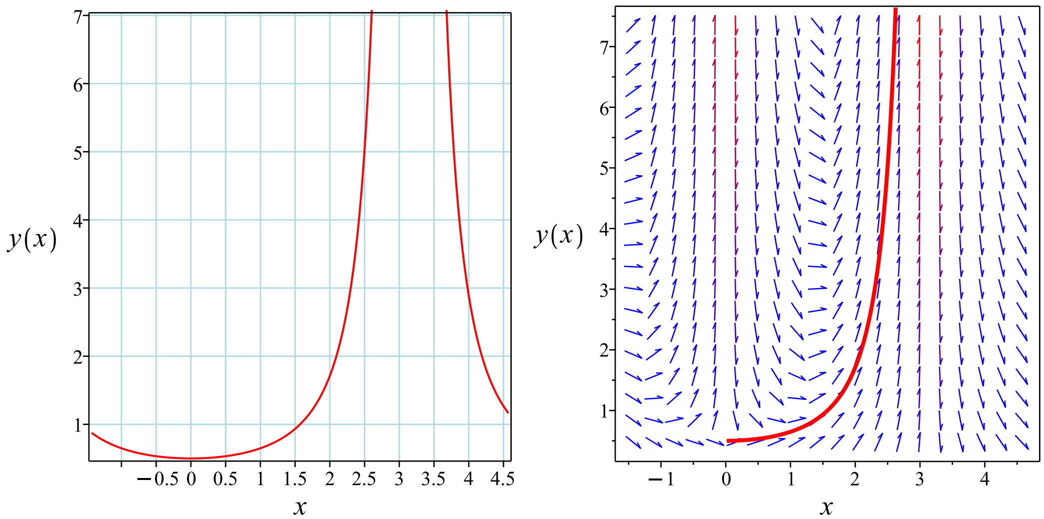
Substituting c_1 found above in the general solution gives

$$y = -\csc(x)^2 \cos(x) + \csc(x)^2$$

Summary

The solution(s) found are the following

$$y = -\csc(x)^2 \cos(x) + \csc(x)^2 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\csc(x)^2 \cos(x) + \csc(x)^2$$

Verified OK.

1.22.5 Maple step by step solution

Let's solve

$\left[y' \sin(x) + 2\cos(x)\, y = 1, y\left(\frac{\pi}{2}\right) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2\cos(x)y}{\sin(x)} + \frac{1}{\sin(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2\cos(x)y}{\sin(x)} = \frac{1}{\sin(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' + \frac{2\cos(x)y}{\sin(x)}\right) = \frac{\mu(x)}{\sin(x)}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x) \left(y' + \frac{2\cos(x)y}{\sin(x)}\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{2\mu(x)\cos(x)}{\sin(x)}$$

- Solve to find the integrating factor

$$\mu(x) = \sin(x)^2$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{\sin(x)} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \frac{\mu(x)}{\sin(x)} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x)}{\sin(x)} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \sin(x)^2$

$$y = \frac{\int \sin(x) dx + c_1}{\sin(x)^2}$$

- Evaluate the integrals on the rhs

$$y = \frac{-\cos(x) + c_1}{\sin(x)^2}$$

- Simplify

$$y = \csc(x)^2 (-\cos(x) + c_1)$$

- Use initial condition $y\left(\frac{\pi}{2}\right) = 1$

$$1 = c_1$$

- Solve for c_1

$$c_1 = 1$$

- Substitute $c_1 = 1$ into general solution and simplify

$$y = \frac{1}{\cos(x) + 1}$$

- Solution to the IVP

$$y = \frac{1}{\cos(x) + 1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 10

```
dsolve([sin(x)*diff(y(x),x)+2*y(x)*cos(x)=1,y(1/2*Pi) = 1],y(x), singsol=all)
```

$$y(x) = \frac{1}{\cos(x) + 1}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 14

```
DSolve[{Sin[x]*y'[x]+2*y[x]*Cos[x]==1,y[Pi/2]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan\left(\frac{x}{2}\right)\csc(x)$$

1.23 problem Problem 14.28

Internal problem ID [2508]
Internal file name [`OUTPUT/2000_Sunday_June_05_2022_02_43_24_AM_76495833/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(5x + y - 7)\, y' - 3y = 3x + 3}$$

1.23.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{3X + 3x_0 + 3Y(X) + 3y_0 + 3}{5X + 5x_0 + Y(X) + y_0 - 7}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 2$$
$$y_0 = -3$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{3X + 3Y(X)}{5X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{3X+3Y}{5X+Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = 3X+3Y$ and $N = 5X+Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{3u+3}{u+5} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{3u(X)+3}{u(X)+5} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{3u(X)+3}{u(X)+5} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) + 5\left(\frac{d}{dX}u(X)\right)X + u(X)^2 + 2u(X) - 3 = 0$$

Or

$$X(u(X)+5)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 2u(X) - 3 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{u^2+2u-3}{X(u+5)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+2u-3}{u+5}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^2+2u-3}{u+5}}\, du &= -\frac{1}{X}\, dX \\
\int \frac{1}{\frac{u^2+2u-3}{u+5}}\, du &= \int -\frac{1}{X}\, dX \\
-\frac{\ln (u+3)}{2} + \frac{3\ln (u-1)}{2} &= -\ln (X) + c_2
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
\frac{-\ln (u+3) + 3\ln (u-1)}{2} &= -\ln (X) + c_2 \\
-\ln (u+3) + 3\ln (u-1) &= (2)\left(-\ln (X) + c_2\right) \\
&= -2\ln (X) + 2c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{-\ln(u+3)+3\ln(u-1)} = \mathrm{e}^{-2\ln(X)+2c_2}
$$

Which simplifies to

$$
\begin{aligned}
\frac{(u-1)^3}{u+3} &= \frac{2c_2}{X^2} \\
&= \frac{c_3}{X^2}
\end{aligned}
$$

Which simplifies to

$$
\frac{(u(X)-1)^3}{u(X)+3} = \frac{c_3 \mathrm{e}^{2c_2}}{X^2}
$$

The solution is

$$
\frac{(u(X)-1)^3}{u(X)+3} = \frac{c_3 \mathrm{e}^{2c_2}}{X^2}
$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$
\frac{\left(\frac{Y(X)}{X} - 1\right)^3}{\frac{Y(X)}{X} + 3} = \frac{c_3 \mathrm{e}^{2c_2}}{X^2}
$$

Which simplifies to

$$-\frac{(-Y(X)+X)^3}{Y(X)+3X} = c_3 e^{2c_2}$$

Using the solution for $Y(X)$

$$-\frac{(-Y(X)+X)^3}{Y(X)+3X} = c_3 e^{2c_2}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 3$$
$$X = x + 2$$

Then the solution in y becomes

$$-\frac{(-y-5+x)^3}{y-3+3x} = c_3 e^{2c_2}$$

Summary

The solution(s) found are the following

$$-\frac{(-y-5+x)^3}{y-3+3x} = c_3 e^{2c_2} \tag{1}$$

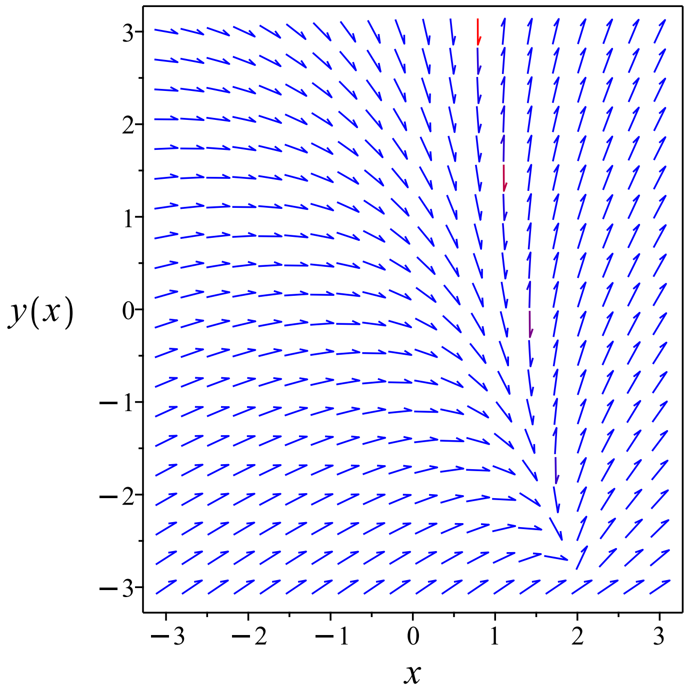


Figure 57: Slope field plot

Verification of solutions

$$-\frac{(-y-5+x)^3}{y-3+3x} = c_3 e^{2c_2}$$

Verified OK.

1.23.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{3x+3y+3}{5x+y-7}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{3(x+y+1)(b_3-a_2)}{5x+y-7} - \frac{9(x+y+1)^2 a_3}{(5x+y-7)^2} \\ &- \left(\frac{3}{5x+y-7} - \frac{15(x+y+1)}{(5x+y-7)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(\frac{3}{5x+y-7} - \frac{3(x+y+1)}{(5x+y-7)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{15x^2a_2 + 9x^2a_3 - 13x^2b_2 - 15x^2b_3 + 6xya_2 + 18xya_3 - 10xyb_2 - 6xyb_3 + 3y^2a_2 - 3y^2a_3 - y^2b_2 - 3y^2b_3}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-15x^2a_2 - 9x^2a_3 + 13x^2b_2 + 15x^2b_3 - 6xya_2 - 18xya_3 + 10xyb_2 + 6xyb_3 \\ &-3y^2a_2 + 3y^2a_3 + y^2b_2 + 3y^2b_3 + 42xa_2 - 18xa_3 - 12xb_1 - 46xb_2 - 6xb_3 + 12ya_1 \\ &+ 18ya_2 + 18ya_3 - 14yb_2 + 6yb_3 + 36a_1 + 21a_2 - 9a_3 + 24b_1 + 49b_2 - 21b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-15a_2v_1^2 - 6a_2v_1v_2 - 3a_2v_2^2 - 9a_3v_1^2 - 18a_3v_1v_2 + 3a_3v_2^2 + 13b_2v_1^2 \\ &+ 10b_2v_1v_2 + b_2v_2^2 + 15b_3v_1^2 + 6b_3v_1v_2 + 3b_3v_2^2 + 12a_1v_2 + 42a_2v_1 \\ &+ 18a_2v_2 - 18a_3v_1 + 18a_3v_2 - 12b_1v_1 - 46b_2v_1 - 14b_2v_2 \\ &- 6b_3v_1 + 6b_3v_2 + 36a_1 + 21a_2 - 9a_3 + 24b_1 + 49b_2 - 21b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-15a_2 - 9a_3 + 13b_2 + 15b_3)\,v_1^2 + (-6a_2 - 18a_3 + 10b_2 + 6b_3)\,v_1v_2 \\ + (42a_2 - 18a_3 - 12b_1 - 46b_2 - 6b_3)\,v_1 + (-3a_2 + 3a_3 + b_2 + 3b_3)\,v_2^2 \\ + (12a_1 + 18a_2 + 18a_3 - 14b_2 + 6b_3)\,v_2 + 36a_1 \\ + 21a_2 - 9a_3 + 24b_1 + 49b_2 - 21b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-15a_2 - 9a_3 + 13b_2 + 15b_3 &= 0\\ -6a_2 - 18a_3 + 10b_2 + 6b_3 &= 0\\ -3a_2 + 3a_3 + b_2 + 3b_3 &= 0\\ 12a_1 + 18a_2 + 18a_3 - 14b_2 + 6b_3 &= 0\\ 42a_2 - 18a_3 - 12b_1 - 46b_2 - 6b_3 &= 0\\ 36a_1 + 21a_2 - 9a_3 + 24b_1 + 49b_2 - 21b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -a_3 - 2b_3\\ a_2 &= 2a_3 + b_3\\ a_3 &= a_3\\ b_1 &= -6a_3 + 3b_3\\ b_2 &= 3a_3\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x - 2\\ \eta &= y + 3\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y + 3 - \left(\frac{3x+3y+3}{5x+y-7}\right)(x-2) \\ &= \frac{-3x^2+2xy+y^2+18x+2y-15}{5x+y-7} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-3x^2+2xy+y^2+18x+2y-15}{5x+y-7}} dy\end{aligned}$$

Which results in

$$S = \frac{3\ln(y+5-x)}{2} - \frac{\ln(3x+y-3)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{3x+3y+3}{5x+y-7}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{3x + 3y + 3}{(3x + y - 3)(x - y - 5)} \\
S_y &= \frac{-5x - y + 7}{(3x + y - 3)(x - y - 5)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

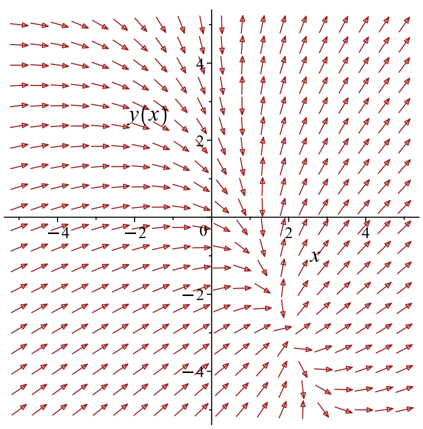
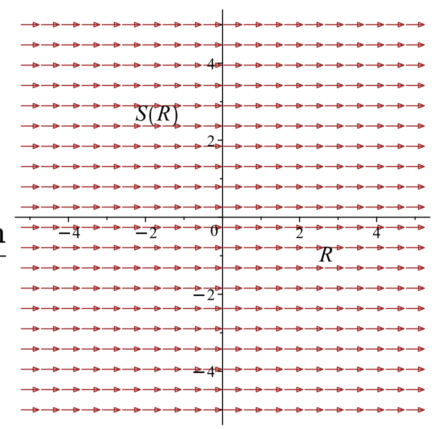
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{3\ln(y + 5 - x)}{2} - \frac{\ln(y - 3 + 3x)}{2} = c_1
$$

Which simplifies to

$$
\frac{3\ln(y + 5 - x)}{2} - \frac{\ln(y - 3 + 3x)}{2} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x+3y+3}{5x+y-7}$ 	$R = x$ $S = \frac{3\ln(y+5-x)}{2} - \ln$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{3\ln(y+5-x)}{2} - \frac{\ln(y-3+3x)}{2} = c_1 \tag{1}$$

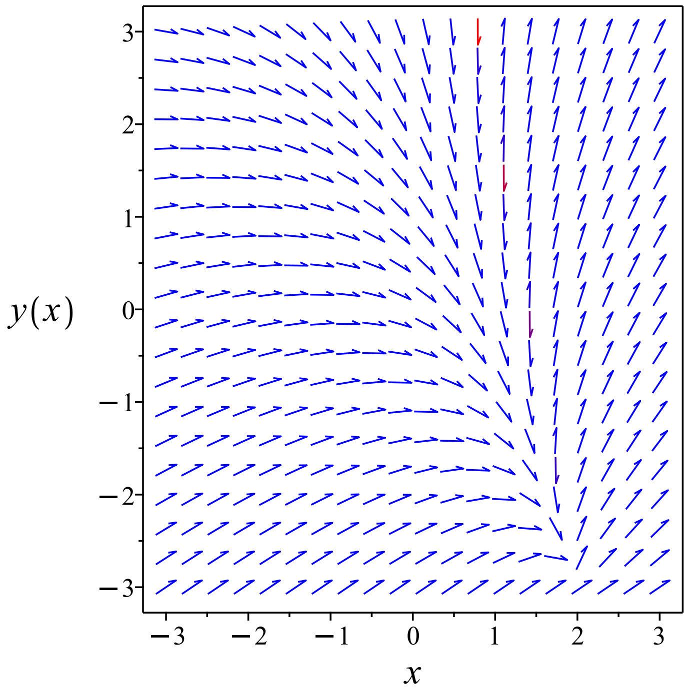


Figure 58: Slope field plot

<u>Verification of solutions</u>

$$\frac{3\ln(y+5-x)}{2}-\frac{\ln(y-3+3x)}{2}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.609 (sec). Leaf size: 217

```
dsolve((5*x+y(x)-7)*diff(y(x),x)=3*(x+y(x)+1),y(x), singsol=all)
```

$$y(x) = \frac{(x-5)\left(i\sqrt{3}-1\right)\left(216\sqrt{c_1\left(-2+x\right)^2\left(-\frac{1}{108}+\left(-2+x\right)^2c_1\right)}+1-216(-2+x)^2c_1}{i\sqrt{3}\left(216\sqrt{c_1\left(-2+x\right)^2\left(-\frac{1}{108}+\left(-2+x\right)^2c_1\right)}+1-216\left(-2+x\right)^2c_1\right)^{\frac{2}{3}}-i\sqrt{3}-\left(216\sqrt{c_1\left(-2+x\right)}\right.}$$

✓ Solution by Mathematica

Time used: 60.172 (sec). Leaf size: 1626

```
DSolve[(5*x+y[x]-7)*y'[x]==3*(x+y[x]+1),y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

1.24 problem Problem 14.29

Internal problem ID [2509]
Internal file name [`OUTPUT/2001_Sunday_June_05_2022_02_43_29_AM_8726393/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{xy' + y - \frac{y^2}{x^{\frac{3}{2}}} = 0}$$

With initial conditions

$$[y(1) = 1]$$

1.24.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{y\left(x^{\frac{3}{2}} - y\right)}{x^{\frac{5}{2}}} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{0<x\}$$

And the point $x_0=1$ is inside this domain. The y domain of $f(x,y)$ when $x=1$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y\left(x^{\frac{3}{2}}-y\right)}{x^{\frac{5}{2}}}\right)\\ &= -\frac{x^{\frac{3}{2}}-y}{x^{\frac{5}{2}}}+\frac{y}{x^{\frac{5}{2}}}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=1$ is

$$\{0<x\}$$

And the point $x_0=1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=1$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=1$ is inside this domain. Therefore solution exists and is unique.

1.24.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{y\left(x^{\frac{3}{2}}-y\right)}{x^{\frac{5}{2}}}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 43: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x\, y^2 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x\,y^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{xy}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y\left(x^{\frac{3}{2}} - y\right)}{x^{\frac{5}{2}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x^2 y} \\ S_y &= \frac{1}{x\,y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^{\frac{7}{2}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^{\frac{7}{2}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{2}{5R^{\frac{5}{2}}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{xy} = -\frac{2}{5x^{\frac{5}{2}}} + c_1$$

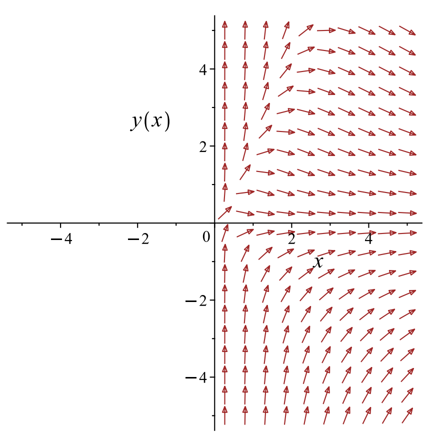
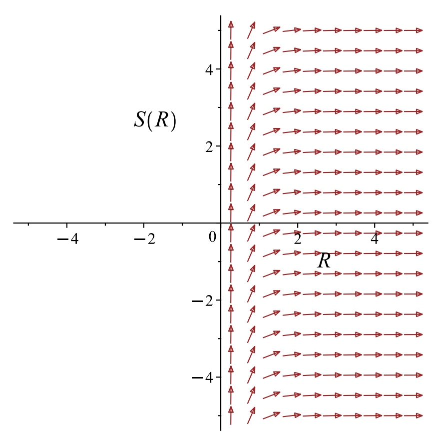
Which simplifies to

$$-\frac{1}{xy} = -\frac{2}{5x^{\frac{5}{2}}} + c_1$$

Which gives

$$y = -\frac{5x^{\frac{5}{2}}}{-2x + 5c_1 x^{\frac{7}{2}}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y\left(x^{\frac{3}{2}} - y\right)}{x^{\frac{5}{2}}}$ 	$R = x$ $S = -\frac{1}{xy}$	$\frac{dS}{dR} = \frac{1}{R^{\frac{7}{2}}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{5}{-2 + 5c_1}$$

$$c_1 = -\frac{3}{5}$$

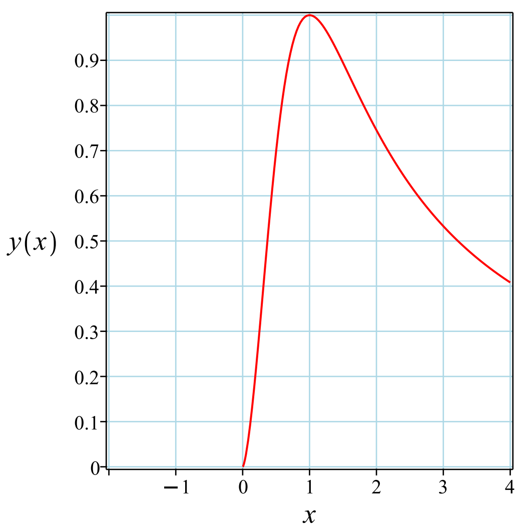
Substituting c_1 found above in the general solution gives

$$y = \frac{5x^{\frac{3}{2}}}{3x^{\frac{5}{2}} + 2}$$

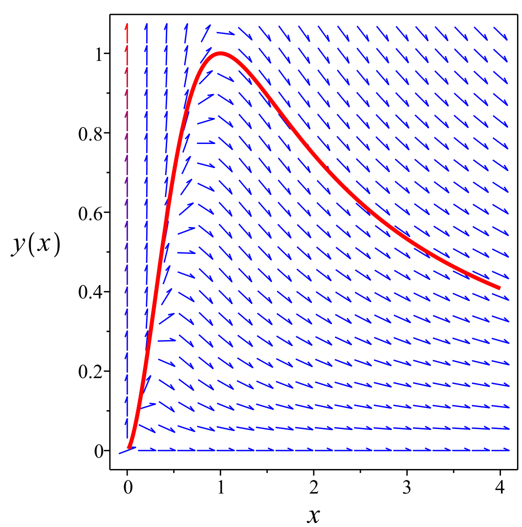
Summary

The solution(s) found are the following

$$y = \frac{5x^{\frac{3}{2}}}{3x^{\frac{5}{2}} + 2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{5x^{\frac{3}{2}}}{3x^{\frac{5}{2}} + 2}$$

Verified OK.

1.24.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y\left(-x^{\frac{3}{2}} + y\right)}{x^{\frac{5}{2}}} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + \frac{1}{x^{\frac{5}{2}}}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= \frac{1}{x^{\frac{5}{2}}} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{xy} + \frac{1}{x^{\frac{5}{2}}} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= -\frac{w(x)}{x}+\frac{1}{x^{\frac{5}{2}}} \\
w' &= \frac{w}{x}-\frac{1}{x^{\frac{5}{2}}}
\end{aligned}
\tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x)+p(x)w(x)=q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= -\frac{1}{x^{\frac{5}{2}}}
\end{aligned}
$$

Hence the ode is

$$
w'(x)-\frac{w(x)}{x}=-\frac{1}{x^{\frac{5}{2}}}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-\frac{1}{x^{\frac{5}{2}}}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(-\frac{1}{x^{\frac{5}{2}}}\right) \\
\mathrm{d}\left(\frac{w}{x}\right) &= \left(-\frac{1}{x^{\frac{7}{2}}}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x} &= \int -\frac{1}{x^{\frac{7}{2}}}\,\mathrm{d}x \\
\frac{w}{x} &= \frac{2}{5x^{\frac{5}{2}}}+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = \frac{2}{5x^{\frac{3}{2}}} + c_1 x$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{2}{5x^{\frac{3}{2}}} + c_1 x$$

Or

$$y = \frac{1}{\frac{2}{5x^{\frac{3}{2}}} + c_1 x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{5}{5c_1 + 2}$$

$$c_1 = \frac{3}{5}$$

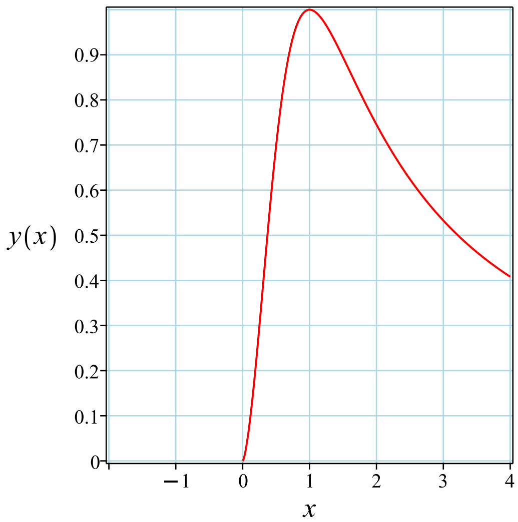
Substituting c_1 found above in the general solution gives

$$y = \frac{5x^{\frac{3}{2}}}{3x^{\frac{5}{2}} + 2}$$

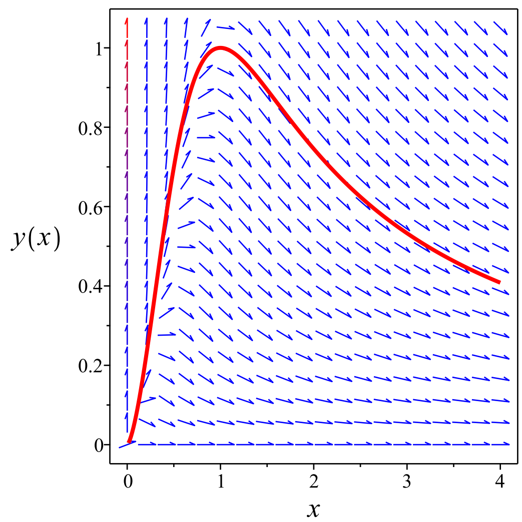
Summary

The solution(s) found are the following

$$y = \frac{5x^{\frac{3}{2}}}{3x^{\frac{5}{2}} + 2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{5x^{\frac{3}{2}}}{3x^{\frac{5}{2}} + 2}$$

Verified OK.

1.24.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(-y + \frac{y^2}{x^{\frac{3}{2}}}\right)\mathrm{d}x\\ \left(y - \frac{y^2}{x^{\frac{3}{2}}}\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y - \frac{y^2}{x^{\frac{3}{2}}}\\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y - \frac{y^2}{x^{\frac{3}{2}}}\right)\\ &= 1 - \frac{2y}{x^{\frac{3}{2}}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}\left(\left(1 - \frac{2y}{x^{\frac{3}{2}}}\right) - (1)\right) \\
&= -\frac{2y}{x^{\frac{5}{2}}}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{x^{\frac{3}{2}}}{y\left(x^{\frac{3}{2}} - y\right)}\left((1) - \left(1 - \frac{2y}{x^{\frac{3}{2}}}\right)\right) \\
&= \frac{2}{x^{\frac{3}{2}} - y}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - \left(1 - \frac{2y}{x^{\frac{3}{2}}}\right)}{x\left(y - \frac{y^2}{x^{\frac{3}{2}}}\right) - y\,(x)} \\
&= -\frac{2}{xy}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{2}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$\mu = e^{-2\ln(t)}$$
$$= \frac{1}{t^2}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^2y^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^2y^2}\left(y - \frac{y^2}{x^{\frac{3}{2}}}\right) \\ &= \frac{x^{\frac{3}{2}} - y}{x^{\frac{7}{2}}y}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^2y^2}(x) \\ &= \frac{1}{x\,y^2}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{x^{\frac{3}{2}} - y}{x^{\frac{7}{2}}y}\right) + \left(\frac{1}{x\,y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{x^{\frac{3}{2}} - y}{x^{\frac{7}{2}} y} \, \mathrm{d}x$$

$$\phi = \frac{-\frac{1}{x} + \frac{2y}{5x^{\frac{5}{2}}}}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{-\frac{1}{x} + \frac{2y}{5x^{\frac{5}{2}}}}{y^2} + \frac{2}{5y\, x^{\frac{5}{2}}} + f'(y) \tag{4}$$

$$= \frac{1}{x\, y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x\, y^2}$. Therefore equation (4) becomes

$$\frac{1}{x\, y^2} = \frac{1}{x\, y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-\frac{1}{x} + \frac{2y}{5x^{\frac{5}{2}}}}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-\frac{1}{x} + \frac{2y}{5x^{\frac{5}{2}}}}{y}$$

The solution becomes

$$y = -\frac{5x^{\frac{5}{2}}}{-2x + 5c_1 x^{\frac{7}{2}}}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{5}{-2 + 5c_1}$$

$$c_1 = -\frac{3}{5}$$

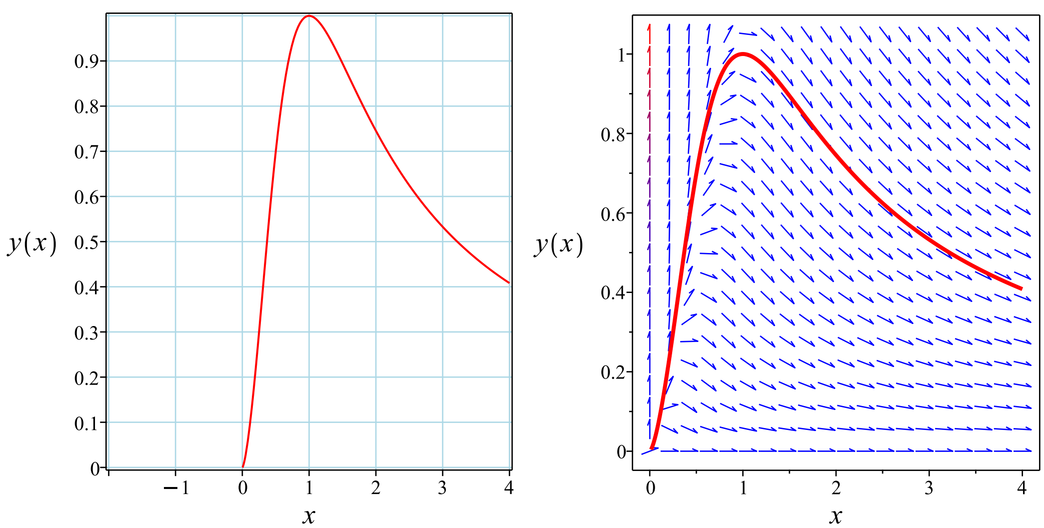
Substituting c_1 found above in the general solution gives

$$y = \frac{5x^{\frac{3}{2}}}{3x^{\frac{5}{2}} + 2}$$

Summary

The solution(s) found are the following

$$y = \frac{5x^{\frac{3}{2}}}{3x^{\frac{5}{2}} + 2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{5x^{\frac{3}{2}}}{3x^{\frac{5}{2}} + 2}$$

Verified OK.

1.24.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= \frac{y\left(-x^{\frac{3}{2}}+y\right)}{x^{\frac{5}{2}}}
\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y}{x} + \frac{y^2}{x^{\frac{5}{2}}}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = \frac{1}{x^{\frac{5}{2}}}$. Let

$$\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{x^{\frac{5}{2}}}} \qquad (1)
\end{aligned}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)$$

But

$$\begin{aligned}
f_2' &= -\frac{5}{2x^{\frac{7}{2}}} \\
f_1 f_2 &= -\frac{1}{x^{\frac{7}{2}}} \\
f_2^2 f_0 &= 0
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^{\frac{5}{2}}} + \frac{7u'(x)}{2x^{\frac{7}{2}}} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{x^{\frac{5}{2}}}$$

The above shows that

$$u'(x) = -\frac{5c_2}{2x^{\frac{7}{2}}}$$

Using the above in (1) gives the solution

$$y = \frac{5c_2}{2x\left(c_1 + \frac{c_2}{x^{\frac{5}{2}}}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{5}{2x\left(c_3 + \frac{1}{x^{\frac{5}{2}}}\right)}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{5}{2c_3 + 2}$$

$$c_3 = \frac{3}{2}$$

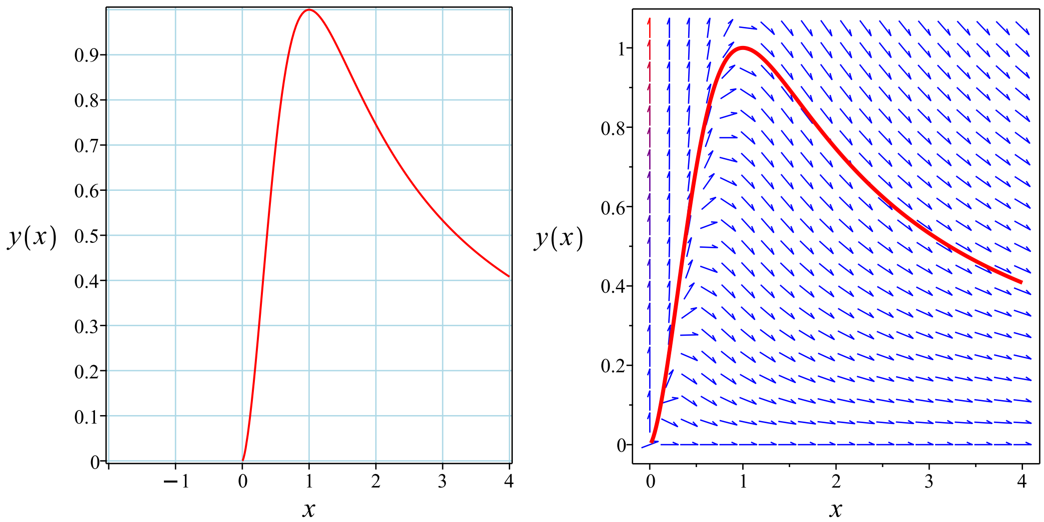
Substituting c_3 found above in the general solution gives

$$y = \frac{5x^{\frac{3}{2}}}{3x^{\frac{5}{2}} + 2}$$

Summary

The solution(s) found are the following

$$y = \frac{5x^{\frac{3}{2}}}{3x^{\frac{5}{2}} + 2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{5x^{\frac{3}{2}}}{3x^{\frac{5}{2}} + 2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 18

```
dsolve([x*diff(y(x),x)+y(x)-y(x)^2/x^(3/2)=0,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{5x^{\frac{3}{2}}}{3x^{\frac{5}{2}} + 2}$$

✓ Solution by Mathematica

Time used: 0.162 (sec). Leaf size: 23

```
DSolve[{x*y'[x]+y[x]-y[x]^2/x^(3/2)==0,y[1]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{5x^{3/2}}{3x^{5/2} + 2}$$

1.25 problem Problem 14.30 (a)

Internal problem ID [2510]
Internal file name [OUTPUT/2002_Sunday_June_05_2022_02_43_33_AM_81818908/index.tex]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.30 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{(2\sin(y)-x)\,y'-\tan(y)=0}$$

With initial conditions

$$[y(0)=0]$$

1.25.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{\tan(y)}{2\sin(y)-x} \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0=0$ is inside this domain. The y domain of $f(x,y)$ when $x=0$ is

$$\left\{-\infty \le y < \pi\_Z139, \pi\_Z139 < y < \frac{1}{2}\pi + \pi\_Z138, \frac{1}{2}\pi + \pi\_Z138 < y \le \infty\right\}$$

But the point $y_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

1.25.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2\sin(y) - x)\,\mathrm{d}y &= (\tan(y))\,\mathrm{d}x \\ (-\tan(y))\,\mathrm{d}x + (2\sin(y) - x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\tan(y) \\
N(x,y) &= 2\sin(y) - x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\tan(y)) \\
&= -\sec(y)^2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2\sin(y) - x) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{2\sin(y) - x}\left(\left(-1 - \tan(y)^2\right) - (-1)\right) \\
&= \frac{\tan(y)^2}{-2\sin(y) + x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\cot(y)\left((-1) - \left(-1 - \tan(y)^2\right)\right) \\
&= -\tan(y)
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\tan(y) \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(\cos(y))} \\
&= \cos(y)
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \cos(y)\,(-\tan(y)) \\
&= -\sin(y)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \cos(y)\,(2\sin(y) - x) \\
&= -(-2\sin(y) + x)\cos(y)
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(-\sin(y)) + (-(-2\sin(y) + x)\cos(y))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int -\sin(y)\,\mathrm{d}x \\
\phi &= -\sin(y)\,x + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\cos(y)\,x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -(-2\sin(y) + x)\cos(y)$. Therefore equation (4) becomes

$$-(-2\sin(y) + x)\cos(y) = -\cos(y)\,x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) &= 2\cos(y)\sin(y) \\ &= \sin(2y) \end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned} \int f'(y)\,\mathrm{d}y &= \int (\sin(2y))\,\mathrm{d}y \\ f(y) &= -\frac{\cos(2y)}{2} + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(y)\,x - \frac{\cos(2y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(y)\,x - \frac{\cos(2y)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = c_1$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-\sin(y)\, x - \frac{\cos(2y)}{2} = -\frac{1}{2}$$

Summary

The solution(s) found are the following

$$-\sin(y)\, x - \frac{\cos(2y)}{2} = -\frac{1}{2} \tag{1}$$

Verification of solutions

$$-\sin(y)\, x - \frac{\cos(2y)}{2} = -\frac{1}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 5

```
dsolve([(2*sin(y(x))-x)*diff(y(x),x)=tan(y(x)),y(0) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 6

```
DSolve[{(2*Sin[y[x]]-x)*y'[x]==Tan[y[x]],y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

1.26 problem Problem 14.30 (b)

Internal problem ID [2511]
Internal file name [`OUTPUT/2003_Sunday_June_05_2022_02_43_39_AM_72750872/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.30 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{(2\sin(y) - x)\, y' - \tan(y) = 0}$$

With initial conditions

$$\left[y(0) = \frac{\pi}{2}\right]$$

1.26.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{\tan(y)}{2\sin(y) - x} \end{aligned}$$

$f(x,y)$ is not defined at $y = \frac{\pi}{2}$ therefore existence and uniqueness theorem do not apply.

1.26.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2\sin(y)-x)\,\mathrm{d}y&=(\tan(y))\,\mathrm{d}x\\ (-\tan(y))\,\mathrm{d}x+(2\sin(y)-x)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\tan(y)\\ N(x,y)&=2\sin(y)-x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\tan(y)) \\ &= -\sec(y)^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2\sin(y) - x) \\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{2\sin(y) - x}\left(\left(-1 - \tan(y)^2\right) - (-1)\right) \\ &= \frac{\tan(y)^2}{-2\sin(y) + x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\cot(y)\left((-1) - \left(-1 - \tan(y)^2\right)\right) \\ &= -\tan(y)\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\tan(y) \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(\cos(y))} \\ &= \cos(y)\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \cos(y)(-\tan(y)) \\ &= -\sin(y)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \cos(y)(2\sin(y) - x) \\ &= -(-2\sin(y) + x)\cos(y)\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (-\sin(y)) + (-(-2\sin(y) + x)\cos(y))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\sin(y)\,\mathrm{d}x \\ \phi &= -\sin(y)\,x + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\cos(y)\,x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -(-2\sin(y) + x)\cos(y)$. Therefore equation (4) becomes

$$-(-2\sin(y) + x)\cos(y) = -\cos(y)\,x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) &= 2\cos(y)\sin(y) \\ &= \sin(2y) \end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int (\sin(2y))\,\mathrm{d}y$$

$$f(y) = -\frac{\cos(2y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(y)\,x - \frac{\cos(2y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(y)\,x - \frac{\cos(2y)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{\pi}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_1$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-\sin(y)\,x - \frac{\cos(2y)}{2} = \frac{1}{2}$$

Summary
The solution(s) found are the following

$$-\sin(y)\,x-\frac{\cos(2y)}{2}=\frac{1}{2} \tag{1}$$

Verification of solutions

$$-\sin(y)\,x-\frac{\cos(2y)}{2}=\frac{1}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple
Time used: 10.359 (sec). Leaf size: 18

```
dsolve([(2*sin(y(x))-x)*diff(y(x),x)=tan(y(x)),y(0) = 1/2*Pi],y(x), singsol=all)
```

$$y(x)=\arcsin\left(\frac{x}{2}+\frac{\sqrt{x^2+4}}{2}\right)$$

✓ Solution by Mathematica

Time used: 18.018 (sec). Leaf size: 67

```
DSolve[{(2*Sin[y[x]]-x)*y'[x]==Tan[y[x]],y[0]==Pi/2},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \cot^{-1}\left(\sqrt{\frac{x^2}{2} - \frac{1}{2}\sqrt{x^4 + 4x^2}}\right)$$

$$y(x) \to \cot^{-1}\left(\frac{\sqrt{x^2 + \sqrt{x^2(x^2 + 4)}}}{\sqrt{2}}\right)$$

1.27 problem Problem 14.31

Internal problem ID [2512]
Internal file name [`OUTPUT/2004_Sunday_June_05_2022_02_43_53_AM_6454647/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 14, First order ordinary differential equations. 14.4 Exercises, page 490
Problem number: Problem 14.31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y", "second_order_nonlinear_solved_by_mainardi_lioville_method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_xy]]
```

$$\boxed{y'' + {y'}^2 + y' = 0}$$

With initial conditions

$$[y(0) = 0]$$

1.27.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + (p(x) + 1)\, p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int -\frac{1}{(p+1)\, p} dp = \int dx$$
$$\ln (p+1) - \ln (p) = x + c_1$$

Raising both side to exponential gives

$$e^{\ln(p+1)-\ln(p)} = e^{x+c_1}$$

Which simplifies to

$$\frac{p+1}{p} = c_2 e^{x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{1}{-1 + c_2 e^{x}}$$

Integrating both sides gives

$$y = \int \frac{1}{-1 + c_2 e^{x}}\, dx$$
$$= \ln (-1 + c_2 e^{x}) - \ln (e^{x}) + c_3$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln (c_2 - 1) + c_3$$

$$c_2 = (e^{c_3} + 1)\, e^{-c_3}$$

Substituting c_2 found above in the general solution gives

$$y = \ln\left(\left(e^{x+c_3} + e^{x} - e^{c_3}\right) e^{-c_3}\right) - \ln\left(e^{x}\right) + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \ln\left(\left(e^{x+c_3} + e^{x} - e^{c_3}\right) e^{-c_3}\right) - \ln\left(e^{x}\right) + c_3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \ln\left(e^{-c_3}\right) + c_3 \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_3\}$. Solving for the constants gives

Substituting these values back in above solution results in

$$y = \ln\left(\left(e^{x+c_3} + e^{x} - e^{c_3}\right) e^{-c_3}\right) - \ln\left(e^{x}\right) + c_3$$

Which simplifies to

$$y = \ln\left(e^{x} + e^{-c_3+x} - 1\right) - \ln\left(e^{x}\right) + c_3$$

Summary

The solution(s) found are the following

$$y = \ln\left(e^{x} + e^{-c_3+x} - 1\right) - \ln\left(e^{x}\right) + c_3 \tag{1}$$

Verification of solutions

$$y = \ln\left(e^{x} + e^{-c_3+x} - 1\right) - \ln\left(e^{x}\right) + c_3$$

Verified OK.

1.27.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y)\left(\frac{d}{dy}p(y)\right) + (p(y)+1)\,p(y) = 0
$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{-p-1}dp &= \int dy \\
-\ln(-p-1) &= y + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
\frac{1}{-p-1} = \mathrm{e}^{y+c_1}
$$

Which simplifies to

$$
\frac{1}{-p-1} = c_2\mathrm{e}^{y}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = -\frac{\mathrm{e}^{-y}}{c_2} - 1
$$

Integrating both sides gives

$$
\begin{aligned}
\int -\frac{c_2\mathrm{e}^{y}}{c_2\mathrm{e}^{y}+1}dy &= \int dx \\
-\ln(c_2\mathrm{e}^{y}+1) &= x + c_3
\end{aligned}
$$

Raising both side to exponential gives

$$
\frac{1}{c_2\mathrm{e}^{y}+1} = \mathrm{e}^{x+c_3}
$$

Which simplifies to

$$\frac{1}{c_2 e^{y}+1} = c_4 e^{x}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln\left(\frac{1-c_4}{c_4 c_2}\right)$$

$$c_2 = -\frac{-1+c_4}{c_4}$$

Substituting c_2 found above in the general solution gives

$$y = \ln\left(\frac{-1+c_4 e^{x}}{-1+c_4}\right) - x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \ln\left(\frac{-1+c_4 e^{x}}{-1+c_4}\right) - x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = 0 \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_4\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

1.27.3 Solving as second order nonlinear solved by mainardi lioville method ode

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y)y'^2 = 0 \tag{1A}$$

Where in this problem

$$f(x) = 1$$
$$g(y) = 1$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx} \ln (y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned} g\frac{dy}{dx} &= \left(\frac{d}{dy} \int g dy\right) \frac{dy}{dx} \\ &= \frac{d}{dx} \int g dy \end{aligned} \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx} \ln (y') + \frac{d}{dx} \int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln (y') + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy} e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = 1$ and $f = 1$, then

$$\begin{aligned} \int -g dy &= \int (-1) \, dy \\ &= -y \\ \int -f dx &= \int (-1) \, dx \\ &= -x \end{aligned}$$

Substituting the above into Eq(6A) gives

$$y' = c_2 \mathrm{e}^{-y} \mathrm{e}^{-x}$$

Which is now solved as first order separable ode. In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x) g(y) \\ &= c_2 \mathrm{e}^{-y} \mathrm{e}^{-x} \end{aligned}$$

Where $f(x) = c_2 e^{-x}$ and $g(y) = e^{-y}$. Integrating both sides gives

$$\frac{1}{e^{-y}} dy = c_2 e^{-x} dx$$
$$\int \frac{1}{e^{-y}} dy = \int c_2 e^{-x} dx$$
$$e^{y} = -c_2 e^{-x} + c_3$$

The solution is

$$e^{y} + c_2 e^{-x} - c_3 = 0$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$e^{y} + c_2 e^{-x} - c_3 = 0 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$1 + c_2 - c_3 = 0 \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_2, c_3\}$. Solving for the constants gives

$$c_2 = -1 + c_3$$

Substituting these values back in above solution results in

$$c_3 e^{-x} - e^{-x} + e^{y} - c_3 = 0$$

Which can be written as

$$e^{y} + (-1 + c_3) e^{-x} - c_3 = 0$$

Summary
The solution(s) found are the following

$$e^{y} + (-1 + c_3) e^{-x} - c_3 = 0 \tag{1}$$

Verification of solutions

$$e^{y} + (-1 + c_3) e^{-x} - c_3 = 0$$

Verified OK.

1.27.4 Maple step by step solution

Let's solve

$[y'' + (y'+1)\, y' = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) + (u(x)+1)\, u(x) = 0$

- Separate variables

  $\frac{u'(x)}{(u(x)+1)u(x)} = -1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{(u(x)+1)u(x)} dx = \int (-1)\, dx + c_1$

- Evaluate integral

  $-\ln (u(x)+1) + \ln (u(x)) = -x + c_1$

- Solve for $u(x)$

  $u(x) = -\frac{e^{-x+c_1}}{e^{-x+c_1}-1}$

- Solve 1st ODE for $u(x)$

  $u(x) = -\frac{e^{-x+c_1}}{e^{-x+c_1}-1}$

- Make substitution $u = y'$

  $y' = -\frac{e^{-x+c_1}}{e^{-x+c_1}-1}$

- Integrate both sides to solve for y

  $\int y' dx = \int -\frac{e^{-x+c_1}}{e^{-x+c_1}-1} dx + c_2$

- Compute integrals

  $y = \ln (e^{-x+c_1} - 1) + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 18

```
dsolve([diff(y(x),x$2)+ (diff(y(x),x))^2+diff(y(x),x)=0,y(0) = 0],y(x), singsol=all)
```

$$y(x) = \ln(c_2 e^x - c_2 + 1) - x$$

✓ Solution by Mathematica

Time used: 0.395 (sec). Leaf size: 54

```
DSolve[{y''[x]+(y'[x])^2+y'[x]==0,y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(-e^x) - \log(e^x) - i\pi$$
$$y(x) \to -\log(e^x) + \log(-e^x + e^{c_1}) - \log(-1 + e^{c_1})$$

2 Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523

2.1 problem Problem 15.1

Internal problem ID [2513]
Internal file name [`OUTPUT/2005_Sunday_June_05_2022_02_43_57_AM_2113077/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + \omega_0^2 x = a \cos(\omega t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

2.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned}
p(t) &= 0 \\
q(t) &= \omega_0^2 \\
F &= a \cos(\omega t)
\end{aligned}$$

Hence the ode is

$$x'' + \omega_0^2 x = a \cos(\omega t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \omega_0^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = a \cos(\omega t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.1.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = \omega_0^2, f(t) = a \cos(\omega t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + \omega_0^2 x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = \omega_0^2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \omega_0^2 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \omega_0^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \omega_0^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(\omega_0^2)} \\ &= \pm\sqrt{-\omega_0^2}\end{aligned}$$

Hence

$$\lambda_1 = +\sqrt{-\omega_0^2}$$

$$\lambda_2 = -\sqrt{-\omega_0^2}$$

Which simplifies to

$$\lambda_1 = \sqrt{-\omega_0^2}$$

$$\lambda_2 = -\sqrt{-\omega_0^2}$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$x = c_1 e^{\left(\sqrt{-\omega_0^2}\right)t} + c_2 e^{\left(-\sqrt{-\omega_0^2}\right)t}$$

Or

$$x = c_1 \mathrm{e}^{\sqrt{-\omega_0^2}\,t} + c_2 \mathrm{e}^{-\sqrt{-\omega_0^2}\,t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^{\sqrt{-\omega_0^2}\,t} + c_2 \mathrm{e}^{-\sqrt{-\omega_0^2}\,t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$a \cos(\omega t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(\omega t), \sin(\omega t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\sqrt{-\omega_0^2}\, t}, e^{-\sqrt{-\omega_0^2}\, t} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(\omega t) + A_2 \sin(\omega t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\omega^2 \cos(\omega t) - A_2\omega^2 \sin(\omega t) + \omega_0^2(A_1 \cos(\omega t) + A_2 \sin(\omega t)) = a \cos(\omega t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{a}{\omega^2 - \omega_0^2}, A_2 = 0 \right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{a \cos(\omega t)}{\omega^2 - \omega_0^2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{\sqrt{-\omega_0^2}\, t} + c_2 e^{-\sqrt{-\omega_0^2}\, t} \right) + \left(-\frac{a \cos(\omega t)}{\omega^2 - \omega_0^2} \right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{\sqrt{-\omega_0^2}\, t} + c_2 e^{-\sqrt{-\omega_0^2}\, t} - \frac{a \cos(\omega t)}{\omega^2 - \omega_0^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = \frac{(-c_1 - c_2)\,\omega_0^2 + (c_1 + c_2)\,\omega^2 - a}{\omega^2 - \omega_0^2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1\sqrt{-\omega_0^2}\,\mathrm{e}^{\sqrt{-\omega_0^2}\,t} - c_2\sqrt{-\omega_0^2}\,\mathrm{e}^{-\sqrt{-\omega_0^2}\,t} + \frac{a\omega\sin(\omega t)}{\omega^2 - \omega_0^2}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = (c_1 - c_2)\sqrt{-\omega_0^2} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{a}{2\omega^2 - 2\omega_0^2}$$
$$c_2 = \frac{a}{2\omega^2 - 2\omega_0^2}$$

Substituting these values back in above solution results in

$$x = \frac{-2a\cos(\omega t) + \mathrm{e}^{\sqrt{-\omega_0^2}\,t}a + \mathrm{e}^{-\sqrt{-\omega_0^2}\,t}a}{2\omega^2 - 2\omega_0^2}$$

Which simplifies to

$$x = \frac{a\left(-2\cos(\omega t) + \mathrm{e}^{\sqrt{-\omega_0^2}\,t} + \mathrm{e}^{-\sqrt{-\omega_0^2}\,t}\right)}{2\omega^2 - 2\omega_0^2}$$

Summary

The solution(s) found are the following

$$x = \frac{a\left(-2\cos(\omega t) + \mathrm{e}^{\sqrt{-\omega_0^2}\,t} + \mathrm{e}^{-\sqrt{-\omega_0^2}\,t}\right)}{2\omega^2 - 2\omega_0^2} \tag{1}$$

Verification of solutions

$$x = \frac{a\left(-2\cos(\omega t) + \mathrm{e}^{\sqrt{-\omega_0^2}\,t} + \mathrm{e}^{-\sqrt{-\omega_0^2}\,t}\right)}{2\omega^2 - 2\omega_0^2}$$

Verified OK.

2.1.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + \omega_0^2 x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \omega_0^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\omega_0^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\omega_0^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\omega_0^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 46: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\omega_0^2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{\sqrt{-\omega_0^2}\, t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= e^{\sqrt{-\omega_0^2}\, t} \end{aligned}$$

Which simplifies to

$$x_1 = e^{\sqrt{-\omega_0^2}\, t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\ &= e^{\sqrt{-\omega_0^2}\, t} \int \frac{1}{e^{2\sqrt{-\omega_0^2}\, t}}\, dt \\ &= e^{\sqrt{-\omega_0^2}\, t} \left(\frac{\sqrt{-\omega_0^2}\, e^{-2\sqrt{-\omega_0^2}\, t}}{2\omega_0^2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1 \left(e^{\sqrt{-\omega_0^2}\, t} \right) + c_2 \left(e^{\sqrt{-\omega_0^2}\, t} \left(\frac{\sqrt{-\omega_0^2}\, e^{-2\sqrt{-\omega_0^2}\, t}}{2\omega_0^2} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + \omega_0^2 x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{\sqrt{-\omega_0^2}\, t} + \frac{c_2 \sqrt{-\omega_0^2}\, \mathrm{e}^{-\sqrt{-\omega_0^2}\, t}}{2\omega_0^2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$a \cos(\omega t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(\omega t), \sin(\omega t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{-\omega_0^2}\, \mathrm{e}^{-\sqrt{-\omega_0^2}\, t}}{2\omega_0^2}, \mathrm{e}^{\sqrt{-\omega_0^2}\, t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(\omega t) + A_2 \sin(\omega t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\omega^2 \cos(\omega t) - A_2\omega^2 \sin(\omega t) + \omega_0^2(A_1 \cos(\omega t) + A_2 \sin(\omega t)) = a \cos(\omega t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{a}{\omega^2 - \omega_0^2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{a \cos(\omega t)}{\omega^2 - \omega_0^2}$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(c_1 \mathrm{e}^{\sqrt{-\omega_0^2}\, t} + \frac{c_2 \sqrt{-\omega_0^2}\, \mathrm{e}^{-\sqrt{-\omega_0^2}\, t}}{2\omega_0^2}\right) + \left(-\frac{a \cos(\omega t)}{\omega^2 - \omega_0^2}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 \mathrm{e}^{\sqrt{-\omega_0^2}\, t} + \frac{c_2 \sqrt{-\omega_0^2}\, \mathrm{e}^{-\sqrt{-\omega_0^2}\, t}}{2\omega_0^2} - \frac{a \cos(\omega t)}{\omega^2 - \omega_0^2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$
0 = \frac{(c_2\omega^2 - c_2\omega_0^2)\sqrt{-\omega_0^2} - 2\omega_0^2(-c_1\omega^2 + c_1\omega_0^2 + a)}{2\omega^2\omega_0^2 - 2\omega_0^4} \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = c_1 \sqrt{-\omega_0^2}\, \mathrm{e}^{\sqrt{-\omega_0^2}\, t} + \frac{c_2 \mathrm{e}^{-\sqrt{-\omega_0^2}\, t}}{2} + \frac{a\omega \sin(\omega t)}{\omega^2 - \omega_0^2}
$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$
0 = \sqrt{-\omega_0^2}\, c_1 + \frac{c_2}{2} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{a}{2\omega^2 - 2\omega_0^2} \\
c_2 &= -\frac{\sqrt{-\omega_0^2}\, a}{\omega^2 - \omega_0^2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
x = \frac{-2a \cos(\omega t) + \mathrm{e}^{\sqrt{-\omega_0^2}\, t} a + \mathrm{e}^{-\sqrt{-\omega_0^2}\, t} a}{2\omega^2 - 2\omega_0^2}
$$

Which simplifies to

$$x = \frac{a\left(-2\cos\left(\omega t\right)+{\mathrm e}^{\sqrt{-\omega_0^2}\, t}+{\mathrm e}^{-\sqrt{-\omega_0^2}\, t}\right)}{2\omega^2-2\omega_0^2}$$

Summary
The solution(s) found are the following

$$x = \frac{a\left(-2\cos\left(\omega t\right)+{\mathrm e}^{\sqrt{-\omega_0^2}\, t}+{\mathrm e}^{-\sqrt{-\omega_0^2}\, t}\right)}{2\omega^2-2\omega_0^2} \tag{1}$$

Verification of solutions

$$x = \frac{a\left(-2\cos\left(\omega t\right)+{\mathrm e}^{\sqrt{-\omega_0^2}\, t}+{\mathrm e}^{-\sqrt{-\omega_0^2}\, t}\right)}{2\omega^2-2\omega_0^2}$$

Verified OK.

2.1.4 Maple step by step solution

Let's solve

$\left[x''+\omega_0^2 x=a\cos\left(\omega t\right), x(0)=0, x'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2+\omega_0^2=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm\left(\sqrt{-4\omega_0^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(\sqrt{-\omega_0^2}, -\sqrt{-\omega_0^2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t)={\mathrm e}^{\sqrt{-\omega_0^2}\, t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-\sqrt{-\omega_0^2}\,t}$
- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$
- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{\sqrt{-\omega_0^2}\,t} + c_2 \mathrm{e}^{-\sqrt{-\omega_0^2}\,t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = a\cos(\omega t)\right]$
  - Wronskian of solutions of the homogeneous equation

    $$W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{\sqrt{-\omega_0^2}\,t} & \mathrm{e}^{-\sqrt{-\omega_0^2}\,t} \\ \sqrt{-\omega_0^2}\,\mathrm{e}^{\sqrt{-\omega_0^2}\,t} & -\sqrt{-\omega_0^2}\,\mathrm{e}^{-\sqrt{-\omega_0^2}\,t} \end{bmatrix}$$
  - Compute Wronskian

    $W(x_1(t), x_2(t)) = -2\sqrt{-\omega_0^2}$
  - Substitute functions into equation for $x_p(t)$

    $$x_p(t) = \frac{a\left(\mathrm{e}^{\sqrt{-\omega_0^2}\,t}\left(\int \mathrm{e}^{-\sqrt{-\omega_0^2}\,t}\cos(\omega t)dt\right) - \mathrm{e}^{-\sqrt{-\omega_0^2}\,t}\left(\int \cos(\omega t)\mathrm{e}^{\sqrt{-\omega_0^2}\,t}dt\right)\right)}{2\sqrt{-\omega_0^2}}$$
  - Compute integrals

    $x_p(t) = -\frac{a\cos(\omega t)}{\omega^2 - \omega_0^2}$
- Substitute particular solution into general solution to ODE

  $x = c_1 \mathrm{e}^{\sqrt{-\omega_0^2}\,t} + c_2 \mathrm{e}^{-\sqrt{-\omega_0^2}\,t} - \frac{a\cos(\omega t)}{\omega^2 - \omega_0^2}$

☐ Check validity of solution $x = c_1 \mathrm{e}^{\sqrt{-\omega_0^2}\,t} + c_2 \mathrm{e}^{-\sqrt{-\omega_0^2}\,t} - \frac{a\cos(\omega t)}{\omega^2 - \omega_0^2}$

  - Use initial condition $x(0) = 0$

    $0 = c_1 + c_2 - \frac{a}{\omega^2 - \omega_0^2}$
  - Compute derivative of the solution

    $x' = c_1\sqrt{-\omega_0^2}\,\mathrm{e}^{\sqrt{-\omega_0^2}\,t} - c_2\sqrt{-\omega_0^2}\,\mathrm{e}^{-\sqrt{-\omega_0^2}\,t} + \frac{a\omega\sin(\omega t)}{\omega^2 - \omega_0^2}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = \sqrt{-\omega_0^2}\, c_1 - \sqrt{-\omega_0^2}\, c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{a}{2(\omega^2-\omega_0^2)}, c_2 = \frac{a}{2(\omega^2-\omega_0^2)}\right\}$

- Substitute constant values into general solution and simplify

  $x = \frac{a\left(-2\cos(\omega t)+\mathrm{e}^{\sqrt{-\omega_0^2}\, t}+\mathrm{e}^{-\sqrt{-\omega_0^2}\, t}\right)}{2\omega^2-2\omega_0^2}$

- Solution to the IVP

  $x = \frac{a\left(-2\cos(\omega t)+\mathrm{e}^{\sqrt{-\omega_0^2}\, t}+\mathrm{e}^{-\sqrt{-\omega_0^2}\, t}\right)}{2\omega^2-2\omega_0^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 28

```
dsolve([diff(x(t),t$2)+ (omega__0)^2*x(t)=a*cos(omega*t),x(0) = 0, D(x)(0) = 0],x(t), singso
```

$$x(t) = \frac{a(\cos(\omega_0 t) - \cos(\omega t))}{\omega^2 - \omega_0^2}$$

✓ Solution by Mathematica

Time used: 0.371 (sec). Leaf size: 33

```
DSolve[{x''[t]+(Subscript[\[Omega],0])^2*x[t]==a*Cos[\[Omega]*t],{x[0]==0,x'[0]==0}},x[t],t,
```

$$x(t) \to \frac{a(\cos(t\omega_0) - \cos(t\omega))}{\omega^2 - \omega_0^2}$$

2.2 problem Problem 15.2(a)

Internal problem ID [2514]
Internal file name [`OUTPUT/2006_Sunday_June_05_2022_02_44_01_AM_63714365/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.2(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{f'' + 2f' + 5f = 0}$$

With initial conditions

$$[f(0) = 1, f'(0) = 0]$$

2.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$f'' + p(t)f' + q(t)f = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 5 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$f'' + 2f' + 5f = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.2.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Af''(t) + Bf'(t) + Cf(t) = 0$$

Where in the above $A = 1, B = 2, C = 5$. Let the solution be $f = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda\, e^{\lambda t} + 5\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(5)}\\ &= -1 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 2i\\ \lambda_2 &= -1 - 2i\end{aligned}$$

Which simplifies to

$$\lambda_1 = -1 + 2i$$
$$\lambda_2 = -1 - 2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$f = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$f = e^{-t}(c_1 \cos(2t) + c_2 \sin(2t))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$f = \mathrm{e}^{-t}(c_1 \cos(2t) + c_2 \sin(2t)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $f = 1$ and $t = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$f' = -\mathrm{e}^{-t}(c_1 \cos(2t) + c_2 \sin(2t)) + \mathrm{e}^{-t}(-2c_1 \sin(2t) + 2c_2 \cos(2t))$$

substituting $f' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = \frac{1}{2}$$

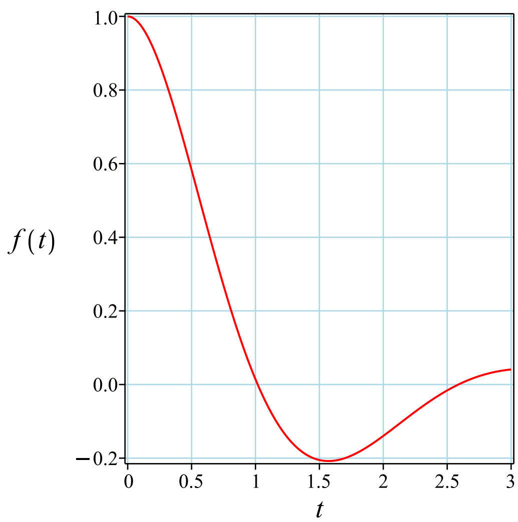
Substituting these values back in above solution results in

$$f = \frac{e^{-t}(2\cos(2t) + \sin(2t))}{2}$$

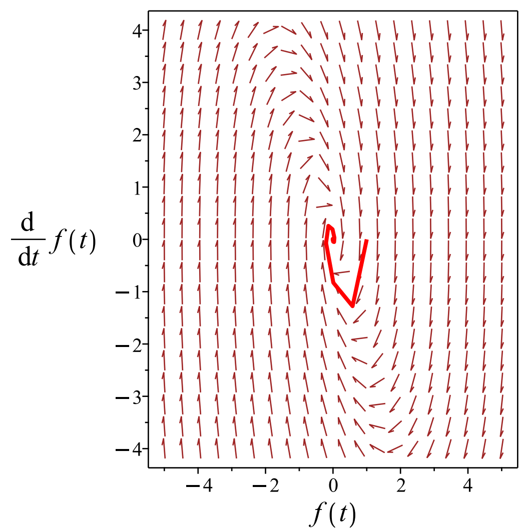
Summary

The solution(s) found are the following

$$f = \frac{e^{-t}(2\cos(2t) + \sin(2t))}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$f = \frac{e^{-t}(2\cos(2t) + \sin(2t))}{2}$$

Verified OK.

2.2.3 Solving using Kovacic algorithm

Writing the ode as

$$f'' + 2f' + 5f = 0 \tag{1}$$
$$Af'' + Bf' + Cf = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 2 \\
C &= 5
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = f e^{\int \frac{B}{2A} \, dt}
$$

Then (2) becomes

$$
z''(t) = r z(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = -4z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then f is found using the inverse transformation

$$
f = z(t) \, e^{-\int \frac{B}{2A} \, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 48: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(2t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in f is found from

$$
\begin{aligned}
f_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\, dt} \\
&= z_1 e^{-t} \\
&= z_1 \left(\mathrm{e}^{-t}\right)
\end{aligned}
$$

Which simplifies to

$$f_1 = \mathrm{e}^{-t} \cos(2t)$$

The second solution f_2 to the original ode is found using reduction of order

$$f_2 = f_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{f_1^2}\, dt$$

Substituting gives

$$\begin{aligned} f_2 &= f_1 \int \frac{e^{\int -\frac{2}{1}\, dt}}{(f_1)^2}\, dt \\ &= f_1 \int \frac{e^{-2t}}{(f_1)^2}\, dt \\ &= f_1 \left(\frac{\tan(2t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} f &= c_1 f_1 + c_2 f_2 \\ &= c_1 \left(\mathrm{e}^{-t} \cos(2t)\right) + c_2 \left(\mathrm{e}^{-t} \cos(2t) \left(\frac{\tan(2t)}{2}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$f = c_1 \mathrm{e}^{-t} \cos(2t) + \frac{c_2 \mathrm{e}^{-t} \sin(2t)}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $f = 1$ and $t = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$f' = -c_1 \mathrm{e}^{-t} \cos(2t) - 2c_1 \mathrm{e}^{-t} \sin(2t) - \frac{c_2 \mathrm{e}^{-t} \sin(2t)}{2} + c_2 \mathrm{e}^{-t} \cos(2t)$$

substituting $f' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$f = e^{-t} \cos(2t) + \frac{e^{-t} \sin(2t)}{2}$$

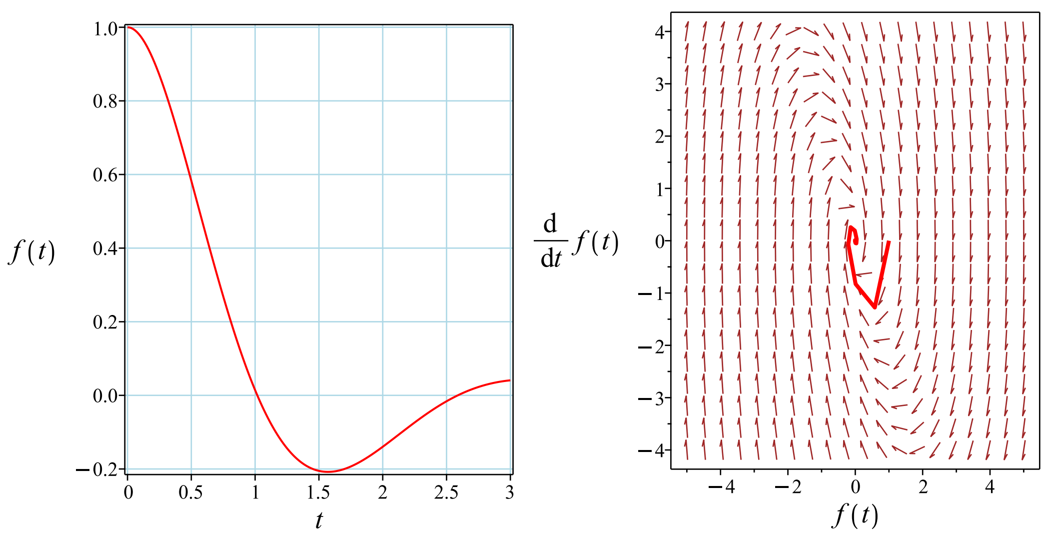
Which simplifies to

$$f = \frac{e^{-t}(2\cos(2t) + \sin(2t))}{2}$$

Summary

The solution(s) found are the following

$$f = \frac{e^{-t}(2\cos(2t) + \sin(2t))}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$f = \frac{e^{-t}(2\cos(2t) + \sin(2t))}{2}$$

Verified OK.

2.2.4 Maple step by step solution

Let's solve

$\left[f'' + 2f' + 5f = 0, f(0) = 1, f'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  f''

- Characteristic polynomial of ODE

  $r^2 + 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - 2\,\mathrm{I}, -1 + 2\,\mathrm{I})$

- 1st solution of the ODE

  $f_1(t) = \mathrm{e}^{-t} \cos(2t)$

- 2nd solution of the ODE

  $f_2(t) = \mathrm{e}^{-t} \sin(2t)$

- General solution of the ODE

  $f = c_1 f_1(t) + c_2 f_2(t)$

- Substitute in solutions

  $f = c_1 \mathrm{e}^{-t} \cos(2t) + c_2 \mathrm{e}^{-t} \sin(2t)$

□ Check validity of solution $f = c_1 \mathrm{e}^{-t} \cos(2t) + c_2 \mathrm{e}^{-t} \sin(2t)$

- Use initial condition $f(0) = 1$

  $1 = c_1$

- Compute derivative of the solution

  $f' = -c_1 \mathrm{e}^{-t} \cos(2t) - 2c_1 \mathrm{e}^{-t} \sin(2t) - c_2 \mathrm{e}^{-t} \sin(2t) + 2c_2 \mathrm{e}^{-t} \cos(2t)$

- Use the initial condition $f'\Big|_{\{t=0\}} = 0$

  $0 = -c_1 + 2c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = 1, c_2 = \frac{1}{2}\right\}$

- Substitute constant values into general solution and simplify

  $f = \frac{e^{-t}(2\cos(2t)+\sin(2t))}{2}$

- Solution to the IVP

  $f = \frac{e^{-t}(2\cos(2t)+\sin(2t))}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([diff(f(t),t$2)+2*diff(f(t),t)+5*f(t)=0,f(0) = 1, D(f)(0) = 0],f(t), singsol=all)
```

$$f(t) = \frac{e^{-t}(\sin(2t) + 2\cos(2t))}{2}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 25

```
DSolve[{f''[t]+2*f'[t]+5*f[t]==0,{f[0]==1,f'[0]==0}},f[t],t,IncludeSingularSolutions -> True
```

$$f(t) \to \frac{1}{2}e^{-t}(\sin(2t) + 2\cos(2t))$$

2.3 problem Problem 15.2(b)

Internal problem ID [2515]
Internal file name [`OUTPUT/2007_Sunday_June_05_2022_02_44_02_AM_61422303/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.2(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{f'' + 2f' + 5f = \mathrm{e}^{-t} \cos(3t)}$$

With initial conditions

$$[f(0) = 0, f'(0) = 0]$$

2.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$f'' + p(t)f' + q(t)f = F$$

Where here

$$\begin{aligned}
p(t) &= 2 \\
q(t) &= 5 \\
F &= \mathrm{e}^{-t} \cos(3t)
\end{aligned}$$

Hence the ode is

$$f'' + 2f' + 5f = e^{-t} \cos(3t)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-t} \cos(3t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.3.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Af''(t) + Bf'(t) + Cf(t) = f(t)$$

Where $A = 1, B = 2, C = 5, f(t) = e^{-t} \cos(3t)$. Let the solution be

$$f = f_h + f_p$$

Where f_h is the solution to the homogeneous ODE $Af''(t)+Bf'(t)+Cf(t) = 0$, and f_p is a particular solution to the non-homogeneous ODE $Af''(t) + Bf'(t) + Cf(t) = f(t)$. f_h is the solution to

$$f'' + 2f' + 5f = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Af''(t) + Bf'(t) + Cf(t) = 0$$

Where in the above $A = 1, B = 2, C = 5$. Let the solution be $f = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + 5 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(5)}\\ &= -1 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 2i\\ \lambda_2 &= -1 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + 2i\\ \lambda_2 &= -1 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$f = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$f = e^{-t}(c_1 \cos(2t) + c_2 \sin(2t))$$

Therefore the homogeneous solution f_h is

$$f_h = \mathrm{e}^{-t}(c_1 \cos(2t) + c_2 \sin(2t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-t}\cos(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t} \cos(3t), e^{-t} \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-t} \cos(2t), e^{-t} \sin(2t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$f_p = A_1 e^{-t} \cos(3t) + A_2 e^{-t} \sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution f_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 e^{-t} \cos(3t) - 5A_2 e^{-t} \sin(3t) = e^{-t} \cos(3t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{5}, A_2 = 0\right]$$

Substituting the above back in the above trial solution f_p, gives the particular solution

$$f_p = -\frac{e^{-t} \cos(3t)}{5}$$

Therefore the general solution is

$$\begin{aligned} f &= f_h + f_p \\ &= \left(e^{-t}(c_1 \cos(2t) + c_2 \sin(2t))\right) + \left(-\frac{e^{-t} \cos(3t)}{5}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$f = e^{-t}(c_1 \cos(2t) + c_2 \sin(2t)) - \frac{e^{-t} \cos(3t)}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $f = 0$ and $t = 0$ in the above gives

$$0 = c_1 - \frac{1}{5} \tag{1A}$$

Taking derivative of the solution gives

$$f' = -e^{-t}(c_1 \cos(2t) + c_2 \sin(2t)) + e^{-t}(-2c_1 \sin(2t) + 2c_2 \cos(2t)) + \frac{e^{-t} \cos(3t)}{5} + \frac{3 e^{-t} \sin(3t)}{5}$$

substituting $f' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 + \frac{1}{5} + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

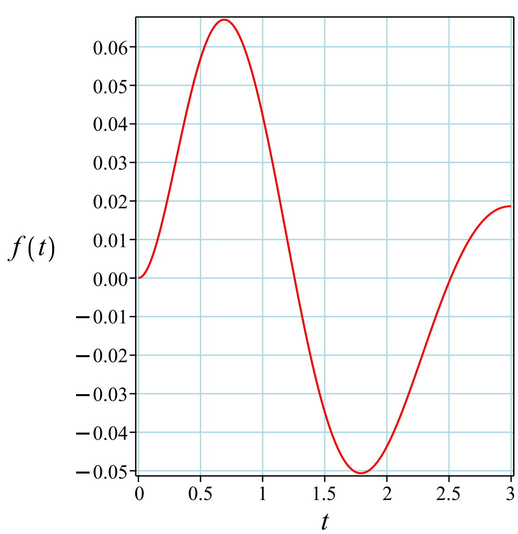
$$c_1 = \frac{1}{5}$$
$$c_2 = 0$$

Substituting these values back in above solution results in

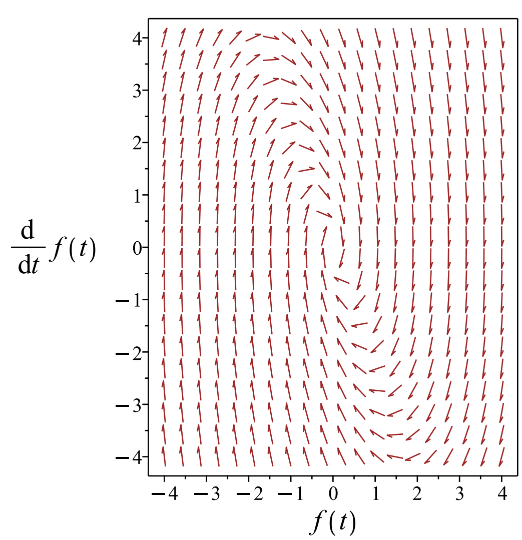
$$f = \frac{e^{-t} \cos(2t)}{5} - \frac{e^{-t} \cos(3t)}{5}$$

<u>Summary</u>
The solution(s) found are the following

$$f = \frac{e^{-t} \cos(2t)}{5} - \frac{e^{-t} \cos(3t)}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$f = \frac{e^{-t} \cos(2t)}{5} - \frac{e^{-t} \cos(3t)}{5}$$

Verified OK.

2.3.3 Solving using Kovacic algorithm

Writing the ode as

$$f'' + 2f' + 5f = 0 \tag{1}$$
$$Af'' + Bf' + Cf = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = fe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then f is found using the inverse transformation

$$f = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 50: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(2t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in f is found from

$$
\begin{aligned}
f_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt} \\
&= z_1 e^{-t} \\
&= z_1 \left(\mathrm{e}^{-t}\right)
\end{aligned}
$$

Which simplifies to

$$
f_1 = \mathrm{e}^{-t} \cos(2t)
$$

The second solution f_2 to the original ode is found using reduction of order

$$
f_2 = f_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{f_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
f_2 &= f_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(f_1)^2}\,dt \\
&= f_1 \int \frac{e^{-2t}}{(f_1)^2}\,dt \\
&= f_1 \left(\frac{\tan(2t)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
f &= c_1 f_1 + c_2 f_2 \\
&= c_1 \left(\mathrm{e}^{-t} \cos(2t)\right) + c_2 \left(\mathrm{e}^{-t} \cos(2t) \left(\frac{\tan(2t)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
f = f_h + f_p
$$

Where f_h is the solution to the homogeneous ODE $Af''(t)+Bf'(t)+Cf(t)=0$, and f_p is a particular solution to the nonhomogeneous ODE $Af''(t)+Bf'(t)+Cf(t)=f(t)$. f_h is the solution to

$$f''+2f'+5f=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$f_h=c_1\mathrm{e}^{-t}\cos(2t)+\frac{c_2\mathrm{e}^{-t}\sin(2t)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-t}\cos(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-t}\cos(3t),\mathrm{e}^{-t}\sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-t}\cos(2t),\frac{\mathrm{e}^{-t}\sin(2t)}{2}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$f_p=A_1\mathrm{e}^{-t}\cos(3t)+A_2\mathrm{e}^{-t}\sin(3t)$$

The unknowns $\{A_1,A_2\}$ are found by substituting the above trial solution f_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1\mathrm{e}^{-t}\cos(3t)-5A_2\mathrm{e}^{-t}\sin(3t)=\mathrm{e}^{-t}\cos(3t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=-\frac{1}{5},A_2=0\right]$$

Substituting the above back in the above trial solution f_p, gives the particular solution

$$f_p = -\frac{e^{-t}\cos(3t)}{5}$$

Therefore the general solution is

$$\begin{aligned} f &= f_h + f_p \\ &= \left(c_1 e^{-t}\cos(2t) + \frac{c_2 e^{-t}\sin(2t)}{2}\right) + \left(-\frac{e^{-t}\cos(3t)}{5}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$f = c_1 e^{-t}\cos(2t) + \frac{c_2 e^{-t}\sin(2t)}{2} - \frac{e^{-t}\cos(3t)}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $f = 0$ and $t = 0$ in the above gives

$$0 = c_1 - \frac{1}{5} \tag{1A}$$

Taking derivative of the solution gives

$$f' = -c_1 e^{-t}\cos(2t) - 2c_1 e^{-t}\sin(2t) - \frac{c_2 e^{-t}\sin(2t)}{2} + c_2 e^{-t}\cos(2t) + \frac{e^{-t}\cos(3t)}{5} + \frac{3\,e^{-t}\sin(3t)}{5}$$

substituting $f' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 + \frac{1}{5} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{1}{5} \\ c_2 &= 0 \end{aligned}$$

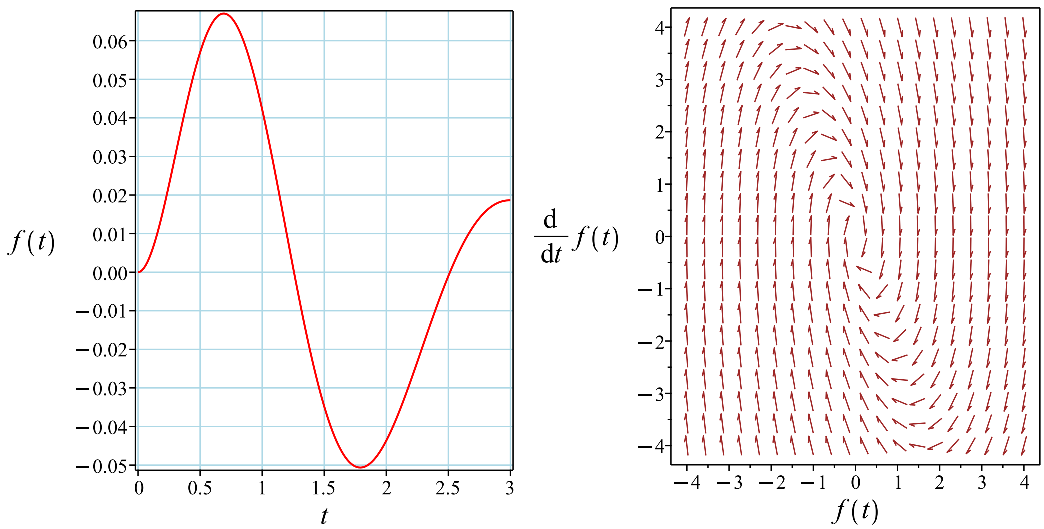
Substituting these values back in above solution results in

$$f = \frac{e^{-t}\cos(2t)}{5} - \frac{e^{-t}\cos(3t)}{5}$$

Summary

The solution(s) found are the following

$$f = \frac{e^{-t} \cos(2t)}{5} - \frac{e^{-t} \cos(3t)}{5} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$f = \frac{e^{-t} \cos(2t)}{5} - \frac{e^{-t} \cos(3t)}{5}$$

Verified OK.

2.3.4 Maple step by step solution

Let's solve

$$\left[f'' + 2f' + 5f = e^{-t} \cos(3t), f(0) = 0, f'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  f''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 5 = 0$

- Use quadratic formula to solve for r

$r = \frac{(-2) \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - 2\,\mathrm{I}, -1 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $f_1(t) = \mathrm{e}^{-t} \cos(2t)$

- 2nd solution of the homogeneous ODE

  $f_2(t) = \mathrm{e}^{-t} \sin(2t)$

- General solution of the ODE

  $f = c_1 f_1(t) + c_2 f_2(t) + f_p(t)$

- Substitute in solutions of the homogeneous ODE

  $f = c_1 \mathrm{e}^{-t} \cos(2t) + c_2 \mathrm{e}^{-t} \sin(2t) + f_p(t)$

☐ Find a particular solution $f_p(t)$ of the ODE

- ○ Use variation of parameters to find f_p here $g(t)$ is the forcing function

  $\left[f_p(t) = -f_1(t) \left(\int \frac{f_2(t) g(t)}{W(f_1(t), f_2(t))} dt\right) + f_2(t) \left(\int \frac{f_1(t) g(t)}{W(f_1(t), f_2(t))} dt\right), g(t) = \mathrm{e}^{-t} \cos(3t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(f_1(t), f_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} \cos(2t) & \mathrm{e}^{-t} \sin(2t) \\ -\mathrm{e}^{-t} \cos(2t) - 2\,\mathrm{e}^{-t} \sin(2t) & -\mathrm{e}^{-t} \sin(2t) + 2\,\mathrm{e}^{-t} \cos(2t) \end{bmatrix}$

- ○ Compute Wronskian

  $W(f_1(t), f_2(t)) = 2\,\mathrm{e}^{-2t}$

- ○ Substitute functions into equation for $f_p(t)$

  $f_p(t) = -\frac{\mathrm{e}^{-t} (\cos(2t) (\int (\sin(5t) - \sin(t)) dt) - \sin(2t) (\int (\cos(t) + \cos(5t)) dt))}{4}$

- ○ Compute integrals

  $f_p(t) = -\frac{\mathrm{e}^{-t} \cos(3t)}{5}$

- Substitute particular solution into general solution to ODE

  $f = c_1 \mathrm{e}^{-t} \cos(2t) + c_2 \mathrm{e}^{-t} \sin(2t) - \frac{\mathrm{e}^{-t} \cos(3t)}{5}$

☐ Check validity of solution $f = c_1 \mathrm{e}^{-t} \cos(2t) + c_2 \mathrm{e}^{-t} \sin(2t) - \frac{\mathrm{e}^{-t} \cos(3t)}{5}$

- ○ Use initial condition $f(0) = 0$

  $0 = c_1 - \frac{1}{5}$

- Compute derivative of the solution

  $f' = -c_1 e^{-t} \cos(2t) - 2c_1 e^{-t} \sin(2t) - c_2 e^{-t} \sin(2t) + 2c_2 e^{-t} \cos(2t) + \frac{e^{-t}\cos(3t)}{5} + \frac{3\,e^{-t}\sin(3t)}{5}$

- Use the initial condition $f'\Big|_{\{t=0\}} = 0$

  $0 = -c_1 + \frac{1}{5} + 2c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{5}, c_2 = 0\right\}$

- Substitute constant values into general solution and simplify

  $f = -\frac{\left(-2\cos(t)^2 + 1 + 4\cos(t)^3 - 3\cos(t)\right)e^{-t}}{5}$

- Solution to the IVP

  $f = -\frac{\left(-2\cos(t)^2 + 1 + 4\cos(t)^3 - 3\cos(t)\right)e^{-t}}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 25

```
dsolve([diff(f(t),t$2)+2*diff(f(t),t)+5*f(t)=exp(-t)*cos(3*t),f(0) = 0, D(f)(0) = 0],f(t), s
```

$$f(t) = -\frac{\left(-2\cos(t)^2 + 1 + 4\cos(t)^3 - 3\cos(t)\right)e^{-t}}{5}$$

✓ Solution by Mathematica

Time used: 0.118 (sec). Leaf size: 34

```
DSolve[{f''[t]+2*f'[t]+5*f[t]==Exp[-t]*Cos[3*t],{f[0]==0,f'[0]==0}},f[t],t,IncludeSingularSo
```

$$f(t) \to \frac{2}{5} e^{-t} \sin^2\left(\frac{t}{2}\right) (2\cos(t) + 2\cos(2t) + 1)$$

2.4 problem Problem 15.4

Internal problem ID [2516]
Internal file name [`OUTPUT/2008_Sunday_June_05_2022_02_44_05_AM_15357516/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{f'' + 6f' + 9f = \mathrm{e}^{-t}}$$

With initial conditions

$$[f(0) = 0, f'(0) = \lambda]$$

2.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$f'' + p(t)f' + q(t)f = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 9 \\ F &= \mathrm{e}^{-t} \end{aligned}$$

Hence the ode is

$$f'' + 6f' + 9f = \mathrm{e}^{-t}$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \mathrm{e}^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.4.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Af''(t) + Bf'(t) + Cf(t) = f(t)$$

Where $A = 1, B = 6, C = 9, f(t) = \mathrm{e}^{-t}$. Let the solution be

$$f = f_h + f_p$$

Where f_h is the solution to the homogeneous ODE $Af''(t) + Bf'(t) + Cf(t) = 0$, and f_p is a particular solution to the non-homogeneous ODE $Af''(t) + Bf'(t) + Cf(t) = f(t)$. f_h is the solution to

$$f'' + 6f' + 9f = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Af''(t) + Bf'(t) + Cf(t) = 0$$

Where in the above $A = 1, B = 6, C = 9$. Let the solution be $f = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 6\lambda\, e^{\lambda t} + 9\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(6)^2 - (4)(1)(9)}\\ &= -3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 3$. Therefore the solution is

$$f = c_1 e^{-3t} + c_2 t\, e^{-3t} \tag{1}$$

Therefore the homogeneous solution f_h is

$$f_h = c_1 e^{-3t} + c_2 t\, e^{-3t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{t\, e^{-3t}, e^{-3t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$f_p = A_1 \mathrm{e}^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution f_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 \mathrm{e}^{-t} = \mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution f_p, gives the particular solution

$$f_p = \frac{\mathrm{e}^{-t}}{4}$$

Therefore the general solution is

$$\begin{aligned} f &= f_h + f_p \\ &= \left(c_1 \mathrm{e}^{-3t} + c_2 t\, \mathrm{e}^{-3t}\right) + \left(\frac{\mathrm{e}^{-t}}{4}\right) \end{aligned}$$

Which simplifies to

$$f = \mathrm{e}^{-3t}(c_2 t + c_1) + \frac{\mathrm{e}^{-t}}{4}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$f = \mathrm{e}^{-3t}(c_2 t + c_1) + \frac{\mathrm{e}^{-t}}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $f = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{1}{4} \tag{1A}$$

Taking derivative of the solution gives

$$f' = -3\,\mathrm{e}^{-3t}(c_2 t + c_1) + \mathrm{e}^{-3t} c_2 - \frac{\mathrm{e}^{-t}}{4}$$

substituting $f' = \lambda$ and $t = 0$ in the above gives

$$\lambda = -\frac{1}{4} - 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{1}{4} \\ c_2 &= \lambda - \frac{1}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$f = \left(\lambda - \frac{1}{2}\right) t\,\mathrm{e}^{-3t} - \frac{\mathrm{e}^{-3t}}{4} + \frac{\mathrm{e}^{-t}}{4}$$

Summary

The solution(s) found are the following

$$f = \left(\lambda - \frac{1}{2}\right) t\,\mathrm{e}^{-3t} - \frac{\mathrm{e}^{-3t}}{4} + \frac{\mathrm{e}^{-t}}{4} \tag{1}$$

Verification of solutions

$$f = \left(\lambda - \frac{1}{2}\right) t\,\mathrm{e}^{-3t} - \frac{\mathrm{e}^{-3t}}{4} + \frac{\mathrm{e}^{-t}}{4}$$

Verified OK.

2.4.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$f'' + p(t)\,f' + \frac{\left(p(t)^2 + p'(t)\right) f}{2} = f(t)$$

Where $p(t) = 6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 6\,dx} \\ &= \mathrm{e}^{3t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)f)'' = e^{3t}e^{-t}$$
$$\left(e^{3t}f\right)'' = e^{3t}e^{-t}$$

Integrating once gives

$$\left(e^{3t}f\right)' = \frac{e^{2t}}{2} + c_1$$

Integrating again gives

$$\left(e^{3t}f\right) = c_1t + \frac{e^{2t}}{4} + c_2$$

Hence the solution is

$$f = \frac{c_1t + \frac{e^{2t}}{4} + c_2}{e^{3t}}$$

Or

$$f = c_1t\,e^{-3t} + \frac{e^{-t}}{4} + e^{-3t}c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$f = c_1t\,e^{-3t} + \frac{e^{-t}}{4} + e^{-3t}c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $f = 0$ and $t = 0$ in the above gives

$$0 = \frac{1}{4} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$f' = c_1e^{-3t} - 3c_1t\,e^{-3t} - \frac{e^{-t}}{4} - 3\,e^{-3t}c_2$$

substituting $f' = \lambda$ and $t = 0$ in the above gives

$$\lambda = c_1 - \frac{1}{4} - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \lambda - \frac{1}{2} \\
c_2 &= -\frac{1}{4}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
f = t\,\mathrm{e}^{-3t}\lambda - \frac{t\,\mathrm{e}^{-3t}}{2} - \frac{\mathrm{e}^{-3t}}{4} + \frac{\mathrm{e}^{-t}}{4}
$$

Which simplifies to

$$
f = \frac{(-1 + (4\lambda - 2)\,t)\,\mathrm{e}^{-3t}}{4} + \frac{\mathrm{e}^{-t}}{4}
$$

Summary
The solution(s) found are the following

$$
f = \frac{(-1 + (4\lambda - 2)\,t)\,\mathrm{e}^{-3t}}{4} + \frac{\mathrm{e}^{-t}}{4} \tag{1}
$$

Verification of solutions

$$
f = \frac{(-1 + (4\lambda - 2)\,t)\,\mathrm{e}^{-3t}}{4} + \frac{\mathrm{e}^{-t}}{4}
$$

Verified OK.

2.4.4 Solving using Kovacic algorithm

Writing the ode as

$$
f'' + 6f' + 9f = 0 \tag{1}
$$
$$
Af'' + Bf' + Cf = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 6 \\
C &= 9
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = f e^{\int \frac{B}{2A}\, dt}
$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then f is found using the inverse transformation

$$f = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 52: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in f is found from

$$\begin{aligned}f_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\, dt} \\ &= z_1 e^{-3t} \\ &= z_1\left(\mathrm{e}^{-3t}\right)\end{aligned}$$

Which simplifies to

$$f_1 = \mathrm{e}^{-3t}$$

The second solution f_2 to the original ode is found using reduction of order

$$f_2 = f_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{f_1^2}\, dt$$

Substituting gives

$$\begin{aligned}f_2 &= f_1 \int \frac{e^{\int -\frac{6}{1}\, dt}}{(f_1)^2}\, dt \\ &= f_1 \int \frac{e^{-6t}}{(f_1)^2}\, dt \\ &= f_1(t)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
f &= c_1 f_1 + c_2 f_2 \\
&= c_1 \left(\mathrm{e}^{-3t}\right) + c_2 \left(\mathrm{e}^{-3t}(t)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
f = f_h + f_p
$$

Where f_h is the solution to the homogeneous ODE $Af''(t)+Bf'(t)+Cf(t) = 0$, and f_p is a particular solution to the nonhomogeneous ODE $Af''(t) + Bf'(t) + Cf(t) = f(t)$. f_h is the solution to

$$
f'' + 6f' + 9f = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
f_h = c_1 \mathrm{e}^{-3t} + c_2 t\,\mathrm{e}^{-3t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{-t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{t\,\mathrm{e}^{-3t}, \mathrm{e}^{-3t}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
f_p = A_1 \mathrm{e}^{-t}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution f_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
4A_1 \mathrm{e}^{-t} = \mathrm{e}^{-t}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution f_p, gives the particular solution

$$f_p = \frac{e^{-t}}{4}$$

Therefore the general solution is

$$\begin{aligned} f &= f_h + f_p \\ &= \left(c_1 e^{-3t} + c_2 t\, e^{-3t}\right) + \left(\frac{e^{-t}}{4}\right) \end{aligned}$$

Which simplifies to

$$f = e^{-3t}(c_2 t + c_1) + \frac{e^{-t}}{4}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$f = e^{-3t}(c_2 t + c_1) + \frac{e^{-t}}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $f = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{1}{4} \tag{1A}$$

Taking derivative of the solution gives

$$f' = -3\, e^{-3t}(c_2 t + c_1) + e^{-3t} c_2 - \frac{e^{-t}}{4}$$

substituting $f' = \lambda$ and $t = 0$ in the above gives

$$\lambda = -\frac{1}{4} - 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{4}$$
$$c_2 = \lambda - \frac{1}{2}$$

Substituting these values back in above solution results in

$$f = \left(\lambda - \frac{1}{2}\right) t\,\mathrm{e}^{-3t} - \frac{\mathrm{e}^{-3t}}{4} + \frac{\mathrm{e}^{-t}}{4}$$

Summary
The solution(s) found are the following

$$f = \left(\lambda - \frac{1}{2}\right) t\,\mathrm{e}^{-3t} - \frac{\mathrm{e}^{-3t}}{4} + \frac{\mathrm{e}^{-t}}{4} \tag{1}$$

Verification of solutions

$$f = \left(\lambda - \frac{1}{2}\right) t\,\mathrm{e}^{-3t} - \frac{\mathrm{e}^{-3t}}{4} + \frac{\mathrm{e}^{-t}}{4}$$

Verified OK.

2.4.5 Maple step by step solution

Let's solve

$$\left[f'' + 6f' + 9f = \mathrm{e}^{-t}, f(0) = 0, f'\Big|_{\{t=0\}} = \lambda\right]$$

- Highest derivative means the order of the ODE is 2

  f''
- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 9 = 0$
- Factor the characteristic polynomial

  $(r+3)^2 = 0$
- Root of the characteristic polynomial

  $r = -3$
- 1st solution of the homogeneous ODE

  $f_1(t) = \mathrm{e}^{-3t}$

- Repeated root, multiply $f_1(t)$ by t to ensure linear independence

  $f_2(t) = t\,\mathrm{e}^{-3t}$
- General solution of the ODE

  $f = c_1 f_1(t) + c_2 f_2(t) + f_p(t)$
- Substitute in solutions of the homogeneous ODE

  $f = c_1\mathrm{e}^{-3t} + c_2 t\,\mathrm{e}^{-3t} + f_p(t)$

☐ Find a particular solution $f_p(t)$ of the ODE

  - Use variation of parameters to find f_p here $g(t)$ is the forcing function

    $\left[f_p(t) = -f_1(t)\left(\int \frac{f_2(t)g(t)}{W(f_1(t),f_2(t))}dt\right) + f_2(t)\left(\int \frac{f_1(t)g(t)}{W(f_1(t),f_2(t))}dt\right), g(t) = \mathrm{e}^{-t}\right]$
  - Wronskian of solutions of the homogeneous equation

    $$W(f_1(t)\,, f_2(t)) = \begin{bmatrix} \mathrm{e}^{-3t} & t\,\mathrm{e}^{-3t} \\ -3\,\mathrm{e}^{-3t} & \mathrm{e}^{-3t} - 3t\,\mathrm{e}^{-3t} \end{bmatrix}$$
  - Compute Wronskian

    $W(f_1(t)\,, f_2(t)) = \mathrm{e}^{-6t}$
  - Substitute functions into equation for $f_p(t)$

    $f_p(t) = \mathrm{e}^{-3t}\left(-\left(\int \mathrm{e}^{2t} t\,dt\right) + \left(\int \mathrm{e}^{2t} dt\right) t\right)$
  - Compute integrals

    $f_p(t) = \frac{\mathrm{e}^{-t}}{4}$

- Substitute particular solution into general solution to ODE

  $f = c_1\mathrm{e}^{-3t} + c_2 t\,\mathrm{e}^{-3t} + \frac{\mathrm{e}^{-t}}{4}$

☐ Check validity of solution $f = c_1\mathrm{e}^{-3t} + c_2 t\mathrm{e}^{-3t} + \frac{\mathrm{e}^{-t}}{4}$

  - Use initial condition $f(0) = 0$

    $0 = c_1 + \frac{1}{4}$
  - Compute derivative of the solution

    $f' = -3c_1\mathrm{e}^{-3t} + \mathrm{e}^{-3t}c_2 - 3c_2 t\,\mathrm{e}^{-3t} - \frac{\mathrm{e}^{-t}}{4}$
  - Use the initial condition $f'\Big|_{\{t=0\}} = \lambda$

    $\lambda = -\frac{1}{4} - 3c_1 + c_2$
  - Solve for c_1 and c_2

$$\left\{c_1 = -\frac{1}{4}, c_2 = \lambda - \frac{1}{2}\right\}$$

- Substitute constant values into general solution and simplify

$$f = \frac{(-1+(4\lambda-2)t)\,\mathrm{e}^{-3t}}{4} + \frac{\mathrm{e}^{-t}}{4}$$

- Solution to the IVP

$$f = \frac{(-1+(4\lambda-2)t)\,\mathrm{e}^{-3t}}{4} + \frac{\mathrm{e}^{-t}}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 26

```
dsolve([diff(f(t),t$2)+6*diff(f(t),t)+9*f(t)=exp(-t),f(0) = 0, D(f)(0) = lambda],f(t), sings
```

$$f(t) = \frac{(-1 + (4\lambda - 2)\,t)\,\mathrm{e}^{-3t}}{4} + \frac{\mathrm{e}^{-t}}{4}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 28

```
DSolve[{f''[t]+6*f'[t]+9*f[t]==Exp[-t],{f[0]==0,f'[0]==\[Lambda]}},f[t],t,IncludeSingularSol
```

$$f(t) \to \frac{1}{4} e^{-3t} \left((4\lambda - 2)t + e^{2t} - 1 \right)$$

2.5 problem Problem 15.5(a)

Internal problem ID [2517]
Internal file name [`OUTPUT/2009_Sunday_June_05_2022_02_44_08_AM_82005271/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.5(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{f'' + 8f' + 12f = 12\,\mathrm{e}^{-4t}}$$

With initial conditions

$$[f(0) = 0, f'(0) = 0]$$

2.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$f'' + p(t)f' + q(t)f = F$$

Where here

$$\begin{aligned} p(t) &= 8 \\ q(t) &= 12 \\ F &= 12\,\mathrm{e}^{-4t} \end{aligned}$$

Hence the ode is

$$f'' + 8f' + 12f = 12\,\mathrm{e}^{-4t}$$

The domain of $p(t) = 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 12$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 12\,\mathrm{e}^{-4t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.5.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Af''(t) + Bf'(t) + Cf(t) = f(t)$$

Where $A = 1, B = 8, C = 12, f(t) = 12\,\mathrm{e}^{-4t}$. Let the solution be

$$f = f_h + f_p$$

Where f_h is the solution to the homogeneous ODE $Af''(t)+Bf'(t)+Cf(t) = 0$, and f_p is a particular solution to the non-homogeneous ODE $Af''(t) + Bf'(t) + Cf(t) = f(t)$. f_h is the solution to

$$f'' + 8f' + 12f = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Af''(t) + Bf'(t) + Cf(t) = 0$$

Where in the above $A = 1, B = 8, C = 12$. Let the solution be $f = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 8\lambda\,\mathrm{e}^{\lambda t} + 12\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 8\lambda + 12 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 8, C = 12$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-8}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{8^2 - (4)\,(1)\,(12)}\\ &= -4 \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -4 + 2\\ \lambda_2 &= -4 - 2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2\\ \lambda_2 &= -6\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}f &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ f &= c_1 e^{(-2)t} + c_2 e^{(-6)t}\end{aligned}$$

Or

$$f = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-6t}$$

Therefore the homogeneous solution f_h is

$$f_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-6t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12\,\mathrm{e}^{-4t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-4t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-6t}, e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$f_p = A_1 e^{-4t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution f_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{-4t} = 12\,e^{-4t}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -3]$$

Substituting the above back in the above trial solution f_p, gives the particular solution

$$f_p = -3\,e^{-4t}$$

Therefore the general solution is

$$\begin{aligned} f &= f_h + f_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-6t}\right) + \left(-3\,e^{-4t}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$f = c_1 e^{-2t} + c_2 e^{-6t} - 3\,e^{-4t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $f = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 - 3 \tag{1A}$$

Taking derivative of the solution gives

$$f' = -2c_1 e^{-2t} - 6c_2 e^{-6t} + 12\,e^{-4t}$$

substituting $f' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 - 6c_2 + 12 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{3}{2}$$
$$c_2 = \frac{3}{2}$$

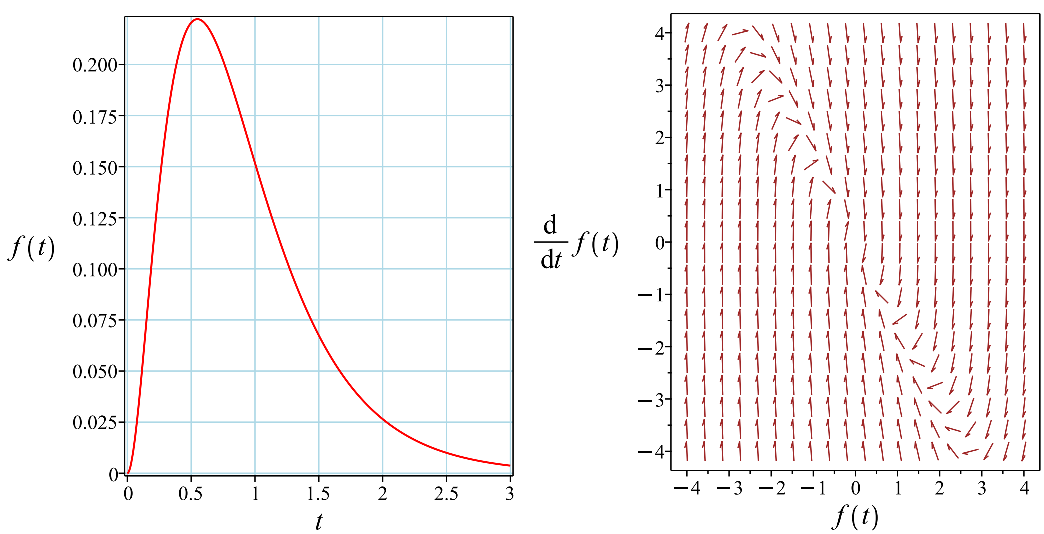
Substituting these values back in above solution results in

$$f = \frac{3\,\mathrm{e}^{-2t}}{2} + \frac{3\,\mathrm{e}^{-6t}}{2} - 3\,\mathrm{e}^{-4t}$$

Summary

The solution(s) found are the following

$$f = \frac{3\,\mathrm{e}^{-2t}}{2} + \frac{3\,\mathrm{e}^{-6t}}{2} - 3\,\mathrm{e}^{-4t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$f = \frac{3\,\mathrm{e}^{-2t}}{2} + \frac{3\,\mathrm{e}^{-6t}}{2} - 3\,\mathrm{e}^{-4t}$$

Verified OK.

2.5.3 Solving using Kovacic algorithm

Writing the ode as

$$f'' + 8f' + 12f = 0 \tag{1}$$
$$Af'' + Bf' + Cf = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 8 \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = fe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then f is found using the inverse transformation

$$f = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 54: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-2t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in f is found from

$$\begin{aligned} f_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8}{1}\,dt} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-4t}\\
&= z_1\left(\mathrm{e}^{-4t}\right)
\end{aligned}
$$

Which simplifies to

$$
f_1 = \mathrm{e}^{-6t}
$$

The second solution f_2 to the original ode is found using reduction of order

$$
f_2 = f_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{f_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
f_2 &= f_1 \int \frac{e^{\int -\frac{8}{1}\, dt}}{(f_1)^2}\, dt\\
&= f_1 \int \frac{e^{-8t}}{(f_1)^2}\, dt\\
&= f_1\left(\frac{\mathrm{e}^{4t}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
f &= c_1 f_1 + c_2 f_2\\
&= c_1\left(\mathrm{e}^{-6t}\right) + c_2\left(\mathrm{e}^{-6t}\left(\frac{\mathrm{e}^{4t}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
f = f_h + f_p
$$

Where f_h is the solution to the homogeneous ODE $Af''(t)+Bf'(t)+Cf(t)=0$, and f_p is a particular solution to the nonhomogeneous ODE $Af''(t)+Bf'(t)+Cf(t)=f(t)$. f_h is the solution to

$$
f'' + 8f' + 12f = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
f_h = c_1\mathrm{e}^{-6t} + \frac{c_2\mathrm{e}^{-2t}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12\,\mathrm{e}^{-4t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-4t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{-2t}}{4}, \mathrm{e}^{-6t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$f_p = A_1\mathrm{e}^{-4t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution f_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1\mathrm{e}^{-4t} = 12\,\mathrm{e}^{-4t}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -3]$$

Substituting the above back in the above trial solution f_p, gives the particular solution

$$f_p = -3\,\mathrm{e}^{-4t}$$

Therefore the general solution is

$$\begin{aligned} f &= f_h + f_p \\ &= \left(c_1\mathrm{e}^{-6t} + \frac{c_2\mathrm{e}^{-2t}}{4}\right) + \left(-3\,\mathrm{e}^{-4t}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$f = c_1 \mathrm{e}^{-6t} + \frac{c_2 \mathrm{e}^{-2t}}{4} - 3\,\mathrm{e}^{-4t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $f = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{4} - 3 \tag{1A}$$

Taking derivative of the solution gives

$$f' = -6c_1 \mathrm{e}^{-6t} - \frac{c_2 \mathrm{e}^{-2t}}{2} + 12\,\mathrm{e}^{-4t}$$

substituting $f' = 0$ and $t = 0$ in the above gives

$$0 = -6c_1 - \frac{c_2}{2} + 12 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{3}{2} \\ c_2 &= 6 \end{aligned}$$

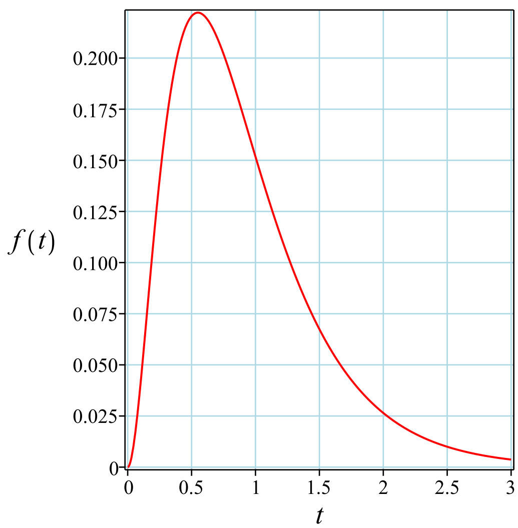
Substituting these values back in above solution results in

$$f = \frac{3\,\mathrm{e}^{-2t}}{2} + \frac{3\,\mathrm{e}^{-6t}}{2} - 3\,\mathrm{e}^{-4t}$$

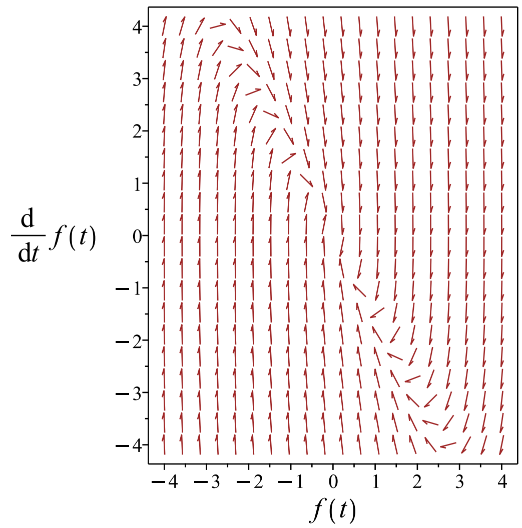
Summary

The solution(s) found are the following

$$f = \frac{3\,\mathrm{e}^{-2t}}{2} + \frac{3\,\mathrm{e}^{-6t}}{2} - 3\,\mathrm{e}^{-4t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$f = \frac{3\,\mathrm{e}^{-2t}}{2} + \frac{3\,\mathrm{e}^{-6t}}{2} - 3\,\mathrm{e}^{-4t}$$

Verified OK.

2.5.4 Maple step by step solution

Let's solve

$$\left[f'' + 8f' + 12f = 12\,\mathrm{e}^{-4t}, f(0) = 0, f'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  f''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 8r + 12 = 0$

- Factor the characteristic polynomial

  $(r+6)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-6, -2)$

- 1st solution of the homogeneous ODE

$f_1(t) = e^{-6t}$

- 2nd solution of the homogeneous ODE

  $f_2(t) = e^{-2t}$

- General solution of the ODE

  $f = c_1 f_1(t) + c_2 f_2(t) + f_p(t)$

- Substitute in solutions of the homogeneous ODE

  $f = c_1 e^{-6t} + c_2 e^{-2t} + f_p(t)$

□ Find a particular solution $f_p(t)$ of the ODE

  - Use variation of parameters to find f_p here $g(t)$ is the forcing function

    $\left[f_p(t) = -f_1(t)\left(\int \frac{f_2(t)g(t)}{W(f_1(t), f_2(t))}dt\right) + f_2(t)\left(\int \frac{f_1(t)g(t)}{W(f_1(t), f_2(t))}dt\right), g(t) = 12\,e^{-4t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(f_1(t), f_2(t)) = \begin{bmatrix} e^{-6t} & e^{-2t} \\ -6\,e^{-6t} & -2\,e^{-2t} \end{bmatrix}$

  - Compute Wronskian

    $W(f_1(t), f_2(t)) = 4\,e^{-8t}$

  - Substitute functions into equation for $f_p(t)$

    $f_p(t) = -3\,e^{-6t}\left(\int e^{2t} dt\right) + 3\,e^{-2t}\left(\int e^{-2t} dt\right)$

  - Compute integrals

    $f_p(t) = -3\,e^{-4t}$

- Substitute particular solution into general solution to ODE

  $f = c_1 e^{-6t} + c_2 e^{-2t} - 3\,e^{-4t}$

□ Check validity of solution $f = c_1 e^{-6t} + c_2 e^{-2t} - 3e^{-4t}$

  - Use initial condition $f(0) = 0$

    $0 = c_1 + c_2 - 3$

  - Compute derivative of the solution

    $f' = -6c_1 e^{-6t} - 2c_2 e^{-2t} + 12\,e^{-4t}$

  - Use the initial condition $f'\Big|_{\{t=0\}} = 0$

    $0 = -6c_1 - 2c_2 + 12$

- Solve for c_1 and c_2

  $\{c_1 = \frac{3}{2}, c_2 = \frac{3}{2}\}$
- Substitute constant values into general solution and simplify

  $f = \frac{3\,e^{-2t}}{2} + \frac{3\,e^{-6t}}{2} - 3\,e^{-4t}$

- Solution to the IVP

  $f = \frac{3\,e^{-2t}}{2} + \frac{3\,e^{-6t}}{2} - 3\,e^{-4t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([diff(f(t),t$2)+8*diff(f(t),t)+12*f(t)=12*exp(-4*t),f(0) = 0, D(f)(0) = 0],f(t), sing
```

$$f(t) = \frac{3\,e^{-2t}}{2} + \frac{3\,e^{-6t}}{2} - 3\,e^{-4t}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 23

```
DSolve[{f''[t]+8*f'[t]+12*f[t]==12*Exp[-4*t],{f[0]==0,f'[0]==0}},f[t],t,IncludeSingularSolut
```

$$f(t) \to \frac{3}{2}e^{-6t}\left(e^{2t} - 1\right)^2$$

2.6 problem Problem 15.5(b)

Internal problem ID [2518]
Internal file name [`OUTPUT/2010_Sunday_June_05_2022_02_44_11_AM_85192585/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.5(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{f'' + 8f' + 12f = 12\,\mathrm{e}^{-4t}}$$

With initial conditions

$$[f(0) = 0, f'(0) = -2]$$

2.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$f'' + p(t)f' + q(t)f = F$$

Where here

$$\begin{aligned} p(t) &= 8 \\ q(t) &= 12 \\ F &= 12\,\mathrm{e}^{-4t} \end{aligned}$$

Hence the ode is

$$f'' + 8f' + 12f = 12\,\mathrm{e}^{-4t}$$

The domain of $p(t) = 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 12$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 12\,\mathrm{e}^{-4t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.6.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Af''(t) + Bf'(t) + Cf(t) = f(t)$$

Where $A = 1, B = 8, C = 12, f(t) = 12\,\mathrm{e}^{-4t}$. Let the solution be

$$f = f_h + f_p$$

Where f_h is the solution to the homogeneous ODE $Af''(t)+Bf'(t)+Cf(t) = 0$, and f_p is a particular solution to the non-homogeneous ODE $Af''(t) + Bf'(t) + Cf(t) = f(t)$. f_h is the solution to

$$f'' + 8f' + 12f = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Af''(t) + Bf'(t) + Cf(t) = 0$$

Where in the above $A = 1, B = 8, C = 12$. Let the solution be $f = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 8\lambda\,\mathrm{e}^{\lambda t} + 12\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 8\lambda + 12 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 8, C = 12$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-8}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{8^2 - (4)\,(1)\,(12)} \\ &= -4 \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -4 + 2 \\ \lambda_2 &= -4 - 2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 \\ \lambda_2 &= -6\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}f &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ f &= c_1 e^{(-2)t} + c_2 e^{(-6)t}\end{aligned}$$

Or

$$f = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-6t}$$

Therefore the homogeneous solution f_h is

$$f_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-6t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12\,\mathrm{e}^{-4t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-4t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-6t}, e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$f_p = A_1 e^{-4t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution f_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{-4t} = 12\, e^{-4t}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -3]$$

Substituting the above back in the above trial solution f_p, gives the particular solution

$$f_p = -3\, e^{-4t}$$

Therefore the general solution is

$$\begin{aligned} f &= f_h + f_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-6t}\right) + \left(-3\, e^{-4t}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$f = c_1 e^{-2t} + c_2 e^{-6t} - 3\, e^{-4t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $f = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 - 3 \tag{1A}$$

Taking derivative of the solution gives

$$f' = -2c_1 e^{-2t} - 6c_2 e^{-6t} + 12\, e^{-4t}$$

substituting $f' = -2$ and $t = 0$ in the above gives

$$-2 = -2c_1 - 6c_2 + 12 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 2$$

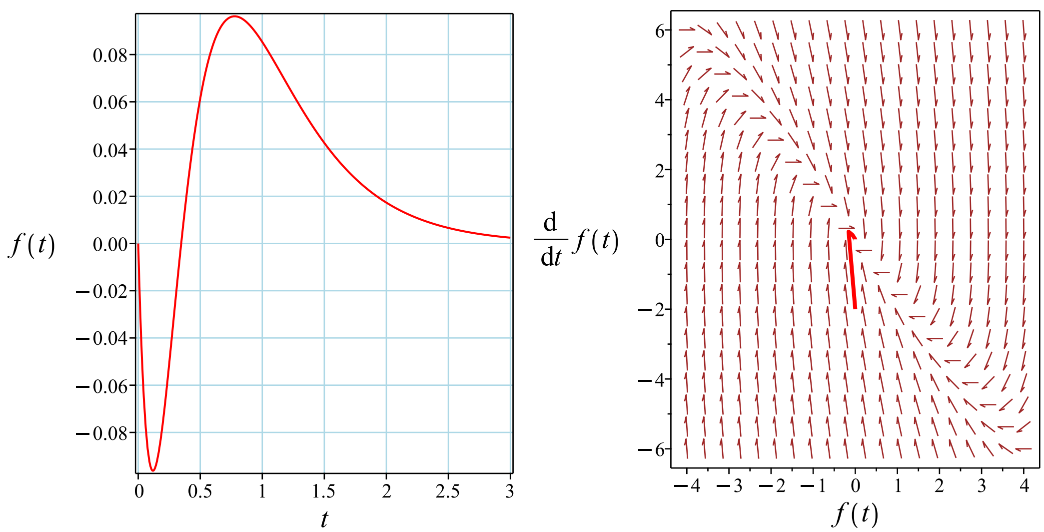
Substituting these values back in above solution results in

$$f = e^{-2t} + 2\,e^{-6t} - 3\,e^{-4t}$$

Summary

The solution(s) found are the following

$$f = e^{-2t} + 2\,e^{-6t} - 3\,e^{-4t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$f = e^{-2t} + 2\,e^{-6t} - 3\,e^{-4t}$$

Verified OK.

2.6.3 Solving using Kovacic algorithm

Writing the ode as

$$f'' + 8f' + 12f = 0 \tag{1}$$
$$Af'' + Bf' + Cf = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 8 \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = f e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then f is found using the inverse transformation

$$f = z(t) \, e^{-\int \frac{B}{2A} \, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 56: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-2t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in f is found from

$$
\begin{aligned}
f_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8}{1}\,dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-4t} \\
&= z_1 \left(\mathrm{e}^{-4t}\right)
\end{aligned}
$$

Which simplifies to

$$
f_1 = \mathrm{e}^{-6t}
$$

The second solution f_2 to the original ode is found using reduction of order

$$
f_2 = f_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{f_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
f_2 &= f_1 \int \frac{e^{\int -\frac{8}{1}\, dt}}{(f_1)^2}\, dt \\
&= f_1 \int \frac{e^{-8t}}{(f_1)^2}\, dt \\
&= f_1 \left(\frac{\mathrm{e}^{4t}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
f &= c_1 f_1 + c_2 f_2 \\
&= c_1 \left(\mathrm{e}^{-6t}\right) + c_2 \left(\mathrm{e}^{-6t} \left(\frac{\mathrm{e}^{4t}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
f = f_h + f_p
$$

Where f_h is the solution to the homogeneous ODE $Af''(t) + Bf'(t) + Cf(t) = 0$, and f_p is a particular solution to the nonhomogeneous ODE $Af''(t) + Bf'(t) + Cf(t) = f(t)$. f_h is the solution to

$$
f'' + 8f' + 12f = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
f_h = c_1 \mathrm{e}^{-6t} + \frac{c_2 \mathrm{e}^{-2t}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12\,\mathrm{e}^{-4t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-4t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{-2t}}{4}, \mathrm{e}^{-6t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$f_p = A_1 \mathrm{e}^{-4t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution f_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 \mathrm{e}^{-4t} = 12\,\mathrm{e}^{-4t}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -3]$$

Substituting the above back in the above trial solution f_p, gives the particular solution

$$f_p = -3\,\mathrm{e}^{-4t}$$

Therefore the general solution is

$$\begin{aligned} f &= f_h + f_p \\ &= \left(c_1 \mathrm{e}^{-6t} + \frac{c_2 \mathrm{e}^{-2t}}{4}\right) + \left(-3\,\mathrm{e}^{-4t}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$f = c_1 e^{-6t} + \frac{c_2 e^{-2t}}{4} - 3\,e^{-4t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $f = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{4} - 3 \tag{1A}$$

Taking derivative of the solution gives

$$f' = -6c_1 e^{-6t} - \frac{c_2 e^{-2t}}{2} + 12\,e^{-4t}$$

substituting $f' = -2$ and $t = 0$ in the above gives

$$-2 = -6c_1 - \frac{c_2}{2} + 12 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 4$$

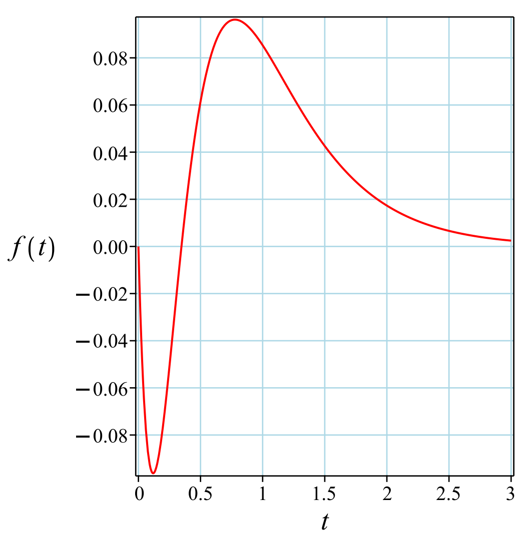
Substituting these values back in above solution results in

$$f = e^{-2t} + 2\,e^{-6t} - 3\,e^{-4t}$$

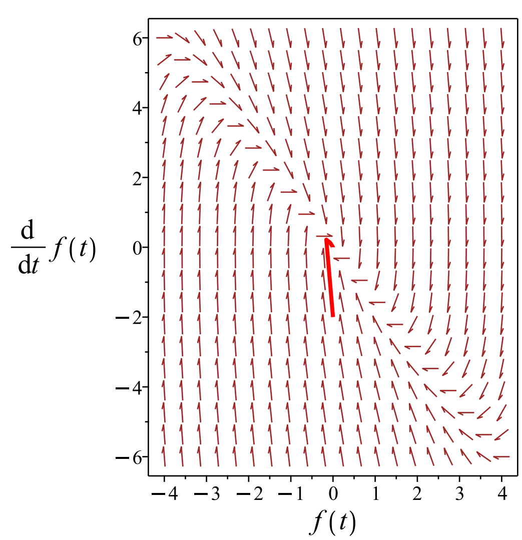
Summary

The solution(s) found are the following

$$f = e^{-2t} + 2\,e^{-6t} - 3\,e^{-4t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$f = e^{-2t} + 2\,e^{-6t} - 3\,e^{-4t}$$

Verified OK.

2.6.4 Maple step by step solution

Let's solve

$\left[f'' + 8f' + 12f = 12\,e^{-4t}, f(0) = 0, f'\Big|_{\{t=0\}} = -2\right]$

- Highest derivative means the order of the ODE is 2

  f''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 8r + 12 = 0$

- Factor the characteristic polynomial

  $(r + 6)\,(r + 2) = 0$

- Roots of the characteristic polynomial

  $r = (-6, -2)$

- 1st solution of the homogeneous ODE

  $f_1(t) = e^{-6t}$

- 2nd solution of the homogeneous ODE

  $f_2(t) = e^{-2t}$

- General solution of the ODE

  $f = c_1 f_1(t) + c_2 f_2(t) + f_p(t)$

- Substitute in solutions of the homogeneous ODE

  $f = c_1 e^{-6t} + c_2 e^{-2t} + f_p(t)$

□ Find a particular solution $f_p(t)$ of the ODE

- ◦ Use variation of parameters to find f_p here $g(t)$ is the forcing function

  $\left[f_p(t) = -f_1(t)\left(\int \frac{f_2(t)g(t)}{W(f_1(t),f_2(t))}dt\right) + f_2(t)\left(\int \frac{f_1(t)g(t)}{W(f_1(t),f_2(t))}dt\right), g(t) = 12\,e^{-4t}\right]$

- ◦ Wronskian of solutions of the homogeneous equation

$$W(f_1(t)\,,f_2(t)) = \begin{bmatrix} e^{-6t} & e^{-2t} \\ -6\,e^{-6t} & -2\,e^{-2t} \end{bmatrix}$$

- Compute Wronskian

  $W(f_1(t)\,,f_2(t)) = 4\,e^{-8t}$

- Substitute functions into equation for $f_p(t)$

  $f_p(t) = -3\,e^{-6t}\left(\int e^{2t}dt\right) + 3\,e^{-2t}\left(\int e^{-2t}dt\right)$

- Compute integrals

  $f_p(t) = -3\,e^{-4t}$

- Substitute particular solution into general solution to ODE

  $f = c_1 e^{-6t} + c_2 e^{-2t} - 3\,e^{-4t}$

□ Check validity of solution $f = c_1 e^{-6t} + c_2 e^{-2t} - 3e^{-4t}$

- Use initial condition $f(0) = 0$

  $0 = c_1 + c_2 - 3$

- Compute derivative of the solution

  $f' = -6c_1 e^{-6t} - 2c_2 e^{-2t} + 12\,e^{-4t}$

- Use the initial condition $f'\Big|_{\{t=0\}} = -2$

  $-2 = -6c_1 - 2c_2 + 12$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $f = e^{-2t} + 2\,e^{-6t} - 3\,e^{-4t}$

- Solution to the IVP

  $f = e^{-2t} + 2\,e^{-6t} - 3\,e^{-4t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([diff(f(t),t$2)+8*diff(f(t),t)+12*f(t)=12*exp(-4*t),f(0) = 0, D(f)(0) = -2],f(t), sin
```

$$f(t) = e^{-2t} + 2\,e^{-6t} - 3\,e^{-4t}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 25

```
DSolve[{f''[t]+8*f'[t]+12*f[t]==12*Exp[-4*t],{f[0]==0,f'[0]==-2}},f[t],t,IncludeSingularSolu
```

$$f(t) \to e^{-6t}\left(-3e^{2t} + e^{4t} + 2\right)$$

2.7 problem Problem 15.7

Internal problem ID [2519]
Internal file name [`OUTPUT/2011_Sunday_June_05_2022_02_44_14_AM_88118312/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y' + y = 4\,\mathrm{e}^{-x}}$$

2.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = 4\,\mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda\,\mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)} \\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}\}]$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\mathrm{e}^{-x}x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1\mathrm{e}^{-x}x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{-x} = 4\,\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{-x}x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + c_2x\,\mathrm{e}^{-x}\right) + \left(2\,\mathrm{e}^{-x}x^2\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2x + c_1) + 2\,\mathrm{e}^{-x}x^2$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2x + c_1) + 2\,\mathrm{e}^{-x}x^2 \tag{1}$$

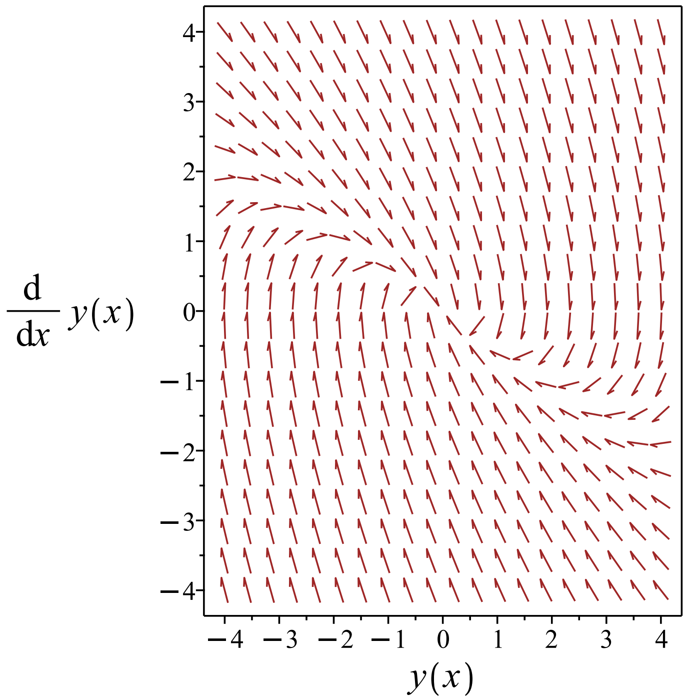


Figure 71: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + 2\,\mathrm{e}^{-x} x^2$$

Verified OK.

2.7.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= \mathrm{e}^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 4\,e^{-x}e^{x}$$
$$(e^{x}y)'' = 4\,e^{-x}e^{x}$$

Integrating once gives

$$(e^{x}y)' = 4x + c_1$$

Integrating again gives

$$(e^{x}y) = x(c_1 + 2x) + c_2$$

Hence the solution is

$$y = \frac{x(c_1 + 2x) + c_2}{e^{x}}$$

Or

$$y = c_1 x\,e^{-x} + 2\,e^{-x}x^2 + c_2 e^{-x}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,e^{-x} + 2\,e^{-x}x^2 + c_2 e^{-x} \tag{1}$$

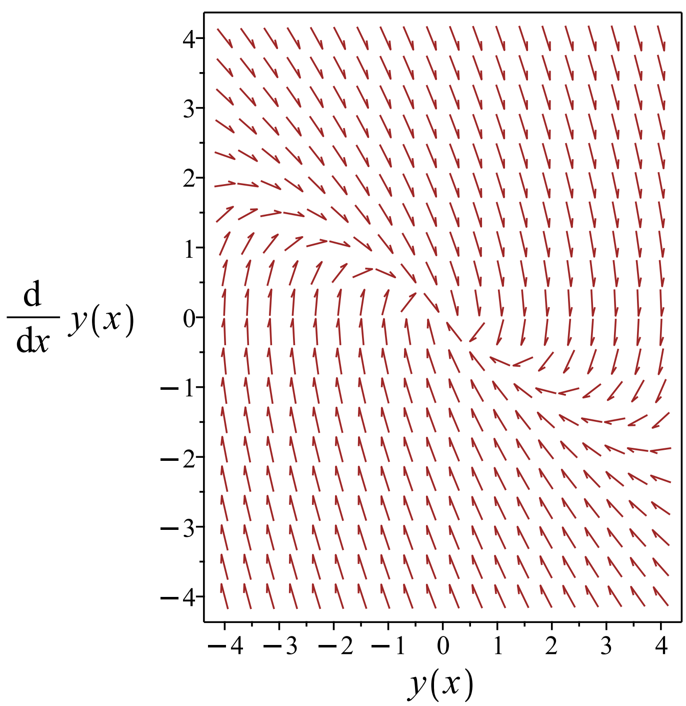


Figure 72: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{-x} + 2\,\mathrm{e}^{-x} x^2 + c_2 \mathrm{e}^{-x}$$

Verified OK.

2.7.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 58: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + c_2 x \,\mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}\}]$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\mathrm{e}^{-x} x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \mathrm{e}^{-x} x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^{-x} = 4\,\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{-x} x^2$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-x} + c_2 x\, e^{-x}\right) + \left(2\, e^{-x} x^2\right)
\end{aligned}
$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) + 2\, e^{-x} x^2$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + 2\, e^{-x} x^2 \tag{1}$$

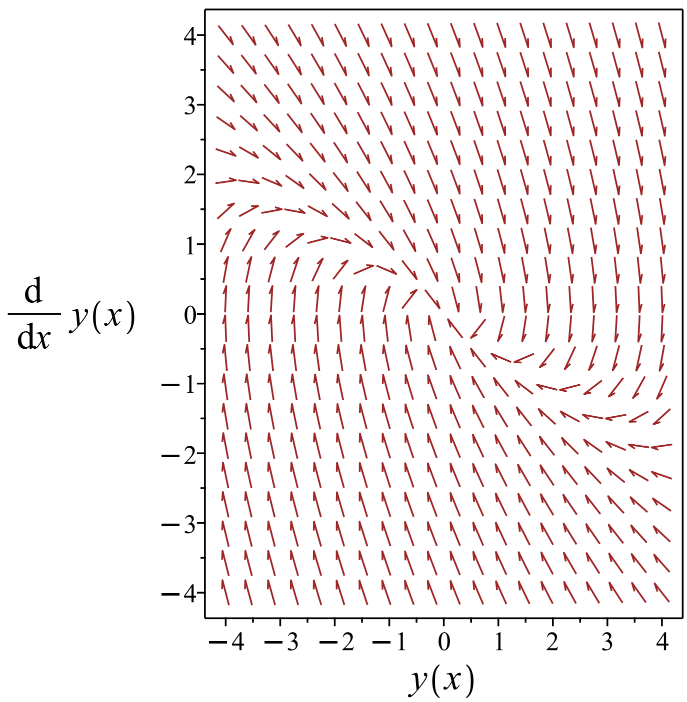


Figure 73: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + 2\, e^{-x} x^2$$

Verified OK.

2.7.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = 4\,\mathrm{e}^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4\,\mathrm{e}^{-x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -4\,\mathrm{e}^{-x}\left(\int x dx - \left(\int 1 dx\right) x\right)$

- Compute integrals

  $y_p(x) = 2\,\mathrm{e}^{-x}x^2$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,\mathrm{e}^{-x} + 2\,\mathrm{e}^{-x}x^2 + c_1\mathrm{e}^{-x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=4*exp(-x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x}\left(c_1 x + 2x^2 + c_2\right)$$

✓ Solution by Mathematica
Time used: 0.026 (sec). Leaf size: 23

```
DSolve[y''[x]+2*y'[x]+y[x]==4*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(2x^2 + c_2 x + c_1\right)$$

2.8 problem Problem 15.9(a)

Internal problem ID [2520]
Internal file name [`OUTPUT/2012_Sunday_June_05_2022_02_44_16_AM_85468804/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.9(a).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' - 12y' + 16y = 32x - 8}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 12y' + 16y = 0$$

The characteristic equation is

$$\lambda^3 - 12\lambda + 16 = 0$$

The roots of the above equation are

$$\lambda_1 = -4$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = e^{2x} c_1 + x\, e^{2x} c_2 + e^{-4x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{2x}$$
$$y_2 = x\, e^{2x}$$
$$y_3 = e^{-4x}$$

Now the particular solution to the given ODE is found

$$y''' - 12y' + 16y = 32x - 8$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{2x}, e^{-4x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$16A_2 x + 16A_1 - 12A_2 = 32x - 8$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1 + 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x}c_1 + x\,e^{2x}c_2 + e^{-4x}c_3\right) + (1 + 2x) \end{aligned}$$

Which simplifies to

$$y = \left((c_2x + c_1)\,e^{6x} + c_3\right)e^{-4x} + 1 + 2x$$

Summary
The solution(s) found are the following

$$y = \left((c_2x + c_1)\,e^{6x} + c_3\right)e^{-4x} + 1 + 2x \tag{1}$$

Verification of solutions

$$y = \left((c_2x + c_1)\,e^{6x} + c_3\right)e^{-4x} + 1 + 2x$$

Verified OK.

2.8.1 Maple step by step solution

Let's solve

$y''' - 12y' + 16y = 32x - 8$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 32x - 8 + 12y_2(x) - 16y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 32x - 8 + 12y_2(x) - 16y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -16 & 12 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 32x - 8 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 32x - 8 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -16 & 12 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-4, \left[\begin{array}{c} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-4x} \cdot \left[\begin{array}{c} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{array}\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 2

$$\overrightarrow{y}_2(x) = \mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, an

$$\overrightarrow{y}_3(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_3(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_3(x)$ to be a solution to the homogeneous system

$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -16 & 12 & 0 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

$$\overrightarrow{y}_3(x) = e^{2x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3(x) + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-4x}}{16} & \frac{e^{2x}}{4} & e^{2x}\left(\frac{x}{4} - \frac{1}{8}\right) \\ -\frac{e^{-4x}}{4} & \frac{e^{2x}}{2} & \frac{x\,e^{2x}}{2} \\ e^{-4x} & e^{2x} & x\,e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-4x}}{16} & \frac{e^{2x}}{4} & e^{2x}\left(\frac{x}{4} - \frac{1}{8}\right) \\ -\frac{e^{-4x}}{4} & \frac{e^{2x}}{2} & \frac{x\,e^{2x}}{2} \\ e^{-4x} & e^{2x} & x\,e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{16} & \frac{1}{4} & -\frac{1}{8} \\ -\frac{1}{4} & \frac{1}{2} & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} (1-2x)\,\mathrm{e}^{2x} & \frac{(6x\,\mathrm{e}^{6x}+\mathrm{e}^{6x}-1)\mathrm{e}^{-4x}}{12} & \frac{(6x-1)\mathrm{e}^{-4x}\mathrm{e}^{6x}}{24}+\frac{\mathrm{e}^{-4x}}{24} \\ -4x\,\mathrm{e}^{2x} & \frac{(2+3x)\mathrm{e}^{-4x}\mathrm{e}^{6x}}{3}+\frac{\mathrm{e}^{-4x}}{3} & \frac{(3x\,\mathrm{e}^{6x}+\mathrm{e}^{6x}-1)\mathrm{e}^{-4x}}{6} \\ -8x\,\mathrm{e}^{2x} & \frac{(4+6x)\mathrm{e}^{-4x}\mathrm{e}^{6x}}{3}-\frac{4\,\mathrm{e}^{-4x}}{3} & \frac{(3x\,\mathrm{e}^{6x}+\mathrm{e}^{6x}+2)\mathrm{e}^{-4x}}{3} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p{}'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}\,'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}\,'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}\,'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}\,'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}\,'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{(6x\,\mathrm{e}^{6x}-10\,\mathrm{e}^{6x}+18x\,\mathrm{e}^{4x}+9\,\mathrm{e}^{4x}+1)\mathrm{e}^{-4x}}{6} \\ \frac{(6x\,\mathrm{e}^{6x}-7\,\mathrm{e}^{6x}+9\,\mathrm{e}^{4x}-2)\mathrm{e}^{-4x}}{3} \\ 2\,\mathrm{e}^{-4x}\left(\frac{4}{3}+(1+4x)\,\mathrm{e}^{4x}+\frac{(6x-7)\mathrm{e}^{6x}}{3}\right) \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \left[\begin{array}{c} \frac{(6x\,e^{6x}-10\,e^{6x}+18x\,e^{4x}+9\,e^{4x}+1)\,e^{-4x}}{6} \\ \frac{(6x\,e^{6x}-7\,e^{6x}+9\,e^{4x}-2)\,e^{-4x}}{3} \\ 2\,e^{-4x}\left(\frac{4}{3}+(1+4x)\,e^{4x}+\frac{(6x-7)e^{6x}}{3}\right) \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{e^{-4x}\left(\left((c_3+4)x+c_2-\frac{c_3}{2}-\frac{20}{3}\right)e^{6x}+(12x+6)e^{4x}+\frac{c_1}{4}+\frac{2}{3}\right)}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$3)-12*diff(y(x),x)+16*y(x)=32*x-8,y(x), singsol=all)
```

$$y(x) = \left((2x+1)\,e^{4x} + (c_3x + c_2)\,e^{6x} + c_1\right)e^{-4x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 35

```
DSolve[y'''[x]-12*y'[x]+16*y[x]==32*x-8,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1e^{-4x} + c_2e^{2x} + x\left(2 + c_3e^{2x}\right) + 1$$

2.9 problem Problem 15.9(b)

Internal problem ID [2521]
Internal file name [`OUTPUT/2013_Sunday_June_05_2022_02_44_19_AM_88354591/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.9(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _reducible
   , _mu_xy]]
```

Unable to solve or complete the solution.

$$0 = -\frac{y''}{y} + \frac{y'^2}{y^2} - \frac{2a\coth(2ax)\, y'}{y} + 2a^2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 53

```
dsolve(diff( 1/y(x)*diff(y(x),x),x)+(2*a*coth(2*a*x))*(1/y(x)*diff(y(x),x))=2*a^2,y(x), sing
```

$$y(x) = \mathrm{e}^{\frac{-x\,a^{2}+c_{1}\operatorname{arctanh}\left(\mathrm{e}^{2ax}\right)-c_{2}}{a}}\sqrt{\mathrm{e}^{ax}-1}\,\sqrt{\mathrm{e}^{ax}+1}\,\sqrt{\mathrm{e}^{2ax}+1}$$

✓ Solution by Mathematica

Time used: 60.504 (sec). Leaf size: 287

```
DSolve[D[1/y[x]*y'[x],x]+(2*a*Coth[1/y[x]*y'[x]])==2*a^2,y[x],x,IncludeSingularSolutions ->
```

$y(x)$

$$\rightarrow c_2 \exp\left(\frac{-\operatorname{PolyLog}\left(2, \frac{(a+1)\exp\left(-2\text{InverseFunction}\left[\frac{-((a+1)\log(1-\tanh(\#1)))+(a-1)\log(\tanh(\#1)+1)+2\log(1-a\tanh(\#1}{2\left(a^2-1\right)}\right.\right.}{a-1}\right.}{}\right.$$

2.10 problem Problem 15.21

Internal problem ID [2522]
Internal file name [`OUTPUT/2014_Sunday_June_05_2022_02_44_38_AM_7022198/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - xy' + y = x}$$

2.10.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = 1, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$x^2y''-xy'+y=0$$

This is Euler second order ODE. Let the solution be $y=x^r$, then $y'=rx^{r-1}$ and $y''=r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2}-xrx^{r-1}+x^r=0$$

Simplifying gives

$$r(r-1)\,x^r-r\,x^r+x^r=0$$

Since $x^r\neq 0$ then dividing throughout by x^r gives

$$r(r-1)-r+1=0$$

Or

$$r^2-2r+1=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1=1$$
$$r_2=1$$

Since the roots are equal, then the general solution is

$$y=c_1y_1+c_2y_2$$

Where $y_1=x^r$ and $y_2=x^r\ln(x)$. Hence

$$y=c_1x+\ln(x)\,c_2x$$

Next, we find the particular solution to the ODE

$$x^2y''-xy'+y=x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \ln(x)\, x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \ln(x)\, x \\ \frac{d}{dx}(x) & \frac{d}{dx}(\ln(x)\, x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \ln(x)\, x \\ 1 & 1 + \ln(x) \end{vmatrix}$$

Therefore

$$W = (x)\,(1 + \ln(x)) - (\ln(x)\, x)\,(1)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\, x^2}{x^3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3} \, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2 x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x\left(\frac{\ln(x)^2}{2} + c_1 + c_2 \ln(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\left(\frac{\ln(x)^2}{2} + c_1 + c_2 \ln(x)\right) \tag{1}$$

Verification of solutions

$$y = x\left(\frac{\ln(x)^2}{2} + c_1 + c_2 \ln(x)\right)$$

Verified OK.

2.10.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - xy' + y = 0$$

In normal form the ode

$$x^2y'' - xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\,(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)} dx \\
&= \int e^{\ln(x)} \, dx \\
&= \int x dx \\
&= \frac{x^2}{2} \qquad (6)
\end{aligned}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2}}{x^2} \\
&= \frac{1}{x^4} \qquad (7)
\end{aligned}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{x^4} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{1}{x^4} = \frac{1}{4\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0
$$

Simplifying gives

$$4r(r-1)\,\tau^{r}+0\,\tau^{r}+\tau^{r}=0$$

Since $\tau^{r}\neq 0$ then dividing throughout by τ^{r} gives

$$4r(r-1)+0+1=0$$

Or

$$4r^{2}-4r+1=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1=\frac{1}{2}$$
$$r_2=\frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau)=c_1y_1+c_2y_2$$

Where $y_1=\tau^{r}$ and $y_2=\tau^{r}\ln(\tau)$. Hence

$$y(\tau)=c_1\sqrt{\tau}+c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y=\frac{x\sqrt{2}\,(c_1-c_2\ln(2)+2c_2\ln(x))}{2}$$

Therefore the homogeneous solution y_h is

$$y_h=\frac{x\sqrt{2}\,(c_1-c_2\ln(2)+2c_2\ln(x))}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = -\frac{\sqrt{2}\,x\ln(2)}{2} + \sqrt{2}\,x\ln(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & -\frac{\sqrt{2}\,x\ln(2)}{2} + \sqrt{2}\,x\ln(x) \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(-\frac{\sqrt{2}\,x\ln(2)}{2} + \sqrt{2}\,x\ln(x)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & -\frac{\sqrt{2}\,x\ln(2)}{2} + \sqrt{2}\,x\ln(x) \\ 1 & -\frac{\sqrt{2}\,\ln(2)}{2} + \sqrt{2}\,\ln(x) + \sqrt{2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{\sqrt{2}\,\ln(2)}{2} + \sqrt{2}\,\ln(x) + \sqrt{2}\right) - \left(-\frac{\sqrt{2}\,x\ln(2)}{2} + \sqrt{2}\,x\ln(x)\right)(1)$$

Which simplifies to

$$W = \sqrt{2}\,x$$

Which simplifies to

$$W = \sqrt{2}\,x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\sqrt{2}\, x \ln(2)}{2} + \sqrt{2}\, x \ln(x)\right) x}{\sqrt{2}\, x^3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{-\ln(2) + 2\ln(x)}{2x} dx$$

Hence

$$u_1 = \frac{\ln(2)\ln(x)}{2} - \frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{\sqrt{2}\, x^3}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{2}}{2x} dx$$

Hence

$$u_2 = \frac{\sqrt{2}\, \ln(x)}{2}$$

Which simplifies to

$$u_1 = \frac{\ln(x)(\ln(2) - \ln(x))}{2}$$

$$u_2 = \frac{\sqrt{2}\, \ln(x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)(\ln(2) - \ln(x))\, x}{2} + \frac{\sqrt{2}\, \ln(x)\left(-\frac{\sqrt{2}\, x \ln(2)}{2} + \sqrt{2}\, x \ln(x)\right)}{2}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x)^2\, x}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{x\sqrt{2}\,(c_1 - c_2 \ln(2) + 2c_2 \ln(x))}{2}\right) + \left(\frac{\ln(x)^2 x}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x\sqrt{2}\,(c_1 - c_2 \ln(2) + 2c_2 \ln(x))}{2} + \frac{\ln(x)^2 x}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{x\sqrt{2}\,(c_1 - c_2 \ln(2) + 2c_2 \ln(x))}{2} + \frac{\ln(x)^2 x}{2}
$$

Verified OK.

2.10.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = x^2, B = -x, C = 1, f(x) = x$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
x^2y'' - xy' + y = 0
$$

In normal form the ode

$$
x^2y'' - xy' + y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{1}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -2c
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{c\tau} c_1$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 x
$$

Now the particular solution to this ODE is found

$$
x^2y'' - xy' + y = x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= -\frac{\sqrt{2}\,x\ln(2)}{2} + \sqrt{2}\,x\ln(x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & -\frac{\sqrt{2}\,x\ln(2)}{2} + \sqrt{2}\,x\ln(x) \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(-\frac{\sqrt{2}\,x\ln(2)}{2} + \sqrt{2}\,x\ln(x)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & -\frac{\sqrt{2}\,x\ln(2)}{2} + \sqrt{2}\,x\ln(x) \\ 1 & -\frac{\sqrt{2}\,\ln(2)}{2} + \sqrt{2}\,\ln(x) + \sqrt{2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{\sqrt{2}\,\ln(2)}{2} + \sqrt{2}\,\ln(x) + \sqrt{2}\right) - \left(-\frac{\sqrt{2}\,x\ln(2)}{2} + \sqrt{2}\,x\ln(x)\right)(1)$$

Which simplifies to

$$W = \sqrt{2}\,x$$

Which simplifies to

$$W = \sqrt{2}\,x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\sqrt{2}\,x\ln(2)}{2} + \sqrt{2}\,x\ln(x)\right)x}{\sqrt{2}\,x^3}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{-\ln(2) + 2\ln(x)}{2x}dx$$

Hence

$$u_1 = \frac{\ln(2)\ln(x)}{2} - \frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{\sqrt{2}\,x^3}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{2}}{2x} dx$$

Hence

$$u_2 = \frac{\sqrt{2}\,\ln(x)}{2}$$

Which simplifies to

$$u_1 = \frac{\ln(x)\left(\ln(2) - \ln(x)\right)}{2}$$
$$u_2 = \frac{\sqrt{2}\,\ln(x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)\left(\ln(2) - \ln(x)\right)x}{2} + \frac{\sqrt{2}\,\ln(x)\left(-\frac{\sqrt{2}\,x\ln(2)}{2} + \sqrt{2}\,x\ln(x)\right)}{2}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x)^2 x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x) + \left(\frac{\ln(x)^2 x}{2}\right) \\ &= \frac{\ln(x)^2 x}{2} + c_1 x \end{aligned}$$

Which simplifies to

$$y = x\left(\frac{\ln(x)^2}{2} + c_1\right)$$

Summary
The solution(s) found are the following

$$y = x\left(\frac{\ln\left(x\right)^{2}}{2}+c_{1}\right) \tag{1}$$

Verification of solutions

$$y = x\left(\frac{\ln\left(x\right)^{2}}{2}+c_{1}\right)$$

Verified OK.

2.10.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A = x^{2}, B = -x, C = 1, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$x^{2}y''-xy'+y=0$$

In normal form the ode

$$x^{2}y''-xy'+y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{1}{x^{2}}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= (c_1 \ln(x) + c_2)\, x \\
&= (c_1 \ln(x) + c_2)\, x
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - x y' + y = x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \ln(x)\, x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \ln(x)\, x \\ \frac{d}{dx}(x) & \frac{d}{dx}(\ln(x)\, x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \ln(x)\, x \\ 1 & 1 + \ln(x) \end{vmatrix}$$

Therefore

$$W = (x)(1 + \ln(x)) - (\ln(x)\, x)(1)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\, x^2}{x^3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2 x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= ((c_1 \ln(x) + c_2)\, x) + \left(\frac{\ln(x)^2 x}{2}\right) \\ &= \frac{\ln(x)^2 x}{2} + (c_1 \ln(x) + c_2)\, x \end{aligned}$$

Which simplifies to

$$y = x\left(\frac{\ln(x)^2}{2} + c_1 \ln(x) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = x\left(\frac{\ln(x)^2}{2} + c_1 \ln(x) + c_2\right) \tag{1}$$

Verification of solutions

$$y = x\left(\frac{\ln(x)^2}{2} + c_1 \ln(x) + c_2\right)$$

Verified OK.

2.10.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \qquad (1)
\end{aligned}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^2 \\
B &= -x \\
C &= 1 \\
F &= x
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left(x^2\right)(0) + (-x)(-1) + (1)(-x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-x^3v'' + (-x^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$-x^2(u'(x)\, x + u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{x} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x}\, \mathrm{d}x \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (-x)\,(c_1 \ln(x) + c_2) \\ &= -(c_1 \ln(x) + c_2)\, x \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= \ln(x)\, x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \ln(x)\, x \\ \frac{d}{dx}(x) & \frac{d}{dx}(\ln(x)\, x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \ln(x)\, x \\ 1 & 1 + \ln(x) \end{vmatrix}$$

Therefore

$$W = (x)(1 + \ln(x)) - (\ln(x)\, x)(1)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\, x^2}{x^3} dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2 x}{2}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (-(c_1 \ln(x) + c_2)\, x) + \left(\frac{\ln(x)^2 x}{2}\right) \\ &= -\left(c_1 \ln(x) + c_2 - \frac{\ln(x)^2}{2}\right) x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\left(c_1 \ln(x) + c_2 - \frac{\ln(x)^2}{2}\right) x \tag{1}$$

Verification of solutions

$$y = -\left(c_1 \ln(x) + c_2 - \frac{\ln(x)^2}{2}\right) x$$

Verified OK.

2.10.6 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - xy' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 61: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int \frac{1}{2x}dx}\\&=\sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{-x}{x^2}\,dx}\\&=z_1e^{\frac{\ln(x)}{2}}\\&=z_1\left(\sqrt{x}\right)\end{aligned}$$

Which simplifies to

$$y_1=x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{e^{\int -\frac{-x}{x^2}\,dx}}{(y_1)^2}\,dx\\&=y_1\int \frac{e^{\ln(x)}}{(y_1)^2}\,dx\\&=y_1(\ln(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2(x(\ln(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' - xy' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 x + \ln(x) c_2 x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \ln(x) x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \ln(x)\, x \\ \frac{d}{dx}(x) & \frac{d}{dx}(\ln(x)\, x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \ln(x)\, x \\ 1 & 1 + \ln(x) \end{vmatrix}$$

Therefore

$$W = (x)(1 + \ln(x)) - (\ln(x)\, x)(1)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\, x^2}{x^3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2 x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x + \ln(x) c_2 x) + \left(\frac{\ln(x)^2 x}{2}\right) \end{aligned}$$

Which simplifies to

$$y = x(c_2 \ln(x) + c_1) + \frac{\ln(x)^2 x}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = x(c_2 \ln(x) + c_1) + \frac{\ln(x)^2 x}{2} \tag{1}$$

<u>Verification of solutions</u>

$$y = x(c_2 \ln(x) + c_1) + \frac{\ln(x)^2 x}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 18

```
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=x,y(x), singsol=all)
```

$$y(x) = x\left(c_2 + \ln\left(x\right) c_1 + \frac{\ln\left(x\right)^2}{2}\right)$$

✓ Solution by Mathematica
Time used: 0.019 (sec). Leaf size: 25

```
DSolve[x^2*y''[x]-x*y'[x]+y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}x\left(\log^2(x) + 2c_2\log(x) + 2c_1\right)$$

2.11 problem Problem 15.22

Internal problem ID [2523]
Internal file name [`OUTPUT/2015_Sunday_June_05_2022_02_44_40_AM_78827088/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{(x+1)^2 y'' + 3(x+1) y' + y = x^2}$$

2.11.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$(x+1)^2 y'' + (3x+3) y' + y = 0$$

In normal form the ode

$$(x+1)^2 y'' + (3x+3) y' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$p(x) = \frac{3}{x+1}$$
$$q(x) = \frac{1}{(x+1)^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-(\int p(x)dx)} \, dx \\
&= \int e^{-\left(\int \frac{3}{x+1} dx\right)} dx \\
&= \int e^{-3\ln(x+1)} \, dx \\
&= \int \frac{1}{(x+1)^3} dx \\
&= -\frac{1}{2(x+1)^2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{(x+1)^2}}{\frac{1}{(x+1)^6}} \\ &= (x+1)^4 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + (x+1)^4 y(\tau) = 0$$

But in terms of τ

$$(x+1)^4 = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\, \ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\left(c_1 - c_2\ln(2) + c_2\ln\left(-\frac{1}{(x+1)^2}\right)\right)}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\left(c_1 - c_2\ln(2) + c_2\ln\left(-\frac{1}{(x+1)^2}\right)\right)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{-\frac{1}{(x+1)^2}}$$

$$y_2 = -\frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\,\ln\left(-\frac{1}{(x+1)^2}\right)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-\frac{1}{(x+1)^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2} \\ \frac{d}{dx}\left(\sqrt{-\frac{1}{(x+1)^2}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-\frac{1}{(x+1)^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2} \\ \frac{1}{\sqrt{-\frac{1}{(x+1)^2}}(x+1)^3} & -\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{(x+1)^2}}(x+1)^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{(x+1)^2}\right)}{2\sqrt{-\frac{1}{(x+1)^2}}(x+1)^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}}{x+1} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{-\frac{1}{(x+1)^2}}\right)\left(-\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{(x+1)^2}}(x+1)^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{(x+1)^2}\right)}{2\sqrt{-\frac{1}{(x+1)^2}}(x+1)^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}}{x+1}\right) \\ - \left(-\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right)\left(\frac{1}{\sqrt{-\frac{1}{(x+1)^2}}(x+1)^3}\right)$$

Which simplifies to

$$W = \frac{\sqrt{2}}{(x+1)^3}$$

Which simplifies to

$$W = \frac{\sqrt{2}}{(x+1)^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\,\ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right) x^2}{\frac{\sqrt{2}}{x+1}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{-\frac{1}{(x+1)^2}}\left(-\ln(2) + \ln\left(-\frac{1}{(x+1)^2}\right)\right) x^2(x+1)}{2} dx$$

Hence

$$\begin{aligned} u_1 = &-\frac{(x+1)\sqrt{-\frac{1}{(x+1)^2}}\,x^3\ln\left(-\frac{1}{(x+1)^2}\right)}{6} + \frac{(x+1)\sqrt{-\frac{1}{(x+1)^2}}\,\ln(2)\,x^3}{6} \\ &- \frac{(x+1)\sqrt{-\frac{1}{(x+1)^2}}\,x^3}{9} + \frac{(x+1)\sqrt{-\frac{1}{(x+1)^2}}\,x^2}{6} \\ &- \frac{(x+1)\sqrt{-\frac{1}{(x+1)^2}}\,x}{3} + \frac{(x+1)\sqrt{-\frac{1}{(x+1)^2}}\,\ln(x+1)}{3} \end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{-\frac{1}{(x+1)^2}}\,x^2}{\frac{\sqrt{2}}{x+1}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{-\frac{1}{(x+1)^2}}\,x^2(x+1)\sqrt{2}}{2} dx$$

Hence

$$u_2 = \frac{x^3\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)\sqrt{2}}{6}$$

Which simplifies to

$$u_1 = \frac{\left(\ln(2)\,x^3 - x^3\ln\left(-\frac{1}{(x+1)^2}\right) - \frac{2x^3}{3} + x^2 - 2x + 2\ln(x+1)\right)(x+1)\sqrt{-\frac{1}{(x+1)^2}}}{6}$$

$$u_2 = \frac{x^3\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)\sqrt{2}}{6}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(2)\,x^3 - x^3\ln\left(-\frac{1}{(x+1)^2}\right) - \frac{2x^3}{3} + x^2 - 2x + 2\ln(x+1)}{6\,(x+1)} + \frac{x^3\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)\,\sqrt{2}\left(-\frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\,\ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right)}{6}$$

Which simplifies to

$$y_p(x) = \frac{2x^3 - 3x^2 - 6\ln(x+1) + 6x}{18x + 18}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\left(c_1 - c_2\ln(2) + c_2\ln\left(-\frac{1}{(x+1)^2}\right)\right)}{2}\right) + \left(\frac{2x^3 - 3x^2 - 6\ln(x+1) + 6x}{18x + 18}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\left(c_1 - c_2\ln(2) + c_2\ln\left(-\frac{1}{(x+1)^2}\right)\right)}{2} + \frac{2x^3 - 3x^2 - 6\ln(x+1) + 6x}{18x + 18} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\left(c_1 - c_2\ln(2) + c_2\ln\left(-\frac{1}{(x+1)^2}\right)\right)}{2} + \frac{2x^3 - 3x^2 - 6\ln(x+1) + 6x}{18x + 18}$$

Verified OK.

2.11.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = (x+1)^2, B = 3x+3, C = 1, f(x) = x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$(x+1)^2 y'' + (3x+3) y' + y = 0$$

In normal form the ode

$$(x+1)^2 y'' + (3x+3) y' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{3}{x+1} \\
q(x) &= \frac{1}{(x+1)^2}
\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{\frac{1}{(x+1)^2}}}{c} \qquad (6) \\
\tau'' &= -\frac{1}{c\sqrt{\frac{1}{(x+1)^2}}\,(x+1)^3}
\end{aligned}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{c\sqrt{\frac{1}{(x+1)^2}}\,(x+1)^3} + \frac{3\sqrt{\frac{1}{(x+1)^2}}}{(x+1)\,c}}{\left(\frac{\sqrt{\frac{1}{(x+1)^2}}}{c}\right)^2} \\
&= 2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{\frac{1}{(x+1)^2}}dx}{c} \\
&= \frac{\sqrt{\frac{1}{(x+1)^2}}\,(x+1)\ln(x+1)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1}{x+1}
$$

Now the particular solution to this ODE is found

$$
(x+1)^2 y'' + (3x+3)\,y' + y = x^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{-\frac{1}{(x+1)^2}}$$

$$y_2 = -\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\,\ln\left(-\frac{1}{(x+1)^2}\right)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-\frac{1}{(x+1)^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2} \\ \frac{d}{dx}\left(\sqrt{-\frac{1}{(x+1)^2}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-\frac{1}{(x+1)^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2} \\ \frac{1}{\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3} & -\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{(x+1)^2}\right)}{2\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}}{x+1} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{-\frac{1}{(x+1)^2}}\right)\left(-\frac{\sqrt{2}\,\ln(2)}{2\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3}+\frac{\sqrt{2}\,\ln\left(-\frac{1}{(x+1)^2}\right)}{2\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3}-\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}}{x+1}\right) - \left(-\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\,\ln(2)}{2}+\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\,\ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right)\left(\frac{1}{\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3}\right)$$

Which simplifies to

$$W = \frac{\sqrt{2}}{(x+1)^3}$$

Which simplifies to

$$W = \frac{\sqrt{2}}{(x+1)^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\,\ln(2)}{2}+\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\,\ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right)x^2}{\frac{\sqrt{2}}{x+1}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{-\frac{1}{(x+1)^2}}\left(-\ln(2)+\ln\left(-\frac{1}{(x+1)^2}\right)\right)x^2(x+1)}{2}dx$$

Hence

$$u_1 = -\frac{(x+1)\sqrt{-\frac{1}{(x+1)^2}}\,x^3\ln\left(-\frac{1}{(x+1)^2}\right)}{6}+\frac{(x+1)\sqrt{-\frac{1}{(x+1)^2}}\,\ln(2)\,x^3}{6} - \frac{(x+1)\sqrt{-\frac{1}{(x+1)^2}}\,x^3}{9}+\frac{(x+1)\sqrt{-\frac{1}{(x+1)^2}}\,x^2}{6} - \frac{(x+1)\sqrt{-\frac{1}{(x+1)^2}}\,x}{3}+\frac{(x+1)\sqrt{-\frac{1}{(x+1)^2}}\,\ln(x+1)}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{-\frac{1}{(x+1)^2}}\, x^2}{\frac{\sqrt{2}}{x+1}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{-\frac{1}{(x+1)^2}}\, x^2 (x+1)\sqrt{2}}{2} dx$$

Hence

$$u_2 = \frac{x^3 \sqrt{-\frac{1}{(x+1)^2}}\, (x+1)\sqrt{2}}{6}$$

Which simplifies to

$$u_1 = \frac{\left(\ln (2)\, x^3 - x^3 \ln \left(-\frac{1}{(x+1)^2}\right) - \frac{2x^3}{3} + x^2 - 2x + 2\ln (x+1)\right) (x+1) \sqrt{-\frac{1}{(x+1)^2}}}{6}$$

$$u_2 = \frac{x^3 \sqrt{-\frac{1}{(x+1)^2}}\, (x+1)\sqrt{2}}{6}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln (2)\, x^3 - x^3 \ln \left(-\frac{1}{(x+1)^2}\right) - \frac{2x^3}{3} + x^2 - 2x + 2\ln (x+1)}{6\,(x+1)} + \frac{x^3 \sqrt{-\frac{1}{(x+1)^2}}\, (x+1)\sqrt{2} \left(-\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\, \ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\, \ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right)}{6}$$

Which simplifies to

$$y_p(x) = \frac{2x^3 - 3x^2 - 6\ln (x+1) + 6x}{18x + 18}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x+1}\right) + \left(\frac{2x^3 - 3x^2 - 6\ln (x+1) + 6x}{18x + 18}\right) \\ &= \frac{2x^3 - 3x^2 - 6\ln (x+1) + 6x}{18x + 18} + \frac{c_1}{x+1} \end{aligned}$$

Which simplifies to

$$y = \frac{2x^3 - 3x^2 - 6\ln(x+1) + 18c_1 + 6x}{18x + 18}$$

Summary

The solution(s) found are the following

$$y = \frac{2x^3 - 3x^2 - 6\ln(x+1) + 18c_1 + 6x}{18x + 18} \tag{1}$$

Verification of solutions

$$y = \frac{2x^3 - 3x^2 - 6\ln(x+1) + 18c_1 + 6x}{18x + 18}$$

Verified OK.

2.11.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int \left((x+1)^2 y'' + (3x+3)\, y' + y\right) dx &= \int x^2 dx \\ y(x+1) + \left(x^2 + 2x + 1\right) y' &= \frac{x^3}{3} + c_1\end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{1}{x+1} \\ q(x) &= \frac{x^3 + 3c_1}{3\,(x+1)^2}\end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x+1} = \frac{x^3 + 3c_1}{3\,(x+1)^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x+1}dx} \\ &= x + 1\end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^3+3c_1}{3\left(x+1\right)^2}\right) \\
\frac{d}{dx}((x+1)\,y) &= (x+1)\left(\frac{x^3+3c_1}{3\left(x+1\right)^2}\right) \\
d((x+1)\,y) &= \left(\frac{x^3+3c_1}{3x+3}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
(x+1)\,y &= \int \frac{x^3+3c_1}{3x+3}\, dx \\
(x+1)\,y &= \frac{x^3}{9} - \frac{x^2}{6} + \frac{x}{3} + \frac{(3c_1-1)\ln(x+1)}{3} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x+1$ results in

$$
y = \frac{\frac{x^3}{9} - \frac{x^2}{6} + \frac{x}{3} + \frac{(3c_1-1)\ln(x+1)}{3}}{x+1} + \frac{c_2}{x+1}
$$

which simplifies to

$$
y = \frac{(18c_1-6)\ln(x+1) + 2x^3 - 3x^2 + 6x + 18c_2}{18x+18}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(18c_1-6)\ln(x+1) + 2x^3 - 3x^2 + 6x + 18c_2}{18x+18} \tag{1}
$$

Verification of solutions

$$
y = \frac{(18c_1-6)\ln(x+1) + 2x^3 - 3x^2 + 6x + 18c_2}{18x+18}
$$

Verified OK.

2.11.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
\left(x+1\right)^2 y'' + (3x+3)\,y' + y = x^2
$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left((x+1)^2 y'' + (3x+3)\, y' + y\right) dx = \int x^2 dx$$

$$y(x+1) + \left(x^2+2x+1\right) y' = \frac{x^3}{3} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x+1}$$
$$q(x) = \frac{x^3+3c_1}{3\,(x+1)^2}$$

Hence the ode is

$$y' + \frac{y}{x+1} = \frac{x^3+3c_1}{3\,(x+1)^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{x+1}dx}$$
$$= x+1$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^3+3c_1}{3\,(x+1)^2}\right)$$
$$\frac{d}{dx}((x+1)\, y) = (x+1)\left(\frac{x^3+3c_1}{3\,(x+1)^2}\right)$$
$$d((x+1)\, y) = \left(\frac{x^3+3c_1}{3x+3}\right) dx$$

Integrating gives

$$(x+1)\, y = \int \frac{x^3+3c_1}{3x+3}\, dx$$
$$(x+1)\, y = \frac{x^3}{9} - \frac{x^2}{6} + \frac{x}{3} + \frac{(3c_1-1)\ln(x+1)}{3} + c_2$$

Dividing both sides by the integrating factor $\mu = x + 1$ results in

$$y = \frac{\frac{x^3}{9} - \frac{x^2}{6} + \frac{x}{3} + \frac{(3c_1-1)\ln(x+1)}{3}}{x+1} + \frac{c_2}{x+1}$$

which simplifies to

$$y = \frac{(18c_1 - 6)\ln(x+1) + 2x^3 - 3x^2 + 6x + 18c_2}{18x + 18}$$

Summary

The solution(s) found are the following

$$y = \frac{(18c_1 - 6)\ln(x+1) + 2x^3 - 3x^2 + 6x + 18c_2}{18x + 18} \tag{1}$$

Verification of solutions

$$y = \frac{(18c_1 - 6)\ln(x+1) + 2x^3 - 3x^2 + 6x + 18c_2}{18x + 18}$$

Verified OK.

2.11.5 Solving using Kovacic algorithm

Writing the ode as

$$(x+1)^2 y'' + (3x+3) y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (x+1)^2 \\ B &= 3x + 3 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4(x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4(x+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4(x+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 62: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+1)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x+1)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4(x+1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4(x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2+2x} + (-)\,(0) \\
&= \frac{1}{2+2x} \\
&= \frac{1}{2+2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2+2x}\right)(0) + \left(\left(-\frac{1}{2(x+1)^2}\right) + \left(\frac{1}{2+2x}\right)^2 - \left(-\frac{1}{4(x+1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1}{2+2x} dx} \\ &= \sqrt{x+1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x+3}{(x+1)^2}\, dx} \\ &= z_1 e^{-\frac{3\ln(x+1)}{2}} \\ &= z_1 \left(\frac{1}{(x+1)^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x+3}{(x+1)^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3\ln(x+1)}}{(y_1)^2}\, dx \\ &= y_1(\ln(x+1)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x+1}\right) + c_2 \left(\frac{1}{x+1} (\ln(x+1))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
(x+1)^2 y'' + (3x+3) y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1}{x+1} + \frac{c_2 \ln(x+1)}{x+1}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x+1} \\
y_2 &= \frac{\ln(x+1)}{x+1}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{\ln(x+1)}{x+1} \\ \frac{d}{dx}\left(\frac{1}{x+1}\right) & \frac{d}{dx}\left(\frac{\ln(x+1)}{x+1}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{\ln(x+1)}{x+1} \\ -\frac{1}{(x+1)^2} & -\frac{\ln(x+1)}{(x+1)^2} + \frac{1}{(x+1)^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x+1}\right)\left(-\frac{\ln(x+1)}{(x+1)^2} + \frac{1}{(x+1)^2}\right) - \left(\frac{\ln(x+1)}{x+1}\right)\left(-\frac{1}{(x+1)^2}\right)$$

Which simplifies to

$$W = \frac{1}{(x+1)^3}$$

Which simplifies to

$$W = \frac{1}{(x+1)^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\ln(x+1)x^2}{x+1}}{\frac{1}{x+1}}\,dx$$

Which simplifies to

$$u_1 = -\int \ln(x+1)\,x^2 dx$$

Hence

$$u_1 = -\frac{(x+1)^3 \ln(x+1)}{3} + \frac{x^3}{9} - \frac{x^2}{6} + \frac{x}{3} + \frac{11}{18} + (x+1)^2 \ln(x+1) - (x+1)\ln(x+1)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x^2}{x+1}}{\frac{1}{x+1}}\, dx$$

Which simplifies to

$$u_2 = \int x^2 dx$$

Hence

$$u_2 = \frac{x^3}{3}$$

Which simplifies to

$$\begin{aligned} u_1 &= -\frac{\ln\left(x+1\right) x^3}{3} + \frac{x^3}{9} - \frac{x^2}{6} - \frac{\ln\left(x+1\right)}{3} + \frac{x}{3} + \frac{11}{18} \\ u_2 &= \frac{x^3}{3} \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{\ln(x+1)x^3}{3} + \frac{x^3}{9} - \frac{x^2}{6} - \frac{\ln(x+1)}{3} + \frac{x}{3} + \frac{11}{18}}{x+1} + \frac{\ln\left(x+1\right) x^3}{3x+3}$$

Which simplifies to

$$y_p(x) = \frac{2x^3 - 3x^2 - 6\ln\left(x+1\right) + 6x + 11}{18x + 18}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x+1} + \frac{c_2 \ln\left(x+1\right)}{x+1}\right) + \left(\frac{2x^3 - 3x^2 - 6\ln\left(x+1\right) + 6x + 11}{18x + 18}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_2 \ln\left(x+1\right) + c_1}{x+1} + \frac{2x^3 - 3x^2 - 6\ln\left(x+1\right) + 6x + 11}{18x + 18}$$

Summary
The solution(s) found are the following

$$y = \frac{c_2 \ln(x+1) + c_1}{x+1} + \frac{2x^3 - 3x^2 - 6\ln(x+1) + 6x + 11}{18x + 18} \tag{1}$$

Verification of solutions

$$y = \frac{c_2 \ln(x+1) + c_1}{x+1} + \frac{2x^3 - 3x^2 - 6\ln(x+1) + 6x + 11}{18x + 18}$$

Verified OK.

2.11.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= (x+1)^2 \\ q(x) &= 3x + 3 \\ r(x) &= 1 \\ s(x) &= x^2 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 3 \end{aligned}$$

Therefore (1) becomes

$$2 - (3) + (1) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\, y' + (q(x) - p'(x))\, y\right)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$(x+1)^2 y' + y(x+1) = \int x^2 \, dx$$

We now have a first order ode to solve which is

$$(x+1)^2 y' + y(x+1) = \frac{x^3}{3} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x+1} \\ q(x) &= \frac{x^3 + 3c_1}{3(x+1)^2} \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x+1} = \frac{x^3 + 3c_1}{3(x+1)^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{1}{x+1} dx} \\ &= x+1 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^3 + 3c_1}{3(x+1)^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}((x+1)\,y) &= (x+1)\left(\frac{x^3 + 3c_1}{3(x+1)^2}\right) \\ \mathrm{d}((x+1)\,y) &= \left(\frac{x^3 + 3c_1}{3x+3}\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} (x+1)\,y &= \int \frac{x^3 + 3c_1}{3x+3} \,\mathrm{d}x \\ (x+1)\,y &= \frac{x^3}{9} - \frac{x^2}{6} + \frac{x}{3} + \frac{(3c_1 - 1)\ln(x+1)}{3} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x + 1$ results in

$$y = \frac{\frac{x^3}{9} - \frac{x^2}{6} + \frac{x}{3} + \frac{(3c_1-1)\ln(x+1)}{3}}{x+1} + \frac{c_2}{x+1}$$

which simplifies to

$$y = \frac{(18c_1 - 6)\ln(x+1) + 2x^3 - 3x^2 + 6x + 18c_2}{18x + 18}$$

Summary

The solution(s) found are the following

$$y = \frac{(18c_1 - 6)\ln(x+1) + 2x^3 - 3x^2 + 6x + 18c_2}{18x + 18} \tag{1}$$

Verification of solutions

$$y = \frac{(18c_1 - 6)\ln(x+1) + 2x^3 - 3x^2 + 6x + 18c_2}{18x + 18}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
dsolve((x+1)^2*diff(y(x),x$2)+3*(x+1)*diff(y(x),x)+y(x)=x^2,y(x), singsol=all)
```

$$y(x) = \frac{(18c_1 - 6)\ln(x+1) + 2x^3 - 3x^2 + 6x + 18c_2}{18x + 18}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 44

```
DSolve[(x+1)^2*y''[x]+3*(x+1)*y'[x]+y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2x^3 - 3x^2 + 6x + 6(-1 + 3c_2)\log(x+1) + 18c_1}{18(x+1)}$$

2.12 problem Problem 15.23

Internal problem ID [2524]
Internal file name [`OUTPUT/2016_Sunday_June_05_2022_02_44_43_AM_15692357/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{(x-2)\,y''+3y'+\frac{4y}{x^2}=0}$$

2.12.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(y''(x-2)\,x^2+3y'x^2+4y\right)dx=0$$

$$4yx+\left(x^3-2x^2\right)y'=c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x) = \frac{4}{(x-2)\,x}$$
$$q(x) = \frac{c_1}{(x-2)\,x^2}$$

Hence the ode is

$$y' + \frac{4y}{(x-2)\,x} = \frac{c_1}{(x-2)\,x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{4}{(x-2)x}dx}\\ &= \mathrm{e}^{2\ln(x-2)-2\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{(x-2)^2}{x^2}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{(x-2)\,x^2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(x-2)^2\,y}{x^2}\right) &= \left(\frac{(x-2)^2}{x^2}\right)\left(\frac{c_1}{(x-2)\,x^2}\right)\\ \mathrm{d}\left(\frac{(x-2)^2\,y}{x^2}\right) &= \left(\frac{(x-2)\,c_1}{x^4}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{(x-2)^2\,y}{x^2} &= \int \frac{(x-2)\,c_1}{x^4}\,\mathrm{d}x\\ \frac{(x-2)^2\,y}{x^2} &= c_1\left(\frac{2}{3x^3} - \frac{1}{2x^2}\right) + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{(x-2)^2}{x^2}$ results in

$$y = \frac{x^2 c_1\left(\frac{2}{3x^3} - \frac{1}{2x^2}\right)}{(x-2)^2} + \frac{c_2 x^2}{(x-2)^2}$$

which simplifies to

$$y = \frac{6c_2x^3 - 3c_1x + 4c_1}{6(x-2)^2 x}$$

Summary

The solution(s) found are the following

$$y = \frac{6c_2x^3 - 3c_1x + 4c_1}{6(x-2)^2 x} \tag{1}$$

Verification of solutions

$$y = \frac{6c_2x^3 - 3c_1x + 4c_1}{6(x-2)^2 x}$$

Verified OK.

2.12.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + 3xy' + \frac{4y}{x} = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\, xy' + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= -1\\ \beta &= 4\\ n &= 2\\ \gamma &= -\frac{1}{2}\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \operatorname{BesselJ}\left(2, \frac{4}{\sqrt{x}}\right)}{x} + \frac{c_2 \operatorname{BesselY}\left(2, \frac{4}{\sqrt{x}}\right)}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 \operatorname{BesselJ}\left(2, \frac{4}{\sqrt{x}}\right)}{x} + \frac{c_2 \operatorname{BesselY}\left(2, \frac{4}{\sqrt{x}}\right)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \operatorname{BesselJ}\left(2, \frac{4}{\sqrt{x}}\right)}{x} + \frac{c_2 \operatorname{BesselY}\left(2, \frac{4}{\sqrt{x}}\right)}{x}$$

Verified OK.

2.12.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y''(x-2)\,x^2 + 3y'x^2 + 4y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(y''(x-2)\,x^2 + 3y'x^2 + 4y\right) dx = 0$$
$$4yx + \left(x^3 - 2x^2\right) y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{4}{(x-2)\,x}$$
$$q(x) = \frac{c_1}{(x-2)\,x^2}$$

Hence the ode is

$$y' + \frac{4y}{(x-2)\,x} = \frac{c_1}{(x-2)\,x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{4}{(x-2)x}dx}$$
$$= e^{2\ln(x-2)-2\ln(x)}$$

Which simplifies to

$$\mu = \frac{(x-2)^2}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{(x-2)\,x^2}\right)$$
$$\frac{d}{dx}\left(\frac{(x-2)^2\,y}{x^2}\right) = \left(\frac{(x-2)^2}{x^2}\right)\left(\frac{c_1}{(x-2)\,x^2}\right)$$
$$d\left(\frac{(x-2)^2\,y}{x^2}\right) = \left(\frac{(x-2)\,c_1}{x^4}\right)\,dx$$

Integrating gives

$$\frac{(x-2)^2\,y}{x^2} = \int \frac{(x-2)\,c_1}{x^4}\,dx$$
$$\frac{(x-2)^2\,y}{x^2} = c_1\left(\frac{2}{3x^3} - \frac{1}{2x^2}\right) + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{(x-2)^2}{x^2}$ results in

$$y = \frac{x^2 c_1\left(\frac{2}{3x^3} - \frac{1}{2x^2}\right)}{(x-2)^2} + \frac{c_2 x^2}{(x-2)^2}$$

which simplifies to

$$y = \frac{6c_2x^3 - 3c_1x + 4c_1}{6\,(x-2)^2\,x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{6c_2x^3 - 3c_1x + 4c_1}{6\,(x-2)^2\,x} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{6c_2x^3 - 3c_1x + 4c_1}{6\,(x-2)^2\,x}$$

Verified OK.

2.12.4 Solving using Kovacic algorithm

Writing the ode as

$$y''(x-2)\,x^2 + 3y'x^2 + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (x-2)\,x^2 \\ B &= 3x^2 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 16x + 32}{4\,(x^2 - 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2 - 16x + 32 \\ t &= 4\big(x^2 - 2x\big)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 16x + 32}{4\,(x^2 - 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 63: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + \frac{3}{4(x-2)^2} - \frac{1}{x-2} + \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^2 - 16x + 32}{4\left(x^2 - 2x\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^2 - 16x + 32}{4\left(x^2 - 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-}\right) \\
&= \frac{3}{2} - \left(\frac{3}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{2}{x} - \frac{1}{2\,(x-2)} + (0) \\
&= \frac{2}{x} - \frac{1}{2\,(x-2)} \\
&= \frac{3x-8}{2x\,(x-2)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{2}{x} - \frac{1}{2\,(x-2)}\right)(0) + \left(\left(-\frac{2}{x^2} + \frac{1}{2\,(x-2)^2}\right) + \left(\frac{2}{x} - \frac{1}{2\,(x-2)}\right)^2 - \left(\frac{3x^2-16x+32}{4\,(x^2-2x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{x} - \frac{1}{2(x-2)}\right) dx} \\
&= \frac{x^2}{\sqrt{x-2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x^2}{(x-2)x^2} \, dx} \\
&= z_1 e^{-\frac{3\ln(x-2)}{2}} \\
&= z_1 \left(\frac{1}{(x-2)^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2}{(x-2)^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2}{(x-2)x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-3\ln(x-2)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-3x+4}{6x^3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2}{(x-2)^2}\right) + c_2\left(\frac{x^2}{(x-2)^2}\left(\frac{-3x+4}{6x^3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^2}{(x-2)^2} + \frac{c_2(-3x+4)}{6\,(x-2)^2\, x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^2}{(x-2)^2} + \frac{c_2(-3x+4)}{6\,(x-2)^2\, x}
$$

Verified OK.

2.12.5 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= (x-2)\, x^2 \\
q(x) &= 3x^2 \\
r(x) &= 4 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 6x - 4 \\
q'(x) &= 6x
\end{aligned}
$$

Therefore (1) becomes

$$
6x - 4 - (6x) + (4) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$(x-2)\,x^2 y' + \left(2x^2 - 2x(x-2)\right) y = c_1$$

We now have a first order ode to solve which is

$$(x-2)\,x^2 y' + \left(2x^2 - 2x(x-2)\right) y = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{4}{(x-2)\,x} \\ q(x) &= \frac{c_1}{(x-2)\,x^2} \end{aligned}$$

Hence the ode is

$$y' + \frac{4y}{(x-2)\,x} = \frac{c_1}{(x-2)\,x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{4}{(x-2)x}dx} \\ &= e^{2\ln(x-2) - 2\ln(x)} \end{aligned}$$

Which simplifies to

$$\mu = \frac{(x-2)^2}{x^2}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{(x-2)\,x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(x-2)^2\,y}{x^2}\right) &= \left(\frac{(x-2)^2}{x^2}\right)\left(\frac{c_1}{(x-2)\,x^2}\right) \\ \mathrm{d}\left(\frac{(x-2)^2\,y}{x^2}\right) &= \left(\frac{(x-2)\,c_1}{x^4}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\frac{(x-2)^2 y}{x^2} = \int \frac{(x-2)\, c_1}{x^4}\, \mathrm{d}x$$
$$\frac{(x-2)^2 y}{x^2} = c_1 \left(\frac{2}{3x^3} - \frac{1}{2x^2}\right) + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{(x-2)^2}{x^2}$ results in

$$y = \frac{x^2 c_1 \left(\frac{2}{3x^3} - \frac{1}{2x^2}\right)}{(x-2)^2} + \frac{c_2 x^2}{(x-2)^2}$$

which simplifies to

$$y = \frac{6c_2x^3 - 3c_1x + 4c_1}{6\,(x-2)^2\, x}$$

Summary

The solution(s) found are the following

$$y = \frac{6c_2x^3 - 3c_1x + 4c_1}{6\,(x-2)^2\, x} \tag{1}$$

Verification of solutions

$$y = \frac{6c_2x^3 - 3c_1x + 4c_1}{6\,(x-2)^2\, x}$$

Verified OK.

2.12.6 Maple step by step solution

Let's solve

$y''(x-2)\, x^2 + 3y'x^2 + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x-2} - \frac{4y}{(x-2)x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x-2} + \frac{4y}{(x-2)x^2} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3}{x-2}, P_3(x) = \frac{4}{(x-2)x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -2$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''(x-2)\, x^2 + 3y'x^2 + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$

- Shift index using $k->k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

$$-2a_0(1+r)\left(-2+r\right)x^r+\left(\sum_{k=1}^{\infty}\left(-2a_k(k+r+1)\left(k+r-2\right)+a_{k-1}(k-1+r)\left(k+r+1\right)\right)x^{k+}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2(1+r)\left(-2+r\right)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-1,2\}$

- Each term in the series must be 0, giving the recursion relation

  $-2(k+r+1)\left(\frac{(-k-r+1)a_{k-1}}{2}+a_k(k+r-2)\right)=0$

- Shift index using $k->k+1$

  $-2(k+r+2)\left(\frac{(-k-r)a_k}{2}+a_{k+1}(k-1+r)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{(k+r)a_k}{2(k-1+r)}$

- Recursion relation for $r=-1$; series terminates at $k=1$

  $a_{k+1}=\frac{(k-1)a_k}{2(k-2)}$

- Apply recursion relation for $k=0$

  $a_1=\frac{a_0}{4}$

- Terminating series solution of the ODE for $r=-1$. Use reduction of order to find the second

  $y=a_0\cdot\left(1+\frac{x}{4}\right)$

- Recursion relation for $r=2$

  $a_{k+1}=\frac{(k+2)a_k}{2(k+1)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+2},a_{k+1}=\frac{(k+2)a_k}{2(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y=a_0\cdot\left(1+\frac{x}{4}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+2}\right),b_{k+1}=\frac{(k+2)b_k}{2(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve((x-2)*diff(y(x),x$2)+3*diff(y(x),x)+4*y(x)/x^2=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2x^3 + 3c_1x - 4c_1}{x\left(-2+x\right)^2}$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 45

```
DSolve[(x-2)*y''[x]+3*y'[x]+4*y[x]/x^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{6c_1x^3 + 3c_2x - 4c_2}{6\sqrt{2-x}(x-2)^{3/2}x}$$

2.13 problem Problem 15.24(a)

Internal problem ID [2525]
Internal file name [`OUTPUT/2017_Sunday_June_05_2022_02_44_47_AM_48309473/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.24(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = x^n}$$

2.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = x^n$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x$$
$$y_2 = e^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & e^{-x} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

Therefore

$$W = (e^x)(-e^{-x}) - (e^{-x})(e^x)$$

Which simplifies to

$$W = -2 e^{-x} e^x$$

Which simplifies to

$$W = -2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-x} x^n}{-2}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-x} x^n}{2} dx$$

Hence

$$u_1 = \frac{x^{\frac{n}{2}} e^{-\frac{x}{2}} \operatorname{WhittakerM}\left(\frac{n}{2}, \frac{n}{2} + \frac{1}{2}, x\right)}{2n + 2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{x} x^n}{-2}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{x} x^n}{2} dx$$

Hence

$$u_2 = \frac{(-1)^{-n} \left(x^n (-1)^n n\Gamma(n) \left(-x\right)^{-n} - x^n (-1)^n e^{x} - x^n (-1)^n n(-x)^{-n} \Gamma(n, -x)\right)}{2}$$

Which simplifies to

$$u_1 = \frac{x^{\frac{n}{2}} e^{-\frac{x}{2}} \operatorname{WhittakerM}\left(\frac{n}{2}, \frac{n}{2} + \frac{1}{2}, x\right)}{2n + 2}$$

$$u_2 = -\frac{x^n \left(\left(\Gamma(n, -x)\, n - \Gamma(n+1)\right) (-x)^{-n} + e^{x}\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^{\frac{n}{2}} e^{-\frac{x}{2}} \operatorname{WhittakerM}\left(\frac{n}{2}, \frac{n}{2} + \frac{1}{2}, x\right) e^{x}}{2n + 2} - \frac{x^n \left(\left(\Gamma(n, -x)\, n - \Gamma(n+1)\right) (-x)^{-n} + e^{x}\right) e^{-x}}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}\right) + \left(\frac{x^{\frac{n}{2}} \mathrm{e}^{-\frac{x}{2}} \operatorname{WhittakerM}\left(\frac{n}{2}, \frac{n}{2}+\frac{1}{2}, x\right) \mathrm{e}^{x}}{2n+2} \right. \\
&\qquad \left. - \frac{x^{n}\left(\left(\Gamma(n,-x)\, n - \Gamma(n+1)\right)(-x)^{-n} + \mathrm{e}^{x}\right) \mathrm{e}^{-x}}{2}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} + \frac{x^{\frac{n}{2}} \mathrm{e}^{-\frac{x}{2}} \operatorname{WhittakerM}\left(\frac{n}{2}, \frac{n}{2}+\frac{1}{2}, x\right) \mathrm{e}^{x}}{2n+2} \\
- \frac{x^{n}\left(\left(\Gamma(n,-x)\, n - \Gamma(n+1)\right)(-x)^{-n} + \mathrm{e}^{x}\right) \mathrm{e}^{-x}}{2}
\end{aligned}
\tag{1}
$$

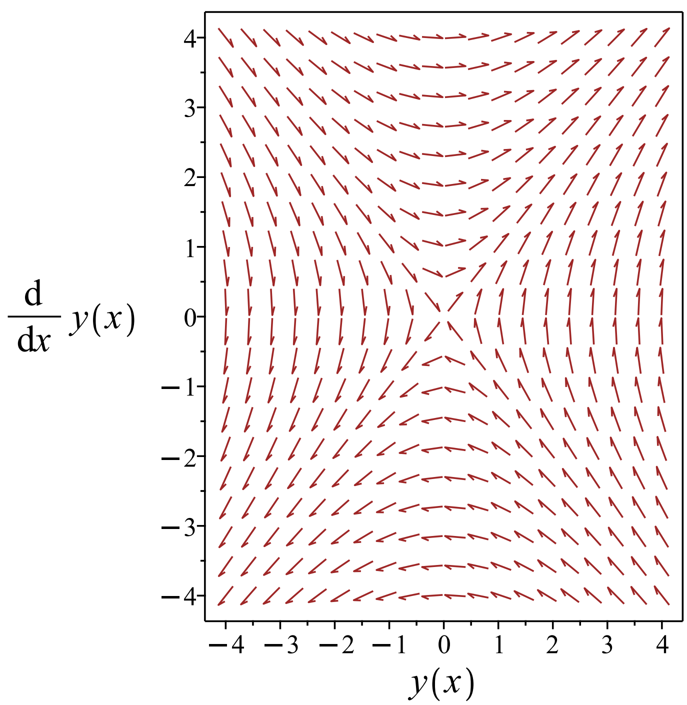


Figure 74: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} + \frac{x^{\frac{n}{2}} \mathrm{e}^{-\frac{x}{2}} \operatorname{WhittakerM}\left(\frac{n}{2}, \frac{n}{2}+\frac{1}{2}, x\right) \mathrm{e}^{x}}{2n+2} - \frac{x^{n}\left(\left(\Gamma(n,-x)\, n - \Gamma(n+1)\right)(-x)^{-n} + \mathrm{e}^{x}\right) \mathrm{e}^{-x}}{2}
$$

Verified OK.

2.13.2 Solving using Kovacic algorithm

Writing the ode as

$$
y'' - y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= -1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 65: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\,dx \\ &= \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^{x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-x}$$
$$y_2 = \frac{e^{x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & \frac{e^{x}}{2} \\ \frac{d}{dx}\left(e^{-x}\right) & \frac{d}{dx}\left(\frac{e^{x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ -e^{-x} & \frac{e^x}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-x}\right)\left(\frac{e^x}{2}\right) - \left(\frac{e^x}{2}\right)\left(-e^{-x}\right)$$

Which simplifies to

$$W = e^{-x} e^x$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^x x^n}{2}}{1}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{e^x x^n}{2} dx$$

Hence

$$u_1 = \frac{(-1)^{-n}\left(x^n(-1)^n n\Gamma(n)\,(-x)^{-n} - x^n(-1)^n e^x - x^n(-1)^n n(-x)^{-n}\,\Gamma(n,-x)\right)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-x} x^n}{1}\, dx$$

Which simplifies to

$$u_2 = \int e^{-x} x^n dx$$

Hence

$$u_2 = \frac{x^{\frac{n}{2}} e^{-\frac{x}{2}} \operatorname{WhittakerM}\left(\frac{n}{2}, \frac{n}{2}+\frac{1}{2}, x\right)}{n+1}$$

Which simplifies to

$$u_1 = -\frac{x^n\left(\left(\Gamma(n,-x)\,n-\Gamma(n+1)\right)(-x)^{-n}+\mathrm{e}^x\right)}{2}$$

$$u_2 = \frac{x^{\frac{n}{2}}\mathrm{e}^{-\frac{x}{2}}\,\mathrm{WhittakerM}\left(\frac{n}{2},\frac{n}{2}+\frac{1}{2},x\right)}{n+1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^{\frac{n}{2}}\mathrm{e}^{-\frac{x}{2}}\,\mathrm{WhittakerM}\left(\frac{n}{2},\frac{n}{2}+\frac{1}{2},x\right)\mathrm{e}^x}{2n+2} - \frac{x^n\left(\left(\Gamma(n,-x)\,n-\Gamma(n+1)\right)(-x)^{-n}+\mathrm{e}^x\right)\mathrm{e}^{-x}}{2}$$

Which simplifies to

$$y_p(x) = \frac{\left(\left(-\mathrm{e}^x+\left(-\Gamma(n,-x)\,n+\Gamma(n+1)\right)(-x)^{-n}\right)(n+1)\,x^n+\mathrm{e}^{\frac{3x}{2}}x^{\frac{n}{2}}\,\mathrm{WhittakerM}\left(\frac{n}{2},\frac{n}{2}+\frac{1}{2},x\right)\right)\mathrm{e}^{-x}}{2n+2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x}+\frac{c_2\mathrm{e}^x}{2}\right) \\ &\quad + \left(\frac{\left(\left(-\mathrm{e}^x+\left(-\Gamma(n,-x)\,n+\Gamma(n+1)\right)(-x)^{-n}\right)(n+1)\,x^n+\mathrm{e}^{\frac{3x}{2}}x^{\frac{n}{2}}\,\mathrm{WhittakerM}\left(\frac{n}{2},\frac{n}{2}+\frac{1}{2},x\right)\right)\mathrm{e}^{-x}}{2n+2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-x}+\frac{c_2\mathrm{e}^x}{2} + \frac{\left(\left(-\mathrm{e}^x+\left(-\Gamma(n,-x)\,n+\Gamma(n+1)\right)(-x)^{-n}\right)(n+1)\,x^n+\mathrm{e}^{\frac{3x}{2}}x^{\frac{n}{2}}\,\mathrm{WhittakerM}\left(\frac{n}{2},\frac{n}{2}+\frac{1}{2},x\right)\right)\mathrm{e}^{-x}}{2n+2} \tag{1}$$

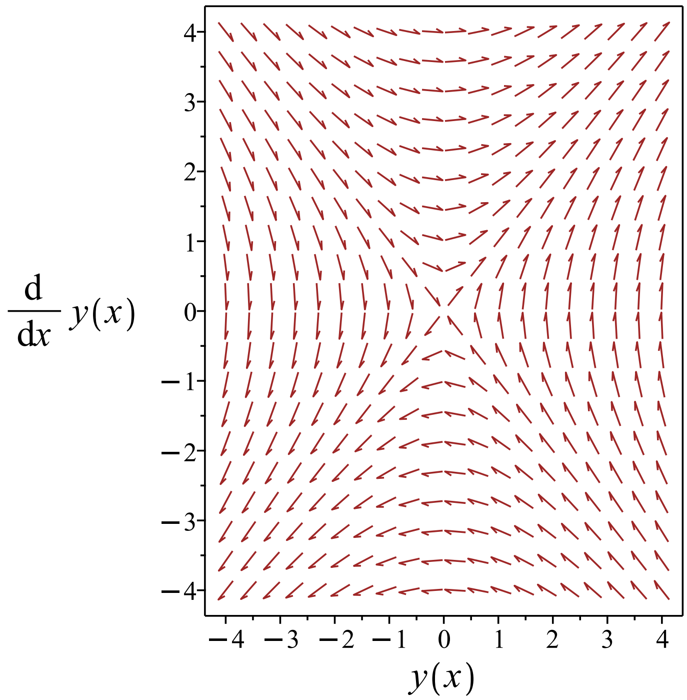


Figure 75: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} + \frac{\left(\left(-e^x + \left(-\Gamma(n,-x)\, n + \Gamma(n+1)\right)(-x)^{-n}\right)(n+1)\, x^n + e^{\frac{3x}{2}} x^{\frac{n}{2}} \,\text{WhittakerM}\left(\frac{n}{2}, \frac{n}{2}+\frac{1}{2}, x\right)\right) e^{-x}}{2n+2}$$

Verified OK.

2.13.3 Maple step by step solution

Let's solve

$y'' - y = x^n$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

$(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x^n\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\mathrm{e}^{-x}\left(\int \mathrm{e}^{x} x^n dx\right)}{2} + \frac{\mathrm{e}^{x}\left(\int \mathrm{e}^{-x} x^n dx\right)}{2}$

  ◦ Compute integrals

  $y_p(x) = \frac{\left(\left(-\mathrm{e}^{x} + (-\Gamma(n,-x)n + \Gamma(n+1))(-x)^{-n}\right)(n+1)x^n + \mathrm{e}^{\frac{3x}{2}} x^{\frac{n}{2}}\, \mathit{WhittakerM}\left(\frac{n}{2}, \frac{n}{2}+\frac{1}{2}, x\right)\right)\mathrm{e}^{-x}}{2n+2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \frac{\left(\left(-\mathrm{e}^{x} + (-\Gamma(n,-x)n + \Gamma(n+1))(-x)^{-n}\right)(n+1)x^n + \mathrm{e}^{\frac{3x}{2}} x^{\frac{n}{2}}\, \mathit{WhittakerM}\left(\frac{n}{2}, \frac{n}{2}+\frac{1}{2}, x\right)\right)\mathrm{e}^{-x}}{2n+2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 85

```
dsolve(diff(y(x),x$2)-y(x)=x^n,y(x), singsol=all)
```

$$y(x) = -\frac{\left(-e^{\frac{3x}{2}} x^{\frac{n}{2}} \operatorname{WhittakerM}\left(\frac{n}{2}, \frac{n}{2}+\frac{1}{2}, x\right) + \left(x^n (n\Gamma(n,-x) - \Gamma(n+1)) (-x)^{-n} - 2c_1 e^{2x} + e^x x^n - 2c_2\right) (n}{2n+2}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 58

```
DSolve[y''[x]-y[x]==x^n,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}e^{-x}x^n(-x)^{-n}\Gamma(n+1,-x) - \frac{1}{2}e^x\Gamma(n+1,x) + c_1e^x + c_2e^{-x}$$

2.14 problem Problem 15.24(b)

Internal problem ID [2526]
Internal file name [`OUTPUT/2018_Sunday_June_05_2022_02_44_50_AM_12582292/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.24(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = 2x\,\mathrm{e}^{x}}$$

2.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = 2x\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)}\\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^{x} + c_2 x\, e^{x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 x\, e^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2x\, e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^{x}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{x}, e^{x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{x}, x^{2}\mathrm{e}^{x}\}]$$

Since $x\,\mathrm{e}^{x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^{2}\mathrm{e}^{x}, x^{3}\mathrm{e}^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^{2}\mathrm{e}^{x} + A_2 x^{3}\mathrm{e}^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{x} + 6A_2 x\,\mathrm{e}^{x} = 2x\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^{3}\mathrm{e}^{x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x}\right) + \left(\frac{x^{3}\mathrm{e}^{x}}{3}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{x}(c_2 x + c_1) + \frac{x^{3}\mathrm{e}^{x}}{3}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x}(c_2 x + c_1) + \frac{x^{3}\mathrm{e}^{x}}{3} \tag{1}$$

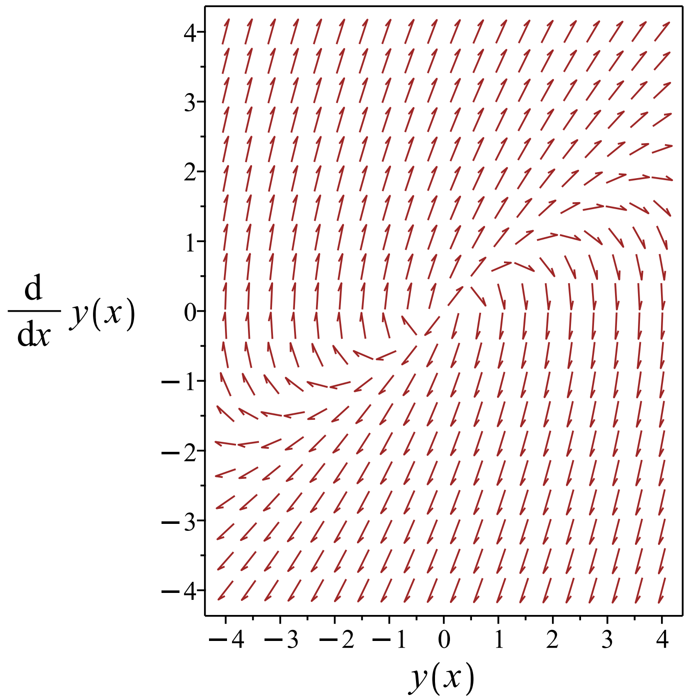


Figure 76: Slope field plot

Verification of solutions

$$y = e^{x}(c_2 x + c_1) + \frac{x^3 e^{x}}{3}$$

Verified OK.

2.14.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -2\, dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 2\,\mathrm{e}^{-x}x\,\mathrm{e}^{x} \\
\left(\mathrm{e}^{-x}y\right)'' &= 2\,\mathrm{e}^{-x}x\,\mathrm{e}^{x}
\end{aligned}
$$

Integrating once gives

$$\left(\mathrm{e}^{-x}y\right)' = x^2 + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-x}y\right) = \frac{1}{3}x^3 + c_1x + c_2$$

Hence the solution is

$$y = \frac{\frac{1}{3}x^3 + c_1x + c_2}{\mathrm{e}^{-x}}$$

Or

$$y = \frac{x^3\mathrm{e}^{x}}{3} + c_1x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3\mathrm{e}^{x}}{3} + c_1x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x} \tag{1}$$

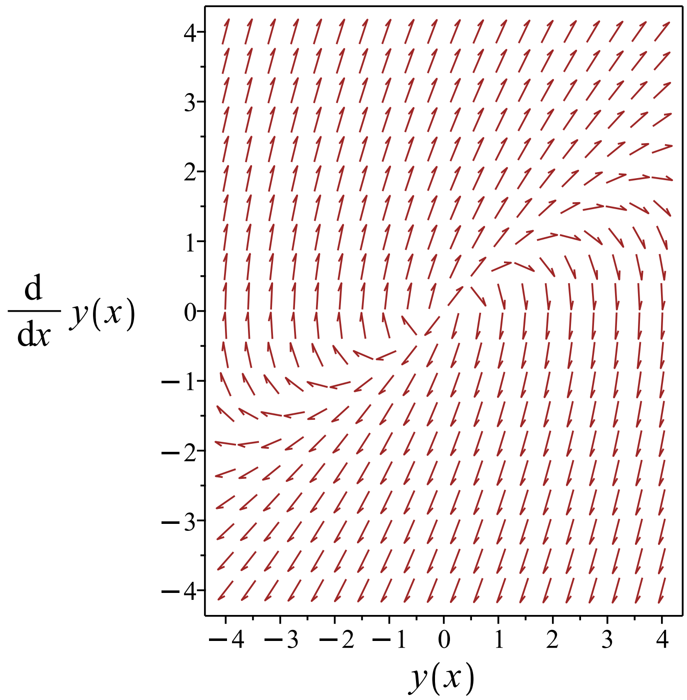


Figure 77: Slope field plot

Verification of solutions

$$y = \frac{x^3 e^x}{3} + c_1 x\, e^x + c_2 e^x$$

Verified OK.

2.14.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 67: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\, dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^x + c_2 x\,\mathrm{e}^x$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2x\,\mathrm{e}^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^x, \mathrm{e}^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^x, \mathrm{e}^x\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^x, x^2\mathrm{e}^x\}]$$

Since $x\,\mathrm{e}^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\mathrm{e}^x, x^3\mathrm{e}^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^x + A_2 x^3 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x + 6A_2 x\, e^x = 2x\, e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^3 e^x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 x\, e^x) + \left(\frac{x^3 e^x}{3}\right) \end{aligned}$$

Which simplifies to

$$y = e^x (c_2 x + c_1) + \frac{x^3 e^x}{3}$$

Summary

The solution(s) found are the following

$$y = e^x (c_2 x + c_1) + \frac{x^3 e^x}{3} \tag{1}$$

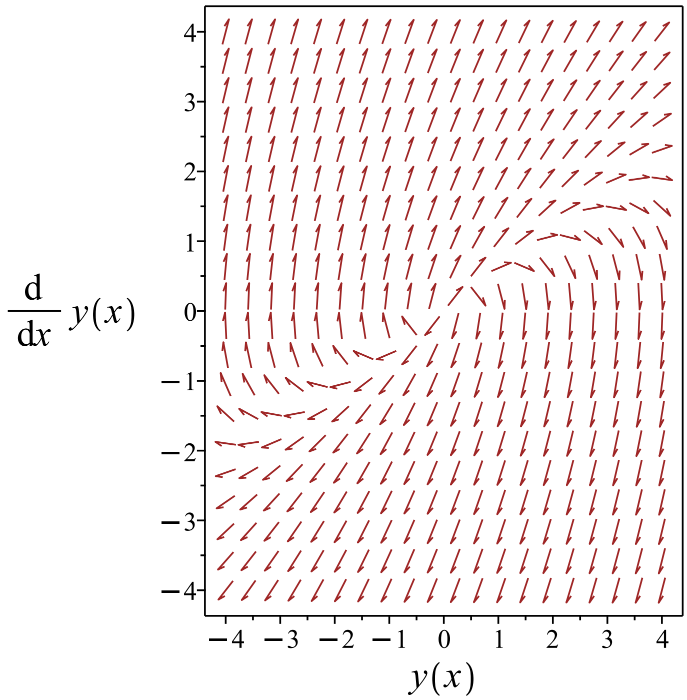


Figure 78: Slope field plot

Verification of solutions

$$y = e^x(c_2x + c_1) + \frac{x^3 e^x}{3}$$

Verified OK.

2.14.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = 2x\,e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

$r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^x$

- General solution of the ODE

  $y = c_1y_1(x) + c_2y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\mathrm{e}^x + c_2x\,\mathrm{e}^x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2x\,\mathrm{e}^x\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^x & x\,\mathrm{e}^x \\ \mathrm{e}^x & x\,\mathrm{e}^x + \mathrm{e}^x \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\,\mathrm{e}^x\left(\int x^2dx - \left(\int xdx\right)x\right)$

  - Compute integrals

    $y_p(x) = \frac{x^3\mathrm{e}^x}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_2x\,\mathrm{e}^x + c_1\mathrm{e}^x + \frac{x^3\mathrm{e}^x}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=2*x*exp(x),y(x), singsol=all)
```

$$y(x) = e^{x}\left(c_2 + c_1 x + \frac{1}{3}x^3\right)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 25

```
DSolve[y''[x]-2*y'[x]+y[x]==2*x*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}e^{x}\left(x^3 + 3c_2 x + 3c_1\right)$$

2.15 problem Problem 15.33

Internal problem ID [2527]
Internal file name [`OUTPUT/2019_Sunday_June_05_2022_02_44_52_AM_32039854/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.33.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _nonlinear]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying 3rd order ODE linearizable_by_differentiation
differential order: 3; trying a linearization to 4th order
trying differential order: 3; missing variables
trying differential order: 3; exact nonlinear
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))^2+(diff(_b(_a), _a))*_b(_a)+(diff(diff(
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying 2nd order Liouville
   trying 2nd order WeierstrassP
   trying 2nd order JacobiSN
   differential order: 2; trying a linearization to 3rd order
   trying 2nd order ODE linearizable_by_differentiation
   trying 2nd order, 2 integrating factors of the form mu(x,y)
   trying a quadrature
   checking if the LODE has constant coefficients
   <- constant coefficients successful
   <- 2nd order, 2 integrating factors of the form mu(x,y) successful
<- differential order: 3; exact nonlinear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 81

```
dsolve(2*y(x)*diff(y(x),x$3)+2*(y(x)+3*diff(y(x),x))*diff(y(x),x$2)+2*(diff(y(x),x))^2=sin(x
```

$$y(x) = -\frac{\sqrt{2}\,\sqrt{-4\left(\left(-\frac{\cos(x)}{4}+\frac{\sin(x)}{4}+c_1(x-1)+c_3\right)\mathrm{e}^{x}-c_2\right)\mathrm{e}^{x}\,\mathrm{e}^{-x}}}{2}$$

$$y(x) = \frac{\sqrt{2}\,\sqrt{-4\left(\left(-\frac{\cos(x)}{4}+\frac{\sin(x)}{4}+c_1(x-1)+c_3\right)\mathrm{e}^{x}-c_2\right)\mathrm{e}^{x}\,\mathrm{e}^{-x}}}{2}$$

✓ Solution by Mathematica

Time used: 0.473 (sec). Leaf size: 88

```
DSolve[2*y[x]*y'''[x]+2*(y[x]+3*y'[x])*y''[x]+2*(y'[x])^2==Sin[x],y[x],x,IncludeSingularSolu
```

$$y(x) \to -\frac{\sqrt{-\sin(x) + \cos(x) + 2c_1x + 2c_3e^{-x} - 2c_1 - 4c_2}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{-\sin(x) + \cos(x) + 2c_1x + 2c_3e^{-x} - 2c_1 - 4c_2}}{\sqrt{2}}$$

2.16 problem Problem 15.34

Internal problem ID [2528]
Internal file name [`OUTPUT/2020_Sunday_June_05_2022_02_44_54_AM_88138497/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.34.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

`[[_3rd_order, _missing_y]]`

$$\boxed{xy''' + 2y'' = Ax}$$

Since y is missing from the ode then we can use the substitution $y' = v(x)$ to reduce the order by one. The ODE becomes

$$xv''(x) + 2v'(x) = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (xv''(x) + 2v'(x))\, dx = 0$$
$$v'(x)\, x + v(x) = c_1$$

Which is now solved for $v(x)$. In canonical form the ODE is

$$\begin{aligned} v' &= F(x, v) \\ &= f(x)g(v) \\ &= \frac{-v + c_1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(v) = -v + c_1$. Integrating both sides gives

$$\frac{1}{-v + c_1} dv = \frac{1}{x} dx$$
$$\int \frac{1}{-v + c_1} dv = \int \frac{1}{x} dx$$
$$-\ln(-v + c_1) = \ln(x) + c_2$$

Raising both side to exponential gives

$$\frac{1}{-v + c_1} = e^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{1}{-v + c_1} = c_3 x$$

Which simplifies to

$$v(x) = \frac{(c_3 e^{c_2} x c_1 - 1) e^{-c_2}}{c_3 x}$$

But since $y' = v(x)$ then we now need to solve the ode $y' = \frac{(c_3 e^{c_2} x c_1 - 1)e^{-c_2}}{c_3 x}$. Integrating both sides gives

$$\begin{aligned} y &= \int \frac{(c_3 e^{c_2} x c_1 - 1) e^{-c_2}}{c_3 x} \mathrm{d}x \\ &= c_1 x - \frac{e^{-c_2} \ln(x)}{c_3} + c_4 \end{aligned}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$xy''' + 2y'' = 0$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} 1 & x & \ln(x) \\ 0 & 1 & \frac{1}{x} \\ 0 & 0 & -\frac{1}{x^2} \end{bmatrix}$$
$$|W| = -\frac{1}{x^2}$$

The determinant simplifies to

$$|W| = -\frac{1}{x^2}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} x & \ln(x) \\ 1 & \frac{1}{x} \end{bmatrix} \\ &= 1 - \ln(x) \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} 1 & \ln(x) \\ 0 & \frac{1}{x} \end{bmatrix} \\ &= \frac{1}{x} \end{aligned}$$

$$W_3(x) = \det \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$
$$= 1$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{(Ax)\,(1-\ln(x))}{(x)\left(-\frac{1}{x^2}\right)}\,dx \\
&= \int \frac{Ax(1-\ln(x))}{-\frac{1}{x}}\,dx \\
&= \int \left(A\,x^2(\ln(x)-1)\right)\,dx \\
&= \frac{A\,x^3\ln(x)}{3} - \frac{4A\,x^3}{9} \\
&= \frac{A\,x^3\ln(x)}{3} - \frac{4A\,x^3}{9}
\end{aligned}$$

$$\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(Ax)\left(\frac{1}{x}\right)}{(x)\left(-\frac{1}{x^2}\right)}\,dx \\
&= -\int \frac{A}{-\frac{1}{x}}\,dx \\
&= -\int (-Ax)\,dx \\
&= \frac{x^2A}{2}
\end{aligned}$$

$$\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(Ax)\,(1)}{(x)\left(-\frac{1}{x^2}\right)}\,dx \\
&= \int \frac{Ax}{-\frac{1}{x}}\,dx \\
&= \int \left(-x^2A\right)\,dx \\
&= -\frac{A\,x^3}{3}
\end{aligned}$$

Now that all the U_i functions have been determined, the particular solution is found from

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Hence

$$\begin{aligned} y_p = &\left(\frac{A\,x^3 \ln\left(x\right)}{3} - \frac{4A\,x^3}{9}\right) \\ &+ \left(\frac{x^2 A}{2}\right)(x) \\ &+ \left(-\frac{A\,x^3}{3}\right)(\ln\left(x\right)) \end{aligned}$$

Therefore the particular solution is

$$y_p = \frac{A\,x^3}{18}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \bigg(y \\ &= c_1 x - \frac{\mathrm{e}^{-c_2} \ln\left(x\right)}{c_3} + c_4\bigg) + \left(\frac{A\,x^3}{18}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x - \frac{\mathrm{e}^{-c_2} \ln\left(x\right)}{c_3} + c_4 + \frac{A\,x^3}{18} \tag{1}$$

Verification of solutions

$$y = c_1 x - \frac{\mathrm{e}^{-c_2} \ln\left(x\right)}{c_3} + c_4 + \frac{A\,x^3}{18}$$

Verified OK.

2.16.1 Maple step by step solution

Let's solve

$xy''' + 2y'' = Ax$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (A*_a-2*_b(_a))/_a, _b(_a)`   *** Suble
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x*diff(y(x),x$3)+2*diff(y(x),x$2)=A*x,y(x), singsol=all)
```

$$y(x) = \frac{A\,x^3}{18} - \ln(x)\,c_1 + c_2 x + c_3$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 26

```
DSolve[x*y'''[x]+2*y''[x]==A*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{Ax^3}{18} + c_3 x - c_1 \log(x) + c_2$$

2.17 problem Problem 15.35

Internal problem ID [2529]
Internal file name [`OUTPUT/2021_Sunday_June_05_2022_02_44_56_AM_61700375/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 15, Higher order ordinary differential equations. 15.4 Exercises, page 523
Problem number: Problem 15.35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4xy' + (4x^2 + 6)\, y = e^{-x^2} \sin(2x)}$$

2.17.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4xy' + (4x^2 + 6)\, y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= 4x \\
q(x) &= 4x^2 + 6
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= 4x^2 + 6 - \frac{(4x)'}{2} - \frac{(4x)^2}{4} \\
&= 4x^2 + 6 - \frac{(4)}{2} - \frac{(16x^2)}{4} \\
&= 4x^2 + 6 - (2) - 4x^2 \\
&= 4
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{4x}{2}} \\
&= \mathrm{e}^{-x^2}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, \mathrm{e}^{-x^2} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
4v(x) + v''(x) = \sin(2x)
$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = f(x)
$$

Where $A = 1, B = 0, C = 4, f(x) = \sin(2x)$. Let the solution be

$$
v(x) = v_h + v_p
$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x)=0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x)=f(x)$. v_h is the solution to

$$4v(x)+v''(x)=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x)+Bv'(x)+Cv(x)=0$$

Where in the above $A=1, B=0, C=4$. Let the solution be $v(x)=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x}+4\,e^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+4=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=0, C=4$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{0}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{0^2-(4)\,(1)\,(4)}\\&=\pm 2i\end{aligned}$$

Hence

$$\lambda_1=+2i$$
$$\lambda_2=-2i$$

Which simplifies to

$$\lambda_1=2i$$
$$\lambda_2=-2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2}=\alpha\pm i\beta$$

Where $\alpha=0$ and $\beta=2$. Therefore the final solution, when using Euler relation, can be written as

$$v(x)=e^{\alpha x}(c_1\cos(\beta x)+c_2\sin(\beta x))$$

Which becomes

$$v(x) = e^{0}(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$v(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(2x), x\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$v_p = A_1 x \cos(2x) + A_2 x \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 \sin(2x) + 4A_2 \cos(2x) = \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = 0\right]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = -\frac{x \cos(2x)}{4}$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(-\frac{x \cos(2x)}{4}\right) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 \cos(2x) + c_2 \sin(2x) - \frac{x \cos(2x)}{4}\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \mathrm{e}^{-x^2}$$

Hence (7) becomes

$$y = \left(c_1 \cos(2x) + c_2 \sin(2x) - \frac{x \cos(2x)}{4}\right) \mathrm{e}^{-x^2}$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 \cos(2x) + c_2 \sin(2x) - \frac{x \cos(2x)}{4}\right) \mathrm{e}^{-x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(2x)\, \mathrm{e}^{-x^2} \\ y_2 &= \mathrm{e}^{-x^2} \sin(2x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x)\,\mathrm{e}^{-x^2} & \mathrm{e}^{-x^2}\sin(2x) \\ \frac{d}{dx}\left(\cos(2x)\,\mathrm{e}^{-x^2}\right) & \frac{d}{dx}\left(\mathrm{e}^{-x^2}\sin(2x)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x)\,\mathrm{e}^{-x^2} & \mathrm{e}^{-x^2}\sin(2x) \\ -2\,\mathrm{e}^{-x^2}\sin(2x) - 2\cos(2x)\,x\,\mathrm{e}^{-x^2} & -2x\,\mathrm{e}^{-x^2}\sin(2x) + 2\cos(2x)\,\mathrm{e}^{-x^2} \end{vmatrix}$$

Therefore

$$W = \left(\cos(2x)\,\mathrm{e}^{-x^2}\right)\left(-2x\,\mathrm{e}^{-x^2}\sin(2x) + 2\cos(2x)\,\mathrm{e}^{-x^2}\right) - \left(\mathrm{e}^{-x^2}\sin(2x)\right)\left(-2\,\mathrm{e}^{-x^2}\sin(2x) - 2\cos(2x)\,x\,\mathrm{e}^{-x^2}\right)$$

Which simplifies to

$$W = 2\,\mathrm{e}^{-2x^2}\sin(2x)^2 + 2\,\mathrm{e}^{-2x^2}\cos(2x)^2$$

Which simplifies to

$$W = 2\,\mathrm{e}^{-2x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{-2x^2}\sin(2x)^2}{2\,\mathrm{e}^{-2x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(2x)^2}{2} dx$$

Hence

$$u_1 = \frac{\sin(2x)\cos(2x)}{8} - \frac{x}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\,e^{-2x^2}\sin(2x)}{2\,e^{-2x^2}} dx$$

Which simplifies to

$$u_2 = \int \frac{\sin(4x)}{4} dx$$

Hence

$$u_2 = -\frac{\cos(4x)}{16}$$

Which simplifies to

$$u_1 = \frac{\sin(4x)}{16} - \frac{x}{4}$$

$$u_2 = -\frac{\cos(4x)}{16}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\sin(4x)}{16} - \frac{x}{4}\right)\cos(2x)\,e^{-x^2} - \frac{\cos(4x)\,e^{-x^2}\sin(2x)}{16}$$

Which simplifies to

$$y_p(x) = \frac{e^{-x^2}(\sin(2x) - 4x\cos(2x))}{16}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1\cos(2x) + c_2\sin(2x) - \frac{x\cos(2x)}{4}\right)e^{-x^2}\right) + \left(\frac{e^{-x^2}(\sin(2x) - 4x\cos(2x))}{16}\right) \end{aligned}$$

Which simplifies to

$$y = -\frac{\left(\left(x-4c_1\right)\cos\left(2x\right)-4c_2\sin\left(2x\right)\right)\mathrm{e}^{-x^2}}{4}+\frac{\mathrm{e}^{-x^2}\left(\sin\left(2x\right)-4x\cos\left(2x\right)\right)}{16}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(\left(x-4c_1\right)\cos\left(2x\right)-4c_2\sin\left(2x\right)\right)\mathrm{e}^{-x^2}}{4}+\frac{\mathrm{e}^{-x^2}\left(\sin\left(2x\right)-4x\cos\left(2x\right)\right)}{16} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\left(x-4c_1\right)\cos\left(2x\right)-4c_2\sin\left(2x\right)\right)\mathrm{e}^{-x^2}}{4}+\frac{\mathrm{e}^{-x^2}\left(\sin\left(2x\right)-4x\cos\left(2x\right)\right)}{16}$$

Verified OK.

2.17.2 Solving using Kovacic algorithm

Writing the ode as

$$y''+4xy'+\left(4x^2+6\right)y = 0 \tag{1}$$
$$Ay''+By'+Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 70: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\
&= z_1 e^{-x^2} \\
&= z_1 \left(\mathrm{e}^{-x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)\,\mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos(2x)\, \mathrm{e}^{-x^2}\right) + c_2 \left(\cos(2x)\, \mathrm{e}^{-x^2} \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4xy' + \left(4x^2 + 6\right) y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{-x^2} \cos(2x)\, c_1 + \frac{\mathrm{e}^{-x^2} \sin(2x)\, c_2}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \cos(2x)\, \mathrm{e}^{-x^2} \\
y_2 &= \frac{\mathrm{e}^{-x^2} \sin(2x)}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x)\,\mathrm{e}^{-x^2} & \frac{\mathrm{e}^{-x^2}\sin(2x)}{2} \\ \frac{d}{dx}\left(\cos(2x)\,\mathrm{e}^{-x^2}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{-x^2}\sin(2x)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x)\,\mathrm{e}^{-x^2} & \frac{\mathrm{e}^{-x^2}\sin(2x)}{2} \\ -2\,\mathrm{e}^{-x^2}\sin(2x) - 2\cos(2x)\,x\,\mathrm{e}^{-x^2} & -x\,\mathrm{e}^{-x^2}\sin(2x) + \cos(2x)\,\mathrm{e}^{-x^2} \end{vmatrix}$$

Therefore

$$W = \left(\cos(2x)\,\mathrm{e}^{-x^2}\right)\left(-x\,\mathrm{e}^{-x^2}\sin(2x) + \cos(2x)\,\mathrm{e}^{-x^2}\right) - \left(\frac{\mathrm{e}^{-x^2}\sin(2x)}{2}\right)\left(-2\,\mathrm{e}^{-x^2}\sin(2x) - 2\cos(2x)\,x\,\mathrm{e}^{-x^2}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-2x^2}\sin(2x)^2 + \mathrm{e}^{-2x^2}\cos(2x)^2$$

Which simplifies to

$$W = \mathrm{e}^{-2x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{-2x^2}\sin(2x)^2}{2}}{\mathrm{e}^{-2x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(2x)^2}{2}\,dx$$

Hence

$$u_1 = \frac{\sin(2x)\cos(2x)}{8} - \frac{x}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\,\mathrm{e}^{-2x^2}\sin(2x)}{\mathrm{e}^{-2x^2}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sin(4x)}{2}dx$$

Hence

$$u_2 = -\frac{\cos(4x)}{8}$$

Which simplifies to

$$u_1 = \frac{\sin(4x)}{16} - \frac{x}{4}$$
$$u_2 = -\frac{\cos(4x)}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\sin(4x)}{16} - \frac{x}{4}\right)\cos(2x)\,\mathrm{e}^{-x^2} - \frac{\cos(4x)\,\mathrm{e}^{-x^2}\sin(2x)}{16}$$

Which simplifies to

$$y_p(x) = \frac{\mathrm{e}^{-x^2}(\sin(2x) - 4x\cos(2x))}{16}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-x^2}\cos(2x)\,c_1 + \frac{\mathrm{e}^{-x^2}\sin(2x)\,c_2}{2}\right) + \left(\frac{\mathrm{e}^{-x^2}(\sin(2x) - 4x\cos(2x))}{16}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{\mathrm{e}^{-x^2}(c_2\sin(2x) + 2c_1\cos(2x))}{2} + \frac{\mathrm{e}^{-x^2}(\sin(2x) - 4x\cos(2x))}{16}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-x^2}(c_2 \sin(2x) + 2c_1 \cos(2x))}{2} + \frac{e^{-x^2}(\sin(2x) - 4x \cos(2x))}{16} \tag{1}$$

Verification of solutions

$$y = \frac{e^{-x^2}(c_2 \sin(2x) + 2c_1 \cos(2x))}{2} + \frac{e^{-x^2}(\sin(2x) - 4x \cos(2x))}{16}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2+6)*y(x)=exp(-x^2)*sin(2*x),y(x), singsol=all)
```

$$y(x) = -\frac{((x - 4c_2)\cos(2x) - 4\sin(2x)\,c_1)\,e^{-x^2}}{4}$$

✓ Solution by Mathematica

Time used: 0.13 (sec). Leaf size: 52

```
DSolve[y''[x]+4*x*y'[x]+(4*x^2+6)*y[x]==Exp[-x^2]*Sin[2*x],y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{1}{32}e^{-x(x+2i)}\left(-4x - e^{4ix}(4x + i + 8ic_2) + i + 32c_1\right)$$

3 Chapter 16, Series solutions of ODEs. Section 16.6 Exercises, page 550

3.1 problem Problem 16.1

Internal problem ID [2530]
Internal file name [OUTPUT/2022_Sunday_June_05_2022_02_44_59_AM_81078342/index.tex]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 16, Series solutions of ODEs. Section 16.6 Exercises, page 550
Problem number: Problem 16.1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[_Gegenbauer]
```

$$\boxed{(-z^2+1)\,y''-3zy'+\lambda y=0}$$

With the expansion point for the power series method at $z=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots\\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{115}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{116}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{3zy' - \lambda y}{z^2 - 1} \\
F_1 &= \frac{dF_0}{dz} \\
&= \frac{\partial F_0}{\partial z} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{((\lambda + 12) z^2 - \lambda + 3) y' - 5y\lambda z}{(z^2 - 1)^2} \\
F_2 &= \frac{dF_1}{dz} \\
&= \frac{\partial F_1}{\partial z} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-10z\left((\lambda + 6) z^2 - \lambda + \frac{9}{2}\right) y' + y\lambda((\lambda + 27) z^2 - \lambda + 8)}{(z^2 - 1)^3} \\
F_3 &= \frac{dF_2}{dz} \\
&= \frac{\partial F_2}{\partial z} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(z - 1)\left(((\lambda^2 + 87\lambda + 360) z^4 + (-2\lambda^2 - 69\lambda + 540) z^2 + \lambda^2 - 18\lambda + 45) y' - 14\left((\lambda + 12) z^2 - \lambda + \right.\right.}{(z^2 - 1)^5} \\
F_4 &= \frac{dF_3}{dz} \\
&= \frac{\partial F_3}{\partial z} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{21(z - 1)\left(((\lambda^2 + 37\lambda + 120) z^4 + (-2\lambda^2 - 14\lambda + 300) z^2 + \lambda^2 - 23\lambda + 75) zy' - \frac{y\lambda((\lambda^2 + 157\lambda + 1200)}{}\right.}{(z^2 - 1)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $z = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \lambda \\
F_1 &= -y'(0) \lambda + 3y'(0) \\
F_2 &= y(0) \lambda^2 - 8y(0) \lambda \\
F_3 &= y'(0) \lambda^2 - 18y'(0) \lambda + 45y'(0) \\
F_4 &= -y(0) \lambda^3 + 32y(0) \lambda^2 - 192y(0) \lambda
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1-\frac{1}{2}\lambda\,z^{2}+\frac{1}{24}\lambda^{2} z^{4}-\frac{1}{3}\lambda\,z^{4}-\frac{1}{720}z^{6} \lambda^{3}+\frac{2}{45}z^{6} \lambda^{2}-\frac{4}{15}z^{6} \lambda \right) y(0)$$
$$+\left(z-\frac{1}{6}z^{3} \lambda +\frac{1}{2}z^{3}+\frac{1}{120}\lambda^{2} z^{5}-\frac{3}{20}\lambda\,z^{5}+\frac{3}{8}z^{5}\right) y^{\prime}(0)+O\left(z^{6}\right)$$

Since the expansion point $z = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(-z^{2}+1\right) y''-3zy'+\lambda y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_{n} z^{n}$$

Then

$$y' = \sum_{n=1}^{\infty} n a_{n} z^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_{n} z^{n-2}$$

Substituting the above back into the ode gives

$$\left(-z^{2}+1\right) \left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} z^{n-2}\right)-3z\left(\sum_{n=1}^{\infty} n a_{n} z^{n-1}\right)+\lambda \left(\sum_{n=0}^{\infty} a_{n} z^{n}\right) = 0 \quad (1)$$

Which simplifies to

$$\sum_{n =2}^{\infty}\left(-z^{n} a_{n} n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} z^{n-2}\right)+\sum_{n =1}^{\infty}\left(-3n a_{n} z^{n}\right)+\left(\sum_{n=0}^{\infty} \lambda a_{n} z^{n}\right) = 0 \quad (2)$$

The next step is to make all powers of z be n in each summation term. Going over each summation term above with power of z in it which is not already z^{n} and adjusting the power and the corresponding index gives

$$\sum_{n =2}^{\infty} n(n-1)\, a_{n} z^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, z^{n}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of z are the same and equal to n.

$$\sum_{n=2}^{\infty} \left(-z^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, z^n\right) + \sum_{n=1}^{\infty} \left(-3n a_n z^n\right) + \left(\sum_{n=0}^{\infty} \lambda a_n z^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$\lambda a_0 + 2a_2 = 0$$

$$a_2 = -\frac{\lambda a_0}{2}$$

$n = 1$ gives

$$\lambda a_1 - 3a_1 + 6a_3 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{1}{6}\lambda a_1 + \frac{1}{2}a_1$$

For $2 \leq n$, the recurrence equation is

$$-n a_n(n-1) + (n+2)\, a_{n+2}(n+1) - 3n a_n + \lambda a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(-n^2 + \lambda - 2n)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$\lambda a_2 - 8a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{24}\lambda^2 a_0 - \frac{1}{3}\lambda a_0$$

For $n = 3$ the recurrence equation gives

$$\lambda a_3 - 15a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{120}\lambda^2 a_1 - \frac{3}{20}\lambda a_1 + \frac{3}{8}a_1$$

For $n = 4$ the recurrence equation gives

$$\lambda a_4 - 24a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{1}{720}\lambda^3 a_0 + \frac{2}{45}\lambda^2 a_0 - \frac{4}{15}\lambda a_0$$

For $n = 5$ the recurrence equation gives

$$\lambda a_5 - 35a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{1}{5040}\lambda^3 a_1 + \frac{53}{5040}\lambda^2 a_1 - \frac{15}{112}\lambda a_1 + \frac{5}{16}a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n z^n \\ &= a_3 z^3 + a_2 z^2 + a_1 z + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 z - \frac{\lambda a_0 z^2}{2} + \left(-\frac{1}{6}\lambda a_1 + \frac{1}{2}a_1\right) z^3 \\ + \left(\frac{1}{24}\lambda^2 a_0 - \frac{1}{3}\lambda a_0\right) z^4 + \left(\frac{1}{120}\lambda^2 a_1 - \frac{3}{20}\lambda a_1 + \frac{3}{8}a_1\right) z^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y = \left(1-\frac{\lambda z^2}{2}+\left(\frac{1}{24}\lambda^2-\frac{1}{3}\lambda\right) z^4\right) a_0 + \left(z+\left(-\frac{\lambda}{6}+\frac{1}{2}\right) z^3+\left(\frac{1}{120}\lambda^2-\frac{3}{20}\lambda+\frac{3}{8}\right) z^5\right) a_1 + O(z^6) \tag{3}$$

At $z=0$ the solution above becomes

$$y = \left(1-\frac{\lambda z^2}{2}+\left(\frac{1}{24}\lambda^2-\frac{1}{3}\lambda\right) z^4\right) c_1 + \left(z+\left(-\frac{\lambda}{6}+\frac{1}{2}\right) z^3+\left(\frac{1}{120}\lambda^2-\frac{3}{20}\lambda+\frac{3}{8}\right) z^5\right) c_2 + O(z^6)$$

Summary

The solution(s) found are the following

$$y = \left(1-\frac{1}{2}\lambda z^2+\frac{1}{24}\lambda^2 z^4-\frac{1}{3}\lambda z^4-\frac{1}{720}z^6\lambda^3+\frac{2}{45}z^6\lambda^2-\frac{4}{15}z^6\lambda\right) y(0) + \left(z-\frac{1}{6}z^3\lambda+\frac{1}{2}z^3+\frac{1}{120}\lambda^2 z^5-\frac{3}{20}\lambda z^5+\frac{3}{8}z^5\right) y'(0) + O(z^6) \tag{1}$$

$$y = \left(1-\frac{\lambda z^2}{2}+\left(\frac{1}{24}\lambda^2-\frac{1}{3}\lambda\right) z^4\right) c_1 + \left(z+\left(-\frac{\lambda}{6}+\frac{1}{2}\right) z^3+\left(\frac{1}{120}\lambda^2-\frac{3}{20}\lambda+\frac{3}{8}\right) z^5\right) c_2 + O(z^6) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{1}{2}\lambda z^2+\frac{1}{24}\lambda^2 z^4-\frac{1}{3}\lambda z^4-\frac{1}{720}z^6\lambda^3+\frac{2}{45}z^6\lambda^2-\frac{4}{15}z^6\lambda\right) y(0) + \left(z-\frac{1}{6}z^3\lambda+\frac{1}{2}z^3+\frac{1}{120}\lambda^2 z^5-\frac{3}{20}\lambda z^5+\frac{3}{8}z^5\right) y'(0) + O(z^6)$$

Verified OK.

$$y = \left(1-\frac{\lambda z^2}{2}+\left(\frac{1}{24}\lambda^2-\frac{1}{3}\lambda\right) z^4\right) c_1 + \left(z+\left(-\frac{\lambda}{6}+\frac{1}{2}\right) z^3+\left(\frac{1}{120}\lambda^2-\frac{3}{20}\lambda+\frac{3}{8}\right) z^5\right) c_2 + O(z^6)$$

Verified OK.

3.1.1 Maple step by step solution

Let's solve

$(-z^2+1)\,y''-3zy'+\lambda y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3zy'}{z^2-1}+\frac{\lambda y}{z^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3zy'}{z^2-1}-\frac{\lambda y}{z^2-1}=0$

□ Check to see if z_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(z)=\frac{3z}{z^2-1}, P_3(z)=-\frac{\lambda}{z^2-1}\right]$

  ◦ $(z+1)\cdot P_2(z)$ is analytic at $z=-1$

  $\left((z+1)\cdot P_2(z)\right)\Big|_{z=-1}=\frac{3}{2}$

  ◦ $(z+1)^2\cdot P_3(z)$ is analytic at $z=-1$

  $\left((z+1)^2\cdot P_3(z)\right)\Big|_{z=-1}=0$

  ◦ $z=-1$is a regular singular point

  Check to see if z_0 is a regular singular point

  $z_0=-1$

- Multiply by denominators

  $y''(z^2-1)+3zy'-\lambda y=0$

- Change variables using $z=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(3u-3)\left(\frac{d}{du}y(u)\right)-\lambda y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} (-a_{k+1}(k+1+r)\,(2k+3+2r) + a_k(k^2+2kr+r^2+2k-\lambda+2r))\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2(k+1+r)\left(k+\frac{3}{2}+r\right) a_{k+1} + \left(k^2+(2r+2)\,k+r^2+2r-\lambda\right) a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(k^2+2kr+r^2+2k-\lambda+2r)\,a_k}{(k+1+r)(2k+3+2r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{(k^2+2k-\lambda)\,a_k}{(k+1)(2k+3)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{(k^2+2k-\lambda)\,a_k}{(k+1)(2k+3)}\right]$$

- Revert the change of variables $u = z+1$

$$\left[y = \sum_{k=0}^{\infty} a_k(z+1)^k, a_{k+1} = \frac{(k^2+2k-\lambda)\,a_k}{(k+1)(2k+3)}\right]$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = \frac{\left(k^2+k-\lambda-\frac{3}{4}\right)a_k}{\left(k+\frac{1}{2}\right)(2k+2)}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1} = \frac{\left(k^2+k-\lambda-\frac{3}{4}\right)a_k}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

- Revert the change of variables $u = z + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (z+1)^{k-\frac{1}{2}}, a_{k+1} = \frac{\left(k^2+k-\lambda-\frac{3}{4}\right)a_k}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (z+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (z+1)^{k-\frac{1}{2}}\right), a_{k+1} = \frac{(k^2+2k-\lambda)a_k}{(k+1)(2k+3)}, b_{k+1} = \frac{\left(k^2+k-\lambda-\frac{3}{4}\right)b_k}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 63

```
Order:=6;
dsolve((1-z^2)*diff(y(z),z$2)-3*z*diff(y(z),z)+lambda*y(z)=0,y(z),type='series',z=0);
```

$$y(z) = \left(1 - \frac{\lambda z^2}{2} + \frac{\lambda(\lambda-8) z^4}{24}\right) y(0) + \left(z - \frac{(\lambda-3) z^3}{6} + \frac{(\lambda-3)(\lambda-15) z^5}{120}\right) D(y)(0) + O(z^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 80

```
AsymptoticDSolveValue[(1-z^2)*y''[z]-3*z*y'[z]+\[Lambda]*y[z]==0,y[z],{z,0,5}]
```

$$y(z) \to c_2 \left(\frac{\lambda^2 z^5}{120} - \frac{3\lambda z^5}{20} + \frac{3z^5}{8} - \frac{\lambda z^3}{6} + \frac{z^3}{2} + z \right) + c_1 \left(\frac{\lambda^2 z^4}{24} - \frac{\lambda z^4}{3} - \frac{\lambda z^2}{2} + 1 \right)$$

3.2 problem Problem 16.2

Internal problem ID [2531]
Internal file name [`OUTPUT/2023_Sunday_June_05_2022_02_45_01_AM_49575234/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 16, Series solutions of ODEs. Section 16.6 Exercises, page 550
Problem number: Problem 16.2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4zy'' + 2(1 - z)\, y' - y = 0}$$

With the expansion point for the power series method at $z = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4zy'' + (-2z + 2)\, y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(z)y' + q(z)y = 0$$

Where

$$p(z) = -\frac{z - 1}{2z}$$
$$q(z) = -\frac{1}{4z}$$

Table 72: Table $p(z), q(z)$ singularites.

$p(z) = -\frac{z-1}{2z}$	
singularity	type
$z = 0$	"regular"

$q(z) = -\frac{1}{4z}$	
singularity	type
$z = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $z = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4zy'' + (-2z + 2)\, y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n z^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n z^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n z^{n+r-2}$$

Substituting the above back into the ode gives

$$4z \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n z^{n+r-2}\right) + (-2z + 2) \left(\sum_{n=0}^{\infty} (n + r)\, a_n z^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n z^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4z^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2z^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n z^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-a_n z^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of z be $n+r-1$ in each summation term. Going over each summation term above with power of z in it which is not already z^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2z^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1) z^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-a_n z^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} z^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of z are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 4z^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1) z^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n z^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} z^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4z^{n+r-1} a_n (n+r)(n+r-1) + 2(n+r) a_n z^{n+r-1} = 0$$

When $n=0$ the above becomes

$$4z^{-1+r} a_0 r(-1+r) + 2r a_0 z^{-1+r} = 0$$

Or

$$\left(4z^{-1+r} r(-1+r) + 2r\, z^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 2r\right) z^{-1+r} = 0$$

Since the above is true for all z then the indicial equation becomes

$$4r^2 - 2r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 2r\right) z^{-1+r} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(z) = z^{r_1} \left(\sum_{n=0}^{\infty} a_n z^n\right)$$
$$y_2(z) = z^{r_2} \left(\sum_{n=0}^{\infty} b_n z^n\right)$$

Or

$$y_1(z) = \sum_{n=0}^{\infty} a_n z^{n+\frac{1}{2}}$$
$$y_2(z) = \sum_{n=0}^{\infty} b_n z^n$$

We start by finding $y_1(z)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) + 2a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n+2r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}}{2n+1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2 + 2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+2r}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4 (1 + r) (2 + r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+2r}$	$\frac{1}{3}$
a_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{15}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8 (1 + r) (2 + r) (3 + r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{1}{105}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+2r}$	$\frac{1}{3}$
a_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{15}$
a_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{105}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16(1+r)(2+r)(3+r)(4+r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{945}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+2r}$	$\frac{1}{3}$
a_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{15}$
a_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{105}$
a_4	$\frac{1}{16(1+r)(2+r)(3+r)(4+r)}$	$\frac{1}{945}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32(1+r)(2+r)(3+r)(4+r)(5+r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{1}{10395}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+2r}$	$\frac{1}{3}$
a_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{15}$
a_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{105}$
a_4	$\frac{1}{16(1+r)(2+r)(3+r)(4+r)}$	$\frac{1}{945}$
a_5	$\frac{1}{32(1+r)(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{10395}$

Using the above table, then the solution $y_1(z)$ is

$$\begin{aligned} y_1(z) &= \sqrt{z}\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6 \dots\right) \\ &= \sqrt{z}\left(1 + \frac{z}{3} + \frac{z^2}{15} + \frac{z^3}{105} + \frac{z^4}{945} + \frac{z^5}{10395} + O(z^6)\right) \end{aligned}$$

Now the second solution $y_2(z)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$4b_n(n+r)(n+r-1) - 2b_{n-1}(n+r-1) + 2(n+r)b_n - b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}}{2n+2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-1}}{2n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{2+2r}$$

Which for the root $r = 0$ becomes

$$b_1 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+2r}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4(1+r)(2+r)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{8(1+r)(2+r)(3+r)}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{48}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{8}$
b_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{48}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16\left(1+r\right)\left(2+r\right)\left(3+r\right)\left(4+r\right)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{384}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{8}$
b_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{48}$
b_4	$\frac{1}{16(1+r)(2+r)(3+r)(4+r)}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{32\left(1+r\right)\left(2+r\right)\left(3+r\right)\left(4+r\right)\left(5+r\right)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{1}{3840}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{8}$
b_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{48}$
b_4	$\frac{1}{16(1+r)(2+r)(3+r)(4+r)}$	$\frac{1}{384}$
b_5	$\frac{1}{32(1+r)(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{3840}$

Using the above table, then the solution $y_2(z)$ is

$$\begin{aligned} y_2(z) &= b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + b_5 z^5 + b_6 z^6 \dots \\ &= 1 + \frac{z}{2} + \frac{z^2}{8} + \frac{z^3}{48} + \frac{z^4}{384} + \frac{z^5}{3840} + O(z^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(z) &= c_1 y_1(z) + c_2 y_2(z) \\ &= c_1 \sqrt{z}\left(1 + \frac{z}{3} + \frac{z^2}{15} + \frac{z^3}{105} + \frac{z^4}{945} + \frac{z^5}{10395} + O(z^6)\right) \\ &\quad + c_2\left(1 + \frac{z}{2} + \frac{z^2}{8} + \frac{z^3}{48} + \frac{z^4}{384} + \frac{z^5}{3840} + O(z^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \sqrt{z}\left(1 + \frac{z}{3} + \frac{z^2}{15} + \frac{z^3}{105} + \frac{z^4}{945} + \frac{z^5}{10395} + O(z^6)\right) \\ &\quad + c_2\left(1 + \frac{z}{2} + \frac{z^2}{8} + \frac{z^3}{48} + \frac{z^4}{384} + \frac{z^5}{3840} + O(z^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1 \sqrt{z}\left(1 + \frac{z}{3} + \frac{z^2}{15} + \frac{z^3}{105} + \frac{z^4}{945} + \frac{z^5}{10395} + O(z^6)\right) \\ + c_2\left(1 + \frac{z}{2} + \frac{z^2}{8} + \frac{z^3}{48} + \frac{z^4}{384} + \frac{z^5}{3840} + O(z^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{z}\left(1+\frac{z}{3}+\frac{z^2}{15}+\frac{z^3}{105}+\frac{z^4}{945}+\frac{z^5}{10395}+O(z^6)\right) + c_2\left(1+\frac{z}{2}+\frac{z^2}{8}+\frac{z^3}{48}+\frac{z^4}{384}+\frac{z^5}{3840}+O(z^6)\right)$$

Verified OK.

3.2.1 Maple step by step solution

Let's solve

$4zy''+(-2z+2)\,y'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{4z}+\frac{(z-1)y'}{2z}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(z-1)y'}{2z}-\frac{y}{4z}=0$

□ Check to see if $z_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(z)=-\frac{z-1}{2z}, P_3(z)=-\frac{1}{4z}\right]$

  - $z\cdot P_2(z)$ is analytic at $z=0$

    $(z\cdot P_2(z))\Big|_{z=0}=\frac{1}{2}$

  - $z^2\cdot P_3(z)$ is analytic at $z=0$

    $(z^2\cdot P_3(z))\Big|_{z=0}=0$

  - $z=0$is a regular singular point

    Check to see if $z_0=0$ is a regular singular point

    $z_0=0$

- Multiply by denominators

  $4zy''+(-2z+2)\,y'-y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k z^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $z^m \cdot y'$ to series expansion for $m=0..1$

    $z^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, z^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $z^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, z^{k+r}$
  - Convert $z \cdot y''$ to series expansion

    $z \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, z^{k+r-1}$
  - Shift index using $k->k+1$

    $z \cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, z^{k+r}$

  Rewrite ODE with series expansions

  $2a_0 r(-1+2r)\, z^{-1+r}+\left(\sum_{k=0}^{\infty}(2a_{k+1}(k+1+r)\,(2k+2r+1)-a_k(2k+2r+1))\, z^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-1+2r)=0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r+\frac{1}{2}\right)\left(a_{k+1}(k+1+r)-\frac{a_k}{2}\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{2(k+1+r)}$
- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{2(k+1)}$
- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k z^k, a_{k+1}=\frac{a_k}{2(k+1)}\right]$
- Recursion relation for $r=\frac{1}{2}$

$a_{k+1} = \frac{a_k}{2(k+\frac{3}{2})}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k z^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k}{2(k+\frac{3}{2})}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k z^k\right) + \left(\sum_{k=0}^{\infty} b_k z^{k+\frac{1}{2}}\right), a_{k+1} = \frac{a_k}{2(k+1)}, b_{k+1} = \frac{b_k}{2(k+\frac{3}{2})}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
Order:=6;
dsolve(4*z*diff(y(z),z$2)+2*(1-z)*diff(y(z),z)-y(z)=0,y(z),type='series',z=0);
```

$$y(z) = c_1\sqrt{z}\left(1+\frac{1}{3}z+\frac{1}{15}z^2+\frac{1}{105}z^3+\frac{1}{945}z^4+\frac{1}{10395}z^5+\mathrm{O}\left(z^6\right)\right) + c_2\left(1+\frac{1}{2}z+\frac{1}{8}z^2+\frac{1}{48}z^3+\frac{1}{384}z^4+\frac{1}{3840}z^5+\mathrm{O}\left(z^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 85

```
AsymptoticDSolveValue[4*z*y''[z]+2*(1-z)*y'[z]-y[z]==0,y[z],{z,0,5}]
```

$$y(z) \to c_1\sqrt{z}\left(\frac{z^5}{10395}+\frac{z^4}{945}+\frac{z^3}{105}+\frac{z^2}{15}+\frac{z}{3}+1\right)+c_2\left(\frac{z^5}{3840}+\frac{z^4}{384}+\frac{z^3}{48}+\frac{z^2}{8}+\frac{z}{2}+1\right)$$

3.3 problem Problem 16.3

Internal problem ID [2532]
Internal file name [`OUTPUT/2024_Sunday_June_05_2022_02_45_07_AM_63936993/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 16, Series solutions of ODEs. Section 16.6 Exercises, page 550
Problem number: Problem 16.3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{zy'' - 2y' + 9z^5y = 0}$$

With the expansion point for the power series method at $z = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$zy'' - 2y' + 9z^5y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(z)y' + q(z)y = 0$$

Where

$$p(z) = -\frac{2}{z}$$
$$q(z) = 9z^4$$

Table 74: Table $p(z), q(z)$ singularites.

$p(z) = -\frac{2}{z}$	
singularity	type
$z = 0$	"regular"

$q(z) = 9z^4$	
singularity	type
$z = \infty$	"regular"
$z = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $z = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$zy'' - 2y' + 9z^5y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n z^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n z^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n z^{n+r-2}$$

Substituting the above back into the ode gives

$$z\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n z^{n+r-2}\right) - 2\left(\sum_{n=0}^{\infty} (n+r)\, a_n z^{n+r-1}\right) + 9z^5\left(\sum_{n=0}^{\infty} a_n z^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} z^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2(n+r)\, a_n z^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 9z^{5+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of z be $n+r-1$ in each summation term. Going over each summation term above with power of z in it which is not already z^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 9z^{5+n+r} a_n = \sum_{n=6}^{\infty} 9a_{n-6} z^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of z are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} z^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2(n+r)\, a_n z^{n+r-1}\right) + \left(\sum_{n=6}^{\infty} 9a_{n-6} z^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$z^{n+r-1} a_n (n+r)\,(n+r-1) - 2(n+r)\, a_n z^{n+r-1} = 0$$

When $n=0$ the above becomes

$$z^{-1+r} a_0 r(-1+r) - 2r a_0 z^{-1+r} = 0$$

Or

$$\left(z^{-1+r} r(-1+r) - 2r\, z^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, z^{-1+r}(-3+r) = 0$$

Since the above is true for all z then the indicial equation becomes

$$r(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, z^{-1+r}(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(z) &= z^{r_1}\left(\sum_{n=0}^{\infty} a_n z^n\right) \\
y_2(z) &= C y_1(z) \ln(z) + z^{r_2}\left(\sum_{n=0}^{\infty} b_n z^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(z) &= z^{3}\left(\sum_{n=0}^{\infty} a_n z^n\right) \\
y_2(z) &= C y_1(z) \ln(z) + \left(\sum_{n=0}^{\infty} b_n z^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(z) &= \sum_{n=0}^{\infty} a_n z^{n+3} \\
y_2(z) &= C y_1(z) \ln(z) + \left(\sum_{n=0}^{\infty} b_n z^n\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

Substituting $n = 2$ in Eq. (2B) gives

$$
a_2 = 0
$$

Substituting $n = 3$ in Eq. (2B) gives

$$
a_3 = 0
$$

Substituting $n = 4$ in Eq. (2B) gives

$$
a_4 = 0
$$

Substituting $n = 5$ in Eq. (2B) gives

$$
a_5 = 0
$$

For $6 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 2a_n(n+r) + 9a_{n-6} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{9a_{n-6}}{n^2 + 2nr + r^2 - 3n - 3r} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{9a_{n-6}}{n(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{9}{r^2 + 9r + 18}$$

Which for the root $r = 3$ becomes

$$a_6 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0
a_6	$-\frac{9}{r^2+9r+18}$	$-\frac{1}{6}$

Using the above table, then the solution $y_1(z)$ is

$$
\begin{aligned}
y_1(z) &= z^3\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6 + a_7 z^7 \dots\right) \\
&= z^3\left(1 - \frac{z^6}{6} + O(z^7)\right)
\end{aligned}
$$

Now the second solution $y_2(z)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_3 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} 0 &= \lim_{r \to 0} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(z) &= \sum_{n=0}^{\infty} b_n z^{n+r} \\
&= \sum_{n=0}^{\infty} b_n z^n
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = 0
$$

Substituting $n = 2$ in Eq(3) gives

$$
b_2 = 0
$$

Substituting $n = 3$ in Eq(3) gives

$$
b_3 = 0
$$

Substituting $n = 4$ in Eq(3) gives

$$
b_4 = 0
$$

Substituting $n = 5$ in Eq(3) gives

$$b_5 = 0$$

For $6 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - 2(n+r) b_n + 9b_{n-6} = 0 \tag{4}$$

Which for for the root $r = 0$ becomes

$$b_n n(n-1) - 2nb_n + 9b_{n-6} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{9b_{n-6}}{n^2 + 2nr + r^2 - 3n - 3r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{9b_{n-6}}{n^2 - 3n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = -\frac{9}{r^2 + 9r + 18}$$

Which for the root $r = 0$ becomes

$$b_6 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0
b_6	$-\frac{9}{r^2+9r+18}$	$-\frac{1}{2}$

Using the above table, then the solution $y_2(z)$ is

$$
\begin{aligned}
y_2(z) &= b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + b_5 z^5 + b_6 z^6 + b_7 z^7 \dots \\
&= 1 - \frac{z^6}{2} + O(z^7)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(z) &= c_1 y_1(z) + c_2 y_2(z) \\
&= c_1 z^3 \left(1 - \frac{z^6}{6} + O(z^7)\right) + c_2 \left(1 - \frac{z^6}{2} + O(z^7)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 z^3 \left(1 - \frac{z^6}{6} + O(z^7)\right) + c_2 \left(1 - \frac{z^6}{2} + O(z^7)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 z^3 \left(1 - \frac{z^6}{6} + O(z^7)\right) + c_2 \left(1 - \frac{z^6}{2} + O(z^7)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 z^3 \left(1 - \frac{z^6}{6} + O(z^7)\right) + c_2 \left(1 - \frac{z^6}{2} + O(z^7)\right)
$$

Verified OK.

3.3.1 Maple step by step solution

Let's solve

$zy'' - 2y' + 9z^5y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y'}{z} - 9z^4y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{z} + 9z^4y = 0$

□ Check to see if $z_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(z) = -\frac{2}{z}, P_3(z) = 9z^4\right]$

  ◦ $z \cdot P_2(z)$ is analytic at $z = 0$

  $(z \cdot P_2(z))\bigg|_{z=0} = -2$

  ◦ $z^2 \cdot P_3(z)$ is analytic at $z = 0$

  $(z^2 \cdot P_3(z))\bigg|_{z=0} = 0$

  ◦ $z = 0$is a regular singular point

  Check to see if $z_0 = 0$ is a regular singular point

  $z_0 = 0$

- Multiply by denominators

  $zy'' - 2y' + 9z^5y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k z^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $z^5 \cdot y$ to series expansion

  $z^5 \cdot y = \sum_{k=0}^{\infty} a_k z^{k+r+5}$

  ◦ Shift index using $k- > k - 5$

$$z^5 \cdot y = \sum_{k=5}^{\infty} a_{k-5} z^{k+r}$$

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k (k+r) z^{k+r-1}$$

- Shift index using $k->k+1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r) z^{k+r}$$

- Convert $z \cdot y''$ to series expansion

  $$z \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) z^{k+r-1}$$

- Shift index using $k->k+1$

  $$z \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)(k+r) z^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-3+r) z^{-1+r} + a_1(1+r)(-2+r) z^r + a_2(2+r)(-1+r) z^{1+r} + a_3(3+r) r z^{2+r} + a_4(4+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 3\}$$

- The coefficients of each power of z must be 0

  $$[a_1(1+r)(-2+r) = 0, a_2(2+r)(-1+r) = 0, a_3(3+r) r = 0, a_4(4+r)(1+r) = 0, a_5(5+r)$$

- Solve for the dependent coefficient(s)

  $$\{a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)(k-2+r) + 9a_{k-5} = 0$$

- Shift index using $k->k+5$

  $$a_{k+6}(k+6+r)(k+3+r) + 9a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+6} = -\frac{9a_k}{(k+6+r)(k+3+r)}$$

- Recursion relation for $r = 0$

$a_{k+6} = -\frac{9a_k}{(k+6)(k+3)}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k z^k, a_{k+6} = -\frac{9a_k}{(k+6)(k+3)}, a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\right]$$

- Recursion relation for $r = 3$

$a_{k+6} = -\frac{9a_k}{(k+9)(k+6)}$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k z^{k+3}, a_{k+6} = -\frac{9a_k}{(k+9)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k z^k\right) + \left(\sum_{k=0}^{\infty} b_k z^{k+3}\right), a_{k+6} = -\frac{9a_k}{(k+6)(k+3)}, a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, b_{k+6}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
Order:=7;
dsolve(z*diff(y(z),z$2)-2*diff(y(z),z)+9*z^5*y(z)=0,y(z),type='series',z=0);
```

$$y(z) = c_1 z^3 \left(1 - \frac{1}{6} z^6 + O\left(z^7\right)\right) + c_2 \left(12 - 6z^6 + O\left(z^7\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 12

```
AsymptoticDSolveValue[z*y''[z]-2*y'[z]+9*z^5*y[z]==0,y[z],{z,0,6}]
```

$$y(z) \to c_2 z^3 + c_1$$

3.4 problem Problem 16.4

Internal problem ID [2533]
Internal file name [`OUTPUT/2025_Sunday_June_05_2022_02_45_10_AM_82650485/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 16, Series solutions of ODEs. Section 16.6 Exercises, page 550
Problem number: Problem 16.4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{f'' + 2(z-1) f' + 4f = 0}$$

With the expansion point for the power series method at $z = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{120}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{121}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -2f'z + 2f' - 4f \\
F_1 &= \frac{dF_0}{dz} \\
&= \frac{\partial F_0}{\partial z} + \frac{\partial F_0}{\partial f} f' + \frac{\partial F_0}{\partial f'} F_0 \\
&= \left(4z^2 - 8z - 2\right) f' + 8(z-1) f \\
F_2 &= \frac{dF_1}{dz} \\
&= \frac{\partial F_1}{\partial z} + \frac{\partial F_1}{\partial f} f' + \frac{\partial F_1}{\partial f'} F_1 \\
&= 4\left(-2z^3 + 6z^2 + z - 5\right) f' - 16f\left(z^2 - 2z - 1\right) \\
F_3 &= \frac{dF_2}{dz} \\
&= \frac{\partial F_2}{\partial z} + \frac{\partial F_2}{\partial f} f' + \frac{\partial F_2}{\partial f'} F_2 \\
&= \left(16z^4 - 64z^3 + 128z - 20\right) f' + 32(z-1)\left(z^2 - 2z - \frac{7}{2}\right) f \\
F_4 &= \frac{dF_3}{dz} \\
&= \frac{\partial F_3}{\partial z} + \frac{\partial F_3}{\partial f} f' + \frac{\partial F_3}{\partial f'} F_3 \\
&= \left(-32z^5 + 160z^4 - 32z^3 - 544z^2 + 248z + 200\right) f' - 64\left(z^4 - 4z^3 - \frac{3}{2}z^2 + 11z - \frac{1}{2}\right) f
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $z = 0$ and $f(0) = f(0)$ and $f'(0) = f'(0)$ gives

$$
\begin{aligned}
F_0 &= -4f(0) + 2f'(0) \\
F_1 &= -2f'(0) - 8f(0) \\
F_2 &= -20f'(0) + 16f(0) \\
F_3 &= -20f'(0) + 112f(0) \\
F_4 &= 200f'(0) + 32f(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
f = &\left(1 - 2z^2 - \frac{4}{3}z^3 + \frac{2}{3}z^4 + \frac{14}{15}z^5 + \frac{2}{45}z^6\right) f(0) \\
&+ \left(z + z^2 - \frac{1}{3}z^3 - \frac{5}{6}z^4 - \frac{1}{6}z^5 + \frac{5}{18}z^6\right) f'(0) + O\left(z^6\right)
\end{aligned}
$$

Since the expansion point $z = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$f = \sum_{n=0}^{\infty} a_n z^n$$

Then

$$f' = \sum_{n=1}^{\infty} n a_n z^{n-1}$$
$$f'' = \sum_{n=2}^{\infty} n(n-1)\, a_n z^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n z^{n-2} = -2\left(\sum_{n=1}^{\infty} n a_n z^{n-1}\right) z + 2\left(\sum_{n=1}^{\infty} n a_n z^{n-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n z^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n z^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n a_n z^n\right) + \sum_{n=1}^{\infty} \left(-2n a_n z^{n-1}\right) + \left(\sum_{n=0}^{\infty} 4 a_n z^n\right) = 0 \tag{2}$$

The next step is to make all powers of z be n in each summation term. Going over each summation term above with power of z in it which is not already z^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n z^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, z^n$$
$$\sum_{n=1}^{\infty} \left(-2n a_n z^{n-1}\right) = \sum_{n=0}^{\infty} \left(-2(n+1)\, a_{n+1} z^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of z are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, z^n\right) + \left(\sum_{n=1}^{\infty} 2n a_n z^n\right) + \sum_{n=0}^{\infty} \left(-2(n+1)\, a_{n+1} z^n\right) + \left(\sum_{n=0}^{\infty} 4 a_n z^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 2a_1 + 4a_0 = 0$$

$$a_2 = -2a_0 + a_1$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 2na_n - 2(n+1)\, a_{n+1} + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2(na_n - na_{n+1} + 2a_n - a_{n+1})}{(n+2)\,(n+1)} \\ &= -\frac{2a_n}{n+1} - \frac{2(-n-1)\, a_{n+1}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 6a_1 - 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{3} - \frac{4a_0}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 8a_2 - 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{2a_0}{3} - \frac{5a_1}{6}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 10a_3 - 8a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{6} + \frac{14a_0}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 12a_4 - 10a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{2a_0}{45} + \frac{5a_1}{18}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 14a_5 - 12a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{17a_1}{126} - \frac{94a_0}{315}$$

And so on. Therefore the solution is

$$\begin{aligned} f &= \sum_{n=0}^{\infty} a_n z^n \\ &= a_3 z^3 + a_2 z^2 + a_1 z + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$f = a_0 + a_1 z + (-2a_0 + a_1) z^2 + \left(-\frac{a_1}{3} - \frac{4a_0}{3}\right) z^3 + \left(\frac{2a_0}{3} - \frac{5a_1}{6}\right) z^4 + \left(-\frac{a_1}{6} + \frac{14a_0}{15}\right) z^5 + \dots$$

Collecting terms, the solution becomes

$$f = \left(1 - 2z^2 - \frac{4}{3}z^3 + \frac{2}{3}z^4 + \frac{14}{15}z^5\right) a_0 + \left(z + z^2 - \frac{1}{3}z^3 - \frac{5}{6}z^4 - \frac{1}{6}z^5\right) a_1 + O(z^6) \quad (3)$$

At $z = 0$ the solution above becomes

$$f = \left(1 - 2z^2 - \frac{4}{3}z^3 + \frac{2}{3}z^4 + \frac{14}{15}z^5\right) c_1 + \left(z + z^2 - \frac{1}{3}z^3 - \frac{5}{6}z^4 - \frac{1}{6}z^5\right) c_2 + O(z^6)$$

Summary

The solution(s) found are the following

$$\begin{aligned} f = &\left(1 - 2z^2 - \frac{4}{3}z^3 + \frac{2}{3}z^4 + \frac{14}{15}z^5 + \frac{2}{45}z^6\right) f(0) \\ &+ \left(z + z^2 - \frac{1}{3}z^3 - \frac{5}{6}z^4 - \frac{1}{6}z^5 + \frac{5}{18}z^6\right) f'(0) + O(z^6) \end{aligned} \tag{1}$$

$$f = \left(1 - 2z^2 - \frac{4}{3}z^3 + \frac{2}{3}z^4 + \frac{14}{15}z^5\right) c_1 + \left(z + z^2 - \frac{1}{3}z^3 - \frac{5}{6}z^4 - \frac{1}{6}z^5\right) c_2 + O(z^6) \tag{2}$$

Verification of solutions

$$\begin{aligned} f = &\left(1 - 2z^2 - \frac{4}{3}z^3 + \frac{2}{3}z^4 + \frac{14}{15}z^5 + \frac{2}{45}z^6\right) f(0) \\ &+ \left(z + z^2 - \frac{1}{3}z^3 - \frac{5}{6}z^4 - \frac{1}{6}z^5 + \frac{5}{18}z^6\right) f'(0) + O(z^6) \end{aligned}$$

Verified OK.

$$f = \left(1 - 2z^2 - \frac{4}{3}z^3 + \frac{2}{3}z^4 + \frac{14}{15}z^5\right) c_1 + \left(z + z^2 - \frac{1}{3}z^3 - \frac{5}{6}z^4 - \frac{1}{6}z^5\right) c_2 + O(z^6)$$

Verified OK.

3.4.1 Maple step by step solution

Let's solve

$f'' = -2f'z + 2f' - 4f$

- Highest derivative means the order of the ODE is 2

  f''

- Group terms with f on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $f'' + (2z - 2) f' + 4f = 0$

- Assume series solution for f

  $f = \sum_{k=0}^{\infty} a_k z^k$

☐ Rewrite DE with series expansions

- Convert $z^m \cdot f'$ to series expansion for $m = 0..1$

$$z^m \cdot f' = \sum_{k=\max(0,1-m)}^{\infty} a_k k \, z^{k-1+m}$$

- Shift index using $k- >k+1-m$

$$z^m \cdot f' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m) \, z^k$$

- Convert f'' to series expansion

$$f'' = \sum_{k=2}^{\infty} a_k k(k-1) \, z^{k-2}$$

- Shift index using $k- >k+2$

$$f'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, z^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) - 2a_{k+1}(k+1) + 2a_k(k+2)\right) z^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$k^2 a_{k+2} + (2a_k - 2a_{k+1} + 3a_{k+2}) k + 4a_k - 2a_{k+1} + 2a_{k+2} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[f = \sum_{k=0}^{\infty} a_k z^k, a_{k+2} = -\frac{2(a_k k - a_{k+1} k + 2a_k - a_{k+1})}{k^2+3k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 52

```
Order:=6;
dsolve(diff(f(z),z$2)+2*(z-1)*diff(f(z),z)+4*f(z)=0,f(z),type='series',z=0);
```

$$f(z) = \left(1-2z^2-\frac{4}{3}z^3+\frac{2}{3}z^4+\frac{14}{15}z^5\right) f(0)+\left(z+z^2-\frac{1}{3}z^3-\frac{5}{6}z^4-\frac{1}{6}z^5\right) D(f)(0)+O(z^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 127

```
AsymptoticDSolveValue[f''[z]+2*(z-a)*f'[z]+4*f[z]==0,f[z],{z,0,5}]
```

$$f(z) \to c_1\left(-\frac{4}{15}a^3z^5-\frac{2a^2z^4}{3}+\frac{6az^5}{5}-\frac{4az^3}{3}+\frac{4z^4}{3}-2z^2+1\right) + c_2\left(\frac{2a^4z^5}{15}+\frac{a^3z^4}{3}-\frac{4a^2z^5}{5}+\frac{2a^2z^3}{3}-\frac{7az^4}{6}+az^2+\frac{z^5}{2}-z^3+z\right)$$

3.5 problem Problem 16.6

Internal problem ID [2534]
Internal file name [`OUTPUT/2026_Sunday_June_05_2022_02_45_13_AM_50657971/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 16, Series solutions of ODEs. Section 16.6 Exercises, page 550
Problem number: Problem 16.6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{z^2y'' - \frac{3zy'}{2} + (z+1)\,y = 0}$$

With the expansion point for the power series method at $z = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$z^2y'' - \frac{3zy'}{2} + (z+1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(z)y' + q(z)y = 0$$

Where

$$p(z) = -\frac{3}{2z}$$
$$q(z) = \frac{z+1}{z^2}$$

Table 77: Table $p(z), q(z)$ singularites.

$p(z) = -\frac{3}{2z}$	
singularity	type
$z = 0$	"regular"

$q(z) = \frac{z+1}{z^2}$	
singularity	type
$z = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $z = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$z^2y'' - \frac{3zy'}{2} + (z+1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n z^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n z^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n z^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n z^{n+r-2}\right) z^2 - \frac{3z\left(\sum_{n=0}^{\infty} (n+r)\,a_n z^{n+r-1}\right)}{2} + (z+1)\left(\sum_{n=0}^{\infty} a_n z^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} z^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty}\left(-\frac{3z^{n+r} a_n (n+r)}{2}\right) + \left(\sum_{n=0}^{\infty} z^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n z^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of z be $n+r$ in each summation term. Going over each summation term above with power of z in it which is not already z^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} z^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} z^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of z are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} z^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty}\left(-\frac{3z^{n+r} a_n (n+r)}{2}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} z^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n z^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$z^{n+r} a_n (n+r)(n+r-1) - \frac{3z^{n+r} a_n (n+r)}{2} + a_n z^{n+r} = 0$$

When $n=0$ the above becomes

$$z^r a_0 r(-1+r) - \frac{3z^r a_0 r}{2} + a_0 z^r = 0$$

Or

$$\left(z^r r(-1+r) - \frac{3z^r r}{2} + z^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(2r^2 - 5r + 2) z^r}{2} = 0$$

Since the above is true for all z then the indicial equation becomes

$$r^2 - \frac{5}{2}r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(2r^2 - 5r + 2)\, z^r}{2} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(z) = z^{r_1}\left(\sum_{n=0}^{\infty} a_n z^n\right)$$
$$y_2(z) = z^{r_2}\left(\sum_{n=0}^{\infty} b_n z^n\right)$$

Or

$$y_1(z) = \sum_{n=0}^{\infty} a_n z^{n+2}$$
$$y_2(z) = \sum_{n=0}^{\infty} b_n z^{n+\frac{1}{2}}$$

We start by finding $y_1(z)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - \frac{3a_n(n+r)}{2} + a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}}{2n^2 + 4nr + 2r^2 - 5n - 5r + 2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{2a_{n-1}}{n(2n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2}{2r^2 - r - 1}$$

Which for the root $r = 2$ becomes

$$a_1 = -\frac{2}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2-r-1}$	$-\frac{2}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{4r^4 + 4r^3 - 5r^2 - 3r}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{2}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2-r-1}$	$-\frac{2}{5}$
a_2	$\frac{4}{4r^4+4r^3-5r^2-3r}$	$\frac{2}{35}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{r\,(4r^3 + 4r^2 - 5r - 3)\,(2r^2 + 7r + 5)}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{4}{945}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2-r-1}$	$-\frac{2}{5}$
a_2	$\frac{4}{4r^4+4r^3-5r^2-3r}$	$\frac{2}{35}$
a_3	$-\frac{8}{r(4r^3+4r^2-5r-3)(2r^2+7r+5)}$	$-\frac{4}{945}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{r(4r^3 + 4r^2 - 5r - 3)(2r^2 + 7r + 5)(2r^2 + 11r + 14)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{2}{10395}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2-r-1}$	$-\frac{2}{5}$
a_2	$\frac{4}{4r^4+4r^3-5r^2-3r}$	$\frac{2}{35}$
a_3	$-\frac{8}{r(4r^3+4r^2-5r-3)(2r^2+7r+5)}$	$-\frac{4}{945}$
a_4	$\frac{16}{r(4r^3+4r^2-5r-3)(2r^2+7r+5)(2r^2+11r+14)}$	$\frac{2}{10395}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{r(4r^3 + 4r^2 - 5r - 3)(2r^2 + 7r + 5)(2r^2 + 11r + 14)(2r^2 + 15r + 27)}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{4}{675675}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2-r-1}$	$-\frac{2}{5}$
a_2	$\frac{4}{4r^4+4r^3-5r^2-3r}$	$\frac{2}{35}$
a_3	$-\frac{8}{r(4r^3+4r^2-5r-3)(2r^2+7r+5)}$	$-\frac{4}{945}$
a_4	$\frac{16}{r(4r^3+4r^2-5r-3)(2r^2+7r+5)(2r^2+11r+14)}$	$\frac{2}{10395}$
a_5	$-\frac{32}{r(4r^3+4r^2-5r-3)(2r^2+7r+5)(2r^2+11r+14)(2r^2+15r+27)}$	$-\frac{4}{675675}$

Using the above table, then the solution $y_1(z)$ is

$$
\begin{aligned}
y_1(z) &= z^2\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6 \dots\right) \\
&= z^2\left(1 - \frac{2z}{5} + \frac{2z^2}{35} - \frac{4z^3}{945} + \frac{2z^4}{10395} - \frac{4z^5}{675675} + O(z^6)\right)
\end{aligned}
$$

Now the second solution $y_2(z)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) - \frac{3b_n(n+r)}{2} + b_{n-1} + b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{2b_{n-1}}{2n^2 + 4nr + 2r^2 - 5n - 5r + 2} \tag{4}
$$

Which for the root $r = \frac{1}{2}$ becomes

$$
b_n = -\frac{2b_{n-1}}{n(2n-3)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{2}{2r^2 - r - 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_1 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2-r-1}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{4r^4 + 4r^3 - 5r^2 - 3r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2-r-1}$	2
b_2	$\frac{4}{4r^4+4r^3-5r^2-3r}$	-2

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8}{r\left(4r^3 + 4r^2 - 5r - 3\right)\left(2r^2 + 7r + 5\right)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = \frac{4}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2-r-1}$	2
b_2	$\frac{4}{4r^4+4r^3-5r^2-3r}$	-2
b_3	$-\frac{8}{r(4r^3+4r^2-5r-3)(2r^2+7r+5)}$	$\frac{4}{9}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{r\left(4r^3+4r^2-5r-3\right)\left(2r^2+7r+5\right)\left(2r^2+11r+14\right)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = -\frac{2}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2-r-1}$	2
b_2	$\frac{4}{4r^4+4r^3-5r^2-3r}$	-2
b_3	$-\frac{8}{r(4r^3+4r^2-5r-3)(2r^2+7r+5)}$	$\frac{4}{9}$
b_4	$\frac{16}{r(4r^3+4r^2-5r-3)(2r^2+7r+5)(2r^2+11r+14)}$	$-\frac{2}{45}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32}{r\left(4r^3+4r^2-5r-3\right)\left(2r^2+7r+5\right)\left(2r^2+11r+14\right)\left(2r^2+15r+27\right)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = \frac{4}{1575}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2-r-1}$	2
b_2	$\frac{4}{4r^4+4r^3-5r^2-3r}$	-2
b_3	$-\frac{8}{r(4r^3+4r^2-5r-3)(2r^2+7r+5)}$	$\frac{4}{9}$
b_4	$\frac{16}{r(4r^3+4r^2-5r-3)(2r^2+7r+5)(2r^2+11r+14)}$	$-\frac{2}{45}$
b_5	$-\frac{32}{r(4r^3+4r^2-5r-3)(2r^2+7r+5)(2r^2+11r+14)(2r^2+15r+27)}$	$\frac{4}{1575}$

Using the above table, then the solution $y_2(z)$ is

$$\begin{aligned} y_2(z) &= z^2\left(b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + b_5 z^5 + b_6 z^6 \dots\right) \\ &= \sqrt{z}\left(1 + 2z - 2z^2 + \frac{4z^3}{9} - \frac{2z^4}{45} + \frac{4z^5}{1575} + O(z^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(z) &= c_1 y_1(z) + c_2 y_2(z) \\ &= c_1 z^2\left(1 - \frac{2z}{5} + \frac{2z^2}{35} - \frac{4z^3}{945} + \frac{2z^4}{10395} - \frac{4z^5}{675675} + O(z^6)\right) \\ &\quad + c_2\sqrt{z}\left(1 + 2z - 2z^2 + \frac{4z^3}{9} - \frac{2z^4}{45} + \frac{4z^5}{1575} + O(z^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 z^2\left(1 - \frac{2z}{5} + \frac{2z^2}{35} - \frac{4z^3}{945} + \frac{2z^4}{10395} - \frac{4z^5}{675675} + O(z^6)\right) \\ &\quad + c_2\sqrt{z}\left(1 + 2z - 2z^2 + \frac{4z^3}{9} - \frac{2z^4}{45} + \frac{4z^5}{1575} + O(z^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 z^2\left(1 - \frac{2z}{5} + \frac{2z^2}{35} - \frac{4z^3}{945} + \frac{2z^4}{10395} - \frac{4z^5}{675675} + O(z^6)\right) \\ &\quad + c_2\sqrt{z}\left(1 + 2z - 2z^2 + \frac{4z^3}{9} - \frac{2z^4}{45} + \frac{4z^5}{1575} + O(z^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 z^2 \left(1-\frac{2z}{5}+\frac{2z^2}{35}-\frac{4z^3}{945}+\frac{2z^4}{10395}-\frac{4z^5}{675675}+O(z^6)\right) + c_2\sqrt{z}\left(1+2z-2z^2+\frac{4z^3}{9}-\frac{2z^4}{45}+\frac{4z^5}{1575}+O(z^6)\right)$$

Verified OK.

3.5.1 Maple step by step solution

Let's solve

$y''z^2 - \frac{3zy'}{2} + (z+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{2z} - \frac{(z+1)y}{z^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{2z} + \frac{(z+1)y}{z^2} = 0$

☐ Check to see if $z_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(z) = -\frac{3}{2z}, P_3(z) = \frac{z+1}{z^2}\right]$

  ◦ $z \cdot P_2(z)$ is analytic at $z = 0$

  $(z \cdot P_2(z))\Big|_{z=0} = -\frac{3}{2}$

  ◦ $z^2 \cdot P_3(z)$ is analytic at $z = 0$

  $(z^2 \cdot P_3(z))\Big|_{z=0} = 1$

  ◦ $z = 0$is a regular singular point

  Check to see if $z_0 = 0$ is a regular singular point

  $z_0 = 0$

- Multiply by denominators

  $2y''z^2 - 3zy' + (2z+2)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k z^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $z^m \cdot y$ to series expansion for $m = 0..1$

$$z^m \cdot y = \sum_{k=0}^{\infty} a_k z^{k+r+m}$$

- Shift index using $k- >k - m$

$$z^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} z^{k+r}$$

- Convert $z \cdot y'$ to series expansion

$$z \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, z^{k+r}$$

- Convert $z^2 \cdot y''$ to series expansion

$$z^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, z^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)\,(-2+r)\, z^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-2) + 2a_{k-1}\right) z^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{2, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+r-\frac{1}{2}\right)(k+r-2)\, a_k + 2a_{k-1} = 0$$

- Shift index using $k- >k + 1$

$$2\left(k+\frac{1}{2}+r\right)(k+r-1)\, a_{k+1} + 2a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{2a_k}{(2k+1+2r)(k+r-1)}$$

- Recursion relation for $r = 2$

$$a_{k+1} = -\frac{2a_k}{(2k+5)(k+1)}$$

- Solution for $r = 2$

$$\left[y=\sum_{k=0}^{\infty} a_k z^{k+2}, a_{k+1}=-\frac{2a_k}{(2k+5)(k+1)}\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+1}=-\frac{2a_k}{(2k+2)\left(k-\frac{1}{2}\right)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k z^{k+\frac{1}{2}}, a_{k+1}=-\frac{2a_k}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k z^{k+2}\right)+\left(\sum_{k=0}^{\infty} b_k z^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{2a_k}{(2k+5)(k+1)}, b_{k+1}=-\frac{2b_k}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
Order:=6;
dsolve(z^2*diff(y(z),z$2)-3/2*z*diff(y(z),z)+(1+z)*y(z)=0,y(z),type='series',z=0);
```

$$y(z)=c_1\sqrt{z}\left(1+2z-2z^2+\frac{4}{9}z^3-\frac{2}{45}z^4+\frac{4}{1575}z^5+\mathrm{O}\left(z^6\right)\right)+c_2z^2\left(1-\frac{2}{5}z+\frac{2}{35}z^2-\frac{4}{945}z^3+\frac{2}{10395}z^4-\frac{4}{675675}z^5+\mathrm{O}\left(z^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 84

```
AsymptoticDSolveValue[z^2*y''[z]-3/2*z*y'[z]+(1+z)*y[z]==0,y[z],{z,0,5}]
```

$$y(z) \to c_1 \left(-\frac{4z^5}{675675} + \frac{2z^4}{10395} - \frac{4z^3}{945} + \frac{2z^2}{35} - \frac{2z}{5} + 1\right) z^2 + c_2 \left(\frac{4z^5}{1575} - \frac{2z^4}{45} + \frac{4z^3}{9} - 2z^2 + 2z + 1\right) \sqrt{z}$$

3.6 problem Problem 16.8

Internal problem ID [2535]
Internal file name [`OUTPUT/2027_Sunday_June_05_2022_02_45_18_AM_81816058/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 16, Series solutions of ODEs. Section 16.6 Exercises, page 550
Problem number: Problem 16.8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Lienard]
```

$$\boxed{zy'' - 2y' + zy = 0}$$

With the expansion point for the power series method at $z = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$zy'' - 2y' + zy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(z)y' + q(z)y = 0$$

Where

$$p(z) = -\frac{2}{z}$$
$$q(z) = 1$$

Table 79: Table $p(z), q(z)$ singularites.

$p(z) = -\frac{2}{z}$	
singularity	type
$z = 0$	"regular"

$q(z) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $z = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$zy'' - 2y' + zy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n z^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n z^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n z^{n+r-2}$$

Substituting the above back into the ode gives

$$z\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n z^{n+r-2}\right) - 2\left(\sum_{n=0}^{\infty} (n+r)\, a_n z^{n+r-1}\right) + z\left(\sum_{n=0}^{\infty} a_n z^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} z^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2(n+r)\, a_n z^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} z^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of z be $n+r-1$ in each summation term. Going over each summation term above with power of z in it which is not already z^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} z^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} z^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of z are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} z^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2(n+r)\, a_n z^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} z^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$z^{n+r-1} a_n (n+r)(n+r-1) - 2(n+r)\, a_n z^{n+r-1} = 0$$

When $n=0$ the above becomes

$$z^{-1+r} a_0 r(-1+r) - 2r a_0 z^{-1+r} = 0$$

Or

$$\left(z^{-1+r} r(-1+r) - 2r\, z^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, z^{-1+r}(-3+r) = 0$$

Since the above is true for all z then the indicial equation becomes

$$r(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, z^{-1+r}(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(z) = z^{r_1}\left(\sum_{n=0}^{\infty} a_n z^n\right)$$
$$y_2(z) = C y_1(z) \ln(z) + z^{r_2}\left(\sum_{n=0}^{\infty} b_n z^n\right)$$

Or

$$y_1(z) = z^3\left(\sum_{n=0}^{\infty} a_n z^n\right)$$
$$y_2(z) = C y_1(z) \ln(z) + \left(\sum_{n=0}^{\infty} b_n z^n\right)$$

Or

$$y_1(z) = \sum_{n=0}^{\infty} a_n z^{n+3}$$
$$y_2(z) = C y_1(z) \ln(z) + \left(\sum_{n=0}^{\infty} b_n z^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 2a_n(n+r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 3n - 3r} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + r - 2}$$

Which for the root $r = 3$ becomes

$$a_2 = -\frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+r-2}$	$-\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+r-2}$	$-\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 + 6r^3 + 7r^2 - 6r - 8}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{1}{280}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+r-2}$	$-\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{r^4+6r^3+7r^2-6r-8}$	$\frac{1}{280}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+r-2}$	$-\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{r^4+6r^3+7r^2-6r-8}$	$\frac{1}{280}$
a_5	0	0

Using the above table, then the solution $y_1(z)$ is

$$\begin{aligned} y_1(z) &= z^3 \left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6 \dots\right) \\ &= z^3 \left(1 - \frac{z^2}{10} + \frac{z^4}{280} + O(z^6)\right) \end{aligned}$$

Now the second solution $y_2(z)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_3 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} 0 &= \lim_{r \to 0} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(z) &= \sum_{n=0}^{\infty} b_n z^{n+r} \\
&= \sum_{n=0}^{\infty} b_n z^n
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) - 2(n+r)\, b_n + b_{n-2} = 0 \tag{4}
$$

Which for for the root $r = 0$ becomes

$$
b_n n(n-1) - 2n b_n + b_{n-2} = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{b_{n-2}}{n^2 + 2nr + r^2 - 3n - 3r} \tag{5}
$$

Which for the root $r = 0$ becomes

$$
b_n = -\frac{b_{n-2}}{n^2 - 3n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{r^2 + r - 2}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+r-2}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+r-2}$	$\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + r - 2)(r^2 + 5r + 4)}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+r-2}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+6r^3+7r^2-6r-8}$	$-\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+r-2}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+6r^3+7r^2-6r-8}$	$-\frac{1}{8}$
b_5	0	0

Using the above table, then the solution $y_2(z)$ is

$$\begin{aligned} y_2(z) &= b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + b_5 z^5 + b_6 z^6 \dots \\ &= 1 + \frac{z^2}{2} - \frac{z^4}{8} + O(z^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(z) &= c_1 y_1(z) + c_2 y_2(z) \\ &= c_1 z^3 \left(1 - \frac{z^2}{10} + \frac{z^4}{280} + O(z^6)\right) + c_2 \left(1 + \frac{z^2}{2} - \frac{z^4}{8} + O(z^6)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 z^3 \left(1 - \frac{z^2}{10} + \frac{z^4}{280} + O(z^6)\right) + c_2 \left(1 + \frac{z^2}{2} - \frac{z^4}{8} + O(z^6)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 z^3 \left(1 - \frac{z^2}{10} + \frac{z^4}{280} + O(z^6)\right) + c_2 \left(1 + \frac{z^2}{2} - \frac{z^4}{8} + O(z^6)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 z^3 \left(1 - \frac{z^2}{10} + \frac{z^4}{280} + O(z^6)\right) + c_2 \left(1 + \frac{z^2}{2} - \frac{z^4}{8} + O(z^6)\right)
$$

Verified OK.

3.6.1 Maple step by step solution

Let's solve

$zy'' - 2y' + zy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y'}{z} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{z} + y = 0$

☐ Check to see if $z_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(z) = -\frac{2}{z}, P_3(z) = 1\right]$

- ○ $z \cdot P_2(z)$ is analytic at $z = 0$

  $\left.(z \cdot P_2(z))\right|_{z=0} = -2$

- ○ $z^2 \cdot P_3(z)$ is analytic at $z = 0$

$\left(z^2 \cdot P_3(z)\right)\Big|_{z=0} = 0$

- $z = 0$is a regular singular point

  Check to see if $z_0 = 0$ is a regular singular point

  $z_0 = 0$

- Multiply by denominators

  $zy'' - 2y' + zy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k z^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $z \cdot y$ to series expansion

    $z \cdot y = \sum_{k=0}^{\infty} a_k z^{k+r+1}$

  - Shift index using $k- > k-1$

    $z \cdot y = \sum_{k=1}^{\infty} a_{k-1} z^{k+r}$

  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r)\, z^{k+r-1}$

  - Shift index using $k- > k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, z^{k+r}$

  - Convert $z \cdot y''$ to series expansion

    $z \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, z^{k+r-1}$

  - Shift index using $k- > k+1$

    $z \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r)\, z^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\, z^{-1+r} + a_1(1+r)(-2+r)\, z^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)(k-2+r) + a_{k-1}\right) z^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term must be 0

  $a_1(1+r)(-2+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k-2+r)+a_{k-1}=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+r-1)+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+2+r)(k+r-1)}$

- Recursion relation for $r=0$

  $a_{k+2}=-\frac{a_k}{(k+2)(k-1)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k z^k, a_{k+2}=-\frac{a_k}{(k+2)(k-1)}, -2a_1=0\right]$

- Recursion relation for $r=3$

  $a_{k+2}=-\frac{a_k}{(k+5)(k+2)}$

- Solution for $r=3$

  $\left[y=\sum_{k=0}^{\infty} a_k z^{k+3}, a_{k+2}=-\frac{a_k}{(k+5)(k+2)}, 4a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k z^k\right)+\left(\sum_{k=0}^{\infty} b_k z^{k+3}\right), a_{k+2}=-\frac{a_k}{(k+2)(k-1)}, -2a_1=0, b_{k+2}=-\frac{b_k}{(k+5)(k+2)}, 4b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 32

```
Order:=6;
dsolve(z*diff(y(z),z$2)-2*diff(y(z),z)+z*y(z)=0,y(z),type='series',z=0);
```

$$y(z) = c_1 z^3 \left(1 - \frac{1}{10} z^2 + \frac{1}{280} z^4 + \mathrm{O}\left(z^6\right)\right) + c_2 \left(12 + 6z^2 - \frac{3}{2} z^4 + \mathrm{O}\left(z^6\right)\right)$$

✓ Solution by Mathematica
Time used: 0.009 (sec). Leaf size: 44

```
AsymptoticDSolveValue[z*y''[z]-2*y'[z]+z*y[z]==0,y[z],{z,0,5}]
```

$$y(z) \to c_1 \left(-\frac{z^4}{8} + \frac{z^2}{2} + 1\right) + c_2 \left(\frac{z^7}{280} - \frac{z^5}{10} + z^3\right)$$

3.7 problem Problem 16.9

Internal problem ID [2536]
Internal file name [`OUTPUT/2028_Sunday_June_05_2022_02_45_22_AM_21992497/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 16, Series solutions of ODEs. Section 16.6 Exercises, page 550
Problem number: Problem 16.9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' - 2zy' - 2y = 0}$$

With the expansion point for the power series method at $z = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (125)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (126)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\Big|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 2zy' + 2y \\
F_1 &= \frac{dF_0}{dz} \\
&= \frac{\partial F_0}{\partial z} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4y'z^2 + 4zy + 4y' \\
F_2 &= \frac{dF_1}{dz} \\
&= \frac{\partial F_1}{\partial z} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 8y'z^3 + 8yz^2 + 20zy' + 12y \\
F_3 &= \frac{dF_2}{dz} \\
&= \frac{\partial F_2}{\partial z} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(16z^4 + 72z^2 + 32\right) y' + \left(16z^3 + 56z\right) y \\
F_4 &= \frac{dF_3}{dz} \\
&= \frac{\partial F_3}{\partial z} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(32z^5 + 224z^3 + 264z\right) y' + 32\left(z^4 + 6z^2 + \frac{15}{4}\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $z = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2y(0) \\
F_1 &= 4y'(0) \\
F_2 &= 12y(0) \\
F_3 &= 32y'(0) \\
F_4 &= 120y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + z^2 + \frac{1}{2}z^4 + \frac{1}{6}z^6\right) y(0) + \left(z + \frac{2}{3}z^3 + \frac{4}{15}z^5\right) y'(0) + O\left(z^6\right)
$$

Since the expansion point $z = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n z^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n z^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n z^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n z^{n-2} = 2z\left(\sum_{n=1}^{\infty} n a_n z^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n z^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n z^{n-2}\right) + \sum_{n=1}^{\infty} (-2n\, z^n a_n) + \sum_{n=0}^{\infty} (-2a_n z^n) = 0 \tag{2}$$

The next step is to make all powers of z be n in each summation term. Going over each summation term above with power of z in it which is not already z^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n z^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, z^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of z are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, z^n\right) + \sum_{n=1}^{\infty} (-2n\, z^n a_n) + \sum_{n=0}^{\infty} (-2a_n z^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 2a_0 = 0$$

$$a_2 = a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2na_n - 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{2a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 6a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 8a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{4a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 10a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{6}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 12a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{8a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n z^n \\ &= a_3 z^3 + a_2 z^2 + a_1 z + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 z + a_0 z^2 + \frac{2}{3} a_1 z^3 + \frac{1}{2} a_0 z^4 + \frac{4}{15} a_1 z^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + z^2 + \frac{1}{2} z^4\right) a_0 + \left(z + \frac{2}{3} z^3 + \frac{4}{15} z^5\right) a_1 + O(z^6) \tag{3}$$

At $z = 0$ the solution above becomes

$$y = \left(1 + z^2 + \frac{1}{2} z^4\right) c_1 + \left(z + \frac{2}{3} z^3 + \frac{4}{15} z^5\right) c_2 + O(z^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + z^2 + \frac{1}{2} z^4 + \frac{1}{6} z^6\right) y(0) + \left(z + \frac{2}{3} z^3 + \frac{4}{15} z^5\right) y'(0) + O(z^6) \tag{1}$$

$$y = \left(1 + z^2 + \frac{1}{2} z^4\right) c_1 + \left(z + \frac{2}{3} z^3 + \frac{4}{15} z^5\right) c_2 + O(z^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + z^2 + \frac{1}{2} z^4 + \frac{1}{6} z^6\right) y(0) + \left(z + \frac{2}{3} z^3 + \frac{4}{15} z^5\right) y'(0) + O(z^6)$$

Verified OK.

$$y = \left(1 + z^2 + \frac{1}{2} z^4\right) c_1 + \left(z + \frac{2}{3} z^3 + \frac{4}{15} z^5\right) c_2 + O(z^6)$$

Verified OK.

3.7.1 Maple step by step solution

Let's solve

$y'' = 2zy' + 2y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 2zy' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k z^k$

□ Rewrite DE with series expansions

  ○ Convert $z \cdot y'$ to series expansion

  $z \cdot y' = \sum_{k=0}^{\infty} a_k k \, z^k$

  ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, z^{k-2}$

  ○ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, z^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) - 2a_k(k+1)) \, z^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)(a_{k+2}(k+2) - 2a_k) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k z^k, a_{k+2} = \frac{2a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 32

```
Order:=6;
dsolve(diff(y(z),z$2)-2*z*diff(y(z),z)-2*y(z)=0,y(z),type='series',z=0);
```

$$y(z) = \left(1 + z^2 + \frac{1}{2}z^4\right) y(0) + \left(z + \frac{2}{3}z^3 + \frac{4}{15}z^5\right) D(y)(0) + O(z^6)$$

✓ Solution by Mathematica
Time used: 0.001 (sec). Leaf size: 38

```
AsymptoticDSolveValue[y''[z]-2*z*y'[z]-2*y[z]==0,y[z],{z,0,5}]
```

$$y(z) \to c_2\left(\frac{4z^5}{15} + \frac{2z^3}{3} + z\right) + c_1\left(\frac{z^4}{2} + z^2 + 1\right)$$

3.8 problem Problem 16.10

Internal problem ID [2537]
Internal file name [`OUTPUT/2029_Sunday_June_05_2022_02_45_24_AM_69342567/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 16, Series solutions of ODEs. Section 16.6 Exercises, page 550
Problem number: Problem 16.10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[_Jacobi]`

$$\boxed{z(1-z)\,y''+(1-z)\,y'+\lambda y=0}$$

With the expansion point for the power series method at $z=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-z^2+z\right)y''+(1-z)\,y'+\lambda y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(z)y'+q(z)y=0$$

Where

$$p(z)=\frac{1}{z}$$
$$q(z)=-\frac{\lambda}{z\,(z-1)}$$

Table 82: Table $p(z), q(z)$ singularites.

$p(z) = \frac{1}{z}$	
singularity	type
$z = 0$	"regular"

$q(z) = -\frac{\lambda}{z(z-1)}$	
singularity	type
$z = 0$	"regular"
$z = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : []

Since $z = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''z(z-1) + (1-z)\,y' + \lambda y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n z^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n z^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n z^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n z^{n+r-2}\right) z(z-1) + (1-z)\left(\sum_{n=0}^{\infty} (n+r)\, a_n z^{n+r-1}\right) + \lambda\left(\sum_{n=0}^{\infty} a_n z^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-z^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} z^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-z^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n z^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} \lambda a_n z^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of z be $n+r-1$ in each summation term. Going over each summation term above with power of z in it which is not already z^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-z^{n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)(n+r-2) z^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-z^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) z^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \lambda a_n z^{n+r} = \sum_{n=1}^{\infty} \lambda a_{n-1} z^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of z are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)(n+r-2) z^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} z^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) z^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n z^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} \lambda a_{n-1} z^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$z^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n z^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$z^{-1+r} a_0 r(-1+r) + r a_0 z^{-1+r} = 0$$

Or

$$\left(z^{-1+r} r(-1+r) + r\, z^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$z^{-1+r} r^2 = 0$$

Since the above is true for all z then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$z^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(z) = \sum_{n=0}^{\infty} a_n z^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(z) = y_1(z) \ln(z) + \left(\sum_{n=1}^{\infty} b_n z^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(z) + c_2 y_2(z)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(z)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} -a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ - a_{n-1}(n+r-1) + a_n(n+r) + \lambda a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(-n^2 - 2nr - r^2 + \lambda + 2n + 2r - 1)}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}(n^2 - \lambda - 2n + 1)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r^2 - \lambda}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -\lambda$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-\lambda}{(r+1)^2}$	$-\lambda$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(-r^2 + \lambda - 2r - 1)(-r^2 + \lambda)}{(r+1)^2 (2+r)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{(\lambda - 1)\lambda}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-\lambda}{(r+1)^2}$	$-\lambda$
a_2	$\frac{(-r^2+\lambda-2r-1)(-r^2+\lambda)}{(r+1)^2(2+r)^2}$	$\frac{(\lambda-1)\lambda}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(r^2 - \lambda + 4r + 4)(r^2 - \lambda + 2r + 1)(r^2 - \lambda)}{(r+1)^2 (2+r)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{(\lambda - 4)(\lambda - 1)\lambda}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-\lambda}{(r+1)^2}$	$-\lambda$
a_2	$\frac{(-r^2+\lambda-2r-1)(-r^2+\lambda)}{(r+1)^2(2+r)^2}$	$\frac{(\lambda-1)\lambda}{4}$
a_3	$\frac{(r^2-\lambda+4r+4)(r^2-\lambda+2r+1)(r^2-\lambda)}{(r+1)^2(2+r)^2(r+3)^2}$	$-\frac{(\lambda-4)(\lambda-1)\lambda}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(-r^2 + \lambda - 6r - 9)(-r^2 + \lambda - 4r - 4)(-r^2 + \lambda - 2r - 1)(-r^2 + \lambda)}{(r+1)^2 (2+r)^2 (r+3)^2 (4+r)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{(\lambda - 9)(\lambda - 4)(\lambda - 1)\lambda}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-\lambda}{(r+1)^2}$	$-\lambda$
a_2	$\frac{(-r^2+\lambda-2r-1)(-r^2+\lambda)}{(r+1)^2(2+r)^2}$	$\frac{(\lambda-1)\lambda}{4}$
a_3	$\frac{(r^2-\lambda+4r+4)(r^2-\lambda+2r+1)(r^2-\lambda)}{(r+1)^2(2+r)^2(r+3)^2}$	$-\frac{(\lambda-4)(\lambda-1)\lambda}{36}$
a_4	$\frac{(-r^2+\lambda-6r-9)(-r^2+\lambda-4r-4)(-r^2+\lambda-2r-1)(-r^2+\lambda)}{(r+1)^2(2+r)^2(r+3)^2(4+r)^2}$	$\frac{(\lambda-9)(\lambda-4)(\lambda-1)\lambda}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(r^2 - \lambda + 8r + 16)(r^2 - \lambda + 6r + 9)(r^2 - \lambda + 4r + 4)(r^2 - \lambda + 2r + 1)(r^2 - \lambda)}{(r+1)^2 (2+r)^2 (r+3)^2 (4+r)^2 (5+r)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{(\lambda - 16)(\lambda - 9)(\lambda - 4)(\lambda - 1)\lambda}{14400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-\lambda}{(r+1)^2}$	$-\lambda$
a_2	$\frac{(-r^2+\lambda-2r-1)(-r^2+\lambda)}{(r+1)^2(2+r)^2}$	$\frac{(\lambda-1)\lambda}{4}$
a_3	$\frac{(r^2-\lambda+4r+4)(r^2-\lambda+2r+1)(r^2-\lambda)}{(r+1)^2(2+r)^2(r+3)^2}$	$-\frac{(\lambda-4)(\lambda-1)\lambda}{36}$
a_4	$\frac{(-r^2+\lambda-6r-9)(-r^2+\lambda-4r-4)(-r^2+\lambda-2r-1)(-r^2+\lambda)}{(r+1)^2(2+r)^2(r+3)^2(4+r)^2}$	$\frac{(\lambda-9)(\lambda-4)(\lambda-1)\lambda}{576}$
a_5	$\frac{(r^2-\lambda+8r+16)(r^2-\lambda+6r+9)(r^2-\lambda+4r+4)(r^2-\lambda+2r+1)(r^2-\lambda)}{(r+1)^2(2+r)^2(r+3)^2(4+r)^2(5+r)^2}$	$-\frac{(\lambda-16)(\lambda-9)(\lambda-4)(\lambda-1)\lambda}{14400}$

Using the above table, then the first solution $y_1(z)$ becomes

$$\begin{aligned} y_1(z) &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6 \dots \\ &= -\lambda z + 1 + \frac{(\lambda - 1)\lambda z^2}{4} - \frac{(\lambda - 4)(\lambda - 1)\lambda z^3}{36} + \frac{(\lambda - 9)(\lambda - 4)(\lambda - 1)\lambda z^4}{576} \\ &\quad - \frac{(\lambda - 16)(\lambda - 9)(\lambda - 4)(\lambda - 1)\lambda z^5}{14400} + O(z^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(z) = y_1(z) \ln(z) + \left(\sum_{n=1}^{\infty} b_n z^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts fro
b_1	$\frac{r^2-\lambda}{(r+1)^2}$	$-\lambda$	$\frac{2\lambda+2r}{(r+1)^3}$
b_2	$\frac{(-r^2+\lambda-2r-1)(-r^2+\lambda)}{(r+1)^2(2+r)^2}$	$\frac{(\lambda-1)\lambda}{4}$	$4r^4+(4\lambda+12)r^3+(6\lambda+12)r^2+($ / $(r+1)^3(2+$
b_3	$\frac{(r^2-\lambda+4r+4)(r^2-\lambda+2r+1)(r^2-\lambda)}{(r+1)^2(2+r)^2(r+3)^2}$	$-\frac{(\lambda-4)(\lambda-1)\lambda}{36}$	$6r^7+(6\lambda+54)r^6+(36\lambda+198)r^5$
b_4	$\frac{(-r^2+\lambda-6r-9)(-r^2+\lambda-4r-4)(-r^2+\lambda-2r-1)(-r^2+\lambda)}{(r+1)^2(2+r)^2(r+3)^2(4+r)^2}$	$\frac{(\lambda-9)(\lambda-4)(\lambda-1)\lambda}{576}$	$8r^{10}+(8\lambda+144)r^9+(108\lambda+112$
b_5	$\frac{(r^2-\lambda+8r+16)(r^2-\lambda+6r+9)(r^2-\lambda+4r+4)(r^2-\lambda+2r+1)(r^2-\lambda)}{(r+1)^2(2+r)^2(r+3)^2(4+r)^2(5+r)^2}$	$-\frac{(\lambda-16)(\lambda-9)(\lambda-4)(\lambda-1)\lambda}{14400}$	$10r^{13}+(10\lambda+300)r^{12}+(240\lambda+$

The above table gives all values of b_n needed. Hence the second solution is

$$y_2(z) = y_1(z) \ln(z) + b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + b_5 z^5 + b_6 z^6 \dots$$

$$
\begin{aligned}
&= \left(-\lambda z + 1 + \frac{(\lambda-1)\,\lambda\, z^2}{4} - \frac{(\lambda-4)\,(\lambda-1)\,\lambda\, z^3}{36} + \frac{(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\, z^4}{576} \right.\\
&\qquad \left. - \frac{(\lambda-16)\,(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\, z^5}{14400} + O\left(z^6\right)\right) \ln(z) \\
&\quad + 2\lambda z + \left(-\frac{\lambda}{2} - \frac{3(\lambda-1)\,\lambda}{4}\right) z^2 \\
&\quad + \left(-\frac{(-\lambda+1)\,\lambda}{9} - \frac{(-\lambda+4)\,\lambda}{18} + \frac{11(-\lambda+4)\,(-\lambda+1)\,\lambda}{108}\right) z^3 \\
&\quad + \left(-\frac{(\lambda-4)\,(\lambda-1)\,\lambda}{96} - \frac{(\lambda-9)\,(\lambda-1)\,\lambda}{144} - \frac{(\lambda-9)\,(\lambda-4)\,\lambda}{288} \right.\\
&\qquad \left. - \frac{25(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda}{3456}\right) z^4 \\
&\quad + \left(-\frac{(-\lambda+9)\,(-\lambda+4)\,(-\lambda+1)\,\lambda}{1800} - \frac{(-\lambda+16)\,(-\lambda+4)\,(-\lambda+1)\,\lambda}{2400} \right.\\
&\qquad - \frac{(-\lambda+16)\,(-\lambda+9)\,(-\lambda+1)\,\lambda}{3600} - \frac{(-\lambda+16)\,(-\lambda+9)\,(-\lambda+4)\,\lambda}{7200} \\
&\qquad \left. + \frac{137(-\lambda+16)\,(-\lambda+9)\,(-\lambda+4)\,(-\lambda+1)\,\lambda}{432000}\right) z^5 + O\left(z^6\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$y_h(z) = c_1 y_1(z) + c_2 y_2(z)$$

$$
\begin{aligned}
&= c_1\left(-\lambda z+1+\frac{(\lambda-1)\,\lambda\,z^2}{4}-\frac{(\lambda-4)\,(\lambda-1)\,\lambda\,z^3}{36}+\frac{(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^4}{576}\right.\\
&\qquad\left.-\frac{(\lambda-16)\,(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^5}{14400}+O\left(z^6\right)\right)\\
&+c_2\left(\left(-\lambda z+1+\frac{(\lambda-1)\,\lambda\,z^2}{4}-\frac{(\lambda-4)\,(\lambda-1)\,\lambda\,z^3}{36}\right.\right.\\
&\qquad+\frac{(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^4}{576}-\frac{(\lambda-16)\,(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^5}{14400}\\
&\qquad\left.+O\left(z^6\right)\right)\ln(z)+2\lambda z+\left(-\frac{\lambda}{2}-\frac{3(\lambda-1)\,\lambda}{4}\right)z^2\\
&\qquad+\left(-\frac{(-\lambda+1)\,\lambda}{9}-\frac{(-\lambda+4)\,\lambda}{18}+\frac{11(-\lambda+4)\,(-\lambda+1)\,\lambda}{108}\right)z^3\\
&\qquad+\left(-\frac{(\lambda-4)\,(\lambda-1)\,\lambda}{96}-\frac{(\lambda-9)\,(\lambda-1)\,\lambda}{144}-\frac{(\lambda-9)\,(\lambda-4)\,\lambda}{288}\right.\\
&\qquad\left.-\frac{25(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda}{3456}\right)z^4\\
&\qquad+\left(-\frac{(-\lambda+9)\,(-\lambda+4)\,(-\lambda+1)\,\lambda}{1800}-\frac{(-\lambda+16)\,(-\lambda+4)\,(-\lambda+1)\,\lambda}{2400}\right.\\
&\qquad-\frac{(-\lambda+16)\,(-\lambda+9)\,(-\lambda+1)\,\lambda}{3600}-\frac{(-\lambda+16)\,(-\lambda+9)\,(-\lambda+4)\,\lambda}{7200}\\
&\qquad\left.\left.+\frac{137(-\lambda+16)\,(-\lambda+9)\,(-\lambda+4)\,(-\lambda+1)\,\lambda}{432000}\right)z^5+O\left(z^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$y = y_h$

$$
\begin{aligned}
&= c_1\Bigg(-\lambda z + 1 + \frac{(\lambda-1)\,\lambda\,z^2}{4} - \frac{(\lambda-4)\,(\lambda-1)\,\lambda\,z^3}{36} + \frac{(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^4}{576} \\
&\qquad - \frac{(\lambda-16)\,(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^5}{14400} + O\big(z^6\big)\Bigg) \\
&\quad + c_2\Bigg(\Bigg(-\lambda z + 1 + \frac{(\lambda-1)\,\lambda\,z^2}{4} - \frac{(\lambda-4)\,(\lambda-1)\,\lambda\,z^3}{36} + \frac{(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^4}{576} \\
&\qquad - \frac{(\lambda-16)\,(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^5}{14400} + O\big(z^6\big)\Bigg)\ln(z) + 2\lambda z \\
&\qquad + \left(-\frac{\lambda}{2} - \frac{3(\lambda-1)\,\lambda}{4}\right) z^2 \\
&\quad + \left(-\frac{(-\lambda+1)\,\lambda}{9} - \frac{(-\lambda+4)\,\lambda}{18} + \frac{11(-\lambda+4)\,(-\lambda+1)\,\lambda}{108}\right) z^3 + \Bigg(-\frac{(\lambda-4)\,(\lambda-1)\,\lambda}{96} \\
&\qquad - \frac{(\lambda-9)\,(\lambda-1)\,\lambda}{144} - \frac{(\lambda-9)\,(\lambda-4)\,\lambda}{288} - \frac{25(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda}{3456}\Bigg) z^4 \\
&\qquad + \Bigg(-\frac{(-\lambda+9)\,(-\lambda+4)\,(-\lambda+1)\,\lambda}{1800} - \frac{(-\lambda+16)\,(-\lambda+4)\,(-\lambda+1)\,\lambda}{2400} \\
&\qquad - \frac{(-\lambda+16)\,(-\lambda+9)\,(-\lambda+1)\,\lambda}{3600} - \frac{(-\lambda+16)\,(-\lambda+9)\,(-\lambda+4)\,\lambda}{7200} \\
&\qquad + \frac{137(-\lambda+16)\,(-\lambda+9)\,(-\lambda+4)\,(-\lambda+1)\,\lambda}{432000}\Bigg) z^5 + O\big(z^6\big)\Bigg)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 \left(-\lambda z + 1 + \frac{(\lambda-1)\,\lambda\,z^2}{4} - \frac{(\lambda-4)\,(\lambda-1)\,\lambda\,z^3}{36} + \frac{(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^4}{576} \right. \\
\left. - \frac{(\lambda-16)\,(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^5}{14400} + O\left(z^6\right)\right) \\
+ c_2 \left(\left(-\lambda z + 1 + \frac{(\lambda-1)\,\lambda\,z^2}{4} - \frac{(\lambda-4)\,(\lambda-1)\,\lambda\,z^3}{36} + \frac{(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^4}{576} \right.\right. \\
\left. - \frac{(\lambda-16)\,(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^5}{14400} + O\left(z^6\right)\right) \ln(z) + 2\lambda z \\
+ \left(-\frac{\lambda}{2} - \frac{3(\lambda-1)\,\lambda}{4}\right) z^2 \\
+ \left(-\frac{(-\lambda+1)\,\lambda}{9} - \frac{(-\lambda+4)\,\lambda}{18} + \frac{11(-\lambda+4)\,(-\lambda+1)\,\lambda}{108}\right) z^3 \\
+ \left(-\frac{(\lambda-4)\,(\lambda-1)\,\lambda}{96} - \frac{(\lambda-9)\,(\lambda-1)\,\lambda}{144} - \frac{(\lambda-9)\,(\lambda-4)\,\lambda}{288} \right. \\
\left. - \frac{25(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda}{3456}\right) z^4 \\
+ \left(-\frac{(-\lambda+9)\,(-\lambda+4)\,(-\lambda+1)\,\lambda}{1800} - \frac{(-\lambda+16)\,(-\lambda+4)\,(-\lambda+1)\,\lambda}{2400} \right. \\
- \frac{(-\lambda+16)\,(-\lambda+9)\,(-\lambda+1)\,\lambda}{3600} - \frac{(-\lambda+16)\,(-\lambda+9)\,(-\lambda+4)\,\lambda}{7200} \\
\left.\left. + \frac{137(-\lambda+16)\,(-\lambda+9)\,(-\lambda+4)\,(-\lambda+1)\,\lambda}{432000}\right) z^5 + O\left(z^6\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
y = c_1\left(-\lambda z + 1 + \frac{(\lambda-1)\,\lambda\,z^2}{4} - \frac{(\lambda-4)\,(\lambda-1)\,\lambda\,z^3}{36} + \frac{(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^4}{576} - \frac{(\lambda-16)\,(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^5}{14400} + O\left(z^6\right)\right)
$$
$$
+ c_2\left(\left(-\lambda z + 1 + \frac{(\lambda-1)\,\lambda\,z^2}{4} - \frac{(\lambda-4)\,(\lambda-1)\,\lambda\,z^3}{36} + \frac{(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^4}{576} - \frac{(\lambda-16)\,(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda\,z^5}{14400} + O\left(z^6\right)\right)\ln(z) + 2\lambda z + \left(-\frac{\lambda}{2} - \frac{3(\lambda-1)\,\lambda}{4}\right)z^2 + \left(-\frac{(-\lambda+1)\,\lambda}{9} - \frac{(-\lambda+4)\,\lambda}{18} + \frac{11(-\lambda+4)\,(-\lambda+1)\,\lambda}{108}\right)z^3 + \left(-\frac{(\lambda-4)\,(\lambda-1)\,\lambda}{96} - \frac{(\lambda-9)\,(\lambda-1)\,\lambda}{144} - \frac{(\lambda-9)\,(\lambda-4)\,\lambda}{288} - \frac{25(\lambda-9)\,(\lambda-4)\,(\lambda-1)\,\lambda}{3456}\right)z^4 + \left(-\frac{(-\lambda+9)\,(-\lambda+4)\,(-\lambda+1)\,\lambda}{1800} - \frac{(-\lambda+16)\,(-\lambda+4)\,(-\lambda+1)\,\lambda}{2400} - \frac{(-\lambda+16)\,(-\lambda+9)\,(-\lambda+1)\,\lambda}{3600} - \frac{(-\lambda+16)\,(-\lambda+9)\,(-\lambda+4)\,\lambda}{7200} + \frac{137(-\lambda+16)\,(-\lambda+9)\,(-\lambda+4)\,(-\lambda+1)\,\lambda}{432000}\right)z^5 + O\left(z^6\right)\right)
$$

Verified OK.

3.8.1 Maple step by step solution

Let's solve

$-y''z(z-1) + (1-z)\,y' + \lambda y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{\lambda y}{z(z-1)} - \frac{y'}{z}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{z} - \frac{\lambda y}{z(z-1)} = 0$

☐ Check to see if z_0 is a regular singular point

  ◦ Define functions

$\left[P_2(z) = \frac{1}{z}, P_3(z) = -\frac{\lambda}{z(z-1)}\right]$

- ◦ $z \cdot P_2(z)$ is analytic at $z = 0$

  $(z \cdot P_2(z))\bigg|_{z=0} = 1$

- ◦ $z^2 \cdot P_3(z)$ is analytic at $z = 0$

  $(z^2 \cdot P_3(z))\bigg|_{z=0} = 0$

- ◦ $z = 0$is a regular singular point

  Check to see if z_0 is a regular singular point

  $z_0 = 0$

- Multiply by denominators

  $y''z(z-1) + (z-1)\,y' - \lambda y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k z^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $z^m \cdot y'$ to series expansion for $m = 0..1$

  $z^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, z^{k+r-1+m}$

- ◦ Shift index using $k- > k+1-m$

  $z^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, z^{k+r}$

- ◦ Convert $z^m \cdot y''$ to series expansion for $m = 1..2$

  $z^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, z^{k+r-2+m}$

- ◦ Shift index using $k- > k+2-m$

  $z^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, z^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r^2 z^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)^2 + a_k(k^2+2kr+r^2-\lambda)\right) z^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1)^2 + a_k(k^2 - \lambda) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k^2-\lambda)}{(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k^2-\lambda)}{(k+1)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k z^k, a_{k+1} = \frac{a_k(k^2-\lambda)}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
      -> solution has integrals; searching for one without integrals...
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric solution without integrals succesful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 261

```
Order:=6;
dsolve(z*(1-z)*diff(y(z),z$2)+(1-z)*diff(y(z),z)+lambda*y(z)=0,y(z),type='series',z=0);
```

$$
\begin{aligned}
y(z) = {} & \left(2\lambda z+\left(\frac{1}{4}\lambda-\frac{3}{4}\lambda^{2}\right)z^{2}+\left(-\frac{37}{108}\lambda^{2}+\frac{2}{27}\lambda+\frac{11}{108}\lambda^{3}\right)z^{3}\right. \\
& +\left(\frac{139}{1728}\lambda^{3}-\frac{649}{3456}\lambda^{2}+\frac{1}{32}\lambda-\frac{25}{3456}\lambda^{4}\right)z^{4} \\
& \left.+\left(-\frac{13}{1600}\lambda^{4}+\frac{8467}{144000}\lambda^{3}-\frac{2527}{21600}\lambda^{2}+\frac{2}{125}\lambda+\frac{137}{432000}\lambda^{5}\right)z^{5}+\mathrm{O}\left(z^{6}\right)\right)c_{2} \\
& +\left(1-\lambda z+\frac{1}{4}(-1+\lambda)\,\lambda z^{2}-\frac{1}{36}\lambda\left(\lambda^{2}-5\lambda+4\right)z^{3}+\frac{1}{576}\lambda\left(\lambda^{3}-14\lambda^{2}+49\lambda-36\right)z^{4}\right. \\
& \left.-\frac{1}{14400}\lambda(-1+\lambda)\,(\lambda-4)\,(\lambda-16)\,(\lambda-9)\,z^{5}+\mathrm{O}\left(z^{6}\right)\right)\left(c_{2}\ln\left(z\right)+c_{1}\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 940

```
AsymptoticDSolveValue[z*(1-z)*y''[z]+(1-z)*y'[z]+\[Lambda]*y[z]==0,y[z],{z,0,5}]
```

$$
\begin{aligned}
y(z) \to & \left(\frac{1}{25}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\frac{1}{9}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)\lambda\right.\right.\\
& -\frac{1}{16}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\frac{1}{9}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)\lambda-\lambda\right)\lambda-\lambda\Bigg)z^5\\
& +\frac{1}{16}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\frac{1}{9}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)\lambda-\lambda\right)z^4\\
& \left.+\frac{1}{9}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)z^3+\frac{1}{4}(\lambda^2-\lambda)\,z^2-\lambda z+1\right)c_1\\
& +c_2\left(-\frac{2}{125}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\frac{1}{9}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)\lambda\right.\right.\\
& -\frac{1}{16}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\frac{1}{9}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)\lambda-\lambda\right)\lambda-\lambda\Bigg)z^5+\frac{1}{25}\left(\frac{\lambda^3}{2}\right.\\
& -2\lambda^2+\frac{1}{4}(\lambda^2-\lambda)\,\lambda+\frac{2}{27}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)\lambda-\frac{1}{9}\left(\frac{\lambda^3}{2}-2\lambda^2+\frac{1}{4}(\lambda^2-\lambda)\,\lambda\right)\lambda\\
& +\frac{1}{32}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\frac{1}{9}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)\lambda-\lambda\right)\lambda\\
& -\frac{1}{16}\left(\frac{\lambda^3}{2}-2\lambda^2+\frac{1}{4}(\lambda^2-\lambda)\,\lambda+\frac{2}{27}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)\lambda-\frac{1}{9}\left(\frac{\lambda^3}{2}-2\lambda^2+\frac{1}{4}(\lambda^2-\lambda)\,\lambda\right)\lambda\right)\lambda\Bigg)z^5\\
& -\frac{1}{32}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\frac{1}{9}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)\lambda-\lambda\right)z^4+\frac{1}{16}\left(\frac{\lambda^3}{2}-2\lambda^2\right.\\
& \left.+\frac{1}{4}(\lambda^2-\lambda)\,\lambda+\frac{2}{27}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)\lambda-\frac{1}{9}\left(\frac{\lambda^3}{2}-2\lambda^2+\frac{1}{4}(\lambda^2-\lambda)\,\lambda\right)\lambda\right)z^4\\
& -\frac{2}{27}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)z^3+\frac{1}{9}\left(\frac{\lambda^3}{2}-2\lambda^2+\frac{1}{4}(\lambda^2-\lambda)\,\lambda\right)z^3-\frac{\lambda^2z^2}{2}\\
& -\frac{1}{4}(\lambda^2-\lambda)\,z^2+2\lambda z\\
& +\left(\frac{1}{25}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\frac{1}{9}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)\lambda-\frac{1}{16}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\frac{1}{9}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)\right.\right.\right.\\
& +\frac{1}{16}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\frac{1}{9}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)\lambda-\lambda\right)z^4\\
& \left.\left.+\frac{1}{9}\left(\lambda^2-\frac{1}{4}(\lambda^2-\lambda)\,\lambda-\lambda\right)z^3+\frac{1}{4}(\lambda^2-\lambda)\,z^2-\lambda z+1\right)\log(z)\right)
\end{aligned}
$$

3.9 problem Problem 16.11

Internal problem ID [2538]
Internal file name [`OUTPUT/2030_Sunday_June_05_2022_02_45_29_AM_29896145/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 16, Series solutions of ODEs. Section 16.6 Exercises, page 550
Problem number: Problem 16.11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{zy'' + (2z - 3)\, y' + \frac{4y}{z} = 0}$$

With the expansion point for the power series method at $z = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$zy'' + (2z - 3)\, y' + \frac{4y}{z} = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(z)y' + q(z)y = 0$$

Where

$$p(z) = \frac{2z - 3}{z}$$
$$q(z) = \frac{4}{z^2}$$

Table 84: Table $p(z), q(z)$ singularites.

$p(z) = \frac{2z-3}{z}$	
singularity	type
$z = 0$	"regular"

$q(z) = \frac{4}{z^2}$	
singularity	type
$z = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $z = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$z^2 y'' + (2z^2 - 3z)\, y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n z^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n z^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n z^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n z^{n+r-2}\right) z^2 + (2z^2 - 3z) \left(\sum_{n=0}^{\infty} (n+r)\, a_n z^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n z^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} z^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2 z^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3 z^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4 a_n z^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of z be $n+r$ in each summation term. Going over each summation term above with power of z in it which is not already z^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2 z^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2 a_{n-1} (n+r-1) z^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of z are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} z^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2 a_{n-1} (n+r-1) z^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3 z^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4 a_n z^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$z^{n+r} a_n (n+r)(n+r-1) - 3 z^{n+r} a_n (n+r) + 4 a_n z^{n+r} = 0$$

When $n=0$ the above becomes

$$z^r a_0 r(-1+r) - 3 z^r a_0 r + 4 a_0 z^r = 0$$

Or

$$\left(z^r r(-1+r) - 3 z^r r + 4 z^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)^2 z^r = 0$$

Since the above is true for all z then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)^2 z^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(z) = \sum_{n=0}^{\infty} a_n z^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(z) = y_1(z) \ln(z) + \left(\sum_{n=1}^{\infty} b_n z^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(z) + c_2 y_2(z)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$y_1(z) = \sum_{n=0}^{\infty} a_n z^{n+2}$$
$$y_2(z) = y_1(z) \ln(z) + \left(\sum_{n=1}^{\infty} b_n z^{n+2}\right)$$

We start by finding the first solution $y_1(z)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) - 3a_n(n+r) + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}(n+r-1)}{n^2 + 2nr + r^2 - 4n - 4r + 4} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{2a_{n-1}(1+n)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2r}{(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_1 = -4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{(-1+r)^2}$	-4

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4+4r}{r(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_2 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{(-1+r)^2}$	-4
a_2	$\frac{4+4r}{r(-1+r)^2}$	6

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-16 - 8r}{(1 + r) r (-1 + r)^2}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{16}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{(-1+r)^2}$	-4
a_2	$\frac{4+4r}{r(-1+r)^2}$	6
a_3	$\frac{-16-8r}{(1+r)r(-1+r)^2}$	$-\frac{16}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{48 + 16r}{(2 + r)(1 + r) r (-1 + r)^2}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{10}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{(-1+r)^2}$	-4
a_2	$\frac{4+4r}{r(-1+r)^2}$	6
a_3	$\frac{-16-8r}{(1+r)r(-1+r)^2}$	$-\frac{16}{3}$
a_4	$\frac{48+16r}{(2+r)(1+r)r(-1+r)^2}$	$\frac{10}{3}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-128 - 32r}{(3 + r)(2 + r)(1 + r) r (-1 + r)^2}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{8}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{(-1+r)^2}$	-4
a_2	$\frac{4+4r}{r(-1+r)^2}$	6
a_3	$\frac{-16-8r}{(1+r)r(-1+r)^2}$	$-\frac{16}{3}$
a_4	$\frac{48+16r}{(2+r)(1+r)r(-1+r)^2}$	$\frac{10}{3}$
a_5	$\frac{-128-32r}{(3+r)(2+r)(1+r)r(-1+r)^2}$	$-\frac{8}{5}$

Using the above table, then the first solution $y_1(z)$ is

$$\begin{aligned} y_1(z) &= z^2 \left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6 \dots\right) \\ &= z^2 \left(6z^2 - 4z + 1 - \frac{16z^3}{3} + \frac{10z^4}{3} - \frac{8z^5}{5} + O(z^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(z) = y_1(z) \ln(z) + \left(\sum_{n=1}^{\infty} b_n z^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=2)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{2r}{(-1+r)^2}$	-4	$\frac{2r+2}{(-1+r)^3}$	6
b_2	$\frac{4+4r}{r(-1+r)^2}$	6	$\frac{-8r^2-12r+4}{r^2(-1+r)^3}$	-13
b_3	$\frac{-16-8r}{(1+r)r(-1+r)^2}$	$-\frac{16}{3}$	$\frac{24r^3+72r^2+16r-16}{(1+r)^2r^2(-1+r)^3}$	$\frac{124}{9}$
b_4	$\frac{48+16r}{(2+r)(1+r)r(-1+r)^2}$	$\frac{10}{3}$	$\frac{-64r^4-352r^3-448r^2+96}{(2+r)^2(1+r)^2r^2(-1+r)^3}$	$-\frac{173}{18}$
b_5	$\frac{-128-32r}{(3+r)(2+r)(1+r)r(-1+r)^2}$	$-\frac{8}{5}$	$\frac{160r^5+1440r^4+4000r^3+3360r^2-512r-768}{(3+r)^2(2+r)^2(1+r)^2r^2(-1+r)^3}$	$\frac{374}{75}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(z) &= y_1(z)\ln(z) + b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + b_5 z^5 + b_6 z^6 \dots \\
&= z^2\left(6z^2 - 4z + 1 - \frac{16z^3}{3} + \frac{10z^4}{3} - \frac{8z^5}{5} + O(z^6)\right)\ln(z) \\
&\quad + z^2\left(-13z^2 + 6z + \frac{124z^3}{9} - \frac{173z^4}{18} + \frac{374z^5}{75} + O(z^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(z) &= c_1 y_1(z) + c_2 y_2(z) \\
&= c_1 z^2\left(6z^2 - 4z + 1 - \frac{16z^3}{3} + \frac{10z^4}{3} - \frac{8z^5}{5} + O(z^6)\right) \\
&\quad + c_2\left(z^2\left(6z^2 - 4z + 1 - \frac{16z^3}{3} + \frac{10z^4}{3} - \frac{8z^5}{5} + O(z^6)\right)\ln(z)\right. \\
&\qquad \left. + z^2\left(-13z^2 + 6z + \frac{124z^3}{9} - \frac{173z^4}{18} + \frac{374z^5}{75} + O(z^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 z^2\left(6z^2 - 4z + 1 - \frac{16z^3}{3} + \frac{10z^4}{3} - \frac{8z^5}{5} + O(z^6)\right) \\
&\quad + c_2\left(z^2\left(6z^2 - 4z + 1 - \frac{16z^3}{3} + \frac{10z^4}{3} - \frac{8z^5}{5} + O(z^6)\right)\ln(z)\right. \\
&\qquad \left. + z^2\left(-13z^2 + 6z + \frac{124z^3}{9} - \frac{173z^4}{18} + \frac{374z^5}{75} + O(z^6)\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 z^2\left(6z^2-4z+1-\frac{16z^3}{3}+\frac{10z^4}{3}-\frac{8z^5}{5}+O\left(z^6\right)\right) \\
& + c_2\left(z^2\left(6z^2-4z+1-\frac{16z^3}{3}+\frac{10z^4}{3}-\frac{8z^5}{5}+O\left(z^6\right)\right)\ln\left(z\right)\right. \\
& \left. + z^2\left(-13z^2+6z+\frac{124z^3}{9}-\frac{173z^4}{18}+\frac{374z^5}{75}+O\left(z^6\right)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 z^2\left(6z^2-4z+1-\frac{16z^3}{3}+\frac{10z^4}{3}-\frac{8z^5}{5}+O\left(z^6\right)\right) \\
& + c_2\left(z^2\left(6z^2-4z+1-\frac{16z^3}{3}+\frac{10z^4}{3}-\frac{8z^5}{5}+O\left(z^6\right)\right)\ln\left(z\right)\right. \\
& \left. + z^2\left(-13z^2+6z+\frac{124z^3}{9}-\frac{173z^4}{18}+\frac{374z^5}{75}+O\left(z^6\right)\right)\right)
\end{aligned}
$$

Verified OK.

3.9.1 Maple step by step solution

Let's solve

$y''z^2+\left(2z^2-3z\right)y'+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4y}{z^2}-\frac{(2z-3)y'}{z}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2z-3)y'}{z}+\frac{4y}{z^2}=0$

□ Check to see if $z_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(z)=\frac{2z-3}{z}, P_3(z)=\frac{4}{z^2}\right]$

  ◦ $z\cdot P_2(z)$ is analytic at $z=0$

    $\left(z\cdot P_2(z)\right)\bigg|_{z=0}=-3$

- $z^2 \cdot P_3(z)$ is analytic at $z = 0$

  $\left(z^2 \cdot P_3(z)\right)\bigg|_{z=0} = 4$

- $z = 0$is a regular singular point

  Check to see if $z_0 = 0$ is a regular singular point

  $z_0 = 0$

- Multiply by denominators

  $y''z^2 + (2z - 3)\, y'z + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k z^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $z^m \cdot y'$ to series expansion for $m = 1..2$

    $z^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, z^{k+r-1+m}$

  - Shift index using $k- > k+1-m$

    $z^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, z^{k+r}$

  - Convert $z^2 \cdot y''$ to series expansion

    $z^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, z^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2\, z^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 + 2a_{k-1}(k+r-1)\right) z^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 + 2a_{k-1}(k+r-1) = 0$

- Shift index using $k- > k+1$

  $a_{k+1}(k+r-1)^2 + 2a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k(k+r)}{(k+r-1)^2}$
- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{2a_k(k+2)}{(k+1)^2}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k z^{k+2}, a_{k+1} = -\frac{2a_k(k+2)}{(k+1)^2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 69

```
Order:=6;
dsolve(z*diff(y(z),z$2)+(2*z-3)*diff(y(z),z)+4/z*y(z)=0,y(z),type='series',z=0);
```

$$y(z) = \left((c_2 \ln(z) + c_1)\left(1 - 4z + 6z^2 - \frac{16}{3}z^3 + \frac{10}{3}z^4 - \frac{8}{5}z^5 + \mathrm{O}\left(z^6\right)\right) + \left(6z - 13z^2 + \frac{124}{9}z^3 - \frac{173}{18}z^4 + \frac{374}{75}z^5 + \mathrm{O}\left(z^6\right)\right)c_2\right)z^2$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 116

```
AsymptoticDSolveValue[z*y''[z]+(2*z-3)*y'[z]+4/z*y[z]==0,y[z],{z,0,5}]
```

$$
\begin{aligned}
y(z) \to\, & c_1\left(-\frac{8z^5}{5}+\frac{10z^4}{3}-\frac{16z^3}{3}+6z^2-4z+1\right)z^2 \\
& + c_2\left(\left(\frac{374z^5}{75}-\frac{173z^4}{18}+\frac{124z^3}{9}-13z^2+6z\right)z^2\right. \\
& \left. + \left(-\frac{8z^5}{5}+\frac{10z^4}{3}-\frac{16z^3}{3}+6z^2-4z+1\right)z^2\log(z)\right)
\end{aligned}
$$

3.10 problem Problem 16.12 (a)

Internal problem ID [2539]
Internal file name [OUTPUT/2031_Sunday_June_05_2022_02_45_33_AM_11785169/index.tex]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 16, Series solutions of ODEs. Section 16.6 Exercises, page 550
Problem number: Problem 16.12 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(z^2+5z+6)\,y''+2y=0}$$

With the expansion point for the power series method at $z=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy_0'+\frac{x^2}{2}\,f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y_0'}+\cdots \\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{130}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{131}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2y}{z^2+5z+6} \\
F_1 &= \frac{dF_0}{dz} \\
&= \frac{\partial F_0}{\partial z} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{\left(-2z^2-10z-12\right) y' + \left(4z+10\right) y}{\left(z^2+5z+6\right)^2} \\
F_2 &= \frac{dF_1}{dz} \\
&= \frac{\partial F_1}{\partial z} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{\left(8z^3+60z^2+148z+120\right) y' - 8\left(z^2+5z+\frac{13}{2}\right) y}{\left(z^2+5z+6\right)^3} \\
F_3 &= \frac{dF_2}{dz} \\
&= \frac{\partial F_2}{\partial z} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(-32z^4-320z^3-1196z^2-1980z-1224\right) y' + 16\left(z^2+5z+\frac{15}{2}\right) y \left(z+\frac{5}{2}\right)}{\left(z^2+5z+6\right)^4} \\
F_4 &= \frac{dF_3}{dz} \\
&= \frac{\partial F_3}{\partial z} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(144z^5+1800z^4+9024z^3+22680z^2+28560z+14400\right) y' - 16\left(z^4+10z^3+\frac{95}{2}z^2+\frac{225}{2}z+102\right) y}{\left(z^2+5z+6\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $z = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{y(0)}{3} \\
F_1 &= \frac{5y(0)}{18} - \frac{y'(0)}{3} \\
F_2 &= -\frac{13y(0)}{54} + \frac{5y'(0)}{9} \\
F_3 &= \frac{25y(0)}{108} - \frac{17y'(0)}{18} \\
F_4 &= -\frac{17y(0)}{81} + \frac{50y'(0)}{27}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{1}{6}z^{2}+\frac{5}{108}z^{3}-\frac{13}{1296}z^{4}+\frac{5}{2592}z^{5}-\frac{17}{58320}z^{6}\right) y(0) \\
&+\left(z-\frac{1}{18}z^{3}+\frac{5}{216}z^{4}-\frac{17}{2160}z^{5}+\frac{5}{1944}z^{6}\right) y'(0)+O\left(z^{6}\right)
\end{aligned}
$$

Since the expansion point $z=0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(z^{2}+5z+6\right) y''+2y=0
$$

Let the solution be represented as power series of the form

$$
y=\sum_{n=0}^{\infty} a_{n} z^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_{n} z^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_{n} z^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(z^{2}+5z+6\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} z^{n-2}\right)+2\left(\sum_{n=0}^{\infty} a_{n} z^{n}\right)=0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} z^{n} a_{n} n(n-1)\right)+\left(\sum_{n=2}^{\infty} 5n\, z^{n-1} a_{n}(n-1)\right) \\
&+\left(\sum_{n=2}^{\infty} 6n(n-1)\, a_{n} z^{n-2}\right)+\left(\sum_{n=0}^{\infty} 2a_{n} z^{n}\right)=0
\end{aligned} \tag{2}
$$

The next step is to make all powers of z be n in each summation term. Going over each summation term above with power of z in it which is not already z^{n} and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 5n\, z^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 5(n+1)\, a_{n+1} n\, z^n$$

$$\sum_{n=2}^{\infty} 6n(n-1)\, a_n z^{n-2} = \sum_{n=0}^{\infty} 6(n+2)\, a_{n+2}(n+1)\, z^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of z are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} z^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 5(n+1)\, a_{n+1} n\, z^n\right) + \left(\sum_{n=0}^{\infty} 6(n+2)\, a_{n+2}(n+1)\, z^n\right) + \left(\sum_{n=0}^{\infty} 2a_n z^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$12a_2 + 2a_0 = 0$$

$$a_2 = -\frac{a_0}{6}$$

$n = 1$ gives

$$10a_2 + 36a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{5a_0}{108} - \frac{a_1}{18}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + 5(n+1)\, a_{n+1} n + 6(n+2)\, a_{n+2}(n+1) + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_n + 5n^2 a_{n+1} - na_n + 5na_{n+1} + 2a_n}{6\,(n+2)\,(n+1)} \\ &= -\frac{(n^2 - n + 2)\, a_n}{6\,(n+2)\,(n+1)} - \frac{(5n^2 + 5n)\, a_{n+1}}{6\,(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$4a_2 + 30a_3 + 72a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{13a_0}{1296} + \frac{5a_1}{216}$$

For $n = 3$ the recurrence equation gives

$$8a_3 + 60a_4 + 120a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{5a_0}{2592} - \frac{17a_1}{2160}$$

For $n = 4$ the recurrence equation gives

$$14a_4 + 100a_5 + 180a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{17a_0}{58320} + \frac{5a_1}{1944}$$

For $n = 5$ the recurrence equation gives

$$22a_5 + 150a_6 + 252a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{5a_0}{979776} - \frac{689a_1}{816480}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n z^n \\ &= a_3 z^3 + a_2 z^2 + a_1 z + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 z - \frac{a_0 z^2}{6} + \left(\frac{5a_0}{108} - \frac{a_1}{18}\right) z^3 + \left(-\frac{13a_0}{1296} + \frac{5a_1}{216}\right) z^4 + \left(\frac{5a_0}{2592} - \frac{17a_1}{2160}\right) z^5 + \dots$$

Collecting terms, the solution becomes

$$\begin{aligned} y = {} & \left(1 - \frac{1}{6}z^2 + \frac{5}{108}z^3 - \frac{13}{1296}z^4 + \frac{5}{2592}z^5\right) a_0 \\ & + \left(z - \frac{1}{18}z^3 + \frac{5}{216}z^4 - \frac{17}{2160}z^5\right) a_1 + O(z^6) \end{aligned} \tag{3}$$

At $z = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{6}z^2 + \frac{5}{108}z^3 - \frac{13}{1296}z^4 + \frac{5}{2592}z^5\right) c_1 + \left(z - \frac{1}{18}z^3 + \frac{5}{216}z^4 - \frac{17}{2160}z^5\right) c_2 + O(z^6)$$

<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1 - \frac{1}{6}z^2 + \frac{5}{108}z^3 - \frac{13}{1296}z^4 + \frac{5}{2592}z^5 - \frac{17}{58320}z^6\right) y(0) \\ & + \left(z - \frac{1}{18}z^3 + \frac{5}{216}z^4 - \frac{17}{2160}z^5 + \frac{5}{1944}z^6\right) y'(0) + O(z^6) \end{aligned} \tag{1}$$

$$\begin{aligned} y = {} & \left(1 - \frac{1}{6}z^2 + \frac{5}{108}z^3 - \frac{13}{1296}z^4 + \frac{5}{2592}z^5\right) c_1 \\ & + \left(z - \frac{1}{18}z^3 + \frac{5}{216}z^4 - \frac{17}{2160}z^5\right) c_2 + O(z^6) \end{aligned} \tag{2}$$

<u>Verification of solutions</u>

$$\begin{aligned} y = {} & \left(1 - \frac{1}{6}z^2 + \frac{5}{108}z^3 - \frac{13}{1296}z^4 + \frac{5}{2592}z^5 - \frac{17}{58320}z^6\right) y(0) \\ & + \left(z - \frac{1}{18}z^3 + \frac{5}{216}z^4 - \frac{17}{2160}z^5 + \frac{5}{1944}z^6\right) y'(0) + O(z^6) \end{aligned}$$

Verified OK.

$$y = \left(1 - \frac{1}{6}z^2 + \frac{5}{108}z^3 - \frac{13}{1296}z^4 + \frac{5}{2592}z^5\right) c_1 + \left(z - \frac{1}{18}z^3 + \frac{5}{216}z^4 - \frac{17}{2160}z^5\right) c_2 + O(z^6)$$

Verified OK.

3.10.1 Maple step by step solution

Let's solve

$(z^2+5z+6)\,y''+2y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{2y}{z^2+5z+6}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{2y}{z^2+5z+6}=0$

□ Check to see if z_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(z)=0, P_3(z)=\frac{2}{z^2+5z+6}\right]$

  ◦ $(z+3)\cdot P_2(z)$ is analytic at $z=-3$

  $\left((z+3)\cdot P_2(z)\right)\bigg|_{z=-3}=0$

  ◦ $(z+3)^2\cdot P_3(z)$ is analytic at $z=-3$

  $\left((z+3)^2\cdot P_3(z)\right)\bigg|_{z=-3}=0$

  ◦ $z=-3$is a regular singular point

Check to see if z_0 is a regular singular point

$z_0=-3$

- Multiply by denominators

$(z^2+5z+6)\,y''+2y=0$

- Change variables using $z=u-3$ so that the regular singular point is at $u=0$

$(u^2-u)\left(\frac{d^2}{du^2}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0r(-1+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)(k+r) + a_k(k^2+2kr+r^2-k-r+2)\right)u^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 1\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + (2r-1)k + r^2 - r + 2)\,a_k - a_{k+1}(k+1+r)(k+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{(k^2+2kr+r^2-k-r+2)a_k}{(k+1+r)(k+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{(k^2-k+2)a_k}{(k+1)k}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{(k^2-k+2)a_k}{(k+1)k}\right]$$

- Revert the change of variables $u = z + 3$

  $$\left[y = \sum_{k=0}^{\infty} a_k(z+3)^k, a_{k+1} = \frac{(k^2-k+2)a_k}{(k+1)k}\right]$$

- Recursion relation for $r = 1$

  $$a_{k+1} = \frac{(k^2+k+2)a_k}{(k+2)(k+1)}$$

- Solution for $r = 1$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+1} = \frac{(k^2+k+2)a_k}{(k+2)(k+1)}\right]$$

- Revert the change of variables $u = z + 3$

$$\left[y = \sum_{k=0}^{\infty} a_k (z+3)^{k+1}, a_{k+1} = \frac{(k^2+k+2) a_k}{(k+2)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (z+3)^k\right) + \left(\sum_{k=0}^{\infty} b_k (z+3)^{k+1}\right), a_{k+1} = \frac{(k^2-k+2) a_k}{(k+1)k}, b_{k+1} = \frac{(k^2+k+2) b_k}{(k+2)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
Order:=6;
dsolve((z^2+5*z+6)*diff(y(z),z$2)+2*y(z)=0,y(z),type='series',z=0);
```

$$y(z) = \left(1 - \frac{1}{6}z^2 + \frac{5}{108}z^3 - \frac{13}{1296}z^4 + \frac{5}{2592}z^5\right) y(0) + \left(z - \frac{1}{18}z^3 + \frac{5}{216}z^4 - \frac{17}{2160}z^5\right) D(y)(0) + O(z^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[(z^2+5*z+6)*y''[z]+2*y[z]==0,y[z],{z,0,5}]
```

$$y(z) \to c_2\left(-\frac{17z^5}{2160} + \frac{5z^4}{216} - \frac{z^3}{18} + z\right) + c_1\left(\frac{5z^5}{2592} - \frac{13z^4}{1296} + \frac{5z^3}{108} - \frac{z^2}{6} + 1\right)$$

3.11 problem Problem 16.12 (b)

Internal problem ID [2540]
Internal file name [OUTPUT/2032_Sunday_June_05_2022_02_45_35_AM_62567743/index.tex]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 16, Series solutions of ODEs. Section 16.6 Exercises, page 550
Problem number: Problem 16.12 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{(z^2+5z+7)\,y''+2y=0}$$

With the expansion point for the power series method at $z=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{133}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{134}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2y}{z^2+5z+7}\\
F_1 &= \frac{dF_0}{dz}\\
&= \frac{\partial F_0}{\partial z} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
&= \frac{(-2z^2-10z-14)\,y' + (4z+10)\,y}{(z^2+5z+7)^2}\\
F_2 &= \frac{dF_1}{dz}\\
&= \frac{\partial F_1}{\partial z} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\
&= \frac{(8z^3+60z^2+156z+140)\,y' - 8\left(z^2+5z+\frac{11}{2}\right)y}{(z^2+5z+7)^3}\\
F_3 &= \frac{dF_2}{dz}\\
&= \frac{\partial F_2}{\partial z} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\
&= \frac{(-32z^4-320z^3-1212z^2-2060z-1316)\,y' + 16y\left(z^2+5z+\frac{5}{2}\right)\left(z+\frac{5}{2}\right)}{(z^2+5z+7)^4}\\
F_4 &= \frac{dF_3}{dz}\\
&= \frac{\partial F_3}{\partial z} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\
&= \frac{144(z^2+5z+7)\,(z^2+5z+5)\left(z+\frac{5}{2}\right)y' - 16\left(z^4+10z^3+\frac{15}{2}z^2-\frac{175}{2}z-\frac{289}{2}\right)y}{(z^2+5z+7)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $z=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{2y(0)}{7}\\
F_1 &= \frac{10y(0)}{49} - \frac{2y'(0)}{7}\\
F_2 &= -\frac{44y(0)}{343} + \frac{20y'(0)}{49}\\
F_3 &= \frac{100y(0)}{2401} - \frac{188y'(0)}{343}\\
F_4 &= \frac{2312y(0)}{16807} + \frac{1800y'(0)}{2401}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{1}{7}z^2 + \frac{5}{147}z^3 - \frac{11}{2058}z^4 + \frac{5}{14406}z^5 + \frac{289}{1512630}z^6\right) y(0)$$
$$+ \left(z - \frac{1}{21}z^3 + \frac{5}{294}z^4 - \frac{47}{10290}z^5 + \frac{5}{4802}z^6\right) y'(0) + O(z^6)$$

Since the expansion point $z = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(z^2 + 5z + 7\right) y'' + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n z^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n z^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n z^{n-2}$$

Substituting the above back into the ode gives

$$\left(z^2 + 5z + 7\right) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n z^{n-2}\right) + 2\left(\sum_{n=0}^{\infty} a_n z^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} z^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 5n\, z^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 7n(n-1)\, a_n z^{n-2}\right) + \left(\sum_{n=0}^{\infty} 2a_n z^n\right) = 0 \tag{2}$$

The next step is to make all powers of z be n in each summation term. Going over each summation term above with power of z in it which is not already z^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 5n\, z^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 5(n+1)\, a_{n+1} n\, z^n$$

$$\sum_{n=2}^{\infty} 7n(n-1)\, a_n z^{n-2} = \sum_{n=0}^{\infty} 7(n+2)\, a_{n+2}(n+1)\, z^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of z are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} z^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 5(n+1)\, a_{n+1} n\, z^n\right) + \left(\sum_{n=0}^{\infty} 7(n+2)\, a_{n+2}(n+1)\, z^n\right) + \left(\sum_{n=0}^{\infty} 2a_n z^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$14a_2 + 2a_0 = 0$$

$$a_2 = -\frac{a_0}{7}$$

$n = 1$ gives

$$10a_2 + 42a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{5a_0}{147} - \frac{a_1}{21}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + 5(n+1)\, a_{n+1} n + 7(n+2)\, a_{n+2}(n+1) + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_n + 5n^2 a_{n+1} - na_n + 5na_{n+1} + 2a_n}{7\,(n+2)\,(n+1)} \\ &= -\frac{(n^2 - n + 2)\, a_n}{7\,(n+2)\,(n+1)} - \frac{(5n^2 + 5n)\, a_{n+1}}{7\,(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$4a_2 + 30a_3 + 84a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{11a_0}{2058} + \frac{5a_1}{294}$$

For $n = 3$ the recurrence equation gives

$$8a_3 + 60a_4 + 140a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{5a_0}{14406} - \frac{47a_1}{10290}$$

For $n = 4$ the recurrence equation gives

$$14a_4 + 100a_5 + 210a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{289a_0}{1512630} + \frac{5a_1}{4802}$$

For $n = 5$ the recurrence equation gives

$$22a_5 + 150a_6 + 294a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{305a_0}{2470629} - \frac{1003a_1}{5294205}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n z^n \\ &= a_3 z^3 + a_2 z^2 + a_1 z + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 z - \frac{a_0 z^2}{7} + \left(\frac{5a_0}{147} - \frac{a_1}{21}\right) z^3 + \left(-\frac{11a_0}{2058} + \frac{5a_1}{294}\right) z^4 + \left(\frac{5a_0}{14406} - \frac{47a_1}{10290}\right) z^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{7}z^2 + \frac{5}{147}z^3 - \frac{11}{2058}z^4 + \frac{5}{14406}z^5\right) a_0 + \left(z - \frac{1}{21}z^3 + \frac{5}{294}z^4 - \frac{47}{10290}z^5\right) a_1 + O(z^6) \tag{3}$$

At $z = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{7}z^2 + \frac{5}{147}z^3 - \frac{11}{2058}z^4 + \frac{5}{14406}z^5\right) c_1 + \left(z - \frac{1}{21}z^3 + \frac{5}{294}z^4 - \frac{47}{10290}z^5\right) c_2 + O(z^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{7}z^2 + \frac{5}{147}z^3 - \frac{11}{2058}z^4 + \frac{5}{14406}z^5 + \frac{289}{1512630}z^6\right) y(0) + \left(z - \frac{1}{21}z^3 + \frac{5}{294}z^4 - \frac{47}{10290}z^5 + \frac{5}{4802}z^6\right) y'(0) + O(z^6) \tag{1}$$

$$y = \left(1 - \frac{1}{7}z^2 + \frac{5}{147}z^3 - \frac{11}{2058}z^4 + \frac{5}{14406}z^5\right) c_1 + \left(z - \frac{1}{21}z^3 + \frac{5}{294}z^4 - \frac{47}{10290}z^5\right) c_2 + O(z^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{7}z^2 + \frac{5}{147}z^3 - \frac{11}{2058}z^4 + \frac{5}{14406}z^5 + \frac{289}{1512630}z^6\right) y(0) + \left(z - \frac{1}{21}z^3 + \frac{5}{294}z^4 - \frac{47}{10290}z^5 + \frac{5}{4802}z^6\right) y'(0) + O(z^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{7}z^2 + \frac{5}{147}z^3 - \frac{11}{2058}z^4 + \frac{5}{14406}z^5\right) c_1 + \left(z - \frac{1}{21}z^3 + \frac{5}{294}z^4 - \frac{47}{10290}z^5\right) c_2 + O(z^6)$$

Verified OK.

3.11.1 Maple step by step solution

Let's solve

$(z^2+5z+7)\,y''+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y}{z^2+5z+7}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2y}{z^2+5z+7}=0$

□ Check to see if z_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(z)=0, P_3(z)=\frac{2}{z^2+5z+7}\right]$

  ◦ $\left(z+\frac{5}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)\cdot P_2(z)$ is analytic at $z=-\frac{5}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$

  $\left(\left(z+\frac{5}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)\cdot P_2(z)\right)\Bigg|_{z=-\frac{5}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}=0$

  ◦ $\left(z+\frac{5}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2\cdot P_3(z)$ is analytic at $z=-\frac{5}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$

  $\left(\left(z+\frac{5}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2\cdot P_3(z)\right)\Bigg|_{z=-\frac{5}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}=0$

  ◦ $z=-\frac{5}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$is a regular singular point

  Check to see if z_0 is a regular singular point

  $z_0=-\frac{5}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$

- Multiply by denominators

  $(z^2+5z+7)\,y''+2y=0$

- Change variables using $z=u-\frac{5}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$ so that the regular singular point is at $u=0$

  $(u^2-\mathrm{I}u\sqrt{3})\left(\frac{d^2}{du^2}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - ○ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$
  - ○ Shift index using $k- >k+2-m$

    $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$-\mathrm{I}\sqrt{3}\,r(r-1)\,a_0u^{r-1} + \left(\sum_{k=0}^{\infty}\left(-\mathrm{I}\sqrt{3}\,(k+1+r)(k+r)\,a_{k+1} + a_k(k^2+2kr+r^2-k-r+2)\right)u$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-\mathrm{I}\sqrt{3}\,r(r-1) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 1\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-\mathrm{I}\sqrt{3}\,(k+1+r)(k+r)\,a_{k+1} + (k^2+(2r-1)\,k+r^2-r+2)\,a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{-\frac{\mathrm{I}}{3}a_k\,(k^2+2kr+r^2-k-r+2)\sqrt{3}}{k^2+2kr+r^2+k+r}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{-\frac{\mathrm{I}}{3}a_k\,(k^2-k+2)\sqrt{3}}{k^2+k}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{3}a_k\,(k^2-k+2)\sqrt{3}}{k^2+k}\right]$$

- Revert the change of variables $u = z + \frac{5}{2} + \frac{\mathrm{I}\sqrt{3}}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k\left(z+\frac{5}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{3}a_k\,(k^2-k+2)\sqrt{3}}{k^2+k}\right]$$

- Recursion relation for $r = 1$

  $$a_{k+1} = \frac{-\frac{\mathrm{I}}{3}a_k\,(k^2+k+2)\sqrt{3}}{k^2+3k+2}$$

- Solution for $r = 1$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+1} = \frac{-\frac{\mathrm{I}}{3}a_k\,(k^2+k+2)\sqrt{3}}{k^2+3k+2}\right]$$

- Revert the change of variables $u = z + \frac{5}{2} + \frac{I\sqrt{3}}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(z + \frac{5}{2} + \frac{I\sqrt{3}}{2}\right)^{k+1}, a_{k+1} = \frac{-\frac{I}{3} a_k (k^2+k+2)\sqrt{3}}{k^2+3k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(z + \frac{5}{2} + \frac{I\sqrt{3}}{2}\right)^{k}\right) + \left(\sum_{k=0}^{\infty} b_k \left(z + \frac{5}{2} + \frac{I\sqrt{3}}{2}\right)^{k+1}\right), a_{k+1} = \frac{-\frac{I}{3} a_k (k^2-k+2)\sqrt{3}}{k^2+k}, b_{k+1} = \frac{-\frac{I}{3} b_k$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve((z^2+5*z+7)*diff(y(z),z$2)+2*y(z)=0,y(z),type='series',z=0);
```

$$y(z) = \left(1-\frac{1}{7}z^2+\frac{5}{147}z^3-\frac{11}{2058}z^4+\frac{5}{14406}z^5\right)y(0) + \left(z-\frac{1}{21}z^3+\frac{5}{294}z^4-\frac{47}{10290}z^5\right)D(y)(0)+O(z^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[(z^2+5*z+7)*y''[z]+2*y[z]==0,y[z],{z,0,5}]
```

$$y(z) \to c_2\left(-\frac{47z^5}{10290}+\frac{5z^4}{294}-\frac{z^3}{21}+z\right)+c_1\left(\frac{5z^5}{14406}-\frac{11z^4}{2058}+\frac{5z^3}{147}-\frac{z^2}{7}+1\right)$$

3.12 problem Problem 16.13

Internal problem ID [2541]
Internal file name [`OUTPUT/2033_Sunday_June_05_2022_02_45_37_AM_84665228/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 16, Series solutions of ODEs. Section 16.6 Exercises, page 550
Problem number: Problem 16.13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + \frac{y}{z^3} = 0}$$

With the expansion point for the power series method at $z = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' + \frac{y}{z^3} = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(z)y' + q(z)y = 0$$

Where

$$p(z) = 0$$
$$q(z) = \frac{1}{z^3}$$

Table 88: Table $p(z), q(z)$ singularites.

$p(z) = 0$	
singularity	type

$q(z) = \frac{1}{z^3}$	
singularity	type
$z = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $z = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $z = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(diff(y(z),z$2)+1/z^3*y(z)=0,y(z),type='series',z=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 222

```
AsymptoticDSolveValue[y''[z]+1/z^3*y[z]==0,y[z],{z,0,5}]
```

$$y(z) \to c_1 e^{-\frac{2i}{\sqrt{z}}} z^{3/4} \left(-\frac{468131288625 i z^{9/2}}{8796093022208} + \frac{66891825 i z^{7/2}}{4294967296} - \frac{72765 i z^{5/2}}{8388608} + \frac{105 i z^{3/2}}{8192} + \frac{33424574007825 z^5}{281474976710656} - \frac{14783093325 z^4}{549755813888} + \frac{2837835 z^3}{268435456} - \frac{4725 z^2}{524288} + \frac{15 z}{512} - \frac{3 i \sqrt{z}}{16} + 1 \right) + c_2 e^{\frac{2i}{\sqrt{z}}} z^{3/4} \left(\frac{468131288625 i z^{9/2}}{8796093022208} - \frac{66891825 i z^{7/2}}{4294967296} + \frac{72765 i z^{5/2}}{8388608} - \frac{105 i z^{3/2}}{8192} + \frac{33424574007825 z^5}{281474976710656} - \right.$$

3.13 problem Problem 16.14

Internal problem ID [2542]
Internal file name [`OUTPUT/2034_Sunday_June_05_2022_02_45_39_AM_65723490/index.tex`]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 16, Series solutions of ODEs. Section 16.6 Exercises, page 550
Problem number: Problem 16.14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{zy'' + (1 - z)\, y' + \lambda y = 0}$$

With the expansion point for the power series method at $z = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$zy'' + (1 - z)\, y' + \lambda y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(z)y' + q(z)y = 0$$

Where

$$p(z) = -\frac{z - 1}{z}$$
$$q(z) = \frac{\lambda}{z}$$

Table 89: Table $p(z), q(z)$ singularites.

$p(z) = -\frac{z-1}{z}$	
singularity	type
$z = 0$	"regular"

$q(z) = \frac{\lambda}{z}$	
singularity	type
$z = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $z = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$zy'' + (1 - z)\, y' + \lambda y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n z^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n z^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n z^{n+r-2}$$

Substituting the above back into the ode gives

$$z \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n z^{n+r-2}\right) + (1 - z) \left(\sum_{n=0}^{\infty} (n + r)\, a_n z^{n+r-1}\right) + \lambda \left(\sum_{n=0}^{\infty} a_n z^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} z^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-z^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n z^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} \lambda a_n z^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of z be $n+r-1$ in each summation term. Going over each summation term above with power of z in it which is not already z^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-z^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) z^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \lambda a_n z^{n+r} = \sum_{n=1}^{\infty} \lambda a_{n-1} z^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of z are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} z^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) z^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n z^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} \lambda a_{n-1} z^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$z^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n z^{n+r-1} = 0$$

When $n=0$ the above becomes

$$z^{-1+r} a_0 r(-1+r) + r a_0 z^{-1+r} = 0$$

Or

$$\left(z^{-1+r} r(-1+r) + r\, z^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$z^{-1+r} r^2 = 0$$

Since the above is true for all z then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$z^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(z) = \sum_{n=0}^{\infty} a_n z^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(z) = y_1(z) \ln(z) + \left(\sum_{n=1}^{\infty} b_n z^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(z) + c_2 y_2(z)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(z)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_n(n+r) + \lambda a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(\lambda - n - r + 1)}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}(-\lambda + n - 1)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r - \lambda}{(r + 1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -\lambda$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-\lambda}{(r+1)^2}$	$-\lambda$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(\lambda - 1 - r)(\lambda - r)}{(r + 1)^2 (2 + r)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{(\lambda - 1)\lambda}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-\lambda}{(r+1)^2}$	$-\lambda$
a_2	$\frac{(\lambda-1-r)(\lambda-r)}{(r+1)^2(2+r)^2}$	$\frac{(\lambda-1)\lambda}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(-\lambda + 2 + r)(-\lambda + 1 + r)(r - \lambda)}{(r + 1)^2 (2 + r)^2 (r + 3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{(\lambda - 2)(\lambda - 1)\lambda}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-\lambda}{(r+1)^2}$	$-\lambda$
a_2	$\frac{(\lambda-1-r)(\lambda-r)}{(r+1)^2(2+r)^2}$	$\frac{(\lambda-1)\lambda}{4}$
a_3	$\frac{(-\lambda+2+r)(-\lambda+1+r)(r-\lambda)}{(r+1)^2(2+r)^2(r+3)^2}$	$-\frac{(\lambda-2)(\lambda-1)\lambda}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(\lambda - 3 - r)(\lambda - 2 - r)(\lambda - 1 - r)(\lambda - r)}{(r + 1)^2 (2 + r)^2 (r + 3)^2 (4 + r)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{(\lambda - 3)(\lambda - 2)(\lambda - 1)\lambda}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-\lambda}{(r+1)^2}$	$-\lambda$
a_2	$\frac{(\lambda-1-r)(\lambda-r)}{(r+1)^2(2+r)^2}$	$\frac{(\lambda-1)\lambda}{4}$
a_3	$\frac{(-\lambda+2+r)(-\lambda+1+r)(r-\lambda)}{(r+1)^2(2+r)^2(r+3)^2}$	$-\frac{(\lambda-2)(\lambda-1)\lambda}{36}$
a_4	$\frac{(\lambda-3-r)(\lambda-2-r)(\lambda-1-r)(\lambda-r)}{(r+1)^2(2+r)^2(r+3)^2(4+r)^2}$	$\frac{(\lambda-3)(\lambda-2)(\lambda-1)\lambda}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(-\lambda + 4 + r)(-\lambda + 3 + r)(-\lambda + 2 + r)(-\lambda + 1 + r)(r - \lambda)}{(r + 1)^2 (2 + r)^2 (r + 3)^2 (4 + r)^2 (5 + r)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{(\lambda-4)(\lambda-3)(\lambda-2)(\lambda-1)\lambda}{14400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-\lambda}{(r+1)^2}$	$-\lambda$
a_2	$\frac{(\lambda-1-r)(\lambda-r)}{(r+1)^2(2+r)^2}$	$\frac{(\lambda-1)\lambda}{4}$
a_3	$\frac{(-\lambda+2+r)(-\lambda+1+r)(r-\lambda)}{(r+1)^2(2+r)^2(r+3)^2}$	$-\frac{(\lambda-2)(\lambda-1)\lambda}{36}$
a_4	$\frac{(\lambda-3-r)(\lambda-2-r)(\lambda-1-r)(\lambda-r)}{(r+1)^2(2+r)^2(r+3)^2(4+r)^2}$	$\frac{(\lambda-3)(\lambda-2)(\lambda-1)\lambda}{576}$
a_5	$\frac{(-\lambda+4+r)(-\lambda+3+r)(-\lambda+2+r)(-\lambda+1+r)(r-\lambda)}{(r+1)^2(2+r)^2(r+3)^2(4+r)^2(5+r)^2}$	$-\frac{(\lambda-4)(\lambda-3)(\lambda-2)(\lambda-1)\lambda}{14400}$

Using the above table, then the first solution $y_1(z)$ becomes

$$\begin{aligned} y_1(z) &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6 \dots \\ &= -\lambda z + 1 + \frac{(\lambda-1)\lambda z^2}{4} - \frac{(\lambda-2)(\lambda-1)\lambda z^3}{36} + \frac{(\lambda-3)(\lambda-2)(\lambda-1)\lambda z^4}{576} \\ &\quad - \frac{(\lambda-4)(\lambda-3)(\lambda-2)(\lambda-1)\lambda z^5}{14400} + O(z^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(z) = y_1(z)\ln(z) + \left(\sum_{n=1}^{\infty} b_n z^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{r-\lambda}{(r+1)^2}$	$-\lambda$	$\frac{-r+1+2\lambda}{(r+1)^3}$
b_2	$\frac{(\lambda-1-r)(\lambda-r)}{(r+1)^2(2+r)^2}$	$\frac{(\lambda-1)\lambda}{4}$	$\frac{-2r^3+(6\lambda-3)r^2+(-4\lambda^2+10\lambda+1)r-6\lambda^2+2\lambda+2}{(r+1)^3(2+r)^3}$
b_3	$\frac{(-\lambda+2+r)(-\lambda+1+r)(r-\lambda)}{(r+1)^2(2+r)^2(r+3)^2}$	$-\frac{(\lambda-2)(\lambda-1)\lambda}{36}$	$\frac{12-3r^5+6(-3+2\lambda)r^4+(-15\lambda^2+66\lambda-35)r^3+6(\lambda}{(r+}$
b_4	$\frac{(\lambda-3-r)(\lambda-2-r)(\lambda-1-r)(\lambda-r)}{(r+1)^2(2+r)^2(r+3)^2(4+r)^2}$	$\frac{(\lambda-3)(\lambda-2)(\lambda-1)\lambda}{576}$	$\frac{-4r^7+(20\lambda-50)r^6+(-36\lambda^2+228\lambda-246)r^5+(28\lambda}{}$
b_5	$\frac{(-\lambda+4+r)(-\lambda+3+r)(-\lambda+2+r)(-\lambda+1+r)(r-\lambda)}{(r+1)^2(2+r)^2(r+3)^2(4+r)^2(5+r)^2}$	$-\frac{(\lambda-4)(\lambda-3)(\lambda-2)(\lambda-1)\lambda}{14400}$	$\frac{-5r^9+(30\lambda-105)r^8+(-70\lambda^2+580\lambda-930)r^7+(80}{}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(z) &= y_1(z)\ln(z) + b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + b_5 z^5 + b_6 z^6 \dots \\
&= \left(-\lambda z + 1 + \frac{(\lambda-1)\lambda z^2}{4} - \frac{(\lambda-2)(\lambda-1)\lambda z^3}{36} + \frac{(\lambda-3)(\lambda-2)(\lambda-1)\lambda z^4}{576}\right. \\
&\quad \left. - \frac{(\lambda-4)(\lambda-3)(\lambda-2)(\lambda-1)\lambda z^5}{14400} + O(z^6)\right)\ln(z) + (1+2\lambda) z \\
&\quad + \left(-\frac{\lambda}{2} + \frac{1}{4} - \frac{3(\lambda-1)\lambda}{4}\right) z^2 + \left(-\frac{(-\lambda+1)\lambda}{36} - \frac{(-\lambda+2)\lambda}{36} + \frac{(-\lambda+2)(-\lambda+1)}{36}\right. \\
&\quad \left. + \frac{11(-\lambda+2)(-\lambda+1)\lambda}{108}\right) z^3 + \left(-\frac{(\lambda-2)(\lambda-1)\lambda}{576} - \frac{(\lambda-3)(\lambda-1)\lambda}{576}\right. \\
&\quad \left. - \frac{(\lambda-3)(\lambda-2)\lambda}{576} - \frac{(\lambda-3)(\lambda-2)(\lambda-1)}{576} - \frac{25(\lambda-3)(\lambda-2)(\lambda-1)\lambda}{3456}\right) z^4 \\
&\quad + \left(-\frac{(-\lambda+3)(-\lambda+2)(-\lambda+1)\lambda}{14400} - \frac{(-\lambda+4)(-\lambda+2)(-\lambda+1)\lambda}{14400}\right. \\
&\quad - \frac{(-\lambda+4)(-\lambda+3)(-\lambda+1)\lambda}{14400} - \frac{(-\lambda+4)(-\lambda+3)(-\lambda+2)\lambda}{14400} \\
&\quad + \frac{(-\lambda+4)(-\lambda+3)(-\lambda+2)(-\lambda+1)}{14400} \\
&\quad \left. + \frac{137(-\lambda+4)(-\lambda+3)(-\lambda+2)(-\lambda+1)\lambda}{432000}\right) z^5 + O(z^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
y_h(z) = c_1 y_1(z) + c_2 y_2(z)
$$

$$
\begin{aligned}
&= c_1\left(-\lambda z+1+\frac{(\lambda-1)\,\lambda\,z^2}{4}-\frac{(\lambda-2)\,(\lambda-1)\,\lambda\,z^3}{36}+\frac{(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^4}{576}\right.\\
&\quad\left.-\frac{(\lambda-4)\,(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^5}{14400}+O\left(z^6\right)\right)+c_2\left(\left(-\lambda z+1\right.\right.\\
&\quad+\frac{(\lambda-1)\,\lambda\,z^2}{4}-\frac{(\lambda-2)\,(\lambda-1)\,\lambda\,z^3}{36}+\frac{(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^4}{576}\\
&\quad\left.-\frac{(\lambda-4)\,(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^5}{14400}+O\left(z^6\right)\right)\ln\left(z\right)+\left(1+2\lambda\right)z\\
&\quad+\left(-\frac{\lambda}{2}+\frac{1}{4}-\frac{3(\lambda-1)\,\lambda}{4}\right)z^2+\left(-\frac{(-\lambda+1)\,\lambda}{36}-\frac{(-\lambda+2)\,\lambda}{36}+\frac{(-\lambda+2)\,(-\lambda+1)}{36}\right.\\
&\quad\left.+\frac{11(-\lambda+2)\,(-\lambda+1)\,\lambda}{108}\right)z^3+\left(-\frac{(\lambda-2)\,(\lambda-1)\,\lambda}{576}-\frac{(\lambda-3)\,(\lambda-1)\,\lambda}{576}\right.\\
&\quad\left.-\frac{(\lambda-3)\,(\lambda-2)\,\lambda}{576}-\frac{(\lambda-3)\,(\lambda-2)\,(\lambda-1)}{576}-\frac{25(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda}{3456}\right)z^4\\
&\quad+\left(-\frac{(-\lambda+3)\,(-\lambda+2)\,(-\lambda+1)\,\lambda}{14400}-\frac{(-\lambda+4)\,(-\lambda+2)\,(-\lambda+1)\,\lambda}{14400}\right.\\
&\quad-\frac{(-\lambda+4)\,(-\lambda+3)\,(-\lambda+1)\,\lambda}{14400}-\frac{(-\lambda+4)\,(-\lambda+3)\,(-\lambda+2)\,\lambda}{14400}\\
&\quad+\frac{(-\lambda+4)\,(-\lambda+3)\,(-\lambda+2)\,(-\lambda+1)}{14400}\\
&\quad\left.\left.+\frac{137(-\lambda+4)\,(-\lambda+3)\,(-\lambda+2)\,(-\lambda+1)\,\lambda}{432000}\right)z^5+O\left(z^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$y = y_h$

$$
\begin{aligned}
&= c_1\left(-\lambda z+1+\frac{(\lambda-1)\,\lambda\,z^2}{4}-\frac{(\lambda-2)\,(\lambda-1)\,\lambda\,z^3}{36}+\frac{(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^4}{576}\right.\\
&\qquad\left.-\frac{(\lambda-4)\,(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^5}{14400}+O\left(z^6\right)\right)\\
&\quad+c_2\left(\left(-\lambda z+1+\frac{(\lambda-1)\,\lambda\,z^2}{4}-\frac{(\lambda-2)\,(\lambda-1)\,\lambda\,z^3}{36}+\frac{(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^4}{576}\right.\right.\\
&\qquad\left.-\frac{(\lambda-4)\,(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^5}{14400}+O\left(z^6\right)\right)\ln(z)+(1+2\lambda)\,z\\
&\qquad+\left(-\frac{\lambda}{2}+\frac{1}{4}-\frac{3(\lambda-1)\,\lambda}{4}\right)z^2\\
&\quad+\left(-\frac{(-\lambda+1)\,\lambda}{36}-\frac{(-\lambda+2)\,\lambda}{36}+\frac{(-\lambda+2)\,(-\lambda+1)}{36}+\frac{11(-\lambda+2)\,(-\lambda+1)\,\lambda}{108}\right)z^3\\
&\qquad+\left(-\frac{(\lambda-2)\,(\lambda-1)\,\lambda}{576}-\frac{(\lambda-3)\,(\lambda-1)\,\lambda}{576}-\frac{(\lambda-3)\,(\lambda-2)\,\lambda}{576}\right.\\
&\qquad\left.-\frac{(\lambda-3)\,(\lambda-2)\,(\lambda-1)}{576}-\frac{25(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda}{3456}\right)z^4\\
&\qquad+\left(-\frac{(-\lambda+3)\,(-\lambda+2)\,(-\lambda+1)\,\lambda}{14400}-\frac{(-\lambda+4)\,(-\lambda+2)\,(-\lambda+1)\,\lambda}{14400}\right.\\
&\qquad-\frac{(-\lambda+4)\,(-\lambda+3)\,(-\lambda+1)\,\lambda}{14400}-\frac{(-\lambda+4)\,(-\lambda+3)\,(-\lambda+2)\,\lambda}{14400}\\
&\qquad+\frac{(-\lambda+4)\,(-\lambda+3)\,(-\lambda+2)\,(-\lambda+1)}{14400}\\
&\qquad\left.\left.+\frac{137(-\lambda+4)\,(-\lambda+3)\,(-\lambda+2)\,(-\lambda+1)\,\lambda}{432000}\right)z^5+O\left(z^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1\left(-\lambda z+1+\frac{(\lambda-1)\,\lambda\,z^2}{4}-\frac{(\lambda-2)\,(\lambda-1)\,\lambda\,z^3}{36}+\frac{(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^4}{576}\right. \\
& \left.-\frac{(\lambda-4)\,(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^5}{14400}+O\!\left(z^6\right)\right) \\
& +c_2\left(\left(-\lambda z+1+\frac{(\lambda-1)\,\lambda\,z^2}{4}-\frac{(\lambda-2)\,(\lambda-1)\,\lambda\,z^3}{36}+\frac{(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^4}{576}\right.\right. \\
& \left.-\frac{(\lambda-4)\,(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^5}{14400}+O\!\left(z^6\right)\right)\ln(z)+(1+2\lambda)\,z \\
& +\left(-\frac{\lambda}{2}+\frac{1}{4}-\frac{3(\lambda-1)\,\lambda}{4}\right)z^2+\left(-\frac{(-\lambda+1)\,\lambda}{36}-\frac{(-\lambda+2)\,\lambda}{36}+\frac{(-\lambda+2)\,(-\lambda+1)}{36}\right. \\
& \left.+\frac{11(-\lambda+2)\,(-\lambda+1)\,\lambda}{108}\right)z^3+\left(-\frac{(\lambda-2)\,(\lambda-1)\,\lambda}{576}-\frac{(\lambda-3)\,(\lambda-1)\,\lambda}{576}\right. \\
& \left.-\frac{(\lambda-3)\,(\lambda-2)\,\lambda}{576}-\frac{(\lambda-3)\,(\lambda-2)\,(\lambda-1)}{576}-\frac{25(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda}{3456}\right)z^4 \\
& +\left(-\frac{(-\lambda+3)\,(-\lambda+2)\,(-\lambda+1)\,\lambda}{14400}-\frac{(-\lambda+4)\,(-\lambda+2)\,(-\lambda+1)\,\lambda}{14400}\right. \\
& -\frac{(-\lambda+4)\,(-\lambda+3)\,(-\lambda+1)\,\lambda}{14400}-\frac{(-\lambda+4)\,(-\lambda+3)\,(-\lambda+2)\,\lambda}{14400} \\
& +\frac{(-\lambda+4)\,(-\lambda+3)\,(-\lambda+2)\,(-\lambda+1)}{14400} \\
& \left.\left.+\frac{137(-\lambda+4)\,(-\lambda+3)\,(-\lambda+2)\,(-\lambda+1)\,\lambda}{432000}\right)z^5+O\!\left(z^6\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y = c_1\left(-\lambda z+1+\frac{(\lambda-1)\,\lambda\,z^2}{4}-\frac{(\lambda-2)\,(\lambda-1)\,\lambda\,z^3}{36}+\frac{(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^4}{576} - \frac{(\lambda-4)\,(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^5}{14400}+O(z^6)\right)$$
$$+ c_2\left(\left(-\lambda z+1+\frac{(\lambda-1)\,\lambda\,z^2}{4}-\frac{(\lambda-2)\,(\lambda-1)\,\lambda\,z^3}{36}+\frac{(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^4}{576} - \frac{(\lambda-4)\,(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda\,z^5}{14400}+O(z^6)\right)\ln(z)+(1+2\lambda)\,z + \left(-\frac{\lambda}{2}+\frac{1}{4}-\frac{3(\lambda-1)\,\lambda}{4}\right)z^2 + \left(-\frac{(-\lambda+1)\,\lambda}{36}-\frac{(-\lambda+2)\,\lambda}{36}+\frac{(-\lambda+2)\,(-\lambda+1)}{36}+\frac{11(-\lambda+2)\,(-\lambda+1)\,\lambda}{108}\right)z^3 + \left(-\frac{(\lambda-2)\,(\lambda-1)\,\lambda}{576}-\frac{(\lambda-3)\,(\lambda-1)\,\lambda}{576}-\frac{(\lambda-3)\,(\lambda-2)\,\lambda}{576} - \frac{(\lambda-3)\,(\lambda-2)\,(\lambda-1)}{576}-\frac{25(\lambda-3)\,(\lambda-2)\,(\lambda-1)\,\lambda}{3456}\right)z^4 + \left(-\frac{(-\lambda+3)\,(-\lambda+2)\,(-\lambda+1)\,\lambda}{14400}-\frac{(-\lambda+4)\,(-\lambda+2)\,(-\lambda+1)\,\lambda}{14400} - \frac{(-\lambda+4)\,(-\lambda+3)\,(-\lambda+1)\,\lambda}{14400}-\frac{(-\lambda+4)\,(-\lambda+3)\,(-\lambda+2)\,\lambda}{14400} + \frac{(-\lambda+4)\,(-\lambda+3)\,(-\lambda+2)\,(-\lambda+1)}{14400} + \frac{137(-\lambda+4)\,(-\lambda+3)\,(-\lambda+2)\,(-\lambda+1)\,\lambda}{432000}\right)z^5+O(z^6)\right)$$

Verified OK.

3.13.1 Maple step by step solution

Let's solve

$zy'' + (1-z)\,y' + \lambda y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(z-1)y'}{z} - \frac{\lambda y}{z}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{(z-1)y'}{z} + \frac{\lambda y}{z} = 0$

□ Check to see if $z_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(z) = -\frac{z-1}{z}, P_3(z) = \frac{\lambda}{z}\right]$

- $z \cdot P_2(z)$ is analytic at $z = 0$

  $\left(z \cdot P_2(z)\right)\Big|_{z=0} = 1$

- $z^2 \cdot P_3(z)$ is analytic at $z = 0$

  $\left(z^2 \cdot P_3(z)\right)\Big|_{z=0} = 0$

- $z = 0$is a regular singular point

  Check to see if $z_0 = 0$ is a regular singular point

  $z_0 = 0$

• Multiply by denominators

  $zy'' + (1 - z)\, y' + \lambda y = 0$

• Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k z^{k+r}$

□ Rewrite ODE with series expansions

- Convert $z^m \cdot y'$ to series expansion for $m = 0..1$

  $z^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, z^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $z^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, z^{k+r}$

- Convert $z \cdot y''$ to series expansion

  $z \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, z^{k+r-1}$

- Shift index using $k- >k+1$

  $z \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, z^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r^2 z^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(k+r-\lambda)\right) z^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 - a_k(k-\lambda) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k-\lambda)}{(k+1)^2}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k-\lambda)}{(k+1)^2}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k z^k, a_{k+1} = \frac{a_k(k-\lambda)}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 309

```
Order:=6;
dsolve(z*diff(y(z),z$2)+(1-z)*diff(y(z),z)+lambda*y(z)=0,y(z),type='series',z=0);
```

$$
\begin{aligned}
y(z) = {} & \left((2\lambda+1)\,z+\left(\frac{1}{4}\lambda+\frac{1}{4}-\frac{3}{4}\lambda^2\right)z^2+\left(-\frac{2}{9}\lambda^2+\frac{1}{27}\lambda+\frac{1}{18}+\frac{11}{108}\lambda^3\right)z^3\right. \\
& +\left(\frac{7}{192}\lambda^3-\frac{167}{3456}\lambda^2+\frac{1}{192}\lambda+\frac{1}{96}-\frac{25}{3456}\lambda^4\right)z^4 \\
& \left.+\left(\frac{1}{1500}\lambda-\frac{37}{4320}\lambda^2+\frac{719}{86400}\lambda^3+\frac{1}{600}-\frac{61}{21600}\lambda^4+\frac{137}{432000}\lambda^5\right)z^5+\mathrm{O}\left(z^6\right)\right)c_2 \\
& +\left(1-\lambda z+\frac{1}{4}(-1+\lambda)\,\lambda z^2-\frac{1}{36}(\lambda-2)\,(-1+\lambda)\,\lambda z^3\right. \\
& +\frac{1}{576}(\lambda-3)\,(\lambda-2)\,(-1+\lambda)\,\lambda z^4-\frac{1}{14400}(\lambda-4)\,(\lambda-3)\,(\lambda-2)\,(-1+\lambda)\,\lambda z^5 \\
& \left.+\mathrm{O}\left(z^6\right)\right)(c_2\ln(z)+c_1)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 415

```
AsymptoticDSolveValue[z*y''[z]+(1-z)*y'[z]+\[Lambda]*y[z]==0,y[z],{z,0,5}]
```

$$y(z) \to c_1\left(-\frac{(\lambda-4)(\lambda-3)(\lambda-2)(\lambda-1)\lambda z^5}{14400}+\frac{1}{576}(\lambda-3)(\lambda-2)(\lambda-1)\lambda z^4-\frac{1}{36}(\lambda-2)(\lambda-1)\lambda z^3+\frac{1}{4}(\lambda-1)\lambda z^2-\lambda z+1\right)+c_2\left(\frac{(\lambda-4)(\lambda-3)(\lambda-2)(\lambda-1)z^5}{14400}+\frac{(\lambda-4)(\lambda-3)(\lambda-2)\lambda z^5}{14400}+\frac{(\lambda-4)(\lambda-3)(\lambda-1)\lambda z^5}{14400}+\frac{(\lambda-4)(\lambda-2)(\lambda-1)\lambda z^5}{14400}+\frac{137(\lambda-4)(\lambda-3)(\lambda-2)(\lambda-1)\lambda z^5}{432000}+\frac{(\lambda-3)(\lambda-2)(\lambda-1)\lambda z^5}{14400}-\frac{1}{576}(\lambda-3)(\lambda-2)(\lambda-1)z^4-\frac{1}{576}(\lambda-3)(\lambda-2)\lambda z^4-\frac{1}{576}(\lambda-3)(\lambda-1)\lambda z^4-\frac{25(\lambda-3)(\lambda-2)(\lambda-1)\lambda z^4}{3456}-\frac{1}{576}(\lambda-2)(\lambda-1)\lambda z^4+\frac{1}{36}(\lambda-2)(\lambda-1)z^3+\frac{1}{36}(\lambda-2)\lambda z^3+\frac{11}{108}(\lambda-2)(\lambda-1)\lambda z^3+\frac{1}{36}(\lambda-1)\lambda z^3-\frac{1}{4}(\lambda-1)z^2-\frac{3}{4}(\lambda-1)\lambda z^2-\frac{\lambda z^2}{4}+\left(-\frac{(\lambda-4)(\lambda-3)(\lambda-2)(\lambda-1)\lambda z^5}{14400}+\frac{1}{576}(\lambda-3)(\lambda-2)(\lambda-1)\lambda z^4-\frac{1}{36}(\lambda-2)(\lambda-1)\lambda z^3+\frac{1}{4}(\lambda-1)\lambda z^2-\lambda z+1\right)\log(z)+2\lambda z+z\right)$$

3.14 problem Problem 16.15

Internal problem ID [2543]
Internal file name [OUTPUT/2035_Sunday_June_05_2022_02_45_43_AM_58917765/index.tex]

Book: Mathematical methods for physics and engineering, Riley, Hobson, Bence, second edition, 2002
Section: Chapter 16, Series solutions of ODEs. Section 16.6 Exercises, page 550
Problem number: Problem 16.15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[_Gegenbauer, [_2nd_order, _linear, `_with_symmetry_[0,F(x)]`]]
```

$$\boxed{(-z^2+1)\, y'' - zy' + m^2 y = 0}$$

With the expansion point for the power series method at $z = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (137) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (138) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{m^2 y - z y'}{z^2-1}\\
F_1 &= \frac{dF_0}{dz}\\
&= \frac{\partial F_0}{\partial z} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
&= \frac{\left(\left(m^2+2\right) z^2 - m^2 + 1\right) y' - 3 y m^2 z}{\left(z^2-1\right)^2}\\
F_2 &= \frac{dF_1}{dz}\\
&= \frac{\partial F_1}{\partial z} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1\\
&= \frac{\left(-6m^2 z^3 + 6m^2 z - 6z^3 - 9z\right) y' + y\left(\left(m^2+11\right) z^2 - m^2 + 4\right) m^2}{\left(z^2-1\right)^3}\\
F_3 &= \frac{dF_2}{dz}\\
&= \frac{\partial F_2}{\partial z} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2\\
&= \frac{\left(\left(m^4+35m^2+24\right) z^4 + \left(-2m^4 - 25m^2 + 72\right) z^2 + m^4 - 10m^2 + 9\right) y' - 10 y z\left(\left(m^2+5\right) z^2 - m^2 +}{\left(z^2-1\right)^4}\\
F_4 &= \frac{dF_3}{dz}\\
&= \frac{\partial F_3}{\partial z} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3\\
&= \frac{\left(-15z\left(\left(m^4+15m^2+8\right) z^4 + \left(-2m^4 - 2m^2 + 40\right) z^2 + m^4 - 13m^2 + 15\right) y' + y m^2\left(\left(m^4 + 85m^2 + 2\right.\right.\right.}{\left(z^2-1\right)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $z=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0)\, m^2\\
F_1 &= -y'(0)\, m^2 + y'(0)\\
F_2 &= y(0)\, m^4 - 4y(0)\, m^2\\
F_3 &= y'(0)\, m^4 - 10y'(0)\, m^2 + 9y'(0)\\
F_4 &= -y(0)\, m^6 + 20y(0)\, m^4 - 64y(0)\, m^2
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}m^{2}z^{2}+\frac{1}{24}m^{4}z^{4}-\frac{1}{6}m^{2}z^{4}-\frac{1}{720}z^{6}m^{6}+\frac{1}{36}z^{6}m^{4}-\frac{4}{45}z^{6}m^{2}\right)y(0)\\
&+\left(z-\frac{1}{6}m^{2}z^{3}+\frac{1}{6}z^{3}+\frac{1}{120}m^{4}z^{5}-\frac{1}{12}m^{2}z^{5}+\frac{3}{40}z^{5}\right)y'(0)+O(z^{6})
\end{aligned}
$$

Since the expansion point $z = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(-z^{2}+1)\,y''-zy'+m^{2}y=0
$$

Let the solution be represented as power series of the form

$$
y=\sum_{n=0}^{\infty}a_{n}z^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_{n} z^{n-1}\\
y'' &= \sum_{n=2}^{\infty} n(n-1)\,a_{n} z^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(-z^{2}+1)\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}z^{n-2}\right)-z\left(\sum_{n=1}^{\infty}na_{n}z^{n-1}\right)+m^{2}\left(\sum_{n=0}^{\infty}a_{n}z^{n}\right)=0 \quad (1)
$$

Which simplifies to

$$
\sum_{n=2}^{\infty}\left(-z^{n}a_{n}n(n-1)\right)+\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}z^{n-2}\right)+\sum_{n=1}^{\infty}\left(-na_{n}z^{n}\right)+\left(\sum_{n=0}^{\infty}m^{2}a_{n}z^{n}\right)=0 \quad (2)
$$

The next step is to make all powers of z be n in each summation term. Going over each summation term above with power of z in it which is not already z^{n} and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty}n(n-1)\,a_{n}z^{n-2}=\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,z^{n}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of z are the same and equal to n.

$$\sum_{n=2}^{\infty} \left(-z^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) z^n\right) + \sum_{n=1}^{\infty} (-n a_n z^n) + \left(\sum_{n=0}^{\infty} m^2 a_n z^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$a_0 m^2 + 2a_2 = 0$$

$$a_2 = -\frac{a_0 m^2}{2}$$

$n = 1$ gives

$$a_1 m^2 - a_1 + 6a_3 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{1}{6} a_1 m^2 + \frac{1}{6} a_1$$

For $2 \leq n$, the recurrence equation is

$$-n a_n (n-1) + (n+2) a_{n+2}(n+1) - n a_n + a_n m^2 = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n (m^2 - n^2)}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$a_2 m^2 - 4a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{24} m^4 a_0 - \frac{1}{6} a_0 m^2$$

For $n = 3$ the recurrence equation gives

$$a_3 m^2 - 9a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{120} m^4 a_1 - \frac{1}{12} a_1 m^2 + \frac{3}{40} a_1$$

For $n = 4$ the recurrence equation gives

$$a_4 m^2 - 16a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{1}{720} m^6 a_0 + \frac{1}{36} m^4 a_0 - \frac{4}{45} a_0 m^2$$

For $n = 5$ the recurrence equation gives

$$a_5 m^2 - 25a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{1}{5040} m^6 a_1 + \frac{1}{144} m^4 a_1 - \frac{37}{720} a_1 m^2 + \frac{5}{112} a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n z^n \\ &= a_3 z^3 + a_2 z^2 + a_1 z + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 z - \frac{a_0 m^2 z^2}{2} + \left(-\frac{1}{6} a_1 m^2 + \frac{1}{6} a_1\right) z^3 \\ + \left(\frac{1}{24} m^4 a_0 - \frac{1}{6} a_0 m^2\right) z^4 + \left(\frac{1}{120} m^4 a_1 - \frac{1}{12} a_1 m^2 + \frac{3}{40} a_1\right) z^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{m^2 z^2}{2} + \left(\frac{1}{24}m^4 - \frac{1}{6}m^2\right) z^4\right) a_0 + \left(z + \left(-\frac{m^2}{6} + \frac{1}{6}\right) z^3 + \left(\frac{1}{120}m^4 - \frac{1}{12}m^2 + \frac{3}{40}\right) z^5\right) a_1 + O(z^6) \tag{3}$$

At $z = 0$ the solution above becomes

$$y = \left(1 - \frac{m^2 z^2}{2} + \left(\frac{1}{24}m^4 - \frac{1}{6}m^2\right) z^4\right) c_1 + \left(z + \left(-\frac{m^2}{6} + \frac{1}{6}\right) z^3 + \left(\frac{1}{120}m^4 - \frac{1}{12}m^2 + \frac{3}{40}\right) z^5\right) c_2 + O(z^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}m^2 z^2 + \frac{1}{24}m^4 z^4 - \frac{1}{6}m^2 z^4 - \frac{1}{720}z^6 m^6 + \frac{1}{36}z^6 m^4 - \frac{4}{45}z^6 m^2\right) y(0) + \left(z - \frac{1}{6}m^2 z^3 + \frac{1}{6}z^3 + \frac{1}{120}m^4 z^5 - \frac{1}{12}m^2 z^5 + \frac{3}{40}z^5\right) y'(0) + O(z^6) \tag{1}$$

$$y = \left(1 - \frac{m^2 z^2}{2} + \left(\frac{1}{24}m^4 - \frac{1}{6}m^2\right) z^4\right) c_1 + \left(z + \left(-\frac{m^2}{6} + \frac{1}{6}\right) z^3 + \left(\frac{1}{120}m^4 - \frac{1}{12}m^2 + \frac{3}{40}\right) z^5\right) c_2 + O(z^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}m^2 z^2 + \frac{1}{24}m^4 z^4 - \frac{1}{6}m^2 z^4 - \frac{1}{720}z^6 m^6 + \frac{1}{36}z^6 m^4 - \frac{4}{45}z^6 m^2\right) y(0) + \left(z - \frac{1}{6}m^2 z^3 + \frac{1}{6}z^3 + \frac{1}{120}m^4 z^5 - \frac{1}{12}m^2 z^5 + \frac{3}{40}z^5\right) y'(0) + O(z^6)$$

Verified OK.

$$y = \left(1 - \frac{m^2 z^2}{2} + \left(\frac{1}{24}m^4 - \frac{1}{6}m^2\right) z^4\right) c_1 + \left(z + \left(-\frac{m^2}{6} + \frac{1}{6}\right) z^3 + \left(\frac{1}{120}m^4 - \frac{1}{12}m^2 + \frac{3}{40}\right) z^5\right) c_2 + O(z^6)$$

Verified OK.

3.14.1 Maple step by step solution

Let's solve

$(-z^2+1)\, y''-z y'+m^2 y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{z y'}{z^2-1}+\frac{m^2 y}{z^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{z y'}{z^2-1}-\frac{m^2 y}{z^2-1}=0$

☐ Check to see if z_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(z)=\frac{z}{z^2-1}, P_3(z)=-\frac{m^2}{z^2-1}\right]$

  ◦ $(z+1)\cdot P_2(z)$ is analytic at $z=-1$

  $\left((z+1)\cdot P_2(z)\right)\bigg|_{z=-1}=\frac{1}{2}$

  ◦ $(z+1)^2\cdot P_3(z)$ is analytic at $z=-1$

  $\left((z+1)^2\cdot P_3(z)\right)\bigg|_{z=-1}=0$

  ◦ $z=-1$is a regular singular point

  Check to see if z_0 is a regular singular point

  $z_0=-1$

- Multiply by denominators

  $y''(z^2-1)+z y'-m^2 y=0$

- Change variables using $z=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)-m^2 y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^{m}\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=1..2$

  $$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_{0}r(-1+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(2k+1+2r)+a_{k}(k+m+r)\,(k-m+r)\right)u^{k+}\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$-r(-1+2r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{0,\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-2(k+1+r)\left(k+\frac{1}{2}+r\right)a_{k+1}+a_{k}(k+m+r)\,(k-m+r)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1}=\frac{a_{k}(k+m+r)(k-m+r)}{(k+1+r)(2k+1+2r)}$$

- Recursion relation for $r=0$

  $$a_{k+1}=\frac{a_{k}(k+m)(k-m)}{(k+1)(2k+1)}$$

- Solution for $r=0$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k},a_{k+1}=\frac{a_{k}(k+m)(k-m)}{(k+1)(2k+1)}\right]$$

- Revert the change of variables $u=z+1$

  $$\left[y=\sum_{k=0}^{\infty}a_{k}(z+1)^{k},a_{k+1}=\frac{a_{k}(k+m)(k-m)}{(k+1)(2k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k \left(k+m+\frac{1}{2}\right) \left(k-m+\frac{1}{2}\right)}{\left(k+\frac{3}{2}\right) (2k+2)}$

- Solution for $r = \frac{1}{2}$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k \left(k+m+\frac{1}{2}\right) \left(k-m+\frac{1}{2}\right)}{\left(k+\frac{3}{2}\right) (2k+2)}\right]$$

- Revert the change of variables $u = z + 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (z+1)^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k \left(k+m+\frac{1}{2}\right) \left(k-m+\frac{1}{2}\right)}{\left(k+\frac{3}{2}\right) (2k+2)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k (z+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (z+1)^{k+\frac{1}{2}}\right), a_{k+1} = \frac{a_k (k+m) (k-m)}{(k+1) (2k+1)}, b_{k+1} = \frac{b_k \left(k+m+\frac{1}{2}\right) \left(k-m+\frac{1}{2}\right)}{\left(k+\frac{3}{2}\right) (2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 71

```
Order:=6;
dsolve((1-z^2)*diff(y(z),z$2)-z*diff(y(z),z)+m^2*y(z)=0,y(z),type='series',z=0);
```

$$y(z) = \left(1 - \frac{m^2 z^2}{2} + \frac{m^2 (m^2-4) z^4}{24}\right) y(0) + \left(z - \frac{(m^2-1) z^3}{6} + \frac{(m^4 - 10m^2 + 9) z^5}{120}\right) D(y)(0) + O(z^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 88

```
AsymptoticDSolveValue[(1-z^2)*y''[z]-z*y'[z]+m^2*y[z]==0,y[z],{z,0,5}]
```

$$y(z) \to c_2\left(\frac{m^4z^5}{120} - \frac{m^2z^5}{12} - \frac{m^2z^3}{6} + \frac{3z^5}{40} + \frac{z^3}{6} + z\right) + c_1\left(\frac{m^4z^4}{24} - \frac{m^2z^4}{6} - \frac{m^2z^2}{2} + 1\right)$$